Hajek–Renyi-type inequality for associated sequences

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Abstract

Let $\{\Omega, \mathscr{F}, P\}$ be a probability space and $\{X_n,\ n \geqslant 1\}$ be a sequence of random variables defined on it. A finite sequence $\{X_1, \ldots, X_n\}$ is said to be *associated* if for any two component wise non-decreasing functions f and g on R^n, $\mathrm{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geqslant 0$. A Hajek–Renyi-type inequality for associated sequences is proved. Some applications are given.

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1. Introduction

Let $\{\Omega, \mathscr{F}, P\}$ be a probability space and $\{X_n,\ n \geqslant 1\}$ be a sequence of random variables defined on it. A finite sequence $\{X_1, \ldots, X_n\}$ is said to be *associated* if for any two component wise non-decreasing functions f and g on R^n,

$$\mathrm{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geqslant 0$$

assuming of course that the covariance exists. The infinite sequence $\{X_n,\ n \geqslant 1\}$ is said to be *associated* if every finite sub-family is associated. The concept of association was introduced by Esary et al. (1967). Comprehensive reviews of probabilistic properties of associated sequences and statistical inference for such sequences are given in Roussas (1999) and Prakasa Rao and Dewan (2001).

We now develop a Hajek–Renyi-type inequality (Hajek and Renyi, 1955) for associated sequences and give some applications.

2. Hajek–Renyi-type inequality

Theorem 2.1. *Let $\{X_n,\ n \geqslant 1\}$ be an associated sequence of random variables with* $\mathrm{Var}(X_j) = \sigma_j^2$ *and $\{b_n,\ n \geqslant 1\}$ be a positive non-decreasing sequence of real numbers. Then, for any $\varepsilon > 0$,*

$$P\left(\max_{1 \leqslant k \leqslant n} \left|\frac{1}{b_n}\sum_{i=1}^{k}(X_i - EX_i)\right| \geqslant \varepsilon\right) \leqslant 4\varepsilon^{-2}\left\{\sum_{j=1}^{n}\frac{\mathrm{Var}(X_j)}{b_j^2} + \sum_{1 \leqslant j \neq k \leqslant n}\frac{\mathrm{Cov}(X_j, X_k)}{b_j b_k}\right\}.$$

Proof. Let $Y_j = b_j^{-1}(X_j - EX_j)$. It is clear from Esary et al. (1967) that $\{Y_n,\ n \geqslant 1\}$ is a zero mean associated sequence.

Let $S_n = \sum_{j=i}^{n}(X_j - EX_j),\ n \geqslant 1$. Let $b_0 = 0$. Note that

$$S_k = \sum_{j=1}^{k} b_j Y_j = \sum_{j=1}^{k}\left(\sum_{i=1}^{j}(b_i - b_{i-1})\right)Y_j$$

$$= \sum_{i=1}^{k}(b_i - b_{i-1})\left(\sum_{j=i}^{k} Y_j\right).$$

Since $b_k^{-1}\sum_{i=1}^{k}(b_i - b_{i-1}) = 1$, it follows that

$$\left[\left|\frac{S_k}{b_k}\right| \geqslant \varepsilon\right] \subset \left[\max_{1 \leqslant i \leqslant k}\left|\sum_{j=i}^{k} Y_j\right| \geqslant \varepsilon\right]$$

and hence

$$\left[\max_{1 \leqslant k \leqslant n}\left|\frac{S_k}{b_k}\right| \geqslant \varepsilon\right] \subset \left[\max_{1 \leqslant k \leqslant n}\max_{1 \leqslant i \leqslant k}\left|\sum_{j=i}^{k} Y_j\right| \geqslant \varepsilon\right]$$

$$= \left[\max_{1 \leqslant i \leqslant k \leqslant n}\left|\sum_{j=1}^{k} Y_j - \sum_{j=1}^{i} Y_j\right| \geqslant \varepsilon\right] \subset \left[\max_{1 \leqslant i \leqslant n}\left|\sum_{j=1}^{i} Y_j\right| \geqslant \frac{\varepsilon}{2}\right].$$

Therefore,

$$P\left(\max_{1 \leqslant k \leqslant n}\left|\frac{S_k}{b_k}\right| \geqslant \varepsilon\right) \leqslant P\left(\max_{1 \leqslant i \leqslant n}\left|\sum_{j=1}^{i} Y_j\right| \geqslant \frac{\varepsilon}{2}\right).$$

Applying the Chebyschev's inequality, we get that

$$P\left(\max_{1 \leqslant k \leqslant n}\left|\frac{S_k}{b_k}\right| \geqslant \varepsilon\right) \leqslant 4\varepsilon^{-2}E\left(\max_{1 \leqslant i \leqslant n}\left|\sum_{j=1}^{i} Y_j\right|^2\right).$$

We now apply the Kolmogorov-type inequality, for the expression on the right-hand side of the above inequality, valid for partial sums of associated random variables $\{Y_j,\ 1 \leqslant j \leqslant n\}$ with mean zero (cf. Theorem 2, Newman and Wright, 1981).

Hence, we have

$$
\begin{aligned}
P\left(\max_{1\leqslant k\leqslant n}\left|\frac{S_k}{b_k}\right| \geqslant \varepsilon\right) &\leqslant 4\varepsilon^{-2}E\left[\sum_{j=1}^{n} Y_j\right]^2 = 4\varepsilon^{-2}\operatorname{Var}\left[\sum_{j=1}^{n} Y_j\right] \\
&= 4\varepsilon^{-2}\left\{\sum_{j=1}^{n}\operatorname{Var}(Y_j) + \sum_{1\leqslant j\neq k\leqslant n}\operatorname{Cov}(Y_j,Y_k)\right\} \\
&= 4\varepsilon^{-2}\left\{\sum_{j=1}^{n}\frac{\operatorname{Var}(X_j)}{b_j^2} + \sum_{1\leqslant j\neq k\leqslant n}\frac{\operatorname{Cov}(X_j,X_k)}{b_jb_k}\right\}. \qquad (2.1)
\end{aligned}
$$

From the non-decreasing positive property of the sequence $\{b_n,\ n \geqslant 1\}$, it follows that

$$
P\left(\max_{1\leqslant k\leqslant n}\left|\frac{1}{b_n}\sum_{i=1}^{k}(X_i - EX_i)\right| \geqslant \varepsilon\right) \leqslant 4\varepsilon^{-2}\left\{\sum_{j=1}^{n}\frac{\operatorname{Var}(X_j)}{b_j^2} + \sum_{1\leqslant j\neq k\leqslant n}\frac{\operatorname{Cov}(X_j,X_k)}{b_jb_k}\right\} \qquad (2.2)
$$

proving the Hajek–Renyi-type inequality. □

Remarks. Under the conditions of Theorem 2.1, it is easy to see that for any positive integer $m \leqslant n$ and for any $\varepsilon \geqslant 0$,

$$
\begin{aligned}
P\left(\max_{m\leqslant k\leqslant n}\left|\frac{1}{b_n}\sum_{i=1}^{k}(X_i - EX_i)\right| \geqslant \varepsilon\right) &\leqslant 4\varepsilon^{-2}\left\{\sum_{j=1}^{m}\frac{\operatorname{Var}(X_j)}{b_m^2} + \sum_{1\leqslant j\neq k\leqslant m}\frac{\operatorname{Cov}(X_j,X_k)}{b_m^2}\right. \\
&\quad \left. + \sum_{j=m+1}^{n}\frac{\operatorname{Var}(X_j)}{b_j^2} + \sum_{m+1\leqslant j\neq k\leqslant n}\frac{\operatorname{Cov}(X_j,X_k)}{b_jb_k}\right\}. \qquad (2.3)
\end{aligned}
$$

3. Applications

Theorem 3.1. *Let $\{X_n,\ n \geqslant 1\}$ be an associated sequence of random variables with*

$$
\sum_{j=1}^{\infty}\operatorname{Var}(X_j) + \sum_{1\leqslant j\neq k}^{\infty}\operatorname{Cov}(X_j,X_k) < \infty.
$$

Then $\sum_{j=1}^{\infty}(X_j - EX_j)$ converges almost surely.

Proof. Without loss of generality, assume that $EX_j = 0$ for all $j \geqslant 1$. Let $\varepsilon > 0$.

Note that,

$$P\left\{\sup_{k,m\geqslant n}|S_k-S_m|\geqslant\varepsilon\right\}\leqslant P\left\{\sup_{k\geqslant n}|S_k-S_n|\geqslant\frac{1}{2}\,\varepsilon\right\}+P\left\{\sup_{m\geqslant n}|S_m-S_n|\geqslant\frac{1}{2}\,\varepsilon\right\}$$
$$\leqslant 2\lim_{N\to\infty}P\left\{\sup_{n\leqslant k\leqslant N}|S_k-S_n|\geqslant\frac{1}{2}\,\varepsilon\right\}$$
$$\leqslant 8\varepsilon^{-2}\lim_{N\to\infty}E\left\{\sup_{n\leqslant k\leqslant N}|S_k-S_n|^2\right\}$$
$$\leqslant 8\varepsilon^{-2}\left\{\sum_{j=n}^{\infty}\mathrm{Var}(X_j)+\sum_{n\leqslant j\neq k}^{\infty}\mathrm{Cov}(X_j,X_k)\right\}$$

and the last term tends to zero by the hypothesis. The last inequality follows either from the result of Newman and Wright (1981) or from the Hajek–Renyi-type inequality proved above. Hence, the sequence of random variables $\{S_n,\ n\geqslant 1\}$ is Cauchy almost surely which implies that S_n converges almost surely proving the theorem. □

For any random variable X and for any constant $c>0$, define $X^c=X$ if $|X|\leqslant c$, $X^c=-c$ if $X<-c$, and $X^c=c$ if $X>-c$. Note that x^c is an increasing function of x. Hence, if $\{X_n,\ n\geqslant 1\}$ is an associated sequence of random variables, then $\{X_n^c,\ n\geqslant 1\}$ is an associated sequence of random variables for any constant $c>0$. As a consequence of Theorem 3.1 and the standard techniques, we obtain the following analogue of the three series theorem for associated random variables.

Theorem 3.2. *Let $\{X_n,\ n\geqslant 1\}$ be an associated sequence of random variables with*

$$\sum_{n=1}^{\infty}EX_n^c<\infty, \tag{3.1}$$

$$\sum_{j=1}^{\infty}\mathrm{Var}(X_j^c)+\sum_{1\leqslant j\neq k}^{\infty}\mathrm{Cov}(X_j^c,X_k^c)<\infty, \tag{3.2}$$

$$\sum_{n=1}^{\infty}P[|X_n|\geqslant c]<\infty \tag{3.3}$$

for some constant $c>0$. Then $\sum_{n=1}^{\infty}X_n$ converges almost surely.

As a consequence of Theorem 3.1. and the Kronecker Lemma (Loeve, 1963), one can obtain the following theorem.

Theorem 3.3. *Let $\{X_n,\ n\geqslant 1\}$ be an associated sequence of random variables with*

$$\sum_{j=1}^{\infty}\frac{\mathrm{Var}(X_j)}{b_j^2}+\sum_{1\leqslant j\neq k}^{\infty}\frac{\mathrm{Cov}(X_j,X_k)}{b_jb_k}<\infty.$$

Then $b_n^{-1}\sum_{j=1}^{n}(X_j-EX_j)$ converges to zero almost surely as $n\to\infty$.

It is easy to see that this result extends the Strong law of large numbers for associated sequences proved in Birkel (1988) for general norming.

Theorem 3.4. *Let* $\{X_n,\ n \geqslant 1\}$ *be an associated sequence of random variables with*

$$\sum_{j=1}^{\infty} \frac{\mathrm{Var}(X_j)}{b_j^2} + \sum_{1 \leqslant j \neq k}^{\infty} \frac{\mathrm{Cov}(X_j, X_k)}{b_j b_k} < \infty.$$

Then, for any $0 < r < 2$,

$$E\left[\sup_n \left(\frac{|S_n|}{b_n}\right)^r\right] < \infty.$$

Proof. Note that

$$E\left[\sup_n \left(\frac{|S_n|}{b_n}\right)^r\right] < \infty$$

if and only if

$$\int_1^{\infty} P\left(\sup_n \frac{|S_n|}{b_n} > t^{1/r}\right) \mathrm{d}t < \infty.$$

By the Hajek–Renyi-type inequality proved above, it follows that

$$\int_1^{\infty} P\left(\sup_n \frac{|S_n|}{b_n} > t^{1/r}\right) \mathrm{d}t \leqslant 4 \int_1^{\infty} t^{-2/r} \left\{\sum_{j=1}^{\infty} \frac{\mathrm{Var}(X_j)}{b_j^2} + \sum_{1 \leqslant j \neq k}^{\infty} \frac{\mathrm{Cov}(X_j, X_k)}{b_j b_k}\right\} \mathrm{d}t$$

$$= 4 \left\{\sum_{j=1}^{\infty} \frac{\mathrm{Var}(X_j)}{b_j^2} + \sum_{1 \leqslant j \neq k}^{\infty} \frac{\mathrm{Cov}(X_j, X_k)}{b_j b_k}\right\} \int_1^{\infty} t^{-2/r}\, \mathrm{d}t < \infty. \qquad \square$$

References

Birkel, T., 1988. A note on the strong law of large numbers for positively dependent random variables. Statist. Probab. Lett. 7, 17–20.

Esary, J., Proschan, F., Walkup, D., 1967. Association of random variables with applications. Ann. Math. Statist. 38, 1466–1474.

Hajek, J., Renyi, A., 1955. A generalization of an inequality of Kolomogorov. Acta Math. Acad. Sci. Hungar. 6, 281–284.

Loeve, M., 1963. Probability Theory. Van Nostrand, Princeton, NJ.

Newman, C., Wright, A.L., 1981. An invariance principle for certain dependent sequences. Ann. Probab. 9, 671–675.

Prakasa Rao, B.L.S., Dewan, I., 2001. Associated sequences and related inference problems. In: Rao, C.R. Shanbhag, D.N., (Eds.), Handbook of Statistics 19: Stochastic Proceseses: Theory and Methods. Elsevier, Amsterdam, pp. 693–731.

Roussas, G., 1999. Positive and negative dependence with some statistical applications. In: Ghosh, S. (Ed.), Asymptotics, Nonparametrics, and Time Series. Marcel Dekker, New York, pp. 757–788.