On a property of orthogonal arrays and optimal blocking of fractional factorial plans

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Abstract. It is shown that fractional factorial plans represented by orthogonal arrays of strength three are universally optimal under a model that includes the mean, all main effects and all two-factor interactions between a specified factor and each of the other factors. Thus, such plans exhibit a kind of model robustness in being universally optimal under two different models. Procedures for obtaining universally optimal block designs for fractional factorial plans represented by orthogonal arrays are also discussed.

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1. Introduction

The study of optimal fractional factorial plans has received considerable attention over the last two decades; see Dey and Mukerjee ((1999a); Chapters 2, 6 and 7) for a review. Most of these results however relate to situations where all factorial effects involving the same number of factors are considered equally important and, as such, the underlying model involves the general mean and all effects involving up to a specified number of factors. Also, most of these have been obtained in unblocked situations.

In practice however, the presumption of equality in the importance of all factorial effects involving the same number of factors may not be an appropriate one. For example, there may be a situation where it is known *a priori* that only one of the factors can possibly interact with each of the factors, all other two-factor and higher order interactions being absent. The model then includes the general mean, all main effects and only *some* but not all two-factor interactions. The issue of estimability and optimality in situations of this kind in the context of two-level factorials has been addressed by Hedayat and Pesotan (1992, 1997) and Chiu and John (1998). Continuing with this line of research, Dey and Mukerjee (1999b) have shown that under a hierarchical model, if a fractional factorial plan for an arbitrary factorial has inter-effect orthogonality, then it is universally optimal in a relevant class of competing designs. Here the term universal optimality is used in the sense of Kiefer (1975) and Sinha and Mukerjee (1982). Dey and Mukerjee (1999b) also obtain a necessary and sufficient combinatorial condition that ensures inter-effect orthogonality and hence, universal optimality under a hierarchical model.

In this paper, further results on the optimality of fractional factorial plans for arbitrary factorials are obtained. In Section 2, some preliminaries are introduced. In Section 3, it is shown that fractional factorial plans represented by orthogonal arrays of strength three are universally optimal under a model that includes the general mean, all main effects and all two-factor interactions between a specified factor and each of the other factors. Recall that fractional factorial plans represented by orthogonal arrays of strength three are also universally optimal under a model that includes the mean, all main effects and all two-factor interactions and the parameters of interest are contrasts belonging to the main effects, the two-factor interactions in the model acting as nuisance parameters. Our result thus shows that such plans are universally optimal under two different models. Finally in Section 4, procedures for blocking fractional factorial plans based on orthogonal arrays are discussed. These blocked designs are also seen to be universally optimal.

2. Preliminaries

Consider the set up of an $m_1 \times \cdots \times m_n$ factorial experiment involving n factors $F_1, \ldots, F_n$ appearing at $m_1, \ldots, m_n$ levels respectively ($m_i \geq 2$, $i = 1, \ldots, n$). The $v = \prod_{i=1}^{n} m_i$ treatment combinations are represented by ordered n-tuples $j_1 \ldots j_n$ ($j_i = 0, \ldots, m_i - 1; i = 1, \ldots, n$). Let $\boldsymbol{\tau}$ denote the $v \times 1$ vector with elements $\tau(j_1 \ldots j_n)$ arranged in the lexicographic order, where $\tau(j_1 \ldots j_n)$ is the fixed effect of the treatment combination $j_1 \ldots j_n$. Also, let Ω denote the set of all binary n-tuples. For each $\boldsymbol{x} = x_1 \ldots x_n \in \Omega$, define $\alpha(\boldsymbol{x}) = \prod_{i=1}^{n} (m_i - 1)^{x_i}$.

For a positive integer s, we denote the $s \times 1$ vector of all ones by $\mathbf{1}_s$ and the identity matrix of order s by I_s. For $i = 1, \ldots, n$, let P_i be an $(m_i - 1) \times m_i$ matrix such that the $m_i \times m_i$ matrix $(m_i^{-1/2}\mathbf{1}_{m_i}, P_i')$ is orthogonal. For each $\boldsymbol{x} = x_1 \ldots x_n \in \Omega$, let the $\alpha(\boldsymbol{x}) \times v$ matrix $P^{\boldsymbol{x}}$ be defined as

$$P^{\boldsymbol{x}} = P_1^{x_1} \otimes \cdots \otimes P_n^{x_n}, \tag{2.1}$$

where for $i = 1, \ldots, n$,

$$P_i^{x_i} = \begin{cases} m_i^{-1/2}\mathbf{1}_{m_i}', & \text{if } x_i = 0 \\ P_i, & \text{if } x_i = 1, \end{cases} \tag{2.2}$$

and $\otimes$ denotes the Kronecker product. Then it is well known that for each $\boldsymbol{x} = x_1 \ldots x_n \in \Omega$, $\boldsymbol{x} \neq 00 \ldots 0$, the elements of $P^{\boldsymbol{x}}\boldsymbol{\tau}$ represent a complete set of orthonormal contrasts belonging to the factorial effect $F_1^{x_1} \ldots F_n^{x_n} \equiv F^{\boldsymbol{x}}$, say. Also $P^{00\ldots0}\boldsymbol{\tau} = v^{1/2}\bar{\tau}$, where $\bar{\tau}$ is the general mean, and in this sense the general mean will be represented by $F^{00\ldots0}$.

In this paper, we consider only hierarchical factorial models. Recall that hierarchical factorial models are such that if a factorial effect $F^{\boldsymbol{x}}$ is included in the model then so is $F^{\boldsymbol{y}}$ for every $\boldsymbol{y} \in \Omega$ satisfying $\boldsymbol{y} \leq \boldsymbol{x}$, where $\boldsymbol{y} \leq \boldsymbol{x}$ means $y_i \leq x_i$ for $i = 1, \ldots, n$.

Let $\Gamma \subset \Omega$ be such that $F^{\boldsymbol{x}}$ is included in the model if and only if $\boldsymbol{x} \in \Gamma$. The parametric functions of interest are then $P\boldsymbol{\tau}$, where

$$P = (\ldots, (P^{\boldsymbol{x}})', \ldots)'_{\boldsymbol{x} \in \Gamma}. \tag{2.3}$$

Assuming that the observations are homoscedastic and uncorrelated, the information matrix for $P\boldsymbol{\tau}$, under a plan d, is given by

$$\mathscr{I}_d = PR_dP', \tag{2.4}$$

where R_d is the $v \times v$ diagonal matrix with diagonal elements $r_d(j_1 \ldots j_n)$, arranged lexicographically and $r_d(j_1 \ldots j_n)$ is the replication of the treatment combination $j_1 \ldots j_n$ in d.

Now consider a hierarchical model specified by $\Gamma \subset \Omega$. For any $\boldsymbol{x} = x_1 \ldots x_n$ and $\boldsymbol{z} = z_1 \ldots z_n$, both members of Γ, let

$$S(\boldsymbol{x}, \boldsymbol{z}) = \{i\text{: either } x_i = 1 \text{ or } z_i = 1\}.$$

Define

$$\bar{\Gamma} = \{\boldsymbol{x} : \boldsymbol{x} \in \Gamma, \text{ there does not exist } \boldsymbol{y} \in \Gamma \text{ such that } \boldsymbol{x} \leq \boldsymbol{y} \text{ and } \boldsymbol{x} \neq \boldsymbol{y}\}.$$

Let $\mathscr{D}$ be the class of all N-run plans for an $m_1 \times \cdots \times m_n$ factorial such that all effects in the model specified by Γ are estimable via any $d \in \mathscr{D}$. The following result from Dey and Mukerjee (1999b), giving a combinatorial characterization for a plan to have inter-effect orthogonality under a hierarchical model, will be needed in the sequel.

Theorem 2.1. *Under a hierarchical model specified by Γ, a fractional factorial plan $d \in \mathscr{D}$ has inter-effect orthogonality (and hence, d is universally optimal over $\mathscr{D}$) if and only if for every $\boldsymbol{x}, \boldsymbol{z} \in \bar{\Gamma}$, all level combinations of the factors $\{F_i : i \in S(\boldsymbol{x}, \boldsymbol{z})\}$ appear equally often in d.*

3. A property of orthogonal arrays of strength three

An orthogonal array $OA(N, n, m_1 \times \cdots \times m_n, g)$, having N rows, n columns, $m_1, \ldots, m_n$ symbols and strength g is an $N \times n$ matrix with elements in the ith column from a set of $m_i \geq 2$ distinct symbols $(i = 1, \ldots, n)$ in which all possible combinations of symbols appear equally often as rows in every $N \times g$ subarray. In this section, we deal with orthogonal arrays of strength three, i.e., $g = 3$.

Consider a factorial experiment with n factors $F_1, \ldots, F_n$, the ith factor F_i appearing at m_i (≥ 2) levels, $i = 1, \ldots, n$. Suppose it is desired to estimate (i) the mean, (ii) complete sets of orthonormal treatment contrasts belonging to all main effects and (iii) all two-factor interactions between a chosen factor and all the rest, i.e., two-factor interactions of the type F_1F_j, $j = 2, \ldots, n$, where, without loss of generality, we consider F_1 as the chosen factor. All other effects are assumed to be zero. The model then is clearly a hierarchical model. From Theorem 2.1, it follows then that a fractional factorial plan d, involving N runs under this model is universally optimal in the class of all N-run plans if and only if in d, all level combinations of the following sets of factors appear equally often:

$$\{F_1F_jF_k\}, \quad 2 \le j < k \le n.$$

Now, if d is represented by an orthogonal array, $OA(N, n, m_1 \times \cdots \times m_n, 3)$, then clearly d satisfies the requirement above and thus, d is a universally optimal plan under the stated model.

Furthermore, it is known that in an $OA(N, n, m_1 \times \cdots \times m_n, 3)$,

$$N \ge 1 + \sum_{i=1}^{n}(m_i - 1) + (m^* - 1)\left\{\sum_{i=1}^{n}(m_i - 1) - (m^* - 1)\right\}, \tag{3.1}$$

where $m^* = \max_i m_i$. Arrays for which N attains the lower bound (3.1) are known as tight. Tight orthogonal arrays of strength 3 are available in the literature; see e.g. Dey and Mukerjee (1999a, Chapters 3 and 4). If $m_1 = \max_i m_i$, and the $OA(N, n, m_1 \times \cdots \times m_n, 3)$ is tight, then the plan represented by the tight array $OA(N, n, m_1 \times \cdots \times m_n, 3)$ is a universally optimal *saturated* plan in the sense that the number of experimental units in the plan equals the number of parameters in the model. We thus have the following result.

Theorem 3.1. *Under a factorial model that includes the mean, all main effects and all two-factor interactions among a specified factor, say F_1, and all the other factors, a fractional factorial plan represented by an orthogonal array of strength three is universally optimal. Furthermore, if the orthogonal array is tight and F_1 has the largest number of levels, then the plan is also saturated.*

Remark. Theorem 3.1 shows that fractional factorial plans, represented by orthogonal arrays are universally optimal under two different models. Also, since the choice of the chosen factor F_1 of Theorem 3.1 is arbitrary, the same plan remains universally optimal no matter which of the n factors is considered as the specified factor. Thus, fractional factorial plans represented by orthogonal arrays of strength three are superior to the ones reported by Dey and Mukerjee (1999b) in their Example 4.

4. Blocking of plans represented by orthogonal arrays

When the number of runs N in a fractional factorial plan is large, the sensitivity of the experiment can be increased by grouping the experimental units into blocks, so that the units within each block are homogeneous, though there may be variation from one block to another. Suppose there exists an N-run fractional factorial plan for an $m_1 \times \cdots \times m_n$ factorial, where $N = bk$ for some positive integers $b \ge 2$ and $k \ge 2$. Further, suppose the plan ensures the estimability of all factorial effects involving f factors or less under the assumption that all effects involving $t+1$ factors or more are absent, where f, t are integers, $1 \le f \le t \le n-1$; that is, such a plan is a Resolution (f, t) plan (*cf.* Dey and Mukerjee (1999a)). It is desired to obtain a block design, involving b blocks of size k each, such that the resultant block design is universally optimal.

Suppose an N-run fractional factorial plan d is represented by an orthogonal array $OA(N, n, m_1 \times \cdots \times m_n, g)$, where $2 \le g \le n$. Such a plan is universally optimal in the class of all N-run Resolution (f, t) plans for every choice of integers f, t such that $f+t=g$ and $1 \le f \le t \le n-1$. Assume that the runs of the plan are grouped into blocks. Let $\mathscr{I}_d^{block}$ denote the information matrix (for factorial effects) of the Resolution (f, t) plan d under a model that includes the block parameters along with the factorial effects and, $\mathscr{I}_d$, the information matrix of d under a model without the block effects. It can be seen that $\mathscr{I}_d^{block} \le \mathscr{I}_d$, where for two nonnegative definite matrices A and B, $A \le B$ means that $B-A$ is nonnegative definite. Under certain conditions (see Theorem 4.1 below), it is possible to have $\mathscr{I}_d^{block} = \mathscr{I}_d$.

Let $\mathscr{D}(b, k)$ be the class of all plans for an $m_1 \times \cdots \times m_n$ factorial involving N $(=bk)$ experimental units grouped into b (≥ 2) blocks each containing k (≥ 2) units. The following result gives a necessary and sufficient condition for $\mathscr{I}_d^{block}$ to be equal to $\mathscr{I}_d$ (see Dey and Mukerjee (1999a), p. 159).

Theorem 4.1. *Let $d_0 \in \mathscr{D}(b, k)$ be a fractional factorial plan represented by an orthogonal array $OA(N, n, m_1 \times \cdots \times m_n, f+t)$.*

(a) *Then $\mathscr{I}_{d_0}^{block} = \mathscr{I}_{d_0}$ if and only if for every $i_1, \ldots, i_f$ and $j_{i_1}, \ldots, j_{i_f}$ $(1 \le i_1 < \cdots < i_f \le n; 0 \le j_{i_1} \le m_{i_1}-1, \ldots, 0 \le j_{i_f} \le m_{i_f}-1)$, the level combination $j_{i_1} \ldots j_{i_f}$ of the i_1th, $\ldots$, i_fth factors appears equally often in the b blocks under the plan d_0.*

(b) *If the condition stated in (a) above holds, then d_0 is a universally optimal Resolution (f, t) plan in $\mathscr{D}(b, k)$.*

We may call the blocking arrangement satisfying Theorem 4.1(a) as *orthogonal* blocking. A method to achieve orthogonal blocking is as follows. With $N = bk$, suppose that an orthogonal array $L_1 \equiv OA(N, n+1, m_1 \times \cdots \times m_n \times b, g)$ is available. One may identify each row of the subarray defined by the first n columns with a treatment combination of an $m_1 \times \cdots \times m_n$ factorial and each symbol in the last column with a block. If a typical row of L_1 is $j_1 \ldots j_n u$, then a (block) design, say d_0, is obtained by assigning the treatment combination $j_1 \ldots j_n$ to the uth block. It can be verified that d_0 satisfies Theorem 4.1(a) for every choice of integers f, t such that $f+t=g$ and $1 \le f \le t \le n-1$. Hence by Theorem 4.1(b), d_0 is a universally optimal Resolution (f, t) plan in $\mathscr{D}(b, k)$ for every choice of integers f, t such that $f+t=g$ and $1 \le f \le t \le n-1$.

We now discuss some alternatives for achieving orthogonal blocking of plans based on orthogonal arrays. We first have the following result, whose proof follows from Theorem 4.1.

Theorem 4.2. *For given integers f_0, t_0, $1 \le f_0 \le t_0 \le n-1$, let there exist a fractional factorial plan represented by an orthogonal array $L \equiv OA(N, n, m_1 \times \cdots \times m_n, f_0+t_0)$. Suppose $d_0 \in \mathscr{D}(b, k)$ is a block design for this plan where $bk = N$. Then,*

(a) $\mathscr{I}_{d_0}^{block} = \mathscr{I}_{d_0}$ *if and only if there exists an orthogonal array* $L_1 \equiv OA(N, n+1, m_1 \times \cdots \times m_n \times b, f_0 + 1)$, *obtained by augmenting* L *by a column with* b *distinct symbols, and in such a case, if a typical row of* L_1 *is* $j_1 \ldots j_n u$, *then the block design* d_0 *is obtained by assigning the treatment combination* $j_1 \ldots j_n$ *to the uth block,* $u = 1, \ldots, b$.

(b) *If the condition stated in (a) above holds, then* d_0 *is a universally optimal Resolution* (f, t) *plan in* $\mathscr{D}(b, k)$ *for every choice of integers* f, t *such that* $f + t = f_0 + t_0$ *and* $1 \leq f \leq f_0$.

Corollary 4.1. *For a given integer* g, $2 \leq g \leq n$, *let there exist a fractional factorial plan represented by an orthogonal array* $L \equiv OA(N, n, m_1 \times \cdots \times m_n, g)$. *Suppose* $d_0 \in \mathscr{D}(b, k)$ *is a block design for this plan where* $bk = N$. *Then,*

(a) $\mathscr{I}_{d_0}^{block} = \mathscr{I}_{d_0}$ *if and only if there exists an orthogonal array* $L_1 \equiv OA(N, n+1, m_1 \times \cdots \times m_n \times b, [g/2] + 1)$, *obtained by augmenting* L *by a column with* b *distinct symbols, where* $[\cdot]$ *is the greatest integer function and in such a case, if a typical row of* L_1 *is* $j_1 \ldots j_n u$, *then the block design* d_0 *is obtained by assigning the treatment combination* $j_1 \ldots j_n$ *to the uth block,* $u = 1, \ldots, b$.

(b) *If the condition stated in (a) above holds, then* d_0 *is a universally optimal Resolution* (f, t) *plan in* $\mathscr{D}(b, k)$ *for every choice of integers* f, t *such that* $f + t = g$ *and* $1 \leq f \leq t \leq n - 1$.

We illustrate the above discussion via two examples.

Example 4.1. Consider the orthogonal array $L_1 \equiv OA(8, 5, 2^4 \times 4, 2)$, displayed below.

$$L_1 = OA(8, 5, 2^4 \times 4, 2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 3 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

Here, $m_1 = \cdots = m_4 = 2$, $b = 4$ and observe that the first four columns form a (symmetric) $OA(8, 4, 2, 3) \equiv L$. By following Theorem 4.2 or Corollary 4.1, we get the following design involving 4 blocks of size 2 each, which is a universally optimal Resolution $(1, 2)$ plan in $\mathscr{D}(4, 2)$:

Block 1	Block 2	Block 3	Block 4
0000	0011	1010	0110
1111	1100	0101	1001

Example 4.2. Consider an orthogonal array $L_1 \equiv OA(16, 6, 2^5 \times 4, 2)$ given below (in transposed form)

$$L_1 = \begin{bmatrix} 0101 & 0110 & 0110 & 0011 \\ 0011 & 0011 & 0011 & 0101 \\ 0011 & 0011 & 1100 & 1010 \\ 0011 & 1100 & 0011 & 1010 \\ 0110 & 1010 & 1010 & 0110 \\ 0000 & 1111 & 2222 & 3333 \end{bmatrix}'.$$

Here, $N = 16$ and $f_0 = 1$. The first five columns of L_1 form an $OA(16, 5, 2, 4) \equiv L$ with $f_0 = 1$, $t_0 = 3$. Blocking with respect to the sixth column, we get an universally optimal Resolution $(1, 3)$ plan $d_0 \in \mathscr{D}(4, 4)$ involving five factors each at two levels. Note that d_0 is not a universally optimal Resolution $(2, 2)$ plan. In fact, it is not possible to obtain an orthogonally blocked Resolution $(2, 2)$ plan in $\mathscr{D}(4, 4)$ for a 2^5 factorial represented by an orthogonal array.

Some further methods of orthogonal blocking of plans represented by orthogonal arrays are discussed next. Let $N = bk$ where $b = \prod_{i=1}^{s} b_i$. Suppose that an orthogonal array $L_1 \equiv OA(N, n+s, m_1 \times \cdots \times m_n \times b_1 \times \cdots \times b_s, f_0 + t_0)$, where $1 \leq s \leq t_0 \leq n-1$, $1 \leq f_0 \leq t_0$, is available. One may identify each row of the subarray defined by the first n columns with a treatment combination of an $m_1 \times \cdots \times m_n$ factorial and each symbol combination in the last s columns with a block. If a typical row of L_1 is $j_1 \ldots j_n u_1 \ldots u_s$, then a (block) design, say d_0, is obtained by assigning the treatment combination $j_1 \ldots j_n$ to the $(u_1 \ldots u_s)$th block. It can be verified that d_0 satisfies Theorem 4.2 and hence, for an $m_1 \times \cdots \times m_n$ factorial, d_0 is a universally optimal Resolution (f, t) plan in $\mathscr{D}(b, k)$ for every choice of f, t satisfying $f + t = f_0 + t_0$ and $1 \leq f \leq f_0$. Thus we have

Theorem 4.3. *Let* $L_1 \equiv OA(N, n+s, m_1 \times \cdots \times m_n \times b_1 \times \cdots \times b_s, f_0 + t_0)$, $1 \leq f_0 \leq t_0 \leq n-1$, *be an orthogonal array with* $N = bk$ *where* $b = \prod_{i=1}^{s} b_i$ *and* s *is an integer,* $1 \leq s \leq t_0$. *Then, for a* $m_1 \times \cdots \times m_n$ *factorial, one can obtain a universally optimal Resolution* (f, t) *plan in* $\mathscr{D}(b, k)$ *for every choice of* f, t *satisfying* $f + t = f_0 + t_0$ *and* $1 \leq f \leq f_0$.

Corollary 4.2. *Let* $L_1 \equiv OA(N, n+s, m_1 \times \cdots \times m_n \times b_1 \times \cdots \times b_s, g)$ *be an orthogonal array with* $N = bk$ *where* $b = \prod_{i=1}^{s} b_i$ *and* s *is an integer,* $1 \leq s \leq g - [g/2]$. *Then, for a* $m_1 \times \cdots \times m_n$ *factorial, one can obtain a universally optimal Resolution* (f, t) *plan in* $\mathscr{D}(b, k)$ *for every choice of integers* f, t *such that* $f + t = g$ *and* $1 \leq f \leq t \leq n-1$.

Next, let $N = bk$ where $b = \prod_{i=1}^{p} b_i$. Suppose that an orthogonal array $L_1 \equiv OA(N, n+p, m_1 \times \cdots \times m_n \times b_1 \times \cdots \times b_p, f_0 + t_0)$, is available, where f_0, t_0 are integers satisfying $1 \leq f_0 \leq t_0 \leq n-1$ and $p > t_0$. Suppose for $f \leq f_0$, L_1 has the additional property that for every $i_1, \ldots, i_f$ and $j_{i_1}, \ldots, j_{i_f}, j_{n+1}, \ldots, j_{n+p}$ $(1 \leq i_1 < \cdots < i_f \leq n; 0 \leq j_{i_1} \leq m_{i_1} - 1, \ldots, 0 \leq j_{i_f} \leq m_{i_f} - 1; 0 \leq j_{n+1} \leq m_{n+1} - 1, \ldots, 0 \leq j_{n+p} \leq m_{n+p} - 1)$, the combination $j_{i_1} \ldots j_{i_f} j_{n+1} \ldots j_{n+p}$ under the i_1th, $\ldots, i_f$th, $(n+1)$th, $\ldots, (n+p)$th columns appears equally often as a row in L_1. One may identify each row of the subarray defined by the first n columns with a treatment combination of an $m_1 \times \cdots \times m_n$ factorial and each symbol combination in the last p columns with a block. If a typical row of L_1 is $j_1 \ldots j_n u_1 \ldots u_p$, then a (block) design, say d_0, is obtained by assigning the treatment combination $j_1 \dots j_n$ to the $(u_1 \dots u_p)$th block. It can be verified that d_0 satisfies Theorem 4.2 and hence, d_0 is a universally optimal Resolution (f,t) plan in $\mathscr{D}(b,k)$ for every choice of f,t satisfying $f+t=f_0+t_0$ and $1 \le f \le f_0$. Thus we have

Theorem 4.4. *Let $L_1 \equiv OA(N, n+p, m_1 \times \cdots \times m_n \times b_1 \times \cdots \times b_p, f_0+t_0)$ be an orthogonal array, with $N = bk$ where $b = \prod_{i=1}^{p} b_i$, $p > t_0$ and f_0, t_0 are integers satisfying $1 \le f_0 \le t_0 \le n-1$. Suppose L_1 has the additional property that for every $i_1, \dots, i_f$ and $j_{i_1}, \dots, j_{i_f}, j_{n+1}, \dots, j_{n+p}$ $(1 \le i_1 < \cdots < i_f \le n$; $0 \le j_{i_1} \le m_{i_1}-1, \dots, 0 \le j_{i_f} \le m_{i_f}-1$; $0 \le j_{n+1} \le m_{n+1}-1, \dots, 0 \le j_{n+p} \le m_{n+p}-1)$, the combination $j_{i_1} \dots j_{i_f} j_{n+1} \dots j_{n+p}$ under the $i_1th, \dots, i_fth$, $(n+1)th, \dots, (n+p)th$ columns appears equally often as a row in L_1. Then, one can obtain a universally optimal Resolution (f,t) plan in $\mathscr{D}(b,k)$ for every choice of f,t satisfying $f+t=f_0+t_0$ and $1 \le f \le f_0$.*

Example 4.3. Consider the orthogonal array $L_1 \equiv OA(16,7,2,3)$. This array can be constructed as follows: Let A be an orthogonal array $OA(8,7,2,2)$ and let the symbols in A be 0 and 1. Then L_1 is given by $L_1 = \binom{A}{J-A}$, where J is matrix of all ones. The array L_1, in transposed form, is displayed below.

$$L_1 = \begin{bmatrix} 1010 & 1010 & 0101 & 0101 \\ 1100 & 1100 & 0011 & 0011 \\ 1001 & 1001 & 0110 & 0110 \\ 1111 & 0000 & 0000 & 1111 \\ 1010 & 0101 & 0101 & 1010 \\ 1100 & 0011 & 0011 & 1100 \\ 1001 & 0110 & 0110 & 1001 \end{bmatrix}'.$$

Here, $N = 16$, $f_0 = 1$ and $t_0 = 2$. Then for $s = 1,2$, using Theorem 4.3 we get a plan for 2^{7-s} factorial d_0 in 2^s blocks of size $16/2^s$ each by blocking with respect to the last s columns. Furthermore, d_0 is a universally optimal Resolution $(1,2)$ plan in $\mathscr{D}(2^s, 2^{4-s})$.

Now, observe that L_1 as given above has the additional property that the combinations under the first three columns and each one of the remaining four columns appears equally often as a row in L_1. Thus, with $p = 3 > 2 = t_0$, from Theorem 4.4 we get a universally optimal Resolution $(1,2)$ plan in $\mathscr{D}(8,2)$ involving four factors each at two levels:

Block 1	Block 2	Block 3	Block 4
0000	1001	1010	0011
1111	0110	0101	1100
Block 5	**Block 6**	**Block 7**	**Block 8**
1100	0101	0110	1111
0011	1010	1001	0000

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