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ON THE INJECTIVITY OF TWISTED SPHERICAL MEANS ON $\mathbb{C}^n$

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ABSTRACT

In this paper we study the injectivity and injectivity sets for the twisted spherical means on $\mathbb{C}^n$. These results are then used to study the same problems on the reduced Hiesenberg group.

1. Introduction

Injectivity of the spherical mean value operator has been studied by several authors [1], [8], [12]. In [12] the second author has considered the problem in the set-up of L^p spaces on $\mathbb{R}^n$ and on the Heisenberg group H^n. Let us briefly recall the known results. Let μ_r be the normalised surface measure on the sphere $|x| = r$ in $\mathbb{R}^n$ and consider the spherical means of a continuous function

$$f * \mu_r(x) = \int_{|y|=r} f(x-y)d\mu_r(y).$$

If we take

$$f(x) = \varphi_\lambda(x) = c_n \frac{J_{n/2-1}(\lambda|x|)}{(\lambda|x|)^{n/2-1}}$$

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where J_α is the Bessel function of order α, then it is well known that

$$f * \mu_r(x) = \varphi_\lambda(r)\varphi_\lambda(x)$$

where $\varphi_\lambda(r)$ stands for $\varphi_\lambda(y)$ with $|y| = r$. Therefore, if r is a zero of $\varphi_\lambda(t)$ then $f * \mu_r = 0$. Note that $\varphi_\lambda \in L^p(\mathbb{R}^n)$, for $p > 2n/(n-1)$ and therefore, on $L^p(\mathbb{R}^n), p > 2n/(n-1)$ the spherical mean value operator is not injective. In [12] it was shown that when $1 \le p \le 2n/(n-1)$ the spherical mean value operator is indeed injective on $L^p(\mathbb{R}^n)$.

The situation on the Heisenberg group is different. Let $H^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the group law

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}\operatorname{Im}(z.\bar{w}))$$

and let μ_r be the normalised surface measure on $\{(z,0) : |z| = r\}$ and consider

$$f * \mu_r(z,t) = \int_{|w|=r} f((z,t)(-w,-s))d\mu_r(w).$$

Let

$$\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$$

be the Laguerre functions of type $(n-1)$. Define $e_k^\lambda(z,t) = e^{i\lambda t}\varphi_k^\lambda(z)$ where $\varphi_k^\lambda(z) = \varphi_k(\sqrt{|\lambda|}z)$ and consider the equation

$$e_k^\lambda * \mu_r(z,t) = \frac{k!(n-1)!}{(k+n-1)!}e_k^\lambda(r,0)e_k^\lambda(z,t)$$

where

$$e_k^\lambda(r,0) = \varphi_k^\lambda(r) = L_k^{n-1}(\frac{1}{2}|\lambda|r^2)e^{-\frac{1}{4}|\lambda|r^2}.$$

This follows from the fact that e_k^λ are spherical functions on the Heisenberg group (see [14]). If we choose λ and r so that $\varphi_k^\lambda(r) = 0$ then we have $e_k^\lambda * f(z,t) = 0$. Note that $e_k^\lambda \in L^p(H^n)$ only when $p = \infty$. As one can expect, it was shown in [12] that on $L^p(H^n)$ with $p < \infty$ the spherical mean value operator is injective. For $L^\infty(H^n)$ there is a two radius theorem (see [1]).

In this paper we look at the same problem in the context of the reduced Heisenberg group H^n_{red}. This is simply the set $\mathbb{C}^n \times S^1$ equipped with the group law

$$(z,e^{it})(w,e^{is}) = (z+w, e^{i(t+s+\frac{1}{2}\operatorname{Im}(z.\bar{w}))}).$$

If μ_r is as before and if λ is an integer, then e_k^λ is a function on H^n_{red} which satisfies $e_k^\lambda * \mu_r(z,e^{it}) = 0$ whenever $\varphi_k^\lambda(r) = 0$. Note that the functions e_k^λ are now in all

$L^p(H^n_{red})$ and so the spherical means are not injective on any L^p space. However, we will show that on suitable subclasses of functions the spherical means will be injective. In view of the counter examples these classes seem to be optimal.

Recently, a slightly different problem has received considerable attention. Given a set $S \subset \mathbb{R}^n$ consider the condition

$$f * \mu_r(x) = 0, \quad x \in S$$

for all $r > 0$. We say that the set S is a set of injectivity for the spherical mean value operator ("spherical Radon transform") if the above condition implies that $f = 0$. Note that we are assuming $f * \mu_r(x) = 0$ for all values of $r > 0$ but only on the set S. In the case of $\mathbb{R}^2$ this problem has been studied by Agranovsky and Quinto [3, 4]. In the set-up of L^p spaces Agranovsky et al. [2] have shown that the boundary of any bounded domain in $\mathbb{R}^n$ is a set of injectivity for L^p spaces as long as $1 \leq p \leq 2n/(n-1)$. They have also considered the case of symmetric spaces of non-compact type. See also the recent work of Rawat and Sitaram [9].

In this article we look at the problem of injectivity sets for the spherical mean value operator on the Heisenberg group. The problem naturally leads to the study of twisted spherical means on $\mathbb{C}^n$. Recently, Agranovsky and Rawat [5] have studied this problem and they have shown that the boundary of any bounded domain in $\mathbb{C}^n$ is a set of injectivity for the twisted spherical means in a class of functions having some exponential decay. When $n = 1$ we obtain an optimal class of functions for which spheres are sets of injectivity for the twisted spherical means. It would be interesting to get a complete characterisation of such sets; but the problem needs new ideas and techniques.

The organisation of this paper is as follows. In the next section we study injectivity of twisted spherical means and also treat the problem of inectivity sets for them. In the last section we use these results to study analogous problems in the context of Heisenberg groups. We study the twisted spherical means by using various properties of special Hermite and Laguerre functions. For the facts used we refer to the monographs [13] and [14].

2. Twisted spherical means

Let μ_r be the normalised surface measure on the sphere $|z| = r$ and let $f \times g$ stand for the twisted convolution

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}\operatorname{Im}(z.\bar{w})}dw.$$

The twisted spherical means of a continuous function f is then defined by

$$f \times \mu_r(z) = \int_{|w|=r} f(z-w) e^{\frac{i}{2}\operatorname{Im}(z.\bar{w})} d\mu_r(w).$$

If φ_k are the Laguerre functions introduced in the previous section, then it follows that

$$\varphi_k \times \mu_r(z) = \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r) \varphi_k(z) \tag{2.1}$$

and therefore $\varphi_k \times \mu_r(z) = 0$ whenever $\varphi_k(r) = 0$. Note that the functions φ_k are all Schwartz class functions.

We need to recall several facts concerning the special Hermite functions. These functions $\{\Phi_{\alpha,\beta}\}$ indexed by α, $\beta \in N^n$ form an orthonormal basis for $L^2(\mathbb{C}^n)$. The special Hermite expansion

$$f(z) = \sum_\alpha \sum_\beta (f, \Phi_{\alpha,\beta}) \Phi_{\alpha,\beta}(z)$$

of an L^2 function f can be put in the compact form

$$f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k(z).$$

We have the orthogonality relations

$$\Phi_{\alpha,\beta} \times \Phi_{\mu,\nu} = (2\pi)^{n/2} \delta_{\beta,\mu} \Phi_{\alpha,\nu}$$

and

$$\varphi_k \times \varphi_j = (2\pi)^n \delta_{k,j} \varphi_k$$

which will be used in the sequel. The functions φ_k can be represented as

$$\varphi_k(z) = (2\pi)^{n/2} \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(z).$$

Since the special Hermite functions are all Schwartz class functions, the special Hermite expansion of any tempered distribution f is well defined and it converges in the topology of tempered distributions. Note that $\varphi_k \times f$ are also well defined for tempered distributions.

Consider the action of the torus T^n on $\mathbb{C}^n$ given by $e^{i\theta} z = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$ where $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n})$. We say that a function f is m-homogeneous where

$m = (m_1, \ldots, m_n)$ is an n-tuple of integers if $f(e^{i\theta}z) = e^{im.\theta} f(z)$ for all $\theta \in \mathbb{R}^n$. It is known that the special Hermite functions $\Phi_{\alpha,\beta}$ are homogeneous:

$$\Phi_{\alpha,\beta}(e^{i\theta}z) = e^{i(\beta-\alpha).\theta}\Phi_{\alpha,\beta}(z).$$

Given a continuous function f on $\mathbb{C}^n$ we can form the multiple Fourier series

$$f(e^{i\theta}z) = \sum_m R_m f(z) e^{im.\theta}$$

where

$$R_m f(z) = \int_{T^n} f(e^{i\theta}z) e^{-im.\theta} d\theta$$

are the Fourier coefficients of $f(e^{i\theta}z)$. Note that $R_m f$ are m-homogeneous; they are called the m-homogenisation of the function f.

We say that a function f on $\mathbb{C}^n$ is of polynomial type if the coefficients $f_m(z)$ are homogeneous polynomials. Examples of such functions are $\varphi_k(z)e^{\frac{1}{4}|z|^2}$. We also define the symplectic Fourier transform of a function f on $\mathbb{C}^n$ by

$$\hat{f}_S(z) = f \times 1(z) = \int_{\mathbb{C}^n} f(z-w) e^{\frac{i}{2}\operatorname{Im}(z.\bar{w})} dw.$$

The functions $\Phi_{\alpha,\beta}$ and φ_k are all eigenfunctions of the symplectic Fourier transform with eigenvalues $(-1)^{|\beta|}$ and $(-1)^k$ respectively. Since we have

$$f \times \varphi_k(z) = \sum_\alpha \sum_{|\beta|=k} (f, \Phi_{\alpha,\beta})\Phi_{\alpha,\beta}(z)$$

the projections $f \times \varphi_k$ are eigenfunctions of the symplectic Fourier transform. We also note that the symplectic Fourier transform and the ordinary Fourier transform on $\mathbb{R}^{2n}$ are related by $\hat{f}_S(z) = \hat{f}(-y/2, x/2)$ where $z = x + iy$. We are now ready to state and prove our results.

THEOREM 2.1: *Let f be a tempered continuous function such that $f \times \mu_r = 0$ for some $r > 0$. Then* (i) *f is an eigenfunction of the symplectic Fourier transform, and* (ii) *$f(z)e^{\frac{1}{4}|z|^2}$ is a function of polynomial type.*

Proof: Taking the twisted convolution of φ_k with the equation $f \times \mu_r = 0$ and using the property (2.1) we get

$$\varphi_k(r) f \times \varphi_k(z) = 0$$

for all $k = 0, 1, 2, \ldots$. Since the Laguerre polynomials $L_k^{n-1}(t)$ have distinct zeros, in the above equation $\varphi_k(r) = 0$ for at most one value of k, say for $k = j$. This means that $f \times \varphi_k = 0$ for all values of k other than j. But then f will be proportional to $f \times \varphi_j$ and so it is an eigenfunction of the symplectic Fourier transform.

Now, we claim that $f(z)e^{\frac{1}{4}|z|^2}$ is of polynomial type. First we observe that

$$R_m(f \times \varphi_j) = R_m(f) \times \varphi_j$$

and therefore, as $f = (2\pi)^{-n} f \times \varphi_j$, we have the relation

$$R_m(f) = (2\pi)^{-n} R_m(f) \times \varphi_j.$$

But then

$$R_m(f) \times \varphi_j(z) = \sum_{|\mu|=j} R_m(f) \times \Phi_{\mu,\mu}(z)$$

and, since $R_m(f)$ is m-homogeneous, it follows that

$$R_m(f) \times \varphi_j(z) = \sum_{|\mu|=j} (f, \Phi_{\mu-m,\mu})\Phi_{\mu-m,\mu}(z).$$

This shows that

$$R_m(f(z)e^{\frac{1}{4}|z|^2}) = \sum_{|\mu|=j} C_\mu \Phi_{\mu-m,\mu}(z)e^{\frac{1}{4}|z|^2}.$$

Finally, since $\Phi_{\alpha,\beta}(z)e^{\frac{1}{4}|z|^2}$ are polynomials we get part (ii) of the theorem. ∎

COROLLARY 2.2: *Suppose f is a continuous function such that $f(z)e^{\frac{1}{4}|z|^2}$ is in $L^p(\mathbb{C}^n)$ for some $1 \le p \le \infty$. If $f \times \mu_r = 0$ for some $r > 0$ then $f = 0$.*

Proof: By the theorem f is a function of polynomial type:

$$R_m(f)(z) = p_m(z)e^{-\frac{1}{4}|z|^2}.$$

Now

$$p_m(z) = \int_{T^n} f(e^{i\theta}z)e^{\frac{1}{4}|z|^2}e^{-i\theta.m}d\theta$$

shows that $p_m \in L^p(\mathbb{C}^n)$ whenever $f(z)e^{\frac{1}{4}|z|^2} \in L^p(\mathbb{C}^n)$. This forces $p_m = 0$ for all m and consequently $f = 0$. ∎

Using conclusion (i) of the theorem we can identify several classes of functions on which the twisted spherical mean value operator is injective. Let us call a space V of tempered functions a Paley–Wiener class if $f, \hat{f}_S \in V$ implies $f = 0$.

COROLLARY 2.3: *The twisted spherical mean value operator is injective on any Paley–Wiener class of functions.*

We have several examples of Paley–Wiener classes. By the classical Paley–Wiener theorem for the Fourier transform $C_0^\infty(\mathbb{C}^n)$ is a Paley–Wiener class. By a theorem of Benedicks [6] on the Fourier transform pairs the class of functions whose supports have finite Lebesgue measures is a Paley–Wiener class. Another example is given by Hardy's theorem [10]: the class of functions such that $|f(z)| \leq Ce^{-a|z|^2}$ for some $a > \frac{1}{4}$. On all these spaces the spherical mean value operator is injective.

Another class of functions with the Paley–Wiener property can be constructed using the following result of Beurling; see Hörmander [8].

THEOREM 2.4: *Let f be an integrable function such that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)\hat{f}(y)|e^{|xy|}dxdy < \infty.$$

Then $f = 0$.

THEOREM 2.5: *Define $V = \{f(z)\colon |f(z)| \leq g(z)e^{-\frac{1}{4}|z|^2}, g \in L^1(\mathbb{C}^n)\}$. Then V is a Paley–Wiener class.*

Proof: We prove this theorem by applying Beurling's theorem to the Radon transform of f. Recall that the Radon transform of a function f on $\mathbb{R}^n$ is defined by (see Folland [7])

$$Rf(s,\omega) = \int_{x.\omega=s} f(x)dx$$

where $s \in \mathbb{R}$, $\omega \in S^{n-1}$ and dx is the $(n-1)$-dimensional Lebesgue measure on the hyperplane $x.\omega = s$. The relation between the Fourier transform and the Radon transform is given by $\hat{f}(\rho\omega) = \tilde{Rf}(\rho,\omega)$, where $\tilde{Rf}$ is the one-dimensional Fourier transform of Rf in the s variable.

If we assume that $f(z)e^{\frac{1}{4}|z|^2}$ is integrable, then it follows that $Rf(s,\omega)e^{\frac{1}{4}s^2} \in L^1(\mathbb{R})$ for almost every $\omega \in S^{2n-1}$. If $\hat{f}_S \in V$ then we have

$$|\hat{f}(z)| = |\hat{f}_S(-2y,-2x)| \leq g(z)e^{-|z|^2},$$

where g is integrable on $\mathbb{C}$. This translates into the condition $|\tilde{Rf}(\rho,\omega)|e^{\rho^2} \in L^1(\mathbb{R})$. Finally

$$\int\int |Rf(s,\omega)||\tilde{Rf}(\rho,\omega)|e^{|s\rho|}dsd\rho \leq \int\int |Rf(s,\omega)|e^{\frac{1}{4}s^2}|\tilde{Rf}(\rho,\omega)|e^{\rho^2}dsd\rho < \infty.$$

Therefore, $Rf(s,\omega) = 0$ for almost all s and ω. By the inversion theorem for the Radon transform we get $f = 0$. ∎

We now consider injectivity sets for the twisted spherical means. Recently, the folowing theorem has been proved by Agranovsky and Rawat [5]:

THEOREM 2.6: *For $\epsilon > 0$ and $1 \le p \le \infty$ let*

$$V_{p,\epsilon} = \{f(z)\colon f(z)e^{(\frac{1}{4}+\epsilon)|z|^2} \in L^p(\mathbb{C}^n)\}.$$

Then the boundary Γ of any bounded domain Ω in $\mathbb{C}^n$ is a set of injectivity for the twisted spherical mean value operator on $V_{p,\epsilon}$.

This theorem is proved by studying radial eigenfunctions of the special Hermite operator L. Since the only functions known to satisfy $f \times \mu_r(z) = 0$ for all $r > 0$ on a set Γ are the Laguerre functions φ_k which have the property that $\varphi_k(z)e^{\frac{1}{4}|z|^2}$ are polynomials, it is natural to conjecture that the above theorem is true even when $\epsilon = 0$. However, an attempt to use the techniques developed in [2] (which uses the wave equation) failed as the condition $f(z)e^{\frac{1}{4}|z|^2} \in L^p(\mathbb{C}^n)$ is not invariant under twisted translations. There is at least one special case in which the conjecture is true; see Theorem 2.9 below.

The following simple observation leads to a strengthening of the above result. The symplectic Fourier transform of a function can be written as $\hat{f}_S = f \times 1$ and hence

$$\hat{f}_S \times \mu_r(z) = f \times 1 \times \mu_r(z).$$

But

$$1 \times \mu_r(z) = c_n \frac{J_{n-1}(r|z|)}{(r|z|)^{n-1}}$$

and therefore

$$\hat{f}_S \times \mu_r(z) = c_n \int_{\mathbb{C}^n} f(z-w)e^{\frac{i}{2}\operatorname{Im} z.\bar{w}} \frac{J_{n-1}(r|w|)}{(r|w|)^{n-1}} dw.$$

The above reduces to the integral

$$C_n \int_0^\infty f \times \mu_s(z) \frac{J_{n-1}(rs)}{(rs)^{n-1}} s^{2n-1} ds.$$

From this it follows that

$$\hat{f}_S \times \mu_r(z) = 0$$

on Γ whenever $f \times \mu_s(z) = 0$ on Γ for all $s > 0$. So, if either f or $\hat{f}_S$ satisfies the conditions of Theorem 2.6 then Γ will be an injectivity set.

COROLLARY 2.7: *Suppose V is a class of functions such that either $f(z)e^{a|z|^2}$ or $\hat{f}_S(z)e^{a|z|^2}$ belongs to $L^p(\mathbb{C}^n)$ for some $a > \frac{1}{4}$. Then the boundary Γ of any bounded domain Ω is a set of injectivity for the twisted spherical means on V.*

We remark that the corollary treats a certain class of functions which cannot have any exponential decay. This is the case when V is the image under Fourier transform of the class $C_0^\infty(\mathbb{C}^n)$. Indeed, any f in this class cannot have any exponential decay in view of Hardy's theorem [10]. But for this class the boundary of any bounded domain is a set of injectivity.

When Ω is a ball in $\mathbb{C}^n$, the following theorem gives a class of functions for which $f \times \mu_r(z)$ cannot vanish on the boundary for all values of r.

THEOREM 2.8: *Let $V_p(\zeta)$ be the space of continuous functions f on $\mathbb{C}^n$ for which*

$$f(z)e^{\frac{1}{4}|z-\zeta|^2} \in L^p(\mathbb{C}^n)$$

and $f(\zeta) \neq 0$. Then $f \times \mu_r(z)$ cannot vanish on any sphere $|z-\zeta| = R$ for all values of $r > 0$.

Proof: Without loss of generality we can very well assume that $\zeta = 0$. Consider the radialisation

$$Rf(z) = \int_{U(n)} f(\sigma z)\, d\sigma$$

of f. Since $|z| = R$ is invariant under the action of $U(n)$ it follows that $f \times \mu_r(z) = 0$ on $|z| = R$ implies $Rf \times \mu_r(z) = 0$ on $|z| = R$ as well. But now Rf is a radial function and therefore

$$Rf \times \mu_r(z) = Rf \times \mu_{|z|}(w), \quad |w| = r.$$

This means that $Rf \times \mu_R(w) = 0$ for all w. Note that $Rf(z)e^{\frac{1}{4}|z|^2} \in L^p(\mathbb{C}^n)$ and by Corollary 2.2 we conclude that $Rf = 0$. But then $f(0) = Rf(0) \neq 0$, which is a contradiction. ∎

In the special case when $n = 1$ and Ω is a ball we prove the above mentioned conjecture in the affirmative. In the proof of the theorem below we make use of the specific knowledge of the zeros of Laguerre polynomials. In the higher dimensional case we have to deal with the zero sets of special Hermite functions which are not well understood.

THEOREM 2.9: *Let $n = 1$ and let $V_p(\zeta)$ be as in the previous theorem. Then any sphere $|z-\zeta| = R$ is a set of injectivity for the twisted spherical means on $V_p(\zeta)$.*

Proof: Again we may assume that $\zeta = 0$. Let $R_m f$ be the m-radialisation of f. Then $f \times \mu_r(z) = 0$ on the sphere $|z| = R$ implies that $R_m f \times \mu_r(z) = 0$ on the same sphere. From this it follows that $R_m f \times \varphi_k(z) = 0$ on $|z| = R$ for any k. But now $R_m f$ is m-homogeneous and therefore

$$R_m f \times \varphi_k(z) = (R_m f, \Phi_{k-m,k})\Phi_{k-m,k}(z).$$

We know that (see Theorem 1.3.5, [13])

$$\Phi_{k-m,k}(z) = c_k z^m L_{k-m}^m(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$$

for $k \geq m \geq 0$, and when $m \leq 0$

$$\Phi_{k-m,k}(z) = c_k' \bar{z}^{-m} L_k^{-m}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}.$$

We use the fact that the zeros of $L_k^\alpha(t)$ are all distinct (see Szego [11]). Therefore, $R_m f \times \mu_r(z) = 0$ on the sphere $|z| = R$ implies that $(R_m f, \Phi_{k-m,k}) = 0$ for all values of k except possibly for one value, say $k = j$. This means that

$$R_m f(z) = (R_m f, \Phi_{j-m,j})\Phi_{j-m,j}(z).$$

But now, on the one hand, $R_m f(z)e^{\frac{1}{4}|z|^2}$ is in $L^p(\mathbb{C})$ and, on the other hand, $\Phi_{j-m,j}(z)e^{\frac{1}{4}|z|^2}$ cannot be in $L^p(\mathbb{C})$. This forces $R_m f = 0$ for all values of m and hence $f = 0$. ∎

An examination of the above proof shows that the following is true.

THEOREM 2.10: *Let Γ be any unbounded rotation invariant subset of $\mathbb{C}$. Then it is a set of injectivity for the twisted spherical means on $L^p(\mathbb{C}), 1 \leq p \leq \infty$.*

Proof: Proceeding as above, we arrive at

$$R_m f \times \varphi_k(z) = c_k(R_m f, \Phi_{k-m,k})\Phi_{k-m,k}(z).$$

Since $\Phi_{k-m,k}$ can vanish only on a finite number of spheres, we conclude that $R_m f \times \varphi_k = 0$ for all m and k. Hence $f = 0$. ∎

3. Spherical means on the Heisenberg group

In this section we prove some results concerning the spherical means on the Heisenberg groups. First we consider the injectivity of the spherical mean value operator on the reduced Heisenberg group H^n_{red}. Recall that the spherical means of a function on the reduced Heisenberg group is given by

$$f * \mu_r(z,t) = \int_{|w|=r} f(z-w, e^{i(t-\frac{1}{2}\operatorname{Im} z.\bar{w})})d\mu_r(w).$$

As before, μ_r is the normalised surface measure on the sphere

$$S_r = \{(z,0): |z| = r\}.$$

Using the results of the previous section we can prove the following

THEOREM 3.1: *Let f be a continuous function on the reduced Heisenberg group such that $f(z,t)e^{a|z|^2}$ is integrable for all $a > 0$. If $f * \mu_r = 0$ for some $r > 0$ then $f = 0$.*

Proof: The convolution equation $f * \mu_r = 0$ is transformed into a family of twisted convolution equations by taking the Fourier series in the t variable. In fact,

$$\int_0^{2\pi} f * \mu_r(z,t)e^{-ikt}dt = 0$$

gives, after a calculation, the equations

$$f^k *_k \mu_r(z) = 0$$

on $\mathbb{C}^n$ where f^k are the Fourier coefficients

$$f^k(z) = \int_0^{2\pi} f(z,t)e^{-ikt}dt$$

and $f^k *_k \mu_r(z)$ is the k-twisted convolution

$$f^k *_k \mu_r(z) = \int_{\mathbb{C}^n} f^k(z-w)e^{\frac{i}{2}k\operatorname{Im}(z.\bar{w})}d\mu_r(w).$$

Note that when $k = 0$ the k-twisted convolution reduces to the ordinary convolution on $\mathbb{C}^n$. When $k \neq 0$, the equation $f^k *_k \mu_r(z) = 0$ together with the condition $f^k(z)e^{\frac{|k|}{4}|z|^2} \in L^1(\mathbb{C}^n)$ implies that $f^k = 0$. The case $k = 1$ was treated in the previous section and the proof in the general case is similar. When $k = 0$ we have $f^0 * \mu_r = 0$ and, by taking the Fourier transform and noting that $\hat{\mu_r}$ is an entire function, we conclude that $f^0 = 0$. Thus, all the Fourier coefficients of f are zero and hence $f = 0$. ∎

We now consider the problem of injectivity sets for the spherical means. We first prove an analogue of Theorem 2.8 for the reduced Heisenberg group.

THEOREM 3.2: *Let $\Gamma = S_R(\zeta) \times S^1$ where $S_R(\zeta)$ is the sphere of radius R centered at ζ. Suppose f is a continuous function on H^1_{red} which satisfies the condition $|f(z,t)| \leq Ce^{-a|z-\zeta|^2}$ for every $a > 0$. Then $f * \mu_r(z,t) = 0$ on Γ for all r implies $f = 0$.*

Proof: Again by taking the Fourier series in the t variable we get

$$f^k *_k \mu_r(z,t) = 0, \quad (z,t) \in S_R(\zeta), \quad r > 0.$$

Since $|f^k(z)| \leq Ce^{-\frac{|k|}{4}|z|^2}$ we can appeal to an analogue of Theorem 2.8 for k-twisted convolutions to conclude that $f^k = 0$. By the result of [2], $f^0 = 0$ as well. Hence the theorem. ∎

On the Heisenberg group H^n, spherical means of the function e^λ_k vanish on sets of the form $S_R \times \mathbb{R}$. Here the radius R cannot be arbitrary; it is related to the zeros of the Laguerre polynomials $L^{n-1}_k(t)$. So, only for finitely many values of R does the above property hold. By a clever choice of σ and τ we can show that sets of the form

$$\Gamma = (S_\sigma \times \mathbb{R}) \cup (S_\tau \times \mathbb{R})$$

are sets of injectivity for the spherical means on the Heisenberg group for L^p spaces.

THEOREM 3.3: *Let $n = 1$ and let Γ be as above. Assume that σ^2/τ^2 is not a quotient of zeros of $L^\alpha_k(t)$ for any k and $\alpha \geq 0$. Suppose $f(z,t)$ is integrable in the t-variable and $f^\lambda \in L^p(\mathbb{C}^n)$ for any λ. If $f * \mu_r(z,t) = 0$ on Γ for all $r > 0$ then $f = 0$.*

Proof: The integrability assumption in the t variable allows us to take the Fourier transform in that variable which reduces the equation $f * \mu_r(z,t) = 0$ to a family of twisted convolution equations. Thus we have

$$f^\lambda *_\lambda \mu_r(z) = 0, \quad z \in S_\sigma \cup S_\tau$$

for all values of $r > 0$ and $\lambda \in \mathbb{R}$. Now proceeding as in the proof of Theorem 2.8 we arrive at

$$R_m f^\lambda *_\lambda \varphi^\lambda_k(z) = 0, \quad z \in S_\sigma \cup S_\tau.$$

This leads to the equations

$$(R_m f^\lambda, \Phi^\lambda_{k-m,k}) L^m_{k-m}\Big(\frac{|\lambda|}{2}s^2\Big) = 0, \quad s = \tau, \sigma$$

for $m \geq 0$, and for $m < 0$

$$(R_m f^\lambda, \Phi^\lambda_{k-m,k}) L_k^{-m}\Big(\frac{|\lambda|}{2}s^2\Big) = 0, \quad s = \tau, \sigma.$$

Since σ^2/τ^2 is not a quotient of zeros of $L_k^\alpha(t)$, the above equations force

$$R_m f^\lambda *_\lambda \varphi_k^\lambda(z) = (\mathrm{Const})(R_m f^\lambda, \Phi_{k-m,k})\Phi_{k-m,k}(z) = 0$$

for any k and m. So we get $f^\lambda = 0$ for all $\lambda \neq 0$. Hence $f = 0$, which proves the theorem. ∎

It would be interesting to see if an analogue of the above theorem holds in the higher dimensional case. We are in the same situation as in the n-dimensional analogue of Theorem 2.8.

Note added in Proof: For the n-dimensional analogue of Theorem 2.9 see: E. K. Narayanan and S. Thangavelu, *Injectivity sets for the spherical means on the Heisenberg group*, preprint (2000).

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