BOSONIZATION IN THE NONCOMMUTATIVE PLANE

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Abstract:
In this Note, we study bosonization of the noncommutative massive Thirring model in $2 + 1$- dimensions. We show that, contrary to the duality between massive Thirring model and Maxwell-Chern-Simons model in ordinary spacetime, in the low energy (or large fermion mass) limit, their noncommutative versions are not equivalent, in the same approximation.

Keywords: $2 + 1$-Dimensional bosonization, Noncommutative field theory.

The string inspired Non-Commutative (NC) spacetime [1] and the subsequent noncommutative field theories living in the D-branes [2] have remodelled a number of established ideas of quantum field theories in ordinary spacetime. From a string theory perspective, NC field theories (and NC gauge theories in particular) yield an effective theory for strings in the presence of a large background B-field. However, the advantage of the NC field theory formalism is that it deviates very little from the field theory in ordinary spacetime, as far as computational techniques are concerned. Hence, working in NC field theory, some string theoretic results, albeit in certain limits, are recovered by conventional quantum field theoretic computations. Generically it will be much harder to obtain analogous results from a string theoretic analysis.

Quite apart from the above mentioned string theory connection, NC quantum field theories are being intensly studied because of some surprising consequences of noncommutativity, such as the ultraviolet-infrared mixing [3], dipolar behaviour of the excitations in electromagnetic interaction [4], a specific type of non-locality [5], etc. to name a few. Another interesting observation is that the (noncommutativity parameter) $\theta \to 0$ limit is not always smooth [6], that is results in NC spacetime do *not* always reduce to ordinary spacetime results for $\theta \to 0$. This might seem unexpected since the dynamical variables of NC and ordinary spacetimes are related explicitly through the Seiberg-Witen Map [1], through a perturbative expansion in θ. Actually the singularity in θ appears in the quantum theory when the $\theta \to 0$ limit and regularization prescription limits do not commute. We will come to the last point later.

A powerful tool in the conventional quantum field theory is the concept of duality (or equivalence) between apparantly dissimilar theories. Apart from the esthetic satisfaction of unifying various theories under one idea, a tangible outcome of duality is that the dual models can represent a physical phenomenon in different limiting domains, such as strong and weak coupling limit etc. It is quite natural to question the fate of a particular duality when the spacetime becomes noncommutative. In the present Note, we will discuss one such duality, *i.e.* Bosonization or the fermion-boson duality in $2+1$-dimensional NC spacetime.

Bosonization in $1+1$-dimensions dates back to Coleman [7] who showed that that the Sine-Gordon model of a scalar field is dual to the massive Thirring model of self interacting fermions. Subsequently the explicit operator realization of the fermion-boson mapping was provided by Mandelstam [8]. An interesting and useful feature of bosonization is that quantum effects corresponding to the fermionic theory get included in the bosonized effective action, which can be studied classically.

However, generalization of bosonization to higher dimensions is not as complete as in $1+1$-dimensions, (where some simplifications occure due to the topology of the single spatial coordinate). In $2+1$-dimensions, following the ideas in [9], Deser and Redlich [10] first studied the equivalence between effective electromagnetic interaction of the CP^1 model and a charged massive fermion in powers of inverse fermion mass. Bosonization of the massive Thirring model in the long wavelength regime - the case relevant to us - was considered later by Fradkin and Schaposnik and by Banerjee [11]. The Thirring model, in the lowest non-trivial order in inverse fermion mass, becomes equivalent to the topologically massive $U(1)$ gauge theory, the Maxwell-Chern-Simons theory. The latter model is of interest [1] in its own right and has been studied exhaustively in [13, 14]. It possesses a single, parity violating, massive, spin one excitation. In the above instances, the parity violation in the bosonic theory, in the form of the Chern-Simons

[1] The vector and tensor gauge theories in $2+1$-dimensions are related to high temperature behaviour of four-dimensional models [12].

term, comes from the parity violating fermion mass term. Notice that the model enjoys gauge invariance even though the gauge boson is massive, the reason being the topological generation of the mass.

This motivates us to the present work - the study of bosonization of the massive Thirring model, in $2+1$-dimensional NC spacetime. [2] In ordinary spacetime, the Massive Thirring (MT) model - a massive fermionic theory with four fermion current-current (Thirring) self interaction - is equivalent to the Self-Dual (SD) model - a massive bosonic theory with Chern-Simons term as the kinetic term - in the lowest non-trivial order of the inverse fermion mass [11]. The large fermion mass approximation is equivalent to the low energy or long wave length limit of the massive Thirring model, where the fermi-bose transmutation is valid. The Thirring coupling constant gets related to the inverse of the gauge boson mass. Furthermore, due to the equivalence between the SD and Maxwell-Chern-Simons (MCS) models [14, 13], the MT model (with no manifest local gauge invariance) becomes equivalent to the (manifestly gauge invariant) MCS model. Our aim is to study, (a): whether bosonization of the Non-Commutative MT (NCMT) theory along the lines of ordinary spacetime is possible and (b): if it is so, whether the NCMT-NCMCS duality is preserved.

Our results consist of a good news and a bad news. The good news is that, (in analogy with ordinary spacetime [11]), *the NCMT model can be bosonized in powers of the inverse fermion mass*. The bad news is that, (contrary to the ordinary spacetime [11]), *the duality between the NCMT and NCMCS models is lost, even in the large fermion mass limit*. The reason is the following. Recently it was shown by us [17] that the NCSD-NCMCS duality survives. However, here we show that bosonization of the NCMT theory induces a theory which differs from the NCSD theory studied previously [17]. Hence the NCMT-NCMCS duality is lost. These constitute the main results of this paper, schematically summarized below:

$$Ordinary\ spacetime \qquad MT\ (fermion) \approx SD\ (boson) \approx MCS\ (boson)$$

$$\Rightarrow MT\ (fermion) \approx MCS\ (boson),$$

$$NC\ spacetime \quad NCMT\ (fermion) \approx Bosonic\ model \neq NCSD\ (boson) \approx NCMCS\ (boson)$$

$$\Rightarrow NCMT\ (fermion) \neq NCMCS\ (boson).$$

After putting our work in its proper perspective, we now move on to explicit computations. The spacetime is characterized by a noncommutativity of the form,

$$[x^\rho, x^\sigma]_* = i\theta^{\rho\sigma}, \tag{1}$$

where the ordinary product is replaced by the Moyal-Weyl or $*$ product,

$$\hat{p}(x) * \hat{q}(x) = e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\sigma_\mu}\partial_{\xi_\nu}} \hat{p}(x+\sigma)\hat{q}(x+\xi)\mid_{\sigma=\xi=0} = \hat{p}(x)\hat{q}(x) + \frac{i}{2}\theta^{\rho\sigma}\partial_\rho \hat{p}(x)\partial_\sigma \hat{q}(x) + O(\theta^2). \tag{2}$$

The *hatted* variables live in NC spacetime. Generally $\theta^{\rho\sigma}$ is taken to be a constant tensor, but this need not always be the case [16]. In this paper we will focus on $2+1$-dimensional NC spacetime.

[2] The analogue [15] of our work in $1+1$-dimensions reveals that the strong-weak coupling duality, similar to the ordinary spacetime result [7, 8], remains intact. The noncommutativity induces a Wess-Zumino-Witten term in the effective $U(1)$ theory.

Let us discuss NC bosonization first. The NCMT model is,

$$\hat{S}_{Th} = \int d^3x\ [\bar{\hat{\psi}}(x) * (i\gamma^\mu \partial_\mu + m)\hat{\psi}(x) - \frac{g^2}{2}\hat{j}^\mu(x) * \hat{j}_\mu(x)]$$

$$= \int d^3x\ [\bar{\hat{\psi}}(x)(i\gamma^\mu \partial_\mu + m)\hat{\psi}(x) - \frac{g^2}{2}\hat{j}^\mu(x)\hat{j}_\mu(x)], \quad (3)$$

where the fermion current $\hat{j}_\mu$is defined as

$$\hat{j}^\mu(x) = \bar{\hat{\psi}}(x) * \gamma^\mu \hat{\psi}(x). \quad (4)$$

The second equality in (3) follows from the property of $*$-product under integral. The next step is to compute the effective action by integrating out the fermions. We consider an alternative action,

$$\hat{S} = \int d^3x\ [\bar{\hat{\psi}}(x) * (i\gamma^\mu \partial_\mu + m)\hat{\psi}(x) + \hat{j}^\mu(x) * \hat{B}_\mu(x) + \frac{1}{2g^2}\hat{B}^\mu * \hat{B}_\mu]$$

$$= \int d^3x\ [\bar{\hat{\psi}}(x)(i\gamma^\mu \partial_\mu + m)\hat{\psi}(x) + \hat{j}^\mu(x)\hat{B}_\mu(x) + \frac{1}{2g^2}\hat{B}^\mu \hat{B}_\mu], \quad (5)$$

where the Thirring interaction is linearized by introducing a field $\hat{B}_\mu$. This is similar to the formalism followed in ordinary spacetime [11]. However, there is a subtlety involved, which is peculiar to NC spacetime. Depending on the positioning of $\hat{B}_\mu$ and $\hat{\psi}$, the covariant derivative can act in three ways [18],

$$\begin{aligned} \hat{D}_\mu \hat{\psi} &= \partial_\mu \hat{\psi} + i\hat{B}_\mu * \hat{\psi} \\ &= \partial_\mu \hat{\psi} - i\hat{\psi} * \hat{B}_\mu \\ &= \partial_\mu \hat{\psi} + i(\hat{\psi} * \hat{B}_\mu - \hat{B}_\mu * \hat{\psi}) \end{aligned} \quad (6)$$

which are termed respectively as fundamental, anti-fundamental and adjoint representations. Notice that we have chosen the anti-fundamental one in (5), since this will reduce to the original Thirring model (3) once B_μ is integrated out.

We now follow the work of Grandi and Silva [18] in computing the fermion determinant and write down the effective action,

$$\hat{S}[\hat{B}] = \int d^3x[-\frac{1}{8\pi}\epsilon^{\mu\nu\lambda}(\hat{B}_\mu * \partial_\nu \hat{B}_\lambda + \frac{2i}{3}\hat{B}_\mu * \hat{B}_\nu * \hat{B}_\lambda) + \frac{1}{2g^2}\hat{B}^\mu \hat{B}_\mu] + O(\frac{1}{m}). \quad (7)$$

Pauli-Villars regularization has been invoked and only the parity violating contribution is exhibited. The first term is the NC Chern-Simons term. The details of the derivation are to be found in [18]. This completes the first part of our work, that is bosonization of NCMT [3].

To understand the non-existance of NCMT-NCMCS duality, we briefly recall earlier works. The (ordinary spacetime) duality between the SD model, otained by bosonizing the MT model,

$$S_{SD} = \int d^3x[\frac{1}{2g^2}B^\mu B_\mu - \frac{1}{8\pi}\epsilon^{\alpha\beta\gamma}B_\alpha \partial_\beta B_\gamma] \quad (8)$$

[3] Referring to our earlier comment on the smoothness of the $\theta \to 0$ limit, notice that even though the coupling in the adjoint representation vanishes for $\theta = 0$, the effective action is non-zero [18]. This is relevant for Majorana fermions which are neutral in ordinary spacetime. However, this does not concern the present work.

and the MCS model,

$$S_{MCS} = \int d^3x[-\frac{1}{2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\partial^\alpha A^\beta + \frac{2\pi}{g^2}\epsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma]. \quad (9)$$

discovered by Deser and Jackiw [14, 13], was surprising since the latter is a gauge theory whereas the former is (naively) not. However constraints of the theories induce identical spectra and a mapping between degrees of freedom of the two theories [13]. This equivalence was further corroborated in [14] where a "'Master"' Lagrangian was constructed, which was capable of generating both the SD and MCS models.

The duality between the following NC versions of SD and MCS theories, shown in [17] by exploiting the "'Master"' Lagrangian technique,

$$\hat{S}_{SD} = \int d^3x[\frac{1}{2g^2}\hat{B}^\mu * \hat{B}_\mu - \frac{1}{8\pi}\epsilon^{\alpha\beta\gamma}\hat{B}_\alpha * \partial_\beta\hat{B}_\gamma] \quad (10)$$

$$\hat{S}_{MCS} = \int d^3x[-\frac{1}{2}(\partial_\alpha(\hat{A}+\hat{a})_\beta - \partial_\beta(\hat{A}+\hat{a})_\alpha) * \partial^\alpha(\hat{A}+\hat{a})^\beta + \frac{2\pi}{g^2}\epsilon^{\alpha\beta\gamma}(\hat{A}+\hat{a})_\alpha * \partial_\beta(\hat{A}+\hat{a})_\gamma]. \quad (11)$$

is all the more non-trivial since (11) is generated via the (inverse) Seiberg-Witten map [4] [1], which, valid to the first non-trivial order in θ, is

$$A_\mu = \hat{A}_\mu - \theta^{\sigma\rho}\hat{A}_\rho(\partial_\sigma\hat{A}_\mu - \frac{1}{2}\partial_\mu\hat{A}_\sigma) \equiv \hat{A}_\mu + \hat{a}_\mu(\hat{A}_\nu, \theta)$$

$$\lambda = \hat{\lambda} + \frac{1}{2}\theta^{\rho\sigma}\hat{A}_\rho\partial_\sigma\hat{\lambda}. \quad (12)$$

As stated before, the "'hatted"' variables on the right are NC degrees of freedom and gauge transformation parameter. It should be mentioned that the $O(\theta)$ expression of the Seiberg-Witten map is used only because the higher order terms in θ can not be obtained uniquely [19]. However, it is important to note that the equivalence result remains valid to all orders in θ as the explicit form of the map is not required in this proof. This is discussed in [20]. Indeed, the $O(\theta)$ analysis plays a central role since in NC spacetime physics as most of the results till date refer to $O(\theta)$ corrections over the results in normal spacetime.

In fact, (11) is nothing but the sum of NC Maxwell term and NC Chern-Simons term, correct up to the first non-trivial order in θ. S_{MCS} in (9) is transformed to $\hat{S}_{MCS}$ in (11) by using the Seiberg-Witten map [1] given in (12). On the other hand, $\hat{S}_{SD}$ (10) is gotten from S_{SD} (8) simply by replacing the ordinary products by $*$-product and using $B_\mu \equiv \hat{B}_\mu$. It does not require the Seiberg-Witen map. Notice that for this reason, the parity-odd term in (10) is not the NC extension of the Chern-Simons term. The non-abelian extension of these ideas are discussed in [21] in ordinary spacetime and in [22] in NC spacetime.

[4] The significance of the Seiberg-Witten map is that under an NC or $*$-gauge transformation of $\hat{A}_\mu$ by,

$$\hat{\delta}\hat{A}_\mu = \partial_\mu\hat{\lambda} + i[\hat{\lambda}, \hat{A}_\mu]_*,$$

A_μ will undergo the transformation

$$\delta A_\mu = \partial_\mu\lambda.$$

Subsequently, under this mapping, a gauge invariant object in conventional spacetime will be mapped to its NC counterpart, which will be $*$-gauge invariant.

The difference between the mechanisms by which $S_{SD} \rightarrow \hat{S}_{SD}$ ((8) $\rightarrow$ (10)) and $S_{MCS} \rightarrow \hat{S}_{MCS}$ ((9) $\rightarrow$ (11)) are obtained, is due to the fact that since S_{MCS} has a gauge invariance, $\hat{S}_{MCS}$ must have the corresponding $*$-gauge invariance. This is ensured by invoking the Seiberg-Witten map. On the other hand, S_{SD} is not a manifestly gauge invariant theory the Seiberg-Witten map does not come into play in the generation of its NC version.

It is now straightforward to see that even in the lowest non-trivial order in θ, the NCSD model given in (10) and the theory (7) obtained from bosonization of the NCMT model are different, because of the triple B_μ term in (7). Note that this difference vanishes in ordinary spacetime. This proves that the NCMCS theory is not the dual of NCMT model. This constitutes the second part of our statement advertised before.

A comment about nomenclature is possibly in order. This pertains to the fact that which of the models between (10) and (7) should be referred to as the NCSD. One can argue in favor of (7) since starting from the ordinary spacetime SD model, in (7), the Chern-Simons term is generalized to its NC version and the mass term remains as such. On the other hand we contend [17] that (10) should be termed as the NCSD model since the SD model being gauge variant, its NC version should be obtainable by only converting the ordinary products to $*$-products. Also, (10) obeys the self dual equation whereas (7) does not. All the same, keeping this ambiguity aside, the fact remains that the model (7) obtained by bosonizing the NCMT is different from the model (10) and in [17] the latter was shown to be equivalent to the NCMCS model.

To conclude, we have shown that, (keeping in mind the subtleties involved in noncommutative field theories), the noncommutative massive Thirring model can be bosonized in the large fermion mass limit, along the lines of its ordinary spacetime version. However, unlike the ordinary spacetime result, the duality between noncommutative massive Thirring model and noncommutative Maxwell-Chern-Simons model does not survive. Intuitivly, the reason for this failure is also not hard to guess. The noncommutative gauge theories that appear here are structurally akin to non-abelian gauge theories in ordinary spacetime and from previous experience [21] we know that these dualities are to be understood in a more restricted sense in the non-abelian setup.

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