Probabilistic Super Dense Coding

A. K. Pati[1], P. Parashar[2] and P. Agrawal[1]
[1]Institute of Physics, Sainik School Post,
Bhubaneswar-751005, India
[2] Physics and Applied Mathematics Unit,
Indian Statistical Institute, Kolkata-700108, India

February 1, 2008

Abstract

We explore the possibility of performing super dense coding with non-maximally entangled states as a resource. Using this we find that one can send two classical bits in a probabilistic manner by sending a qubit. We generalize our scheme to higher dimensions and show that one can communicate $2\log_2 d$ classical bits by sending a d-dimensional quantum state with a certain probability of success. The success probability in super dense coding is related to the success probability of distinguishing non-orthogonal states. The optimal average success probabilities are explicitly calculated. We consider the possibility of sending $2\log_2 d$ classical bits with a shared resource of a higher dimensional entangled state $(D \times D, D > d)$. It is found that more entanglement does not necessarily lead to higher success probability. This also answers the question as to why we need $\log_2 d$ ebits to send $2\log_2 d$ classical bits in a deterministic fashion.

PACS NO: 03.67.-a, 03.65.Bz

Emails : akpati@iopb.res.in, parashar@isical.ac.in, agrawal@iopb.res.in

1 Introduction

It is by now, well demonstrated that entangled states are at the heart of quantum information theory. One can do many surprising tasks using entangled states which are otherwise impossible, e.g., super dense coding [1], quantum teleportation [2], remote state preparation [3], quantum cryptography [4] and so on. In the case of super dense coding, Bennett and Wisner have shown that it is possible to send two classical bits of information by sending just a single qubit [1]. Ordinarily by sending a single qubit one would extract only one bit of classical information. However, prior sharing of entangled state enhances the classical communication capacity, hence the name super dense coding. In a similar fashion, if one shares $\log_2 d$ ebits of entanglement then one can extract $2\log_2 d$ classical bits of information by sending a d-level quantum system (a qudit).

In recent years, super dense coding has been generalized in various directions. For example, it is possible to generalize the super dense coding for multi-parties [5]. Also, one can perform super dense coding not only with quantum states in finite dimensional Hilbert spaces but also with quantum states in infinite dimensional Hilbert spaces [6, 7]. All these cases deal with maximally entangled (ME) states. But suppose Alice and Bob share a non-maximally entangled (NME) state, then what can they do? This question was first addressed by Barenco and Ekert [8]. However, their scheme is not a conclusive one. It was shown by Hausladen *et al* [9] that if one has a non-maximally entangled state then the classical capacity of dense coding scheme is not $2\log_2 d$ but equal to $H_E + \log_2 d$ bits of information in the asymptotic limit, where H_E is the entropy of entanglement of the shared state. Here, $0 \leq H_E \leq \log_2 d$. However, the above scheme is a deterministic one. So this result tells us that deterministically we cannot send $2\log_2 d$ bits using NME states. The super dense coding protocol has been generalized for mixed entangled states and the classical capacity has been related to various measures of entanglement [10]. Very recently, Mozes *et al* [11] have investigated the relationship between the entanglement of a given NME state and the maximum number of alphabets which can be perfectly transmitted in a deterministic fashion (this is called 'not so super dense coding').

All the previous work are primarily on deterministic super dense coding. If one does not demand that the scheme works in a deterministic manner, then it should be possible to send $2\log_2 d$ bits of information with certain probability of success by sending a qudit. This is the aim of the present investigation. The paper is organized as follows. First, we illustrate the protocol for exact but probabilistic super dense coding for qubits in section 2. In section 3 we generalize the scheme to higher dimensions. We find that the success probability of performing super dense coding is exactly same as the success probability of distinguishing a set of non-orthogonal states. It is indeed interesting to identify the problem of probabilistic super dense coding with unambiguous state discrimination. Alternately, one may think that this problem is related to unambiguous discrimination among unitary operators with an entangled probe state. It has been shown that a set of unitary operators can be unambiguously discriminated iff they are linearly independent [12]. This is true for any Hilbert space dimension. Furthermore, any probe state with maximum Schmidt rank is sufficient to enable us to do the discrimination. Therefore, we can say that one can do probabilistic dense coding with any maximum Schmidt rank pure entangled state if you encode the information using a set of d^2 linearly independent unitary operators. This shows that the ability to perform super dense coding is not only determined by the amount of entanglement shared between the sender and the receiver but also depends on

the extent to which the states encoding the message can be distinguished. In section 4, we investigate if the use of more prior entangled state can enhance the success probability of performing dense coding. In particular, we have asked if by sharing a $(D \times D, D > d)$ entangled state and by encoding d^2 messages in a D-state system, one can send $2\log_2 d$ classical bits in a deterministic fashion? The answer to this can be negative sometimes. We find that more entanglement may not be useful in the sense that it may or may not enhance the success probability of performing dense coding. On the contrary, if we use a $(D \times D, D > d)$ maximally entangled state and try to send $2\log_2 d$ classical bits, then surprisingly the success probability decreases with increasing D. When $D = d$, then the optimal probability of performing the dense coding is exactly unity, which is the standard case. This also helps us to understand why we need $\log_2 d$ ebits to send $2\log_2 d$ classical bits in a deterministic fsahion. We end the paper with discussions, conclusions and some future directions in section 5.

2 Probabilistic dense coding with a qubit

In this section we describe how to send two classical bits $(2\log_2 2)$ of information in a probabilistic manner using a partially entangled state. First we give the most general set of basis vectors for two qubit Hilbert space. This was introduced in [13] in the context of probabilistic teleportation. We can define a set of mutually orthogonal NME basis vectors $\{|\psi_i\rangle\}(i = 1, 2, 3, 4) \in \mathcal{H}^2 \otimes \mathcal{H}^2$ as follows

$$\begin{aligned} |\psi_1\rangle &= |\varphi_\ell^+\rangle = L\,(|00\rangle + \ell\,|11\rangle) \\ |\psi_2\rangle &= |\varphi_\ell^-\rangle = L\,(\ell^*|00\rangle - |11\rangle) \\ |\psi_3\rangle &= |\psi_p^+\rangle = P\,(|01\rangle + p\,|10\rangle) \\ |\psi_4\rangle &= |\psi_p^-\rangle = P\,(p^*|01\rangle - |10\rangle) \end{aligned} \tag{1}$$

Here ℓ and p can be complex numbers in general and $L = \frac{1}{\sqrt{1+|\ell|^2}}$ and $P = \frac{1}{\sqrt{1+|p|^2}}$ are real numbers. We notice that when $\ell = p = 0$, this basis reduces to the computational basis which is not entangled. For $\ell = p = 1$, it reduces to the Bell basis which is maximally entangled. Therefore this set interpolates between unentangled and maximally entangled set of basis vectors. Also note that the set $|\varphi_\ell^\pm\rangle$ and $|\psi_p^\pm\rangle$ have different amount of entanglement for $0 < \ell, p < 1$. As measured by von Neumann entropy [14], the entanglement of $E(|\varphi_\ell^\pm\rangle) = (-L^2\log_2 L^2 - L^2\,|\ell|^2\log_2 L^2|\ell|^2)$ and of $E(|\psi_p^\pm\rangle) = (-P^2\log_2 P^2 - P^2\,|p|^2\log_2 P^2|p|^2)$, respectively are different for these sets. However, when $\ell = p$, then all basis vectors have identical von Neumann entropy. Even though $|\varphi_\ell^\pm\rangle$ and $|\psi_p^\pm\rangle$ have different amount of entanglement they satisfy the completeness condition, i.e., $\sum_i |\psi_i\rangle\langle\psi_i| = I$ for all ℓ and p.

For the purpose of super dense coding one may use any one of the NME basis vectors as a shared resource. Let Alice and Bob share a non-maximally entangled state $|\phi_\ell^+\rangle$ as a quantum channel which is given by

$$|\phi_\ell^+\rangle = L\,(|00\rangle + \ell|11\rangle). \tag{2}$$

Here, without loss of generality ℓ can be chosen to be a real number. Notice that because of the existence of Schmidt decomposition [15, 16] any two qubit entangled state $|\Psi\rangle \in \mathcal{H}^2 \otimes \mathcal{H}^2$ such as

$$|\Psi\rangle = a|00\rangle + b|11\rangle + c|01\rangle + d|10\rangle \tag{3}$$

can be written as a superposition of two basis vectors. In general, the computational basis states such as $|0\rangle$ and $|1\rangle$ need not be the Schmidt basis, but we assume that Alice and Bob know the Schmidt basis and coefficients. Then (2) is the most general non-maximally entangled state up to local unitary transformations relating Schmidt basis and computational basis states. By local unitary transformation, Eqn.(3) can be brought to Eqn.(2).

Let Alice apply on her particle, any one of the four unitary operators $\{I, \sigma_x, i\sigma_y, \sigma_z\}$ that encodes two bits of classical information. Then, depending on the applied unitary transformation the shared state undergoes the following transformation

$$\begin{aligned}
|\phi_\ell^+\rangle &\rightarrow (I \otimes I)\phi_\ell^+|\rangle = |\phi_\ell^+\rangle \\
\phi_\ell^+\rangle| &\rightarrow (\sigma_x \otimes I)|\phi_\ell^+\rangle = L(||10\rangle + \ell|01\rangle) = \tilde{\psi}_\ell^+\rangle \\
|\phi_\ell^+\rangle &\rightarrow (i\sigma_y \otimes I)||\phi_\ell^+\rangle = L(-|10\rangle + \ell|01\rangle) = |\psi_\ell^-\rangle \\
\phi_\ell^+\rangle &\rightarrow |(\sigma_z \otimes I)|\phi_\ell^+\rangle = L(|00\rangle - \ell||11\rangle) = \tilde{\phi}_\ell^-\rangle. \qquad (4)
\end{aligned}$$

Now Alice sends her qubit to Bob. Bob has at his disposal two qubits which could be in any one of the four possible states $\{|\phi_\ell^+\rangle, |\tilde{\psi}_\ell^+\rangle, |\psi_\ell^-\rangle, |\tilde{\phi}_\ell^-\rangle\}$. If Bob is able to distinguish all the four states deterministically then he can extract two classical bits of information. However, the above four states are not mutually orthogonal. In quantum theory, non-orthogonal states cannot be distinguished with certainty. Note that if the shared state is a ME state, then all the above four states are mutually orthogonal and the protocol reduces to the standard one [1].

However, it is known that if a set contains non-orthogonal states that are linearly independent then they can be distinguished with some probability of success [19, 20, 21, 22, 23]. Now in our case, it is easy to check that the above set $\{|\phi_\ell^+\rangle, |\tilde{\psi}_\ell^+\rangle, |\psi_\ell^-\rangle, |\tilde{\phi}_\ell^-\rangle\}$ is actually *linearly independent*. The basic idea is that once Bob is able to distinguish these states with some probability of success, then he can know which unitary operation Alice has applied, hence he can extract two classical bits of information. The optimal probability of distinguishing these linearly independent states is then the optimal success probability of performing the super dense coding with a partially entangled state.

The way it works is that first Bob performs a projection onto the subspaces spanned by the basis states $\{|00\rangle, |11\rangle\}$ and $\{|01\rangle, |10\rangle\}$. The corresponding projection operators are $P_1 = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $P_2 = |01\rangle\langle 01| + |10\rangle\langle 10|$, where P_1 and P_2 are mutually orthogonal. If he projects onto P_1, then he knows that the state is either $|\phi_\ell^+\rangle$ or $|\tilde{\phi}_\ell^-\rangle$. Similarly, if he projects onto P_2, then he knows that the state is either $|\tilde{\psi}_\ell^+\rangle$ or $|\psi_\ell^-\rangle$. Now the task at Bob's hand is to further distinguish between these two states within the given subspace. To achieve this, he performs a generalized measurement described by Positive Operator Valued Measurements (POVMs) on his two qubit states. POVMs are nothing but the generalized measurement operators which can be realized by enlarging the Hilbert space of the quantum system and performing orthogonal projections on the ancilla system. They are described by a set of positive operators $\{A_\mu\}$ that sum to unity, i.e, $\sum_\mu A_\mu = I$. Here, the number of outcomes can be much larger than the Hilbert space dimension of the quantum system, i.e. $\mu \geq d$. Upon measurement, the probability of observing μth outcome in a quantum state ρ is given by $p_\mu = \mathrm{tr}(A_\mu \rho)$. In general these POVM's are not necessarily orthogonal. If they are orthogonal then they reduce to the standard von Neumann projection operators.

Now the corresponding POVM elements for the two qubit case in the subspace

$\{|00\rangle, |11\rangle\}$ are given by

$$A_1 = \frac{1}{2}\begin{pmatrix} \ell^2 & \ell \\ \ell & 1 \end{pmatrix}, \quad A_2 = \frac{1}{2}\begin{pmatrix} \ell^2 & -\ell \\ -\ell & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1-\ell^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5}$$

This was first given in [17] and also used in conclusive teleportation [18]. One can check that $A_1 + A_2 + A_3 = I$. Here if Bob gets A_1 then the state is $|\phi_\ell^+\rangle$, if he gets A_2 then it is $|\tilde{\phi}_\ell^-\rangle$ and if he gets A_3 then the result is inconclusive. The success probability of distinguishing $|\phi_\ell^+\rangle$ and $|\tilde{\phi}_\ell^-\rangle$ is $1 - \langle\phi_\ell^+|A_3|\phi_\ell^+\rangle$ which is same as $1 - \langle\tilde{\phi}_\ell^-|A_3|\tilde{\phi}_\ell^-\rangle$. This turns out to be equal to $\frac{2\ell^2}{1+\ell^2}$. Similarly, for the other two cases one can show that the success probability is given by the above expression. Hence, we can say that Bob can extract two bits of classical information with a success probability given by $\frac{2\ell^2}{1+\ell^2}$. For the maximally entangled case, $\ell = 1$ and so probability becomes one. This is then the standard super dense coding protocol that works in a deterministic fashion. This completes the probabilistic super dense coding protocol with a qubit.

3 Probabilistic dense coding for qudit

We know that if Alice and Bob share a $(d \times d)$ maximally entangled state then by sending a qudit Alice can communicate $2\log_2 d$ bits of classical information. Can she send the same amount of classical information in a probabilistic manner if they share a non-maximally entangled state? The answer is yes. Interestingly, this problem is also directly related to the problem of distinguishing a set of non-orthogonal states with a certain probability of success.

In this section we generalize our protocol when Alice and Bob share a NME state in higher dimensions (say a two-qudit state in $d \times d$). The shared NME state is expressed as

$$|\Psi\rangle = \sum_{k=0}^{d-1} \sqrt{p_k}|k\rangle|k\rangle, \tag{6}$$

where p_k's are the Schmidt coefficients and $|k\rangle$'s are the Schmidt bases vectors. Alice and Bob possess one particle each. Now Alice encodes her d^2 possible choices or $2\log_2 d$ bits of classical information using unitary operators $\mathcal{U}_{mn}$, where $m, n = 0, 1, \ldots d-1$. These unitary operators are given by

$$\mathcal{U}_{mn} = (U)^m (V)^n, \tag{7}$$

where U is the shift operator and V is the rotation operator whose action on the basis states are defined as follows

$$\begin{aligned} U|k\rangle &= |(k \oplus 1)\rangle \\ V|k\rangle &= e^{2\pi i k/d}|k\rangle \end{aligned} \tag{8}$$

and $\oplus$ is addition modulo d. After Alice applies $\mathcal{U}_{mn}$ to her particle the two-qudit state transforms as

$$|\Psi\rangle \to (\mathcal{U}_{mn} \otimes I)|\Psi\rangle = \sum_{k=0}^{d-1} \sqrt{p_k} e^{2\pi i n k/d}|k \oplus m\rangle|k\rangle = |\Psi_{mn}\rangle. \tag{9}$$

Next, Alice sends her qudit to Bob who has the two qudit state $|\Psi_{mn}\rangle$ at his disposal. If Bob is able to perform a measurement and distinguish all d^2 states perfectly then he can

extract $2\log_2 d$ bits of information deterministically. However, these d^2 states given above are not orthogonal. Indeed, they satisfy the following relation

$$\langle \Psi_{mn}|\Psi_{m'n'}\rangle = \sum_{k=0}^{d-1} p_k e^{-2\pi i k(n-n')/d}\delta_{mm'}. \tag{10}$$

Only when all p_k's are same (i.e. the shared state is ME) the above d^2 states are orthogonal. Now Bob has to find a strategy to distinguish these states. His ability to distinguish them will decide the success or failure to extract $2\log_2 d$ bits of classical information. Of course, he cannot do so perfectly. But he can succeed in distinguishing the above states with some probability. Then the probability of distinguishing these non-orthogonal states will be the probability of successful dense coding for a qudit.

Here, we are going to use ideas about discriminating non-orthogonal, but linearly indepedent quantum states and present a closed form expression for average success probability of distinguishing a collection of such quantum states. This is another direction of research by itself, so we do not intend to review its status here [19, 20, 21, 22]. Rather we will be using some of the results. The pertinent question in the present context is that if we have a set that contains a collection of quantum states $\{|\Psi_i\rangle\}(i = 1, 2, \ldots N)$ in some Hilbert space, then can we perform some measurement and tell in which state the system is? If these states are orthogonal then the standard von Neumann projection can give us an answer with certainty. However, if they are non-orthogonal then no von Neumann type measurement can unambiguously identify the states. Then one has to take recourse to the idea of generalized or POVM measurements which can help us in discriminating non-orthogonal states with some probability if and only if the states are linearly independent [21]. A more convenient approach was suggested by a theorem of Duan-Guo [23] which tells us that there is a unitary operator together with post selection of measurement action which can identify a set of linearly independent states with some success probability. More precisely it states that the set $\{|\Psi_i\rangle\}(i = 1, 2, \ldots N)$ can be identified, respectively, with efficiency γ_i if and only if the matrix $X^{(1)} - \Gamma$ is positive definite [23] where $X^{(1)} = [\langle\Psi_i|\Psi_j\rangle]$ is the Gram matrix and $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots \gamma_N)$. In terms of the unitary operator on the input and probe state the process takes the form

$$U(|\Psi_i\rangle|P\rangle) = \sqrt{\gamma_i}|\Psi_i'\rangle|P_i\rangle + \sqrt{1-\gamma_i}|\Phi_i\rangle|P_{N+1}\rangle \tag{11}$$

where $|P\rangle$ is the initial state of the probe, $|P_1\rangle, |P_2\rangle, \cdots |P_{N+1}\rangle$ are orthonormal basis of the probe Hilbert space, $|\Psi_i'\rangle$ is the final state of the system, and $|\Phi_i\rangle$ is the failure component. After the unitary evolution, if we perform a von Neumann projection on the ancilla system and get $|P_i\rangle, i = 1, 2, \ldots N$, then we are able to identify the state. But if we get $|P_{N+1}\rangle$, then we discard it. The success probability of identifying these states is γ_i. Using Eqn.(11) we derive the optimal bound on the success probability of distinguishing any two non-orthogonal but linearly indepedent states. Taking the inter inner product we have

$$\langle\Psi_i|\Psi_j\rangle = \sqrt{\gamma_i\gamma_j}\langle\Psi_i'|\Psi_j'\rangle\langle P_i|P_j\rangle + \sqrt{(1-\gamma_i)(1-\gamma_j)}\langle\Phi_i|\Phi_j\rangle. \tag{12}$$

Using the above equation we can obtain the tight inequality for distinguishing any two non-orthogonal states from the set. It is given by

$$\frac{1}{2}(\gamma_i + \gamma_i)(1 - \delta_{ij}) \leq 1 - |\langle\Psi_i|\Psi_j\rangle|. \tag{13}$$

This holds for all i, j. For $i = j$ we have $\gamma_i = 0$. One may solve a series of inequalities to obtain individual success probabilities. However, we are interested in the average success probability. This may be obtained as follows. Define the total success probability as $\gamma = \sum_i \gamma_i$ and the average success probability as $\bar{\gamma} = \frac{\sum_i \gamma_i}{N}$, where N is the number of linearly independent vectors and $N \leq \dim(\mathcal{H})$. Then performing a double sum in the above inequality, we have the average success probability as

$$\bar{\gamma} \leq \frac{N}{N-1} - \frac{1}{N(N-1)} \sum_{i,j=1}^{N} |\langle \Psi_i | \Psi_j \rangle|. \tag{14}$$

Alternately, this can be expressed as

$$\bar{\gamma} \leq 1 - \frac{1}{N(N-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} |\langle \Psi_i | \Psi_j \rangle|. \tag{15}$$

This shows that if the set contains states that are orthogonal then there is no error, the average success probability will be always unity. The second term in the optimal success probability represents the deviation due to the non-orthogonal nature of the states involved. To our knowledge such a closed form expression for total or average success probability of distinguishing N non-orthogonal states has not been obtained before. This is another key result of our paper.

Coming back to the super dense coding scheme, once Alice applies d^2 unitary operators and sends the qudit to Bob, Bob has d^2 non-orthogonal states $\{|\Psi_{mn}\rangle\}$. The task for Bob is how well he can distinguish these states. First, Bob performs d orthogonal projections $P_m = \sum_k |k \oplus m\rangle\langle k \oplus m| \otimes |k\rangle\langle k|$, $m = 0, 1, \ldots d-1$ that projects these states onto d mutually orthogonal subspaces. Now within each subspace there are d non-orthogonal but linearly indepedent states that Bob has to distinguish. For example, if Bob projects onto P_0, then this subspace has $\{|\Psi_{0n}\rangle\}$ states which are all non-orthogonal. He can perform a unitary operation on two qudits and an ancilla state. After post selection of measurement outcome (in other words he is performing a POVM) he can extract $2 \log_2 d$ bits of information with certain non-zero probability of success. The average success probability of distinguishing d states within a subspace (let us say for $m = 0$) can be obtained from eqn. (14) by putting $N = d$

$$\bar{\gamma} \leq \frac{d}{(d-1)} - \frac{1}{d(d-1)} \sum_{n,n'=0}^{d-1} |\langle \Psi_{0n} | \Psi_{0n'} \rangle| \tag{16}$$

Alternately, the average success probability with which he can distinguish d non-orthogonal states is given by

$$\bar{\gamma} \leq 1 - \frac{1}{d(d-1)} \sum_{\substack{n,n'=0 \\ n \neq n'}}^{d-1} |\sum_{k=0}^{d-1} p_k e^{-2\pi i k(n-n')/d}| \tag{17}$$

The protocol works for other subspaces also with the average success probability as given in (16). Thus by sharing a partially entangled state Alice can communicate $2 \log_2 d$ classical bits to Bob with a non-zero success probability. This completes the super dense coding scheme with any higher dimensional entangled state.

Just as a consistency check one can also obtain the average success probability of performing super dense coding with qubits. Recalling from previous section, we note that after

Bob performs projection onto two subspaces he has only two non-orthogonal states within each subspace, so $N = 2$. Then the above relation reduces to $\bar{\gamma} \leq 1 - (p_0 - p_1)$. Identifying $p_0 = L^2$ and $p_1 = L^2\ell^2$ we have $\bar{\gamma} \leq 1 - L^2(1-\ell^2) = 2\ell^2/(1+\ell^2)$ which was obtained in the section 2.

As a further illustration of the general result for $d \times d$, let us consider probabilistic dense coding for qutrits, i.e., for $d = 3$. In this case Alice and Bob possess one qutrit each. Alice can encode $2\log_2 3$ bits of information using unitary operators $\mathcal{U}_{mn}$, where $m, n = 0, 1, 2$. These operators will lead to nine linearly independent states all of which are not orthogonal. Although these nine states are not mutually orthogonal, they can be divided into three subspaces, which are mutually orthogonal. The states in these subspaces are spanned by basis states $\{|00\rangle, |11\rangle, |22\rangle\}$, $\{|10\rangle, |21\rangle, |02\rangle\}$ and $\{|20\rangle, |01\rangle, |12\rangle\}$ respectively. By making appropriate Von-Neumann measurements, Bob can distinguish these three classes. But he cannot perfectly distinguish the states within a class, since those states are not orthogonal. However as the states within a particular class are linearly independent, we can use formula (15) to find the probability for Bob to be able to distinguish these states within a class. This probability will be the same for all the three subspaces. Let us consider the states within the class $\{|\Psi_{0n}\rangle\}$, $(n = 0, 1, 2)$. These states can be distinguished with a success probability

$$\bar{\gamma} \leq 1 - \sqrt{(\frac{3}{2}p_0 - \frac{1}{2})^2 + \frac{3}{4}(p_1 - p_2)^2} \tag{18}$$

Thus, by sharing a 3×3 NME state Alice can communicate $2\log_2 3$ classical bits with a success probability given in (18). As expected for ME states, $p_0 = p_1 = p_2 = 1/3$ and hence $\bar{\gamma}_{opt} = 1$ which reduces to the standard case.

4 Super dense coding with more entanglement

Since the classical capacity of the communication channel enhances due to the presence of prior entanglement, one may wonder if the presence of more entanglement can help to enhance the probability of successful dense coding when Alice and Bob share a NME. Specifically, as a result of the above discussion, we ask the question whether one can send $2\log_2 d$ bits of classical information by encoding d^2 messages in a quDit (a quantum system with D-dimensional Hilbert space), and sharing a $D \times D$ partially entangled state where $D > d$. It may be recalled that recently Gour [24] has investigated the question of teleporting a d level quantum system faithfully using a higher dimensional (say $D \times D$ with $D > d$) partially entangled state. He has found that the classical communication cost of teleporting a qudit is at least $\log_2(dD)$ bits.

Let the state that Alice and Bob have shared is given by

$$|\Phi\rangle = \sum_{\mu=0}^{D-1} \sqrt{p_\mu}|\mu\rangle|\mu\rangle \tag{19}$$

Alice encodes her d^2 messages by applying the unitary operators $\mathcal{U}_{mn}$. Here we have to enlarge the definition of these operators. The unitary operators $\mathcal{U}_{mn}$ act as it is given earlier by Eqns. (7 - 8) for $m, n = 0, 1, ..., d-1$, while for the rest of the indices, they act as identity operators. In other words, U acts on the first d-dimensional subspace as defined

earlier and as an identity for the rest of the subspace. After Alice's operation the two-quDit state transforms as

$$|\Phi\rangle \rightarrow \mathcal{U}_{mn}|\Phi\rangle = |\Phi_{mn}\rangle = |\tilde{\Psi}_{mn}\rangle + \sum_{\mu=d}^{D-1} \sqrt{p_\mu}|\mu\rangle|\mu\rangle \tag{20}$$

It can be shown that $|\tilde{\Psi}_{mn}\rangle$ are not orthogonal to each other and also they are not normalized. Similarly the $|\Phi_{mn}\rangle$ are non-orthogonal and satisfy

$$\langle\Phi_{mn}|\Phi_{m'n'}\rangle = \sum_{k=0}^{d-1} p_k e^{-2\pi ik(n-n')/d}\delta_{mm'} + \sum_{\mu=d}^{D-1} p_\mu. \tag{21}$$

Let us just note that if the shared entangled state $|\Phi\rangle$ is ME, then $p_\mu = 1/D$ and the above orthogonality relation reduces to

$$\langle\Phi_{mn}|\Phi_{m'n'}\rangle = \frac{d}{D}\delta_{mm'}\delta_{nn'} + \frac{(D-d)}{D}. \tag{22}$$

Now Alice sends her D dimensional particle which encodes her d^2 messages. So basically she has not utilized the total Hilbert space of her particle. Bob after receiving Alice's particle has the task of distinguishing effectively d^2 quantum states $\{|\Phi_{mn}\rangle\}$. Since the states are not orthogonal, we can conclude that he cannot discriminate them with certainty and so deterministic dense coding is not possible. He can however extract $2\log_2 d$ bits of information in a probabilistic manner. First, Bob performs the von Neumann projection onto the d subspaces. Then, he performs POVM's within each subspace to distinguish d non-orthogonal states with an average success probability

$$\bar{\gamma} \leq 1 - \frac{1}{d(d-1)} \sum_{\substack{n,n'=0 \\ n\neq n'}}^{d-1} |\sum_{k=0}^{d-1} p_k e^{-2\pi ik(n-n')/d} + \sum_{\mu=d}^{D-1} p_\mu| \tag{23}$$

Thus, with a non-zero success probability as given in (23) Bob can extract $2\log_2 d$ bits of information. But we cannot compare here whether it is smaller or larger than the previous one (when Alice and Bob share a $d \times d$ partial entangled state) in general.

The situation is more interesting with maximally entangled states. Let us concentrate on the case where Alice and Bob have shared a $D \times D$ maximally entangled state. Then we have a simple expression for average success probability which is given by

$$\bar{\gamma} \leq \frac{d}{D}. \tag{24}$$

This shows that if we use a $D \times D$ maximally entangled state and want to communicate $2\log_2 d$ classical bits then we can do so with an optimal probability d/D. This simple expression gives many new insights indeed. Note that when $d = D$, we have $\bar{\gamma}_{opt} = 1$ which is the standard case. However, if we use higher dimensional entangled states as shared resource, then the average success probability is less than one. As we go to higher dimensions i.e., $D > d$, then the average success probability of distinguishing non orthogonal states decreases. Thus we can say that the presence of more entanglement in shared states may not be always useful (as shown in this scheme). However, it may be too early to claim that "the use of more entanglement does not help". One could imagine other different

schemes where more entanglement may be useful for dense coding. However, this is not clear at the moment. For example, Alice could apply D^2 unitary operations and choose d^2 of them to encode her messages that are best distinguishable for Bob. But this is something which needs to be explored in future. Also, (24) shows that in order to send $2\log_2 d$ classical bits in a deterministic fashion (i.e., with probability one) we must have $\log_2 d$ ebits as a shared resource.

We reiterate that the motivation for Section 4 was to see if the mere presence of more entanglement can enhance the success probability. It is not the question of using or not using those extra dimensions. (One may recall that in quantum information processing, sometimes mere presence of the entanglement can act as a catalyst.) Of course, we know that if we use all the entanglement then we can send more classical bits as entanglement increases in determinsitci setting. The interesting finding here is that just the mere presence of more entanglement may *decrease* the probability of the success. One would have expected this probability to remain at least the same.

5 Discussions and Conclusions

Ideal EPR pairs are very useful for deterministic dense coding, teleportation, remote state preparation and so on. However, given arbitrary entangled state of two qubits or qudits can we do these important tasks. In this paper we have investigated the possibility of performing super dense coding in a probabilistic manner using a non-maximally entangled state as a resource. We have shown that $2\log_2 2$ classical bits can be sent probabilistically by sharing an entangled state that has less than $\log_2 2$ ebits of entanglement. Generalizing to higher dimensions, we have shown that $2\log_2 d$ classical bits can be sent in a probabilistic manner using a shared entangled state that has less than $\log_2 d$ amount of ebits. The success probability of performing super dense coding is related to the optimal success probability of distinguishing linearly independent non-orthogonal states. The expressions for average success probability are given for qubit, qutrit as well as for qudit cases. As far as we know the results presented are new and do not exist in the literature.

It may be remarked that we could first convert a NME state to a ME state with certain probability of success [25] and then follow Bennett-Wisner (BW) protocol [1]. In this scenario the probability of successful conversion of NME to ME state will be the success probability of dense coding. However there are two subtle differences between this case and our protocol. If we first perform local filtering and follow BW protocol, the probability enters at the stage of conversion of NME state to a ME state. Then two classical bits are transferred deterministically. Whereas in our case, probability enters at a different stage. In our protocol two classical bits are transfered but recovered probabilistically (due to non-orthogonal nature of the states). Another important difference is the following. In the first case, if we fail in converting NME state to a ME state, we cannot use the Bennett-Wisner protocol. Whereas in our case, whatever is the NME state, we can send two classical bits to Bob probabilistically. That is, in our case we will always be able to run the protocol. Therefore, our protocol is a single shot super dense coding protocol for non-maximally entangled resource without first converting to a maximally entangled pair.

In addition to superdense coding for qubit and qudit, we have also asked that if one uses a non-maximally entangled state in higher dimensions (say $D \times D$), then can one send $2\log_2 d, (D > d)$ classical bits with a higher probability of success? Interestingly, we find

that the answer to the above question may be negative: more may not be always better. But there can be other schemes where entanglement might help. We have shown that if we use a maximally entangled state in $D \times D$ dimensions, then surprisingly the success probability of performing super dense coding decreases with increasing D. Our analysis also explains that to send $2 \log_2 d$ classical bits in a deterministic fashion why one needs exactly $\log_2 d$ ebits and not more, not less.

In future it will be interesting to investigate if one can send $\log_2(dD)$ classical bits by sharing a $D \times D$ partial entangled state and sending a qudit. This would be reverse of teleportation described in [24]. A priori it does not look like possible, but it is worth exploring. Also one can see if the probabilistic super dense coding scheme can be generalized for mixed entangled states. That will shed light on the relation between the classical communication capacity and ability to distinguish mixed entangled states. It will be of great value to generalize our protocol for continuous variable quantum systems. We hope that the probabilistic super dense coding protocol can be verified experimentally with the present technology.

Acknowledgments: AKP thanks A. Chefles for useful remarks. PP thanks G. Kar for discussions and acknowledges financial assistance from DST under the SERC Fast Track Proposal scheme for young scientists.

References

[1] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[3] A. K. Pati, Phys. Rev. A, **63** , 014320 (2001).

[4] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).

[5] S. Bose, V. Vedral and P.L. Knight, Phys. Rev. A **57**, 822 (1998).

[6] S. L. Braunstein and H. J. Kimble, Phys. Rev. A **61**, 042302 (2002).

[7] S. L. Braunstein and A. K. Pati, Edt. *Quantum Information with Continuous Variables*, Kluwer Academic Publisher, The Netherlands, 2003.

[8] A. Barenco and A. Ekert, J. Mod. Opt. **42**, 1253 (1995).

[9] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland and W.K. Wootters, Phys. Rev. A **54**, 1869 (1996).

[10] S. Bose, M. B. Plenio, and V. Vedral, J. Mod. Opt. **47**, 291 (2000).

[11] S. Mozes, B. Reznik and J. Oppenheim, quant-ph/0403189.

[12] A. Chefles, M. Sasaki, Phys. Rev. A **67**, 032112 (2003).

[13] P. Agrawal and A. K. Pati, Phys. Lett. A **305**, 12 (2002).

[14] S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).

[15] A. Ekert and P. Knight, Am. J. Phys. **63**, 415 (1995);

[16] A. K. Pati, Phys. Lett. A **278**, 118 (2000).

[17] A. K. Ekert, B. Huttner, G. M. Palma, and A. Peres, Phys. Rev. A **50**, 1047 (1994).

[18] T. Mor and P. Horodecki, quant-ph/9906039.

[19] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987).

[20] A. Peres, Phys. Lett. A **128**, 19 (1988).

[21] A. Chefles, Phys. Lett. A **239**, 339 (1998).

[22] A. Chefles, Contemp. Phys. **41**, 401 (2000).

[23] L-M Duan and G-C Guo, Phys. Rev. Lett. **80**, 4999 (1998).

[24] G. Gour, Phys. Rev. A **70**, 042301 (2004).

[25] N. Gisin, Phys. Lett. A **210**, 151 (1996).