ON SYSTEM RELIABILITY UNDER STRESS-STRENGTH MODELING

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ABSTRACT

In this paper, we consider stress-strength modeling for calculation of system reliability for a general coherent system under different stress-strength scenarios. We first consider a random system level stress common to all the components. Then we deal with the stress present locally at component level in addition to the common system level stress.

Key Words: Coherent system; Association; Reliability bounds

1. INTRODUCTION

Literature on stress-strength modeling has been generally concerned with reliability of a unit (e.g., a component) with strength Y, which is subject to stress X, both Y and X being independent random variables.

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The reliability of the unit is given by the probability $P[X < Y]$, which has been calculated for different distributions of Y and X. See [1] for detailed discussion on this and calculation of reliability for different types of distributional assumptions (See also [2] and [3]). Stress-strength modeling for system reliability has also received some attention ([4]–[7]; see also [8]). However, their attention has been mostly focussed on k-out-of-n systems. Hanagal [9], apparently for the first time, considered dependent strengths for the two components in a parallel system subject to common stress and calculated system reliability assuming bivariate exponential distribution for the two strength variables. However, there does not seem to be much work on stress-strength modeling for more complicated coherent systems. Also, all the work has been with a random stress common to all the components. There has been no allowance for different component level stresses in addition to one common system level stress.

It is to be noted that if there are only component level stresses (no common stress), then the calculation of system reliability is as routine as those in Chapter 2 of [10] with the component reliabilities replaced by $P[X_i < Y_i]$, where X_i and Y_i denote the stress to and strength of, respectively, the ith component, for $i = 1, \ldots, n$, with n being the number of components. If the components act independently, then the calculation is easy; otherwise, one has to work with the bounds. This case of only component level stress is, therefore, not interesting.

In this work, we address some relevant issues on calculation of system reliability under different stress-strength scenarios. In Section 2, we discuss system reliability with a common system level stress for a general coherent system including some results on bounds. Section 3 considers both system and component level stresses. Section 4 ends with a discussion.

2. SYSTEM RELIABILITY WITH COMMON STRESS

This case has been well-studied with independent strengths of the components for simple systems as discussed in the previous section. Let us consider a general coherent system ϕ with n components with random strengths $T_1, \ldots, T_n$. Suppose they are subject to the common system level stress S. Suppose that the T_i's have marginal distributions given by $F_i(t) = P[T_i \leq t]$ for $i = 1, \ldots, n$, and S follows a distribution given by $F_S(s) = P[S \leq s]$ with range $\mathcal{S}$. It is easy to see that the system reliability h_ϕ can be written as

$$h_\phi = \int_{\mathcal{S}} h_\phi(s)\, f_S(s)\, ds, \tag{1}$$

where $h_\phi(s)$ denotes the system reliability for given $S = s$ and $f_S(\cdot)$ denotes the density of S. Suppose now that the coherent system has p minimal path sets given by $P_1, \dots, P_p$ and k minimal cut sets given by $K_1, \dots, K_k$. Then, assuming S to be independent of the T_i's,

$$h_\phi(s) = P\left[\max_{1 \le i \le p} \min_{j \in P_i} T_j > s\right] = P\left[\min_{1 \le i \le k} \max_{j \in K_i} T_j > s\right].$$

One can make use of the inclusion-exclusion principle to compute $h_\phi(s)$ as follows.

$$\begin{aligned} h_\phi(s) &= P\left[\max_{1 \le i \le p} \min_{j \in P_i} T_j > s\right] \\ &= P\left[\bigcup_{i=1}^{p}\left\{\min_{j \in P_i} T_j > s\right\}\right] \\ &= \sum_{i=1}^{p} P\left[\min_{j \in P_i} T_j > s\right] - \sum_{i < i'} P\left[\min_{j \in P_i \cup P_{i'}} T_j > s\right] + \cdots \end{aligned} \quad (2)$$

If the T_j's are independent, all the probabilities in (2) can be easily computed as, for a given set A of components, $P[\min_{j \in A} T_j > s] = \prod_{j \in A} \bar{F}_j(s)$. One can also calculate $h_\phi(s)$ using the minimal cut sets as

$$\begin{aligned} h_\phi(s) &= P\left[\min_{1 \le i \le k} \max_{j \in K_i} T_j > s\right] \\ &= P\left[\bigcap_{i=1}^{k}\left\{\max_{j \in K_i} T_j > s\right\}\right] \\ &= \sum_{i=1}^{k} P\left[\max_{j \in K_i} T_j > s\right] - \sum_{i < i'} P\left[\max_{j \in K_i \cup K_{i'}} T_j > s\right] + \cdots, \end{aligned} \quad (3)$$

where, for a given set A and independent T_j's, $P[\max_{j \in A} T_j > s] = 1 - \prod_{j \in A} F_j(s)$.

For independent T_j's, there is a third approach to calculate $h_\phi(s)$ by exploiting a modular decomposition of ϕ given by, say, $\phi = \psi(\chi_1, \dots, \chi_r)$. Then, $h_\phi(s)$ can be written as $h_\psi(h_{\chi_1}(s), \dots, h_{\chi_r}(s))$, where $h(s)$ with a system as subscript denotes reliability of the corresponding system, given $S = s$, which can be routinely calculated using standard methods for independent components. Then, the system reliability can be calculated using (1).

Let us assume independence of T_i's for the time. For the two simplest systems, series and parallel, we get, using (2), $h_\phi(s) = \prod_{j=1}^{n} \bar{F}_j(s)$, and, using (3), $h_\phi(s) = 1 - \prod_{j=1}^{n} F_j(s)$, respectively. When T_j follows exponential (λ_j) distribution, for $j = 1, \ldots, n$, the above two $h_\phi(s)$'s reduce to $\exp[-s\sum_{j=1}^{n}\lambda_j]$ and $1 - \prod_{j=1}^{n}(1 - e^{-\lambda_j s})$, respectively. If S also follows an exponential distribution with parameter α, say, then the two system reliabilities can be obtained as

$$h_\phi = \frac{\alpha}{\alpha + \sum_{j=1}^{n}\lambda_j},$$

and

$$h_\phi = \sum_{j=1}^{n}\frac{\alpha}{\alpha+\lambda_j} - \sum_{j<j'}\frac{\alpha}{\alpha+\lambda_j+\lambda_{j'}} + \cdots + (-1)^{n+1}\frac{\alpha}{\alpha+\sum_{j=1}^{n}\lambda_j},$$

for series and parallel system, respectively. We consider three examples of coherent system taken from Chapter 1 of [10]: (*i*) 2-out-of-3 system, (*ii*) bridge system and (*iii*) stereo system. Briefly, a 2-out-of-3 system functions if at least two of its three components function. A bridge system has a centrally located component {3}, with two starting components {1, 2} and two terminal components {4, 5}, in such a way that the system can function either through one starting component and the corresponding terminal one ({1, 4} or {2, 5}) or passing through the central component and the opposite terminal one ({1, 3, 5} or {2, 3, 4}). A stereo system has two players {1, 2} in parallel, one amplifier {3}, and two speakers {4, 5} in parallel; the system functions if at least one player, the amplifier and at least one speaker work. For the first two systems, we demonstrate the use of (2) or (3), and (1), to compute system reliability. Since there exists a non-trivial modular decomposition only for the stereo system, we demonstrate the use of the third approach for this example.

For a 2-out-of-3 system, the minimal path sets are {1, 2}, {2, 3} and {1, 3}; they are also the minimal cut sets. Using (2), we have

$$h_\phi(s) = \bar{F}_1(s)\bar{F}_2(s) + \bar{F}_2(s)\bar{F}_3(s) + \bar{F}_1(s)\bar{F}_3(s) - 2\bar{F}_1(s)\bar{F}_2(s)\bar{F}_3(s). \quad (4)$$

Using (3) also, one can derive (4). For T_j following exponential (λ_j) distribution, for $j = 1, 2, 3$, and S following exponential (α) distribution, (4) and (1) give

$$h_\phi = \frac{\alpha}{\alpha+\lambda_1+\lambda_2} + \frac{\alpha}{\alpha+\lambda_2+\lambda_3} + \frac{\alpha}{\alpha+\lambda_1+\lambda_3} - \frac{2\alpha}{\alpha+\lambda_1+\lambda_2+\lambda_3}.$$

For the bridge system, the minimal path sets are {1, 4}, {2, 5}, {1, 3, 5} and {2, 3, 4}, and the minimal cut sets are {1, 2}, {4, 5}, {1, 3, 5} and {2, 3, 4}.

Using the minimal path sets and (2), one can find $h_\phi(s)$ as

$$\begin{aligned} h_\phi(s) = {}& \bar{F}_1(s)\bar{F}_4(s) + \bar{F}_2(s)\bar{F}_5(s) + \bar{F}_1(s)\bar{F}_3(s)\bar{F}_5(s) + \bar{F}_2(s)\bar{F}_3(s)\bar{F}_4(s) \\ & - \bar{F}_1(s)\bar{F}_2(s)\bar{F}_3(s)\bar{F}_4(s) - \bar{F}_1(s)\bar{F}_2(s)\bar{F}_3(s)\bar{F}_5(s) \\ & - \bar{F}_1(s)\bar{F}_2(s)\bar{F}_4(s)\bar{F}_5(s) - \bar{F}_1(s)\bar{F}_3(s)\bar{F}_4(s)\bar{F}_5(s) \\ & - \bar{F}_2(s)\bar{F}_3(s)\bar{F}_4(s)\bar{F}_5(s) + 2\bar{F}_1(s)\bar{F}_2(s)\bar{F}_3(s)\bar{F}_4(s)\bar{F}_5(s). \end{aligned} \quad (5)$$

This can also be derived using the minimal cut sets and (3). For the T_j's having exponential(λ_j) distributions and S having exponential(α) distribution, we get, from (5) and (1),

$$\begin{aligned} h_\phi = {}& \frac{\alpha}{\alpha+\lambda_1+\lambda_4} + \frac{\alpha}{\alpha+\lambda_2+\lambda_5} + \frac{\alpha}{\alpha+\lambda_1+\lambda_3+\lambda_5} + \frac{\alpha}{\alpha+\lambda_2+\lambda_3+\lambda_4} \\ & - \sum_{j=1}^{5} \frac{\alpha}{\alpha+(\lambda-\lambda_j)} + \frac{2\alpha}{\alpha+\lambda}, \end{aligned}$$

where $\lambda = \sum_{j=1}^{5} \lambda_j$.

In general, for a coherent system with minimal path sets $P_1, \ldots, P_p$, with the components strengths T_j's following independent exponential(λ_j) distributions and the common stress S following exponential(α) distribution, the system reliability can be written as

$$h_\phi = \sum_{i=1}^{p} \frac{\alpha}{\alpha + \sum_{j \in P_i} \lambda_j} - \sum_{i<i'} \frac{\alpha}{\alpha + \sum_{j \in P_i \cup P_{i'}} \lambda_j} + \cdots.$$

For the stereo system, one can use the minimal path sets and use the above general formula to find h_ϕ directly. The number of minimal cut sets is, however, less than that of minimal path sets. Therefore, it may be easier to use (3) and (1). We consider a modular decomposition with modules {1, 2}, {3} and {4, 5}, all connected in series, so that $h_{\chi_1}(s) = 1 - F_1(s)F_2(s) = \bar{F}_1(s) + \bar{F}_2(s) - \bar{F}_1(s)\bar{F}_2(s)$, $h_{\chi_2}(s) = \bar{F}_3(s)$ and $h_{\chi_3}(s) = 1 - F_4(s)F_5(s) = \bar{F}_4(s) + \bar{F}_5(s) - \bar{F}_4(s)\bar{F}_5(s)$, respectively. Then,

$$h_\phi(s) = \left(\bar{F}_1(s) + \bar{F}_2(s) - \bar{F}_1(s)\bar{F}_2(s)\right)\bar{F}_3(s)\left(\bar{F}_4(s) + \bar{F}_5(s) - \bar{F}_4(s)\bar{F}_5(s)\right),$$

which can be expanded and checked with those obtained by using (2) or (3). The expression for the special case of exponential distributions are now routine.

All this illustrations requires the assumption of independence between the T_i's. However, as mentioned earlier, the formula (1), with the help of (2) or (3), is very general to include dependent components. For a simple illustration, let us consider a system with two components (series or parallel) with (T_1, T_2) following a bivariate exponential distribution of [11] with parameters $(\lambda_1, \lambda_2, \lambda_{12})$ as given by

$$\bar{F}(t_1, t_2) = P[T_1 > t_1, T_2 > t_2] = \exp[-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_{12} \max(t_1, t_2)]. \tag{6}$$

If the system is series, then $h_\phi(s)$ is given by, using (2),

$$h_\phi(s) = P[\min(T_1, T_2) > s] = \exp[-(\lambda_1 + \lambda_2 + \lambda_{12})s],$$

so that h_ϕ, from (1), is obtained as $\alpha/(\alpha + \lambda_1 + \lambda_2 + \lambda_{12})$. If the system is parallel, then, using (2), we have

$$h_\phi(s) = P[\max(T_1, T_2) > s] = e^{-\lambda_{12} s}\left[e^{-\lambda_1 s} + e^{-\lambda_2 s} - e^{-(\lambda_1+\lambda_2)s}\right],$$

so that h_ϕ is obtained as

$$\alpha\left[\frac{1}{\alpha + \lambda_1 + \lambda_{12}} + \frac{1}{\alpha + \lambda_2 + \lambda_{12}} - \frac{1}{\alpha + \lambda_1 + \lambda_2 + \lambda_{12}}\right].$$

In general, if the T_i's are associated ([10], p29–31), one can have bounds on $h_\phi(s)$ in the usual way. For given $S = s$, the binary variables $X_i = I\{T_i > s\}$, for $i = 1, \ldots, n$, are increasing functions of T_i's, hence, associated. The structure function, for given s, can be written as $\phi(X_1, \ldots, X_n)$ and, then, $h_\phi(s) = P[\phi(X_1, \ldots, X_n) = 1]$, for which the bounds are readily available in [10] (p34–45) in terms of marginal distributions of X_i's (or T_i's). By integrating with respect to the distribution of S, one can routinely find bounds for h_ϕ, with arbitrary dependence (association) between the T_i's, in terms of their marginal distributions and that of S. For a simple illustration, let us consider the stereo system, as described before, but with associated components (that is, the T_i's are associated). Also, assume that the individual T_i's have exponential marginal distributions with parameters λ_i, $i = 1, \ldots, 5$. Then, given $S = s$, the component reliabilities are given by $p_i = e^{-\lambda_i s}$, for $i = 1, \ldots, 5$, which can be used to obtain bounds for the system reliability, given $S = s$. For example, the max-min bounds are given by

$$\max_i \left\{\prod_{j \in P_i} p_j\right\} \leq h_\phi(s) \leq \min_i \left\{1 - \prod_{j \in K_i} (1 - p_j)\right\}, \tag{7}$$

where the P_i's (K_i's) are the minimal path (cut) sets of the system. We know, for the stereo system, the minimal path sets are $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 3, 5\}$ and $\{2, 3, 5\}$, and the minimal cut sets are $\{1, 2\}$, $\{3\}$ and $\{4, 5\}$. The bounds in (7), therefore, depends on the relative values of the λ_i's. For simplicity, we assume $\lambda_i = \lambda$ for all i. Then (7) reduces to $e^{-3\lambda s} \le h_\phi(s) \le e^{-\lambda s}$ so that the bounds for the system reliability are obtained as

$$\frac{\alpha}{\alpha + 3\lambda} \le h_\phi \le \frac{\alpha}{\alpha + \lambda}.$$

Note that, if the T_i's are identically (but not independently) distributed, the max-min bounds for the stereo system, given $S = s$, is $\left(\bar{F}(s)\right)^3 \le h_\phi(s) \le \bar{F}(s)$, where $\bar{F}(s) = P[T_i > s]$. This can be integrated with respect to the distribution of S to obtain bounds for h_ϕ.

3. SYSTEM AND COMPONENT LEVEL STRESS

In practice, the components of a system may be subjected to some localized stress (said to be at component level) applicable to only the corresponding component, in addition to the common system level stress S. Let these component level stresses be denoted by the random variables S_i, $i = 1, \ldots, n$. While computing reliability, it is therefore necessary to compare the individual component strengths T_i's with the resultant stress at the ith component position which is a combination of S and S_i. It will be useful to incorporate any knowledge regarding the stress mechanism, in specific examples, to obtain the resultant stress. In the absence of any such knowledge, and for simplicity, we assume, in this work, the resultant stress to be $U_i = S + S_i$ at the ith component level, for $i = 1, \ldots, n$.

Writing $X_i = I\{U_i < T_i\}$, for $i = 1, \ldots, n$, the system reliability h_ϕ can be written as

$$h_\phi = P\left[\max_{1 \le i \le p} \min_{j \in P_i} X_j = 1\right] = P\left[\min_{1 \le i \le k} \max_{j \in K_i} X_j = 1\right].$$

Note that, if there is no component level stress (i.e., $S_i = 0$ for all i), then the above expression is same as that obtained by using (2), or (3), and (1). However, with no further assumption, one can find bounds of h_ϕ using the marginal distributions of X_i's, or $P[X_i = 1] = P[U_i < T_i]$. We shall see later in this section how to obtain this probability. The system reliability, in theory, can be obtained using the inclusion-exclusion principle, as in (2) and (3). It is easy to see from the above expression that

$$\begin{aligned} h_\phi &= P\left[\bigcup_{i=1}^{p}\left\{U_j < T_j, \quad j \in P_i\right\}\right] \\ &= \sum_{i=1}^{p} P\left[U_j < T_j, \quad j \in P_i\right] - \sum_{i<i'} P\left[U_j < T_j, \quad j \in P_i \cup P_{i'}\right] + \cdots \quad (8) \end{aligned}$$

Also,

$$h_\phi = P\left[\bigcap_{i=1}^{k}\{U_j < T_j, \text{ for at least one } j \in K_i\}\right]$$
$$= \sum_{i=1}^{k} P\left[U_j < T_j, \text{ for at least one } j \in K_i\right]$$
$$- \sum_{i<i'} P[U_j < T_j, \text{ for at least one } j \in K_i \cup K_{i'}] + \cdots \tag{9}$$

Note that, for a set A of components, $P[U_j < T_j, \text{for at least one } j \in A] = 1 - P[U_j > T_j, j \in A]$. Without further assumption, it is difficult to obtain the probabilities in (8) and (9), that is, the probabilities of the forms $P[U_j < T_j, j \in A]$ and $P[U_j > T_j, j \in A]$. Let us assume independence between S, the S_i's and the T_i's; that is, each random variable is independent of any other. Then,

$$P[U_j < T_j, \quad j \in A]$$
$$= \int_S P\left[S_j < T_j - s, \ j \in A\right] f_S(s)\,ds$$
$$= \int_S \left\{\prod_{j \in A} P\left[S_j < T_j - s\right]\right\} f_S(s)\,ds$$
$$= \int_S \left\{\prod_{j \in A} \int F_{S_j}(t-s) f_{T_j}(t)\,dt\right\} f_S(s)\,ds, \tag{10}$$

where F and f with a random variable as subscript denote the corresponding distribution function and density, respectively. The other probability can be found similarly as

$$P[U_j > T_j, j \in A] = \int_S \left\{\prod_{j \in A} \int \bar{F}_{S_j}(t-s) f_{T_j}(t)\,dt\right\} f_S(s)\,ds. \tag{11}$$

Note that, when $S_j = 0$ for all j (no component level stress), the integrands (terms inside the braces) in (10) and (11) reduce to $\prod_{j \in A} \bar{F}_{T_j}(s)$ and $\prod_{j \in A} F_{T_j}(s)$, respectively, as in Section 2. For a singleton set $A = \{i\}$, $P[U_i < T_i]$ can be found from (10) as $\int[\int F_{S_i}(t-s) f_{T_i}(t)\,dt] f_S(s)\,ds$. For $i = 1, \ldots, n$, these are component reliabilities and can be used to obtain bounds for h_ϕ, when the T_i's (and S_i's) themselves are associated, as in the previous section. Although the formulae (8)–(11) look very complicated, they can be worked with for a given system and given the distributions. We illustrate here for three simplest systems, namely, series, parallel and 2-out-of-3, assuming independence between the different variables. Let S follow exponential(α), S_i follow exponential(μ_i) and T_i follow exponential(λ_i) distribution. Since the series system has only one minimal path set, the system reliability can be written as, using (8) and (10),

$$\begin{aligned} h_\phi &= P[U_j < T_j,\ j = 1, \ldots, n] \\ &= \int_0^\infty \left\{ \prod_{j=1}^n \int_s^\infty (1 - e^{-\mu_j(t-s)})\lambda_j e^{-\lambda_j t}\, dt \right\} \alpha e^{-\alpha s} ds \\ &= \int_0^\infty \left\{ \prod_{j=1}^n \left[\frac{\mu_j e^{-\lambda_j s}}{\lambda_j + \mu_j} \right] \right\} \alpha e^{-\alpha s} ds \\ &= \frac{\alpha}{\alpha + \sum_{j=1}^n \lambda_j} \times \prod_{j=1}^n \left(\frac{\mu_j}{\lambda_j + \mu_j} \right). \end{aligned}$$

Similarly, for a parallel system, since there is only one minimal cut set, the system reliability can be written as, using (9) and (11),

$$\begin{aligned} h_\phi &= 1 - \int_0^\infty \left\{ \prod_{j=1}^n \left[1 - e^{-\lambda_j s} + \int_s^\infty e^{-\mu_j(t-s)}\lambda_j e^{-\lambda_j t}\, dt \right] \right\} \alpha e^{-\alpha s}\, ds \\ &= 1 - \int_0^\infty \left\{ \prod_{j=1}^n \left[1 - \frac{\mu_j e^{-\lambda_j s}}{\lambda_j + \mu_j} \right] \right\} \alpha e^{-\alpha s}\, ds \\ &= \sum_{j=1}^n \frac{\alpha \mu_j}{(\lambda_j + \mu_j)(\lambda_j + \alpha)} - \sum_{j<j'} \frac{\alpha \mu_j \mu_{j'}}{(\lambda_j + \mu_j)(\lambda_{j'} + \mu_{j'})(\lambda_j + \lambda_{j'} + \alpha)} + \cdots \end{aligned}$$

For a 2-out-of-3 system, recall that there are three minimal path sets: $P_1 = \{1, 2\}$, $P_2 = \{2, 3\}$ and $P_3 = \{1, 3\}$. Hence, using (8) and (10), the system reliability is

$$\begin{aligned} h_\phi &= \sum_{i=1}^3 P[U_j < T_j,\ j \in P_i] - 2P[U_j < T_j,\ j = 1, 2, 3] \\ &= \sum_{i=1}^3 \left[\frac{\alpha}{\alpha + \sum_{j \in P_i} \lambda_j} \prod_{j \in P_i} \left(\frac{\mu_j}{\lambda_j + \mu_j} \right) \right] - 2 \frac{\alpha}{\alpha + \sum_{j=1}^3 \lambda_j} \prod_{j=1}^3 \left(\frac{\mu_j}{\lambda_j + \mu_j} \right). \end{aligned}$$

This can also be derived using the minimal cut sets, and (9) and (11); however, that will be tedious. In general, for a coherent system ϕ with minimal path sets $P_1, \ldots, P_p$ and having independent exponential stress and strength distributions as described above, the system reliability can be written as

$$h_\phi = \sum_{i=1}^{p}\left[\frac{\alpha}{\alpha + \sum_{j\in P_i}\lambda_j}\prod_{j\in P_i}\left(\frac{\mu_j}{\lambda_j+\mu_j}\right)\right] - \sum_{i<i'}\left[\frac{\alpha}{\alpha + \sum_{j\in P_i\cup P_{i'}}\lambda_j}\prod_{j\in P_i\cup P_{i'}}\left(\frac{\mu_j}{\lambda_j+\mu_j}\right)\right] + \cdots$$

Similar general formula can also be given using the minimal cut sets.

4. DISCUSSION

It is clear from (8) and (9) that, even under most general kind of dependence, one only needs to obtain probability of the type $P[U_j < T_j, j \in A]$ or $P[U_j > T_j, j \in A]$ in order to calculate the system reliability. This, however, requires a very strong modeling for the joint distribution of $\{S, S_1, \ldots, S_n, T_1, \ldots, T_n\}$ and then a complicated integral algebra. The situation is somewhat easier if we assume S, the vector of S_i's and the vector of T_i's to be independent of each other, although the S_i's (and the T_i's) themselves may be dependent. Following the derivation of (10), we then have

$$P[U_j < T_j,\ j \in A] = \int_0^\infty \left\{ \int P[T_j > s_j + s,\ j \in A] f_{\{S_j,\, j\in A\}}(s_j,\ j \in A) \prod_{j\in A} ds_j \right\} f_S(s)\, ds, \tag{12}$$

where the notation and the range of integration is as usual. Similarly, we have

$$P[U_j > T_j,\ j \in A] = \int_0^\infty \left\{ \int P\big[T_j < s_j + s,\ j \in A\big] f_{\{S_j,\, j\in A\}}(s_j,\ j \in A) \prod_{j\in A} ds_j \right\} f_S(s)\, ds. \tag{13}$$

Given the joint distributions of the S_i's and the T_i's and distribution of S, one can, in theory, calculate (12) or (13). Although the algebra may be very tedious, the calculation is feasible with modern computer. To illustrate with a simple example, let us consider a series system with two components. Assume that (T_1, T_2) follows a bivariate exponential distribution of [11] as given by (6). However, (S_1, S_2) is assumed to follow an absolutely continuous bivariate exponential distribution with parameters (μ_1, μ_2, μ_{12}) ([12]) given by the density function

$$f(s_1, s_2) = \begin{cases} \dfrac{\mu_1 \mu (\mu_2 + \mu_{12})}{\mu_1 + \mu_2} e^{-\mu_1 s_1 - (\mu_2 + \mu_{12}) s_2} & \text{for } s_1 < s_2, \\ \dfrac{\mu_2 \mu (\mu_1 + \mu_{12})}{\mu_1 + \mu_2} e^{-(\mu_1 + \mu_{12}) s_1 - \mu_2 s_2} & \text{for } s_1 > s_2, \end{cases}$$

where $\mu = \mu_1 + \mu_2 + \mu_{12}$. Then, assuming S to follow an exponential distribution with parameter α, the system reliability can be calculated as

$$\begin{aligned} h_\phi &= P[U_1 < T_1, \; U_2 < T_2] \\ &= \frac{\alpha\mu}{(\mu_1 + \mu_2)(\lambda + \mu)(\lambda + \alpha)} \\ &\quad \times \left[\frac{\mu_2(\mu_1 + \mu_{12})}{\lambda_1 + \mu_1 + \lambda_{12} + \mu_{12}} + \frac{\mu_1(\mu_2 + \mu_{12})}{\lambda_2 + \mu_2 + \lambda_{12} + \mu_{12}} \right], \end{aligned}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.

If the resultant stress U_i is non-linear in S_i and S, one can, in theory, proceed by using (8) or (9), but it involves more difficult algebra. It is to be noted that the formulae (8) and (9) hold for any general system with any general stress-strength relationship. This allows one to exploit any physical knowledge regarding the stress mechanism in writing U_i as a function of S_i and S (and may be other S_j's).

In general, both stress and strength may be time dependent. The strength of a unit may decrease in time because of natural decay; the stress on the unit may also change with time because of change in environmental conditions. Some work consider stresses being applied as loads at discrete time points possibly following a cyclic pattern (Chapter 8 of [1]; Chapter 4.2 of [13]) when the strengths may also be different. More generally, stress $X(t)$ and strength $Y(t)$ at time t may be modeled as independent stochastic processes and reliability at time t_0 may be defined as (Johnson, 1988)

$$P[Y(t) > X(t), \;\text{ for all } t \le t_0].$$

See also [14] dealing with degrading strengths. It may be of interest to extend this stochastic process formulation for system reliability and also for both system and component level stress, which will be taken up later.

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REFERENCES

1. Kapur, K. C.; Lamberson, L. R. *Reliability in Engineering Design*. John Wiley & Sons, New York, 1977.
2. Maiti, S. S. Estimation of $P(X \leq Y)$ in the geometric case. Journal of the Indian Statistical Association **1995**, 33, 87–91.
3. Mukherjee, S. P.; Maiti, S. S. Reliability from damaged stress and strength data. Calcutta Statistical Association Bulletin **1996**, 46, 135–141.
4. Bhattacharyya, G. K.; Johnson, R. A. Estimation of reliability in a multicomponent stress-strength model. *Journal of the American Statistical Association* **1974**, 69, 966–970.
5. Bhattacharyya, G. K.; Johnson, R. A. Stress-strength models for system reliability. *Proceedings of the symposium on Reliability and Fault Tree Analysis* **1975**, SIAM, 509–532.
6. Bhattacharyya, G. K.; Johnson, R. A. Estimation of system reliability by nonparametric techniques. *Bulletin of the Mathematical Society of Greece* **1977**, (memorial Volume), 94–105.
7. Ebrahimi, N. Estimation of reliability for a series stress-strength system. *IEEE Transactions on Reliability* **1982**, 31, 202–205.
8. Johnson, R. A. Stress-strength models for reliability. In *Handbook of Statistics: Quality Control and Reliability*; Krishnaiah, P. R., Rao, C. R., Eds.; Elsevier Science Publishers B. V., 1988, Vol. 7, 27–54.
9. Hanagal, D. D. Estimation of system reliability from stress-strength relationship. *Communication in Statistics: Theory and Methods* **1996**, 25, 1783–1797.
10. Barlow, R. E.; Proschan, F. *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart and Winston Inc., New York, 1975.
11. Marshall, A. W.; Olkin, I. A multivariate exponential distribution. Journal of the American Statistical Association **1967**, 62, 30–44.
12. Block, H. W.; Basu, A. P. A continuous bivariate exponential extension. *Journal of the American Statistical Association* **1974**, 69, 1031–1037.
13. Gnedenko, B.; Ushakov, I. *Probabilistic Reliability Engineering*. John Wiley & Sons, New York, 1995.
14. Xue, J.; Yang, K. Upper and lower bounds of stress-strength interference reliability with random strength-degradation. IEEE Transactions on Reliability **1997**, 46, 142–145.

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