

# GAUGE INVARIANCE AND DUALITY IN THE NONCOMMUTATIVE PLANE

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## **Abstract:**

We show that the duality between the self-dual and Maxwell-Chern-Simons theories in 2+1-dimensions survives when the space-time becomes noncommutative. Existence of the Seiberg-Witten map is crucial in the present analysis. It should be noted that the above models, being manifestly gauge variant and invariant respectively, transform differently under the Seiberg-Witten map. We also discuss this duality in the Stueckelberg formalism where the self-dual model is elevated to a gauge theory. The "‘master’" lagrangian approach has been followed throughout.

Key Words: Noncommutative gauge theory, Seiberg-Witten map, Self-dual model, Maxwell-Chern-Simons model.

# 1 Introduction

Lower dimensional gauge theories have yielded unexpected subtleties in their structure, be it in the form of massive propagating modes in the 1+1-dimensional Schwinger model [1], topologically massive gauge theories in 2+1-dimensions [2] or a "hidden" gauge invariance in Self-Dual (SD) models in 2+1-dimensions [2, 3]. Once again, as in the case of 1+1-dimensions, these bosonic models are connected to fermionic models in the large fermion mass limit [4]. Indeed, the above interrelations are interesting from the point of view of providing unifications among diverse models [5] as well as exploiting the equivalences in explicit computations.

Let us elaborate a little on the SD models in 2+1-dimensions as we will be concerned with them in the present work. Only abelian gauge theories will be considered here. The local lagrangian formulation of them were first provided in [6]. However, their equivalence with the well studied Maxwell-Chern-Simons (MCS) topological gauge theory [2] was revealed in [3]. Intuitively, the duality was expected since both SD and MCS models allow the propagation of a (parity violating) massive mode. But the above correspondence was clearly established by Deser and Jackiw [3] by the construction of a "master" lagrangian from which both SD and MCS models can be generated. The manifest gauge invariance of SD and the parent "master" model is *hidden* in the SD model, where the above symmetry is not manifest.

In recent years, Non-Commutative (NC) field theories [7] and in particular NC Gauge Theories (NCGT) [8] have generated a lot of interest due to their appearance in open string and D-brane physics. This space-time noncommutativity modifies the previous results in a non-trivial way and can lead to physically interesting bounds on the NC parameter  $\theta^{\rho\sigma}$ . For this reason a lot of effort is being given to re-derive existing results in NC space-time.

In this paper, we are going to assess the effect of noncommutativity in the above mentioned SD-MCS duality. This is all the more pertinent since gauge invariance plays a major role here and the noncommutativity affects gauge invariant and non-invariant theories in strikingly different ways [8]. The work of Seiberg and Witten [8] showed that there exists a mapping, the Seiberg-Witten Map (SWM), that lifts a gauge theory to its NC counterpart such that gauge orbits are mapped into NC gauge orbits. The SWM is unique in the lowest nontrivial order of  $\theta^{\rho\sigma}$ . On the other hand, in a non-gauge theory, noncommutativity affects only products of fields (in the action), without changing the individual field structures. This shows that the SD and MCS theories will be generalized to NC field theories in distinct ways due to their *manifest* gauge variance and invariance respectively. We will explicitly demonstrate by way of constructing the "master" lagrangian that in spite of the above complications, *the SD-MCS duality is maintained in the NC space-time*. Incidentally, the NCGT with abelian gauge group is structurally quite akin to a non-abelian gauge theory in conventional space-time. Hence our analysis might shed some light on the SD-MCS duality in conventional space-time with a non-abelian gauge group [3]. This constitutes the first part of our work.

In the second part of our work, we show that there is another way of comparing noncommutative SD and MCS theories where the conventional SD theory is first embedded in a gauge invariant theory via the Stueckelberg formalism. Thus the starting SD and MCS theories, both being manifestly gauge invariant, will be altered in similar ways under NC generalizations. We again establish the SD-MCS duality through the "master" lagrangian.

## 2 Self-dual and Maxwell-Chern-Simons Duality in Non-commutative Space-time

The NC space-time is characterized by,

$$[x^\rho, x^\sigma]_* = i\theta^{\rho\sigma}. \quad (1)$$

The  $*$ -product is given by the Moyal-Weyl formula,

$$p(x) * q(x) = pq + \frac{i}{2}\theta^{\rho\sigma}\partial_\rho p\partial_\sigma q + O(\theta^2). \quad (2)$$

All our discussions will be valid up to the first non-trivial order in  $\theta$ . The (inverse) SWM [8] to  $O(\theta)$  is,

$$\begin{aligned} A_\mu &= \hat{A}_\mu - \theta^{\sigma\rho}\hat{A}_\rho(\partial_\sigma\hat{A}_\mu - \frac{1}{2}\partial_\mu\hat{A}_\sigma) \equiv \hat{A}_\mu + \hat{a}_\mu(\hat{A}_\nu, \theta) \\ \lambda &= \hat{\lambda} + \frac{1}{2}\theta^{\rho\sigma}\hat{A}_\rho\partial_\sigma\hat{\lambda}, \end{aligned} \quad (3)$$

where the ‘‘hatted’’ variables on the right live in NC space-time. The significance of the above map is that under an NC or  $*$ -gauge transformation of  $\hat{A}_\mu$  by,

$$\hat{\delta}\hat{A}_\mu = \partial_\mu\hat{\lambda} + i[\hat{\lambda}, \hat{A}_\mu]_*,$$

$A_\mu$  will undergo the transformation

$$\delta A_\mu = \partial_\mu\lambda$$

given in (6). Subsequently, under this mapping, a gauge invariant action in conventional space-time will be mapped to its NC counterpart, which will be  $*$ -gauge invariant.

Let us start by introducing the SD-MCS duality in conventional space-time. For convenience we follow the notations and metric ( $g^{\mu\nu} = \text{diag}(1, -1, -1)$ ) of [3]. The self-dual lagrangian, consisting of the ordinary and topological mass terms is [6],

$$\mathcal{L}_{SD} = \frac{1}{2}f^\mu f_\mu - \frac{1}{2m}\epsilon^{\alpha\beta\gamma}f_\alpha\partial_\beta f_\gamma. \quad (4)$$

On the other hand, the MCS model [2, 3] is described by,

$$\mathcal{L}_{MCS} = -\frac{1}{2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\partial^\alpha A^\beta + \frac{m}{2}\epsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma. \quad (5)$$

Note that total derivative terms in the lagrangian will be dropped throughout the present (classical) discussion. Clearly (5) is invariant under the gauge transformation,

$$A_\mu \rightarrow A_\mu + \partial_\mu\lambda, \quad (6)$$

whereas no such manifest symmetry exists for (4). Indeed, one can solve the equations of motion and constraints for both of the above models and show that there exists the identification [3],

$$f_\mu = \epsilon_{\mu\nu\tau}\partial^\nu A^\tau.$$

An alternative way is the construction of the following "master" lagrangian [3]

$$\mathcal{L}_{DJ} = \frac{1}{2} f^\mu f_\mu - \epsilon^{\alpha\beta\gamma} f_\alpha \partial_\beta A_\gamma + \frac{m}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma, \quad (7)$$

which reduces to (4) or (5) upon integrating out  $A_\mu$  or  $f_\mu$  respectively, by exploiting the equations of motion. Note that (7) is also gauge invariant.

The mapping (3) together with  $f_\mu = \hat{f}_\mu$  lifts the "master" lagrangian in (7) to NC "master" lagrangian,

$$\begin{aligned} \hat{\mathcal{L}}_{DJ} &= \frac{1}{2} \hat{f}^\mu \hat{f}_\mu - \epsilon^{\alpha\beta\gamma} \hat{f}_\alpha \partial_\beta (\hat{A}_\gamma + \hat{a}_\gamma) + \frac{m}{2} \epsilon^{\alpha\beta\gamma} (\hat{A}_\alpha + \hat{a}_\alpha) \partial_\beta (\hat{A}_\gamma + \hat{a}_\gamma) \\ &= \frac{1}{2} \hat{f}^\mu \hat{f}_\mu - \epsilon^{\alpha\beta\gamma} \hat{f}_\alpha \partial_\beta (\hat{A}_\gamma + \hat{a}_\gamma) + \frac{m}{2} \epsilon^{\alpha\beta\gamma} (\hat{A}_\alpha \partial_\beta \hat{A}_\gamma + 2\hat{a}_\alpha \partial_\beta \hat{A}_\gamma) + O(\theta^2). \end{aligned} \quad (8)$$

Let us now follow the computational scheme, (of selective integration of fields), used in the conventional space-time. Variation of  $\hat{f}_\mu$  yields the equation,

$$\hat{f}_\mu = \epsilon_{\mu\nu\tau} \partial^\nu (\hat{A}^\tau + \hat{a}^\tau). \quad (9)$$

Substituting  $\hat{f}_\mu$  from (9) in (8), we obtain,

$$\hat{\mathcal{L}}_{DJ} = -\frac{1}{2} (\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu) \partial^\mu \hat{A}^\nu - (\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu) \partial^\mu \hat{a}^\nu + \frac{m}{2} \epsilon_{\mu\alpha\beta} (\hat{A}^\mu \partial^\alpha \hat{A}^\beta + 2\hat{A}^\mu \partial^\alpha \hat{a}^\beta). \quad (10)$$

To  $O(\theta)$ , this is nothing but  $\hat{\mathcal{L}}_{MCS}$  where

$$\hat{\mathcal{L}}_{MCS} = -\frac{1}{2} [\partial_\alpha (\hat{A}_\beta + \hat{a}_\beta) - \partial_\beta (\hat{A}_\alpha + \hat{a}_\alpha)] [\partial^\alpha (\hat{A}^\beta + \hat{a}^\beta)] + \frac{m}{2} \epsilon^{\alpha\beta\gamma} (\hat{A}_\alpha + \hat{a}_\alpha) \partial_\beta (\hat{A}_\gamma + \hat{a}_\gamma). \quad (11)$$

In a similar manner, variation of  $\hat{A}_\mu$  reproduces the equation of motion,

$$m \epsilon_{\mu\alpha\beta} \partial^\alpha \hat{A}^\beta \approx \epsilon_{\mu\alpha\beta} \partial^\alpha (\hat{f}^\beta - m \hat{a}^\beta) - \epsilon_{\nu\alpha\beta} (m \partial^\alpha \hat{A}^\beta - \partial^\alpha \hat{f}^\beta) \frac{\delta \hat{a}^\nu}{\delta \hat{A}^\mu}. \quad (12)$$

Notice that the last term in the right hand side of (12) is actually of  $O(\theta^2)$  and hence can be omitted. Once again putting back the above relation (12) in (8) we recover  $\hat{\mathcal{L}}_{SD}$ ,

$$\hat{\mathcal{L}}_{SD} = \frac{1}{2} \hat{f}^\mu \hat{f}_\mu - \frac{1}{2m} \epsilon^{\alpha\beta\gamma} \hat{f}_\alpha \partial_\beta \hat{f}_\gamma. \quad (13)$$

This constitutes the demonstration that (to  $O(\theta)$ ) the SD-MCS duality remains intact in noncommutative space-time even though the mappings of the SD and MCS models to their respective noncommutative counterparts are entirely different. It appears that, to the first non-trivial order in  $\theta$  (that we are presently interested in), existence of the perturbative form of the SWM [8] is sufficient to guarantee the SD-MCS duality in noncommutative space-time, irrespective of the explicit structure of the map in (3). We will return to this point briefly at the end.

### 3 Duality in the Stuckelberg Formalism

For various reasons it sometimes turns out to be convenient to embed the particular model under study in a larger phase space in which the extended model has a gauge invariance. The original model and its gauge invariant extension are obviously gauge equivalent so that the physical (*i.e.* gauge invariant) sector of the original model remains intact. Stuckelberg formalism is one of the simplest such prescription where the gauge variant SD model  $\mathcal{L}_{SD}$  in (4) can be extended to the manifestly gauge invariant  $\mathcal{L}_{SD}^S$ ,

$$\mathcal{L}_{SD}^S = \frac{1}{2}(f^\mu - \partial^\mu \phi)(f_\mu - \partial_\mu \phi) - \frac{1}{2m} \epsilon^{\alpha\beta\gamma} f_\alpha \partial_\beta f_\gamma. \quad (14)$$

Here  $\phi$  is an auxiliary degree of freedom, which is introduced to induce the invariance of  $\mathcal{L}_{SD}^S$  under the gauge transformation,

$$f_\mu \rightarrow f_\mu + \partial_\mu \sigma, \quad \phi \rightarrow \phi + \sigma. \quad (15)$$

Obviously  $\phi$  is a gauge degree of freedom and the gauge choice  $\phi = 0$  reproduces  $\mathcal{L}_{SD}$ .

The "master" lagrangian  $\mathcal{L}_{DJ}^S$  corresponding to the pair,  $\mathcal{L}_{SD}^S$  of (14) and the earlier MCS model (5), (which remains unchanged), is given by the following extended version of  $\mathcal{L}_{DJ}$ ,

$$\mathcal{L}_{DJ}^S = \frac{1}{2}(f^\mu - \partial^\mu \phi)(f_\mu - \partial_\mu \phi) - \epsilon^{\alpha\beta\gamma} f_\alpha \partial_\beta A_\gamma + \frac{m}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma. \quad (16)$$

The above model (16) actually has two independent gauge invariances,

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda; \quad f_\mu \rightarrow f_\mu + \partial_\mu \sigma, \quad \phi \rightarrow \phi + \sigma. \quad (17)$$

Now we consider the SWM appropriate for the present gauge theory (16) having two gauge invariances. Thus, the SWM of (3) has to be augmented by the following map,

$$\begin{aligned} f_\mu &= \hat{f}_\mu - \theta^{\alpha\rho} \hat{f}_\rho (\partial_\alpha \hat{f}_\mu - \frac{1}{2} \partial_\mu \hat{f}_\alpha) \equiv \hat{f}_\mu + \hat{h}_\mu(\hat{f}_\nu, \theta) \\ \sigma &= \hat{\sigma} + \frac{1}{2} \theta^{\alpha\rho} \hat{f}_\rho \partial_\alpha \hat{\sigma} \equiv \hat{\sigma} + \hat{\sigma}^1, \end{aligned} \quad (18)$$

The full SWM leads to the NC counterpart of  $\mathcal{L}_{DJ}^S$ ,

$$\begin{aligned} \hat{\mathcal{L}}_{DJ}^S &= \frac{1}{2}(\hat{f}^\mu + \hat{h}^\mu - \partial^\mu \hat{\phi})(\hat{f}_\mu + \hat{h}_\mu - \partial_\mu \hat{\phi}) - \epsilon^{\alpha\beta\gamma} (\hat{f}_\alpha + \hat{h}_\alpha) \partial_\beta (\hat{A}_\gamma + \hat{a}_\gamma) \\ &\quad + \frac{m}{2} \epsilon^{\alpha\beta\gamma} (\hat{A}_\alpha + \hat{a}_\alpha) \partial_\beta (\hat{A}_\gamma + \hat{a}_\gamma) \\ &= \frac{1}{2}(\hat{f}^\mu - \partial^\mu \hat{\phi})(\hat{f}_\mu - \partial_\mu \hat{\phi}) + (\hat{f}^\mu - \partial^\mu \hat{\phi}) \hat{h}_\mu - \epsilon^{\alpha\beta\gamma} (\hat{f}_\alpha \partial_\beta \hat{A}_\gamma + \hat{f}_\alpha \partial_\beta \hat{a}_\gamma + \hat{h}_\alpha \partial_\beta \hat{A}_\gamma) \\ &\quad + \frac{m}{2} \epsilon^{\alpha\beta\gamma} (\hat{A}_\alpha \partial_\beta \hat{A}_\gamma + 2\hat{a}_\alpha \partial_\beta \hat{A}_\gamma) + O(\theta^2). \end{aligned} \quad (19)$$

Two points regarding this model are to be noted. Firstly, although there are two independent gauge invariances with distinct gauge parameters  $\lambda$  and  $\sigma$ , this feature does not show up in  $\hat{\mathcal{L}}_{DJ}^S$ , since the SWM for the gauge field is independent of the gauge transformation function.

Secondly, from the basic tenet of the SWM, that for abelian gauge group, gauge invariant sectors in ordinary spacetime under SWM will be translated to \*-gauge invariant sectors in NC spacetime, we conclude that the combination  $\hat{f}^\mu + \hat{h}^\mu - \partial^\mu \hat{\phi}$  has to be \*-gauge invariant, since  $f^\mu - \partial^\mu \phi$  is gauge invariant. This shows that under \*-gauge transformation,

$$\delta \hat{\phi} = \hat{\sigma} - \frac{1}{2} \theta^{\alpha\beta} \hat{f}_\beta \partial_\alpha \hat{\sigma}. \quad (20)$$

Now, we follow the formal way [9] of extracting the SWM for  $\phi$  and use the following expansions,

$$\hat{\phi} = \phi + \varphi(\theta) + O(\theta^2) \quad ; \quad \hat{f}^\mu = f^\mu + h^\mu(\theta) + O(\theta^2) \quad ; \quad \hat{\sigma} = \sigma + \sigma_1(\theta) + O(\theta^2), \quad (21)$$

in (20).  $\hat{h}^\mu$  and  $\sigma_1$  are defined in (18). Comparing powers of  $\theta$ , we get

$$\delta\phi = \sigma \quad ; \quad \delta\varphi = 0. \quad (22)$$

But such a gauge invariant structure for  $\varphi$  is impossible to construct indicating that  $\varphi = 0$  and so  $\hat{\phi} = \phi + O(\theta^2)$ .

Proceeding in exactly the same way as before, the variational equations of motion are obtained as,

$$\hat{f}_\mu - \partial_\mu \hat{\phi} = \epsilon_{\mu\alpha\beta} \partial^\alpha \hat{A}^\beta + \epsilon_{\mu\alpha\beta} \partial^\alpha \hat{a}^\beta - \hat{h}_\mu + (\epsilon_{\nu\alpha\beta} \partial^\alpha \hat{A}^\beta - \hat{f}^\nu + \partial^\nu \hat{\phi}) \frac{\delta \hat{h}_\nu}{\delta \hat{f}^\mu} + O(\theta^2), \quad (23)$$

$$m\epsilon_{\mu\alpha\beta} \partial^\alpha \hat{A}^\beta = \epsilon_{\mu\alpha\beta} \partial^\alpha (\hat{f}^\beta - m\hat{a}^\beta + \hat{h}^\beta) - \epsilon_{\nu\alpha\beta} (m\partial^\alpha \hat{A}^\beta - \partial^\alpha \hat{f}^\beta) \frac{\delta \hat{a}^\nu}{\delta \hat{A}^\mu} + O(\theta^2). \quad (24)$$

Note that in both the above equations (23) and (24), the last terms in the right hand side containing variational derivatives are actually of  $O(\theta)^2$  and hence can be left out in the present analysis. Substitution of (23) in (19) reproduces (11), the noncommutative version of the MCS model. On the other hand, using (24) in (19) we get,  $\hat{\mathcal{L}}_{SD}^S$ , the noncommutative analogue of  $\mathcal{L}_{SD}^S$  in (14),

$$\hat{\mathcal{L}}_{SD}^S = \frac{1}{2} (\hat{f}^\mu + \hat{h}^\mu - \partial^\mu \hat{\phi}) (\hat{f}_\mu + \hat{h}_\mu - \partial_\mu \hat{\phi}) - \frac{1}{2m} \epsilon^{\alpha\beta\gamma} (\hat{f}_\alpha \partial_\beta \hat{f}_\gamma + 2\hat{h}_\alpha \partial_\beta \hat{f}_\gamma). \quad (25)$$

This exercise shows that the DS-MCS duality remains intact in NC space-time, when the SD model is also treated as a gauge theory in the Stuckelberg formalism.

It should be mentioned that from the variational nature of the problem, it might seem that under any change of the field variable to the lowest non-trivial order, the equivalence or duality relations of this sort will remain intact. However, in general this may not be true. The explicit nature of the "master" lagrangian as well as the induced lagrangians, and the fact that the Seiberg-Witten map does not distinguish between different gauge invariances, (as far as the field variables are concerned), all contribute to the above derivations.

It would be very interesting to establish the equivalence between self dual and Maxwell-Chern-Simons theories for arbitrary orders in  $\theta$ , since that will bring into play the explicit forms of the SWM. However, that has not been included in the present work mainly because (I):  $O(\theta)$  computations and results have a very important role in NC theories, and (II): there are ambiguities [10] in the SWM for higher orders in  $\theta$ . On the other hand, the above duality

may be exploited to favour one expression of the SWM over the other alternatives for say  $O(\theta^2)$ . These issues will be addressed separately.

**Note added:**

It is straightforward to show that the duality between Maxwell-Chern-Simons and self dual theories, in the noncommutative plane, holds to *all* orders in  $\theta$ . We start with the full equation corresponding to (12),

$$\epsilon^{\sigma\beta\gamma}\partial_\beta(\hat{f} - m\hat{A})_\gamma(g_\sigma^\alpha + \frac{\delta\hat{c}_\sigma}{\delta\hat{A}_\alpha}) = m\epsilon^{\sigma\beta\gamma}\partial_\beta\hat{c}_\gamma(g_\sigma^\alpha + \frac{\delta\hat{c}_\sigma}{\delta\hat{A}_\alpha}), \quad (26)$$

where  $\hat{c}^\mu$  denotes the full  $\theta$  correction to  $A^\mu$ . Since the operator  $(g_\sigma^\alpha + \frac{\delta\hat{c}_\sigma}{\delta\hat{A}_\alpha})$  is invertible, one gets the exact relation

$$\epsilon^{\sigma\beta\gamma}\partial_\beta(\hat{f} - m\hat{A})_\gamma = m\epsilon^{\sigma\beta\gamma}\partial_\beta\hat{c}_\gamma. \quad (27)$$

Substituting this in the NC master Lagrangian (8) (with  $\hat{a}^\mu$  replaced by  $\hat{c}^\mu$ ), reproduces the self dual model.

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