

Reflected backward stochastic differential equations in an orthant

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Abstract. We consider RBSDE in an orthant with oblique reflection and with time and space dependent coefficients, viz.

$$Z(t) = \xi + \int_t^T b(s, Z(s)) ds + \int_t^T R(s, Z(s)) dY(s) - \int_t^T \langle U(s), dB(s) \rangle$$

with $Z_i(\cdot) \geq 0$, $Y_i(\cdot)$ nondecreasing and $Y_i(\cdot)$ increasing only when $Z_i(\cdot) = 0$, $1 \leq i \leq d$. Existence of a unique solution is established under Lipschitz continuity of b , R and a uniform spectral radius condition on R . On the way we also prove a result concerning the variational distance between the 'pushing parts' of solutions of auxiliary one-dimensional problem.

Keywords. Backward SDE's; Skorokhod problem; oblique reflection; spectral radius; total variation; local time; contraction map; subsidy-surplus model.

1. Introduction

Since backward stochastic differential equations were introduced about a decade back there has been a lot of interest in them owing to wide applicability in stochastic control, differential games and economics. Recently backward stochastic differential equations with reflecting barrier have been studied by El Karoui *et al* [5] and Cvitanic and Karatzas [1] in the one-dimensional case; and by Gegout-Petit and Pardoux [7] in a convex domain in higher dimensions; these works concern the case of normal reflection at the boundary.

On the other hand, following the impetus given by queueing theory, deterministic as well as stochastic Skorokhod problem in an orthant with oblique reflection at the boundary has been studied by many authors over the last two decades; see the references in [11].

The aim of this article is to study reflected backward stochastic differential equations (RBSDE's) in an orthant with oblique reflection at the boundary. The drift vector and the reflection matrix can be time and space dependent; existence and uniqueness are established under a uniform spectral radius condition on the reflection matrix (plus, of course, a Lipschitz continuity condition on the coefficients); such a condition has proved useful in the study of Skorokhod problem; see [8,9,11,12].

In §2, after describing the set up, we indicate briefly two situations from economics where RBSDE can be used as a model. The first one is a backward stochastic analogue of the subsidy-surplus model considered in Ramasubramanian [11], and the second example is a backward stochastic (oblique) analogue of a projected dynamical system studied in Nagurney and Siokos [10].

An auxiliary one-dimensional RBSDE is discussed in §3. A result concerning the variational distance between the ‘pushing parts’ of solutions of two auxiliary one-dimensional equations is established, the inspiration being a deterministic analogue due to Shashashvili [14]; see also [15]. Existence of a unique solution to RBSDE is proved in §4 by a contraction mapping argument; the metric is given in terms of total variation and L^1 -norm. As in [11] a couple of a priori results help in confining the analysis to a smaller space. It is also shown that it is enough to have the reflection coefficients defined on the boundary.

2. RBSDE in an orthant with oblique reflection

Let $\{B(t) = (B_1(t), \dots, B_d(t)) : 0 \leq t \leq T\}$ be a d -dimensional standard Brownian motion defined on a probability space (Ω, \mathcal{F}, P) ; let $\{\mathcal{F}_t\}$ be the natural filtration generated by $\{B(t)\}$, with \mathcal{F}_0 containing all P -null sets.

Let $G = \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ denote the d -dimensional positive orthant. We are given the following :

ξ is an \mathcal{F}_T -measurable \bar{G} -valued bounded random variable;

$b : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $R : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{M}_d(\mathbb{R})$ are both bounded measurable functions such that for each $z \in \mathbb{R}^d$, $b(\cdot, \cdot, z) = (b_1(\cdot, \cdot, z), \dots, b_d(\cdot, \cdot, z))$, $R(\cdot, \cdot, z) = ((r_{ij}(\cdot, \cdot, z)))_{1 \leq i, j \leq d}$ are $\{\mathcal{F}_t\}$ -predictable processes; it is also assumed that $r_{ii}(\cdot, \cdot, \cdot) \equiv 1$ which is just a suitable normalization. (Here $\mathbb{M}_d(\mathbb{R})$ denotes the class of $d \times d$ matrices with real entries.)

A pair $\{Y(t) = (Y_1(t), \dots, Y_d(t))\}$, $\{Z(t) = (Z_1(t), \dots, Z_d(t))\}$, $0 \leq t \leq T$ of $\{\mathcal{F}_t\}$ -progressively measurable continuous integrable processes is said to solve RBSDE (ξ, b, R) if there is an $\{\mathcal{F}_t\}$ -progressively measurable square integrable process $U(t) = ((U_{ij}(t)))_{1 \leq i, j \leq d}$ such that

(i) for $i = 1, \dots, d, 0 \leq t \leq T$

$$\begin{aligned} Z_i(t) = & \xi_i + \int_t^T b_i(s, Z(s)) ds + Y_i(T) - Y_i(t) \\ & + \sum_{j \neq i} \int_t^T r_{ij}(s, Z(s)) dY_j(s) - \int_t^T \sum_{j=1}^d U_{ij}(s) dB_j(s) \end{aligned} \quad (2.1)$$

(ii) $Z(t) \in \bar{G}$ for all $0 \leq t \leq T$;

(iii) $Y_i(0) = 0$, $Y_i(\cdot)$ continuous, nondecreasing and $Y_i(\cdot)$ can increase only when $Z_i(\cdot) = 0$, $1 \leq i \leq d$; that is,

$$Y_i(t) = \int_0^t I_{\{0\}}(Z_i(s)) dY_i(s). \quad (2.2)$$

Equation (2.1) is the analogue of Skorokhod equation. Note that the process $U(\cdot)$ need not be continuous; b is the drift and R gives the reflection directions.

We now describe briefly two situations where the above model may be applicable.

Remark 2.1. Following Ramasubramanian [11], RBSDE (ξ, b, R) can be viewed upon as a subsidy-surplus model. We consider an economy with d interdependent sectors, with the following interpretations:

- (a) $Z_i(t)$ = current surplus in Sector i at time t ;
- (b) $Y_i(t)$ = cumulative subsidy given to Sector i over $[0, t]$;
- (c) ξ_i = desired surplus in Sector i at time T ;
of course, $Z_i(t), Y_i(t)$ depend on 'information' only up to time t ;
- (d) $\int_s^t b_i(u, Z(u)) du$ = net production of Sector i over $[s, t]$ due to evolution of the system; this being negative indicates there is net consumption;
- (e) $\int_s^t r_{ij}^-(u, Z(u)) dY_j(u)$ = amount of subsidy for Sector j mobilized from Sector i over $[s, t]$;
- (f) $\int_s^t r_{ij}^+(u, Z(u)) dY_j(u)$ = amount of subsidy mobilized for Sector j which is actually used in Sector i (but not as subsidy in Sector i) over $[s, t]$.

The condition (iii) in RBSDE (ξ, b, R) means that subsidy for Sector i can be mobilized only when Sector i has no surplus; this is a *natural minimality condition*. The uniform spectral radius condition (A3) which is imposed in §4 would mean that the subsidy mobilized from external sources is nonzero; so this would be an 'open' system in the jargon of economics; see also §2 of [11]. This suggests that the above situation may be called a *stochastic differential subsidy-surplus model* a la Duffie and Epstein [2].

Remark 2.2. We give another interpretation. Suppose the system represents d traders each specializing in a different commodity. For this model we assume $r_{ij}(\cdot \cdot \cdot) \leq 0, i \neq j$. Here $Z_i(t)$ = current price of Commodity i at time t ; there is a price floor viz. prices cannot be negative;

$Y_i(t)$ = cumulative 'tatonnement' (adjustment) involved in the price of Commodity i over $[0, t]$;

$b_i(t, Z(t)) dt$ = infinitesimal change in price of Commodity i due to evolution of the system;

ξ_i = desired price level of Commodity i at time T .

Condition (iii) (that is, (2.2)) of RBSDE (ξ, b, R) then means that tatonnement/adjustment $dY_i(\cdot)$ can take place only if the price of Commodity i is zero. In such a case $\int_s^t r_{ij}^-(u, Z(u)) dY_j(u)$ = tatonnement from Trader i when price of Commodity j is zero.

Note that $dY_j(\cdot)$ can be viewed upon as a sort of artificial/forced infinitesimal consumption when the price of Commodity j is zero to boost up the price; hence $r_{ij}^-(t, Z(t)) dY_j(t)$ is the contribution of Trader i towards this forced consumption. As before, (A3) implies that there is nonzero 'external tatonnement', like perhaps governmental intervention/consumption to boost prices when prices crash.

In the context of the Skorokhod problem with normal reflection, a similar interpretation is given in ([10] pp. 76–80) in connection with financial networks; these authors call the model as a 'projected dynamical system'; see also [4]. One-dimensional RBSDE (of course, with normal reflection), has been proposed as a model for pricing of American contingent claims in El Karoui and Quenez ([6], pp. 229–231).

Since 'tatonnement' can be viewed upon as a 'subsidy', the above may also be taken as a special case of Remark 2.1.

3. Auxiliary one-dimensional problem

In this section we look at an auxiliary one-dimensional problem needed for studying the d -dimensional problem.

Let $(\Omega, \mathcal{F}, P), \{B(t)\}, \{\mathcal{F}_t\}, 0 \leq t \leq T$ be as in §2. We are given the following:

ζ is an \mathcal{F}_T -measurable bounded nonnegative random variable;

$f : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, g_j : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, 1 \leq j \leq k$, are bounded measurable functions such that for each $z \in \mathbb{R}, f(\cdot, \cdot, z), g_j(\cdot, \cdot, z)$ are $\{\mathcal{F}_t\}$ -predictable;

$A_j, 1 \leq j \leq k$ are $\{\mathcal{F}_t\}$ -progressively measurable integrable continuous nondecreasing processes.

A pair $\{L(t)\}, \{M(t)\}, 0 \leq t \leq T$ of real valued $\{\mathcal{F}_t\}$ -progressively measurable continuous integrable processes is said to solve the auxiliary one-dimensional problem corresponding to (ζ, f, g_j, A_j) if there exists an $\{\mathcal{F}_t\}$ -progressively measurable square integrable process $\{V(t) = (V_1(t), \dots, V_d(t))\}, 0 \leq t \leq T$ such that

(i) the Skorokhod equation holds, viz.

$$\begin{aligned} M(t) = & \zeta + \int_t^T f(s, M(s)) ds + \sum_{j=1}^k \int_t^T g_j(s, M(s)) dA_j(s) \\ & + L(T) - L(t) - \int_t^T \sum_{\ell=1}^k V_\ell(s) dB_\ell(s) \end{aligned} \quad (3.1)$$

(ii) $M(t) \geq 0$ for all $0 \leq t \leq T$;

(iii) $L(0) = 0, L(\cdot)$ nondecreasing, $L(\cdot)$ can increase only when $M(\cdot) = 0$.

Proceeding as in the proof of Proposition 4.2 and Remark 4.3 of [5] the following result can be proved.

Lemma 3.1. Let $\{L(t)\}, \{M(t)\}, 0 \leq t \leq T$ be a solution to the auxiliary one-dimensional problem. Let $\{\ell(t)\}, 0 \leq t \leq T$ denote the local time at 0 of the continuous semimartingale $\{M(t)\}$. Then

$$0 \leq dL(t) \leq I_{\{0\}}(M(t)) \left\{ |f(t, 0)| dt + \sum_{j=1}^k |g_j(t, 0)| dA_j(t) \right\} \quad (3.2)$$

$$0 \leq d\ell(t) \leq I_{\{0\}}(M(t)) \left\{ |f(t, 0)| dt + \sum_{j=1}^k |g_j(t, 0)| dA_j(t) \right\}. \quad (3.3)$$

If in addition $A_j, 1 \leq j \leq k$ are absolutely continuous then

$$dL(t) + \frac{1}{2} d\ell(t) = I_{\{0\}}(M(t)) \left[f(t, 0) + \sum_{j=1}^k g_j(t, 0) \frac{dA_j(t)}{dt} \right]^- dt. \quad (3.4)$$

The next result concerns the variational distance between the L -parts of the solutions of two auxiliary one-dimensional equations; it has been motivated by a deterministic analogue due to Shashashvili [14] in the context of Skorokhod problem. For our purposes it suffices to consider the case when $f : \Omega \times [0, T] \rightarrow \mathbb{R}$ is $\{\mathcal{F}_t\}$ -predictable and $g_j \equiv 0$ for all j .

To be more precise, for $k = 1, 2$ let $f^{(k)} : \Omega \times [0, T] \rightarrow \mathbb{R}$ be bounded $\{\mathcal{F}_t\}$ -predictable process, $\zeta^{(k)}$ bounded nonnegative $\{\mathcal{F}_T\}$ -measurable random variable; let

$\{L^{(k)}(t)\}, \{M^{(k)}(t)\}, 0 \leq t \leq T$ solve the auxiliary one-dimensional problem corresponding to $(\zeta^{(k)}, f^{(k)}, 0, 0)$ so that

$$M^{(k)}(t) = \zeta^{(k)} + \int_t^T f^{(k)}(s) ds + L^{(k)}(T) - L^{(k)}(t) - \int_t^T \left\langle V^{(k)}(s), dB(s) \right\rangle, \tag{3.5}$$

$$\int M^{(k)}(s) dL^{(k)}(s) = 0, \tag{3.6}$$

$M^{(k)}(t) \geq 0, L^{(k)}(0) = 0, L^{(k)}(\cdot)$ continuous nondecreasing, $k = 1, 2, 0 \leq t \leq T$; (all these hold a.s.). By Proposition 5.1 of [5] unique square integrable $M^{(k)}(\cdot), L^{(k)}(\cdot), V^{(k)}(\cdot)$ exist solving the above. Clearly $L^{(1)}(\cdot) - L^{(2)}(\cdot)$ is of bounded variation a.s.; in fact, by the preceding lemma $L^{(1)}, L^{(2)}$ are absolutely continuous; let $\lambda^{(1)}(\cdot), \lambda^{(2)}(\cdot)$ denote their respective derivatives. Let $|d(L^{(1)} - L^{(2)})|(\cdot)$ denote the measure given by the total variation of $(L^{(1)} - L^{(2)})(\cdot)$.

Theorem 3.2. For any $\theta \geq 0, 0 \leq s \leq t \leq T$

$$\begin{aligned} & E \int_s^t (e^{\theta r} - 1) |\lambda^{(1)}(r) - \lambda^{(2)}(r)| dr \\ &= E \int_s^t (e^{\theta r} - 1) |d(L^{(1)} - L^{(2)})|(r) \\ &\leq E[(e^{\theta t} - 1)|M^{(1)}(t) - M^{(2)}(t)| - (e^{\theta s} - 1)|M^{(1)}(s) - M^{(2)}(s)|] \\ &\quad - E \int_s^t e^{\theta r} \theta |M^{(1)}(r) - M^{(2)}(r)| dr \\ &\quad + E \int_s^t (e^{\theta r} - 1) |f^{(1)}(r) - f^{(2)}(r)| dr. \end{aligned} \tag{3.7}$$

Proof. All equalities/inequalities below are satisfied almost surely. We denote $\widehat{\xi}(\cdot) = \xi^{(1)}(\cdot) - \xi^{(2)}(\cdot)$ for $\xi = \lambda, L, f, V$. Proceeding as in the proof of eq. (13) in Shashashvili ([14], pp. 171–173) using

$$I_{(0,\infty)}(\widehat{M}(r)) dL^{(1)}(r) = 0, I_{(-\infty,0)}(\widehat{M}(r)) dL^{(2)}(r) = 0$$

we get for $0 \leq s \leq t \leq T$

$$\int_s^t |d(\widehat{L})|(r) = \int_s^t [-\text{sgn}(\widehat{M}(r)) + I_{\{0\}}(\widehat{M}(r))\chi(r)] d(\widehat{L})(r) \tag{3.8}$$

where $\chi(\cdot)$ is $\{\mathcal{F}_t\}$ -progressively measurable function taking only the values $+1, -1$ and the function sgn is defined by

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0. \\ -1 & \text{if } x < 0 \end{cases}$$

Progressive measurability of $\chi(\cdot)$ follows by the proof of Radon–Nikodym theorem and p. 171 of Shashashvili [14]. Therefore denoting the integrand on the r.h.s. of (3.8) by $J(r)$ and using (3.5) for $k = 1, 2$, we get

$$\begin{aligned}
\int_s^t (e^{\theta r} - 1)|\widehat{\lambda}(r)| \, dr &= \int_s^t (e^{\theta r} - 1)|d(\widehat{L})(r)| \\
&= \int_s^t (e^{\theta r} - 1)J(r) \, d(\widehat{L})(r) \\
&= - \int_s^t (e^{\theta r} - 1)J(r)\widehat{f}(r) \, dr \\
&\quad + \int_s^t (e^{\theta r} - 1)J(r)\langle \widehat{V}(r), dB(r) \rangle \\
&\quad - \int_s^t (e^{\theta r} - 1)J(r) \, d\widehat{M}(r) \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.9}$$

As $|J(r)| \leq 1$ it is clear that

$$E(I_1) \leq E \int_s^t (e^{\theta r} - 1)|\widehat{f}(r)| \, dr, \tag{3.10}$$

and I_2 being an Ito integral

$$E(I_2) = 0. \tag{3.11}$$

Let $t \mapsto \ell(t, a)$ denote the local time of the continuous semimartingale \widehat{M} at $a \in \mathbb{R}$. By the version of Ito–Tanaka–Meyer formula given in Exercise 1.25, Chapter VI (p. 219) of [13] we get

$$\begin{aligned}
d((e^{\theta r} - 1)|\widehat{M}|(r)) &= |\widehat{M}|(r)\theta e^{\theta r} \, dr + (e^{\theta r} - 1)d|\widehat{M}|(r) \\
&= |\widehat{M}|(r)\theta e^{\theta r} \, dr + \frac{(e^{\theta r} - 1)}{2}[d\ell(r, 0) + d\ell(r, 0-)] \\
&\quad + (e^{\theta r} - 1)[\text{sgn}(\widehat{M})(r) \, d(\widehat{M})(r)]
\end{aligned}$$

and consequently

$$\begin{aligned}
&\int_s^t (e^{\theta r} - 1)\text{sgn}(\widehat{M}(r)) \, d(\widehat{M})(r) \\
&= (e^{\theta t} - 1)|\widehat{M}|(t) - (e^{\theta s} - 1)|\widehat{M}|(s) - \int_s^t e^{\theta r}\theta|\widehat{M}|(r) \, dr \\
&\quad - \frac{1}{2} \left[\int_s^t (e^{\theta r} - 1) \, d\ell(r, 0) + \int_s^t (e^{\theta r} - 1) \, d\ell(r, 0-) \right].
\end{aligned} \tag{3.12}$$

By Theorem 1.7, Chapter VI of [13]

$$\begin{aligned}
&\left| \int_s^t -(e^{\theta r} - 1)\chi(r)I_{(0)}(\widehat{M})(r) \, d(\widehat{M})(r) \right| \\
&\leq \frac{1}{2} \int_s^t (e^{\theta r} - 1)|d(\ell(\cdot, 0) - \ell(\cdot, 0-))|(r) \\
&\leq \frac{1}{2} \left[\int_s^t (e^{\theta r} - 1) \, d\ell(r, 0) + \int_s^t (e^{\theta r} - 1) \, d\ell(r, 0-) \right].
\end{aligned} \tag{3.13}$$

By (3.12), (3.13)

$$I_3 \leq (e^{\theta t} - 1)|\widehat{M}(t) - (e^{\theta s} - 1)|\widehat{M}(s) - \int_s^t e^{\theta r} \theta |\widehat{M}(r) \, dr \tag{3.14}$$

Taking expectation in (3.9) and (3.14), and using (3.10), (3.11) the required estimate (3.7) is now immediate. ■

4. Existence and uniqueness

We make the following assumptions on the coefficients b, R .

(A1): For $1 \leq i \leq d, z \mapsto b_i(\omega, t, z)$ is Lipschitz continuous, uniformly over (ω, t) ; there is a constant β_i such that $|b_i(\omega, t, z)| \leq \beta_i$ for all ω, t, z . Denote $\beta = (\beta_1, \dots, \beta_d)$.

(A2): For $1 \leq i, j \leq d, z \mapsto r_{ij}(\omega, t, z)$ is Lipschitz continuous, uniformly over (ω, t) . Also $r_{ii} \equiv 1$ for all i .

(A3): For $i \neq j$ there exists constant v_{ij} such that $|r_{ij}(\omega, t, z)| \leq v_{ij}$. Set $V = (v_{ij})$ with $v_{ii} = 0$; we assume that $\sigma(V) < 1$ where $\sigma(V)$ denotes the spectral radius of V .

If $\sigma(V) < 1$ observe that

$$(I - V)^{-1} = I + V + V^2 + V^3 + \dots$$

is a matrix with nonnegative entries; here I is the $(d \times d)$ identity matrix.

We first establish an a priori estimate.

PROPOSITION 4.1

Assume (A1)–(A3) and let ξ be a bounded \mathcal{F}_T -measurable \overline{G} -valued random variable. Suppose $\{Y(t)\}, \{Z(t)\}, 0 \leq t \leq T$ solve RBSDE (ξ, b, R) . Then

$$0 \leq dY(t) \leq (I - V)^{-1} \beta \, dt \tag{4.1}$$

in the sense that

$$0 \leq dY_i(t) \leq ((I - V)^{-1} \beta)_i \, dt, 1 \leq i \leq d. \tag{4.2}$$

In particular $dY_i(\cdot)$ is absolutely continuous, and hence the local time at 0 of $Z_i(\cdot)$ is also absolutely continuous for each $i = 1, \dots, d$.

Proof. For each fixed $i = 1, \dots, d$ note that $\{L(t) = Y_i(t)\}, \{M(t) = Z_i(t)\}, 0 \leq t \leq T$ is a solution to the auxiliary one-dimensional problem corresponding to $\zeta = \xi_i, f(\omega, s, z) = b_i(\omega, s, \overline{Z}_{i,z}(s, \omega)), g_j(\omega, s, z) = r_{ij}(\omega, s, \overline{Z}_{i,z}(s, \omega)), dA_j(s) = dY_j(s), j \neq i, 1 \leq i, j \leq d$ where $\overline{Z}_{i,z} = (Z_1, \dots, Z_{i-1}, z, Z_{i+1}, \dots, Z_d)$.

By Lemma 3.1 and our hypotheses

$$\begin{aligned} 0 \leq dY_i(t) &\leq I_{\{0\}}(Z_i(t)) \left\{ |b_i(t, Z(t))| \, dt + \sum_{j \neq i} |r_{ij}(t, Z(t))| \, dY_j(t) \right\} \\ &\leq \beta_i \, dt + \sum_{j \neq i} v_{ij} \, dY_j(t). \end{aligned}$$

Consequently

$$dY_i(t) - \sum_{j \neq i} v_{ij} dY_j(t) \leq \beta_i dt, 1 \leq i \leq d.$$

The above can be expressed as

$$((I - V) dY)_i(t) \leq (\beta)_i dt, 1 \leq i \leq d. \tag{4.3}$$

As $\sigma(V) < 1$ we can get (4.1), (4.2) from (4.3). The last assertion is now a consequence of Lemma 3.1. ■

Remark 4.2. As $\sigma(V^\dagger) = \sigma(V)$, where V^\dagger denotes transpose of V , by (A3) it follows that there are constants $a_j > 0, 1 \leq j \leq d$ and $0 < \alpha < 1$ such that

$$\sum_{i \neq j} a_i |r_{ij}(\omega, t, z)| \leq \sum_{i \neq j} a_i v_{ij} \leq \alpha a_j \tag{4.4}$$

for all $j = 1, \dots, d, \omega \in \Omega, 0 \leq t \leq T, z \in \mathbb{R}^d$; see, for example, Dupuis and Ishii [3] for a proof. □

Let $\theta > 0$ be a constant. Let \mathcal{H} denote the space of all (equivalence classes of) $\{\mathcal{F}_t\}$ -progressively measurable processes $\{Y(t) := (Y_1(t), \dots, Y_d(t))\}, \{Z(t) := (Z_1(t), \dots, Z_d(t))\}, 0 \leq t \leq T$ such that

- (i) $Z_i(t) \geq 0, 0 \leq t \leq T, 1 \leq i \leq d,$
- (ii) $Y_i(0) = 0, Y_i(\cdot)$ is nondecreasing, $1 \leq i \leq d,$
- (iii) $E \sum_{i=1}^d \int_0^T e^{\theta t} a_i |Z_i(t)| dt < \infty$
- (iv) $E \sum_{i=1}^d \int_0^T e^{\theta t} a_i \varphi_t(Y_i) dt < \infty$

where $\varphi_t(g)$ denotes the total variation of g over $[t, T]$. The constant $\theta > 0$ will be chosen suitably later; the constants a_i are as in Remark 4.2.

For $(Y, Z), (\widehat{Y}, \widehat{Z}) \in \mathcal{H}$ define the metric

$$\begin{aligned} d((Y, Z), (\widehat{Y}, \widehat{Z})) &= E \sum_{i=1}^d \int_0^T e^{\theta t} a_i |Z_i(t) - \widehat{Z}_i(t)| dt \\ &\quad + E \sum_{i=1}^d \int_0^T e^{\theta t} a_i \varphi_t(Y_i - \widehat{Y}_i) dt. \end{aligned} \tag{4.5}$$

Note that (\mathcal{H}, d) is a complete metric space.

Let $\widetilde{\mathcal{H}}$ denote the collection of all $(Y, Z) \in \mathcal{H}$ such that there is an $\{\mathcal{F}_t\}$ -progressively measurable process $D(t) = (D_1(t), \dots, D_d(t)), 0 \leq t \leq T$ with $0 \leq D_i(t) \leq ((I - V)^{-1} \beta)_i$ a.s. and $Y_i(t) = \int_0^t D_i(s) ds, 0 \leq t \leq T.$

Observe that $\widetilde{\mathcal{H}}$ is a closed subspace of \mathcal{H} and hence $(\widetilde{\mathcal{H}}, d)$ is a complete metric space.