ANALYSIS OF VARIANCE FOR MULTIVARIATE NORMAL POPULATIONS : THE SAMPLING DISTRIBUTION OF THE REQUISITE p-STATISTICS ON THE NULL AND NON-NULL HYPOTHESES

By S. N. ROY

Statistical Laboratory

Introduction

It is well known that (*i*) the problem of discrimination in respect of variances between two univariate normal populations is tackled and solved in practically the same manner as (*ii*) the problem of discrimination in respect of mean values among l $(l>2)$ univariate normal populations supposed to have the same variance. On the null hypothesis the statistic for problem (*i*) has the same form of sampling distribution as the one for problem (*ii*). Into symbols this may translated as follows : Suppose we have (*i*) 2 samples S_1 and S_2 of sizes n_1 and n_2 and standard deviations s_1 and s_2 drawn at random from 2 univariate normal populatiins Σ_1 and Σ_2 with population standard deviations σ_1 and σ_2 ; and further (*ii*) l samples $S_1, S_2, \ldots S_l$ of sizes $n_1, n_2, \ldots n_l$ with means $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_l$ and standard deviations $s_1, s_2, \ldots s_l$ drawn at random from l univariate normal populations $\Sigma_1, \Sigma_2, \ldots \Sigma_l$ with mean values $\xi_1, \xi_2, \ldots \xi_l$ and a common standard deviation σ. For (*i*) the null hypothesis (associated with the process of discrimination in respect of variance) is $\sigma_1=\sigma_2$ and for (*ii*) the null hypothesis which goes with discrimination in respect of mean values is $\xi_1=\xi_2 \ldots=\xi_l$. To test the null hypothesis for (*i*) the usual statistic is s^2_1/s_2^2 and to test the null hypothesis for (*ii*) the usual statistic is B/W, where B is the familiar 'between variance' and W is the familiar 'within variance'. In fact with $N = n_1 + n_2 + \ldots n_l$ and $\bar{x} = (n_1 \bar{x}_1 + n_2 \bar{x}_2 + \ldots . n_l \bar{x}_l)/N$, $B = \{n_1 (\bar{x}_1 - \bar{x})^2 + n_2 (\bar{x}_2 - \bar{x})^2 + \ldots . n_l (\bar{x}_l - \bar{x})^2\}/(l-1)$ and $W = \{(n_1-1)s_1^2+(n_2 - 1)s_2^2 + \ldots . (n_l - 1)s_l^2\}/(N - l)$ and it is well known that s_1^2/s_2^2 of (*i*) has the same form of sampling distributiin when $\sigma_1 = \sigma_2$ as B/W of (*ii*) when $\xi_1 = \xi_2 = \xi_3 = \ldots \xi_l$. The common distribution is Fisher's wellknown 'F' distribution. Whether it be for purposes of classification or for purposes associated with Neyman and Pearson's theory of testing of hypothesis it is important to know the sampling distribution of s_1/s_2 of (*i*) on the non-null hypothesis, that is, when $\sigma_1 \neq \sigma_2$, and similarly the sampling distribution of B/W of (*ii*) when $\xi_1 \neq \xi_2 \neq \ldots \neq \xi_l$. It is also known that now the two distributions are entirely different. In fact on the non-null hypothesis s_1^2/s^2_2 has the distribution

$$\frac{\text{Const. } (s_1^2/s_2^2)^{\frac{n_1-3}{2}} \, d(s_1^2/s_2^2)}{\left\{1 + \frac{n_1-1}{n_2-1}\frac{s_1^2}{s_2^2} \Big/ \frac{\sigma_1^2}{\sigma^2_2}\right\}^{\frac{n_1+n_2-2}{2}}} \qquad \ldots \quad (a)$$

and B/W has the distribution[8]

$$\text{Const.}\ \frac{(B/W)^{\frac{l-3}{2}}\, d(B/W)}{\left\{1+\frac{l-1}{N-l}\frac{B}{W}\Big/\frac{\beta}{\sigma^2}\right\}^{\frac{N-1}{2}}}$$

$$\times\ {}_1F_1\left\{\frac{N-1}{2}\,;\ \frac{l-1}{2}\,;\ \frac{\beta}{\sigma^2}\cdot\frac{l-1}{N-l}\cdot\frac{B}{W}\Big/\left(1+\frac{l-1}{N-l}\cdot\frac{B}{W}\right)\right\} \qquad \ldots (b)$$

where

with

$$\left.\begin{aligned}\beta &= (n_1(\xi_1-\xi)^2 \pm \ldots n_l(\xi_l-\xi)^2/(l-1)\\ \xi &= (n_1\xi_1+\ldots n_l\xi_l)/N\end{aligned}\right\} \qquad \ldots (c)$$

The technique outlined above was developed for the univariate case; but it could be completely generalised for purposes of tackling the corresponding problems in the multivariate. As a matter of fact part of the generalisation has already been made and it is the object of the present paper to complete the scheme. What has been already achieved by others and by the author, and what the present paper proposes to do can be sent forth in technical language as follows: Suppose we have (iii) 2 samples $S(1)$ $S(2)$ of sizes n_1 and n_2, and variances and covariances $a(1, ij)$ and $a(2, ij)$ ($i, j=1, 2, \ldots p$) drawn at random from 2 p-variate normal populations $\Sigma(1)$ and $\Sigma(2)$ with variances and covariances $\alpha(1, ij)$ and $\alpha(2, ij)$ (i, j$=1, 2, \ldots$ p), and further (iv) l samples $S(1), S(2), \ldots S(l)$ of sizes $n_1, n_2, \ldots n_l$, mean values $\bar{x}(1, i), \bar{x}(2, i), \ldots \bar{x}(l, i)$ and variances and covariances $a(1, ij)$, $a(2, ij)$, ... $a(l, ij)$ (i, j$=1, 2, \ldots$ p; the first suffix referring to the sample and the next ones to the character) drawn at random from l p-variate normal populations with mean values $\xi(1, i), \xi(2, i) \ldots \xi(l, i)$ ($i=1, 2, \ldots$ p); the first suffix referring to the population and the next ones to the character) and a common set of variances and covariances $\alpha^*(ij)$ (i, j$=1, 2, \ldots$ p). The situations (ii) and (iv) are respectively the multivariate generalisation, of the univariate situations (i) and (ii) already discussed. For (iii) the null hypothesis (associated with the process of discrimination in respect of the sets of variances and covariances, between the populations $\Sigma(1)$ and $\Sigma(2)$) is $\alpha(1, ij)=\alpha(2, ij)$(i, j$=1, 2, \ldots p$); for (iv) the null hypothesis (associated with discrimination in respect of the set of mean values, between the populations $\Sigma(1), \Sigma(2), \ldots \Sigma(l)$) is $\xi(1, i)=(2, i) \ldots \xi(l, i)$ i$=1, 2, \ldots$ p). To test the null hypothesis for (iii), that is to test $\alpha(1, ij)=\alpha(2, ij)$ (i, j$=1, 2, \ldots$ p) the author constructed about three years ago from certain considerations a set of p-statistics $k^2_1, k^2_2, \ldots k^2_p$ which might be regarded as appropriate generalisation of s^2_1/s^2_2 of (i) and which are the roots of the determinantal equation in k^2

$$|a(1, ij)-k^2\, a(2, ij)|=0 \qquad \ldots (d)$$

The sampling distribution of these p-statistics in the null hypothesis ($\alpha(1, ij)=\alpha(2\, ij)$; i, j$=1, 2, \ldots$ p) was obtained in the form

$$\text{Const.}\ \prod_{i=1}^{p}\frac{k_i^{n_1-p-1}\, dk_i}{\left(1+\frac{n_1-1}{n_2-1}k^2_i\right)^{\frac{n_1+n_2-2}{2}}}$$

$$\times\ \text{mod}\ ((k^2_1-k^2_2)\ldots(k^2_1-k^2_p)\ (k^2_2-k^2_3)\ldots(k^2_2-k^2_p)\ldots(k^2_{p-1}-k^2_p)) \qquad \ldots (e)$$

The sampling distribution of the same set of statistics on the non-null hypothesis ($\alpha(1, ij) \neq \alpha(2, ij)$; $i, j = 1, 2, \ldots p$) was also obtained some time later.

To test the null hypothesis for (iv), that is to test $\xi(1, i) = \xi(2, i) = \ldots \xi(l, i) (i = 1, 2, \ldots p)$, another set of statistics $t^2_1, t^2_2, \ldots t_p^2$ were constructed again about three years ago which might be regarded as appropriate generalisation of B/W and which came out as the p roots of the determinantal equation in t^2

$$|a'(ij) - t^2 a''(ij)| = 0 \qquad (f)$$

where $a'(ij)$ and $a''(ij)$ are quantities defined by

$$\left.\begin{array}{l} \sum_{r=1}^{l} (n_r - 1)\, a(r, ij) = (N - l)\, a''(ij) \\ \sum_{r=1}^{l} n_r (\bar{x}(r, i) - \bar{x}_i)\, (\bar{x}(r, j) - \bar{x}_j) = (l - 1)\, a'(ij), \\ \text{N and } \bar{x}_i \ (i = 1, 2, \ldots p) \text{ being given by} \\ N = \sum_{r=1}^{l} n_r \, ; \ \bar{x}_i = \sum_{r=1}^{l} n_r \ \bar{x}(r, i)/N \\ \text{with } i, j = 1, 2, \ldots p \end{array}\right\} \qquad (g)$$

It was found[1, 2] that the t_i's ($i = 1, 2, \ldots p$) of (f) have the same form of joint sampling distribution on the null hypothesis for (iv) ($\xi(1, i) = \xi(2, i) = \ldots = \xi(l, i)$; $i = 1, 2, \ldots p$) as the set of statistics k_i's($i = 1, 2, \ldots p$) of (a) on the null hypothesis for (iii), which is $\alpha(1, ij) = \alpha(2, ij)$ ($i, j = 1, 2, \ldots p$). In fact the common form of joint distribution in that given by (e).

It is the primary object of the present paper to obtain the joint distribution of the set of statistics t_i's ($i = 1, 2, \ldots p$) of (f) on the non-null hypothesis for (iv), which is $\xi(1, i) \neq \xi(2, i) \neq \ldots \neq \xi(l, i)$ ($i = 1, 2, \ldots p$). This, as has been observed earlier, is important both for purposes of classification (in respect of mean values) of the populations $\Sigma(1), \Sigma(2), \ldots \Sigma(l)$ as well as for purposes connected with Neyman and Pearson's theory of testing of hypothesis. To make this paper self-contained it would be desirable, however, not to take on trust the p-statistics defined by (f) and (g) and proceed forthwith to obtain the joint distribution on the non-null hypothesis, but to go back to an earlier stage, from there to build up the p-statistics of (f) and (g) by properly generalising to the multivariate case the B/W of (ii) which applies to the univariate structure, and then obtain the sampling distribution in question. The distribution of this set of statistics on the null hypothesis ($\xi(1, i) = \xi(2, i) = \ldots = \xi(l, i)$; $i = 1, 2, \ldots p$) would then come out as a special case of this distribution so that the present paper would be self-contained in practically every respect except for one important stage in the derivation of the main distribution where use will be made of an earlier paper[3] by the author.

1. Preliminaries to the Reduction of the Distribution Problem

As we have remarked earlier S(1), S(2), ... S(l) are samples of sizes $n_1, n_2, \ldots n_l$ drawn at random from l p-variate normal populations $\Sigma(1), \Sigma(2) \ldots \Sigma(l)$. $\alpha''(ij)$ ($i, j, = 1, 2 \ldots p$) denote the common set of variances and covariances for the populations $\Sigma(1), \Sigma(2), \ldots \Sigma(l)$ so that $\alpha''(ij) = \rho_{ij}\ \sigma_i\ \sigma_j$, σ_i, σ_j being the common standard deviations for the i^{th} and j^{th} characters for all the populations and ρ_{ij} the common correlation coefficient (for the populations) between the i^{th} and j^{th} characters. $\| a''(ij) \|$ would be called the common dispersion matrix for the populations. Likewise $a(1, ij)$, $a(2, ij)$, ... $a(l, ij)$ denote the sets of variances and co-variance for the samples $S(1)$, $S(2)$, ... $S(l)$ respectively ($i, j, = 1, 2, \ldots p$; the first suffix denotes the sample and the next ones the various characters), so that $\| a(1, ij) \|$, $\| a(2, ij) \|$, ... $\| a(l, ij) \|$ are the dispersion matrices for the various samples.

Let $\xi(1, i)$, $\xi(2, i)$, ... $\xi(l, i)$ ($i = 1, 2, \ldots p$; the first suffix denotes the population and the next one the character) denote the mean values of the various populations, and $\bar{x}(1, i)$, $\bar{x}(2, i)$, ... $\bar{x}(l, i)$ ($i = 1, 2, \ldots p$; the first suffix denotes the sample and the next one the character) stand for the means of the different samples.

Let $x(1, i, \nu_1)$, $x(2, i, \nu_2)$, ... $x(l, i, \nu_l)$ stand for the sample readings of the different samples $S(1)$, $S(2)$, ... $S(l)$ ($i = 1, 2, \ldots p$; $\nu_1 = 1, 2, \ldots n_1$; $\nu_2 = 1, 2, \ldots n_2$; ... $\nu_l = 1, 2, \ldots n_l$; the first suffix denotes the sample, the second suffix denotes the character and the third suffix denotes the order of the individual in the particular sample in question).

We have thus

$$\left.\begin{aligned}
&\bar{x}(r, i) = \frac{1}{n_r} \sum_{\nu_r=1}^{n_r} x(r, i, \nu_r) \\
&a(r, ij) = \frac{1}{n_r - 1} \sum_{\nu_r=1}^{n_r} \{x(r, i, \nu_r) - \bar{x}(r, i)\}\{x(r, j, \nu_r) - \bar{x}(r, j)\} \\
\text{and} \quad &\bar{x}_i = \sum_{r=1}^{l} n_r \bar{x}(r, i)/N \\
\text{with} \quad &i = 1, 2, \ldots\ldots p; \; r = 1, 2, \ldots\ldots l \\
&\nu_r = 1, 2, \ldots; n_r; \; N = \sum_{r=1}^{l} n_r
\end{aligned}\right\} \quad \ldots \quad (1\cdot1)$$

After the technique of Professor Fisher the reduction of the problem to the univariate case can be effected as follows. A compound character built on a linear compound of the variates is taken for the samples $S(1)$, $S(2)$, ... $S(l)$ which are now characterised respectively by readings (for the different individuals)

$$\left.\begin{aligned}
S(1) &\rightarrow \sum_{i=1}^{p} \lambda_i x(1, i, 1), \; \sum_{i=1}^{p} \lambda_i x(1, i, 2), \ldots \sum_{i=1}^{p} \lambda_i x(1, 2, n_1) \\
S(2) &\rightarrow \sum_{i=1}^{p} \lambda_i x(2, i, 1), \; \sum_{i=1}^{p} \lambda_i x(2, i, 2), \ldots \sum_{i=1}^{p} \lambda_i x(2, i, n_2) \\
S(l) &\rightarrow \sum_{i=1}^{p} \lambda_i x(l, i, 1), \; \sum_{i=1}^{p} \lambda_i x(l, i, 2), \ldots \sum_{i=1}^{p} \lambda_i x(l, i, n_l)
\end{aligned}\right\} \quad \ldots \quad (1\cdot11)$$

Denoting now by B and W the 'between variance' and 'within variance' of the different samples for the compound character, and introducing new quantities $a'(ij)$, $a''(ij)$ defined by

$$\left.\begin{aligned}
a'(ij) &= \sum_{r=1}^{l} \{\bar{x}(r, i) - \bar{x}_i\} \; (\bar{x}(r, j) - \bar{x}_j)/(l-1) \\
a''(ij) &= \sum_{r=1}^{l} (n_r - 1) a(r, ij)/(N - l)
\end{aligned}\right\}$$

where $\bar{x}(r, i)$, $\bar{x}(r, ij)$, $\bar{x}_i$, N have been already defined in (1.1),

we have

$$\left.\begin{aligned} B &= \sum_{i,j=1}^{p} \lambda_i \lambda_j a'_{ij} \\ W &= \sum_{i,j=1}^{p} \lambda_i \lambda_j a''_{ij} \end{aligned}\right\} \quad \ldots \quad (1\cdot 3)$$

Setting now $B/W=t^2$, we can so choose the λ_i's ($i=1, 2, \ldots p$) as to maximise t^2 whereby we obtain p stationary values of $t^2(t^2_1, t^2_2, \ldots t^2_p)$ as the roots of the p-fold determinantal equation in t^2

$$|a'(ij)-t^2 a''(ij)|=0 \quad \ldots \quad (1\cdot 4)$$

For each of the populations $\Sigma(1), \Sigma(2), \ldots \Sigma(l)$ start with a similiar linear compound of the p-variates and bring in new quantities ξ_i, $\alpha'(ij)$ defined by

$$\left.\begin{aligned} \xi_i &= \sum_{r=1}^{l} n_r \, \xi(r, i)/N \\ \alpha'(ij) &= \sum_{r=1}^{l} n_r (\xi(r, i)-\xi_i)\,(\xi(r, j)-\xi_j)/(l-1) \end{aligned}\right\} \quad \ldots \quad (1\cdot 5)$$

Having reduced the multivariate problem to the univariate case we can now introduce quantities β and σ^2 (analogous to β and σ^2 of case (ii) of the introduction) defined by

$$\left.\begin{aligned} \beta &= \sum_{i,j=1}^{p} \lambda_i \lambda_j \alpha'(ij) \\ \sigma^2 &= \sum_{i,j=1}^{p} \lambda_i \lambda_j \alpha''(ij) \end{aligned}\right\} \quad \ldots \quad (1\cdot 6)$$

Putting now $\beta/\sigma^2=\tau^2$ and maximising τ^2 with respect to the λ_i's ($i=1, 2, \ldots p$) we obtain the p values of $\tau^2(\tau^2_1, \tau^2_2, \ldots \tau^2_p)$ as the p-roots of the p-fold determinantal equation in τ^2

$$| \alpha'(ij)-\tau^2 \alpha''(ij) | =0 \quad \ldots \quad (1\cdot 7)$$

By considering (1·1), (1·2) (1·4), (1·5) and (1·7) it can be easily proved that all the roots of (1·4)—$t_1^2, t_2^2, \ldots t_p^2$, and all the roots of (1·7)—$t_1^2, t_2^2, \ldots t_p^2$ are zero when and only when for the first case $\bar{x}(1, i)=\bar{x}(2, i)= \ldots \bar{x}(l, i)$ and for the second case $\xi(1, i)=\xi(2, i)=\ldots\xi(l, i)$, ($i=1, 2, \ldots p$), that is, for each character the samples (for the first case) and populations (for the second case) have the same mean value.

As in the case of k_i's and κ_i^2s of the previous paper [2 and 3] the t_i's of (1·4) and τ_i's of (1·7) are invariant under any general linear transformation of the p-variates to p new variates (the set of p^2-transformation coefficients for the samples may not necessarily be the same as the p^2-set for the populations but, of course, for all the samples it must be the same p^2-set and for all the populations there must be the same p^2-set though it may be different from the common set for the samples).

The sample $S(1)$ with readings $x(1, i, \nu_i)$ ($i=1, 2, \ldots p$; $\nu_1=1, 2, \ldots n_1$) can be conveniently represented in the usual Fisherian flat sample space $f(1, n_1)$ of n_1, dimensions by the p points with co-ordinates $x(1, i, 1)$, $x(1, i, 2)$, $\ldots x(1, i, n_1)$ ($i=1, 2,\ldots p$) or by p vectors, $\mathbf{x}(1, i)$ ($i=1, 2, \ldots p$ joining the points to the origin. We may take another flat space $f(2, n_2)$

of n_2 dimensions absolutely orthogonal to $f(1, n_1)$ and in it represent the sample $S(2)$ by p other similar vectors $x(2, i)$ $(i=1, 2, \ldots, p)$; next take another flat space $f(3, n_3)$ of n_3 dimensions absolutely orthogonal to $f(1, n_1)$ and $f(2, n_2)$ and represent in it $S(3)$ by similar vectors $x(3, i)$ $(i=1, 2, \ldots, p)$; continue like this till we come to $S(l)$ which we represent by p similar vectors $x(l, i)$ $(i=1, 2, \ldots, p)$ in a similar flat $f(l, n_l)$ of n_l dimensions. In place of the old variables introduce new variables $y(r, i, \nu_i)$ defined by

$$y(r, i, \nu_r)=x(r, i, \nu_r)-\bar{x}(r, i) \qquad .. \quad (1.8)$$

where $i=1, 2, \ldots p$; $\nu_r=1, 2, \ldots\ldots n_r$; $r=1, 2, \ldots\ldots l$.

the first suffix refers to the sample, the second suffix refers to the character and the third suffix (where it occurs) refers to the individual in a sample.

For the r^{th} sample $(r=1, 2, \ldots l)$ the vector $x(r, i)$ is now conveniently resolved into two orthogonal vectors, one $\bar{x}(r, i)$ of magnitude $\bar{x}(r, i)$ along the equiangular line in the flat $f(r, n_r)$ and the other, say $y(r, i)$ (with components $y(r, i, \nu_r) \rightarrow \nu_r=1, 2, \ldots n_r$) lying obviously in a flat $f(r, n_r-1)$ which is immersed in the flat $f(r, n_r)$ and is orthogonal to the equiangular line. This is evident from (1.8). The end points of the vectors $\bar{x}(r, i)$ and $y(r, i)$ will be denote by $\bar{X}(r, i)$ and $Y(r, i)$ while the vectors $y(r, i)$ $(i=1, 2, \ldots p)$ will be called the p variation vectors for the r^{th} sample. The covariance between the i^{th} and j^{th} variates for the r^{th} sample $(r=1, 2, \ldots l;\ i, j=1, 2, \ldots p)$ is, therefore, given by

$$a(r, ij)=\{y(r, i).\ y(r, j)\}/(n_r-1) \qquad (1.9)$$

where $\{y(r, i).\ y(r, j)\}$ is the scalar product of the vectors $y(r, i)$ and $y(r, j)$. The l equiangular lines in the l flats $f(1, n_1)\, f(r, n_2), \ldots f(l, n_l)$ will be called $oe_1, oe_2, \ldots oe_l$. From the fact that the flats $f(1, n_1), f(2, n_2), \ldots f(l, n_l)$ are absolutely orthogonal to one another and further that any equiangular line oe_r lies in $f(l, n_r)$ and orthogonal to $f(r, n_r-1)$ it is evident that oe_1, $oe_2, \ldots oe_l$ are mutually orthogonal, and all orthogonal to $f(r, n_r-1)$'s$(r=1, 2, \ldots l)$ which latter are themselves mutually orthogonal; thus $oe_r, f(r, n_r-1)$ $(r=1, 2, \ldots l)$ form a mutually orthogonal set. For the r^{th} sample we have p-vectors $\bar{x}(r, i)$ along oe_r and p-vectors $y(r, i)$ in the flat $f(r, n_r-1)$ $(r=1, 2, \ldots l;\ i=1, 2, \ldots p$; the first suffix refers to the sample and the next to the character). Take the resultant of the vectors $\bar{y}(1, i)\, y(2, i)\ldots y(l, i)$ and call it $y^*(i)(i=1, 2, \ldots p)$. Then the p-vectors $y^*(i)$'s $(i=1, 2, \ldots p)$ lie in a flat of $n_1-1+n_2-1+\ldots$ $\ldots n_l-1$ (*i. e.* $N-l$) dimensions composed of the flats $f(1, n_1-1), f(2, n_2-1), \ldots\ldots f(l, n_l-1)$ which let us call the flat $f^*(N-l)$; this $f^*(N-l)$ is itself immersed in a flat composed of $f(1, n_1), f(2, n_2) \ldots\ldots f(l, n_l)$ which let us call $f(N)$. From the foregoing considerations and from (g) of the introduction and (1.1), (1.8) and (1.9) it is evident that

$$a^*(ij)=\{y^*(i).\ y^*(j)\}/(N-l)$$

where $\{y^*(i).\ y^*(j)\}$ is the scalar product of the vectors $y^*(i)$ and $y^*(j)$ Next consider the equiangular lines $oe_1, oe_2, \ldots oe_l$; these form a flat of l-dimensions which let us call $f'(l)$; in this flat take a line oe with direction cosines (referred to $oe_1, oe_2, \ldots oe_l$ as axes) $\sqrt{(n_1/N)}, \sqrt{(n_2/N)}$, $\ldots \sqrt{(n_l/N)}$ with $N=n_1+n_2+\ldots n_l$. Take now in this flat $f'(l)$ vectors $y'(i)$ $(i=1, 2, \ldots p)$ such that the i^{th} vector y'_i has components along the l axes $oe_r (r=1, 2, \ldots\ldots l)$ given by

$$\left.\begin{array}{l}\sqrt{n_1}(\bar{x}(1, i)-\bar{x}_i),\ \sqrt{n_2}(\bar{x}(2, i)-\bar{x}_i), \ldots\ldots \sqrt{n_l}(\bar{x}(l, i)-\bar{x}_i)\\ \text{where} \quad \bar{x}_i=\sum_{r=1}^{l} n_r\bar{x}(r, i)/N\end{array}\right\} \qquad .. \quad (1.92)$$

It is evident, therefore, that these p-vectors $y'(i)$ (i=1, 2, ... p) lie in a flat $f'(l-1)$ which is immersed in the flat $f'(l)$ and is orthogonal to the line oe in $f'(l)$ introduced just now. It is also clear that $a'(ij)$'s (i. j=1, 2, ... p) defined by (g) of the introduction is connected with the $y'(i)$'s by

$$a'(ij)=(y'(i).\ y'(j))/(l-1) \qquad .\quad (1\cdot 93)$$

where $(y'(i).\ y'(j))$ is the scalar product of the vectors $y'(i)$ and $y'(j)$. Denote the end points of the vectors $y'(i)$'s by $Y'(i)$'s, (i=1, 2, ... p). Hence we have altogether p-vectors $y'(i)$, (i=1, 2, ... p) defining a p-flat, say, $f'(p)$ which lies immersed in $f'(l-1)$ ($p\leqslant l-1$) of $l-1$ dimensions, and p-vectors $y''(i)$ (i=1, 2, ... p) which define a p-flat, say, $f''(p)$ whichli es immersed in $f''(N-l)$ of $N-l$ dimensions; the flats $f'(l-1)$ and $f''(N-l)$ have been already described.

Now take the resultant of the vectors $y'(i)$ and $y''(i)$ and call it $y(i)$, the end point being Y_i(i=1, 2, ... p). Then y_i's determine a p-flat, say, f_p. Then from considerations similar to those discussed in the two previous papers [2, 3] it would follow from (f) of the introduction and from (1·91) and (1·93) that if θ_i(i=1, 2, ... p) be the p critical angles between the flat f_p and the flat $f''(N-l)$, then

$$t_i=\tan\ \theta_i\sqrt{(N-l)/(l-1)},\ (i=1,\ 2,\ \ldots .\ p) \qquad ..\quad (1\cdot 94)$$

The t_i's are invariant under any (common) linear transforation of the p-variates to p new variates. This is for the l samples $S(1)$, $S(2)\ldots S(l)$. Likewise for the l populations $\Sigma(1)$, $\Sigma(2)\ldots\Sigma(l)$ take vectors $V[\eta'(i)]$s and $V[\eta''(i)]$'s analogous respectively to vectors $y'(i)$'s and $y''(i)$'s for the l samples (i=1, 2, ... p) such that $\alpha'(ij)$'s of (1·5) and $\alpha''(ij)$'s the common dispersion matrix for the populations are given by

$$\left.\begin{aligned}\alpha'(ij)&=V[\eta''(i)].\ V[\eta'(j)]\\ \alpha''(ij)&=V[\eta''(i)].\ V[\eta''(j)]\end{aligned}\right\} \qquad ..\quad (1\cdot 95)$$

where $V[\eta'(i)].\ V[\eta'(j)$ is the scalar product of $V[\eta'(i)]$ and $V[\eta'(j)]$ and $V[\eta''(i)].\ V[\eta''(j)]$ the scalar product of $V[\eta''(i)$ and $V[\eta''(j)]$. The p-vectors $V[\eta'(i)]$ would consitute a p-flat $F'(p)$ and the p-vectors $V[\eta''(i)]$ would constitute a p-flat $F''(p)$. So arrange matters that $F'(p)$ is absolutely orthogonal to $F''(p)$. If now we form the resultant of the vectors $V[\eta'(i)]$ and $V[\eta''(i)]$ and call it $V(\eta_i)$ then $V(\eta_i)$'s form a p-flat $F(p)$ which makes with the p-flat $F''(p)$ p critical angles which let us call Θ_i's (i=1, 2, ... p). Then it follows from considerations similar to those of the previous papers [2 and 3] that τ_i's of (1·7) are connected with Θ_i's by

$$\tau_i=\tan\ \Theta_i(i=1, 2, \ldots .\ p) \qquad ..\quad (1\cdot 96)$$

The τ_i's are invariant under any linear (common to all populations) transformation of the p-variates to p new variates.

2. The Reduction of the Distribution Problem

The joint probability of the j samples $S(1)$, $S(2)$, $S(l)$ coming as random samples from $\Sigma(1)$, $\Sigma(2)$, ... $\Sigma(l)$ or the probability of sample readings $x(r, i, \nu_r)$ lying between $x(r, i, \nu_r)$ and $x(r,, i\ \nu_r)+dx(r, i, \nu_r)$ (r=1, 2, ...l ; i=1, 2, ...p : ν_r=1, 2,....n_r) is given by

$$\text{Const}\ \exp.[-\tfrac{1}{2}\sum_{r=1}^{l}\ \alpha^{ij}\sum_{i, j=1}^{p}\{n_r(\bar{x}(r, i)-\xi(r, i))\,(\bar{x}(r, j)-\xi(r, j))+(n_r-1)\,a(r, ij)\}]$$
$$\times\prod_{r=1}^{l}\ \prod_{\nu_r=1}^{n_r}\ \prod_{i=1}^{p}\ dx(r, i, \nu_r) \qquad ..\quad (2\cdot 1)$$

where as previously the suffix r refers to the sample, the suffixes i and j refer to the character and the suffix ν_r (where it occurs) refers to the individual in a sample; and α^{*ij} is the co-factor of $\alpha^*(ij)$ in the determinant $|\alpha^*(ij)|$ divided by the determinant itself. The other quantities $\bar{x}(i)$'s, $\xi(r, i)$'s, $a(r, ij)$'s have been already defined in the introduction and in section 1.

Consider now the density factor exp( ) in (2·1); $\Sigma\ (n_r-1)\,a(r, ij)$ is really equal to to $(N-1)\,a^*(ij)$ from (1·2). Furthermore,

$$\sum_{r=1}^{l} n_r(\bar{x}(r, i)-\xi(r, i))\ (\bar{x}(r, j)-\xi(r, j))$$

$$=\sum_{r=1}^{l} n_r\{(\bar{x}(r, i)-\bar{x}_i)+(\bar{x}(i)-\xi(r, i))\}\ \{(\bar{x}(r, j)-\bar{x}_j)+(\bar{x}(j)-\xi(r, j)\}$$

$$=\sum_{r=1}^{l} (\bar{x}(r, i)-\bar{x}_i)\ (\bar{x}(r, j)-\bar{x}_j)-\sum_{r=1}^{l} n_r\ \xi(r, j)\ (\bar{x}(r, i)-\bar{x}_i)$$

$$-\sum_{r=1}^{l} n_r\,\xi(r, i)\ (\bar{x}(r, j)-\bar{x}_j)+N\bar{x}_i\ \bar{x}_j-N\bar{x}_i\ \xi_j-N\bar{x}_j\ \xi_i$$

$$=(l-1)\,a'(i, j)-\sum_{r=1}^{l} n_r\xi(r, i)\ (\bar{x}(r, j)-\bar{x}_j)-\sum_{r=1}^{l} n_r\ \xi(r, j)\ (\bar{x}(r, i)-\bar{x}_i)$$
$$+N\bar{x}_i\ \bar{x}_j-N\bar{x}_i\ \xi_j-N\bar{x}_j\ \xi_i \quad \ldots\ (2{\cdot}2)$$

where $a'(ij)$, $\bar{x}(r, i)$, $\xi(r, i)$ have been already defined respectively by (1·2), (1·1), (1·4) and (1·5).

Hence the density factory in (2.1) becomes

$$\text{Const. exp. } [-\tfrac{1}{2}\sum_{i,j=1}^{p} \alpha^{*ij}\ \{(N-1)a^*(ij)+(l-1)a'(ij)-\sum_{r=1}^{l} n_r\ \xi(r, j)\,(\bar{x}(r, i)-\bar{x}_i)-$$
$$\sum_{r=1}^{l} n_r\ \xi(r, i)(\bar{x}(r, j)-\bar{x}_j)+N\bar{x}_i\,\bar{x}_j-N\bar{x}_i\,\xi_j-N\bar{x}_j\,\xi_i)\}] \quad (2{\cdot}3)$$

As has been observed in Section 1 and in the previous papers the l_i's of (*f*) and τ_i's of (1·7) are invariant under any linear transformation (common to all the samples) of the p-variates to p new variates (for the samples) and any linear transformation (common to all the populations) of the p-variates to p new variates (for the populations). For purposes of invariance it is not essential that the transformation coefficients for the samples should be the same as the set for the populations. Here, however, we take them to be the same. It is known from one previous paper[3] by the author that we can construct a linear transformation such that in the new scheme the changed population parameters have the following properties:

$$\alpha^*(i\,i)=1\ ;\ \alpha^*(ij)=0 \text{ when } i\neq i \quad \ldots\ (2{\cdot}4)$$

from which it easily follows that

$$\alpha^{*ii}=1\ ;\ \alpha^{*ij}(i\neq j)=0 \quad \ldots\ (2{\cdot}41)$$

We can assume now without any loss of generality that our variates are what would be obtained after this transformation and thus in place of (2·3) we can write

$$\text{Const. exp. } [-\tfrac{1}{2}\sum_{i=1}^{p} \{(N-1)\,a^*(i\,i)+(l-1)\,a'(i\,i)-2\sum_{r=1}^{l} n_r\ \xi\,(r, i)(\bar{x}(r, i)-\bar{x}_i)+N\bar{x}_i^2$$
$$-2N\,\bar{x}_i\,\xi_i)\}]$$

or

$$\text{Const. exp. } [-\tfrac{1}{2} \sum_{i=1}^{p} \{(N-1)\,a''(i\,i) + (l-1)\,a'(i\,i) - 2\sum_{r=1}^{l} n_r\,(\xi(r,i)-\xi_i)(\bar{x}(r,i)-\bar{x}_i) + N\bar{x}_i^2 - 2N\bar{x}_i\,\xi_i\}]$$

or

$$\text{Const exp. } [-\tfrac{1}{2} \sum_{i=1}^{p} \{y''(i)^2 + y'(i)^2 - 2V[\eta'(i)].\,y'(i) + Nx_i^2 - 2Nx_i\,\xi_i\}] \qquad .. \quad (2{\cdot}5)$$

from (1·91), 1·93) and denoting by $y''(i)$ and $y'(i)$ the magnitudes of the vectors $\boldsymbol{y}''(i)$ and $\boldsymbol{y}'(i)$ of section 1, and further by turning round the vectors $V[\eta'(i)]$ of section 1 ($i=1, 2, \ldots .\ p$) so as to lie in the flat $f'(l)$ and make projections $\sqrt{n_1}(\xi(1,i)-\xi_i)$, $\sqrt{n_2}(\xi(2,i)-\xi_i) \ldots \sqrt{n_l}(l,i)-\xi_i)$ along the different l axes of $f'(l)$.

$(V[\eta'(i)].\ \boldsymbol{y}'(i))$ is of course, the scalar product of the vectors $V[\eta'(i)]$ and $\boldsymbol{y}'(i)$

Let ψ_i be the angle between the vectors $V[\eta'(i)]$ and $\boldsymbol{y}'(i)$; then $(V[\eta'(i).\ \boldsymbol{y}'(i))$ could be written as $\eta'(i)\ y'(i)\ \cos\psi_i$. Hence in the density factor (2·4)

$$\sum_{i=1}^{p} V[\eta'(i)].\,\boldsymbol{y}'(i) = \sum_{i=1}^{p} \eta'(i).\ y'(i).\ \cos\psi_i \qquad .. \quad (2{\cdot}51)$$

(2·5) can, therefore, be written in the alternative form

$$\text{Const. exp. } [-\tfrac{1}{2} \sum_{i=1}^{p} \{y''^2(i) + y'^2(i) - 2(V(y'(i)].\ \boldsymbol{y}'(i)) + N\bar{x}_i^2 - 2N\bar{x}_i\ \xi_i\}] \qquad .. \quad (2{\cdot}52)$$

Let us go back a little to the geometrical representation in Section 1. The vectors $\boldsymbol{y}_i$'s (with magnitude y_i's), the resultant of $\boldsymbol{y}'(i)$ and $\boldsymbol{y}''(i)$, constitute a p-flat $f(p)$ which make the p-flat $f''(p)$ (and also with the $(N-l)$-flat $f''(N-l)$)p critical angles $\theta_i(i=1, 2, \ldots .p)$; with the flat $f'(p)$ (and also with $f'(l-1)$) the flat $f(p)$ makes p critical angles $\pi/2-\theta_i$ ($i = 1, 2, \ldots .\ p$) ; there are p critical (orthogonal) lines in f_p. Referred to them as axes let the co-ordinates of the end points Y_i of the i-th resultant vector $\boldsymbol{y}_i$ be $y_{ij}(j=1, 2, \ldots .p)$; the first suffix referring to the character and the second to the axis along which the component is taken. Again referred to the p (orthogonal) cirtical lines of $f'(p)$ as axes let the co-ordinates of the end ponits of $\boldsymbol{y}'(i)$ be $y'(ij)$ and referred to the critical lines in $f''(p)$ as axes let the co-ordinates of the end points of $\boldsymbol{y}''(i)$ be $y''(ij)$. The same convention about suffixes holds for the components of vectors$\boldsymbol{y}_i$'s as well.

Hence from considerations similar to those immediately preceding (2·6) of the previous paper (5).

$$\left.\begin{aligned} & y'(ij)=y_{ij}\ \sin\ \theta_j,\ y''(ij)=y_{ij}\ \cos\ \theta_j \\ & y'^2(i)=\sum_{j=1}^{p} y^2_{ij}\ \sin^2\ \theta_j,\ y''^2(i)=\sum_{j=1}^{p} y^2_{ij}\ \cos^2\ \theta_j \\ & y'^2(i)+y''^2(i)=\sum_{j=1}^{p} y^2_{ij}=y^2_i \\ & \text{with } i, j = 1, 2, \ldots\ p. \end{aligned}\right\} \qquad .. \quad (2{\cdot}53)$$

The density factor (2·52) now reduces to

$$\text{Const. exp. } [-\tfrac{1}{2} \sum_{i=1}^{p} \{y^2_i - 2V[\eta'(i)].\ \boldsymbol{y}'(i) + N\bar{x}^2_i - 2N\bar{x}_i\ \xi_i\}] \qquad .. \quad (2{\cdot}54)$$

Consider now the joint distribution (2·1) which by using (2·54) can be written in the form

$$\text{Const. exp. } \left[-\tfrac{1}{2}\sum_{i=1}^{p}\left(y^2_i - 2Y'[Y'(i)].\ y'(i) + N\bar{x}^2_i - 2N\bar{x}_i\ \xi_i\right)\right] \times \prod_{i=1}^{p}\ \prod_{r=1}^{l}\ \prod_{\nu_r=1}^{n_r} dx(r, i, \nu_r) \qquad \text{.. (2·55)}$$

In Section 1 vectors $y(r, i)$ were introduced whose components referred to n_r axes of the r^{th} sample space $f(r, n_r)$ were defined by (1·8). As observed there these vectors $y(r, i)$'s lie in a flat $f(r, n_r-1)$ immersed in $f(r, n_r)$ and perpendicular to the equiangular line in the flat $f(r, n_r)$. Refer the vectors $y(r, i)$ now to any arbitray orthogonal n_r-1 axes in $f(r, n_r-1)$ and let the components be $z(r, i, \nu_r)$ $(\nu_r = 1, 2, \ldots\ldots n_r-1)$.

By arguments axactly similar to those of Section 2 of the previous paper [3] we can write down the volume element of (2·55) in the form

$$\text{Const. } \prod_{i=1}^{p}\ \prod_{r=1}^{l}\ d\bar{x}(r, i)\ \prod_{i=1}^{p}\ \prod_{r=1}^{l}\ \prod_{\nu_r=1}^{n_r-1} d\,z(r, i, \nu_r) \qquad \text{.. (2·56)}$$

The vectors $y''(i)$'s the resultant of the vectors $y''(1, i), y''(2, i) \ldots . y''(l, i)$ $(i=1, 2, \ldots . p)$, introduced in Section 1 in the lines after (1·9) lie, as observed there, in a flat $f''(N-l)$ absolutely orthogonal to the flat of the equiangular lines $f'(l)$ or the derived flat $f'(l-1)$. Referring the vectors $y''(i)$'s to $N-l$ arbitrary orthogonal axes in $f''(N-l)$ and denoting the components by $z''(i, \nu'')$ $(\nu''=1, 2, \ldots\ldots N-l)$ we immediately see that

$$\prod_{i=1}^{p}\ \prod_{r=1}^{l}\ \prod_{\nu_r=1}^{n_r} dx(r, i, \nu_r) \text{ reduces to } \prod_{i=1}^{p} d\bar{x}_i \prod_{\nu''=1}^{N-l}\ \prod_{i=1}^{p} dz''(i, \nu'') \qquad \text{.. (2·57)}$$

Again in Section 1 vectors $y'(i)$'s were introduced whose components referred to l axes of the space $f'(l)$ (constituted by the l equiangular lines of the l sample spaces $f(1, n_1)$, $f(2, n_2)$, ... $f(l, n_l)$ defined by (1·92). These vectors $y'(i)$ really lie in a flat $f'(l-1)$ immersed in $f'(l)$ and perpendicular to the line oe defined in the lines after (2·92). Let the components of $y'(i)$ referred to arbitrary $(l-1)$ orthogonal axes in $f'(l-1)$ be $z'(i, \nu')$, $(\nu'=1, 2, \ldots . l-1)$

Then $\prod_{i=1}^{p}\ \prod_{r=1}^{l}\ d\bar{x}_i(r, i)$ is easily transformed to

$$\prod_{i=1}^{p}\ \prod_{\nu'=1}^{l-1} dz'(i, \nu')\ \prod_{i=1}^{p} d\bar{x}_i \qquad \text{.. (2·58)}$$

Altogether, therefore, the volume element transforms to

$$\prod_{i=1}^{p} d\bar{x}_i\ \prod_{i=1}^{p}\ \prod_{\nu'=1}^{l-1} dz'(i, \nu')\ \prod_{i=1}^{p}\ \prod_{\nu''=1}^{N-l} dz''(i, \nu'') \qquad \text{.. 2·59}$$

Each of the variables $\bar{x}_i$, $z'(i, \nu')$, $z''(i, \nu'')$ varies from $-\infty$ to $+\infty$. Turning now to the density factor in (2·55) and taking account of the definitions of $z'(i, \nu')$, $z''(i, \nu'')$ just given and of y_i, $y'(i)$, $y''(i)$ given in (2·53), in lines immediately proceding it, following (2·5) and preceding (1·91), (1·93) and 2·51) we easily find that y^2_i's are pure functions of $z''(i\ \ \nu'')$'s, and $z'(i, \nu')$'s, $y'(i)$. $[Y'(i)]$'s are pure functions of $z'(i, \nu')$'s

Hence we write down (2·55) in the form

$$\text{Const. exp. } [-\tfrac{1}{2}\sum_{i=1}^{p}\{y^2_i - V[\eta'(i)].\ y'(i)\} + N\bar{x}^2_i - 2N\bar{x}_i\xi_i\}]$$

$$\times \prod_{i=1}^{p} d\bar{x}_i \prod_{i=1}^{p}\prod_{\nu'=1}^{l-1} dz'(i, \nu') \prod_{i=1}^{p}\prod_{\nu''=1}^{N-l} dz''(i, \nu'') \quad .. \quad (2·6)$$

Next we integrate out (2·6) over $\bar{x}_i$'s (i=1, 2, ... p) from $-\infty$ to $+\infty$ and get the joint distribution of $z''(i, \nu'')$'s and $z'(i, \nu')$'s in the form

$$\text{Const. exp. } [-\tfrac{1}{2}\sum_{i=1}^{p}\{y^2_i - 2V[\eta'(i).\ y'(i)\}] \times \prod_{i=1}^{p}\prod_{\nu'=1}^{l-1} dz'(i, \nu') \prod_{i=1}^{p}\prod_{\nu''=1}^{N-l} dz''(i, \nu'') \quad .. \quad (2·61)$$

Let us turn to the expression $\sum_{i=1}^{p} V[\eta'(i)].\ y'(i)$ or $\Sigma\eta'(i)\,y'(i)\cos\psi_i$ in the density factor of (2·61). Let $\overrightarrow{OQ}$ be a unit vector lying in the (l−1)-flat $f'(l-1)$ and making angles $\psi_1, \psi_2, \ldots, \psi_p$ respectively with the vectors $y'(1), y'(2) \ldots y'(p)$. Then if θ be the angle between $\overrightarrow{OQ}$ and the vector $\sum_{i=1}^{p}\eta'(i).\ y'(i)$ we have

$$\Sigma(V[\eta'(i)].\ y'(i)) \text{ or } \Sigma\eta'(i).\ y'(i)\cos\psi_i = \eta'(\overrightarrow{OQ}.\ y'(i)) = \eta'\ y'\cos\theta$$

where

$$\left.\begin{aligned} y' &\equiv \text{the vector} \sum_{i=1}^{p}\eta'(i)\,y'(i)/(\sum_{i=1}^{p}\eta^{2\prime}(i))^{1/2} \\ \eta'^2 &= \sum_{i=1}^{p}\eta'^2_i \end{aligned}\right\} \quad .. \quad (2·62)$$

$(\overrightarrow{OQ}.\ y')$ is the scalar product of y' and $\overrightarrow{OQ}$, and y' is the magnitude of the vector y'. In fact

$$y'^2 = \sum_{j=1}^{p}(\sum_{i=1}^{p}\eta'(i)\ y'(ij))^2/\eta'^2 \quad .. \quad (2·63)$$

if we refer the vectors $y'(i)$'s to the p critical lines of the flat $f'(p)$ (constituted by the $y'(i)$'s) as axes. Hence (2·61) now reduces to

$$\text{Const. exp } [-\tfrac{1}{2}\{\sum_{i=1}^{p} y^2_i - 2\eta'\ y'\cos\theta\}] \prod_{i=1}^{p}\prod_{\nu'=1}^{l-1} dz'(i;\ \nu') \prod_{i=1}^{p}\prod_{\nu''=1}^{N-l} d\,z''(i, \nu'') \quad .. \quad (2·64)$$

Vectors y_i(i=1, 2, ...p) as will be evident from the lines following (2·52), form a p-flat $f(p)$ which lies immersed a flat $f(N-1)$ of $N-l+l-1$ i. e. dimensions. Again $y'(i)$ of the density factor in (2·64) is given by (2·63) where again, $y'(ij) = y_{ij}.\ \sin\theta_j$ from (2·53); the y_{ij}'s as has been already observed, are the components of y_i's along the p (orthogonal) critical lines in f(p)

$$\text{Hence} \quad \left.\begin{aligned} \sum_{i=1}^{p} y^2_i &= \sum_{j,i=1}^{p} y^2_{ij} \\ \eta'\ y'\cos\theta &= \cos\theta[\sum_{j=1}^{p}(\sum_{i=1}^{p}\eta'(i)\ y_{ij}.\ \sin\theta)^2]^{\frac{1}{2}} \end{aligned}\right\} \quad .. \quad (2·65)$$

using (2·65), the density factor in (2·64) reduces to

$$\text{Const. exp } [-\tfrac{1}{2}\sum_{i,j=1}^{p} y'^2_{ij} + \cos\theta\{\sum_{j=1}^{p}\sin^2\theta_j(\sum_{i=1}^{p}\eta'(i)\ y_{ij})^2\}^{\frac{1}{2}}] \quad .. \quad (2·66)$$

Taking the joint distribution (2·64) and changing the density factor to (2·66) our business will now be to obtain the joint distribution of θ_j's (j=1, 2, ..., p) and hence of the t_j's which are connected with θ_j's by (1·94), (j=1, 2. ...., p). With this end in view we have first to express the volume element of (2·64) in terms of y_{ij}'s (i=1, 2, ... p), θ_j's (j=1, 2, ... p) and θ *i.e.* we have to translate the volume element of (2·64) consisting of $p(N-1)$ variables $z'(i, \nu')$'s(i=1, 2, .. p; ν'=1, 2, ... l−1), $z''(i\nu'')$'s (i=1, 2, .. p; ν''=1, 2, ... N−l) into a new volume element expressed in terms of the p^2+p+1 variables that occur in the density factor (2·66).

Now the end points of the vectors y_i's have been already denoted by Y_i's(i=1, 2, ... p) in section 1. The volume element in (2·64) can now be regarded as the joint volume element described by the end points Q, Y_1, Y_2, ... Y_p of vectors $\overline{OQ}$, $y_1, y_2, \ldots y_p$. Geometrically speaking, to effect what has been proposed in the last paragraph all that we have to do is to find out the joint volume element described by Q, Y_1, Y_2,... Y_p subject to y_{ij}'s lying between y_{ij} and $y_{ij}+dy_{ij}$, θ_j lying between θ_j and $\theta_j+d\theta_j$, and θ lying between θ and $\theta+d\theta$. Now $\overline{OQ}$ lies in the flat $f'(l-1)$ (derived from the flat of equiangular lines $f'(l)$), and makes an angle θ with a given vector y' in that flat defined by $y'=\sum_{i=1}^{p} \eta'(i)\, y'(i)/\eta'$. Hence keeping vectors $y'(i)$'s (i=1, 2, ... p) fixed, *i.e.* keeping the vector y' fixed, Q describes a volume element

$$\text{Const. } (\sin\theta)^{l-3}\, d\theta \qquad \ldots \quad (2\cdot67)$$

Again, as will be evident from Section 3 of the previous paper (5) the volume element described by Y_i's the end points of the vectors $y(i)$ (i=1, 2, ...p) subject to y_{ij}'s lying between y_{ij} and $y_{ij}+dy_{ij}$(i, j=1, 2, ... p) and θ_j's lying between θ_j and $\theta_j+d\theta_j$(j=1, 2, ... p) would be given by

$$\text{Const. } (\text{mod}|y_{ij}|)^{N-1-p} \prod_{i,j=1}^{p} dy_{ij} \times \prod_{j=1}^{p} \frac{t_j^{l-1-p}\, dt_j}{\left(1+\frac{l-1}{N-l}\, t_j^2\right)^{\frac{N-1}{2}}}$$

$$\times \text{ mod. } \{(t_1^2-t_2^2) \ldots (t_1^2-t_p^2)\,(t_2^2-t_3^2) \ldots (t_2^2-t_p^2) \ldots (t_{p-1}^2-t_p^2)\} \qquad \ldots \quad (2\cdot68)$$

where t_j's are connected with θ_j's by (1·94).

Hence the distribution (2·64) now finally reduces to

$$\text{Const. exp. } [-\tfrac{1}{2} \sum_{i,j=1}^{p} y_{ij}^2 + \cos\theta\, (\sum_{j=1}^{p} \sin^2\theta_j (\sum_{i}^{p} \eta'_i\, y_{ij})^2)^{\frac{1}{2}}]$$

$$\times (\sin\theta)^{l-3}\, d\theta\, (\text{mod}|y_{ij}|)^{N-1-p} \prod_{i,j=1}^{p} dy_{ij}$$

$$\times \text{mod } \{(t_1^2-t_2^2) \ldots (t_1^2-t_p^2)\,(t_2^2-t_3^2) \ldots (t_2^2-t_p^2) \ldots (t_{p-1}^2-t_p^2)\}$$

$$\times \prod_{j=1}^{p} \frac{t_j^{l-1-p}\, dt_j}{\left(1+\frac{l-1}{N-l}\, t_j^2\right)^{\frac{N-1}{2}}} \qquad (2\cdot7)$$

Section 3. The Actual Derivation of the Joint Distribution of p-Statistics

To get the joint distribution of t_j's or θ_j's we have to integrate out (2·7) over y_{ij}'s from $-\infty$ to $+\infty$ and θ from 0 to π

$$\int_0^\pi \exp.\ [\cos\theta\{\qquad\}]\ (\sin\theta)^{l-2}\, d\theta$$

$$=\text{Const}\ \{\sum_{j=1}^{p}\sin^2\theta_j(\sum_{i=1}^{p}\eta'(i)\ y_{ij})^2\}^{-\frac{l-2}{4}} I_{\frac{l-2}{2}}\ \{\sum_{j=1}^{p}\sin^2\theta_j(\sum_{i=1}^{p}\eta'(i)\ y_{ij})^2\}^{\frac{1}{2}} \qquad .. \quad (3·1)$$

Hence from (2·7) we have the joint distribution of y_{ij}'s(i, j=1, 2, ... p) and θ_j's (j=1, 2, .. p) in the form

$$\text{Const. } \exp.[-\tfrac{1}{2}\sum_{i,j=1}^{p}y^2_{ij}].\ \{\sum_{j=1}^{p}\sin^2\theta_j(\sum_{i=1}^{p}\eta'(i)\ y_{ij})^2\}^{-\frac{l-2}{4}}$$

$$\times\ I_{\frac{l-2}{2}}\ \{\sum_{j=1}^{p}\sin^2\theta_j(\sum_{i=1}^{p}\eta'(i)\ y_{ij})^2\}^{\frac{1}{2}}\ \text{mod}(|y_{ij}|)^{N-l-p}\prod_{i,j=1}^{p}dy_{ij}$$

$$\times\ \text{mod}((t^2_1-t^2_2)\ \ldots\ (t^2_1-t^2_p)\ (t^2_2-t^2_3)\ \ldots\ (t^2_2-t^2_p)\ \ldots\ (t^2_{p-1}-t^2_p))$$

$$\times\prod_{j=1}^{p}\frac{t_j^{l-1-p}\,dt_j}{\left(1+\frac{l-1}{N-l}\,t^2_j\right)^{\frac{N-1}{2}}} \qquad .. \quad (3·2)$$

It should be noticed that since r_i's defined by (1·7) are invariant under any linear transformation of the variates of the populations to p new variates, they (r_i's) are necessarily invariant under the special linear transformation considered in Section 2 which makes $\alpha''(i\,j)=0$ $(i\neq j)$, $\alpha''(ii)=1$ and hence $\alpha''^{ij}=0$, $\alpha''^{ii}=1$. But $\eta'^2(i)$'s considered in (3·2) are the values of $\alpha'(ii)$'s (of (1·95)) after this linear transformation. Hence from (1·7) and (1·95) $r_i=\eta'(i)$. Consequently

$$\eta'^2=\sum_{i=1}^{p}\eta'^2(i)=\sum_{i=1}^{p}r^2_i=r^2\ \text{(suppose)} \qquad .. \quad (3·3)$$

Hence $\qquad \eta'=r$

where r is given by (3·3)

Consider now in the determinant $|y_{ij}|$ in (3·2) any column, say, the j^{th}. Then make an orthogonal transformation of the constituents of this column y_{ij}(i=1, 2, ... p) to p new values, say v_{ij} such that

$$\left.\begin{aligned} v_{1j}&=\sum_{i=1}^{p}\eta'(i)\ y_{ij}/\eta' \\ v_{2j}&=\lambda_{12}\ y_{1j}+\lambda_{22}\ y_{2j}+\ \ldots\ \lambda_{p2}\ y_{pj} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ v_{pj}&=\lambda_{1p}\ y_{1j}+\lambda_{2p}\ y_{2j}+\ \ldots\ \lambda_{pp}\ y_{pj} \end{aligned}\right\} \qquad .. \quad (3·32)$$

and further that the p set $(\eta'(1)/\eta',\ \eta'(2)/\eta'\ \ldots\ \eta'(p)/\eta)$, $(\lambda_{12}, \lambda_{22}, \ldots\lambda_{p2})$, ... $(\lambda_{1p}, \lambda_{2p}, \ldots$

λ_{pp}) constitute the sets of co-efficients for an orthogonal transfromation. Apply the same transformation to all the columns of $|y_{ij}|$. Then since in (3.2), $|y_{ij}|$, $\sum\limits_{i}\sum\limits_{j=1}^{p} y^2_{ij}$ and $\prod\limits_{i}\prod\limits_{j=1}^{p} dy_{ij}$ are all invariant under the orthogonal transformation considered in (3·32), the distribution (3.2) changes over into

$$\text{Const. exp.}\left[-\tfrac{1}{2}\sum_{i}\sum_{j=1}^{p} v^2_{ij}\right]\left(\tau^2\sum_{j=1}^{p} v^2_{1j}\sin^2\theta_j\right)^{-\frac{l-2}{4}}$$

$$\times I_{\frac{l-2}{2}}\left(\tau^2\sum_{j=1}^{p} v^2_{1j}\sin^2\theta_j\right)^{\frac{1}{2}}\{\text{mod}|v_{ij}|\}^{N-l-p}\prod_{i}\prod_{j=1}^{p} dv_{ij}$$

$$\times\ \text{mod}\left((t^2_1-t^2_2)\ \ldots\ (t^2_1-t^2_p)\ (t^2_2-t^2_3)\ \ldots\ (t^2_2-t^2_p)\ \ldots\ (t^2_{p-1}-t^2_p)\right)$$

$$\times\prod_{j=1}^{p}\frac{t_j^{l-1-p}\,dt_j}{\left(1+\frac{l-1}{N-l}t^2_j\right)^{\frac{N-1}{2}}} \qquad \ldots\ (3{\cdot}4)$$

In the determinant $|v_{ij}|$ we can conveniently regard any row, say, the i^{th} as a vector v_i with components $v_{ij}(j=1, 2, \ldots p)$ along the different orthogonal axes; $\text{mod}|v_{ij}|$ is the volume of the hyper-par[4] constituted by these p vectors v_i's$(i=1, 2, \ldots p)$ as edges. Let the magnitude of these vectors be v_i's$(i=1, 2, \ldots p)$. Let the angle made by the vector v_p with the $(p-1)$-flat formed by the other vectors $v_1, v_2, \ldots v_{p-1}$ be ϕ_{p-1} the angle between v_{p-1} and the $(p-2)$-flat formed by $v_1, v_2, \ldots v_{p-2}$ be ϕ_{p-2} and so on, and finally the angle between v_2 and v_1 be ϕ_1. Then $\text{mod}\,|v_{ij}| = \text{vol}\,(o_1, v_2, \ldots v_p)$

$$= v_1 . v_2 . \ldots v_p .\ \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{p-1} \qquad \ldots\ (3{\cdot}41)$$

Also $\prod\limits_{i}\prod\limits_{j=1}^{p} dv_{ij}$ transforms to

$$\prod_{j=1}^{p} dv_{1j}\ \prod_{j=2}^{p} v_j^{p-1}\,dv_j(\sin\phi_1)^{p-2}\ (\sin\phi_2)^{p-3}\ \ldots\ (\sin\phi_{p-2})^1\prod_{i=1}^{p-1} d\phi_i \qquad \ldots\ (3{\cdot}42)$$

It should be noticed that we do not tamper with the components $v_{1j}(j=1, 2, \ldots p)$ of the first vector v_1. This is because these components occur separately in the density factor of (3·4). The position is different with the components of the other vectors v_i's$(i=2, 3, \ldots p)$ which occur in convenient lumps in (3·4).

In (3·4) consider next the portion involving v_{ij}'s which we can now conveniently write (by using (3·41) and (3·42)) in the form

$$\text{Const exp}\left[-\tfrac{1}{2}\sum_{j=1}^{p} v^2_{1j}-\tfrac{1}{2}\sum_{i=2}^{p} v^2_i\right]\left(\tau^2\sum_{j=1}^{p} v^2_{1j}\sin^2\theta_j\right)^{-\frac{l-2}{4}}$$

$$\times I_{\frac{l-2}{2}}\left(\tau^2\sum_{j=1}^{p} v^2_{1j}\sin^2\theta_j\right)^{\frac{1}{2}}\prod_{j=1}^{p} v_{1j}.\ v_1^{N-}$$

$$\times\prod_{i=2}^{p} v_i^{N-2}\,dv_i(\sin\phi_1)^{p-2}\ (\sin\phi_2)^{p-3}\ \ldots\ (\sin\phi_{p-2})^1\prod_{i=1}^{p-1} d\phi_i \qquad \ldots\ (3{\cdot}43)$$

The limits of $\varphi_1, \varphi_2, \ldots \varphi_{p-1}$ can evidently be taken from $-\pi/2$ to $\pi/2$. Integrating out (3.43) over $\varphi_1, \varphi_2, \ldots \varphi_{p-1}$ and also over $v_2, v_3, \ldots v_p$ and absorbing into the const. we have the joint distribution of v_{1j}'s and θ_j's $(j=1, 2, \ldots p)$ in the form

$$\text{Const.}\ \exp\Big[-\tfrac{1}{2}\sum_{j=1}^{p} v^2_{1j}\Big]\Big(r^2\sum_{j=1}^{p} v^2_{1j}\sin^2\theta_j\Big)^{-\frac{l-3}{4}}$$

$$\times\ I_{\frac{l-3}{2}}\Big(r^2\sum_{j=1}^{p} v^2_{1j}\sin^2\theta_j\Big)^{\frac{1}{2}}\prod_{j=1}^{p} dv_{1j}\Big(r^2\sum_{j=1}^{p} v^2_{1j}\Big)^{\frac{N-2}{2}} \qquad \ldots \quad (3{\cdot}44)$$

$$\times\ \text{mod}\ \{(t^2_1-t^2_2)\ldots(t^2_1-t^2_p)(t^2_2-t^2_3)\ldots(t^2_2-t^2_p)\ldots(t^2_{p-1}-t^2_p)\}$$

$$\times\prod_{j=1}^{p}\frac{t_j^{l-1-p}\,dt_j}{\Big(1+\frac{l-1}{N-l}\,t^2_j\Big)^{\frac{N-1}{2}}}$$

remembering that $v_1^2 = \sum_{j=1}^{p} v^2_{1j}$ $\qquad \ldots \quad (3{\cdot}45)$

It can be shown that in (3·44)

$$I_{\frac{l-3}{2}}\Big(r^2\sum_{j=1}^{p} v^2_{1j}\sin^2\theta_j\Big)^{\frac{1}{2}}.\ \Big(r^2\sum_{j=1}^{p} v^2_{1j}\sin^2\theta_j\Big)^{-\frac{l-3}{4}}$$

can be thrown into the form

$$\sum_{m_p=0}^{\infty}\ldots\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{\nu+2\sum_{j=1}^{p} m_j}(r^2)^{\sum_{j=1}^{p} m_j}\prod_{j=1}^{p}(\sin^2\theta_j\, v^2_{1j})^{m_j}}{\prod_{j=1}^{p}(m_j!)\,(\nu+\sum_{j=1}^{p} m_j)!} \qquad \ldots \quad (3{\cdot}46)$$

where

$$\nu = \frac{l-3}{2} \qquad \ldots \quad (3{\cdot}47)$$

Hence (3·44) transforms to

$$\text{Const.}\ \exp\Big[-\tfrac{1}{2}\sum_{j=1}^{p} v^2_{1j}\Big)\Big(r^2\sum_{j=1}^{p} v^2_{1j}\Big)^{\frac{N-2}{2}}$$

$$\times\sum_{m_p=0}^{\infty}\ldots\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{\nu+2\sum_{j=1}^{p} m_j}(r^2)^{\sum_{j=1}^{p} m_j}\prod_{j=1}^{p}(\sin^2\theta_j\, v^2_{1j})^{m_j}}{\prod_{j=1}^{p}(m_j!)\,(\nu+\sum_{j=1}^{p} m_j)!} \qquad \ldots \quad (3{\cdot}48)$$

$$\times\ \text{mod}\ \{(t^2_1-t^2_2)\ldots(t^2_1-t^2_p)(t^2_2-t^2_3)\ldots(t^2_2-t^2_p)\ldots(t^2_{p-1}-t^2_p)\}$$

$$\times\prod_{j=1}^{p}\frac{t_j^{l-p-1}\,dt_j}{\Big(1+\frac{l-1}{N-l}\,t^2_j\Big)^{\frac{N-1}{2}}}$$

Remembering that v_{ij}'s(j=1, 2, ... p) vary from $-\infty$ to $+\infty$ and further that

$$\int_{-\infty}^{\infty} \exp.\ (-\tfrac{1}{2}\ v^2_{ij})\ (v_{ij})^{2n}\ dv_{ij}$$

$$= \text{Const}\ \Gamma\left(\frac{2n+1}{2}\right) \qquad .. \quad (3\cdot49)$$

one can easily integrate out (3.48) over v_{ij}'s (j=1, 2, p) from $-\infty$ to $+\infty$ and obtain the distribution of t_j s in the form

$$\text{Const.} \sum_{m_1, m_2, \ldots m_p=0}^{\infty} \frac{(\tfrac{1}{2})^{-\frac{l-3}{2}+2\sum_{j=1}^{p} m_j}\ (\tau^2)^{\sum_{j=1}^{p} m_j}\ \Gamma\left(\frac{N-1}{2}+\sum_{j=1}^{p} m_j\right) \prod_{j=1}^{p} (\sin^2\theta_j)^{m_j}\ \Gamma(m_j+\tfrac{1}{2})}{\Gamma\left(\frac{l-3}{2}+\sum_{j=1}^{p} m_j\right)\Gamma\left(\frac{p}{2}+\sum_{j=1}^{p} m_j\right)\prod_{j=1}^{p} m_j\,!}$$

$$\times \text{ mod } \{(t^2_1-t^2_2) \ldots (t^2_1-t^2_p)\ (t^2_2-t^2_3) \ldots (t^2_2-t^2_p) \ldots (t^2_{p-1}-t^2_p)\}$$

$$\times \prod_{j=1}^{p} \frac{t_j^{l-1-p}\ dt_j}{\left(1+\frac{l-1}{N-1}\,t^2_j\right)^{\frac{N-1}{2}}} \qquad .. \quad (3\cdot5)$$

where

$$\sin\ \theta_j = \sqrt{\frac{l-1}{N-1}}\ t_j/(1+\frac{l-1}{N-1}\ t^2_j)^{\frac{1}{2}} \qquad .. \quad (3\cdot51)$$

When t_i's (i=1, 2, ... p) are all zero, that is, when the populations sampled have the same mean values for each character, (3·51) reduces, as it should to the form (*c*) of the introduction. Also when p=1, that is, in the univariate case, the distributing of t^2 reduces to (*b*) of the introduction.

The function in (3·7) involving the multiple summation can really be regarded as a convenient generalisation to many variables of the ordinary hypergeometric function of one variable. It could, of course, be thrown into a more suitable form amenable to practical computation. This will be considered in the next paper where distributions of symmetric functions of t_j's (more directly useful for purposes of classification as well as for purposes connected with Neyman and Pearson's theory of testing of hypothesis) will also be discussed.

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