Scalar perturbations of the Nagy-Foias characteristic function *

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Abstract

When T is a completely non-unitary (cnu) contraction with Sz.-Nagy–Foias characteristic function θ and μ is a scalar $(0 < \mu < 1)$ then general theory implies the existence of a cnu contraction $T[\mu]$ with characteristic function $\mu\theta$. How to describe $T[\mu]$ in terms of T? In this paper, we find a simple answer in case T is in the class $C_{\cdot 0}$. An operator is called homogeneous if its spectrum is contained in the closed unit disc and all the bi-holomorphic automorphisms of the unit disc lift to automorphisms of the operator T modulo unitary equivalence. When T is homogeneous, so is $T[\mu]$. We find explicit formulae for the characteristic functions of the (homogeneous) twisted Bergman shifts – these have product formulae involving the discrete series projective representations of the Möbius group. These formulae lead to an explicit description of the Sz.-Nagy–Foias models of these weighted shifts. Combining our main results with an analytic continuation argument, we find a three-parameter family of homogeneous operators.

1 Introduction

1.1 All Hilbert spaces in this paper are separable Hilbert spaces over the complex numbers. For positive integers k, $\mathbb{C}^k$ will denote the k-dimensional Hilbert space with the standard inner product. All operators are bounded linear operators between Hilbert spaces. For Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ (respectively $\mathcal{B}(\mathcal{H})$) will denote the Banach space of all operators from $\mathcal{H}$ to $\mathcal{K}$ (respectively from $\mathcal{H}$ to $\mathcal{H}$) with the operator norm. $\mathbb{C}^{k\times k}$ will denote $\mathcal{B}(\mathbb{C}^k)$ viewed as the space of $k \times k$ matrices (with respect to the standard basis). $\mathcal{U}(\mathcal{H})$ will denote the Borel group of all unitary operators on the Hilbert space $\mathcal{H}$ with the Borel structure induced by the strong operator topology. An operator T is called a contraction if $\|T\| \leq 1$. It is called a pure contraction if $\|Tx\| < \|x\|$ for all nonzero vectors x. T is called a completely

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non unitary (cnu) operator if it has no unitary part, i.e., if T has no non-trivial reducing subspace $\mathcal{M}$ such that the restriction $T|_{\mathcal{M}}$ is unitary.

1.2 Let $T \in \mathcal{B}(\mathcal{H})$ be a cnu contraction. Let $\mathcal{K}$ and $\mathcal{L}$ denote the closures of the ranges of $D_T := (I - T^*T)^{1/2}$ and of $D_{T^*} := (I - TT^*)^{1/2}$, respectively. $\mathcal{K}$, $\mathcal{L}$ are called the defect spaces of T, and their dimensions are called its defect indices. D_T and D_{T^*} are called the defect operators associated with T.

1.3 In [7, Chapter VI], Sz.-Nagy and Foias associate with each cnu contraction T a pure contraction valued analytic function θ_T, called the characteristic function of T. Namely, letting $\mathcal{K}$ and $\mathcal{L}$ be the defect spaces of T, one defines $\theta_T : \mathbb{D} \to \mathcal{B}(\mathcal{K}, \mathcal{L})$ as follows. For z in the unit disc $\mathbb{D}$,

$$\theta_T(z) = (-T + zD_{T^*}(I - zT^*)^{-1}D_T)|_{\mathcal{K}} \,. \tag{1.1}$$

It is easy to verify that the right hand side of (1.1) maps $\mathcal{K}$ into $\mathcal{L}$; hence it defines a contraction $\theta_T(z) \in \mathcal{B}(\mathcal{K}, \mathcal{L})$. In particular, $-T$ maps $\mathcal{K}$ into $\mathcal{L}$ and hence defines an element $\theta_T(0)$ of $\mathcal{B}(\mathcal{K}, \mathcal{L})$. It is easy to see that $\theta_T(0)$, thus defined, is a pure contraction. However, if a contraction valued analytic function θ is pure at some point in its domain, then, applying the strong maximum modulus principle to the Hilbert space valued analytic function $z \to \theta(z)x$ for a fixed but arbitrary nonzero vector x, one sees that θ must be pure contraction valued through out its domain. Thus the characteristic function θ_T of T is indeed a pure contraction valued analytic function from $\mathbb{D}$ into $\mathcal{B}(\mathcal{K}, \mathcal{L})$. Note that if T is a cnu contraction then so is its adjoint T^*. If θ is the characteristic function of T, then the characteristic function of T^* is θ^*. Here $*$ is the usual involution on analytic functions defined by $\theta^*(z) = \theta(\bar{z})^*$, $z \in \mathbb{D}$.

1.4 In the terminology of Sz.-Nagy and Foias, two pure contractions C_1 and C_2 *coincide* if there exist unitary operators U, V (between appropriate Hilbert spaces) such that $C_2 = UC_1V$. Two pure contraction valued analytic functions $\theta_1 : \mathbb{D} \to \mathcal{B}(\mathcal{K}_1, \mathcal{L}_1)$ and $\theta_2 : \mathbb{D} \to \mathcal{B}(\mathcal{K}_2, \mathcal{L}_2)$ *coincide* if there exist unitaries $U : \mathcal{L}_1 \to \mathcal{L}_2$, $V : \mathcal{K}_2 \to \mathcal{K}_1$ such that $\theta_2(z) = U\theta_1(z)V$ for all $z \in \mathbb{D}$. (Of course, this is stronger than merely requiring that $\theta_1(z)$ and $\theta_2(z)$ coincide for each $z \in \mathbb{D}$.) It is easy to see that if the cnu contractions T_1 and T_2 are unitarily equivalent then their characteristic functions coincide. The power of the Sz.-Nagy–Foias theory lies in the fact that the converse statement is also correct. Thus the characteristic function, modulo coincidence, is a complete unitary invariant for cnu contractions.

1.5 This indicates that it should be possible to recover a cnu contraction, up to unitary equivalence, from its characteristic function. Sz.-Nagy–Foias showed that this is indeed the case. Indeed, to any pure contraction valued analytic function $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{K}, \mathcal{L})$, they associate a model operator T_θ in $\mathcal{B}(\mathcal{H})$ as follows. Let $H^2 = H^2(\mathbb{D})$ be the usual Hilbert space of analytic functions on the unit disc $\mathbb{D}$ with square integrable boundary values and let $L^2 = L^2(\mathbb{T})$ be the L^2-space of all square integrable (with respect to the normalised arc length measure) measurable functions on the unit circle $\mathbb{T}$. Let $H^2_{\mathcal{L}} = H^2 \otimes \mathcal{L}$ and $L^2_{\mathcal{K}} = L^2 \otimes \mathcal{K}$ be the Hilbert spaces of $\mathcal{L}$ valued analytic functions on $\mathbb{D}$ with square integrable boundary values on $\mathbb{T}$ and $\mathcal{K}$ valued square integrable measurable functions on $\mathbb{T}$ respectively. Define $H^2_{\mathcal{K}}$ similarly. Define $\Theta : H^2_{\mathcal{K}} \to H^2_{\mathcal{L}}$ and $\Delta : L^2_{\mathcal{K}} \to L^2_{\mathcal{K}}$ by

$$(\Theta f)(z) = \theta(z)f(z), \quad (\Delta g)(w) = (I - \theta(w)^*\theta(w))^{1/2}g(w), \tag{1.2}$$

for $z \in \mathbb{D}$, $w \in \mathbb{T}$, $f \in H^2_{\mathcal{K}}$, $g \in L^2_{\mathcal{K}}$. (From the classical theory of scalar valued bounded analytic functions, it is easy to deduce that that the contraction valued analytic function θ has radial limits at almost all points of the unit circle $\mathbb{T}$. In the definition of Δ in (1.2), we have used the same letter θ to denote the boundary value as well.)

Let $\mathrm{ran}(\Delta)$ denote the closure in $L^2_{\mathcal{K}}$ of the range of Δ. Let $\mathcal{V}$ be the closure in $H^2_{\mathcal{L}} \oplus \mathrm{ran}(\Delta)$ of $\{\Theta f \oplus \Delta f : f \in H^2_{\mathcal{K}}\}$ and let $\mathcal{H}$ be the orthocomplement of $\mathcal{V}$ in $H^2_{\mathcal{L}} \oplus \mathrm{ran}(\Delta)$. This defines a subspace $\mathcal{H}$ of $H^2_{\mathcal{L}} \oplus L^2_{\mathcal{K}}$. Let M be the multiplication operator defined on $H^2_{\mathcal{L}} \oplus L^2_{\mathcal{K}}$ by $M(f \oplus g) = f_1 \oplus g_1$, where $f_1(z) = zf(z), z \in \mathbb{D}$ and $g_1(w) = wg(w), w \in \mathbb{T}$. Finally, let T_θ be the compression of M to $\mathcal{H}$. Then it is shown in Theorem VI.3.1 of [7] that T_θ is a cnu contraction and the characteristic function of T_θ coincides with θ. Thus every pure contraction valued analytic function on $\mathbb{D}$ is indeed the characteristic function of a cnu contraction. In view of this bijection between unitary equivalence classes of cnu contractions and coincidence classes of pure contraction valued analytic functions on $\mathbb{D}$, we shall identify two such functions if they coincide.

1.6 For $w \in \mathbb{D}$, let $\varphi_w : \mathbb{D} \to \mathbb{D}$ denote the Möbius map

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}, \quad z \in \mathbb{D}. \tag{1.3}$$

Each φ_w is a biholomorphic map from $\mathbb{D}$ onto $\mathbb{D}$ which is its own inverse. Let Möb denote the collection of all the maps $\alpha\varphi_w$, $\alpha \in \mathbb{T}, w \in D$. It is well known that Möb is a group under composition; indeed, it is the group of all biholomorphic automorphisms of the unit disc. The Möbius group is intimately related to the Sz.-Nagy–Foias theory via the following easy fact (which is essentially the result (1.7) in [7, p. 240]) : if T is a cnu contraction with characteristic function θ then for any $\varphi \in$ Möb, $\varphi(T)$ is a cnu contraction whose characteristic function coincides with $\theta \circ \varphi^{-1}$. In symbols,

$$\theta_{\varphi(T)} = \theta_T \circ \varphi^{-1}. \tag{1.4}$$

1.7 Recall from [1] that T is said to be a *homogeneous operator* if $\varphi(T)$ is unitarily equivalent to T for every φ in Möb for which $\varphi(T)$ is well defined (i.e., for every φ which is analytic in a neighborhood of the spectrum of T). It was shown in [1, Lemma 2.2] that if T is homogeneous then the spectrum of T is either the unit circle or the unit disc. Thus, in either case, $\varphi(T)$ is defined (and unitarily equivalent to T) for all φ in Möb.

The notion of homogeneous operators was introduced informally in [5], and since then it has been studied by many authors. (Complete references may be found in [2].) Our long term goal is to classify all homogeneous operators. Note that any contraction may be written uniquely as the direct sum of its cnu part and the unitary part. Further, it is clear that a contraction is homogeneous if and only if both its parts are homogeneous. Finally, using the spectral theorem, it is easy to determine all homogeneous unitary operators. Namely, if T is a homogeneous unitary, then the scalar spectral measure of T must be quasi invariant under Möb and its multiplicity function must be stable under Möb. This implies that the only homogeneous unitaries are direct integrals of copies of the (unweighted) bilateral shift. This reduces the problem of classifying homogeneous contractions to that of classifying cnu homogeneous contractions.

In view of (1.4), it is immediate that a cnu contraction T is homogeneous if and only if its characteristic function θ coincides with $\theta \circ \varphi^{-1}$ for every φ in Möb. This raises the serious possibility that it may be possible to classify all homogeneous cnu contractions using the theory outlined above. However, this is not the objective of this paper.

1.8 Instead, we exhibit in this paper how the Sz.-Nagy–Foias theory may be applied to create various continuums of new homogeneous operators starting from known ones. One basic construction is the following. Let T be a cnu contraction with characteristic function θ. then, for any scalar μ in the range $0 < \mu \leq 1$, $\mu\theta$ is clearly a pure contraction valued analytic function, and hence it is the characteristic function of a cnu contraction $T[\mu]$. Clearly, the operators $T[\mu], 0 < \mu \leq 1$ are mutually unitarily inequivalent except in the case when θ is the constant function identically equal to zero. From the characterisation of homogeneity in terms of characteristic functions presented in section 1.7 above, it is immediate that if T is a homogeneous cnu contraction then $T[\mu]$ is homogeneous for every μ in $(0, 1)$.

1.9 For the above to be a useful construction, we should have a usable description of $T[\mu]$ in terms of T. Of course, the Sz.-Nagy–Foias model (presented in section 1.5 above) provides an answer. This is, however, not so useful since the model is complicated. However, there is one case in which the Sz.-Nagy–Foias model is remarkably simple, namely, for the cnu operators in class $C_{\cdot 0}$. Recall that the cnu contraction T is said to be in the class $C_{\cdot 0}$ if $T^{*n} \to 0$ strongly as $n \to \infty$. In Proposition VI.3.5 of [7] it is shown that a cnu contraction T is in the class $C_{\cdot 0}$ if and only if its characteristic function θ is *inner* in the sense that θ (more precisely, the boundary value of θ) is isometry valued at almost all points of the unit circle. (Equivalently, the operator Θ of (1.2) is an isometry.) This implies that the operator Δ of (1.2) is the zero operator, and hence the Sz.-Nagy–Foias model for T simplifies as follows. If $T \in C_{\cdot 0}$ with $\theta_T = \theta$, M is the multiplication operator (by the co-ordinate function z) on $H^2_{\mathcal{L}}$ and $\mathcal{M}$ is the M- invariant subspace corresponding to the inner function θ (i.e., $\mathcal{M}$ is the range of Θ) then T is unitarily equivalent to the compression of M to $\mathcal{M}^{\perp}$. Thus the operators in the class $C_{\cdot 0}$ are precisely the compressions of unweighted unilateral shifts (with multiplicity) to co-invariant subspaces. Further, for such operators T, the characteristic function θ_T is the inner function involved in the description of the complementary invariant subspace.

In section 2 of this paper, we show that for $T \in C_{\cdot 0}$ and $0 < \mu < 1$, $T[\mu]$ has a model which is almost as simple as the Sz.-Nagy–Foias model for T. Namely, $T[\mu]$ is a "slight perturbation" of the operator $M \oplus N^*$, where M, N are multiplication operators on $H^2_{\mathcal{L}}$ and $H^2_{\mathcal{K}}$, respectively.

1.10 It is clear that, in order to apply the above to create new examples of homogeneous operators, one must begin with a supply of homogeneous operators in the class $C_{\cdot 0}$ for which the characteristic functions are "known" in a sufficiently nice and usable form. Unfortunately, there is hardly any example in the literature of cnu operators with "known" characteristic functions. (Of course, one could write down the formula (1.1) in any particular case, but this is almost never very useful!) One exception is the unweighted unilateral shift. This is a homogeneous contraction in the class $C_{\cdot 0}$, and its characteristic function is as simple as can be - the identically zero function. But this is too trivial. A slightly less trivial example that we can think of is the operator N on $L^2(\mathbb{D})$ given by $Nf(z) = zf(z)$. By [1, Proposition 2.3], N is homogeneous. Also, clearly, N is in the class $C_{\cdot 0}$. Since N is normal, it is easy to simplify the formula (1.1) in this case to obtain the following simple but interesting formula for the characteristic function $\theta_N : \mathbb{D} \to \mathcal{B}(L^2(\mathbb{D}))$:

$$(\theta_N(z)f)(w) = -\varphi_w(z)f(w), \quad z, w \in \mathbb{D}, f \in L^2(\mathbb{D}). \tag{1.5}$$

Here φ_w is as in (1.3). The formula (1.5) indicates yet another mysterious link between characteristic function and the Möbius group. Unfortunately, we are unable to use formula (1.5) to obtain explicit descriptions of the operators $N[\mu]$, $0 < \mu < 1$.

1.11 *Notation:* For any real number x and integer $n \geq 0$, $\binom{x}{n}$ will denote the coefficient of t^n in the Taylor expansion of $(1+t)^x$ around $t = 0$. For positive integers x, this agrees with the usual notation for the Binomial coefficients.

The earliest known examples (cf. [5]) of homogeneous operators are the twisted Bergman shifts $M^{(\lambda)}$, $\lambda > 0$. $M^{(\lambda)}$ may most simply be described as the unilateral weighted shifts with weight sequence $\sqrt{\frac{n+1}{n+\lambda}}$, $n \geq 0$. Thus in particular, $M^{(1)}$ is the unweighted unilateral shift, while $M^{(2)}$ is the Bergman shift. Clearly, $M^{(\lambda)}$ is a contraction if and only if $\lambda \geq 1$. Let $\mathcal{H}^{(\lambda)}$ denote the functional Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernel $(z, w) \to (1 - z\bar{w})^{-\lambda}$. Then $M^{(\lambda)}$ may alternatively be described as the operator of multiplication by the co-ordinate function z on $\mathcal{H}^{(\lambda)}$. (Indeed, let $e_n^{(\lambda)}$, $n \geq 0$, be the elements of $\mathcal{H}^{(\lambda)}$ defined by $e_n^{(\lambda)}(z) = \sqrt{\binom{n+\lambda-1}{n}} z^n$, $z \in \mathbb{D}$. Then $\{e_n^{(\lambda)} : n \geq 0\}$ is an ortho-normal basis of $\mathcal{H}^{(\lambda)}$ and multiplication by z acts as the weighted shift with the indicated weight sequence with respect to this basis.) This description shows that, for $\lambda \geq 1$, $M^{(\lambda)}$ is actually in the class $C_{\cdot 0}$.

In section 3 of this paper, we explicitly determine the characteristic function of $M^{(\lambda)}$, $\lambda > 1$. To describe these, recall that a projective representation π of Möb on a Hilbert space $\mathcal{H}$ is a Borel map from Möb into the group $\mathcal{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$ such that $\pi(\varphi_1\varphi_2) = m(\varphi_1, \varphi_2)\pi(\varphi_1)\pi(\varphi_2)$ for all φ_1, φ_2 in Möb. Here $m(\varphi_1, \varphi_2)$ is a unimodular constant, depending on φ_1, φ_2. The function $m :$ Möb $\times$ Möb $\to \mathbb{T}$ is called the multiplier associated with π. For $\lambda > 0$, D_λ will denote the holomorphic discrete series (projective) representation of Möb on $\mathcal{H}^{(\lambda)}$ given by

$$(D_\lambda(\varphi^{-1})f)(z) = \varphi'(z)^{(\lambda/2)} f(\varphi(z)) \tag{1.6}$$

for $z \in \mathbb{D}$, $f \in \mathcal{H}^{(\lambda)}$, $\varphi \in$ Möb. (Warning: D_λ is an "ordinary" representation, i.e., the associated multiplier is the constant function 1, only in the cases where λ is an even integer. The adjective "discrete series" is usually reserved for these cases. We are using this word in a generalised sense.)

It is well known, and easy to see, that the representations D_λ are intimately related to the operator $M^{(\lambda)}$. Indeed, D_λ is *associated* with $M^{(\lambda)}$ in the sense that

$$\varphi(M^{(\lambda)}) = D_\lambda(\varphi)^* M^{(\lambda)} D_\lambda(\varphi) \quad \varphi \in \text{Möb} \tag{1.7}$$

for $\lambda > 0$. This is easy to verify and contains a direct verification of the homogeneity of $M^{(\lambda)}$, $\lambda > 0$. One surprising find of this paper is that, for $\lambda > 1$, the representations $D_{\lambda-1}$ and $D_{\lambda+1}$ are also intimately related to $M^{(\lambda)}$. Indeed, the characteristic function of $M^{(\lambda)}$

coincides with the function $\theta_\lambda : \mathbb{D} \to \mathcal{B}(\mathcal{H}^{(\lambda+1)}, \mathcal{H}^{(\lambda-1)})$ given by

$$\theta_\lambda(z) = \frac{1}{\sqrt{\lambda(\lambda-1)}} D_{\lambda-1}(\varphi_z)^* \partial^* D_{\lambda+1}(\varphi_z). \tag{1.8}$$

Here ∂^* is the adjoint of the differentiation operator $\partial : \mathcal{H}^{(\lambda-1)} \to \mathcal{H}^{(\lambda+1)}$ and φ_z is the involutive element of Möb defined in section 1.6.

Using (1.8) it is easy to calculate the Sz.-Nagy–Foias model for $M^{(\lambda)}$. This is done in section 3 and it turns out that the model is as follows. Let's identify, as usual, the tensor product $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ with a space of functions on the bi-disc $\mathbb{D} \times \mathbb{D}$, and let $\mathcal{M}$ be the subspace of functions vanishing on the diagonal $\{(z,z) : z \in \mathbb{D}\}$. Let M_1 be the multiplication by the first co-ordinate on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$. That is, $M_1 = M^{(1)} \otimes I$. Then the Sz.-Nagy–Foias model of $M^{(\lambda)}$ is the compression of M_1 to $\mathcal{M}^\perp$. In particular, this shows that $M^{(\lambda)}$ is unitarily equivalent to this compression; but that is a special case of a well known description (arising from the theory of module tensor products : see Theorem 1.1 in [4]) of $M^{(\lambda_1+\lambda_2)}$ as the compression of $M^{(\lambda_1)} \otimes I$ on $\mathcal{H}^{(\lambda_1)} \otimes \mathcal{H}^{(\lambda_2)}$ to the orthocomplement of functions vanishing on the diagonal. The interesting point is that at least a special case of this construction is implicit in the Sz.-Nagy–Foias theory : the minimal isometric dilation of $M^{(\lambda)}$ ($\lambda > 1$) is $M^{(1)} \otimes I$ on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ in a natural way.

1.12 With an explicit description of the characteristic functions of $M^{(\lambda)}$, $\lambda > 1$, available, the path is now open for the application of the results of section 2 to explicitly determine the homogeneous operators $M^{(\lambda)}[\mu]$, $\lambda > 1$, $0 < \mu < 1$. This is done in section 4. Indeed, in conjunction with Proposition 2.4 of [1] and an analytic continuation argument, this yields a new three parameter family of homogeneous operators. This is a record, the previous record being held by the two parameter family of homogeneous operators found by Wilkins in [8].

2 The models for scalar multiples of inner functions

2.1 Notation: Throughout, T is a cnu contraction in the class $C_{\cdot 0}$, and θ is its characteristic function. Thus, for a suitable Hilbert space $\mathcal{L}$, T may be identified with the compression of M to $\mathcal{M}^\perp$, where $M : H^2_{\mathcal{L}} \to H^2_{\mathcal{L}}$ is multiplication by the co-ordinate function, and $\mathcal{M}$ is an invariant subspace for M (corresponding to the inner function θ). Let $M = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}$ be the block matrix representation of M corresponding to the decomposition $H^2_{\mathcal{L}} = \mathcal{M}^\perp \oplus \mathcal{M}$. (Thus, in particular, $M_{11} = T$ and M_{22} is the restriction of M to $\mathcal{M}$.) Finally, let $\mathcal{K}$ denote the co-kernel (i.e., the orthocomplement of the range) of M_{22}, and let $N : H^2_{\mathcal{K}} \to H^2_{\mathcal{K}}$ be multiplication by the co-ordinate function, and let $E : H^2_{\mathcal{K}} \to \mathcal{M}$ be defined by $Ef = f(0) \in \mathcal{K}$. (Remember $\mathcal{K} \subseteq \mathcal{M}$.)

Theorem 2.1 *Let T be a cnu contraction in $C_{\cdot 0}$ with characteristic function θ. Let μ be a scalar in the range $0 < \mu < 1$, and put $\delta = +\sqrt{1-\mu^2}$. In terms of the above notation, let $T[\mu] : H^2_{\mathcal{L}} \oplus H^2_{\mathcal{K}} \to H^2_{\mathcal{L}} \oplus H^2_{\mathcal{K}}$ be the operator whose block matrix representation with respect to the decomposition $\mathcal{M}^\perp \oplus \mathcal{M} \oplus H^2_{\mathcal{K}}$ is given by*

$$T[\mu] = \begin{pmatrix} M_{11} & 0 & 0 \\ \delta M_{21} & M_{22} & \mu E \\ 0 & 0 & N^* \end{pmatrix}.$$

Then the characteristic function of $T[\mu]$ coincides with $\mu\theta$. (Of course, this includes the fact that $T[\mu]$, thus defined, is a cnu contraction.)

Proof: We begin with the Sz.-Nagy–Foias model $T_{\mu\theta}$ and reduce it to the above description. Let $\tilde{M}$ be the multiplication operator on $L^2_{\mathcal{K}}$, where, for now, $\mathcal{K}$ is the first defect space of $T_{\mu\theta}$ (we shall later identify $\mathcal{K}$ with $\operatorname{coker}(M_{22})$). Recall that $T_{\mu\theta}$ is the compression of $M \oplus \tilde{M}$ to the subspace $\mathcal{H}$ of $H^2_{\mathcal{L}} \oplus L^2_{\mathcal{K}}$, where in our case (since θ is inner), $\mathcal{H}^\perp = \{(\mu\Theta f, \delta f) : f \in H^2_{\mathcal{K}}\}$. Let $\sim$ denote the unitary operator from $H^2_{\mathcal{K}}$ onto $L^2_{\mathcal{K}} \ominus H^2_{\mathcal{K}}$ given by $\tilde{g}(z) = z^{-1}g(z^{-1})$, $z \in \mathbb{T}, g \in H^2_{\mathcal{K}}$. (Here we have used the usual identification of the elements of $H^2_{\mathcal{K}}$ with their boundary values. Thus, for now, we view all Hilbert spaces in sight as Hilbert spaces of functions on the unit circle $\mathbb{T}$.) Then it is trivial to verify that for $g \in H^2_{\mathcal{L}}$ and $h \in H^2_{\mathcal{K}}$, $(g, -\delta^{-1}\mu\Theta^* g + \tilde{h})$ is orthogonal to all elements of $\mathcal{H}^\perp$ and hence is in $\mathcal{H}$. Conversely, if $(g, k) \in H^2_{\mathcal{L}} \oplus L^2_{\mathcal{K}}$ is orthogonal to $(\mu\Theta f, \delta f)$ for every $f \in H^2_{\mathcal{K}}$ then $k + \delta^{-1}\mu\Theta^* g$ is orthogonal to all the elements of $H^2_{\mathcal{K}}$ and hence is of the form $\tilde{h}$ for a uniquely determined $h \in H^2_{\mathcal{K}}$. Thus $k = -\delta^{-1}\mu\Theta^* g + \tilde{h}$ in this case. This shows,

$$\mathcal{H} = \{(g, -\delta^{-1}\mu\Theta^* g + \tilde{h}) : g \in H^2_{\mathcal{L}}, h \in H^2_{\mathcal{K}}\}.$$

It follows that we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3,$$

where $\mathcal{H}_1 = \{(g, -\delta^{-1}\mu\Theta^* g) : g \in \mathcal{M}\}$, $\mathcal{H}_2 = \{(g, 0) : g \in \ker \Theta^*\}$ and $\mathcal{H}_3 = \{(0, \tilde{h}) : h \in H^2_{\mathcal{K}}\}$. (Here, as in notation 2.1, $\mathcal{M}$ is the range closure of Θ. So $\ker \Theta^* = \mathcal{M}^\perp$.) To compute $T_{\mu\theta}$, it suffices to compute its action on these three subspaces.

Notice that for $g \in \mathcal{M}, (\Theta^* g)(z) = \theta(z)^* g(z)$, $z \in \mathbb{T}$. (This follows since θ is isometry valued on $\mathbb{T}$.) It follows that $\tilde{M}\Theta^* g = \Theta^* M g$ for $g \in \mathcal{M}$. Therefore, $\mathcal{H}_1$ is invariant under $M \oplus \tilde{M}$. Hence

$$T_{\mu\theta}(g, -\delta^{-1}\mu\Theta^* g) = (Mg, -\delta^{-1}\mu\Theta^* Mg), \quad g \in \mathcal{M}, \tag{2.1}$$

gives the action of $T_{\mu\theta}$ on $\mathcal{H}_1$.

Next we show that the action of $T_{\mu\theta}$ on $\mathcal{H}_2$ is given by

$$T_{\mu\theta}(g, 0) = (\delta^2 \Theta E \Theta^* Mg, -\delta\mu E\Theta^* Mg) + (Mg - \Theta E\Theta^* Mg, 0), \quad g \in \mathcal{M}^\perp. \tag{2.2}$$

Indeed, since Θ is an isometry, the first term in the right hand side (RHS) of (2.2) is in $\mathcal{H}_1$. Therefore, to show that the RHS is in $\mathcal{H}$, it is enough to establish that its second term is in $\mathcal{H}_2$, i.e., that $\mathcal{M}^\perp$ is invariant under $(I - \Theta E\Theta^*)M$. Since Θ is an isometry, this amounts to showing that $\mathcal{M}^\perp$ is contained in the kernel of $(I - E)\Theta^* M$. Since the kernel of $I - E$ is $\mathcal{K}$ (identified with the subspace of constant functions in $H^2_{\mathcal{K}}$), we need to show that $\Theta^* M$ maps $\mathcal{M}^\perp$ into $\mathcal{K}$. But this is obvious from the fact that, for $g \in H^2_{\mathcal{L}}$, $\Theta^* g$ is the projection to $H^2_{\mathcal{K}}$ of the element $z \mapsto \theta(z)^* g(z)$ of $L^2_{\mathcal{K}}$. Thus the RHS of (2.2) is indeed in $\mathcal{H}$. So, to complete the proof of (2.2), it is enough to verify that the difference between this RHS and $(Mg, 0)$ is in $\mathcal{H}^\perp$. But this difference is $(\mu^2\Theta E\Theta^* Mg, \delta\mu E\Theta^* Mg) = (\mu\Theta f, \delta f)$, where $f = \mu E\Theta^* Mg$. But this is indeed in $\mathcal{H}^\perp$ as is clear from the initial description of $\mathcal{H}^\perp$.

Finally, we claim that the action of $T_{\mu\theta}$ on $\mathcal{H}_3$ is given by

$$T_{\mu\theta}(0, \tilde{h}) = (-\delta\mu\Theta Eh, \mu^2 Eh + \widetilde{N^* h}), \qquad h \in H^2_{\mathcal{K}}. \tag{2.3}$$

Indeed, since $(-\delta\mu\Theta Eh, \mu^2 Eh)$ is in $\mathcal{H}_1$ and $(0, \widetilde{N^*h})$ is in $\mathcal{H}_3$, the RHS of (2.3) is in $\mathcal{H}$. So, to prove (2.3), it is enough to show that the difference of the RHS of (2.3) and $(0, \widetilde{M}\tilde{h})$ is in $\mathcal{H}^\perp$. But this difference equals $(\delta\mu\Theta Eh, \widetilde{M}\tilde{h} - \widetilde{N^*h} - \mu^2 Eh)$. Using $\mu^2 + \delta^2 = 1$ and the trivial identity $\widetilde{M}\tilde{h} - \widetilde{N^*h} = Eh$ for $h \in H^2_{\mathcal{K}}$, this vector equals $(\delta\mu\Theta Eh, \delta^2 Eh) = (\mu\Theta f, \delta f)$, where $f = \delta Eh$. This is in $\mathcal{H}^\perp$ by definition.

This completes our description of $T_{\mu\theta}$. Now define $U : H^2_{\mathcal{L}} \oplus H^2_{\mathcal{K}} \to \mathcal{H}$ by

$$Ug = \begin{cases} (\delta g, -\mu\Theta^* g) & \text{if } g \in \mathcal{M} \\ (g, 0) & \text{if } g \in \mathcal{M}^\perp \\ (0, \tilde{g}) & \text{if } g \in H^2_{\mathcal{K}}. \end{cases}$$

Since Θ is an isometry and $\delta^2 + \mu^2 = 1$, U is clearly a unitary operator. Put $T[\mu] = U^* T_{\mu\theta} U$. Since $T[\mu]$ is unitarily equivalent to $T_{\mu\theta}$, by Sz.-Nagy–Foias theory it is a cnu contraction whose characteristic function coincides with $\mu\theta$. The formule (2.1), (2.2), (2.3) transform to the following description of $T[\mu]$:

$$T[\mu]g = \begin{cases} Mg & \text{for } g \in \mathcal{M} \\ Mg - (1-\delta)\Theta E\Theta^* Mg & \text{for } g \in \mathcal{M}^\perp \\ (-\mu\Theta Eg) \oplus (N^* g) & \text{for } g \in H^2_{\mathcal{K}}. \end{cases}$$

Since $-\Theta$ is an isometry, it defines an unitary operator from $H^2_{\mathcal{K}}$ onto $\mathcal{M}$ which carries the range of E (viz. the subspace of $H^2_{\mathcal{K}}$ consisting of constant functions) onto coker(M_{22}). If we identify $\mathcal{K}$ with coker(M_{22}) via this map, then the above becomes the description of $T[\mu]$ given in the statement of the theorem. □

2.3 Remark: From the description of $T[\mu]$ given in the above theorem it is clear that both $\mathcal{M}$ and $\mathcal{M} \oplus H^2_{\mathcal{K}}$ are invariant for $T[\mu]$. The compression of $T[\mu]$ to the co-invariant subspace $\mathcal{M}^\perp$ is $M_{11} = T$. Also, it is easy to verify that the restriction of $T[\mu]$ to the invariant subspace $\mathcal{M} \oplus H^2_{\mathcal{K}}$ is the direct sum of dim $\mathcal{K}$ copies of the bilateral weighted shift T_μ with weight sequence $\{\ldots, 1, 1, \mu, 1, 1, \ldots\}$. By [1], the characteristic function of $T[\mu]|_{\mathcal{M}\oplus H^2_{\mathcal{K}}}$ is the constant function $\mu I_{\mathcal{K}}$. Since $\mu\theta = (\theta)(\mu I_{\mathcal{K}})$ is an instance of inner outer factorisation, the above theorem may be viewed as an elaboration of Theorem VI.1.1 in [7].

3 The characteristic function of the twisted Bergman shifts

Recall that for $\lambda > 0$, the twisted Bergman shift $M^{(\lambda)}$ is the multiplication operator on $\mathcal{H}^{(\lambda)}$ and D_λ denotes the discrete series representation of Möb on $\mathcal{H}^{(\lambda)}$. We now prove :

Theorem 3.1 *For $\lambda > 1$, the characteristic function of $M^{(\lambda)}$ coincides with the function $\theta_\lambda : \mathbb{D} \to \mathcal{B}(\mathcal{H}^{(\lambda+1)}, \mathcal{H}^{(\lambda-1)})$ defined by $\theta_\lambda(z) = \frac{1}{\sqrt{\lambda(\lambda-1)}} D_{\lambda-1}(\varphi_z)^* \partial^* D_{\lambda+1}(\varphi_z)$. (Here ∂^* is the adjoint of the differentiation operator $\partial : \mathcal{H}^{(\lambda-1)} \to \mathcal{H}^{(\lambda+1)}$.)*

Proof: It is slightly easier to prove that the characteristic function θ^*_λ of the adjoint $M^{(\lambda)*}$ is given by the formula

$$\theta^*_\lambda(z) = \frac{1}{\sqrt{\lambda(\lambda-1)}} D_{\lambda+1}(\varphi^*_z)^* \partial D_{\lambda-1}(\varphi^*_z), \tag{3.1}$$

where $\varphi_z^*(w) = \overline{\varphi_z(\bar{w})}$. Notice that $M^{(\lambda)}$ is a pure contraction, so that its defect spaces are both equal to the whole of $\mathcal{H}^{(\lambda)}$. Define the unitary operators $\Gamma : \mathcal{H}^{(\lambda)} \to \mathcal{H}^{(\lambda+1)}$ and $\Lambda : \mathcal{H}^{(\lambda-1)} \to \mathcal{H}^{(\lambda)}$ by $\Gamma(e_n^{(\lambda)}) = e_n^{(\lambda+1)}$, $\Lambda(e_n^{(\lambda-1)}) = e_n^{(\lambda)}$ for $n \geq 0$. (Here $\{e_n^{(\lambda)}\}$, for instance, is the standard orthonormal basis of $\mathcal{H}^{(\lambda)}$defined in section 1.11.) Define $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{H}^{(\lambda-1)}, \mathcal{H}^{(\lambda+1)})$ by $\theta(z) = -\Gamma\tilde{\theta}(z)\Lambda$, where $\tilde{\theta} : \mathbb{D} \to \mathcal{B}(\mathcal{H}^{(\lambda)})$ is the characteristic function of ${M^{(\lambda)}}^*$. Clearly, θ coincides with $\tilde{\theta}$. Now, one verifies that θ_λ^* defined by (3.1) is also given by

$$(\theta_\lambda^*(z)f)(w) = \frac{1}{\sqrt{\lambda(\lambda-1)}} f'(w) - \sqrt{\frac{\lambda-1}{\lambda}} \frac{z}{1-zw} f(w) \tag{3.2}$$

for $z, w \in \mathbb{D}$, $f \in \mathcal{H}^{(\lambda-1)}$. To verify this one uses the two trivial identities obtained by differentiating $\varphi_z^{*-1} \circ \varphi_z^* = id$ once and twice and the identity $\varphi_z^{*\prime\prime}(w)/\varphi_z^{*\prime}(w) = 2z/(1-zw)$. To verify that $\tilde{\theta}$ is also given by the RHS of (3.2), note that (3.2) holds for $f = e_n^{(\lambda-1)}$, $n \geq 0$ and that both sides of (3.2) are bounded operators in f. So, $\theta_\lambda^* = \tilde{\theta}$. □

Theorem 3.2 *The Sz.-Nagy–Foias model of $M^{(\lambda)}$ ($\lambda > 1$) is precisely the compression of $M^{(1)} \otimes I$ on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ to the orthocomplement of the subspace of functions vanishing on the diagonal. In consequence, the minimal isometric dilation of $M^{(\lambda)}$ is $M^{(1)} \otimes I$.*

Proof: Since $M^{(\lambda)}$, $\lambda \geq 1$, is in the class $C_{\cdot 0}$, its characteristic function θ_λ (given by (1.8)) is an inner function. Hence to describe the model operator T, it suffices to determine the invariant subspace $\mathcal{M}$ of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ corresponding to the inner function θ_λ. Recall that $\mathcal{M}$ is the closure of the range of the operator $\Theta_\lambda : \mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda+1)} \to \mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ given by $(\Theta_\lambda f)(z) = (\theta_\lambda(z)f(z))$, $z \in \mathbb{D}$, $f \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda+1)}$. (Here we are viewing elements of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda+1)}$ as $\mathcal{H}^{(\lambda+1)}$ - valued analytic functions on $\mathbb{D}$, but later in this proof we view the same elements as scalar valued analytic functions on the bidisc $\mathbb{D} \times \mathbb{D}$.) Of course, this amounts calculating $(\ker \Theta_\lambda^*)^\perp$. Note that $\Theta_\lambda^* f$ is the orthogonal projection into $H^{(1)} \otimes H^{(\lambda+1)}$ of the element $(z, \cdot) \mapsto \theta_\lambda^*(\bar{z}) f(z, \cdot)$ of $L^2(\mathbb{T}) \otimes H^{(\lambda+1)}$. Consequently, equation (3.2) implies that

$$(\Theta_\lambda^* f)(z,w) = \frac{1}{\sqrt{\lambda(\lambda-1)}} \frac{\partial}{\partial w} f(z,w) - \sqrt{\frac{\lambda-1}{\lambda}} P\Big((z,w) \mapsto \frac{\bar{z}}{1-\bar{z}w} f(z,w)\Big),$$

where P is the orthogonal projection onto the space $H^{(1)} \otimes H^{(\lambda+1)}$. An easy calculation in terms of the monomial basis leads to the formula

$$(\Theta_\lambda^* f)(z,w) = \frac{1}{\sqrt{\lambda(\lambda-1)}} \frac{\partial}{\partial w} f(z,w) - \sqrt{\frac{\lambda-1}{\lambda}} \frac{f(z,w) - f(w,w)}{z-w}.$$

Therefore the kernel of Θ_λ^* consists of all $f \in H^{(1)} \otimes H^{(\lambda-1)}$ satisfying the differential equation

$$\frac{\partial}{\partial w} f(z,w) = (\lambda - 1) \frac{f(z,w) - f(w,w)}{z-w}.$$

Another calculation with the monomial basis shows that the solutions to this equation form the closed linear span of $\{f_k : k = 0, 1, \ldots\}$ where the elements f_k of $H^{(1)} \otimes H^{(\lambda-1)}$ are as in Equation (4.2) below. Since we show in subsection 4.1 that this is the orthocomplement of the subspace of functions vanishing on the diagonal, we are done. $\square$

4 A continuum of new homogeneous operators from the twisted Bergman shifts

4.1 In this section we analyse the twisted Bergman shift $M^{(\lambda)}$ in the light of results of the previous two sections. Fix $\lambda > 1$, and let $\mathcal{L} = \mathcal{H}^{(\lambda-1)}$. So, $H^2_{\mathcal{L}} \cong \mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ is naturally identified with a Hilbert space of analytic functions on the bi-disc. Let $\mathcal{M}$ be the subspace of all functions in $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ which vanish on the diagonal $\{(z,w) \in \mathbb{D} \times \mathbb{D} : z = w\}$. Let $M = M^{(1)} \otimes I$ be the operation of multiplication by the first co-ordinate on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$. Thus, by Theorem 3.2, $M^{(\lambda)}$ is the compression of M to $\mathcal{M}^{\perp}$. As in notation 2.1, we write $M = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}$ with respect to the decomposition $\mathcal{M}^{\perp} \oplus \mathcal{M}$ of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$. Thus $M_{11} = M^{(\lambda)}$. For integers $k \geq \ell \geq 1$, let $e_{k,\ell} \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ be defined by

$$e_{k,\ell}(z,w) = (\lambda-1)^{1/2}\ell^{-1/2}\binom{\ell+\lambda-1}{\ell}^{-1/2}\left(\sum_{j=0}^{\ell-1}\binom{j+\lambda-2}{j}z^{k-j}w^j - \binom{\ell+\lambda-2}{\ell-1}z^{k-\ell}w^{\ell}\right), \tag{4.1}$$

$z, w \in \mathbb{D}$. For integers $k \geq 0$, define $f_k \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ by

$$f_k(z,w) = \binom{k+\lambda-1}{k}^{-1/2}\sum_{j=0}^{k}\binom{j+\lambda-2}{j}z^{k-j}w^j, \quad z,w \in \mathbb{D}. \tag{4.2}$$

We have the orthogonal decomposition $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)} = \oplus_{k\geq 0}\mathrm{Hom}(k)$, where $\mathrm{Hom}(k)$ is the space of all homogeneous polynomials of degree k (in two variables z, w). In view of the well known identity

$$\sum_{j=0}^{\ell-1}\binom{j+\lambda-2}{j} = \binom{\ell+\lambda-2}{\ell-1}, \tag{4.3}$$

it is easy to verify that $\{e_{k,\ell} : 1 \leq \ell \leq k\}$ is an orthonormal set of vectors in $\mathrm{Hom}(k) \cap \mathcal{M}$ and f_k is a unit vector orhogonal to all $e_{k,\ell}$, $1 \leq \ell \leq k$. Since $\mathrm{Hom}(k) \cap \mathcal{M}$ is clearly a co-dimension 1 subspace of the $(k+1)$-dimensional space $\mathrm{Hom}(k)$, it follows that $\{f_k : k = 0, 1, 2, \ldots\}$ is an orthonormal basis of $\mathcal{M}^{\perp}$ and $\{e_{k,\ell} : k = 1, 2, \ldots, \ell = 1, \ldots, k\}$ is an orthonormal basis of $\mathcal{M}$. (The special case $\lambda = 2$ of this computation occurs in [4].)

(To verify (4.3), note that both sides are polynomials in λ, so it is enough to verify it for positive integral values λ. In this case, $\binom{\ell+\lambda-2}{\ell-1}$ is the number of subsets of $\{1, 2, \ldots, \ell+\lambda-2\}$ of size $\lambda - 1$, while $\binom{j+\lambda-2}{j}$ is the number of such subsets with maximum element $= \lambda + j - 1$.)

It is easy to verify that the action of $M = M^{(1)} \otimes I$ on these basis elements is given by

$$M(e_{k,\ell}) = e_{k+1,\ell}, \quad M(f_k) = \sqrt{\frac{k+1}{k+\lambda}}f_{k+1} + \sqrt{\frac{\lambda-1}{k+\lambda}}e_{k+1,k+1}.$$

This yields the following description of the operators M_{ij}:

$$\begin{aligned} M_{11}: &\qquad f_k \to \sqrt{\frac{k+1}{k+\lambda}} f_{k+1}, \quad k \geq 0, \\ M_{21}: &\qquad f_k \to \sqrt{\frac{\lambda-1}{k+\lambda}} e_{k+1,k+1}, \quad k \geq 0, \\ M_{22}: &\qquad e_{k,\ell} \to e_{k+1,\ell}, \quad \ell \geq 1, k \geq \ell. \end{aligned}$$

(This again identifies M_{11} with $M^{(\lambda)}$.)

In particular, $\mathcal{K} := \operatorname{coker}(M_{22})$ is spanned by $\{e_{\ell,\ell} : \ell \geq 1\}$. Let $\{g_{\ell,m} : \ell \geq 1, m \geq 0\}$ be the orthonormal basis of $H^2_{\mathcal{K}}$ defined by $g_{\ell,m}(w) = w^m e_{\ell,\ell}$, $w \in \mathbb{D}$. With respect to this basis the actions of the operator $E : H^2_{\mathcal{K}} \to \mathcal{L} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(\lambda-1)}$ given by $Ef = f(0)$, and of the adjoint N^* of the multiplication (by co-ordinate function) operator N on $H^2_{\mathcal{K}}$ are as follows

$$\begin{aligned} E: &\qquad g_{\ell,m} \to \begin{cases} e_{\ell,\ell} & \text{if } m = 0, \ell \geq 1 \\ 0 & \text{if } m \geq 1, \ell \geq 1 \end{cases} \\ N^*: &\qquad g_{\ell,m} \to \begin{cases} 0 & \text{if } m = 0, \ell \geq 1 \\ g_{\ell,m-1} & \text{if } m \geq 1, \ell \geq 1. \end{cases} \end{aligned}$$

Thus in view of Theorem 2.1 we have all the ingredients for the description of the operators $M^{(\lambda)}[\mu]$, $\lambda > 1, 0 < \mu < 1$. However, for the sake of clarity, and to be consistent with Remark 2.3, it is better to rename the basis elements as follows. Put

$$h_{k,\ell} = \begin{cases} g_{\ell,-k} & \text{for } k \leq 0, \ell \geq 1 \\ e_{k+\ell-1,\ell} & \text{for } k \geq 1, \ell \geq 1. \end{cases}$$

Thus we get the following description of the operator $M^{(\lambda)}[\mu]$ (with $\delta = \sqrt{1-\mu^2}$) : This operator acts on a Hilbert space with orthonormal basis

$$\{f_k : k = 0, 1, 2, \ldots\} \cup \{h_{k,\ell} : k = 0, \pm 1, \pm 2, \ldots, \ell = 1, 2, \ldots\} \tag{4.4}$$

by the rule

$$M^{(\lambda)}[\mu] : \begin{cases} f_k \to \sqrt{\frac{k+1}{k+\lambda}} f_{k+1} + \delta \sqrt{\frac{\lambda-1}{k+\lambda}} h_{1,k+1} \\ h_{k,\ell} \to \begin{cases} \mu h_{k+1,\ell} & \text{if } k = 0, \ell \geq 1 \\ h_{k+1,\ell} & \text{if } k = \pm 1, \pm 2, \ldots, \ell \geq 1. \end{cases} \end{cases} \tag{4.5}$$

4.2 Recall that a cnu contraction T is said to be in the class $C_{\cdot 0}$ if $T^{*n}x \to 0$ for all x, and it is said to be in the class $C_{\cdot 1}$ if $T^{*n}x \not\to 0$ for all $x \neq 0$. T is in class $C_{\cdot 0}$ if and only if its characteristic function θ is inner (i.e., θ is almost surely isometry valued on the boundary of $\mathbb{D}$) and T is in class $C_{\cdot 1}$ if and only if its characteristic function θ is outer (i.e., the operator Θ given by (1.1) has dense range). For an arbitrary cnu contraction T on a Hilbert space $\mathcal{H}$, let $\mathcal{M}^\perp$ be the subspace of $\mathcal{H}$ consisting of all x for which $T^{*n}x \to 0$. Clearly, $\mathcal{M}$ is invariant for T. Let T_1 be the compression of T to $\mathcal{M}^\perp$ and let T_2 be the restriction T to $\mathcal{M}$. Thus with respect to the decomposition $\mathcal{M}^\perp \oplus \mathcal{M}$ of $\mathcal{H}$, T has a canonical triangularization $\begin{pmatrix} T_1 & 0 \\ X & T_2 \end{pmatrix}$ where T_1 is in $C_{\cdot 0}$ and T_2 is $C_{\cdot 1}$. T_1 and T_2 are called the $C_{\cdot 0}$ part of T and the $C_{\cdot 1}$ part of

T, respectively. Clearly, the bilateral shifts $T_\mu, 0 < \mu < 1$, mentioned in Remark 2.3 are in the class $C_{\cdot 1}$. In view of this remark, it follows that if T is in class $C_{\cdot 0}$ then the $C_{\cdot 0}$ part of the operator $T[\mu]$ of Theorem 2.1 is T while the $C_{\cdot 1}$ part is $\begin{pmatrix} M_{22} & \mu E \\ 0 & N^* \end{pmatrix}$. The latter is the direct sum of $\dim \mathcal{K}$ copies of T_μ.

LEMMA 4.1 *If T is a cnu contraction in the class $C_{\cdot 0}$ then $\varphi(T)$ is a cnu contraction in the same class for every φ in Möb.*

Proof: Let θ be the characteristic function of T. Thus θ is inner. Since φ maps T onto itself and sends null sets (with respect to arc length measure) to null sets, it follows that $\theta \circ \varphi^{-1}$ is also inner. Hence by (1.3) $\varphi(T)$ is in the class $C_{\cdot 0}$. $\square$

4.3 Let T be an operator on a Hilbert space $\mathcal{H}$ and let π be the representation of Möb on $\mathcal{H}$. Recall from [1] that π is said to be associated to T if the spectrum of T is contained in the closed unit disc and $\pi(\varphi)^* T \pi(\varphi) = \varphi(T)$ for all φ in Möb. Clearly, if T has an associated (projective unitary) representation then T is homogeneous. In [6], the converse statement was shown to hold for irreducible operators.

PROPOSITION 4.2 *Let T be a (homogeneous) cnu contraction with associated projective representation π. Let T_1 and T_2 be the $C_{\cdot 0}$ and $C_{\cdot 1}$ part of T respectively. Then there are projective representations π_1 and π_2 of Möb such that $\pi = \pi_1 \oplus \pi_2$ and π_j is associated to T_j, for $j = 1, 2$. In consequence, T_1 and T_2 are homogeneous operators. More generally, letting $T = \begin{pmatrix} T_1 & 0 \\ X & T_2 \end{pmatrix}$ be the canonical triangularisation of T (cf. section 4.2 above), one gets that for any scalar $t \geq 0$, the operator $\begin{pmatrix} T_1 & 0 \\ tX & T_2 \end{pmatrix}$ is homogeneous with associated representation π.*

Proof: Let $\mathcal{M}$ be the domain of T_2. It is sufficient to show that $\mathcal{M}$ (or equivalently $\mathcal{M}^\perp$) is reducing subspace for π, so that π breaks up as $\pi_1 \oplus \pi_2$ along $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. The rest of the statement will then be immediate consequence of [1, Proposition 2.4].

Fix φ in Möb. Since $T_1 \in C_{\cdot 0}$, by Lemma 4.1 we get $\varphi(T_1) \in C_{\cdot 0}$. That is, $\varphi(T_1)^{*n} x = (\varphi^*(T_1))^{*n} x \to 0$ for all $x \in \mathcal{M}^\perp$. But $\varphi(T_1)^*$ is the restriction of $\varphi(T)^*$ to $\mathcal{M}^\perp$. So we get $\varphi(T)^{*n} x \to 0$ for each $x \in \mathcal{M}^\perp$. That is, $\pi(\varphi)^* T^{*n} \pi(\varphi) x \to 0$ as $n \to \infty$. Since $\pi(\varphi)$ is unitary this implies $T^{*n}\pi(\varphi)x \to 0$. Thus $\pi(\varphi)x \in \mathcal{M}^\perp$. Since this holds for all $x \in \mathcal{M}^\perp$ and all φ in Möb, this means that $\mathcal{M}^\perp$ is invariant (hence reducing) under π. $\square$

THEOREM 4.3 *For any three real numbers $\lambda > 0, \mu > 0, \delta > 0$, let $M^{(\lambda)}[\mu, \delta]$ be the operator defined by the right hand side of (4.5). (So we have removed the restrictions $\lambda > 1, \mu < 1$ and $\delta = \sqrt{1 - \mu^2}$).) Then $M^{(\lambda)}[\mu, \delta]$, $\lambda > 0, \mu > 0, \delta > 0$, are mutually unitarily inequivalent homogeneous operators.*

Proof: First take $0 < \mu < 1$, $\delta = \sqrt{1 - \mu^2}$. Then the $C_{\cdot 0}$ part of the operator $M^{(\lambda)}[\mu, \delta] = M^{(\lambda)}[\mu]$ is $M^{(\lambda)}$, while the $C_{\cdot 1}$ part is the direct sum of infinitely many copies of T_μ. But $M^{(\lambda)}$ has associated representation D_λ, while T_μ has associated representation $D_1 \oplus D_2$ by a result in [1]. It follows that the $C_{\cdot 1}$ part has associated representation E, where E is the direct sum of infinitely many copies of $D_1 \oplus D_2$. So, by Proposition 4.2, $M^{(\lambda)}[\mu, \delta]$

has associated representation $\pi = D_\lambda \oplus E$. The point is that there is a projective representation π, independent of all parameters, such that π is associated with $M^{(\lambda)}[\mu, \sqrt{1-\mu^2}]$ for each fix μ in the range $0 < \mu < 1$. but replacing the X part in the canonical triangularization of $M^{(\lambda)}[\mu, \sqrt{1-\mu^2}]$ by an arbitrary positive scalar multiple has the effect of replacing $M^{(\lambda)}[\mu, \sqrt{1-\mu^2}]$ by $M^{(\lambda)}[\mu, \delta]$ for an arbitrary $\delta > 0$. So, by Proposition 4.2, $M^{(\lambda)}[\mu, \delta]$ is homogeneous with the same associated projective representation π for all such μ, δ. That is, $\pi(\varphi)^* M^{(\lambda)}[\mu, \delta]\pi(\varphi) = \varphi(M^{(\lambda)}[\mu, \delta])$ for all μ, δ with $0 < \mu < 1$, $\delta > 0$. But both sides of this equation are real analytic functions of μ for each fixed $\delta > 0$. So, by analytic continuation, the equation holds with the same associated representation π for $\mu > 0, \delta > 0$. (One might object that this argument is incorrect: $\varphi(M^{(\lambda)}[\mu, \delta])$ may not be defined for arbitrary μ, δ. This objection is overruled by noting that, according to [1, Lemma 2.2], to establish homogeneity of an operator T, it is enough to show that $\varphi(T)$ is unitarily equivalent to T for all φ in an arbitrarily small neighborhood of identity in Möb.) We omit the messy proof of the irreducibility and unitary inequivalence of these operators. $\square$

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