A short proof of the uniqueness of Kühnel's 9-vertex complex projective plane

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Abstract. We introduce the notion of amicable partitions for combinatorial manifolds with complementarity. We prove that any 4-dimensional combinatorial manifold X_9^4 satisfying complementarity has an amicable partition and any amicable partition determines X_9^4 up to isomorphism. This gives a short proof of the uniqueness of Kühnel's 9-vertex complex projective plane.

1 Introduction

1.1. In [4], Brehm and Kühnel proved that if X is a non-sphere d-dimensional combinatorial manifold on n vertices then $n \geqslant 3d/2 + 3$. In case of equality, the only possibilities are $d = 2^m$, $m \leqslant 4$, and in these cases $|X|$ is a 'manifold like a projective plane'. Arnoux and Marin showed in [1] that in the cases of equality X must have the following *complementarity* property: exactly one of the two cells in any non-trivial bipartition (of the vertex set of X) must be a face of X. In [6], the second-named author proved the following converse: if X is an n-vertex d-dimensional combinatorial manifold with the complementarity property then $n = 3d/2 + 3$ (and hence $d = 2^m$, $m \leqslant 4$).

1.2. Let us say that a non-sphere combinatorial manifold is a *B-K manifold* (B-K stands for Brehm and Kühnel, of course) if it satisfies $n = 3d/2 + 3$. It is well known (and quite easy to prove, see for instance [3]) that there is a unique 2-dimensional B-K manifold, namely the 6-vertex real projective plane $\mathbb{R}P_6^2$. It is also known (and this is much harder to prove) that there is a unique 4-dimensional B-K manifold, namely Kühnel's 9-vertex complex projective plane $\mathbb{C}P_9^2$. In [5], Brehm and Kühnel constructed three distinct 8-dimensional B-K manifolds. These three are combinatorially equivalent and hence their geometric realizations are PL-homeomorphic. (Recall that two simplicial complexes are called combinatorially equivalent if they have isomorphic subdivisions.) It is not known whether these are the only 8-dimensional B-K manifolds, nor is it known whether the common topological manifold triangulated by them is the quaternionic projective plane. No 16-dimensional example is known at present; presumably such an object would triangulate the Cayley projective plane.

1.3. Several proofs of the existence and uniqueness of $\mathbb{C}P^2_9$ are now known. The first was the computer-aided proof of Kühnel and Laßmann [9]. (A beautiful exposition of this paper may be found in [8].) The second proof, due to Arnoux and Marin [1], uses cohomology theory with $\mathbb{Z}_2$ coefficients. The third, a combinatorial proof, is due to the present authors in [2]. In [11], Morin and Yoshida surveyed the known proofs (and added one of their own) of the fact that the topological space triangulated by $\mathbb{C}P^2_9$ is the complex projective plane. Since then, one more proof of the last-named fact has been found by Madahar and Sarkaria [10]. They constructed a 17-vertex 4-ball D^4_{17} whose boundary is a 12-vertex 3-sphere S^3_{12} and defined a combinatorial analogue $h : S^3_{12} \to S^2_4$ of the Hopf map so that the simplicial complex $S^2_4 \cup_h D^4_{17}$ is precisely $\mathbb{C}P^2_9$.

1.4. In [11], Morin and Yoshida presented arguments in support of having so many proofs identifying the geometric realization of $\mathbb{C}P^2_9$. The gist of their argument is that $\mathbb{C}P^2_9$ is such an important and exotic object that it is certainly worth in-depth studies, and different proofs will throw light on different aspects of this object. We believe that this argument applies equally well to proofs of the uniqueness of $\mathbb{C}P^2_9$. Thus encouraged, we present yet another combinatorial proof of the uniqueness. More precisely we prove:

Theorem. *Up to simplicial isomorphism there is a unique 9-vertex 4-dimensional combinatorial manifold satisfying complementarity.*

1.5. In [7], the second-named author proved that a 4-dimensional weak pseudomanifold (without boundary) satisfying complementarity is automatically a combinatorial manifold on 9 vertices. Therefore, the above theorem may also be stated as saying that: *up to isomorphism there is a unique 4-dimensional weak pseudomanifold without boundary satisfying complementarity.* Our proof, presented below, has the virtue of brevity: it is much shorter than all the previous proofs. The proof is based on the notion of amicable partition: in the language of [2], they are just the partitions of the vertex set into blue triangles. We prove that (a) any combinatorial manifold $X = X^4_9$ satisfying complementarity has an amicable partition, (b) up to isomorphism there are two types of amicable partitions, (c) any amicable partition determines X^4_9 up to isomorphism and (d) both types of amicable partitions determine the same combinatorial manifold. The general theory is developed in Section 2, while we specialize these results to $\mathbb{C}P^2_9$ in Section 3. Thus, Section 3 contains the proof of the main theorem.

2 Amicable partitions

2.1. *Amicable* partitions may be defined for any d-dimensional B-K manifold. These are the partitions of its vertex set into three $(d/2)$-faces A_1, A_2, A_3 such that the link of each A_i is the standard $(d/2-1)$-sphere on A_{i+1} (addition in the suffix is modulo three). We have:

Lemma 1. *Let A be a $(d/2)$-face of a d-dimensional B-K manifold X. Suppose the link of A is a standard sphere. Then A belongs to a unique amicable partition of X.*

Proof. Put $A = A_1$. Let A_2 be the vertex-set of the link of A_1 and let A_3 be the set of vertices outside $A_1 \cup A_2$. Then each A_i contains $d/2+1$ vertices. Note that complementarity implies that any set of $d/2+1$ (or fewer) vertices of X spans a face. In particular, each A_i is a $(d/2)$-face of X. So, to complete the proof, it is sufficient to show that the link of A_2 (respectively A_3) is the standard sphere on A_3 (respectively A_1).

Take any vertex $x \in A_2$. Then $A_3 \cup \{x\}$ is not a face since its complement $A_1 \cup (A_2 \backslash \{x\})$ is a face. Thus no vertex of A_2 belongs to the link of A_3. Therefore, the vertex set of the link of A_3 is contained in A_1. Since this link has at least $d/2+1$ vertices, it follows that the link of A_3 is the standard sphere on A_1. Replacing A_1 by A_3 (and hence A_2 by A_1, A_3 by A_2) in this argument, we see that the link of A_2 is the standard sphere on A_3.

In particular, this lemma shows that each edge of $\mathbb{R}P^2_6$ is a cell of a unique amicable partition. Hence there are five amicable partitions in $\mathbb{R}P^2_6$, and this fact trivialises the existence and uniqueness of $\mathbb{R}P^2_6$. From [2] it can be read off that $\mathbb{C}P^2_9$ has seven amicable partitions. (But this fact will not be used in what follows.) We observe that each of the three known 8-dimensional B-K manifolds has amicable partitions. (In fact, these three B-K manifolds have five, nine and eleven amicable partitions, respectively.) But we see no way to prove (or disprove!) the following:

Conjecture. Every B-K manifold has an amicable partition.

2.2. If U is an n-vertex m-sphere $(n > m+2)$ then clearly each vertex x of U is of degree $\geqslant m+1$ (i.e., x is in at least $m+1$ edges). If x is a vertex of degree $m+1$, we can construct an $(n-1)$-vertex m-sphere V as follows. Delete the vertex x (and all faces through x); introduce the set of neighbours of x as a new facet (i.e., maximal face). We shall say that V is obtained from U by *collapsing* the vertex x. Conversely, U can be recovered from V by starring a vertex x in the new facet.

Let X be a d-dimensional B-K manifold with an amicable partition $\{A_1, A_2, A_3\}$. Say, $A_1 = \{x_0, \dots, x_{d/2}\}$. Then the link in X of the $(d/2-1)$-face $A_1 \backslash \{x_i\}$ is a $(d/2)$-sphere on the vertex set $\{x_i\} \cup A_2 \cup A_3$ wherein x_i is a vertex of degree $d/2+1$ and its neighbours are the vertices in A_2. Let X_i be the $(d/2)$-sphere obtained by collapsing x_i. The set $\{X_i : 0 \leqslant i \leqslant d/2\}$ of $(d/2)$-spheres thus obtained will be called a *layer* of the given amicable partition with respect to the cell A_1. Thus, any amicable partition has three layers of $(d/2)$-spheres corresponding to its three cells.

2.3. For any combinatorial sphere U, we shall use $\Gamma(U)$ to denote the graph with the vertices of U as vertices, such that two distinct vertices x and y are adjacent in

$\Gamma(U)$ if and only if $\{x, y\}$ is not a face of U. In other words, the edges of $\Gamma(U)$ are precisely the missing edges of U. Thus $\Gamma(U)$ is just the graph theoretic complement of the 1-skeleton of U.

The spheres in a layer of an amicable partition are far from arbitrary; they satisfy some strong compatibility requirements:

Lemma 2. *Let $\{X_i : 0 \leqslant i \leqslant d/2\}$ be a layer of an amicable partition $\{A_1, A_2, A_3\}$ of a d-dimensional B-K manifold X, say with respect to the cell A_1. Then*

(a) *A_2 and A_3 are common facets of all the X_i, $0 \leqslant i \leqslant d/2$; and $\{A_2, A_3\}$ gives a partition of the common vertex set of these spheres. It follows that for each i, $\Gamma(X_i)$ is a bipartite graph (with A_2, A_3 as its parts).*

(b) *$\{\Gamma(X_i) : 0 \leqslant i \leqslant d/2\}$ is an edge-partition of the complete bipartite graph $K_{d/2+1,d/2+1}$ with parts A_2, A_3.*

(c) *For $0 \leqslant i \neq j \leqslant d/2$, any facet C of X_i intersects any facet D of X_j, provided $\{C, D\} \neq \{A_2, A_3\}$.*

Proof. A_2 is a facet of each X_i by construction. Since $\mathrm{Lk}_X(A_3)$ is the standard sphere on A_1, $A_3 \cup (A_1 \setminus \{x_i\})$ is a facet of X, and hence A_3 is a facet of X_i. Since A_2 (or A_3) is a facet of X_i, no two vertices in A_2 (or in A_3) are adjacent in $\Gamma(X_i)$. So, $\Gamma(X_i)$ is bipartite. This proves (a).

Let $\{x, y\}$ be an edge of $K_{d/2+1,d/2+1}$. Say $x \in A_2$, $y \in A_3$. Then $(A_2 \setminus \{x\}) \cup (A_3 \setminus \{y\})$ is a $(d-1)$-face of X. One of the two facets of X containing this face is $A_2 \cup (A_3 \setminus \{y\})$. The other facet cannot be $(A_2 \setminus \{x\}) \cup A_3$ (since the vertex set of $\mathrm{Lk}_X(A_3)$ is A_1). So, there is a unique vertex x_i in A_1 such that $(A_2 \setminus \{x\}) \cup (A_3 \setminus \{y\}) \cup \{x_i\}$ is a facet of X. By complementarity, x_i is the unique vertex in A_1 for which $(A_1 \setminus \{x_i\}) \cup \{x, y\}$ is not a face of X. Thus $\{x, y\}$ is not a face of X_i for a uniquely determined index i. This proves (b).

If $C \cap D = \emptyset$, C a facet of X_i, D is a facet of X_j, then $C \cup D = A_2 \cup A_3$. If, further $\{C, D\} \neq \{A_2, A_3\}$ then it follows that $C \neq A_2$ and $D \neq A_2$. Hence $C \cup (A_1 \setminus \{x_i\})$ and $D \cup (A_1 \setminus \{x_j\})$ are two facets of X which together cover the vertex set of X (as $i \neq j$). Therefore, the complement of either of these two facets of X is a face of X—contradicting complementarity. This proves (c).

2.4. If $\{X_i : 0 \leqslant i \leqslant d/2\}$ is one of the layers of an amicable partition, then the set $\{\Gamma(X_i) : 0 \leqslant i \leqslant d/2\}$ will be called the *frame* of the layer. Thus the frame is an edge partition of a complete bipartite graph by spanning subgraphs.

Lemma 3. *Each layer of an amicable partition of a B-K manifold determines the other two frames.*

Proof. Let the cells of the amicable partition be $A_i = \{x_{ij} : 0 \leqslant j \leqslant d/2\}$ with corresponding layer $\{X_{ij} : 0 \leqslant j \leqslant d/2\}$ and frame $\{\Gamma_{ij} = \Gamma(X_{ij}) : 0 \leqslant j \leqslant d/2\}, 1 \leqslant i \leqslant 3$. Suppose the layer $\{X_{1j} : 0 \leqslant j \leqslant d/2\}$ is known. Then $\{x_{1j}, x_{3l}\}$ is an edge of Γ_{2k} if and only if $(A_2 \setminus \{x_{2k}\}) \cup \{x_{1j}, x_{3l}\}$ is not a face of the B-K manifold X; by complementarity this happens if and only if $(A_1 \setminus \{x_{1j}\}) \cup (A_3 \setminus \{x_{3l}\}) \cup \{x_{2k}\}$ is a facet of X, i.e., if and only if $(A_3 \setminus \{x_{3l}\}) \cup \{x_{2k}\}$ is a facet of X_{1j}. Similarly $\{x_{1j}, x_{2k}\}$ is an edge of Γ_{3l} if and only if $(A_2 \setminus \{x_{2k}\}) \cup \{x_{3l}\}$ is a facet of X_{1j}.

3 Uniqueness of $\mathbb{C}P_9^2$

Throughout this section, Y is a 4-dimensional B-K manifold. Hence Y satisfies complementarity. From complementarity and Dehn–Sommerville equations, it readily follows that the number f_i of i-faces of Y are given by: $f_0 = 9$, $f_1 = \binom{9}{2} = 36$, $f_2 = \binom{9}{3} = 84$, $f_3 = 90$ and $f_4 = 36$. Further, we have:

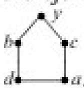
Lemma 4. *Y has an amicable partition.*

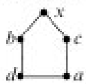
Proof. By Lemma 1, it is sufficient to show that there is at least one triangle (i.e., 2-face) in Y whose link is an S_3^1. Suppose not. Then the link of each triangle has $\geqslant 4$ vertices. Fix any facet σ of Y. By complementarity, the complement of σ induces an S_4^2. Therefore, the link of each of the four triangles in the complement of σ is contained in σ and hence (by our assumption) has four or five vertices. Let a of them have 5-vertex links, hence the remaining $4 - a$ have 4-vertex links. Therefore, the total number of tetrahedra meeting σ in a singleton is $5a + 4(4 - a) = 16 + a$. But, by complementarity, this number is $\binom{5}{1}\binom{4}{3}$ minus the number of facets meeting σ in a 3-face $= 20 - 5 = 15$. Hence $a = -1$, a contradiction.

Next we determine the possibilities for the layers of an amicable partition of Y. Each layer consists of three 6-vertex 2-spheres (S_6^2's). It is well known (and immediate from the classifications of S_{d+4}^d's in [3]) that up to isomorphism there are two S_6^2's. Their Γ-graphs are $3K_2$ (the disjoint union of three edges) and the three path $P_3 =$ •——•——•——• (plus two isolated vertices) respectively. We need the following stronger statement:

Lemma 5. *Given a graph $\Gamma = 3K_2$ or P_3, there is a unique 6-vertex 2-sphere U with $\Gamma(U) = \Gamma$ (not merely unique up to automorphism of Γ).*

Proof. Note that any 6-vertex 2-sphere U has eight facets and they are 3-cocliques of $\Gamma(U)$. (Recall that a coclique in a graph is a set of pairwise non-adjacent vertices.) Since $\Gamma = 3K_2$ has exactly eight 3-cocliques, the lemma is immediate in this case.

In the second case, let $\Gamma =$ a •——• b ——• c ——• d with isolated vertices x and y. Then the link of x in U is a pentagon. This pentagon induces a 3-path on $\{a, b, c, d\}$ which is edge disjoint from Γ. Hence this 3-path is b •——• d ——• a ——• c. Thus, the link of x in U is 

. Similarly, the link of y is 

. This determines all the facets of U.

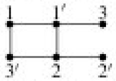
Lemma 6. *Y is uniquely determined (not merely up to isomorphism) by any of the frames of any given amicable partition.*

Proof. Since the graphs in any frame are isomorphic to $3K_2$ or P_3, Lemma 5 shows that the frame determines the corresponding layer. Then Lemma 3 determines all the frames of the given amicable partition. Another appeal to Lemma 5 determines all three layers. The known links of the three cells of the amicable partition give us 9 facets of Y. The known layers give $((8-2) \times 3 \times 3)/2 = 27$ more. We now have all the $9 + 27 = 36$ facets of Y.

Lemma 7. *Up to isomorphism there are two possible types of frames for Y.*

Proof. If two of the graphs in a frame are $3K_2$, they must consist of alternating edges of a hexagon. Then the third graph in the frame is determined as the relative complement of this hexagon with respect to $K_{3,3}$. This third graph is the $3K_2$ whose edges are the long diagonals of the hexagon. This yields the *first type* of frames—consisting of three edge-disjoint copies of $3K_2$.

Next let the frame consist of one $3K_2$ and hence two P_3's. Then the relative complement of the $3K_2$ is a hexagon and each P_3 must consist of three consecutive edges of the hexagon. Say, the edges of the $3K_2$ are $1 - 1'$, $2 - 2'$, $3 - 3'$. Then, without loss, the P_3's in the frame are $1 - 2' - 3 - 1'$ and $1' - 2 - 3' - 1$. Then, from the proof of Lemma 5, we see that $\{1, 1', 2\}$ is a face of the S^2_6 with the first P_3 as Γ-graph while $\{2', 3, 3'\}$ is a face of the S^2_6 with the second P_3 as Γ-graph. Since these two triangles are disjoint and distinct from the parts $\{1, 2, 3\}$, $\{1', 2', 3'\}$ of the $K_{3,3}$, this contradicts Lemma 2 (c).

So, in the remaining case, the frame must be an edge partition of $K_{3,3}$ into three copies of P_3. Let the parts of the $K_{3,3}$ be $\{1, 2, 3\}$ and $\{1', 2', 3'\}$. Without loss, let the first graph in the frame be $1 - 2' - 3 - 3'$. The relative complement (with respect to $K_{3,3}$) of this graph is 

. It is obvious that the last graph has a unique edge partition into two P_3's. So the remaining two graphs in the frame must be $2 - 3' - 1 - 1'$ and $3 - 1' - 2 - 2'$. This gives the *second* isomorphism type of frames, consisting of three copies of P_3.

Lemma 8. *Y has an amicable partition one of whose frames is of the first type (i.e., consists of three copies of $3K_2$).*

Proof. Take an amicable partition $\{1, 2, 3\}$, $\{1', 2', 3'\}$, $\{1'', 2'', 3''\}$ of Y. (This exists by Lemma 4.) If the frame corresponding to the cell $\{1'', 2'', 3''\}$ is not of the first type, then (by Lemma 7) it is of the second type. Hence, without loss, this frame consists of $\Gamma(X_1) = 2 - 1' - 3 - 3'$, $\Gamma(X_2) = 3 - 2' - 1 - 1'$, $\Gamma(X_3) = 1 - 3' - 2 - 2'$, where X_i is the 2-sphere obtained from the link of $\{1'', 2'', 3''\} \setminus \{i''\}$ by collapsing i''. Thus, following the proof of Lemma 6, Y is uniquely determined. Hence one finds that $\{\{1, 1', 1''\}, \{2, 2', 2''\}, \{3, 3', 3''\}\}$ is also an amicable partition of Y and the frame corresponding to the part $\{1, 1', 1''\}$ consists of $\{\overset{2}{\bullet}\!\relbar\!\relbar\!\overset{3'}{\bullet}, \overset{2'}{\bullet}\!\relbar\!\relbar\!\overset{3}{\bullet}, \overset{2''}{\bullet}\!\relbar\!\relbar\!\overset{3''}{\bullet}\}$, $\{\overset{2}{\bullet}\!\relbar\!\relbar\!\overset{3}{\bullet}, \overset{2'}{\bullet}\!\relbar\!\relbar\!\overset{3''}{\bullet}, \overset{2''}{\bullet}\!\relbar\!\relbar\!\overset{3'}{\bullet}\}$ and $\{\overset{2}{\bullet}\!\relbar\!\relbar\!\overset{3''}{\bullet}, \overset{2'}{\bullet}\!\relbar\!\relbar\!\overset{3'}{\bullet}, \overset{2''}{\bullet}\!\relbar\!\relbar\!\overset{3}{\bullet}\}$. This frame is of the first type.

Proof of the theorem. By Lemma 6 and Lemma 8, Y is uniquely determined up to isomorphism.

Remark. It can be seen that all three frames of any amicable partition of $\mathbb{C}P^2_9$ are of the same type. $\mathbb{C}P^2_9$ contains a unique amicable partition of type one and six of type two.

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