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Another Esseen-type inequality for multivariate probability density functions

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Abstract

An upper bound for the supremum of the absolute value of the difference of two multivariate probability density functions is obtained. The upper bound involves integrals of the absolute value of suitable transforms of the characteristic functions of the probability density functions. Results are similar to the work of Gamkrelidze (Theory Probab. Appl. 22 (1977) 877–880) on the Esseen's inequality for multidimensional distribution functions.

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1. Introduction

Roussas (2001) obtained an upper bound for the supremum of the absolute difference of two multivariate probability density functions. The upper bound involves integral of the absolute value of the difference of the corresponding characteristic functions. We now obtain another upper bound involving integrals of absolute values of suitable transforms of the characteristic functions. Our approach is similar to the work in Gamkrelidze (1977) for obtaining upper bounds for the supremum of the difference of two multidimensional distributions functions.

2. Inequalities for the difference of distribution functions

Let F and G be two distribution functions defined on the real line $\mathbb{R}$ and having characteristic functions f and g, respectively. The one-dimensional Esseen inequality states that

$$\sup_x |F(x) - G(x)| \leqslant C_1 \int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right| \mathrm{d}t + C_2 \frac{A}{T}, \tag{2.1}$$

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where T is an arbitrary positive number, C_1 and C_2 are absolute constants and $A = \sup_x\{G'(x)\}$ assuming that the distribution function G has a bounded probability density function G'. Proof of this inequality can be found in Feller (1966, p. 510). Sadikova (1966) obtained a two-dimensional version of the inequality (2.1). Let F and G be two distribution functions defined on $\mathbb{R}^2$ with characteristic functions f and g, respectively. Suppose that the distribution function $G(x,y)$ has partial derivatives with respect to x and y on $\mathbb{R}^2$ and

$$A_1 = \sup_{-\infty<x,y<\infty} \frac{\partial G(x,y)}{\partial x}, \qquad A_2 = \sup_{-\infty<x,y<\infty} \frac{\partial G(x,y)}{\partial y} < \infty.$$

Let

$$\hat{f}(s,t) = f(s,t) - f(s,0)f(0,t)$$

and

$$\hat{g}(s,t) = g(s,t) - g(s,0)g(0,t).$$

Then the following inequality holds.

Theorem 2.1 (Sadikova). *For any* $T > 0$,

$$\begin{aligned}\sup_{-\infty<x,y<\infty} |F(x,y) - G(x,y)| \leqslant {} & \frac{2}{(2\pi)^2} \int_{-T}^{T}\int_{-T}^{T} \left| \frac{\hat{f}(s,t) - \hat{g}(s,t)}{st} \right| \mathrm{d}s\,\mathrm{d}t \\ & + 2 \sup_{-\infty<x<\infty} |F(x,\infty) - G(x,\infty)| \\ & + 2 \sup_{-\infty<y<\infty} |F(\infty,y) - G(\infty,y)| \\ & + \frac{2(A_1+A_2)}{T}(3\sqrt{2} + 4\sqrt{3}).\end{aligned}$$

Gamkrelidze (1977) generalized Theorem 2.1 to multidimensional distribution functions. We now introduce some notation following Gamkrelidze (1977).

Let ξ and v be k-dimensional random vectors with distribution functions $F(\boldsymbol{x})$ and $G(\boldsymbol{x})$ and characteristic functions $f(\boldsymbol{t})$ and $g(\boldsymbol{t})$, respectively, where $\boldsymbol{x} = (x_1,\ldots,x_k)$ and $\boldsymbol{t} = (t_1,\ldots,t_k)$. Suppose that the function $G(\boldsymbol{x})$ is partially differentiable with respect to x_i and

$$A_i = \sup_{\boldsymbol{x}\in\mathbb{R}^k} \frac{\partial G}{\partial x_i} < \infty, \quad 1 \leqslant i \leqslant k.$$

Let $i(j) = \{i_1,\ldots,i_j\}$ where $i_1 < i_2 < \cdots < i_j$, $j < k$ be a combination of size j from the sequence $\{1,\ldots,k\}$. For any function $\psi(\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^k$, define

$$\psi_{i(j)}(\boldsymbol{t}) = \psi(\boldsymbol{t})|_{t_{i_1}=\cdots=t_{i_j}=0}$$

and

$$\psi^{i(j)}(\boldsymbol{t}) = \lim_{t_{i_1}\to\infty,\ldots,t_{i_j}\to\infty} \psi(\boldsymbol{t}).$$

Note that the functions $\psi_{i(j)}(\boldsymbol{t})$ and $\psi^{i(j)}(\boldsymbol{t})$ are defined on $(k-j)$ one-dimensional arguments and both are functions of $(t_1,\ldots,t_{i_1-1},t_{i_1+1},\ldots,t_{i_2-1},t_{i_2+1},\ldots,t_{i_j-1},t_{i_j+1},\ldots,t_k)$. We write

$$\int_{-T}^{T} \psi_{i(j)}(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

and

$$\int_{-T}^{T} \psi^{i(j)}(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t},$$

for integrals of the functions involved over an appropriate $(k-j)$-dimensional cube. By the symbol $\Pi^{i(j)}$, we denote the product of all elements involved in the computation except those which have the indices $i_1,\ldots,i_j$. For any function $\psi(\boldsymbol{t})$, $\boldsymbol{t}\in\mathbb{R}^k$ for any k, we define the transformation

$$L\psi(\boldsymbol{t}) = \psi(\boldsymbol{t}) + \sum_{j=1}^{k-1}(-1)^j \sum_{i(j)} \psi_{i(j)}(\boldsymbol{t}).$$

Let

$$\Delta(\boldsymbol{t}) = Lf(\boldsymbol{t}) - Lg(\boldsymbol{t})$$

and

$$\Delta_{i(j)}(\boldsymbol{t}) = Lf_{i(j)}(\boldsymbol{t}) - Lg_{i(j)}(\boldsymbol{t}).$$

The following inequality is due to Gamkrelidze (1977).

Theorem 2.2 (Gamkrelidze). *For any* $T>0$,

$$\sup_{x\in\mathbb{R}^k} |F(\boldsymbol{x}) - G(\boldsymbol{x})| \leqslant 2\left\{ \frac{1}{(2\pi)^k}\int_{-T}^{T}\left|\frac{\Delta(\boldsymbol{t})}{\Pi_{r=1}^k t_r}\right|\mathrm{d}\boldsymbol{t} + \sum_{j=1}^{k-1}\frac{1}{(2\pi)^{k-j}}\sum_{i(j)}\int_{-T}^{T}\left|\frac{\Delta_{i(j)}(\boldsymbol{t})}{\Pi_{r=1}^{k}{}^{i(j)} t_r}\right|\mathrm{d}\boldsymbol{t}\right\} + C(k)\frac{A}{T}$$

where

$$C(k) = \frac{24\log 2}{\pi} + 8(2\pi\log\tfrac{4}{3})^{-1/3}k^{1/3}$$

and

$$A = \sum_{r=1}^{k} A_r, \quad A_r = \sup_{\boldsymbol{x}} \frac{\partial G}{\partial x_r}, \quad r = 1,\ldots,k.$$

3. Inequalities for the differences of probability density functions

In a recent paper, Roussas (2001), following the ideas in Sadikova (1966), obtained an upper bound for the supremum of the absolute value of the difference of the probability density functions of two k-dimensional random vectors. Let $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_k)$ and $\boldsymbol{\xi}'=(\xi'_1,\ldots,\xi'_k)$ be two k-dimensional random vectors with characteristic functions Q_{ξ} and $Q_{\xi'}$, respectively. Suppose the characteristic functions Q_{ξ} and $Q_{\xi'}$ are absolutely integrable and the probability density functions f_{ξ} and $f_{\xi'}$ of $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$, respectively, are bounded and satisfy the Lipschitz condition of order one, that is, there exists a constant c such that for every $\boldsymbol{x}\in\mathbb{R}^k$ and $\boldsymbol{u}\in\mathbb{R}^k$,

$$|f_{\xi}(\boldsymbol{x}+\boldsymbol{u})-f_{\xi}(\boldsymbol{x})|\leqslant c\sum_{j=1}^{k}|u_j|$$

and

$$|f_{\xi'}(\boldsymbol{x}+\boldsymbol{u})-f_{\xi'}(\boldsymbol{x})|\leqslant c\sum_{j=1}^{k}|u_j|.$$

Theorem 3.1 (Roussas). *There exists constant $C>0$ such that for every $T_j>0$, $1\leqslant j\leqslant k$,*

$$\sup_{\boldsymbol{x}}|f_{\xi}(\boldsymbol{x})-f_{\xi'}(\boldsymbol{x})|\leqslant\frac{1}{(2\pi)^k}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}|Q_{\xi}(\boldsymbol{t})-Q_{\xi'}(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}+4C\sqrt{3}\sum_{j=1}^{k}\frac{1}{T_j}.$$

We will now obtain an alternate inequality for the supremum of the absolute value of the difference of the probability density functions of two k-dimensional random vectors following the ideas in Gamkrelidze (1977). Let

$$\Delta(\boldsymbol{t})=LQ_{\xi}(\boldsymbol{t})-LQ_{\xi'}(\boldsymbol{t})$$

and

$$\Delta_{i(j)}(\boldsymbol{t})=LQ_{\xi,i(j)}(\boldsymbol{t})-LQ_{\xi',i(j)}(\boldsymbol{t}).$$

Consider the probability density function

$$q(x)=\left(\frac{4}{x}\sin\frac{x}{4}\right)^4\left(\frac{x}{8\pi}\right),$$

corresponding to the characteristic function

$$h(t)=\begin{cases}0 & \text{for } |t|\geqslant 1,\\ 2(1-|t|^3) & \text{for } \frac{1}{2}\leqslant|t|<1,\\ 1-6t^2+6|t|^3 & \text{for } 0\leqslant|t|<\frac{1}{2}.\end{cases}$$

It can be checked that

$$\int_{-\infty}^{\infty}x^2q(x)\,\mathrm{d}x=12$$

and

$$\int_{-\infty}^{\infty} |x| q(x)\, dx \leqslant \sqrt{12} = 2\sqrt{3}.$$

For $T_j > 0,\ 1 \leqslant j \leqslant k$, let

$$Q_{\boldsymbol{T}}(\boldsymbol{x}) = T_1 q(x_1) \ldots T_k q(x_k), \quad \boldsymbol{T} = (T_1, \ldots, T_k).$$

Then $Q_{\boldsymbol{T}}$ is the probability density function of a random vector $\boldsymbol{\xi}''$ with independent components. Further more $\boldsymbol{\xi}''$ has the characteristic function

$$H(\boldsymbol{t}) = h\left(\frac{t_1}{T_1}\right) \cdots h\left(\frac{t_k}{T_k}\right)$$

which vanishes outside the cube $(-T_1, T_1) \times \cdots \times (-T_k, T_k)$. We suppose that $\boldsymbol{\xi}''$ is independent of $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$.

(A1) Assume that the probability density functions $f_{\boldsymbol{\xi}}$ and $f_{\boldsymbol{\xi}'}$ are bounded and are Lipschitzian of order one, that is, there exists a constant $C > 0$ such that

$$|f_{\boldsymbol{\xi}}(\boldsymbol{x} + \boldsymbol{u}) - f_{\boldsymbol{\xi}}(\boldsymbol{x})| \leqslant C \sum_{j=1}^{k} |u_j|, \quad \boldsymbol{u} \in \mathbb{R}^k,\ \boldsymbol{x} \in \mathbb{R}^k,$$

$$|f_{\boldsymbol{\xi}'}(\boldsymbol{x} + \boldsymbol{u}) - f_{\boldsymbol{\xi}'}(\boldsymbol{x})| \leqslant C \sum_{j=1}^{k} |u_j|, \quad \boldsymbol{u} \in \mathbb{R}^k,\ \boldsymbol{x} \in \mathbb{R}^k.$$

(A2) Assume that the characteristic functions $Q_{\boldsymbol{\xi}}$ and $Q_{\boldsymbol{\xi}'}$ are absolutely integrable. From the convolution formula, we have

$$\begin{aligned} f_{\boldsymbol{\xi}+\boldsymbol{\xi}''}(\boldsymbol{x}) &= \int_{\mathbb{R}^k} f_{\boldsymbol{\xi}}(\boldsymbol{x} - \boldsymbol{u}) f_{\boldsymbol{\xi}''}(\boldsymbol{u})\, d\boldsymbol{u} \\ &= \int_{\mathbb{R}^k} f_{\boldsymbol{\xi}}(\boldsymbol{x} - \boldsymbol{u}) Q_{\boldsymbol{T}}(\boldsymbol{u})\, d\boldsymbol{u} \end{aligned}$$

and similarly

$$f_{\boldsymbol{\xi}'+\boldsymbol{\xi}''}(\boldsymbol{x}) = \int_{\mathbb{R}^k} f_{\boldsymbol{\xi}'}(\boldsymbol{x} - \boldsymbol{u}) Q_{\boldsymbol{T}}(\boldsymbol{u})\, d\boldsymbol{u}.$$

Hence

$$\begin{aligned} r^*(\boldsymbol{x}) &= f_{\boldsymbol{\xi}+\boldsymbol{\xi}''}(\boldsymbol{x}) - f_{\boldsymbol{\xi}'+\boldsymbol{\xi}''}(\boldsymbol{x}) \\ &= \int_{\mathbb{R}^k} [f_{\boldsymbol{\xi}}(\boldsymbol{x} - \boldsymbol{u}) - f_{\boldsymbol{\xi}'}(\boldsymbol{x} - \boldsymbol{u})] Q_{\boldsymbol{T}}(\boldsymbol{u})\, d\boldsymbol{u} \\ &= \int_{\mathbb{R}^k} r(\boldsymbol{x} - \boldsymbol{u}) Q_{\boldsymbol{T}}(\boldsymbol{u})\, d\boldsymbol{u}, \end{aligned}$$

where

$$r(\boldsymbol{x}) = f_{\xi}(\boldsymbol{x}) - f_{\xi'}(\boldsymbol{x}).$$

Following computations given in Roussas (2001), it follows from the inversion formula (cf. Loéve, 1963, p. 188) that, for all $\boldsymbol{x} \in \mathbb{R}^k$,

$$r^*(\boldsymbol{x}) = \frac{1}{(2\pi)^k} \int_{-T_k}^{T_k} \cdots \int_{-T_1}^{T_1} \mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}}[Q_{\xi}(\boldsymbol{t}) - Q_{\xi'}(\boldsymbol{t})]H(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}.$$

Note that

$$LQ_{\xi}(\boldsymbol{t}) - LQ_{\xi'}(\boldsymbol{t}) = Q_{\xi}(\boldsymbol{t}) - Q_{\xi'}(\boldsymbol{t}) + \sum_{j=1}^{k-1}(-1)^j \sum_{i(j)} [Q_{\xi,i(j)}(\boldsymbol{t}) - Q_{\xi',i(j)}(\boldsymbol{t})].$$

Therefore

$$\begin{aligned} r^*(\boldsymbol{x}) &= \frac{1}{(2\pi)^k} \int_{-T_k}^{T_k} \cdots \int_{-T_1}^{T_1} \mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}}[LQ_{\xi}(\boldsymbol{t}) - LQ_{\xi'}(\boldsymbol{t})]H(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \\ &\quad - \sum_{j=1}^{k-1}(-1)^j \sum_{i(j)} \frac{1}{(2\pi)^k} \int_{-T_k}^{T_k} \cdots \int_{-T_1}^{T_1} \mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}}[Q_{\xi,i(j)}(\boldsymbol{t}) - Q_{\xi',i(j)}(\boldsymbol{t})]H(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \\ &= \frac{1}{(2\pi)^k} \int_{-T_k}^{T_k} \cdots \int_{-T_1}^{T_1} \mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}} \Delta(\boldsymbol{t})H(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \\ &\quad - \sum_{j=1}^{k-1}(-1)^j \sum_{i(j)} \int_{\mathbb{R}^k} r^{i(j)}(\boldsymbol{t} - \boldsymbol{u})Q_{\boldsymbol{T}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \end{aligned}$$

by following the computations in the reverse order on p. 400 of Roussas (2001). We consider the above relation as a recurrence relation. Then

$$\begin{aligned} &\sum_{i(1)} \int_{\mathbb{R}^k} r^{i(1)}(\boldsymbol{x} - \boldsymbol{u})Q_{\boldsymbol{T}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \\ &\quad = \frac{1}{(2\pi)^{k-1}} \sum_{i(1)} \int_{-T_k}^{T_k} \cdots \int_{-T_1}^{T_1} \mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}} \Delta_{i(1)}(\boldsymbol{t})H(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \\ &\quad\quad - \sum_{j=2}^{k-1}(-1)^j \sum_{i(j)} \int_{\mathbb{R}^k} r^{i(j)}(\boldsymbol{x} - \boldsymbol{u})Q_{\boldsymbol{T}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \\ &\quad = \sum_{j=1}^{k-1} \frac{1}{(2\pi)^{k-j}} \sum_{i(j)} \int_{-T_k}^{T_k} \cdots \int_{-T_1}^{T_1} \Delta_{i(j)}(\boldsymbol{t})H(\boldsymbol{t})\mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}}\,\mathrm{d}\boldsymbol{t}. \end{aligned}$$

We obtain an analogous equation for $\sum_{i(2)}\cdots,\sum_{i(3)}\cdots$ and so on leading to

$$\sum_{i(k-1)}\int r^{i(k-1)}(\boldsymbol{x}-\boldsymbol{u})Q_{\boldsymbol{T}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}=\frac{1}{2\pi}\sum_{i(k-1)}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}\Delta_{i(k-1)}(\boldsymbol{t})H(\boldsymbol{t})\mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}}\,\mathrm{d}\boldsymbol{t}.$$

Combining these results, we have

$$r^*(\boldsymbol{x})=\int_{\mathbb{R}^k}r(\boldsymbol{x}-\boldsymbol{u})Q_{\boldsymbol{T}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}=\frac{1}{(2\pi)^k}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}\Delta(\boldsymbol{t})\mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}}H(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$
$$+\sum_{j=1}^{k-1}\frac{1}{(2\pi)^{k-j}}\sum_{i(j)}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}\Delta_{i(j)}(\boldsymbol{t})\mathrm{e}^{-\mathrm{i}\boldsymbol{t}'\boldsymbol{x}}H(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}.$$

Hence

$$\sup_{\boldsymbol{x}}|r^*(\boldsymbol{x})|\leqslant\frac{1}{(2\pi)^k}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}|\Delta(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}+\sum_{j=1}^{k-1}\frac{1}{(2\pi)^{k-j}}\sum_{i(j)}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}|\Delta_{i(j)}(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}.$$

Let $\gamma=\sup\{r(\boldsymbol{x})\colon\ \boldsymbol{x}\in\mathbb{R}^k\}$ and $\gamma^*=\sup\{|r^*(\boldsymbol{x})|\colon\ \boldsymbol{x}\in\mathbb{R}^k\}$. Following arguments given in Roussas (2001, p. 401), it follows that there exists a constant $C>0$ such that

$$\gamma^*\geqslant\gamma-4C\sqrt{3}\sum_{j=1}^{k}\frac{1}{T_j}.$$

A similar bound holds if $\gamma=\sup\{-r(\boldsymbol{x}),\ \boldsymbol{x}\in\mathbb{R}^k\}$ as pointed out by Roussas (2001, p. 401). Combining all the above inequalities, we get that

$$\gamma\leqslant\gamma^*+4C\sqrt{3}\sum_{j=1}^{k}\frac{1}{T_j},$$

where $\gamma=\sup\{|r(\boldsymbol{x})|\colon\ \boldsymbol{x}\in\mathbb{R}^k\}$ and $\gamma^*=\sup\{|r^*(\boldsymbol{x})|\colon\ \boldsymbol{x}\in\mathbb{R}^k\}$. But

$$\gamma^*\leqslant\frac{1}{(2\pi)^k}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}|\Delta(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}+\sum_{j=1}^{k-1}\frac{1}{(2\pi)^{k-j}}\sum_{i(j)}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}|\Delta_{i(j)}(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}$$

and we have the following main theorem of this paper.

Theorem 3.2. *Suppose conditions* (A1) *and* (A2) *hold. Then there exists a constant* $C>0$ *such that*

$$\sup_{\boldsymbol{x}}|f_{\xi}(\boldsymbol{x})-f_{\xi'}(\boldsymbol{x})|\leqslant\frac{1}{(2k)^k}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}|\Delta(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}$$
$$+\sum_{j=1}^{k-1}\frac{1}{(2\pi)^{k-j}}\sum_{i(j)}\int_{-T_k}^{T_k}\cdots\int_{-T_1}^{T_1}|\Delta_{i(j)}(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}$$
$$+4C\sqrt{3}\sum_{j=1}^{k}\frac{1}{T_j}. \tag{3.1}$$

(Note that the integration is carried out on a $(k-j)$-dimensional cube in the jth term in the second expression on the right-hand side of the above inequality.)

3.1. Special case

Let us consider the case $k=2$. Then the inequality given by Theorem 3.2 reduces to

$$\begin{aligned}\sup_{x\in\mathbb{R}^2}|f_\xi(\boldsymbol{x})-f_{\xi'}(\boldsymbol{x})| \leqslant\ & \frac{1}{(2\pi)^2}\int_{-T_2}^{T_2}\int_{-T_1}^{T_1}|\Delta(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}\\ &+\frac{1}{2\pi}\sum_{i(1)}\int_{-T_2}^{T_2}\int_{-T_1}^{T_1}|\Delta_{i(1)}(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}\\ &+4C\sqrt{3}\left(\frac{1}{T_1}+\frac{1}{T_2}\right),\end{aligned}$$

where

$$\begin{aligned}\Delta(\boldsymbol{t}) = &\ Q_\xi(t_1,t_2)-Q_\xi(t_1,0)-Q_\xi(0,t_2)\\ &-\{Q_{\xi'}(t_1,t_2)-Q_{\xi'}(t_1,0)-Q_{\xi'}(0,t_2)\}\end{aligned}$$

and

$$\begin{aligned}\Delta_{i(1)}(\boldsymbol{t}) &= Q_\xi(t_1,0)-Q_{\xi'}(t_1,0) \quad \text{if } i(1)=\{2\},\\ &= Q_\xi(0,t_2)-Q_{\xi'}(0,t_2) \quad \text{if } i(1)=\{1\}.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{x\in\mathbb{R}^2}|f_\xi(\boldsymbol{x})-f_{\xi'}(\boldsymbol{x})| \leqslant\ & \frac{1}{(2\pi)^2}\int_{-T_2}^{T_2}\int_{-T_1}^{T_1}|\Delta(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}\\ &+\frac{1}{2\pi}\int_{-T_1}^{T_1}|Q_\xi(t_1,0)-Q_{\xi'}(t_1,0)|\,\mathrm{d}t_1\\ &+\frac{1}{2\pi}\int_{-T_2}^{T_2}|Q_\xi(0,t_2)-Q_{\xi'}(0,t_2)|\,\mathrm{d}t_2\\ &+4C\sqrt{3}\left(\frac{1}{T_1}+\frac{1}{T_2}\right).\end{aligned}$$

An application of Theorem 3.1 of Roussas (2001) proves that

$$\begin{aligned}\sup_{x\in\mathbb{R}^2}|f_\xi(\boldsymbol{x})-f_{\xi'}(\boldsymbol{x})| \leqslant\ & \frac{1}{(2\pi)^2}\int_{-T_2}^{T_2}\int_{-T_1}^{T_1}|Q_\xi(t_1,t_2)-Q_{\xi'}(t_1,t_2)|\,\mathrm{d}t_1\,\mathrm{d}t_2\\ &+4C\sqrt{3}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\end{aligned}$$

which gives an alternate bound.

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