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OPTIMAL FRACTIONAL FACTORIAL PLANS FOR MAIN EFFECTS AND SPECIFIED TWO-FACTOR INTERACTIONS: A PROJECTIVE GEOMETRIC APPROACH

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Finite projective geometry is used to obtain fractional factorial plans for m-level symmetrical factorial experiments, where m is a prime or a prime power. Under a model that includes the mean, all main effects and a specified set of two-factor interactions, the plans are shown to be universally optimal within the class of all plans involving the same number of runs.

1. Introduction. Considerable amount of work on the optimality of fractional factorial plans has been carried out in the last two decades. For a recent review of optimality of fractional factorial plans, see Dey and Mukerjee [(1999a), Chapters 2, 6 and 7]. Most of these results, however, relate to situations where all factorial effects involving the same number of factors are considered equally important and, as such, the underlying model involves the general mean and all factorial effects involving up to a specified number of factors.

In practice however, the presumption of equality in the importance of all factorial effects involving the same number of factors may not always be an appropriate one. For example, there may be a situation where it is known a priori that only one of the factors can possibly interact with each of the factors, all other two-factor and higher order interactions being absent. The model then includes the general mean, all main effects and only a specified set of two-factor interactions. The issue of estimability and optimality in situations of this kind in the context of two-level factorials has been addressed by Hedayat and Pesotan (1992, 1997), Wu and Chen (1992) and Chiu and John (1998). Further optimality results for arbitrary factorials including the asymmetric ones, were obtained by Dey and Mukerjee (1999b).

In this paper, we obtain optimal fractional factorial plans for factorials of the type m^n, where m is a prime or a prime power, under a model that includes the general mean, all main effects and a specified set of two-factor interactions. All other interactions are assumed to be negligible. Here, and henceforth, the optimality criterion is the universal optimality of Kiefer (1975); see also Sinha and Mukerjee (1982).

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It is well known that a regular fractional factorial plan for an m^n factorial involving m^r runs can be generated by an $r \times n$ matrix A with entries from $GF(m)$, the finite (or, Galois) field of order m. Each column of A represents a factor and each element in the row space of A represents a treatment combination. For instance, if the matrix A is such that every $r \times 2$ submatrix of A has column rank 2 over $GF(m)$, then the plan given by the row space of A can accommodate up to $(m^r - 1)/(m - 1)$ factors, allowing the optimal estimation of the mean and complete sets of orthonormal contrasts belonging to the main effects of all the factors, under the assumption that all interactions involving two or more factors are absent. Such a matrix A can be obtained by choosing points in an $(r - 1)$-dimensional finite projective geometry, $PG(r-1, m)$ as columns of A, such that no two of these points are linearly dependent. For example, let $m = 3,\ r = 2,\ n = 4$. Then, $(0, 1),\ (1, 0),\ (1, 1)$ and $(1, 2)$ are four points in $PG(1, 3)$, such that no two of these are linearly dependent. Thus

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix},$$

and the row space of A gives the following fractional factorial plan for a 3^4 factorial involving nine treatment combinations:

$$(0, 0, 0, 0), (0, 1, 1, 1), (1, 0, 1, 2), (1, 1, 2, 0),$$
$$(1, 2, 0, 1), (0, 2, 2, 2), (2, 0, 2, 1), (2, 2, 1, 0), (2, 1, 0, 2).$$

These treatment combinations, treated as rows of a 9×4 matrix, form an orthogonal array of strength two and thus allow the optimal estimation of the mean and contrasts belonging to the main effects under the assumption that all interactions with two or more factors are absent.

In a plan of the above type, if the number of factors n is less than $(m^r - 1)/(m - 1)$, one might entertain some two-factor interactions in the model. In that situation, it is important to know which of the two-factor interactions can be included in the model, so that the plan remains optimal for the estimation of all the effects in the model involving the mean, all main effects and the specified set of two-factor interactions. We address this problem in this paper. It may be recalled that if *all* two-factor interactions are in the model, along with the mean and all the main effects, then a plan represented by an orthogonal array of strength four is universally optimal. If only a subset of the set of two-factor interactions are important, then using the results of this paper, one can get optimal plans with far fewer number of runs than required by a plan represented by an orthogonal array of strength four. For obtaining universally optimal plans under the above stated model, we use concepts and results from a finite projective geometry. Several families of such optimal plans are reported.

2. Preliminaries. Suppose it is desired to construct an optimal fractional factorial plan for an m^n factorial involving n factors $F_1, \dots, F_n$, each at m (≥ 2) levels and suppose that the number of runs in the plan is m^r for some integer $r < n$. Throughout this paper, we take m to be a prime or a prime power. The model includes the mean, the main effects $F_1, \dots, F_n$ and k specified two-factor interactions $F_{i_1}F_{j_1}, \dots, F_{i_k}F_{j_k}$, where $k \leq \min\{(m^r - mn + n - 1)/(m-1)^2, \binom{n}{2}\}$.

We use an $(r-1)$-dimensional finite projective geometry, $PG(r-1, m)$ to arrive at such plans. Recall that in a $PG(r-1, m)$, a point is represented by an ordered r-tuple $(x_0, x_1, \dots, x_{r-1})$, where for $0 \leq i \leq r-1$, $x_i \in GF(m)$. Two r-tuples represent the same point in $PG(r-1, m)$ if one is a multiple of the other. A t-flat consists of points whose coordinates can be written as a linear combination of $t+1$ independent points. Thus, there are $(m^{t+1}-1)/(m-1)$ distinct points in a t-flat. In particular, a one-flat, consisting of $m+1$ points is referred to as a line in a finite projective geometry. Similarly, a two-flat, consisting of m^2+m+1 points and m^2+m+1 lines is also called a plane. Given integers $s, t,\ s \leq t$, there are

$$\frac{(m^{r-s-1}-1)(m^{r-s-2}-1)\cdots(m^{t-s+1}-1)}{(m^{r-t-1}-1)(m^{r-t-2}-1)\cdots(m-1)}$$

t-flats passing through an s-flat in $PG(r-1, m)$. Hence there are $(m^{r-1}-1)/(m-1)$ lines through a point and $(m^{r-2}-1)/(m-1)$ planes through a line. For more about finite projective geometries, the reader is referred to Hirschfeld (1979).

As a first step towards constructing a fractional factorial plan, we carefully assign a distinct point of $PG(r-1, m)$ to each factor and the interaction of two factors is represented by the $m-1$ other points on the line joining the two factors. If the $n + k(m-1)$ points corresponding to the n main effects and k two-factor interactions are all distinct, then the $r \times n$ matrix A formed by the n column vectors corresponding to the n factors generates a plan given by the row space of A that ensures the estimability of the mean, n main effects and the given k two-factor interactions. We illustrate the above steps through an example.

EXAMPLE 2.1. Consider a 2^4 experiment involving factors F_0, F_1, F_2 and F_3. It is desired to estimate the mean, the four main effects and the three two-factor interactions F_0F_1, F_0F_2, F_0F_3 via a fractional factorial plan involving eight runs. All other interactions are assumed to be absent. Let us assign the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(0,1,1)$ of $PG(2,2)$ to F_0, F_1, F_2 and F_3, respectively. The points corresponding to the two-factor interactions F_0F_1, F_0F_2 and F_0F_3 are then respectively $(1,1,0)$, $(1,0,1)$ and $(1,1,1)$. Since these seven points, corresponding to the four main effects and three two-factor interactions are distinct, the required plan involving eight runs can be generated by the row space of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

REMARK 2.1. Given any n factors and k two-factor interactions, one can always find an r such that all the main effects and the k specified two-factor interactions are estimable via a fractional factorial plan involving m^r runs. It is therefore clear that one should attempt to find a plan with the least value of r given the integers n and k.

3. Optimal plans. Dey and Mukerjee (1999b) recently gave a combinatorial characterization for a plan to be universally optimal under a hierarchical model; for a definition of a hierarchical factorial model see Dey and Mukerjee (1999b). Since the models that we consider in this paper are also hierarchical in nature, we shall make use of this characterization. The following is a modification of the result of Dey and Mukerjee (1999b), suited for our purpose. Throughout, we use the same notation F_i to denote a point in a finite projective geometry as well as a factor or, its main effect.

THEOREM 3.1. *Let $\mathcal{D}$ be the class of all N-run fractional factorial plans for an arbitrary factorial experiment involving n factors, such that each member of $\mathcal{D}$ allows the estimability of the mean, the main effects $F_1, \dots, F_n$ and the k two-factor interactions $F_{i_1}F_{j_1}, \dots, F_{i_k}F_{j_k}$, where $1 \le i_u,\ j_u \le n$ for all $u = 1, \dots, k$. Under a hierarchical model, a plan $d \in \mathcal{D}$ has intereffect orthogonality, and hence is universally optimal over $\mathcal{D}$ if all level combinations of the following sets of factors appear equally often in d*:

(a) $\{F_u, F_v\}$, $1 \le u < v \le n$;
(b) $\{F_u, F_{i_v}, F_{j_v}\}$, $1 \le u \le n,\ 1 \le v \le k$;
(c) $\{F_{i_u}, F_{j_u}, F_{i_v}, F_{j_v}\}$, $1 \le u < v \le k$,

where a factor is counted only once if it is repeated in (b) *or* (c).

The following result shows that the method proposed in the previous section satisfies the condition of Theorem 3.1 and thus the plans constructed by the proposed method leads to universally optimal plans within the class of all plans involving the same number of runs.

THEOREM 3.2. *Let $F_1, \dots, F_n$ be distinct points in a $PG(r-1, m)$ and let A be an $r \times n$ matrix with columns $F_1, \dots, F_n$. For every $u = 1, \dots, k$, let $F_{i_u}F_{j_u}$ $(1 \le i_u,\ j_u \le n)$ denote the set of $m-1$ other points on the line containing the points F_{i_u} and F_{j_u}. If the $n + k(m-1)$ points $F_1, \dots, F_n, F_{i_1}F_{j_1}, \dots, F_{i_k}F_{j_k}$ are all distinct, then the row space of A satisfies the condition of Theorem* 3.1.

PROOF. From a result of Bose and Bush (1952) [see, e.g., Dey and Mukerjee (1999a), Lemma 3.4.1], it suffices to show that each of the matrices,

(a) $[F_u \vdots F_v]$, $1 \le u < v \le n$;

(b) $[F_u \vdots F_{i_v} \vdots F_{j_v}]$, $1 \le u \le n$, $1 \le v \le k$;

(c) $[F_{i_u} \vdots F_{j_u} \vdots F_{i_v} \vdots F_{j_v}]$, $1 \le u < v \le k$,

has full column rank.

The matrices in (a) above clearly have rank 2, as F_u is not a multiple of F_v. For matrices in (b) above, we can distinguish two cases, according as (i) $F_u = F_{i_v}$ or F_{j_v} or, (ii) F_u, F_{i_v}, F_{j_v} are all distinct. In the first case, clearly the matrix $[F_u \vdots F_{i_v} \vdots F_{j_v}]$ reduces to $[F_{i_v} \vdots F_{j_v}]$. It has rank 2 as F_{i_v} is not a multiple of F_{j_v}. In case (ii), since F_u, F_{i_v}, F_{j_v} are distinct, the matrices in (b) above have rank 3 each, since the point F_u is not on the line joining F_{i_v} and F_{j_v}. Finally, for the case (c) above, if $F_{i_u} = F_{i_v}$ or F_{j_v}, then the matrix $[F_{i_u} \vdots F_{j_u} \vdots F_{i_v} \vdots F_{j_v}]$ reduces to $[F_{j_u} \vdots F_{i_v} \vdots F_{j_v}]$, which has rank 3, since F_{j_u} is not on the line joining the points F_{i_v} and F_{j_v}. If $F_{i_u}, F_{j_u}, F_{i_v}, F_{j_v}$ are distinct points, they are not on the same two-flat, since the line through F_{i_u} and F_{j_u} does not intersect the line through F_{i_v} and F_{j_v}. This shows that the matrices under (c) above have rank 4 each. □

Based on Theorem 3.2, we now construct specific families of optimal plans, permitting the estimability of the mean, all main effects and a specified set of two-factor interactions. In order to facilitate the presentation, we introduce the following notations:

1. A plan allowing the optimal estimation of the mean, $2u$ main effects $F_1, \dots, F_{2u}$ and u two-factor interactions $F_1F_2, F_3F_4, \dots, F_{2u-1}F_{2u}$ will be denoted by

$$(F_1, F_2; F_3, F_4; \dots; F_{2u-1}, F_{2u})_1.$$

2. A plan allowing the optimal estimation of the mean, $u + v$ main effects $F_1, \dots, F_{u+v}$ and uv two-factor interactions F_iF_j $(1 \le i \le u,\ u+1 \le j \le u+v)$ will be denoted by

$$(F_1, \dots, F_u; F_{u+1}, \dots, F_{u+v})_2.$$

3. A plan allowing the optimal estimation of the mean, u main effects $F_1, \dots, F_u$ and u two-factor interactions $F_1F_2, \dots, F_{u-1}F_u, F_uF_1$ will be denoted by

$$(F_1, \dots, F_u)_3.$$

Note that the above notation to express the parameters in the model are not unique. For instance, the parameter set of the plan in Example 2.1 can be expressed both as $\{(F_0; F_1, F_2, F_3)_2\}$ or $\{(F_0, F_1)_1, (F_0, F_2)_1, (F_0, F_3)_1\}$. However, in what follows, we attempt to present the parameters in the model in a simple and unabiguous way by using the above notation.

REMARK 3.1. The interactions of type 2 above appear to be most interesting, as such models have applications in robust design or, the so-called Taguchi methods for quality improvement. In a production line, the quality of a product depends on two types of factors, called *control* and *noise* factors. The control factors are those that can be set at specified levels during the production process, while the noise factors can be fixed at selected levels during the experiment but not during the production or, later use of the product. In such experiments, some or all of the control versus noise interactions are of major importance, apart from the main effects of these two types of factors. Often, a cross-array is used for planning such experiments. Alternatives to cross-arrays, requiring fewer runs have also been suggested in the literature [see, e.g., Shoemaker, Tsui and Wu (1991)]. Some of the optimal plans reported in this paper are also not based on cross arrays (see, e.g., Example 3.1).

We now have the following results.

THEOREM 3.3. *For any prime or prime power m and any integer $r \geq 2$, we can construct a universally optimal plan*:

(i) *d_1 for an $m^{2(m^{2r}-1)/(m^2-1)}$ experiment involving m^{2r} runs where*

$$d_1 \equiv \{(F_1, F_2; F_3, F_4; \ldots; F_{2(m^{2r}-1)/(m^2-1)-1}, F_{2(m^{2r}-1)/(m^2-1)})_1\};$$

(ii) *d_2 for an $m^{2(m^{2r+1}-m^3)/(m^2-1)+2}$ experiment involving m^{2r+1} runs where*

$$d_2 \equiv \{(F_1, F_2; F_3, F_4; \ldots; F_{2(m^{2r+1}-m^3)/(m^2-1)+1}, F_{2(m^{2r+1}-m^3)/(m^2-1)+2})_1\}.$$

PROOF. The proof follows from a result of Wu, Zhang and Wang (1992), who show that the maximum number of m^2-level factors in an m^{2r}-run plan and in an m^{2r+1}-run plan are, respectively, $(m^{2r}-1)/(m^2-1)$ and $(m^{2r+1}-m^3)/(m^2-1)+1$. Replacing each m^2-level factor by two m-level factors, the required plans are obtained. □

Observe that the plan d_1 is saturated. In the plan d_2, a further of m^2 factors, each with m levels can be added to make it saturated.

THEOREM 3.4. *For any prime or prime power m and any integers r, u (≥ 1), v (≥ 1) such that $u+v=r$, we can construct a universally optimal plan d for an $m^{(m^u+m^v-2)/(m-1)}$ experiment involving m^r runs where*

$$d \equiv \{(F_1, \ldots, F_{(m^u-1)/(m-1)}; F_{(m^u-1)/(m-1)+1}, \ldots, F_{(m^u+m^v-2)/(m-1)})_2\}.$$

PROOF. Since $u+v=r$, there exist an $(u-1)$-flat and a $(v-1)$-flat which are disjoint in $PG(r-1,m)$. Let $F_1, \ldots, F_{(m^u-1)/(m-1)}$ be the points on the $(u-1)$-flat and $F_{(m^u-1)/(m-1)+1}, \ldots, F_{(m^u+m^v-2)/(m-1)}$ be the points on the $(v-1)$-flat. Then the $(m^u+m^v-2)/(m-1)$ main effects F_i $[1 \leq i \leq (m^u+m^v-2)/(m-1)]$

and the $(m^u - 1)(m^v - 1)/(m - 1)^2$ two-factor interactions $F_j F_k$ $[1 \le j \le (m^u - 1)/(m - 1), (m^u - 1)/(m - 1) + 1 \le k \le (m^u + m^v - 2)/(m - 1)]$ satisfy the condition of Theorem 3.2. Hence the required plan is obtained. □

For $v = 1$ in Theorem 3.4, we obtain the following interesting plan in which only one factor has interactions with all other factors.

COROLLARY 3.1. *For any prime or prime power m and any integer $r \ge 3$, we can construct a universally optimal plan d for an $m^{(m^{r-1}-1)/(m-1)+1}$ experiment involving m^r runs where*

$$d \equiv \{(F_0; F_1, \dots, F_{(m^{r-1}-1)/(m-1)})_2\}.$$

In the next few theorems we construct plans such that the factors can be divided into several groups and have the following properties. Interactions between the factors in different groups are absent, and there is only one factor which interacts with all other factors in the same group.

THEOREM 3.5. *Let m be a prime or a prime power and r, s, u_i's $(u_i \ge 1)$ be integers such that $r/2 \ge s \ge 1$ and $\sum_{i=1}^{(m^s-1)/(m-1)} u_i = \frac{m^{r-s}-1}{m-1}$. Then one can construct a universally optimal plan d for an $m^{(m^{r-1}-1)/(m-1)+m^{s-1}}$ experiment involving m^r runs where*

$$d \equiv \{(F_{0,1}; F_{1,1}, \dots, F_{u_1 m^{s-1},1})_2, (F_{0,2}; F_{1,2}, \dots, F_{u_2 m^{s-1},2})_2, \dots,$$
$$(F_{0,(m^s-1)/(m-1)}; F_{1,(m^s-1)/(m-1)}, \dots, F_{u_{(m^s-1)/(m-1)} m^{s-1},(m^s-1)/(m-1)})_2\}.$$

PROOF. Let $F_{0,1}, F_{0,2}, \dots, F_{0,(m^s-1)/(m-1)}$ be the $(m^s - 1)/(m - 1)$ points on an $(s - 1)$-flat L_0 in $PG(r - 1, m)$. There are $(m^{r-s} - 1)/(m - 1)$ s-flats through the $(s - 1)$-flat L_0, say, $K_{i,j}$ where $i = 1, \dots, (m^s - 1)/(m - 1)$ and $j = 1, \dots, u_i$. Let $L_{i,j}$ be an $(s - 1)$-flat in $K_{i,j}$ which does not pass through $F_{0,i}$. We can now choose $F_{(j-1)m^{s-1}+1,i}, \dots, F_{jm^{s-1},i}$ to be the m^{s-1} points on $L_{i,j}$ but not on L_0. □

EXAMPLE 3.1. For $m = s = 2, r = 4, u_1 = u_2 = u_3 = 1$ in Theorem 3.5, let L_0 be the line containing three points $F_{0,1}(0, 0, 0, 1)$, $F_{0,2}(0, 0, 1, 0)$, and $F_{0,3}(0, 0, 1, 1)$ in $PG(3, 2)$. Let $K_{1,1}$, $K_{2,1}$, $K_{3,1}$ be the planes through L_0 and the points $F_{1,1}(0, 1, 0, 0)$, $F_{1,2}(1, 0, 0, 0)$, $F_{1,3}(1, 1, 0, 0)$, respectively. Now choose $L_{1,1}$ to be the line through the points $F_{1,1}$ and $F_{2,1}(0, 1, 1, 0)$, $L_{2,1}$ to be the line through the points $F_{1,2}$ and $F_{2,2}(1, 0, 0, 1)$, and $L_{3,1}$ to be the line through the points $F_{1,3}$ and $F_{2,3}(1, 1, 1, 0)$. We have thus obtained a plan d for a 2^9 experiment in 16 runs, where

$$d \equiv \{(F_{0,1}; F_{1,1}, F_{2,1})_2, (F_{0,2}; F_{1,2}, F_{2,2})_2, (F_{0,3}; F_{1,3}, F_{2,3})_2\}.$$

The actual plan is given by the row space of a 4×9 matrix with columns as $F_{0,1},\ F_{0,2},\ F_{0,3},\ F_{1,1},\ F_{1,2},\ F_{1,3},\ F_{2,1},\ F_{2,2},\ F_{2,3}$. This plan can be used in the context of robust design with six control factors, say, $C_1, \dots, C_6$ and three noise factors, $N_1,\ N_2,\ N_3$. Identifying $F_{0,i}$ with N_i for $i = 1, 2, 3$ and the remaining factors $F_{i,j}$ $(i = 1,\ 2;\ \ j = 1, 2, 3)$ with the control factors, the above plan allows the optimal estimation of the mean, all main effects and the six noise versus control interactions $C_1N_1,\ C_2N_1,\ C_3N_2,\ C_4N_2,\ C_5N_3,\ C_6N_3$. The plan is clearly saturated. Note that this plan is *not* based on cross-arrays.

THEOREM 3.6. *For any prime or prime power m and any integer r (≥ 4), we can construct a universally optimal plan d for an $m^{2+(m^{r-1}+m^3-2m^2)/(m-1)}$ experiment involving m^r runs where*

$$d \equiv \{(F_1, F_2)_1, (F_{0,1}; F_{1,1}, \dots, F_{(m^{r-3}-1)/(m-1),1})_2, \dots,$$
$$(F_{0,m^2}; F_{1,m^2}, \dots, F_{(m^{r-3}-1)/(m-1),m^2})_2\}.$$

PROOF. Choose F_1 and F_2 to be two points on a line L in $PG(r-1, m)$. Let K_0 be a plane containing L, and let $F_{0,1}, \dots, F_{0,m^2}$ be the m^2 points on the plane K_0 but not on L. There are $(m^{r-3}-1)/(m-1)$ three-flats through the plane K_0 in $PG(r-1, m)$, say, H_i $[i = 1, \dots, (m^{r-3}-1)/(m-1)]$. By Theorem 4.1.1 of Hirschfeld (1979), there exist m^2+1 lines $L, L_{i,1}, \dots, L_{i,m^2}$ which partition the three-flat H_i. For each $j = 1, \dots, m^2$, the line $L_{i,j}$ is not on the plane K_0 (otherwise it intersects L), hence it meets K_0 in a point. We can choose $L_{i,j}$ such that $F_{0,j}$ is on the line $L_{i,j}$. Now for every $i = 1, \dots, (m^{r-3}-1)/(m-1)$ and $j = 1, \dots, m^2$, choose $F_{i,j}$ to be a point other than $F_{0,j}$ on the line $L_{i,j}$. □

THEOREM 3.7. *For any prime or prime power m, we can construct a universally optimal plan d for an $m^{m^3+2m^2+m}$ experiment involving m^5 runs where*

$$d \equiv \{(F_{0,1}; F_{1,1}, \dots, F_{m,1})_2, \dots, (F_{0,m^2+m}; F_{1,m^2+m}, \dots, F_{m,m^2+m})_2\}.$$

PROOF. Let F_0 be a point on a plane K_0 in $PG(4, m)$, and let $L_1, \dots, L_{m+1}$ be the $m+1$ lines through F_0 on the plane K_0. For $i = 1, \dots, m+1$, let $F_{0,(i-1)m+1}, \dots, F_{0,im}$ be the m points other than F_0 on the line L_i. There are $m+1$ three-flats through K_0, say, H_i $(i = 1, \dots, m+1)$. There are m planes $K_{1,i}, \dots, K_{m,i}$ other than K_0 through the line L_i in the three-flat H_i. For $i = 1, \dots, m+1$ and $j = 1, \dots, m$, let $L_{i,j}$ be a line on the plane $K_{j,i}$ which does not pass through $F_{0,(i-1)m+j}$. Now choose $F_{1,(i-1)m+j}, \dots, F_{m,(i-1)m+j}$ to be the m points on the line $L_{i,j}$ but not on L_i. □

EXAMPLE 3.2. For $m = 2$ in Theorem 3.7, choose F_0 to be the point $(0, 0, 0, 0, 1)$ in $PG(4, 2)$. Let $L_1,\ L_2$ and L_3 be the lines that pass through F_0 and the points $F_{0,1}(0,0,0,1,0)$, $F_{0,3}(0,0,1,0,0)$ and $F_{0,5}(0,0,1,1,0)$, respectively. Following the construction procedure given in Theorem 3.7, we get a plan d for a 2^{18} experiment in 32 runs, where

$$d \equiv \{(F_{0,1}; F_{1,1}, F_{2,1})_2, (F_{0,2}; F_{1,2}, F_{2,2})_2, (F_{0,3}; F_{1,3}, F_{2,3})_2,$$
$$(F_{0,4}; F_{1,4}, F_{2,4})_2, (F_{0,5}; F_{1,5}, F_{2,5})_2, (F_{0,6}; F_{1,6}, F_{2,6})_2\}.$$

The required plan is given by the row space of the following 5×18 matrix:

$$\begin{bmatrix} 000 & 000 & 011 & 011 & 011 & 011 \\ 011 & 011 & 000 & 000 & 011 & 011 \\ 000 & 011 & 100 & 100 & 100 & 100 \\ 100 & 100 & 000 & 011 & 100 & 111 \\ 001 & 101 & 001 & 101 & 001 & 101 \end{bmatrix}.$$

Additionally, the above plan allows optimal estimation of one of the interactions among $F_{0,1}F_{0,2}$, $F_{0,3}F_{0,4}$, $F_{0,5}F_{0,6}$, $F_{1,1}F_{2,1}$, $F_{1,2}F_{2,2}$, $F_{1,3}F_{2,3}$, $F_{1,4}F_{2,4}$, $F_{1,5}F_{2,5}$ or $F_{1,6}F_{2,6}$. Thus it can estimate 18 main effects and 13 two-factor interactions and is therefore saturated.

THEOREM 3.8. *For any prime or prime power m, we can construct a universally optimal plan d for an $m^{m^4+2m^3+m^2+2}$ experiment involving m^6 runs where*

$$d \equiv \{(F_1, F_2)_1, (F_{0,1}; F_{1,1}, \dots, F_{m,1})_2, \dots,$$
$$(F_{0,m^3+m^2}; F_{1,m^3+m^2}, \dots, F_{m,m^3+m^2})_2\}.$$

PROOF. Let F_1 and F_2 be two points on a line L in $PG(5,m)$, and let H_0 be a three-flat containing L. There are $m+1$ planes through the line L in H_0, say, K_i $(i = 1,\dots,m+1)$. For each i, there are m lines on the plane K_i other than L through the point F_1, say, $L_{i,j}$ $(j = 1,\dots,m)$. Let $F_{0,(i-1)m^2+(j-1)m+1},\dots,F_{0,(i-1)m^2+jm}$ be the m points other than F_1 on the line $L_{i,j}$. Let $M_1,\dots,M_{m+1}$ be the $m+1$ four-flats through the three-flat H_0 in $PG(5,m)$. There are m three-flats other than H_0 through the plane K_i in the four-flat M_i, say, $H_{i,j}$ $(j = 1,\dots,m)$. There are m planes other than K_i through the line $L_{i,j}$ in the three-flat $H_{i,j}$, say, $K_{i,j,k}$ $(k = 1,\dots,m)$. Now let $L_{i,j,k}$ be a line on the plane $K_{i,j,k}$ which does not pass through the point $F_{0,(i-1)m^2+(j-1)m+k}$, choose $F_{1,(i-1)m^2+(j-1)m+k},\dots,F_{m,(i-1)m^2+(j-1)m+k}$ to be the m points on the line $L_{i,j,k}$ but not on $L_{i,j}$. □

EXAMPLE 3.3. For $m = 2$ in Theorem 3.8, let L be the line through the points $F_1(0,0,0,0,0,1)$ and $F_2(0,0,0,0,1,0)$ in $PG(5,2)$. Let H_0 be the three-flat containing the line L and the points $F_{0,1}(0,0,0,1,0,0)$ and $F_{0,5}(0,0,1,0,0,0)$. Let M_1, M_2 and M_3 be the four-flats that pass through the three-flat H_0

and the points $F_{1,1}(0,1,0,0,0,0)$, $F_{1,5}(1,0,0,0,0,0)$ and $F_{1,9}(1,1,0,0,0,0)$, respectively. Following the construction procedure given in Theorem 3.8, we get a plan d for a 2^{38} experiment in 64 runs, where

$$\begin{aligned} d \equiv \{&(F_1, F_2)_1, (F_{0,1}; F_{1,1}, F_{2,1})_2, (F_{0,2}; F_{1,2}, F_{2,2})_2, (F_{0,3}; F_{1,3}, F_{2,3})_2, \\ &(F_{0,4}; F_{1,4}, F_{2,4})_2, (F_{0,5}; F_{1,5}, F_{2,5})_2, (F_{0,6}; F_{1,6}, F_{2,6})_2, \\ &(F_{0,7}; F_{1,7}, F_{2,7})_2, (F_{0,8}; F_{1,8}, F_{2,8})_2, (F_{0,9}; F_{1,9}, F_{2,9})_2, \\ &(F_{0,10}; F_{1,10}, F_{2,10})_2, (F_{0,11}; F_{1,11}, F_{2,11})_2, (F_{0,12}; F_{1,12}, F_{2,12})_2\}. \end{aligned}$$

The required plan is given by the row space of the following 6×38 matrix:

$$\begin{bmatrix} 00 & 000 & 000 & 000 & 000 & 011 & 011 & 011 & 011 & 011 & 011 & 011 & 011 \\ 00 & 011 & 011 & 011 & 011 & 000 & 000 & 000 & 000 & 011 & 011 & 011 & 011 \\ 00 & 000 & 000 & 011 & 011 & 100 & 100 & 100 & 100 & 100 & 100 & 111 & 111 \\ 00 & 100 & 100 & 100 & 100 & 000 & 000 & 011 & 011 & 100 & 100 & 100 & 100 \\ 01 & 000 & 011 & 100 & 111 & 000 & 011 & 100 & 111 & 000 & 011 & 100 & 111 \\ 10 & 001 & 101 & 001 & 101 & 001 & 101 & 001 & 101 & 001 & 101 & 001 & 101 \end{bmatrix}.$$

This plan can optimally estimate 38 main effects and 25 two-factor interactions, and is therefore saturated.

THEOREM 3.9. *For any prime or prime power m, we can construct a universally optimal plan d for an $m^{(m^3+1)(m+2)}$ experiment involving m^6 runs where*

$$d \equiv \{(F_{0,1}; F_{1,1}, \dots, F_{m+1,1})_2, \dots, (F_{0,m^3+1}; F_{1,m^3+1}, \dots, F_{m+1,m^3+1})_2\}.$$

PROOF. Let $K_1, \dots, K_{m^3+1}$ be m^3+1 planes which partition $PG(5, m)$. For each $i = 1, \dots, m^3+1$, choose $F_{1,i}, \dots, F_{m+1,i}$ to be $m+1$ points of a line and $F_{0,i}$ to be a point not on this line on the plane K_i. $\square$

Many m^r-run plans for $r \geq 7$ can be constructed by considering the geometry of $PG(r-1, m)$. We do not elaborate on these, since these involve too many runs even when $m = 2$.

We finally consider plans that allow estimation of two-factor interactions of the third type.

THEOREM 3.10. *If r (≥ 2) is an integer and $2^r - 1 = uv$ $(1 \leq u < 2^r - 1)$, then one can construct a universally optimal plan d for a 2^{2^r-1} experiment in 2^{r+1} runs where*

$$\begin{aligned} d \equiv \{&(F_0, F_u, \dots, F_{(v-1)u})_3, (F_1, F_{u+1}, \dots, F_{(v-1)u+1})_3, \dots, \\ &(F_{u-1}, F_{2u-1}, \dots, F_{uv-1})_3\}. \end{aligned}$$

PROOF. Let ω be a primitive element of the Galois field $GF(2^r)$ with the minimum polynomial $\omega^r = \alpha_0 + \alpha_1\omega + \cdots + \alpha_{r-1}\omega^{r-1}$, where for $0 \le i \le r-1$, $\alpha_i \in GF(2)$. The elements of $GF(2^r)$ can be represented as $0, 1, \omega, \ldots, \omega^{2^r-2}$. For $0 \le i \le 2^r - 2$, let $\omega^i = \alpha_{0,i} + \alpha_{1,i}\omega + \cdots + \alpha_{r-1,i}\omega^{r-1}$, where for $0 \le j \le r-1$, $\alpha_{j,i} \in GF(2)$. Choose F_i to be the point in $PG(r,2)$ with coordinates $(\alpha_{0,i}, \ldots, \alpha_{r-1,i}, 1)$, $0 \le i \le 2^r - 2$. For $i = 0, \ldots, u-1$ and $j = 0, \ldots, v-1$, the two-factor interaction $F_{i+ju}F_{i+(j+1)u}$ is represented by the point $(\alpha_{0,i+ju} + \alpha_{0,i+(j+1)u}, \alpha_{1,i+ju} + \alpha_{1,i+(j+1)u}, \ldots, \alpha_{r-1,i+ju} + \alpha_{r-1,i+(j+1)u}, 0)$. Since $\omega^{i+ju} + \omega^{i+(j+1)u} = \omega^{i+ju}(\omega^u + 1)$ represents all the nonzero elements of $GF(2^r)$ for $i = 0, \ldots, u-1$ and $j = 0, \ldots, v-1$, points representing $\{F_{i+ju}F_{i+(j+1)u}\}$ are different for distinct pairs (i, j). □

EXAMPLE 3.4. We construct three plans, each involving 32 runs for a 2^{15} experiment. Let ω be a primitive element of $GF(16)$ with the minimum polynomial $\omega^4 = \omega + 1$. The elements of $GF(16)$ are $0, 1, \omega, \omega^2, \omega^3, \omega^4 = \omega + 1$, $\omega^5 = \omega + \omega^2$, $\omega^6 = \omega^2 + \omega^3$, $\omega^7 = 1 + \omega + \omega^3$, $\omega^8 = 1 + \omega^2$, $\omega^9 = \omega + \omega^3$, $\omega^{10} = 1 + \omega + \omega^2$, $\omega^{11} = \omega + \omega^2 + \omega^3$, $\omega^{12} = 1 + \omega + \omega^2 + \omega^3$, $\omega^{13} = 1 + \omega^2 + \omega^3$, $\omega^{14} = 1 + \omega^3$. Choose the points $F_0, \ldots, F_{14}$ as follows:

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
1	0	0	0	1	0	0	1	1	0	1	0	1	1	1
0	1	0	0	1	1	0	1	0	1	1	1	1	0	0
0	0	1	0	0	1	1	0	1	0	1	1	1	1	0
0	0	0	1	0	0	1	1	0	1	0	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

By following the methods discussed above, we get the following three plans, each with 32 runs for a 2^{15} experiment:

$$d_1 \equiv \{(F_0, F_1, \ldots, F_{14})_3\}.$$

$$d_2 \equiv \{(F_0, F_3, F_6, F_9, F_{12})_3, (F_1, F_4, F_7, F_{10}, F_{13})_3, (F_2, F_5, F_8, F_{11}, F_{14})_3\}.$$

$$d_3 \equiv \{(F_0, F_5, F_{10})_3, (F_1, F_6, F_{11})_3, (F_2, F_7, F_{12})_3, (F_3, F_8, F_{13})_3, (F_4, F_9, F_{14})_3\}.$$

The plan d_3 appears to be especially interesting. This plan can be used in a situtation where the 15 factors can be grouped into five sets of three factors each and it is known that the factors within the same set only can interact, all other interactions being absent.

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