No-Flipping as a consequence of No-Signalling and Non-increase of Entanglement under LOCC

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Abstract

Non existence of Universal Flipper for arbitrary quantum states is a fundamental constraint on the allowed operations performed on physical systems. The largest set of qubits that can be flipped by a single machine is a great circle of the Bloch-sphere. In this paper, we show the impossibility of universal exact-flipping operation, first by using the fact that no faster than light communication is possible and then by using the principle of "non-increase of entanglement under LOCC". Interestingly, in both the cases, there is no violation of the two principles if and only if the set of states to be flipped, form a great circle.

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The structure of the allowed operations performed on the quantum systems imposes some restrictions on the systems. Sometimes these restrictions play a crucial role to understand the basic features of the system and naturally our task is to find the fundamental nature of these restrictions in a simple way. It has been shown that an arbitrary state taken from a set of two known, non-orthogonal states can not be copied exactly in a deterministic way [1]. Similarly, one can not delete copy of an unknown state by performing some linear, trace preserving joint operations on two copies of that state [2, 3]. Interestingly, several authors derived these no-cloning and no-deleting theorems by applying some fundamental principles of nature like impossibility of signalling [4, 5, 6, 7], preservation of entanglement for closed systems under local operations [8] or rather increase of entanglement by LOCC [9].

Another interesting feature of quantum system, is the non-existence of universal flipping machine [10] for arbitrary input qubit states, *i.e.,* there exists no universal flipper which can operate on any unknown qubit state $|\psi\rangle$ resulting the orthogonal state $|\psi^{\perp}\rangle$ [11, 12, 13, 14]. This no-flipping theorem has a stark dissimilarity with others, as unlike no-cloning and no-deleting, two non-orthogonal states can always be flipped. Actually the largest set of states (of qubit system) which can be flipped exactly, by a single unitary operator is the set of states lying on a great circle of the Bloch sphere [15, 16, 17].

In this paper our aim is to establish the no-flipping theorem by applying the following established principles of nature:

1. Impossibility of superluminal signalling - It is impossible to communicate any message between some spatially separated parties with a speed greater than the speed of light.

2. The thermodynamical law of Entanglement - Amount of Entanglement shared between some spatially separated parties can not be increased by LOCC, i.e., *by performing local operations on the subsystems and classical communications between them.*

In other words our aim is to show, if exact flipping of even the minimal number (*i.e.,* three) of states, not taken from one great circle, is possible, then one can send instantaneous signal as well as increase entanglement between two distant parties by local operations.

For this purpose we consider three arbitrary states not lying in one great circle in their simplest form as;

$$\begin{aligned}
&|0\rangle, \\
&|\psi\rangle = a|0\rangle + b|1\rangle, \\
&|\phi\rangle = c|0\rangle + d\ e^{i\theta}|1\rangle,
\end{aligned} \tag{1}$$

where a, b, c, d are real numbers satisfying the relation $a^2+b^2 = 1 = c^2+d^2$

and $0 < \theta < \pi, a > 0, c > 0$, and the states $|0\rangle$, $|1\rangle$ are orthogonal to each other.

We assume that a machine exists which can flip at least these three states exactly. The most general flipping operation for these three states can be described as

$$\begin{array}{l} |0\rangle|M\rangle \longrightarrow |1\rangle|M_0\rangle \\ |\psi\rangle|M\rangle \longrightarrow e^{i\mu}|\overline{\psi}\rangle|M_\psi\rangle \\ |\phi\rangle|M\rangle \longrightarrow e^{i\nu}|\overline{\phi}\rangle|M_\phi\rangle \end{array} \quad (2)$$

where μ and ν are some arbitrary phases and $|M\rangle$ is the initial machine state. The flipped states are orthogonal to the original states, *i.e.*,

$$\langle 0|1\rangle = \langle\psi|\overline{\psi}\rangle = \langle\phi|\overline{\phi}\rangle = 0, \quad (3)$$

where $|\overline{\psi}\rangle = b|0\rangle - a|1\rangle$ and $|\overline{\phi}\rangle = d\ e^{-i\theta}|0\rangle - c|1\rangle$ in their usual notations.

The above operation is not assumed to be unitary and the operation acts linearly on one side of an entangled state only when the density matrix has a mixture representation of the states in equation (2).

First we show that the exact flipping machine which we have considered, implies signalling. Consider two spatially separated parties, say, Alice and Bob who initially share an entangled state of the form,

$$|\Psi\rangle^i_{AB} \ = \ \tfrac{1}{\sqrt{3}}\ (|0\rangle_A|0\rangle_B + |1\rangle_A|\psi\rangle_B + |2\rangle_A|\phi\rangle_B) \otimes\ |M\rangle_B \quad (4)$$

where Alice holds a system associated with three dimensional Hilbert space, having a basis, $\{\ |0\rangle,\ |1\rangle,\ |2\rangle\}$(say) and Bob's system consists of a qubit (entangled with Alice's system) and a flipping machine defined as in equation (2). Here one should note that the joint system of Alice and Bob has been chosen in such a manner that the marginal density matrix of Bob's side admits a representation in terms of the three states $|0\rangle,\ |\psi\rangle,\ |\phi\rangle$, on which the flipping machine has been defined.

The reduced density matrix of Alice's side is

$$\begin{array}{rcl} \rho^i_A & = & \frac{1}{3}\ \{\ P[|0\rangle] + P[|1\rangle] + P[|2\rangle] + a(|0\rangle\langle 1| + |1\rangle\langle 0|) \\ & & +c(|0\rangle\langle 2| + |2\rangle\langle 0|) + \langle\phi|\psi\rangle|1\rangle\langle 2| + \langle\psi|\phi\rangle|2\rangle\langle 1|\ \} \end{array} \quad (5)$$

Now assume that Bob applies the flipping machine on his qubit. After the flipping operation the shared state between Alice and Bob takes the following form,

$$|\Psi\rangle^f_{AB} = \ \tfrac{1}{\sqrt{3}}\ \{|0\rangle_A|1\ M_0\rangle_B + e^{i\mu}|1\rangle_A|\overline{\psi}\ M_\psi\rangle_B + e^{i\nu}|2\rangle_A|\overline{\phi}\ M_\phi\rangle_B\} \quad (6)$$

The final density matrix of Alice's side (expanded in the computational basis) is

$$\begin{aligned}\rho_A^f \;=\;& \tfrac{1}{3}\{P[|0\rangle]+P[|1\rangle]+P[|2\rangle]-a(e^{-i\mu}\langle M_\psi|M_0\rangle|0\rangle\langle 1| \\ &+e^{i\mu}\langle M_0|M_\psi\rangle|1\rangle\langle 0|)-c(e^{-i\nu}\langle M_\phi|M_0\rangle|0\rangle\langle 2|+e^{i\nu}\langle M_0|M_\phi\rangle|2\rangle\langle 0|) \\ &+\langle\psi|\phi\rangle e^{i(\mu-\nu)}\langle M_\phi|M_\psi\rangle|1\rangle\langle 2|+\langle\phi|\psi\rangle e^{i(\nu-\mu)}\langle M_\psi|M_\phi\rangle|2\rangle\langle 1|\}\end{aligned} \tag{7}$$

As this is a trace preserving local operation performed entirely on Bob's side and there is also no classical communication between them, so to prevent any violation of the principle of no-signalling, the reduced state on Alice's side must remain unchanged. Equating the reduced density matrices on Alice's side before and after the flipping operation on Bob's side we get the following relations

$$a \;=\; -\,ae^{i\mu}\langle M_0|M_\psi\rangle \;=\; -\,ae^{-i\mu}\;\langle M_\psi|M_0\rangle \tag{8}$$

$$c \;=\; -\,ce^{i\nu}\langle\, M_0|M_\phi\rangle \;=\; -\,ce^{-i\nu}\langle\, M_\phi|M_0\rangle \tag{9}$$

$$\langle\phi|\psi\rangle \;=\; e^{i(\mu-\nu)}\;\langle\psi|\phi\rangle\;\langle M_\phi|M_\psi\rangle \tag{10}$$

$$\langle\psi|\phi\rangle \;=\; e^{i(\nu-\mu)}\;\langle\phi|\psi\rangle\;\langle M_\psi|M_\phi\rangle \tag{11}$$

The above relations imply,

$$\text{either } b=0 \text{ or } d=0 \text{ or } \sin\theta=0,$$

forcing the states to lie on a great circle. So, we observe that as long as the three states do not lie on a great circle, we have, $\rho_A^i \neq \rho_A^f$, and interestingly whenever the equality holds, i.e., $\rho_A^i = \rho_A^f$, the three states actually lie on a great circle. This clearly shows that exact flipping of any three states not lying on a great circle is an impossibility.

Now, we are going to show the impossibility of universal flipping using the principle of non-increase of entanglement by LOCC. Here we must be careful about the fact that in the earlier set-up (equation (4)), eigen values of ρ_A^i and ρ_A^f are equal, implying no change in entanglement even when states are not on the great circle. For this we choose another state, $|\Psi_{AB}^i\rangle$ of five qubits, shared between Alice and Bob, situated at distant locations, where the first qubit is with Alice, and remaining four(B_1, B_2, B_3, B_4) are with Bob.

$$\begin{aligned}|\Psi_{AB}^i\rangle \;=\;& \tfrac{1}{\sqrt{8}}\{(|000\rangle+|111\rangle)_{AB_1B_2}\otimes|10\rangle_{B_3B_4} \\ &-(|010\rangle+|100\rangle+|101\rangle)_{AB_1B_2}\otimes|\overline{\psi}\psi\rangle_{B_3B_4} \\ &-(|011\rangle+|110\rangle+|001\rangle)_{AB_1B_2}\otimes|\overline{\phi}\phi\rangle_{B_3B_4}\}\otimes|M\rangle_{B_M}\end{aligned} \tag{12}$$

Let us assume that Bob has a flipping machine (defined earlier) with him which he applies on his last qubit(B_4). After the flipping operation the joint state between them takes the form,

$$\begin{aligned} |\Psi^f_{AB}\rangle &= \tfrac{1}{\sqrt{8}} \{(|000\rangle + |111\rangle)_{AB_1B_2} \otimes |11M_0\rangle_{B_3B_4B_M} \\ &\quad - e^{i\mu}(|010\rangle + |100\rangle + |101\rangle)_{AB_1B_2} \otimes |\overline{\psi}\ \overline{\psi} M_\psi\rangle_{B_3B_4B_M} \\ &\quad - e^{i\nu}(|011\rangle + |110\rangle + |001\rangle)_{AB_1B_2} \otimes |\overline{\phi}\ \overline{\phi} M_\phi\rangle_{B_3B_4B_M}\}. \end{aligned} \tag{13}$$

In order to compare the amount of entanglement present in $|\Psi^i_{AB}\rangle$ with that present in $|\Psi^f_{AB}\rangle$, we compare the eigenvalues of the marginal density matrices on Alice's side before and after the flipping operation.

The initial state of Alice's subsystem is;

$$\rho^i_A = \tfrac{1}{8}\{4(P[|0\rangle] + P[|1\rangle]) + (a^2 + c^2 + 2|\langle\psi|\phi\rangle|^2)(|0\rangle\langle 1| + |1\rangle\langle 0|)\}. \tag{14}$$

The final state of Alice's subsystem is;

$$\begin{aligned} \rho^f_A &= \tfrac{1}{8}\{4(P[|0\rangle] + P[|1\rangle]) + (2Z - a^2X^* - c^2Y)|0\rangle\langle 1| \\ &\quad + (2Z - a^2X - c^2Y^*)|1\rangle\langle 0|\} \end{aligned} \tag{15}$$

where, $Z = Re[e^{i(\mu-\nu)}(\langle\psi|\phi\rangle)^2\langle M_\phi|M_\psi\rangle]$, $X = e^{i\mu}\ \langle M_0|M_\psi\rangle$ and $Y = e^{i\nu}\ \langle M_0|M_\phi\rangle$.

After constructing the eigenvalue equations for both ρ^i_A and ρ^f_A we find that the largest eigenvalues are respectively

$$\lambda^i = \frac{1}{2} + \frac{1}{8}(\ 2|\langle\psi|\phi\rangle|^2 + a^2 + c^2)$$

and

$$\lambda^f = \frac{1}{2} + \frac{1}{8}|\ 2Z - a^2X^* - c^2Y|$$

It can easily be checked that $\lambda^f \leq \lambda^i$(see appendix). This implies, $E(|\Psi^f_{A:B}\rangle) \geq E(|\Psi^i_{A:B}\rangle)$ where E stands for amount of entanglement. So this establishes the impossibility of universal flipping as here the only operation that has been allowed is local. A close look will reveal that the greatest eigenvalues are equal (i.e., no increase of entanglement), only when, the three states on which we have defined our flipping machine, lie on one great circle.

The above construction seems to be a complicated one due to our linearity assumption. We have considered our flipping operation acts linearly on one side of an entangled state only when the reduced density matrix has a mixture representation of the states in equation (2). However a more simpler proof is possible consisting only a three qubit system if we allow further that the operation is linear on superposition level.

Now consider a three qubit state shared between Alice and Bob where Bob has a two qubit system (B_1 and B_2) as follows,

$$
\begin{aligned}
|\Phi^i\rangle_{AB} &= \tfrac{1}{\sqrt{b^2+d^2}}\{\ |0\rangle_A \tfrac{|0\rangle_{B_1}|\psi\rangle_{B_2}-|\psi\rangle_{B_1}|0\rangle_{B_2}}{\sqrt{2}}\ +\ |1\rangle_A \tfrac{|0\rangle_{B_1}|\phi\rangle_{B_2}-|\phi\rangle_{B_1}|0\rangle_{B_2}}{\sqrt{2}}\} \\
&= \tfrac{1}{\sqrt{b^2+d^2}}\{(b|0\rangle_A + de^{i\theta}|1\rangle_A)\tfrac{|0\rangle_{B_1}|1\rangle_{B_2}-|1\rangle_{B_1}|0\rangle_{B_2}}{\sqrt{2}}\}
\end{aligned} \tag{16}
$$

Clearly $|\Phi^i\rangle_{AB}$ is a separable (pure product) state in $A:B$ cut. Assume Bob has a flipping machine with him which he applies on his last qubit, i.e., on B_2. Then the joint state between them takes the form,

$$
|\Phi^f\rangle_{AB} = \tfrac{1}{\sqrt{N}}\{e^{i\mu}|00\rangle|\overline{\psi}\rangle|M_\psi\rangle + e^{i\nu}|10\rangle|\overline{\phi}\rangle|M_\phi\rangle - (|0\rangle|\psi\rangle + |1\rangle|\phi\rangle)|1\rangle|M_0\rangle\} \tag{17}
$$

where $N = 2 + a^2 Re\{e^{-i\mu}\langle M_\psi \mid M_0\rangle\} + c^2 Re\{e^{-i\nu}\langle M_\phi \mid M_0\rangle\}$. It is easy to check that the state in general is an entangled state in $A:B$ cut, thus in this setting we also observe an increase of entanglement by LOCC.

In the conclusive remarks, we want to mention that, as a constraint on quantum mechanical system, the 'No-Flipping' theorem is weaker than the 'No-Cloning' and 'No-Deleting' theorem, because exact flipping is possible for all states lying in one great circle of the Bloch sphere. Hence, showing the impossibility of universal flipping from the principle like No-signalling and Non increase of entanglement by local operations, is an interesting problem to deal with. We find the nonphysical nature of flipping operation on whole Bloch sphere in two different ways. In this context, recently no-flipping has been established by using the existence of incomparable states where one qutrit and two qubits are required [18].

Appendix

To check that, $\lambda^f \leq \lambda^i$ for the second set-up, we have,
$\lambda^i \geq \lambda^f \Leftrightarrow\ 2|\langle\psi|\phi\rangle|^2 + a^2 + c^2 \geq |\ 2Z - a^2X^* - c^2Y|$
$\Leftrightarrow\ a^4(1-|X|^2)+c^4(1-|Y|^2)+2a^2c^2[1-Re(XY)]+4a^2[|\langle\psi|\phi\rangle|^2+ZRe(X)]+4c^2[|\langle\psi|\phi\rangle|^2 + ZRe(Y)] + 4[|\langle\psi|\phi\rangle|^4 - Z^2] \geq 0$
As, all terms on the L.H.S. of the last inequality are non-negative. Therefore, the above inequality for eigenvalues is satisfied.

The equality holds only when, $|X| = |Y| = 1$, $Re(XY) = 1$, $|\langle\psi|\phi\rangle|^2 + ZRe(X) = 0$, $|\langle\psi|\phi\rangle|^2 + ZRe(Y) = 0$, $Z^2 = |\langle\psi|\phi\rangle|^4$.
Now, $Z = Re[e^{i(\mu-\nu)}(\langle\psi|\phi\rangle)^2\langle M_\phi|M_\psi\rangle]$, $X = e^{i\mu}\ \langle M_0|M_\psi\rangle$ and $Y = e^{i\nu}\ \langle M_0|M_\phi\rangle$, which implies machine states will differ by only some phases and the states $|0\rangle, |\psi\rangle, |\phi\rangle$ will lie on a great circle.

Hence we see that in this case, where $\rho_A^i = \rho_A^f$, the three states of equation (1) on which we have defined our flipping machine, actually lies on a great circle of the Bloch sphere.

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