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Abstract

For nonnegative random variables X and Y we write $X \leq_{\rm ttt} Y$ if $T_X(p) \leq T_Y(p)$ for all $p \in (0,1)$, where $T_X(p) \equiv \int_0^{F^{-1}(p)} (1 - F(x))\, dx$ and $T_Y(p) \equiv \int_0^{G^{-1}(p)} (1 - G(x))\, dx$; here F and G denote the distribution functions of X and Y, respectively. The purpose of this article is to study some properties of this new stochastic order. New properties of the excess wealth (or right spread) order, and of other related stochastic orders, are obtained in the present article as well. Applications in the statistical theory of reliability and in economics are included.

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1 Motivation and Definitions

Consider a distribution function F, of a nonnegative random variable X, which is strictly increasing on its interval support. Let $p \in (0,1)$ and $t \geq 0$ be two values related by $p = F(t)$ or, equivalently, by $t = F^{-1}(p)$, where F^{-1} is the right continuous inverse of F. Every choice of such p and t determines three regions of interest:

$$\begin{aligned}
A_F &\equiv \{(x,u) : u \in (0,p),\ x \in (0, F^{-1}(u))\} = \{(x,u) : x \in (0,t),\ u \in (F(x), F(t))\}, \\
B_F &\equiv \{(x,u) : u \in (p,1),\ x \in (0, F^{-1}(p))\} = \{(x,u) : x \in (0,t),\ u \in (F(t),1)\}, \\
C_F &\equiv \{(x,u) : u \in (p,1),\ x \in (F^{-1}(p), F^{-1}(u))\} = \{(x,u) : x \in (t,\infty),\ u \in (F(x),1)\},
\end{aligned}$$

as depicted in Figure 1. When we want to emphasize the dependence of A_F on $p \in (0,1)$ we write $A_F(p)$. When we want to emphasize the dependence of A_F on $t > 0$ we write $\widetilde{A}_F(t)$. Of course, $A_F(p) = \widetilde{A}_F(t)$ when $p = F(t)$. Similarly we denote $B_F(p)$, $\widetilde{B}_F(t)$, $C_F(p)$, and $\widetilde{C}_F(t)$.

Figure 1: *Depiction of A_F, B_F and C_F*

The areas of the regions depicted in Figure 1 have various intuitive meanings in different applications. For example, if F is the distribution of wealth in some community then $\|C_F(p)\|$ (denoting by $\|D\|$ the area of D for any two-dimensional set D with an area) corresponds to the excess wealth of the richest $(1-p)\cdot 100\%$ individuals in that community (see Shaked and Shanthikumar (1998)). Similarly, $\|A_F(p)\|$ corresponds to the total income of the poorest $p \cdot 100\%$ individuals in that community. If F is the distribution function of the lifetime of a machine then

$$T_X(p) \equiv \|A_F(p) \cup B_F(p)\|, \quad p \in (0,1),$$

corresponds to the total time on test (TTT) transform associated with this distribution (see, for example, Figure 1 in Klefsjö (1991), or Figure 9.2 in Høyland and Rausand (1994), or Figure 2.1 in Hürlimann (2002)). Notice also that

$$\|A_F(p) \cup B_F(p) \cup C_F(p)\| = \|\widetilde{A}_F(t) \cup \widetilde{A}_F(t) \cup \widetilde{A}_F(t)\|$$

is the mean EX of that lifetime, provided the mean exists.

Let G be another distribution function, of a nonnegative random variable Y, which is also strictly increasing on its interval support. Let $\overline{G} \equiv 1-G$ be the corresponding survival function, and analogously define $A_G(p)$, $\widetilde{A}_G(t)$, etc. Assume the existence of the means EX and EY, if necessary. Comparisons of areas of analogous sets of F and G for each $p \in (0,1)$ or $t > 0$, yield and characterize many well known useful stochastic orders. For example,

$$\Big(\|\widetilde{A}_F(t) \cup \widetilde{B}_F(t)\| \le \|\widetilde{A}_G(t) \cup \widetilde{B}_G(t)\|,\ \forall t \in (0,\infty)\Big) \Longleftrightarrow X \le_{\text{icv}} Y, \tag{1.1}$$

where $\le_{\text{icv}}$ denotes the increasing concave order (see Shaked and Shanthikumar (1994, Section 3.A)), whereas

$$\Big(\|\widetilde{C}_F(t)\| \le \|\widetilde{C}_G(t)\|,\ \forall t \in (0,\infty)\Big) \Longleftrightarrow X \le_{\text{icx}} Y,$$

where $\le_{\text{icx}}$ denotes the increasing convex order (again, see Shaked and Shanthikumar (1994, Section 3.A)). The normalized comparison

$$\|\widetilde{C}_F(t)\|/\overline{F}(t) \le \widetilde{\|}C_G(t)\|/\overline{G}(t), \quad t > 0,$$

yields the mean residual life order $\le_{\text{mrl}}$ (see Shaked and Shanthikumar (1994, Section 1.D)). Similarly,

$$\Big(\|A_F(p)\|/EX \le \|A_G(p)\|/EY,\ \forall p \in (0,1)\Big) \Longleftrightarrow X \ge_{\text{Lorenz}} Y,$$

where $\le_{\text{Lorenz}}$ denotes the Lorenz order (see Shaked and Shanthikumar (1994, Section 3.A)). The comparison

$$\|C_F(p)\| \le \|C_G(p)\|, \quad p \in (0,1), \tag{1.2}$$

yields the excess wealth order, that is, $X \le_{\text{ew}} Y$ (see Shaked and Shanthikumar (1998)), or, equivalently, the right spread order $X \le_{RS} Y$ (see Fernandez-Ponce, Kochar, and Muñoz-Perez (1998)). The NBUE (new better than used in expectation) order of Kochar and Wiens (1987) can also be characterized by the sets above as follows

$$\Big(\|A_F(p) \cup B_F(p)\|/EX \le \|A_G(p) \cup B_G(p)\|/EY,\ \forall p \in (0,1)\Big) \Longleftrightarrow X \ge_{\text{nbue}} Y$$

(see (3.5) in Kochar (1989)).

The various stochastic orders mentioned above share some similarities, but they are all distinct, and each is useful in different contexts. For example, the order $\le_{\text{ew}}$ is location independent (and thus it can be used to compare also random variables that are not nonnegative) and it compares the variability of the underlying random variables (see Shaked and Shanthikumar (1998)). Similarly the order $\leq_{\text{Lorenz}}$ is an order which compares variability. On the other hand, the orders $\leq_{\text{icx}}$ and $\leq_{\text{icv}}$ combine comparison of location with comparison of variation. The order $\leq_{\text{nbue}}$ compares aging mechanisms of different items.

One purpose of this article is to study the stochastic order which is defined by

$$T_X(p) \leq T_Y(p), \quad p \in (0,1), \tag{1.3}$$

where $T_Y(p) \equiv \|A_G(p) \cup B_G(p)\|$. When (1.3) holds we write $X \leq_{\text{ttt}} Y$, and we say that X *is smaller than* Y *in the* TTT *transform order*. We investigate in this article some properties of this stochastic order. New properties of the excess wealth (or right spread) order, and of other related stochastic orders, are obtained in the present article as well.

The inequality (1.3) has appeared already in Bartoszewicz (1986), but it has not been studied there as a stochastic order. In fact, Bartoszewicz (1986) has derived (1.3) for the so-called *generalized* TTT *transforms*. In the present paper we only study the order defined in (1.3) for standard TTT transforms, and for such transforms the result obtained in Proposition 1 of Bartoszewicz (1986) is trivial. The inequality (1.3) for the so-called *normalized generalized* TTT *transforms* has appeared in Barlow and Doksum (1972), in Barlow (1979), and in Bartoszewicz (1995, 1998), but, again, it has not been studied there as a stochastic order.

We also devote a section in this article to the excess wealth order. In that section we give some new and useful properties of this order.

Applications in the statistical theory of reliability and in economics illustrate the usefulness of our results.

In this paper "increasing" and "decreasing" stand for "nondecreasing" and "nonincreasing," respectively. For any distribution function F we denote by $\overline{F} \equiv 1-F$ the corresponding survival function.

2 Some Basic Properties of the TTT Transform Order

Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. It is easy to verify that $X \leq_{\text{ttt}} Y$ if, and only if,

$$\int_0^{F^{-1}(p)} \overline{F}(x)\,dx \leq \int_0^{G^{-1}(p)} \overline{G}(x)\,dx, \quad p \in (0,1). \tag{2.1}$$

A simple sufficient condition for the order $\leq_{\text{ttt}}$ is the usual stochastic order:

$$X \leq_{\text{st}} Y \Longrightarrow X \leq_{\text{ttt}} Y, \tag{2.2}$$

where $X \leq_{\text{st}} Y$ means $\overline{F}(x) \leq \overline{G}(x)$ for all $x \in \mathbb{R}$ (see, for example, Shaked and Shanthikumar (1994, Section 1.A)). In order to verify (2.2) one may just notice that if $X \leq_{\text{st}} Y$ then $F^{-1}(p) \leq G^{-1}(p)$ for all $p \in (0,1)$.

Using the fact that for any nonnegative random variable X, and for any $a > 0$, we have

$$T_{aX}(p) = aT_X(p), \quad p \in (0,1),$$

it is easy to see that for any two nonnegative random variables X and Y we have

$$X \le_{\text{ttt}} Y \Longrightarrow aX \le_{\text{ttt}} aY \quad \text{for any } a > 0. \tag{2.3}$$

The implication (2.3) may suggest that if $X \le_{\text{ttt}} Y$ then $\phi(X) \le_{\text{ttt}} \phi(Y)$ whenever ϕ is an increasing function. However, this is not true, as it is shown in the following example.

Example 2.1. In this example we show that

$$X \le_{\text{ttt}} Y \not\Longrightarrow \phi(X) \le_{\text{ttt}} \ phi(Y) \text{ for all increasing functions } \phi.$$

Let X, with distribution function F, be an exponential random variable with rate $\lambda > 0$, and let Y, with distribution function G, be a uniform $(0,1)$ random variable. Then a straightforward computation yields

$$\begin{aligned} T_X(p) &\equiv \|A_F(p) \cup B_F(p)\| = \frac{p}{\lambda}, \quad p \in (0,1), \quad \text{and} \\ T_Y(p) &\equiv \|A_G(p) \cup B_G(p)\| = \frac{p(2-p)}{2}, \quad p \in (0,1). \end{aligned}$$

When $\lambda = 4$ we see that $T_X(p) \le T_Y(p)$ for all $p \in (0,1)$, and thus $X \le_{\text{ttt}} Y$. Let us consider the kth power of both X and Y when $k > 1$. Then

$$T_{X^k}(p) = \frac{k}{\lambda^k} \int_0^{-\log(1-p)} x^{k-1} e^{-x}\, dx, \quad T_{Y^k}(p) = k \cdot \frac{p^k(k+1-kp)}{k(k+1)}, \quad p \in (0,1).$$

Now,

$$\lim_{p \uparrow 1} T_{X^k}(p) = \frac{k}{\lambda^k} \int_0^{\infty} x^{k-1} e^{-x}\, dx = \frac{k!}{\lambda^k} \qquad \text{and} \qquad \lim_{p \uparrow 1} T_{Y^k}(p) = \frac{1}{k+1}.$$

If $\lambda = 4$ and $k = 10$ we get

$$\lim_{p \uparrow 1} T_{X^k}(p) = \frac{10!}{4^{10}} > \frac{1}{11} = \lim_{p \uparrow 1} T_{Y^k}(p).$$

So for some p near 1 we have $T_{X^k}(p) > T_{Y^k}(p)$, and thus $X^k \not\le_{\text{ttt}} Y^k$ when $k = 10$. ◀

It is true, however, that the order $\le_{\text{ttt}}$ is closed under increasing concave transformations. This is shown in the next theorem, the proof of which is given in the Appendix.

Theorem 2.2. *Let X and Y be two continuous nonnegative random variables with interval supports, and with 0 being the common left endpoint of the supports. Then, for any increasing concave function ϕ, such that $\phi(0) = 0$, we have*

$$X \le_{\text{ttt}} Y \Longrightarrow \phi(X) \le_{\text{ttt}} \phi(Y).$$

A stochastic order $\preccurlyeq$ is said to be *location independent* if

$$X \preccurlyeq Y \Longrightarrow X \preccurlyeq Y + c \quad \text{for any } c \in (-\infty, \infty). \tag{2.4}$$

For example, the order $\leq_{\mathrm{ew}}$ is location independent- –see Section 4. The order $\leq_{\mathrm{ttt}}$ is not location independent. However, if Y is a random variable with distribution function G, then

$$T_{Y+c}(p) = \|A_{G(\cdot-c)}(p) \cup B_{G(\cdot-c)}(p)\| = \|A_G(p) \cup B_G(p)\| + c = T_Y(p) + c,$$
$$p \in (0,1),\ c \in (-\infty, \infty).$$

It follows that the order $\leq_{\mathrm{ttt}}$ is *closed under right shifts of the larger variable*; that is,

$$X \leq_{\mathrm{ttt}} Y \Longrightarrow X \leq_{\mathrm{ttt}} Y + c \quad \text{for any } c > 0.$$

Note that

$$X \leq_{\mathrm{ttt}} Y \Longrightarrow EX \leq EY, \tag{2.5}$$

provided the expectations exist.

3 The Relationship of the TTT Transform Order to Other Stochastic Orders

In this section X and Y are continuous nonnegative random variables with interval supports, and with distribution functions F and G, respectively.

When $EX = EY$ then the order $\leq_{\mathrm{ttt}}$ is equivalent to the orders $\leq_{\mathrm{ew}}$ and $\leq_{\mathrm{nbue}}$ (described in Section 1) in the sense

$$X \leq_{\mathrm{ttt}} Y \Longleftrightarrow X \geq_{\mathrm{ew}} Y \Longleftrightarrow X \geq_{\mathrm{nbue}} Y. \tag{3.1}$$

However, these orders are distinct when $EX < EY$—this will be shown later in this section. It is useful to note that for nonnegative random variables X and Y with finite means we have

$$X \geq_{\mathrm{nbue}} Y \Longleftrightarrow \frac{X}{EX} \leq_{\mathrm{ttt}} \frac{Y}{EY}. \tag{3.2}$$

Note that the inequality on the right side of (3.2) is just an inequality between two scaled TTT transforms; such transforms are studied, for example, in Barlow and Campo (1975). This provides an interesting illustration of the $\geq_{\mathrm{nbue}}$ inequality. Furthermore, recall that the scaled TTT transform that is associated with an exponential distribution (with any mean) is just a straight line connecting $(0,0)$ and $(1,1)$. Recall also from Kochar and Wiens (1987) that if X is an exponential random variable, then Y is an NBUE random variable if, and only if, $X \geq_{\mathrm{nbue}} Y$. Thus it is seen from (3.2) that Y is an NBUE random variable if, and only if, its scaled TTT transform is above the diagonal of the unit square; the latter is an observation in Bergman (1979).

The next result, which is a corollary of Theorem 2.2, shows that the order $\leq_{\rm ttt}$ is stronger than the order $\leq_{\rm icv}$. This agrees with the intuitive fact that the order $\leq_{\rm ttt}$ is a stochastic order that combines comparison of location with comparison of variation.

Corollary 3.1. *Let X and Y be two continuous nonnegative random variables with interval supports, and with 0 being the common left endpoint of the supports. Then*

$$X \leq_{\rm ttt} Y \Longrightarrow X \leq_{\rm icv} Y.$$

Proof. Suppose that $X \leq_{\rm ttt} Y$. Let ϕ be an increasing concave function defined on $[0,\infty)$. Define $\tilde{\phi}(\cdot) = \phi(\cdot) - \phi(0)$, so that $\tilde{\phi}(0) = 0$. From Theorem 2.2 we obtain $\tilde{\phi}(X) \leq_{\rm ttt} \tilde{\phi}(Y)$. Hence from (2.5) we get $E[\tilde{\phi}(X)] \leq E[\tilde{\phi}(Y)]$, and this reduces to $E[\phi(X)] \leq E[\phi(Y)]$, provided the expectations exist. □

The order $\leq_{\rm ttt}$ seems to be closely related to the order $\leq_{\rm ew}$, and to the *location independent riskier* (lir) order of Jewitt (1989) which is defined by

$$X \leq_{\rm lir} Y \Longleftrightarrow \Big(\|D_F(p)\| \leq \|D_G(p)\|,\ \forall p \in (0,1)\Big),$$

where, for $p \in (0,1)$ (and $t = F^{-1}(p)$), the set $D_F(p)$, depicted in Figure 2, is defined as

$$D_F(p) \equiv \{(x,u) : u \in (0,p),\, x \in (F^{-1}(u), F^{-1}(p))\} = \{(x,u) : x \in (0,t),\, u \in (0, F(x))\},$$

and $D_G(p)$ is similarly defined. In particular, Kochar and Carrière (1997, Theorem 2.2) and Shaked and Shanthikumar (1998, Theorem 2.1) showed, under the same conditions on the supports of X and of Y as in the present Corollary 3.1, that $X \leq_{\rm ew} Y \Longrightarrow X \leq_{\rm icx} Y$ (see Corollary 4.3 in Section 4 below), and Fagiuoli, Pellerey, and Shaked (1999, Corollary 3.4) showed, under some conditions on the supports of X and of Y, that $X \leq_{\rm lir} Y \Longrightarrow X \leq_{\rm icv} Y$. Thus, one may ask: Can the result of the present Corollary 3.1 be directly derived from the above mentioned facts? We could not prove the present Corollary 3.1 using such an argument. In fact, we argue and show below that the order $\leq_{\rm ttt}$ is strictly different from any one of the orders $\leq_{\rm ew}$ and $\leq_{\rm lir}$.

First we show that none of the orders $\leq_{\rm ew}$ and $\leq_{\rm lir}$ imply the order $\leq_{\rm ttt}$. In order to see this, recall that the order $\leq_{\rm ew}$ is location independent in the sense of (2.4). The order $\leq_{\rm lir}$ is also location independent (an easy way to see it is by using the fact (see Figure 2) that $\|D_{F(\cdot - c)}(p)\| = \|D_F(p)\|$ for any $p \in (0,1)$ and $c \in (-\infty,\infty)$). Thus, if $X \leq_{\rm ew} Y$ (respectively, $X \leq_{\rm lir} Y$) had implied $X \leq_{\rm ttt} Y$ then it would have followed that it would have implied $X + c \leq_{\rm ttt} Y$ for every $c > 0$, and in particular it would have implied, by (2.5), that $E[X+c] \leq EY$ for every $c > 0$. But clearly the last inequality does not hold for $c > EY - EX$. Thus none of the inequalities $X \leq_{\rm ew} Y$ and $X \leq_{\rm lir} Y$ necessarily implies $X \leq_{\rm ttt} Y$. In a similar manner it can be shown that none of the inequalities $Y \leq_{\rm ew} X$ and $Y \leq_{\rm lir} X$ necessarily implies $X \leq_{\rm ttt} Y$.

The following examples show that the converses are also false.

Figure 2: *Depiction of D_F*

Example 3.2. In this example we show that

$$X \leq_{\rm ttt} Y \not\Rightarrow X \geq_{\rm ew} Y.$$

Let X, with distribution function F, be an exponential random variable with rate $\lambda > 0$, and let Y, with distribution function G, be a uniform $(0, 1)$ random variable, as in Example 2.1. We saw there that if $\lambda = 4$ then $X \leq_{\rm ttt} Y$. A straightforward computation yields

$$W_X(p) \equiv \|C_F(p)\| = \frac{1-p}{\lambda}, \quad p \in (0,1), \quad \text{and}$$
$$W_Y(p) \equiv \|C_G(p)\| = \frac{(1-p)2}{2}, \quad p \in (0,1).$$

Note, when $\lambda = 4$, that $W_X(p) \leq W_Y(p)$ if, and only if, $p \in (0, 1/2)$, and thus neither $X \leq_{\rm ew} Y$ nor $Y \leq_{\rm ew} X$ hold. ◀

Note that Example 3.2 also shows that

$$X \leq_{\rm ttt} Y \not\Rightarrow X \leq_{\rm st} Y. \tag{3.3}$$

This is so because for X and Y in Example 3.2 we have $X \not\leq_{\rm st} Y$.

Example 3.3. Let X, with distribution function F, be a uniform $(0, 1)$ random variable, and let Y be a beta$(2, 1)$ random variable, that is, the distribution function of Y is given by $G(x) = x2$, $x \in (0, 1)$. Obviously $X \leq_{\rm st} Y$, and therefore, by (2.2), $X \leq_{\rm ttt} Y$. On the other hand, a straightforward computation yields

$$\|D_F(p)\| = \frac{p2}{2}, \quad p \in (0,1), \quad \text{and}$$

$$\|D_G(p)\| = \frac{p^{3/2}}{3}, \quad p \in (0,1).$$

That is, $\|D_F(p)\| \le \|D_G(p)\|$ if, and only if, $p \le 4/9$, and thus neither $X \le_{\text{lir}} Y$ nor $Y \le_{\text{lir}} X$ hold. ◀

In light of (3.1) it is also of interest to note that without the assumption $EX = EY$, the orders $\le_{\text{ttt}}$ and $\le_{\text{nbue}}$ are distinct. This is shown in the following example.

Example 3.4. First we show that

$$X \ge_{\text{nbue}} Y \not\Rightarrow X \le_{\text{ttt}} Y.$$

In order to see this, first note that for any nondegenerate nonnegative random variable X, we have $X \ge_{\text{nbue}} X$. Since the order $\le_{\text{nbue}}$ is scale independent, it follows that for such a random variable X we have $aX \ge_{\text{nbue}} X$ for any $a > 0$. Now, obviously for $a > 1$ we have $EaX > EX$. Therefore, from (2.5) we get that $aX \not\le_{\text{ttt}} X$ when $a > 1$.

Next we show that

$$X \le_{\text{ttt}} Y \not\Rightarrow X \ge_{\text{nbue}} Y.$$

For this purpose, let X be a uniform $(0,2)$ random variable, and let Y have the distribution function G given by

$$G(x) = \begin{cases} 0, & x < 0; \\ x/2, & x \in [0,1]; \\ (x+1)/4, & x \in [1,3]; \\ 1, & x > 3; \end{cases}$$

that is, G is an equal mixture of the uniform $(0,1)$ and $(1,3)$ distributions. It is easy to see that $X \le_{\text{st}} Y$, and therefore, by (2.2), $X \le_{\text{ttt}} Y$. Actual computations of the TTT transforms give

$$T_X(p) = 2p - p2, \quad p \in (0,1); \quad \text{and}$$
$$T_Y(p) = \begin{cases} 2p - p2, & p \in (0,1/2); \\ 3/4 + (4p-2)(3/4 - p/2), & p \in [1/2,1]. \end{cases}$$

Also, $EX = T_X(1) = 1$ and $EY = T_Y(1) = 5/4$. Therefore $T_X(p)/EX > T_Y(p)/EY$ when $p \in (0,1/2)$. That is, $X/EX \not\le_{\text{ttt}} Y/EY$. It follows from (3.2) that $X \not\ge_{\text{nbue}} Y$. ◀

4 Some New Properties of the Excess Wealth Order

Let X and Y be two random variables with distribution functions F and G, respectively. It is well known (or it can be easily seen from (1.2)) that $X \le_{\text{ew}} Y$, or, equivalently, $X \le_{RS} Y$, if, and only if,

$$\int_{F^{-1}(p)}^{\infty} \overline{F}(x)\,dx \le \int_{G^{-1}(p)}^{\infty} \overline{G}(x)\,dx, \quad p \in (0,1). \tag{4.1}$$

The similarity between (2.1) and (4.1) may suggest that results which involve the order $\le_{\text{ttt}}$ may have analogs that involve the order $\le_{\text{ew}}$. In this section we highlight some similarities and some differences between these two orders. While doing that we also obtain some new results involving the order $\le_{\text{ew}}$.

First we note that the order $\le_{\text{ew}}$ is location independent (see (2.4))—an easy way to see it is to notice (see Figure 1) that

$$\|C_{F(\cdot-c)}(p)\| = \|C_F(p)\|, \quad p \in (0,1),\ c \in (-\infty, \infty).$$

In contrast, the order $\le_{\text{ttt}}$ is not location independent. We recall that the above facts about location independence were used in Section 3 to show that $Y \le_{\text{ew}} X \not\Rightarrow X \ge_{\text{ttt}} Y$.

Because of the location independence property of the order $\le_{\text{ew}}$, when we study this order we do not need to assume that the compared random variables are nonnegative. As a consequence, the random variables that are studied in this section can have any support in $\mathbb{R}$, unless stated otherwise.

Remark 4.1. In light of (3.1) it is of interest to note that without the assumption $EX = EY$, the orders $\le_{\text{ew}}$ and $\le_{\text{nbue}}$ are distinct. This can be seen using the facts that the order $\le_{\text{ew}}$ is location independent, whereas the order $\le_{\text{nbue}}$ is scale independent. Explicitly, for any random variable X we have that $X \le_{\text{ew}} X + a$ for any a. Now, suppose that X is nonnegative, and that $EX > 0$ is finite. Let $p \in (0,1)$ be such that $T_X(p) < EX$. Then, for any $a > 0$ we have

$$\frac{T_X(p)}{EX} < \frac{T_X(p) + a}{EX + a} = \frac{T_{X+a}(p)}{E(X+a)}.$$

Therefore $X/EX \not\ge_{\text{ttt}} (X+a)/E(X+a)$, and hence, by (3.2), $X \not\le_{\text{nbue}} X + a$.

Conversely, for any random variable X we have that $X \le_{\text{nbue}} aX$ for any $a > 0$. However, if X is a uniform $(0,1)$ random variable then, as can be easily verified, $X \not\le_{\text{ew}} aX$ when $a < 1$. ◀

In Theorem 2.2 we showed that the order $\le_{\text{ttt}}$ is closed under increasing concave transformations. In the following theorem it is shown that somewhat similarly the order $\le_{\text{ew}}$ is closed under increasing convex transformations.

Theorem 4.2. *Let X and Y be two continuous random variables with finite means. Then, for any increasing convex function ϕ we have*

$$X \le_{\text{ew}} Y \Longrightarrow \phi(X) \le_{\text{ew}} \phi(Y).$$

The proof of Theorem 4.2 is given the Appendix.

A result which is similar to Theorem 4.2 holds for the dispersive order. It is reported in Rojo and He (1991), but it is already implicit in Bartoszewicz (1985, p. 389).

Theorem 4.2 is a significant extension of Theorem 2.2 of Kochar and Carrière (1997) and of Theorem 2.1 of Shaked and Shanthikumar (1998) (which are stated as Corollary 4.3 below). Explicitly, let X and Y have the same left endpoint of support which, by the location independence property of the order $\le_{\rm ew}$, can be taken to be 0 without loss of generality. Let ϕ be an increasing convex function. Define $\tilde{\phi}(\cdot) \equiv \phi(\cdot) - \phi(0)$, so that $\tilde{\phi}(0) = 0$. Then both $\tilde{\phi}(X)$ and $\tilde{\phi}(Y)$ have 0 as the left endpoint of their supports. By Theorem 4.2 we have $\tilde{\phi}(X) \le_{\rm ew} \tilde{\phi}(Y)$, and from (4.1) with $p \to 0$ we obtain $E[\tilde{\phi}(X)] \le E[\tilde{\phi}(Y)]$, and therefore $E[\phi(X)] \le E[\phi(Y)]$. We thus obtain Theorem 2.2 of Kochar and Carrière (1997) and Theorem 2.1 of Shaked and Shanthikumar (1998) for continuous random variables as the following corollary. This corollary is used later in Section 5.

Corollary 4.3. *Let X and Y be two continuous random variables with finite means, and with a common left endpoint of support. Then $X \le_{\rm ew} Y \Longrightarrow X \le_{\rm icx} Y$.*

The following example shows that the convexity assumption in Theorem 4.2 cannot be dropped.

Example 4.4. In this example we show that

$$X \le_{\rm ew} Y \not\Longrightarrow \phi(X) \le_{\rm ew} \phi(Y) \text{ for all increasing functions } \phi.$$

Let X, with distribution function F, be a uniform $(0,1)$ random variable, and let Y, with distribution function G, be an exponential random variable with rate 2. Then a straightforward computation yields

$$\int_{F^{-1}(p)}^{\infty} \overline{F}(x)\,dx = \frac{(1-p)2}{2}, \quad \int_{G^{-1}(p)}^{\infty} \overline{G}(x)\,dx = \frac{1-p}{2}, \quad p \in (0,1).$$

Therefore $X \le_{\rm ew} Y$. Let $\phi(x) = 1 - e^{-x}$, $x \ge 0$. Then

$$\int_{F^{-1}(p)}^{\infty} \overline{F}(x)\phi'(x)\,dx = e^{-1} - pe^{-p}, \quad \int_{G^{-1}(p)}^{\infty} \overline{G}(x)\phi'(x)\,dx = \frac{(1-p)^{3/2}}{3}, \quad p \in (0,1).$$

The first function is smaller than the second for p in a right neighborhood of 0. Therefore $\phi(X) \not\le_{\rm ew} \phi(Y)$. ◀

5 Some Applications of the TTT Transform and the Excess Wealth Orders

In this section we give various applications of the results that were developed in previous sections. We remind the reader of (3.1); that is, the $\le_{\rm ttt}$ comparison is the same as the $\ge_{\rm ew}$ comparison when the compared random variables have the same means. Below we do not always state the results for both of the above orders, but in some cases (when the means are equal) it should be easy to translate a result involving one order into a result involving the other order (and to the order $\ge_{\rm nbue}$ as well).

The first theorem below shows that if $X \leq_{\rm ttt} Y$ then a series system of n components having independent lifetimes which are copies of Y has a larger lifetime, in the sense of $\leq_{\rm ttt}$, than a similar system of n components having independent lifetimes which are copies of X. A similar result for parallel systems involving the excess wealth order is also given. The proof of the following theorem is given in the Appendix.

Theorem 5.1. *Let $X_1, X_2, \ldots$ be a collection of independent and identically distributed random variables, and let $Y_1, Y_2, \ldots$ be another collection of independent and identically distributed random variables.*

(a) *If X_1 and Y_1 are nonnegative, and if $X_1 \leq_{\rm ttt} Y_1$, then $\min\{X_1, X_2, \ldots, X_n\} \leq_{\rm ttt} \min\{Y_1, Y_2, \ldots, Y_n\}$, $n \geq 1$.*

(b) *If $X_1 \leq_{\rm ew} Y_1$ then $\max\{X_1, X_2, \ldots, X_n\} \leq_{\rm ew} \max\{Y_1, Y_2, \ldots, Y_n\}$, $n \geq 1$.*

Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two collections of independent and identically distributed random variables with 0 being the common left endpoint of the supports. Barlow and Proschan (1975, page 121) proved that if $X_1 \leq_{\rm icv} Y_1$, then $\min\{X_1, X_2, \ldots, X_n\} \leq_{\rm icv} \min\{Y_1, Y_2, \ldots, Y_n\}$, $n \geq 1$. Comparing this to Theorem 5.1(a) we see, using Corollary 3.1, that the latter yields a stronger conclusion, but under a stronger assumption. Barlow and Proschan (1975, page 121) also proved that if $X_1 \leq_{\rm icx} Y_1$, then $\max\{X_1, X_2, \ldots, X_n\} \leq_{\rm icx} \max\{Y_1, Y_2, \ldots, Y_n\}$, $n \geq 1$. Comparing this result to Theorem 5.1(b) we see, this time using Corollary 4.3, that the latter again yields a stronger conclusion, but, again, under a stronger assumption.

Application 5.2 (Reliability). Recall from Belzunce (1999) that if a random variable X with mean μ is NBUE then

$$X \leq_{\rm ew} {\rm Exp}(\mu), \tag{5.1}$$

where ${\rm Exp}(\mu)$ denotes an exponential random variable with mean μ. Consider now a parallel system of n components having independent and identically distributed NBUE lifetimes $X_1, X_2, \ldots, X_n$ with the left endpoint of the common support being 0. Denote the common mean by μ. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed exponential random variables with mean μ. From Theorem 5.1(b) we obtain

$$\max\{X_1, X_2, \ldots, X_n\} \leq_{\rm ew} \max\{Y_1, Y_2, \ldots, Y_n\}. \tag{5.2}$$

Since both $\max\{X_1, X_2, \ldots, X_n\}$ and $\max\{Y_1, Y_2, \ldots, Y_n\}$ have 0 as the left endpoint of their corresponding supports, it follows that

$$\begin{aligned} E[\max\{X_1, X_2, \ldots, X_n\}] &\leq E[\max\{Y_1, Y_2, \ldots, Y_n\}], \quad \text{and} \\ {\rm Var}[\max\{X_1, X_2, \ldots, X_n\}] &\leq {\rm Var}E[\max\{Y_1, Y_2, \ldots, Y_n\}] \end{aligned}$$

(this is so since if two random variables X and Y have 0 as the left endpoint of their respective supports, and if $X \leq_{\rm ew} Y$, then $EX \leq EY$ and ${\rm Var}[X] \leq {\rm Var}[Y]$; the first inequality follows from (4.1) with $p \to 0$, and the second inequality follows from Corollary 3.3 in Shaked and Shanthikumar (1998)). Now, computing

$$E[\max\{Y_1, Y_2, \ldots, Y_n\}] = \int_0^\infty \left[1 - \left(1 - e^{-\frac{x}{\mu}}\right)^n\right] dx = \int_0^\infty \sum_{k=0}^{n-1} e^{-\frac{x}{\mu}} \left(1 - e^{-\frac{x}{\mu}}\right)^k dx = \mu \sum_{k=1}^{n} \frac{1}{k},$$

and

$$E[(\max\{Y_1, Y_2, \ldots, Y_n\})2] = 2 \int_0^\infty x\left[1 - \left(1 - e^{-\frac{x}{\mu}}\right)^n\right] dx = 2\mu 2 \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k2} \binom{n}{k},$$

we obtain the following upper bounds on the mean and on the variance of the lifetime of the parallel system

$$E[\max\{X_1, X_2 \ldots, X_n\}] \le \mu \sum_{k=1}^{n} \frac{1}{k} \tag{5.3}$$

and

$$\mathrm{Var}[\max\{X_1, X_2 \ldots, X_n\}] \le \mu 2 \left[2 \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k2} \binom{n}{k} - \left(\sum_{k=1}^{n} \frac{1}{k}\right) 2\right]. \tag{5.4}$$

It should be remarked that (5.3) (but not (5.4)) can alternatively be obtained also as follows. Let X_i and Y_i be as above, $i = 1, 2, \ldots, n$. If $X_i \le_{\text{ew}} Y_i$, and they both have 0 as the left endpoint of their supports, then $X_i \le_{\text{icx}} Y_i$ (see Corollary 4.3). It follows by Theorem 9 of Li, Li, and Jing (2000) (or by a more general result of Ross (1996, p. 436) which is also given as Theorem 3.A.9 in Shaked and Shanthikumar (1994)) that $\max\{X_1, X_2, \ldots, X_n\} \le_{\text{icx}} \max\{Y_1, Y_2, \ldots, Y_n\}$, and therefore (5.3) holds. In fact, (5.3) even holds if the X_i's are merely HNBUE (harmonic new better than used in expectation, that is, $X_i \le_{\text{icx}} \text{Exp}(\mu)$, where μ is the mean of X_i, $i = 1, 2, \ldots, n$) rather than NBUE.

We also mention that the inequalities (5.3) and (5.4) are reversed if the X_i's are new worse than used in expectation (NWUE).

Finally it is worthwhile to note that from (3.1) and (5.1) it follows that if X is an NBUE random variable with mean μ then $X \ge_{\text{ttt}} \text{Exp}(\mu)$. Therefore, from Theorem 5.1(a) we obtain

$$\min\{X_1, X_2, \ldots, X_n\} \ge_{\text{ttt}} \min\{Y_1, Y_2, \ldots, Y_n\},$$

where the X_i's and the Y_i's are the same as in inequality (5.2). ★

From Theorem 5.1(a) and (2.5) we get the following corollary.

Corollary 5.3. *Let $X_1, X_2, \ldots, X_n$ be a collection of independent and identically distributed random variables, and let $Y_1, Y_2, \ldots, Y_n$ be another collection of independent and identically distributed random variables. If X_1 and Y_1 are nonnegative, and if $X_1 \le_{\text{ttt}} Y_1$, then*

$$E[\min\{X_1, X_2, \ldots, X_n\}] \le E[\min\{Y_1, Y_2, \ldots, Y_n\}].$$

A similar result which compares $E[\max\{X_1, X_2, \ldots, X_n\}]$ and $E[\max\{Y_1, Y_2, \ldots, Y_n\}]$ can be derived under the assumptions that X_1 and Y_1 have the same left endpoint of support, and $X_1 \le_{\mathrm{ew}} Y_1$; see Application 5.2.

It is worthwhile to mention that whereas the conclusion of Corollary 5.3 easily follows from $X \le_{\mathrm{st}} Y$, the assumption $X \le_{\mathrm{ttt}} Y$ of the corollary is strictly weaker than $X \le_{\mathrm{st}} Y$; see (2.2) and (3.3).

A useful identity that involves the TTT transform T_X of a nonnegative random variable X is given in the next lemma.

Lemma 5.4. *Let X be a nonnegative random variable with survival function $\overline{F}$. Then*

$$(n-1)\int_0 1(1-p)^{n-2} T_X(p)\,dp = \int_0^\infty \overline{F}^n(t)\,dt, \quad n \ge 2. \tag{5.5}$$

Proof. We compute

$$\begin{aligned}\int_0 1(1-p)^{n-2} T_X(p)\,dp &= \int_0 1 \int_0^{F^{-1}(p)} (1-p)^{n-2}\overline{F}(t)\,dtdp \\ &= \int_0^\infty \int_0^x \overline{F}^{n-2}(x)\overline{F}(t)\,dtdF(x) \\ &= \int_0^\infty \int_t^\infty \overline{F}^{n-2}(x)\overline{F}(t)\,dF(x)dt \\ &= \int_0^\infty \frac{1}{n-1}\overline{F}^n(t)\,dt,\end{aligned}$$

and the stated result follows. □

The identity (5.5) is used in the following application.

Application 5.5 (Economics). Let F be the wealth distribution of some population. Bhattacharjee and Krishnaji (1984) studied the following Lorenz measure of inequality:

$$L_F = 1 - 2\int_0^\infty F_1(x)\,dF(x),$$

where F_1 is the length-biased distribution associated with F, given by

$$F_1(x) = \mu_F^{-1}\int_0^x t\,dF(t), \quad x \ge 0.$$

A straightforward computation gives

$$L_F = 1 - \mu_F^{-1}\int_0^\infty \overline{F}^2(x)\,dx$$

(this corrects a minor mistake in Klefsjö (1984, page 306)). Now, from (5.5) it is seen that if X and Y are two nonnegative random variables corresponding to wealth distributions F and G, respectively, and if $EX = EY$ and $X \le_{\mathrm{ttt}} Y$, then $L_F \ge L_G$; that is, a wealth distribution that is larger in the $\le_{\mathrm{ttt}}$ order yields a smaller inequality measure. In other words, by (3.1), a wealth distribution that is smaller in the $\le_{\mathrm{ew}}$ order yields a smaller inequality measure. ★

A further application of the orders $\leq_{\rm ttt}$, $\geq_{\rm ew}$, and $\geq_{\rm nbue}$ is the following.

Application 5.6 (Statistical reliability). Let $X_1, X_2, \ldots, X_m$ be a sample (of size m) of independent and identically distributed nonnegative random variables with a finite mean and a common continuous distribution function F, and let $Y_1, Y_2, \ldots, Y_n$ be another sample (of size n) of independent and identically distributed nonnegative random variables with a finite mean and a common continuous distribution function G. We assume that the two samples are independent and we wish to test the null hypothesis

$$H_0 : F =_{\rm nbue} G \quad \text{(that is, } F(\cdot) = G(\theta \cdot) \text{ for some } \theta > 0),$$

against the alternative hypothesis

$$H_1 : G \text{is more NBUE than } F \quad \text{(that is, } Y_1 \leq_{\rm nbue} X_1).$$

Let X and Y denote generic random variables with distributions F and G, respectively. Motivated by (3.2) (that is, $Y \leq_{\rm nbue} X \iff \frac{X}{EX} \leq_{\rm ttt} \frac{Y}{EY}$) it is seen that for testing H_0 versus H_1, one can base a test on an estimate of

$$S \equiv \int_0 1 \left[\frac{T_Y(p)}{EY} - \frac{T_X(p)}{EX} \right] dp.$$

This integral is the difference between the area below the scaled TTT transform of X and that below Y. A practitioner of the test described below should be aware that S may be positive even if these transforms cross each other (that is, if $Y_1 \not\leq_{\rm nbue} X_1$).

Let $0 \equiv X_{0:m} \leq X_{1:m} \leq X_{2:m} \leq \cdots \leq X_{m:m}$ denote the order statistics corresponding to $X_1, X_2, \ldots, X_m$. The corresponding empirical TTT transform, T_m^X, is defined by

$$T_m^X(p) = \int_0^{F_m^{-1}(p)} \overline{F}_m(x)\, dx, \quad 0 \leq p \leq 1, \tag{5.6}$$

where F_m and $\overline{F}_m$ are the corresponding empirical distribution and survival functions. From (5.6) we have

$$T_m^X\left(\frac{i}{m}\right) = \frac{1}{m} \sum_{j=1}^{i} (m - j + 1)(X_{j:m} - X_{j-1:m}), \quad 0 \leq i \leq m.$$

Note that $T_m^X(1) = \overline{X}_m$. Similarly define $T_n^Y(i/n)$, $0 \leq i \leq n$. The cumulative empirical scaled TTT statistics based on the X-sample and on the Y-samples are, respectively,

$$A_m^X = \frac{1}{m} \sum_{i=1}^{m-1} \frac{T_m^X(i/m)}{T_m^X(1)} \quad \text{and} \quad A_n^Y = \frac{1}{n} \sum_{i=1}^{n-1} \frac{T_n^Y(i/n)}{T_n^Y(1)}.$$

Barlow and Doksum (1972) proposed a test based on large values of A_m^X for the one-sample goodness-of-fit problem of testing the exponentiality of F against IFR alternatives. Later

Hollander and Proschan (1975) proved the consistency of the same test for NBUE alternatives. The test was also generalized by Klefsjö (1983) to the larger HNBUE class.

For testing H_0 versus H_1 above, we base out test on large values of the statistic

$$S_{m,n} = A_n^Y - A_m^X.$$

Let $N = m + n$. Denote $\eta(F) = \int_0 1 \frac{T_X(p)}{EX}\, dp$. Note, by (5.5), that $\eta(F) = \int_0^\infty \overline{F}^2(t)\, dt$. Define

$$\nu 2(F) = 2 \iint_{0 \le x \le y} [2\overline{F}(x) - \eta(F)][2\overline{F}(y) - \eta(F)]F(x)\overline{F}(y)\, dxdy. \tag{5.7}$$

Similarly define $\nu 2(G)$. It follows from Theorem 6.6 of Barlow, Bartholomew, Bremner, and Brunk (1972) that, under some regularity conditions, the limiting distribution of

$$N^{1/2}[S_{m,n} - (\eta(G) - \eta(F))]$$

is normal with mean 0 and variance

$$\sigma 2 = \frac{\nu 2(F)}{\lambda(EX)2} + \frac{\nu 2(G)}{(1-\lambda)(EY)2}, \tag{5.8}$$

where $\lambda = \lim_{N\to\infty} \frac{m}{N}$ and $0 < \lambda < 1$.

Let $\hat{\sigma} 2_{m,n}$ be a consistent estimator of $\sigma 2$. Such an estimator can be obtained, for example, by replacing F and G in (5.7) and (5.8) by the corresponding empirical distribution functions. It follows that under the null hypothesis H_0 the limiting distribution of $N^{1/2}S_{m,n}/\hat{\sigma}_{m,n}$ is normal with mean 0 and variance 1. Thus, the two-sample test for testing H_0 versus H_1, which reject H_0 when

$$N^{1/2}S_{m,n}/\hat{\sigma}_{m,n} > z_{1-\alpha},$$

where $z_{1-\alpha}$ is the quantile of order $(1-\alpha)$ of the standard normal distribution, is asymptotically unbiased whenever $\frac{X}{EX} \le_{\text{ttt}} \frac{Y}{EY}$, that is, $X \ge_{\text{nbue}} Y$.

Ideas similar to those used above have been utilized by Gerlach (1988) to propose a test for the two-sample problem of testing that one distribution is "more NBU" than another. ★

A Appendix

In this Appendix we give the proofs of Theorems 2.2, 4.2, and 5.1, as well as lemmas that are used in these proofs.

Proof of Theorem 2.2. Let F and G denote the distribution functions of X and of Y, respectively. First note that if F and G are not identical, and do not cross each other, then, from (2.1) it is seen that $\overline{F} \le \overline{G}$ at a right neighborhood of 0, and therefore $\overline{F}(x) \le \overline{G}(x)$ for all $x \ge 0$; that is, $X \le_{\text{st}} Y$. It then follows that $\phi(X) \le_{\text{st}} \phi(Y)$ for any increasing function ϕ, and from (2.2) we get $\phi(X) \le_{\text{ttt}} \phi(Y)$.

Thus, let us assume that F and G cross each other at least once. Denote the consecutive crossing points by $(0,0) \equiv (t_0, p_0), (t_1, p_1), (t_2, p_2), \ldots$; see an example in Figure 3. Let ϕ be an increasing concave function such that $\phi(0) = 0$. For simplicity we assume that ϕ is differentiable with derivative ϕ'. We note that

$$T_{\phi(X)}(p) = \int_0^{F^{-1}(p)} \overline{F}(x)\phi'(x)\,dx, \quad p \in (0,1), \quad \text{and}$$
$$T_{\phi(Y)}(p) = \int_0^{G^{-1}(p)} \overline{G}(x)\phi'(x)\,dx, \quad p \in (0,1).$$

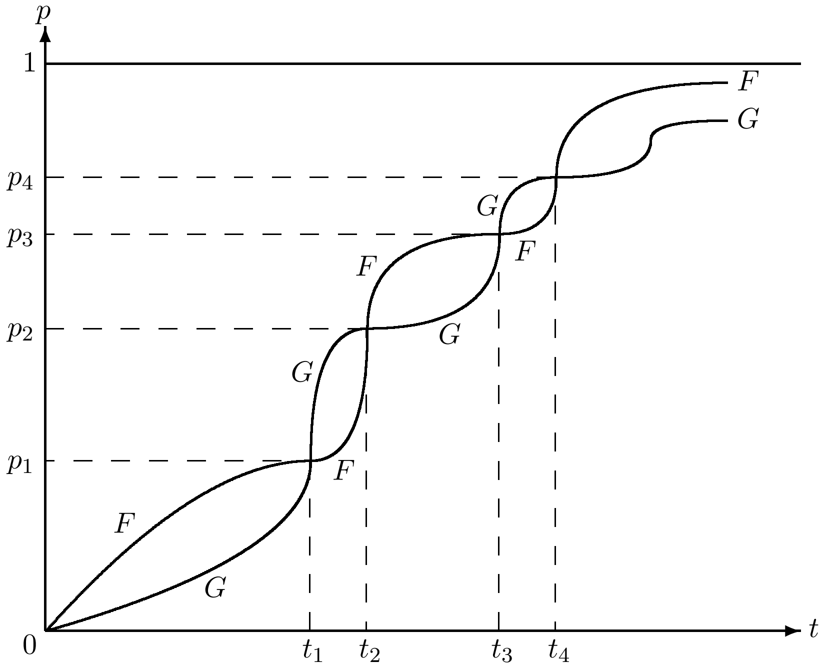

Figure 3: *Typical graphs of the distribution functions F and G (of X and Y, respectively) when $X \leq_{\text{ttt}} Y$*

First consider $p \in (0, p_1]$. Then $G^{-1}(p) \geq F^{-1}(p)$. Also, for $x \in (0, G^{-1}(p))$ we have $\overline{G}(x) - \overline{F}(x) \geq 0$ and $\phi'(x) \geq \phi'(t_1) \geq 0$ (since ϕ is increasing and concave). Thus

$$\begin{aligned} T_{\phi(Y)}(p) - T_{\phi(X)}(p) &\geq \phi'(t_1)\left[\int_0^{F^{-1}(p)} \left[\overline{G}(x) - \overline{F}(x)\right] dx + \int_{F^{-1}(p)}^{G^{-1}(p)} \overline{G}(x)\,dx\right] \\ &= \phi'(t_1)[T_Y(p) - T_X(p)], \quad p \in (0, p_1]. \end{aligned} \tag{A.1}$$

Next let $p \in (p_1, p_2]$ (here $p_2 = 1$ if F and G cross only once). Then $G^{-1}(p) \leq F^{-1}(p)$. Also (recall that $F^{-1}(p_1) = G^{-1}(p_1) = t_1$), for $x \in (t_1, F^{-1}(p))$ we have $\overline{F}(x) - \overline{G}(x) \geq 0$ and $0 \leq \phi'(x) \leq \phi'(t_1)$ (since ϕ is increasing and concave). Thus

$$\begin{aligned} &T_{\phi(Y)}(p) - T_{\phi(X)}(p) \\ &\quad = T_{\phi(Y)}(p_1) - T_{\phi(X)}(p_1) + \int_{t_1}^{G^{-1}(p)} \left[\overline{G}(x) - \overline{F}(x)\right]\phi'(x)\,dx - \int_{G^{-1}(p)}^{F^{-1}(p)} \overline{F}(x)\phi'(x)\,dx \end{aligned}$$

$$\geq T_{\phi(Y)}(p_1) - T_{\phi(X)}(p_1) + \phi'(t_1)\left[\int_{t_1}^{G^{-1}(p)} \left[\overline{G}(x) - \overline{F}(x)\right] dx - \int_{G^{-1}(p)}^{F^{-1}(p)} \overline{F}(x)\, dx\right]$$
$$\geq \phi'(t_1)[T_Y(p_1) - T_X(p_1)] + \phi'(t_1)[T_Y(p) - T_Y(p_1) - T_X(p) + T_X(p_1)],$$

where the last inequality follows from (A.1). That is,

$$T_{\phi(Y)}(p) - T_{\phi(X)}(p) \geq \phi'(t_1)[T_Y(p) - T_X(p)], \quad p \in (p_1, p_2]. \tag{A.2}$$

In a manner similar to the proof of (A.1) it can be shown that if F and G cross at least twice then for $p \in (p_2, p_3]$ we have

$$\begin{aligned}
&T_{\phi(Y)}(p) - T_{\phi(X)}(p) \\
&\quad \geq T_{\phi(Y)}(p_2) - T_{\phi(X)}(p_2) + \phi'(t_3)\big[[T_Y(p) - T_Y(p_2)] - [T_X(p) - T_X(p_2)]\big] \\
&\quad \geq \phi'(t_1)[T_Y(p_2) - T_X(p_2)] + \phi'(t_3)\big[[T_Y(p) - T_Y(p_2)] - [T_X(p) - T_X(p_2)]\big] \\
&\quad \geq \phi'(t_3)[T_Y(p_2) - T_X(p_2)] + \phi'(t_3)\big[[T_Y(p) - T_Y(p_2)] - [T_X(p) - T_X(p_2)]\big]
\end{aligned}$$

(here, if F and G cross exactly twice we set $p_3 = 1$ and $\phi'(t_3) = \lim_{t\to\infty} \phi'(t)$), where the second inequality above follows from (A.2), and the last inequality from the concavity of ϕ and $t_3 \geq t_1$. That is,

$$T_{\phi(Y)}(p) - T_{\phi(X)}(p) \geq \phi'(t_3)[T_Y(p) - T_X(p)], \quad p \in (p_2, p_3]. \tag{A.3}$$

Furthermore, if F and G cross each other at least three times it can be shown, using (A.3) and the ideas in the proof of (A.2), that

$$T_{\phi(Y)}(p) - T_{\phi(X)}(p) \geq \phi'(t_3)[T_Y(p) - T_X(p)], \quad p \in (p_3, p_4];$$

here $p_4 = 1$ if F and G cross exactly three times.

In general, if F and G cross each other at least i times then

$$T_{\phi(Y)}(p) - T_{\phi(X)}(p) \geq \phi'(t_{j(i)})[T_Y(p) - T_X(p)], \quad p \in (p_i, p_{i+1}], \tag{A.4}$$

where $j(i) = i$ if i is odd, and $j(i) = i + 1$ if i is even. If there are exactly i crossings, and i is even, then in (A.4) we take $p_{i+1} = 1$ and $\phi'(t_{j(i)}) = \lim_{t\to\infty} \phi'(t)$. From (A.4) and $X \leq_{\text{ttt}} Y$ we get that

$$T_{\phi(Y)}(p) - T_{\phi(X)}(p) \geq 0, \quad p \in (p_i, p_{(i+1)}]. \tag{A.5}$$

Since (A.5) is true for all relevant i we obtain $T_{\phi(Y)}(p) - T_{\phi(X)}(p) \geq 0$ for all $p \in (0, 1)$, that is, $\phi(X) \leq_{\text{ttt}} \phi(Y)$. □

For the proof of Theorem 4.2 we will need the following two lemmas.

Lemma A.1 (Belzunce (1999)). *Let X and Y be two continuous random variables with distribution functions F and G, respectively. Then $X \leq_{\text{ew}} Y$ if, and only if,*

$$\int_t^\infty \overline{F}(x + F^{-1}(p))\, dx \leq \int_t^\infty \overline{G}(x + G^{-1}(p))\, dx, \quad t \geq 0,\ p \in (0, 1).$$

Lemma A.2 (Barlow and Proschan (1975, p. 120)). *Let W be a measure on the interval (a,b), not necessarily nonnegative. Let h be a nonnegative function defined on (a,b).*

(a) *If $\int_t^b dW(x) \geq 0$ for all $t \in (a,b)$, and if h is increasing, then $\int_a^b h(x)\,dW(x) \geq 0$.*

(b) *If $\int_a^t dW(x) \geq 0$ for all $t \in (a,b)$, and if h is decreasing, then $\int_a^b h(x)\,dW(x) \geq 0$.*

Proof of Theorem 4.2. Let F and G be the distribution functions of X and Y, respectively. Assume that $X \leq_{\rm ew} Y$. Let ϕ be an increasing convex function; for simplicity we assume that ϕ is strictly increasing and differentiable.

Let F_ϕ and G_ϕ denote the distribution functions of $\phi(X)$ and $\phi(Y)$, respectively. Then

$$F_\phi(x) = F(\phi^{-1}(x)), \quad G_\phi(x) = G(\phi^{-1}(x)), \quad x \in \mathbb{R}, \quad \text{and}$$
$$F_\phi^{-1}(p) = \phi(F^{-1}(p)), \quad G_\phi^{-1}(p) = \phi(G^{-1}(p)), \quad p \in (0,1).$$

Therefore

$$\begin{aligned}\int_{F_\phi^{-1}(p)}^\infty \overline{F}_\phi(x)\,dx &= \int_{\phi(F^{-1}(p))}^\infty \overline{F}(\phi^{-1}(x))\,dx \\ &= \int_{F^{-1}(p)}^\infty \overline{F}(y)\phi'(y)\,dy \\ &= \int_0^\infty \overline{F}(y+F^{-1}(p))\phi'(y+F^{-1}(p))\,dy, \quad p \in (0,1).\end{aligned}$$

Similarly,

$$\int_{G_\phi^{-1}(p)}^\infty \overline{G}_\phi(x)\,dx = \int_0^\infty \overline{G}(y+G^{-1}(p))\phi'(y+G^{-1}(p))\,dy, \quad p \in (0,1).$$

Thus, in order to prove the theorem we need to show that

$$\int_{G^{-1}(p)}^\infty \overline{G}(x)\phi'(x)\,dx \geq \int_{F^{-1}(p)}^\infty \overline{F}(x)\phi'(x)\,dx, \quad p \in (0,1). \tag{A.6}$$

or, equivalently, that

$$\int_0^\infty \overline{G}(x+G^{-1}(p))\phi'(x+G^{-1}(p))\,dx \geq \int_0^\infty \overline{F}(x+F^{-1}(p))\phi'(x+F^{-1}(p))\,dx, \quad p \in (0,1). \tag{A.7}$$

First we show that (A.7) holds for all $p \in (0,1)$ such that $G^{-1}(p) \geq F^{-1}(p)$. For such a p, using the increasingness of ϕ', we get

$$\begin{aligned}&\int_0^\infty \Big[\overline{G}(x+G^{-1}(p))\phi'(x+G^{-1}(p)) - \overline{F}(x+F^{-1}(p))\phi'(x+F^{-1}(p))\Big]dx \\ &\qquad \geq \int_0^\infty \Big[\overline{G}(x+G^{-1}(p)) - \overline{F}(x+F^{-1}(p))\Big]\phi'(x+F^{-1}(p))dx, \quad p \in (0,1).\end{aligned} \tag{A.8}$$

By Lemma A.1 we have

$$\int_t^\infty \Big[\overline{G}(x+G^{-1}(p)) - \overline{F}(x+F^{-1}(p))\Big]\,dx \geq 0, \quad t \geq 0.$$

Since $\phi'(x + F^{-1}(p))$ is nonnegative and increasing in x, it follows from Lemma A.2 that

$$\int_0^\infty \left[\overline{G}(x + G^{-1}(p)) - \overline{F}(x + F^{-1}(p))\right]\phi'(x + F^{-1}(p))\,dx \geq 0.$$

This inequality, applied to (A.8), yields (A.7) for all $p \in (0,1)$ such that $G^{-1}(p) \geq F^{-1}(p)$.

Consider now a $p \in (0,1)$ such that $G^{-1}(p) < F^{-1}(p)$. Note that in such a case F and G are distinct and they must cross each other because otherwise (4.1) would not hold in a left neighborhood of 1. In fact, in the last point of crossing F must cross G from below. Therefore there exists a point $p_2 \in (p,1)$ defined by $p_2 = \inf\{u > p : G^{-1}(p) \geq F^{-1}(p)\}$. Define also $p_1 = \sup\{u < p : G^{-1}(p) \geq F^{-1}(p)\}$, where $p_1 \equiv 0$ if $\{u < p : G^{-1}(p) \geq F^{-1}(p)\} = \varnothing$. Denote $t_i = F^{-1}(p_i)$ and note that $t_i = G^{-1}(p_i)$, $i = 1, 2$, by the continuity of F and G; see Figure 4.

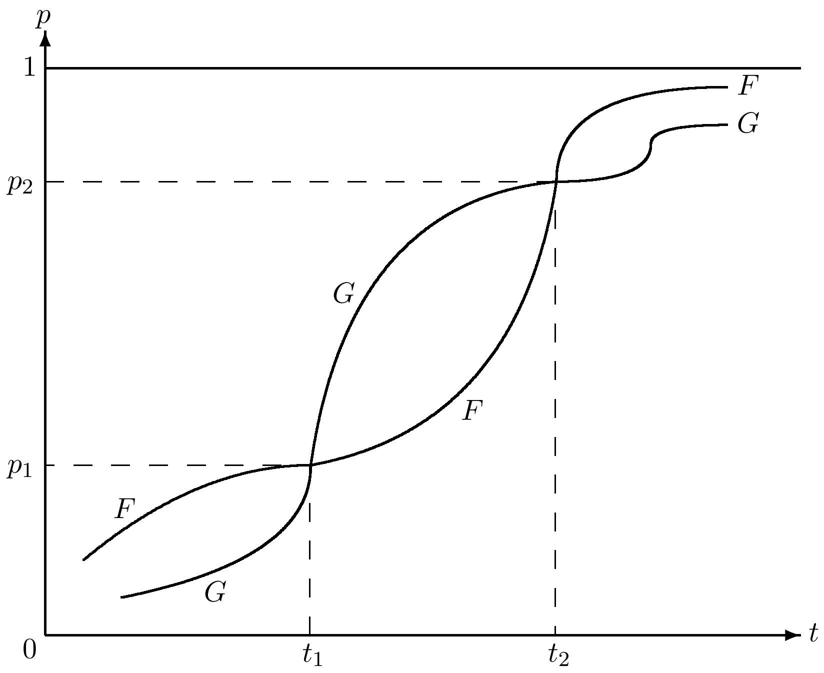

Figure 4: *Typical crossing points of the distribution functions F and G (of X and Y, respectively) when $X \leq_{\text{ew}} Y$*

For $p \in (0,1)$ such that $G^{-1}(p) < F^{-1}(p)$ we have $\overline{G}(x) \leq \overline{F}(x)$ for all $x \in [G^{-1}(p_1), G^{-1}(p)]$. Recall also that $G^{-1}(p_1) = F^{-1}(p_1)$. Therefore

$$\begin{aligned}
\int_{G^{-1}(p)}^\infty \overline{G}(x)\phi'(x)\,dx &= \int_{G^{-1}(p_1)}^\infty \overline{G}(x)\phi'(x)\,dx - \int_{G^{-1}(p_1)}^{G^{-1}(p)} \overline{G}(x)\phi'(x)\,dx \\
&\geq \int_{G^{-1}(p_1)}^\infty \overline{G}(x)\phi'(x)\,dx - \int_{F^{-1}(p_1)}^{F^{-1}(p)} \overline{F}(x)\phi'(x)\,dx \\
&\geq \int_{F^{-1}(p_1)}^\infty \overline{F}(x)\phi'(x)\,dx - \int_{F^{-1}(p_1)}^{F^{-1}(p)} \overline{F}(x)\phi'(x)\,dx \\
&= \int_{F^{-1}(p)}^\infty \overline{F}(x)\phi'(x)\,dx,
\end{aligned}$$

where the second inequality follows from the validity of (A.6) for p_1 proven earlier. This proves that (A.6) holds also for $p \in (0,1)$ such that $G^{-1}(p) < F^{-1}(p)$, and the proof of the theorem is complete. □

Because the orders $\le_{\mathrm{ew}}$ and $\le_{\mathrm{ttt}}$ are essentially different, the proofs of Theorems 2.2 and 4.2 should be contrasted. On one hand, both proofs share the idea of obtaining the desired inequalities on one interval at the time, where the intervals are determined by the points in which F and G cross each other. On the other hand, the proofs differ significantly once the inter-crossing interval is fixed.

We end this Appendix with the proof of Theorem 5.1.

Proof of Theorem 5.1. We only give the proof of part (a) since the proof of part (b) is similar. So assume that $X_1 \le_{\mathrm{ttt}} Y_1$. It suffices to consider only the case $n = 2$. Let $\overline{F}$ and $\overline{G}$ denote the survival functions of X_1 and of Y_1, respectively, and let $\overline{F}_2$ and $\overline{G}_2$ denote the survival functions of $\min\{X_1, X_2\}$ and of $\min\{Y_1, Y_2\}$, respectively. That is,

$$\overline{F}_2(x) = \overline{F}^2(x), \quad x \ge 0,$$

and

$$\overline{G}_2(x) = \overline{G}^2(x), \quad x \ge 0.$$

Now, from the assumed inequality (2.1) it follows that

$$\int_0^p (1-u)d(G^{-1}(u) - F^{-1}(u)) \ge 0, \quad p \in (0,1).$$

By Lemma A.2(b) it is seen that

$$\int_0^p (1-u)^2 d(G^{-1}(u) - F^{-1}(u)) \ge 0, \quad p \in (0,1).$$

That is,

$$\int_0^{F^{-1}(p)} \overline{F}^2(x)\,dx \le \int_0^{G^{-1}(p)} \overline{G}^2(x)\,dx, \quad p \in (0,1).$$

Since $F_2^{-1}(p) = F^{-1}(1-\sqrt{1-p}\,)$ and $G_2^{-1}(p) = G^{-1}(1-\sqrt{1-p}\,)$, $p \in (0,1)$, it follows that

$$\int_0^{F_2^{-1}(p)} \overline{F}_2(x)\,dx \le \int_0^{G_2^{-1}(p)} \overline{G}_2(x)\,dx, \quad p \in (0,1),$$

that is, $\min\{X_1, X_2\} \le_{\mathrm{ttt}} \min\{Y_1, Y_2\}$. □

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