Nonparametric Inference for a Class of Stochastic Partial Differential Equations II

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Abstract. Consider the stochastic partial differential equation

$$\mathrm{d}u_\epsilon(t,x) = \theta(t)\triangle u_\epsilon(t,x)\,\mathrm{d}t + \epsilon\,\mathrm{d}W_Q(t,x), \qquad 0 \leqslant t \leqslant T,$$

where $\triangle = \partial^2/\partial x^2$, $\theta \in \Theta$ and Θ is a class of positive valued functions. We obtain an estimator for the linear multiplier $\theta(t)$ and establish the consistency, rate of convergence and asymptotic normality of this estimator as $\epsilon \to 0$.

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1. Introduction

Stochastic partial differential equations (SPDE) are used for stochastic modelling, for instance, in the study of neuronal behaviour in neurophysiology and in building stochastic models of turbulence (cf. Kallianpur and Xiong [5]. The theory of SPDE is investigated in Ito [4], Rozovskii [13] and De prato and Zabczyk [1] among others.

Huebner et al. [2] started the investigation of maximum likelihood estimation of parameters for a class of SPDE and extended their results to parabolic SPDE in Huebner and Rozovskii [3]. Bernstein-von Mises theorems were developed for such SPDE in Prakasa Rao [8, 12] following the techniques in Prakasa Rao [7]. Asymptotic properties of Bayes estimators of parameters for SPDE were discussed in Prakasa Rao [8, 12]. Statistical inference for diffusion type processes and semimartingales is studied in Prakasa Rao [9, 10].

The problem of estimation of a linear multiplier for a class of SPDE which generate measures which are absolutely continuous with respect to each other is discussed in Prakasa Rao [11] using the methods of nonparametric inference following the approach of Kutoyants [6]. We now discuss a similar problem for a class of SPDE which generate measures which are singular with respect to each other.

2. Stochastic PDE with Linear Multiplier

2.1. PRELIMINARIES

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_\epsilon(t, x), 0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$ governed by the SPDE

$$\mathrm{d}u_\epsilon(t, x) = \theta(t)\Delta u_\epsilon(t, x)\,\mathrm{d}t + \epsilon\,\mathrm{d}W_Q(t, x), \tag{2.1}$$

where $\Delta = \partial^2/\partial x^2$. Suppose that $\epsilon \to 0$ and $\theta \in \Theta$ where Θ is a class of positive valued functions $\theta(t), 0 \leqslant t \leqslant T$ uniformly bounded , k times continuously differentiable and that the k-th derivative $\theta^{(k)}(.)$ satisfies the Lipschitz condition of order $\alpha \in (0, 1]$, that is,

$$|\theta^{(k)}(t) - \theta^{(k)}(s)| \leqslant |t - s|^\alpha, \qquad \beta = k + \alpha. \tag{2.2}$$

Further suppose the initial and the boundary conditions are given by

$$\begin{aligned} u_\epsilon(0, x) &= f(x), \qquad f \in L_2[0, 1], \\ u_\epsilon(t, 0) &= u_\epsilon(t, 1) = 0, \qquad 0 \leqslant t \leqslant T \end{aligned} \tag{2.3}$$

and Q is the nuclear covariance operator for the Wiener process $W_Q(t, x)$ taking values in $L_2[0, 1]$ so that $W_Q(t, x) = Q^{1/2}W(t, x)$ and $W(t, x)$ is a cylindrical Brownian motion in $L_2[0, 1]$. Then, it is known that (cf. Rozovskii [13], Kallianpur and Xiong [5])

$$W_Q(t, x) = \sum_{i=1}^{\infty} q_i^{1/2} e_i(x) W_i(t) \text{ a.s.}, \tag{2.4}$$

where $\{W_i(t), 0 \leqslant t \leqslant T\}, i \geqslant 1$ are independent one-dimensional standard Wiener processes and $\{e_i\}$ is a complete orthonormal system in $L_2[0, 1]$ consisting of eigen vectors of Q and $\{q_i\}$ eigen values of Q.

We assume that the operator Q is a special covariance operator Q with $e_k = \sin(k\pi x), k \geqslant 1$ and $\lambda_k = (\pi k)^2, k \geqslant 1$. Then $\{e_k\}$ is a complete orthonormal system with eigen values $q_i = (1 + \lambda_i)^{-1}, i \geqslant 1$ for the operator Q and $Q = (I - \Delta)^{-1}$. Note that

$$\mathrm{d}W_Q = Q^{1/2}\,\mathrm{d}W. \tag{2.5}$$

We define a solution $u_\epsilon(t, x)$ of (2.1) as a formal sum

$$u_\epsilon(t, x) = \sum_{i=1}^{\infty} u_{i\epsilon}(t) e_i(x) \tag{2.6}$$

(cf. Rozovskii [13]). It can be checked that the Fourier coefficient $u_{i\epsilon}(t)$ satisfies the stochastic differential equation

$$\mathrm{d}u_{i\epsilon}(t) = -\theta(t)\lambda_i u_{i\epsilon}(t)\,\mathrm{d}t + \frac{\epsilon}{\sqrt{\lambda_i + 1}}\,\mathrm{d}W_i(t), \qquad 0 \leqslant t \leqslant T \tag{2.7}$$

with the initial condition

$$u_{i\epsilon}(0) = v_i,\ \ v_i = \int_0^1 f(x)e_i(x)\,\mathrm{d}x. \tag{2.8}$$

We assume that the initial function f in (2.3) is such that

$$v_i = \int_0^1 f(x)e_i(x)\,\mathrm{d}x > 0, \qquad i \geqslant 1.$$

2.2. ESTIMATION

We now consider the problem of estimation of the function $\theta(t)$, $0 \leqslant t \leqslant T$ based on the observation of the Fourier coefficients $u_{i\epsilon}(t)$, $1 \leqslant i \leqslant N$ over $[0, T]$ or equivalently the projection $u_\epsilon^{(N)}(t, x)$ of the process $u_\epsilon(t, x)$ onto the subspace spanned by $\{e_1, \dots, e_N\}$ in $L_2[0, 1]$.

We will at first construct an estimator of $\theta(.)$ based on the path $\{u_{i\epsilon}(t), 0 \leqslant t \leqslant T\}$. Our technique follows the methods in Kutoyants [6], p.155.

Let us suppose that

$$\sup_{\theta\in\Theta} \sup_{0 \leqslant t \leqslant T} \theta(t) \leqslant L_0. \tag{2.9}$$

Consider the differential equation

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\theta(t)\lambda_i u_i(t), \qquad u_i(0) = v_i, \qquad 0 \leqslant t \leqslant T. \tag{2.10}$$

It is easy to see that

$$u_i(t) = v_i e^{-\lambda_i \int_0^t \theta(s)\,\mathrm{d}s}, \qquad 0 \leqslant t \leqslant T$$

and hence

$$u_i(t) \geqslant v_i e^{-M_i t}, \qquad 0 \leqslant t \leqslant T, \tag{2.11}$$

where

$$M_i = L_0 \lambda_i. \tag{2.12}$$

From Lemma 1.13 of Kutoyants [6], it follows that

$$\sup_{0 \leqslant s \leqslant t} |u_{i\epsilon}(s) - u_i(s)| \leqslant \frac{\epsilon}{\sqrt{\lambda_i + 1}} e^{M_i t} \sup_{0 \leqslant s \leqslant t} |W_i(s)| \tag{2.13}$$

almost surely. Let

$$A_t^{(i)} = \left\{\omega\colon \inf_{0 \leqslant s \leqslant t} u_{i\epsilon}(s) \geqslant \tfrac{1}{2} v_i e^{-M_i t}\right\} \tag{2.14}$$

and $A_i = A_T^{(i)}$. Note that $A_t^{(i)}$ contains the set A_i for $0 \leqslant t \leqslant T$.

Define the process $\{Y_{i\epsilon}(t), 0 \leqslant t \leqslant T\}$ by the stochastic differential equation

$$\begin{aligned}\mathrm{d}Y_{i\epsilon}(t) &= -\frac{\epsilon^2}{2(\lambda_i+1)} u_{i\epsilon}^{-2}(t)\chi(A_t^{(i)})\,\mathrm{d}t + \\ &\quad + u_{i\epsilon}^{-1}(t)\chi(A_t^{(i)})\,\mathrm{d}u_{i\epsilon}(t), \qquad 0 \leqslant t \leqslant T, \end{aligned} \tag{2.15}$$

where $\chi(E)$ denotes the indicator function of a set E. Let $\phi_\epsilon \to 0$ as $\epsilon \to 0$ and define

$$\hat{\theta}_{i\epsilon}(t)\lambda_i = -\{\chi(A_i)\phi_\epsilon^{-1} \int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right) \mathrm{d}Y_{i\epsilon}(s)\}, \tag{2.16}$$

where $G(.)$ is a bounded kernel with finite support, that is, there exist constants a and b such that

$$\int_a^b G(u)\,\mathrm{d}u = 1, \qquad G(u) = 0 \quad \text{for} \quad u < a \quad \text{and} \quad u > b. \tag{2.17}$$

We suppose that $a < 0$ and $b > 0$. Further suppose that the kernel $G(.)$ satisfies the additional condition

$$\int_{-\infty}^{\infty} G(u)u^j\,\mathrm{d}u = 0, \qquad j = 1, \ldots, k. \tag{2.18}$$

Note that

$$\begin{aligned}-\hat{\theta}_{i\epsilon}(t)\lambda_i &= \chi(A_i)\phi_\epsilon^{-1} \int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right) \mathrm{d}Y_{i\epsilon}(s), \\ &= \chi(A_i)\phi_\epsilon^{-1} \int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right) \times \\ &\quad \times \left[-\theta(s)\lambda_i - \frac{\epsilon^2}{2(\lambda_i+1)} u_{i\epsilon}^{-2}(s)\right] \chi(A_s^{(i)})\,\mathrm{d}s + \\ &\quad + \chi(A_i)\phi_\epsilon^{-1} \int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right) \times \\ &\quad \times \frac{\epsilon}{\sqrt{\lambda_i+1}} u_{i\epsilon}^{-1}(s)\chi(A_s^{(i)})\,\mathrm{d}W_i(s). \end{aligned} \tag{2.19}$$

Hence

$$\begin{aligned}-E[\hat{\theta}_{i\epsilon}(t)\lambda_i] &= E\left[\chi(A_i)\phi_\epsilon^{-1} \int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)(-\theta(s)\lambda_i)\,\mathrm{d}s\right] - \\ &\quad - E\left[\chi(A_i)\phi_\epsilon^{-1} \int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right) \times \right. \\ &\quad \left. \times \frac{\epsilon^2}{2(\lambda_i+1)} u_{i\epsilon}^{-2}(s)\chi(A_s^{(i)})\,\mathrm{d}s\right] + \end{aligned}$$

$$+ E\left[\chi(A_i)\phi_\epsilon^{-1}\int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)\times \right.$$
$$\left.\times \frac{\epsilon}{\sqrt{\lambda_i+1}}u_{i\epsilon}^{-1}(s)\chi(A_s^{(i)})\,\mathrm{d}W_i(s)\right].$$

Note that

$$\begin{aligned}
&\left(E\left[\chi(A_i)\phi_\epsilon^{-1}\int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)\frac{\epsilon}{\sqrt{\lambda_i+1}}u_{i\epsilon}^{-1}(s)\chi(A_s^{(i)})\,\mathrm{d}W_i(s)\right]\right)^2\\
&\quad\leqslant E\left[\phi_\epsilon^{-2}\left\{\int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)\frac{\epsilon}{\sqrt{\lambda_i+1}}u_{i\epsilon}^{-1}(s)\chi(A_s^{(i)})\,\mathrm{d}W_i(s)\right\}^2\right]\\
&\quad\leqslant \phi_\epsilon^{-2}\int_0^T G^2\left(\frac{t-s}{\phi_\epsilon}\right)\frac{\epsilon^2}{\lambda_i+1}E[u_{i\epsilon}^{-2}(s)\chi(A_s^{(i)})]\,\mathrm{d}s,\\
&\quad= J_{1i\epsilon}^2 \quad \text{(say)}.
\end{aligned}$$

Therefore, for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
&E[-\hat{\theta}_{i\epsilon}(t)\lambda_i + \theta(t)\lambda_i]\\
&\quad= E\left[\chi(A_i)\phi_\epsilon^{-1}\int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)(-\theta(s)\lambda_i+\theta(t)\lambda_i)\,\mathrm{d}s\right]-\\
&\qquad - E[\chi(A_i^c)(-\theta(t)\lambda_i)]-\\
&\qquad - E\left[\chi(A_i)\phi_\epsilon^{-1}\int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)\frac{\epsilon^2}{2(\lambda_i+1)}u_{i\epsilon}^{-2}(s)\chi(A_s^{(i)})\,\mathrm{d}s\right]+\\
&\qquad + \mathrm{O}(J_{1i\epsilon})
\end{aligned}\tag{2.20}$$

since, for $0 < t < T$,

$$\phi_\epsilon^{-1}\int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)\mathrm{d}s = 1 \tag{2.21}$$

for sufficiently small $\epsilon > 0$ by the conditions imposed on the kernel G. Therefore, for $0 < t < T$, for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
&|E[-\hat{\theta}_{i\epsilon}(t)\lambda_i + \theta(t)\lambda_i]|^2\\
&\quad\leqslant 4\left\{\phi_\epsilon^{-1}\int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)(-\theta(s)\lambda_i+\theta(t)\lambda_i)\,\mathrm{d}s\right\}^2+\\
&\qquad + 4[P(A_i^c)]^2(\theta(t)\lambda_i)^2+\\
&\qquad + 4\left(E\left[\chi(A_i)\phi_\epsilon^{-1}\int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right)\frac{\epsilon^2}{2(\lambda_i+1)}u_{i\epsilon}^{-2}(s)\chi(A_s^{(i)})\,\mathrm{d}s\right]\right)^2+\\
&\qquad + \mathrm{O}(J_{1i\epsilon}^2).
\end{aligned}\tag{2.22}$$

Applying the Taylor series expansion and properties of the kernel $G(.)$ and the function $\theta(.)$, it is easy to see that the first term is bounded by

$$C_1\lambda_i^2\phi_\epsilon^{2\beta}\left\{\int_{-\infty}^{\infty}|G(u)u^\beta|\,\mathrm{d}u\right\}^2, \tag{2.23}$$

where C_1 is a constant depending only on the the constants L_0 in (2.9) and the smoothness parameter k of $\theta(.)$. Note that

$$\begin{aligned}
P(A_i^c) &= P\left\{\inf_{0\leqslant t\leqslant T} u_{i\epsilon}(t) < \frac{1}{2}v_i e^{-M_iT}\right\}\\
&\leqslant P\left\{\inf_{0\leqslant t\leqslant T}[u_{i\epsilon}(t)-u_i(t)] + \inf_{0\leqslant t\leqslant T} u_i(t) < \frac{1}{2}v_i e^{-M_iT}\right\}\\
&\leqslant P\left\{\inf_{0\leqslant t\leqslant T}[u_{i\epsilon}(t)-u_i(t)] < -\frac{1}{2}v_i e^{-M_iT}\right\} \qquad \text{(from (2.11))}\\
&\leqslant P\left\{\sup_{0\leqslant t\leqslant T}|u_{i\epsilon}(t)-u_i(t)| > \frac{1}{2}v_i e^{-M_iT}\right\}\\
&\leqslant 2\exp\left\{-\frac{v_i^2e^{-4M_iT}(\lambda_i+1)}{8\epsilon^2T}\right\},
\end{aligned}$$

since

$$P\left(\sup_{0\leqslant t\leqslant T}|W_i(t)|>\alpha\right)\leqslant \min\left(2,\frac{4}{\alpha}\sqrt{\frac{T}{2\pi}}\right)e^{-\frac{\alpha^2}{2T}}$$

from Kutoyants [6], p. 28. The third term is bounded by

$$C_2\frac{\epsilon^4}{(\lambda_i+1)^2}v_i^{-2}e^{2M_iT}\left\{\int_{-\infty}^{\infty}|G(u)|\,\mathrm{d}u\right\}^2, \tag{2.24}$$

where C_2 is an absolute constant. Relations (2.22)–(2.24) show that

$$\begin{aligned}
|\lambda_i^2E[\hat\theta_{i\epsilon}(t)-\theta(t)]|^2 \leqslant\; & C_3\Bigg[\lambda_i^2\phi_\epsilon^{2\beta}\left\{\int_{-\infty}^{\infty}|G(u)u^\beta|\,\mathrm{d}u\right\}^2+\\
&+\exp\left\{-\frac{2v_i^2e^{-4M_iT}(\lambda_i+1)}{8\epsilon^2T}\right\}(\theta(t)\lambda_i)^2+\\
&+\frac{\epsilon^4}{(\lambda_i+1)^2}v_i^{-2}e^{2M_iT}\left\{\int_{-\infty}^{\infty}|G(u)|\,\mathrm{d}u\right\}^2\Bigg]+\\
&+\mathrm{O}(J_{1i\epsilon}^2),
\end{aligned} \tag{2.25}$$

where C_3 is an absolute constant. Hence

$$|E[\hat\theta_{i\epsilon}(t)-\theta(t)]|^2\leqslant C_4[\phi_\epsilon^{2\beta}+e^{-d_i\epsilon^{-2}}\theta(t)^2+\epsilon^4q_i+J_{1i\epsilon}^2], \tag{2.26}$$

where C_4 is a constant depending on the kernel $G(.)$ and the Lipschitz constant L_0,

$$d_i = \frac{v_i^2 e^{-4M_i T}(\lambda_i + 1)}{4T} \tag{2.27}$$

and

$$q_i = \frac{e^{2M_i T}}{(\lambda_i + 1)^2 \lambda_i^2 v_i^2}. \tag{2.28}$$

Following computations as given above (cf. Kutoyants [6], p. 157), we can show that

$$E[\hat{\theta}_{i\epsilon}(t) - \theta(t)]^2 \leqslant C_5[\phi_\epsilon^{2\beta} + e^{-d_i \epsilon^{-2}} \theta(t)^2 + \epsilon^4 q_i + \epsilon^2 \phi_\epsilon^{-1} q_i + J_{1i\epsilon}^2], \tag{2.29}$$

where C_5 is a constant depending on the kernel $G(.)$ and the Lipschitz constant L_0. Choosing ϕ_ϵ such that

$$\phi_\epsilon^{2\beta} = \epsilon^2 \phi_\epsilon^{-1},$$

we obtain that $\phi_\epsilon = \epsilon^{2/(2\beta+1)}$ and we have

$$\begin{aligned} &E[\hat{\theta}_{i\epsilon}(t) - \theta(t)]^2 \\ &\quad \leqslant C_5[\epsilon^{(4\beta/2\beta+1)} + e^{-d_i \epsilon^{-2}} \theta(t)^2 + \epsilon^4 q_i + \epsilon^{(4\beta/2\beta+1)} k_i + J_{1i\epsilon}^2] \end{aligned} \tag{2.30}$$

and

$$(E[\hat{\theta}_{i\epsilon}(t) - \theta(t)])^2 \leqslant C_4[\epsilon^{(4\beta/2\beta+1)} + e^{-d_i \epsilon^{-2}} \theta(t)^2 + \epsilon^4 q_i + J_{1i\epsilon}^2]. \tag{2.31}$$

Note that $\theta_{i\epsilon}(t)$, $1 \leqslant i \leqslant N$ are independent estimators of $\theta(t)$ since the processes W_i, $1 \leqslant i \leqslant N$ are independent Wiener processes. The above inequalities imply that

$$\begin{aligned} &\sup_{1 \leqslant i \leqslant N} E[\hat{\theta}_{i\epsilon}(t) - \theta(t)]^2 \\ &\quad \leqslant C_6[\epsilon^{(4\beta/2\beta+1)} + e^{-(\inf_{1 \leqslant i \leqslant N} d_i)\epsilon^{-2}} \theta(t)^2 + \epsilon^4 \sup_{1 \leqslant i \leqslant N} q_i + \\ &\qquad + \epsilon^{(4\beta/2\beta+1)} \sup_{1 \leqslant i \leqslant N} q_i + \sup_{1 \leqslant i \leqslant N} J_{1i\epsilon}^2]. \end{aligned} \tag{2.32}$$

Note that

$$\sup_{1 \leqslant i \leqslant N} q_i \leqslant \frac{e^{2(L_0 + N^2\pi^2)T}}{\pi^4 \inf_{1 \leqslant i \leqslant N} v_i^2} = \beta_N \text{ (say)} \tag{2.33}$$

and

$$\inf_{1 \leqslant i \leqslant N} d_i \geqslant \left(\inf_{1 \leqslant i \leqslant N} v_i^2 \right) \frac{e^{-4L_0 T}}{4T} = \gamma_N \text{ (say)}. \tag{2.34}$$

Therefore

$$\sup_{1 \leqslant i \leqslant N} E[\hat{\theta}_{i\epsilon}(t) - \theta(t)]^2$$
$$\leqslant C_7 \left[\epsilon^{(4\beta/2\beta+1)} + e^{-\gamma_N \epsilon^{-2}} \theta(t)^2 + \epsilon^4 \beta_N + \epsilon^{(4\beta/2\beta+1)} \beta_N + \sup_{1 \leqslant i \leqslant N} J_{1i\epsilon}^2 \right]. \tag{2.35}$$

In particular

$$\sup_{1 \leqslant i \leqslant N} \operatorname{Var}(\hat{\theta}_{i\epsilon}(t))$$
$$\leqslant C_7 \left[\epsilon^{(4\beta/2\beta+1)} + e^{-\gamma_N \epsilon^{-2}} \theta(t)^2 + \epsilon^4 \beta_N + \epsilon^{(4\beta/2\beta+1)} \beta_N + \sup_{1 \leqslant i \leqslant N} J_{1i\epsilon}^2 \right]. \tag{2.36}$$

We assume that the following conditions hold for $1 \leqslant i \leqslant N$: Let $\gamma_\epsilon = \epsilon^{(-2\beta/2\beta+1)}$. Suppose that

$$(C_1) \gamma_\epsilon^2 J_{1i\epsilon}^2 \to \frac{1}{u_i^2(t)(\lambda_i + 1)} \int_{-\infty}^{\infty} G^2(u)\, \mathrm{d}u \text{ as } \epsilon \to 0;$$

Under the above condition, it follows that the estimators $\theta_{i\epsilon}(t)$, $1 \leqslant i \leqslant N$ are independent estimators of $\theta(t)$ such that

$$\sup_{1 \leqslant i \leqslant N} |E[\hat{\theta}_{i\epsilon}(t) - \theta(t)]| \leqslant C_8 \epsilon^{(2\beta/2\beta+1)} \tag{2.37}$$

and

$$\sup_{1 \leqslant i \leqslant N} E[\hat{\theta}_{i\epsilon}(t) - \theta(t)]^2 \leqslant C_9 \epsilon^{(4\beta/2\beta+1)}, \tag{2.38}$$

where C_8 and C_9 are constants depending on the kernel $G(.)$, the Lipschitz constant L_0 and N. Note that the estimators $\theta_{i\epsilon}(t)$, $1 \leqslant i \leqslant N$ are the best estimators of $\theta(t)$ as far as the rate of mean square error are concerned by Theorem 4.6 in Kutoyants [6]. We now combine these estimators in an optimum fashion to get an estimator using all the information available.

It is easy to check that

$$\gamma_\epsilon[-\hat{\theta}_{i\epsilon}(t)\lambda_i + \theta(t)\lambda_i]$$
$$= \chi(A_i)\gamma_\epsilon \phi_\epsilon^{-1} \int_0^T G\left(\frac{t-s}{\phi_\epsilon}\right) \frac{\epsilon}{\sqrt{\lambda_i + 1}} u_{i\epsilon}^{-1}(s) \chi(A_s^{(i)})\, \mathrm{d}W_i(s) + \tilde{J}_{2i\epsilon}, \tag{2.39}$$

$$= \chi(A_i) \tilde{J}_{1i\epsilon} + \tilde{J}_{2i\epsilon} \text{ (say)}. \tag{2.40}$$

Note that $E(\tilde{J}^2_{1i\epsilon}) = O(\gamma_\epsilon^2 J^2_{1i\epsilon})$ where

$$J^2_{1i\epsilon} = \epsilon^2\phi_\epsilon^{-2}\int_0^T G^2\left(\frac{t-s}{\phi_\epsilon}\right)\frac{1}{\lambda_i+1}E(u_{i\epsilon}^{-2}(s)\chi(A_s^{(i)}))\,\mathrm{d}s \tag{2.41}$$

as defined earlier. In addition to the condition (C_1) , assume that

$$(C_2)\,\tilde{J}_{2i\epsilon} = o_p(1) \quad \text{as} \quad \epsilon \to 0$$

for $1 \leqslant i \leqslant N$.

Since $P(A_i) \to 1$ as $\epsilon \to 0$, it follows by the central limit theorem for stochastic integrals (cf.Kutoyants [6], Prakasa Rao [9]) that

$$\gamma_\epsilon[\hat{\theta}_{i\epsilon}(t) - \theta(t)] \xrightarrow{\mathcal{L}} N\left(0, \frac{1}{u_i^2(t)\lambda_i^2(\lambda_i+1)}\int_{-\infty}^{\infty} G^2(u)\,\mathrm{d}u\right) \tag{2.42}$$

as $\epsilon \to 0$ for $1 \leqslant i \leqslant N$. Define

$$\tilde{\theta}_{N\epsilon}(t) = \frac{\sum_{i=1}^N \hat{\theta}_{i\epsilon}(t)\lambda_i^2(\lambda_i+1)u_i^2(t)}{\sum_{i=1}^N \lambda_i^2(\lambda_i+1)u_i^2(t)}. \tag{2.43}$$

Note that the random variable $\tilde{\theta}_{N\epsilon}(t)$ is not an estimator of $\theta(t)$ as the functions $u_i(t)$ depend on the function $\theta(t)$. However, the random variable $\tilde{\theta}_{N\epsilon}(t)$ is a linear function of independent random variables $\hat{\theta}_{i\epsilon}(t)$, $1 \leqslant i \leqslant N$. From the earlier calculations , it can be checked that

$$\begin{aligned} E(\tilde{\theta}_{N\epsilon}(t) - \theta(t))^2 &= Var(\tilde{\theta}_{N\epsilon}(t)) + (E(\tilde{\theta}_{N\epsilon}(t) - \theta(t)))^2 \\ &\leqslant C_8\epsilon^{(4\beta/2\beta+1)} + C_9\epsilon^{(4\beta/2\beta+1)} \\ &\leqslant C_{10}\epsilon^{(4\beta/2\beta+1)}. \end{aligned} \tag{2.44}$$

As a consequence, we have the following result.

PROPOSITION 2.1. *Under the conditions stated earlier, for* $0 < t < T$,

(i) $\tilde{\theta}_{N\epsilon}(t) \xrightarrow{p} \theta(t)$ *as* $\epsilon \to 0$;
(ii) $E(\tilde{\theta}_{N\epsilon}(t)) \to \theta(t)$ *as* $\epsilon \to 0$;
(iii) $\lim_{\epsilon\to 0} E(\tilde{\theta}_{N\epsilon}(t) - \theta(t))^2 \to 0$ *as* $\epsilon \to 0$;
(iv) $\limsup_{\epsilon\to 0} E(\tilde{\theta}_{N\epsilon}(t) - \theta(t))^2\epsilon^{(-4\beta/2\beta+1)} < \infty$;
(v) $\epsilon^{(-2\beta/2\beta+1)}(\tilde{\theta}_{N\epsilon}(t) - \theta(t)) \xrightarrow{\mathcal{L}} N(0, \sigma^2(t))$ as $\epsilon \to 0$,

where $N(0, \sigma^2(t))$ *denotes the normal distribution with mean zero and variance* $\sigma^2(t))$ *given by*

$$\sigma^2(t) = \frac{1}{\sum_{i=1}^N u_i^2(t)\lambda_i^2(\lambda_i+1)}\int_{-\infty}^{\infty} G^2(u)\,\mathrm{d}u. \tag{2.45}$$

Let

$$\theta^*_{N\epsilon}(t) = \frac{\sum_{i=1}^N \hat{\theta}_{i\epsilon}(t)\lambda_i^2(\lambda_i+1)\hat{u}^2_{i\epsilon}(t)}{\sum_{i=1}^N \lambda_i^2(\lambda_i+1)\hat{u}^2_{i\epsilon}(t)}, \tag{2.46}$$

where

$$\hat{u}_{i\epsilon}(t) = v_i e^{-\lambda_i \int_0^t \hat{\theta}_{i\epsilon}(s)\,\mathrm{d}s}. \tag{2.47}$$

Note that for any $1 \leqslant i \leqslant N$,

$$\begin{aligned} E\left[\int_0^t \hat{\theta}_{i\epsilon}(s)\,\mathrm{d}s - \int_0^t \theta(s)\,\mathrm{d}s\right]^2 &= E\left[\int_0^t (\hat{\theta}_{i\epsilon}(s) - \theta(s))\,\mathrm{d}s\right]^2 \\ &\leqslant E\left[t\int_0^t (\hat{\theta}_{i\epsilon}(s) - \theta(s))^2\,\mathrm{d}s\right] \\ &= t\int_0^t E[(\hat{\theta}_{i\epsilon}(s) - \theta(s))^2]\,\mathrm{d}s \\ &\leqslant C_{10}t^2\epsilon^{(4\beta/2\beta+1)} \end{aligned}$$

and hence

$$\int_0^t \hat{\theta}_{i\epsilon}(s)\,\mathrm{d}s - \int_0^t \theta(s)\,\mathrm{d}s \xrightarrow{p} 0 \quad \text{as} \quad \epsilon \to 0 \tag{2.48}$$

for $1 \leqslant i \leqslant N$. This in turn implies that, for $0 < t < T$,

$$\hat{u}_{i\epsilon}(t) \xrightarrow{p} u_i(t) \quad \text{as} \quad \epsilon \to 0 \tag{2.49}$$

for $1 \leqslant i \leqslant N$. In view of (2.30), it follows that the estimator $\theta^*_{N\epsilon}(t)$ is a consistent estimator of $\theta(t)$ for $0 < t < T$ as $\epsilon \to 0$.

THEOREM 2.2. *Under the conditions stated above, for* $0 < t < T$,

$$\theta^*_{N\epsilon}(t) \xrightarrow{p} \theta(t) \quad \textit{as} \quad \epsilon \to 0. \tag{2.50}$$

Note that

$$\begin{aligned} \gamma_\epsilon[\theta^*_{N\epsilon}(t) - \theta(t)] &= \gamma_\epsilon\left[\frac{\sum_{i=1}^N \hat{\theta}_{i\epsilon}(t)\lambda_i^2(\lambda_i+1)\hat{u}^2_{i\epsilon}(t)}{\sum_{i=1}^N \lambda_i^2(\lambda_i+1)\hat{u}^2_{i\epsilon}(t)} - \theta(t)\right], \\ &= \frac{\sum_{i=1}^N \gamma_\epsilon(\hat{\theta}_{i\epsilon}(t) - \theta(t))\lambda_i^2(\lambda_i+1)\hat{u}^2_{i\epsilon}(t)}{\sum_{i=1}^N \lambda_i^2(\lambda_i+1)\hat{u}^2_{i\epsilon}(t)}. \end{aligned}$$

Since

(i) $\gamma_\epsilon(\hat{\theta}_{i\epsilon}(t) - \theta(t)) \xrightarrow{\mathcal{L}} N(0, \sigma^2(t))$ as $\epsilon \to 0$ for $1 \leqslant i \leqslant N$,

(ii) $\hat{u}_{i\epsilon}(t) \xrightarrow{p} u_i(t) \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad 1 \leqslant i \leqslant N,$

for $0 < t < T$, and since the estimators $\hat{\theta}_{i\epsilon}(t), 1 \leqslant i \leqslant N$ are independent random variables, it follows that the estimator $\theta^*_{N\epsilon}(t)$ is asymptotically normal and we have the following theorem.

THEOREM 2.3. *Under the conditions stated earlier, for* $0 < t < T$,

$$\gamma_\epsilon(\theta^*_{N\epsilon}(t) - \theta(t)) \xrightarrow{\mathcal{L}} N(0, \sigma^2(t)) \quad as \quad \epsilon \to 0, \tag{2.51}$$

where

$$\gamma_\epsilon = \epsilon^{-(2\beta/2\beta+1)} \tag{2.52}$$

and

$$\sigma^2(t) = \frac{1}{\sum_{i=1}^N u_i^2(t)\lambda_i^2(\lambda_i + 1)} \int_{-\infty}^{\infty} G^2(u)\, \mathrm{d}u. \tag{2.53}$$

Remarks. (1) If $k = 0$ and $\beta = 1$,that is, the function $\theta(.) \in \Theta$ where Θ is the class of uniformly bounded nonnegative functions which are Lipschitzian of order one, then it follows that, for $0 < t < T$,

$$\epsilon^{-2/3}(\theta^*_{N\epsilon}(t) - \theta(t)) \xrightarrow{\mathcal{L}} N(0, \sigma^2(t)) \quad \text{as} \quad \epsilon \to 0. \tag{2.54}$$

(2) It is well known that if α_i, $1 \leqslant i \leqslant N$ are independent unbiased estimators of a parameter θ with variances σ_i^2, $1 \leqslant i \leqslant N$ respectively, then a better estimator, in the sense of smaller variance, can be obtained by taking a linear combination of α_i, $1 \leqslant i \leqslant N$ with the coefficient of α_i inversely proportional to the variance σ_i^2 and adjusting the proportionality constant so that the new estimator is also unbiased. Here the estimators $\hat{\theta}_{i\epsilon}(t)$, $1 \leqslant i \leqslant N$ are asymptotically independent unbiased estimators of $\theta(t)$ and the estimator $\theta^*_{N\epsilon}(t)$ is obtained following the above procedure so that this estimator has smaller asymptotic variance compared to the asymptotic variances of $\hat{\theta}_{i\epsilon}(t)$, $1 \leqslant i \leqslant N$.

(3) If the function $\theta(t) = \theta$ is a positive constant, then the problem of estimation of θ becomes a problem in parametric inference. It was shown in Huebner et al. [2] and Prakasa Rao [12] that the parameter θ can be recovered in the limit either by letting $\epsilon \to 0$ or equivalently $T \to \infty$ or by letting $N \to \infty$ keeping ϵ fixed either by the method of maximum likelihood estimation or by a Bayesian approach. It should be possible to study nonparametric estimation of the function $\theta(t)$ in (2.1) and hence in (2.7) by other methods of estimation such as the method of sieves or method of wavelets (cf. Prakasa Rao [9]) and recover the function $\theta(t)$ either by keeping ϵ fixed and letting $N \to \infty$ or by linking ϵ and N such that $N = N(\epsilon) \to \infty$. We hope to come back to this discussion in future.

(4) Conditions (C_1) and (C_2) are the natural conditions to ensure the asymptotic normality of the stochastic integral in (2.39) and to ensure that the remaining part

in (2.39) is negligible. It would be interesting if these conditions can be removed and the results can be proved directly. This will involve asymptotic expansion of the quantity $E(u_{i\epsilon}^{-2}(t)\chi(A_t^{(i)}))$.

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