RELATIONSHIP BETWEEN STRONG MONOTONICITY PROPERTY, P_2-PROPERTY, AND THE GUS-PROPERTY IN SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEMS

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In a recent paper on semidefinite linear complementarity problems, Gowda and Song (2000) introduced and studied the P-property, P_2-property, GUS-property, and strong monotonicity property for linear transformation $L: S^n \to S^n$, where S^n is the space of all symmetric and real $n \times n$ matrices. In an attempt to characterize the P_2-property, they raised the following two questions: (i) Does the strong monotonicity imply the P_2-property? (ii) Does the GUS-property imply the P_2-property? In this paper, we show that the strong monotonicity property implies the P_2-property for any linear transformation and describe an equivalence between these two properties for Lyapunov and other transformations. We show by means of an example that the GUS-property need not imply the P_2-property, even for Lyapunov transformations.

1. Introduction. Let S^n be the space of all symmetric real $n \times n$ matrices and S^n_+ the space of symmetric and real $n \times n$ positive semidefinite matrices. Given a linear transformation $L: S^n \to S^n$ and $Q \in S^n$, the semidefinite linear complementarity problem SDLCP(L, S^n_+, Q) is the problem of finding a matrix $X \in S^n$ such that

$$X \in S^n_+, \qquad Y = L(X) + Q \in S^n_+, \qquad \langle X, Y \rangle = \mathrm{tr}(XY) = 0,$$

where "tr" denotes the trace.

This problem was originally introduced by Kojima et al. (1997), although in a slightly different form. The SDLCP can be considered as a generalization of the linear complementarity problem (LCP); see Cottle et al. (1992). Motivated by various useful results in the linear complementarity theory, Gowda and Song (2000) introduced the P, GUS, and various other properties for the SDLCP. For related results on SDLCP, see Gowda and Parthasarathy (2000). As mentioned in Gowda and Song (2000), the commutativity of X and $L(X)$ makes the analysis of P-property simpler, since X and $L(X)$ can be simultaneously diagonalized. The question that naturally arises is, "What can we say about the linear transformations for which X and $L(X)$ do not commute?" This, as has been pointed out in Gowda and Song (2000), motivated the introduction of the P_1- and P_2-properties. So the P_2-property can be thought of as a variation of the P-property of a linear transformation in SDLCP. We know that when L has the strong monotonicity property, then it satisfies the P-property. We have shown that if a linear transformation L satisfies the strong monotonicity property, then it also satisfies the P_2-property. For some special type of transformations, for example the Lyapunov transformation, we could show that the strong monotonicity property and the P_2-property are equivalent. However, if L is monotone but not strongly monotone, then from Example 1 it is clear that it may not satisfy the P_2-property.

The significance of the P_2-property also lies in the fact that it can be thought of as a generalization of the P-matrix condition of the LCP, since the two conditions are equivalent

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for a matrix M; see Gowda and Song (2000). Hence, an interesting problem would be to derive relationships between the P_2- and GUS-properties. Gowda and Song (2000) showed that the P_2-property always implies the GUS-property. We show here, by means of a counterexample, that the converse is not always true. We have also shown that if the matrix A is positive definite, then the Lyapunov transformation satisfies the P_2-property, and vice versa. Also, if we make the additional assumption that A is symmetric, then for the Lyapunov transformation and for the transformation $M_A(X)$ defined by AXA, we could show that the GUS-property and the P_2-property are equivalent.

1.1. Preliminaries. For a matrix $A \in R^{n\times n}$ we recall the following definitions.

(1) The trace of A is the sum of all diagonal elements of A or, equivalently, the sum of all eigenvalues of A.

(2) A is positive semidefinite (definite) if the usual inner product $\langle Ax, x\rangle \geq 0$ (>0) for all nonzero $x \in R^n$.

(3) A is positive stable if every eigenvalue of A has a positive real part.

(4) A is orthogonal if $AA^T = I = A^TA$, where I is the $n \times n$ identity matrix.

We write $X \succeq 0$ when $X \in S^n_+$.

We list below some well-known matrix theoretic properties: see Bellman (1995) and Zhang (1999).

(1) $X \succeq 0 \Rightarrow PXP^t \succeq 0$ for any nonsingular matrix P.

(2) $X \succeq 0,\ Y \succeq 0 \Rightarrow \langle X, Y\rangle \geq 0$.

(3) $X \succeq 0,\ Y \succeq 0,\ \langle X, Y\rangle = 0 \Rightarrow XY = YX = 0$.

Definition 1. For a linear transformation $L : S^n \to S^n$, we say that

(1) L has the GUS-property if for all $Q \in S^n$, SDLCP(L,Q) has a unique solution.

(2) L has the strong monotonicity property if $\langle L(X), X\rangle > 0$ for all nonzero $X \in S^n$.

(3) L has the monotonicity property if $\langle L(X), X\rangle \geq 0$ for all nonzero $X \in S^n$.

(4) L has the P_2-property if $X \succeq 0,\ Y \succeq 0,\ (X-Y)L(X-Y)(X+Y) \preceq 0 \Rightarrow X = Y$.

(5) L has the P-property if X and $L(X)$ commute, $XL(X) \preceq 0 \Rightarrow X = 0$.

Note that if L has the strong monotonicity property then L has the P-property.

(6) L has the cross commutative property if for every $Q \in S^n$ and solutions X_1 and X_2 of SDLCP(L, Q), the following holds:

$$X_1Y_2 = Y_2X_1 \quad \text{and} \quad X_2Y_1 = Y_1X_2,$$

where $Y_i = L(X_i) + Q$, $\quad i = 1, 2$.

Definition 2. For a matrix $A \in R^{n\times n}$ we define the corresponding Lyapunov transformation L_A: $S^n \to S^n$ by

$$L_A(X) = AX + XA^T.$$

Theorem 1 (Karamardian 1976). *Consider a linear transformation L: $S^n \to S^n$. If the problems SDLCP(L, 0) and SDLCP(L, E) for some E positive definite have unique solutions, then for all $Q \in S^n$, SDLCP(L, Q) has a solution.*

Theorem 2 (Gowda and Song 2000). *For a linear transformation L: $S^n \to S^n$, the following are equivalent:*

(1) *For all $Q \in S^n$, SDLCP(L, Q) has at most one solution.*

(2) *L has the P- and cross-commutative properties.*

(3) *L has the GUS-property.*

Theorem 3 (Gowda and Song 2000). *For a matrix $A \in R^{n\times n}$, consider the Lyapunov transformation L_A. Then the following statements are equivalent:*

(1) *L_A has the GUS-property.*

(2) *A is positive stable and positive semidefinite.*

2. Main results. Gowda and Song (2000) have shown that the P_2-property always implies the GUS-property. The following example shows that the converse need not be true.

EXAMPLE 1. For $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, consider the Lyapunov transformation L_A. Since A is positive semidefinite and positive stable, L_A has the GUS-property by Theorem 3.

Now let $X = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then $X \succeq 0$, $Y \succeq 0$ and $(X - Y)L_A(X - Y) \times (X + Y) = 0$.

Since $X \neq Y$, L_A does not satisfy the P_2-property, we see that L_A, although monotone, does not satisfy the P_2-property. Thus we see that although the P_2-property always implies the GUS-property (Gowda and Song 2000), the converse is not always true.

It is obvious from the definition that for a given $L: S^n \to S^n$, if it satisfies the strong monotonicity property, then it satisfies the P-property. Below we prove a stronger result.

THEOREM 4. *If a linear transformation $L : S^n \to S^n$ has the strong monotonicity property, then it has the P_2-property.*

PROOF. We will prove this by contradiction. Suppose there exists an $X \succeq 0$ and $Y \succeq 0$ such that $(X - Y)L(X - Y)(X + Y) \preceq 0$.

Assume $X \neq Y$ and without loss of generality, let $X + Y \neq 0$. Then there exists an orthogonal matrix U, positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ $(1 \leq r \leq n)$ with

$$U^T(X+Y)U = D\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}D,$$

where $D = \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}, 1, \ldots, 1)$ and I_r is the identity matrix of size $r \times r$.

Let $A = (D)^{-1}U^TXUD^{-1}$ and $B = (D)^{-1}U^TYUD^{-1}$. Then A and B are symmetric positive semidefinite with

$$A + B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix} A_r & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} B_r & 0 \\ 0 & 0 \end{bmatrix},$$

where A_r and B_r are $r \times r$ matrices. Now premultiplying and postmultiplying $(X - Y) \times L(X - Y)(X + Y)$ by $D^{-1}U^T$ and UD^{-1}, respectively, and introducing appropriate matrices between the three factors of $(X - Y)L(X - Y)(X + Y)$, we get

$$(A - B)[\widehat{L}(A) - \widehat{L}(B)](A + B) \preceq 0,$$

where $\widehat{L}(Z) = DU^TL(UDZDU^T)UD$. Note that $\widehat{L}$ is a strongly monotone linear transformation on S^n. Writing

$$\widehat{L}(A) - \widehat{L}(B) = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix},$$

we get

$$\operatorname{tr}\left(\begin{bmatrix} A_r - B_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}\right) = \langle \widehat{L}(A - B), A - B \rangle > 0,$$

i.e., $\mathbf{tr}[(A_r - B_r)P] > 0$. On the other hand,

$$(A - B)[\widehat{L}(A) - \widehat{L}(B)](A + B) \preceq 0$$

gives (after simplification) $\mathbf{tr}[(A_r - B_r)P] \leq 0$, leading to a contradiction. Hence, we must have $X = Y$, giving us the P_2-property. $\square$

Remark. Theorem 4 is false if the transformation is just monotone; see Example 1. However, one can prove the following proposition.

Proposition 1. *Let $L : S^n \to S^n$ be monotone. Suppose L has the following property:*

$$X \succeq 0, \qquad Y \succeq 0, \qquad (X - Y)L(X - Y)(X + Y) = 0 \Rightarrow X = Y.$$

Then L has the P_2-property.

The proof follows along lines similar to those of Theorem 4, and so we have omitted it. This proposition is motivated by Example 1. Regarding the converse statement of Theorem 4, we do not have a complete answer. We can give a partial answer. We show that for the Lyapunov transformation and for $M_A(X) = AXA^T$ when A is symmetric, the strong monotonicity property and the P_2-property are equivalent. Note that in Example 1, A is positive semidefinite and we have shown that L_A does not satisfy the P_2-property. However, if A is positive definite, then the following theorem shows that L_A satisfies the P_2-property and vice versa.

Theorem 5. *The following statements are equivalent for a Lyapunov transformation L_A.*

(i) *A is positive definite.*
(ii) *L_A has the strong monotonicity property.*
(iii) *L_A has the P_2-property.*

Proof. To show (i) $\Rightarrow$ (ii). Suppose A is positive definite. If X is a nonzero matrix in S^n where $x_1, x_2, \ldots, x_n$ are the columns of X, then $\mathrm{tr}L_A(X)X = 2\mathrm{tr}(XAX) = 2\sum_{i=1}^{n} x_i^T A x_i > 0$. This proves (ii).

(ii) $\Rightarrow$ (iii) has already been established in Theorem 4.

(iii) $\Rightarrow$ (i). If L_A satisfies P_2, then it has the GUS-property. From Gowda and Song (2000), we get that A is positive stable and positive semidefinite. Suppose if A is not positive definite; then there exists an $x \neq 0$ such that $x^T A x = 0$. Take $X = xx^T$; then X is symmetric and $XL_A(X)X = xx^T(Axx^T + xx^T A^T)xx^T = 0$ since $x^T A x = 0$. However, since L_A satisfies P_2, this implies that $X = 0$; that is, $x = 0$, which is a contradiction. Thus, A is positive definite. □

Note that if we take L_A to be monotone instead of strongly monotone, then it is clear from Example 1 that the above theorem does not hold good. While P_2 and GUS are not equivalent, they are so for L_A when $(A + A^t)$ is nonsingular.

Corollary 1. *If $\det(A + A^T) \neq 0$, then the following are equivalent for the Lyapunov transformation L_A.*

(i) *L_A has the GUS-property.*
(ii) *L_A satisfies the P_2-property.*

The following theorem shows that when A is symmetric, P_2, GUS, and the strong monotonicity properties are equivalent for $M_A(X) = AXA$.

Theorem 6. *When A is symmetric, the following statements are equivalent for the transformation $M_A(X) = AXA$.*

(i) *A is positive definite or negative definite.*
(ii) *M_A has the strong monotonicity property.*
(iii) *M_A has the P_2-property.*
(iv) *M_A has the GUS-property.*

Proof. (i) $\Rightarrow$ (ii). Since $M_A(X) = M_{-A}(X)$, without loss of generality we assume A to be positive definite. Suppose $\mathrm{tr}\, M_A(X)X \leq 0$ for some $X \in S^n \neq 0$. Then $\mathrm{tr}(AXAX) \leq 0$. Since A is symmetric and positive definite, $\mathrm{tr}(AXAX) \geq 0$ (since XAX is also positive semidefinite). Thus $\mathrm{tr}\, AXAX = 0 \Rightarrow AXAX = 0$; this implies that $XAX = 0 \Rightarrow X = 0$, which is a contradiction. Thus, M_A has strong monotonicity property.

(ii) $\Rightarrow$ (iii) follows from Theorem 4.

(iii) $\Rightarrow$ (iv) follows from Gowda and Song 2000.

(iv) $\Rightarrow$ (i). Suppose M_A has the GUS-property. If the order of the matrix is one, then it is easy to see that (iv) $\Rightarrow$ (i). Assume that the order of the matrix is at least two. If A is not positive definite or negative definite, then there exists an $x \neq 0$ such that $x^T Ax = 0$. Suppose not. Then let us consider the sets $E = \{x : x^T Ax < 0\}$ and $F = \{x : x^T Ax > 0\}$. These two sets are open and since there exists no $x \neq 0$ such that $x^T Ax = 0$, $E \cup F = R^n \setminus 0$. This implies that $R^n \setminus 0$ is disconnected, which is a contradiction. Hence there exists an $x \neq 0$ such that $x^T Ax = 0$. Take $X = xx^T$; then X is symmetric and positive semidefinite $XL_A(X) = xx^T Axx^T A = 0$. So we have two solutions for SDLCP(M_A, 0) contradicting the GUS-property. Thus (iv) $\Rightarrow$ (i). $\square$

Theorem 6 need not hold good if A is not symmetric, as the following example shows.

Example 2. Consider the following transformation: $M_A(X) = AXA^T$ where $A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$.

Note that A is not symmetric but is positive definite. Let $X = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$. Then $AXA^T X = \begin{bmatrix} -20 & 24 \\ -16 & -20 \end{bmatrix}$ and $\operatorname{tr}(AXA^T X) = -40 < 0$.

In other words M_A does not have the strong monotonicity property. In this example M_A also does not have the P_2-property, and this can be seen as follows. Let $X_1 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Now $(X_1 - X_2)M_A(X_1 - X_2)(X_1 + X_2)$ is negative semidefinite but $X_1 \neq X_2$. In other words M_A fails to have the P_2-property. However, this transformation M_A has the GUS-property. In fact, it is shown in Bhimshankaram et al. (2000) that A is positive definite if and only if M_A has the GUS-property. Note that the result is not true for the Lyapunov transformation (see Example 1) unless A is symmetric (see Corollary 1).

Combining Theorems 5, 6, and Corollary 1, we have the following result.

Corollary 2. *Suppose A is symmetric. Then L_A has the GUS-property if and only if M_A has the GUS-property.*

Summarizing, our findings in this paper are as follows. Every strongly monotone linear transformation has the P_2-property. For the Lyapunov transformation and the transformation M_A, strong monotonicity is equivalent to the P_2-property. An example is given to show that the GUS-property need not imply the P_2-property in general (although the P_2-property always implies the GUS-property; see Gowda and Song 2000). The following problem remains open: Does the P_2-property imply the strong monotonicity for a general linear transformation L?

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