A new class of PT-symmetric Hamiltonians with real spectra.

F.Cannata[a,*], M.Ioffe[b,†], R.Roychoudhury[c,‡] and P.Roy[d,§]

[a] *Dipartimento di Fisica and INFN, Via Irnerio 46, 40126 Bologna, Italy*
[b] *Department of Theoretical Physics, Institute of Physics, University of S.-Petersburg, Ulyanovskaya 1, S.-Petersburg 198904, Russia*
[c] *Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700035, India*
[d] *Abdus Salam ICTP, Trieste, Italy.*

Abstract

We investigate complex PT-symmetric potentials, associated with quasi-exactly solvable non-hermitian models involving polynomials and a class of rational functions. We also look for special solutions of intertwining relations of SUSY Quantum Mechanics providing a partnership between a real and a complex PT-symmetric potential of the kind mentioned above. We investigate conditions sufficient to ensure the reality of the full spectrum or, for the quasi-exactly solvable systems, the reality of the energy of the finite number of levels.

PACS: 03.65.Ge;03.65.Fd;11.30

Keywords: Supersymmetry; PT-symmetry; Quasi-solvable models; Complex Hamiltonians with real spectra.

1. Introduction

While it is known that particular complex potentials have real spectra, irrespectively [1], [2], [3] of PT-invariance, it is a widespread belief that PT-symmetric complex potentials have in general real spectra. Within the general ongoing discussion of complex PT-symmetric potentials with real spectrum[1] [4] (and references quoted therein) we study with special attention quartic and sextic polynomial potentials and their generalizations incorporating additional rational functions. In our discussion we will make use of algebraic techniques, of construction of quasi exactly solvable (QES) models [5], [6], [7] and methods of Supersymmetric Quantum Mechanics (SUSY QM) [8], [9].

We will not study exactly solvable non-hermitian models as done for example in [10], [11] but our main goal is to show rigorous examples which are not exactly solvable for which a fixed amount of levels has real energies. In the line of the study of the sextic

* *E-mail:*cannata@bo.infn.it
† *E-mail:*ioffe@snoopy.phys.spbu.ru
‡ *E-mail:*raj@isical.ac.in
§ On leave of absence from Indian Statistical Institute, Calcutta 700035,India;*E-mail:*pinaki@isical.ac.in

[1] Physical motivation for this approach have been presented in [4].

potentials in [12] we investigate both quartic and sextic generalized potentials, also including complex inverse square barriers [13], [14] [2]. We provide also explicit constructions of models, which again are not exactly solvable, but where the full spectrum of a non-hermitian potential is real, for example, due to SUSY isospectrality with a real potential [1], [2], [17]. In Section 2 we discuss generalized quartic potentials. In Section 3 the sextic case is analyzed.

The main new results of this paper are:

- we have obtained a new class of quartic complex PT-symmetric potentials which are quasi-exactly solvable with a finite number of levels with real energies;

- we have also discovered a generalized complex PT-symmetric quartic potential with a real SUSY partner thus ensuring the reality of the full spectrum;

- we found a generalized complex PT-symmetric sextic potential including an inverse square complex barrier with solvable ground state or ground and first excited states with a real SUSY partner and thus with fully real spectrum.

2. Quartic oscillator.

Within the general framework of complex potentials with real spectrum we want to discuss some conditions sufficient to ensure that a complex generalized polynomial potential of fourth order has indeed a completely real spectrum. As outlined in the Introduction, we will try to use different approaches in order to obtain complex potentials with real spectrum.

The elementary algebraic approach, based on a complex shift of coordinate, will apply straightforwardly to a strict polynomial case. More elaborated quasi-exactly solvable PT-symmetric quartic potentials have been investigated in [18] with special attention to the reality of the QES-part (of finite dimensionality) of the spectrum, here we find a much simpler quartic analogue of the celebrated [5], [6], [7], [19], QES sextic potential $V = x^6 - (8j+3)x^2$. Finally, based on the intertwining relations of SUSY QM [1], [2], [17], we can demonstrate that the full spectrum of a modified quartic complex potential is real.

For a pure quartic case a sufficient condition for reality of its spectrum is provided if one can show that the complex potential can be reexpressed as a polynomial of $x_\epsilon \equiv (x+i\epsilon)$ with real coefficients, preserving PT-invariance. When this is the case the complexity of the potential is related only to the complex shift of coordinate.

The general PT-symmetric quartic complex potential reads:

$$V_4(x) = \rho x^4 + iax^3 + bx^2 + icx, \tag{1}$$

where $\rho = \pm 1$ and a, b, c are, in general, arbitrary real parameters. We study the restrictions on the values of these parameters which are sufficient to ensure the real character of the spectrum. The condition can be expressed in a compact form as:

$$V_4(x) = \rho x_\epsilon^4 + Ax_\epsilon^2 + B \tag{2}$$

[2]The PT-symmetric harmonic potential, including a complex barrier, has been investigated in [15]. For a recent discussion of such potential in the contex of shape-invariance, cf. [16].

with $A, B-$ real. The appearance of only even powers follows from the simultaneous requirement of PT-invariance and of the reality of the spectrum. The Eq.(2) leads to the constraints for Eq.(1) :

$$c = \frac{1}{8}a(a^2 + 4\rho b)$$

and ϵ in Eq.(2) takes the value $\epsilon = \frac{1}{4}\rho a$.

We can now compare the potential (1) restricted by the condition above with the quartic QES potential of [18] and find that they are not compatible with quasi-exactly solvability condition[3]. Since the condition we have discussed is only sufficient, this does not imply that the spectrum of [18] can not be completely real.

Let us now consider pure quartic potentials in the framework of SUSY QM [8], [9]. If one considers the simplest ansatz for the superpotential $W = \rho x_\epsilon^2$ one can recognize that the ground state normalizability is lost for real values of ρ. Furthermore ρ can not be anymore real if one requires PT-invariance. Indeed

$$W^2 \pm W' = \rho^2 x_\epsilon^4 \pm 2\rho x_\epsilon$$

is PT-invariant only for pure imaginary ρ and this is inconsistent with Eq.(2).[4] We are thus unable to show that the full spectrum is real, but we can at least show that a QES version (with $(2j+1)$ levels) of this model can be constructed following the approach of [5], [6], [7].

We study the PT-symmetric potential of the form:

$$V(x) = -x_\epsilon^4 - 2iAx_\epsilon + Bj \tag{3}$$

with A, B - real constants and $j = 0, 1/2, 1...$ It is useful to consider a peculiar "gauge" transformation of the wave functions:

$$\Psi(x_\epsilon) \equiv \exp(ix_\epsilon^3/3)\Phi(x_\epsilon) \tag{4}$$

for which the Schroedinger equation takes the form:

$$[-\partial^2 - 2ix_\epsilon^2\partial - 2ix_\epsilon(A+1) + Bj - E]\Phi(x_\epsilon) = 0; \qquad \partial \equiv \frac{d}{dx}. \tag{5}$$

The corresponding non-hermitian operator (5) for $A = -(2j+1)$ can be written in terms of generators [20] of the algebra $sl(2)$:

$$[-(J^-)^2 - 2iJ^+ + Bj - E]\Phi(x_\epsilon) = 0; \tag{6}$$

$$J^- = \partial; \quad J^+ = x_\epsilon^2\partial - 2jx_\epsilon; \quad J^0 = x_\epsilon\partial - j; \tag{7}$$

$$[J^0, J^\pm] = \pm J^\pm; \qquad [J^-, J^+] = 2J^0 \tag{8}$$

[3] With the notations of [18] (1) corresponds to the case for $J = 0$.

[4] One can easily check that with imaginary ρ one can have a normalizable ground state wave function for one of the superpartners.

As usual [5], [6], [7], polynomials $P_{2j+1}(x_\epsilon)$ of degree $(2j+1)$ will provide a basis for the solutions $\Phi(x_\epsilon)$ of the Schroedinger equation (5). In our case the polynomials will be complex and will have coefficients which respect PT-symmetry, i.e. all the odd coefficients are imaginary when all the even are real. One can explicitly check that all $(2j+1)$ lowest eigenvalues are real.

Within SUSY QM one can investigate, independently on the idea of the shift $x \to x_\epsilon$, the partnership between hermitian and non-hermitian Hamiltonians [1], [2], [17]. For reader's convenience we present the relevant condition [1] which expresses the above mentioned partnership with real $V_- = W^2 - W'$ is given by:

$$\exp \int 2(ReW(x))dx = \kappa(ImW(x)), \tag{9}$$

where κ is an abitrary real integration constant. PT-invariance [5] furthermore requires $(ReW(x))$ to be parity odd and $(ImW(x))$ to be even.

We will consider polynomial superpotentials with addition of the logarithmic derivative terms $P'_M(x)/P_M(x)$ with lowest degrees of polynomials P_M $\quad M = 1, 2$, for which the integration in (9) is trivial. Guided by an ansatz discussed, for example, in [21], we investigate the PT-invariant superpotential (with all constants β, f, g - real):

$$W(x) = ix^2 + i\beta + \frac{2fx}{1+fx^2} - \frac{ig}{1+igx} \tag{10}$$

in order to construct two SUSY partner potentials of generalized quartic class. We will pay attention to a suitable choice of parameters which will make one partner real and the other partner complex. One partner potential:

$$\begin{aligned} V_-(x) = W^2(x) - W'(x) = -x^4 - 2\beta x^2 - \beta^2 + \frac{2}{g} + \frac{8f^2x^2}{(1+fx^2)^2} + \\ \frac{1}{(1+fx^2)}[4ix(-1+\beta f - \frac{f^2 g}{f-g^2}) + \frac{2f(3g^2-f)}{f-g^2}] \\ + \frac{2}{1+igx}[-\frac{1}{g} + \beta g - \frac{2fg^2}{f-g^2})], \end{aligned} \tag{11}$$

can be made real by the requirements:

$$-1 + \beta f - \frac{f^2 g}{f-g^2} = 0; \tag{12}$$

$$-\frac{1}{g} + \beta g - \frac{2fg^2}{f-g^2} = 0. \tag{13}$$

The constant f should be positive to avoid singularities of V_- and the constraints (12), (13) can be solved for f and β in terms of the arbitrary negative[6] constant g. It is instructive to verify that equivalent constraints arise if one calculates $\exp \int 2(ReW(x))dx$ and imposes that it should be equal to $\kappa(ImW(x))$.

[5]The superpotential W should be odd under PT transformation, like ∂.

[6]The constant g must be negative because it can be written as $g = -(fg)^2(f-g^2)^{-2}$.

The other partner potential with the above substitutions remains complex:

$$V_+(x) = W^2(x) + W'(x) = -x^4 - 2\beta x^2 - \beta^2 + \frac{2}{g} + 4ix + \frac{2f(f+g^2)}{(f-g^2)(1+fx^2)} - \frac{2g^2}{(1+igx)^2}. \quad (14)$$

Thus we realize an isospectrality between a complex $V_+(x)$ and real $V_-(x)$ potential.

The ground state of the superHamiltonian is the normalizable ground state wave function [7], associated to the real potential V_- :

$$\Psi_0^{(-)}(x) = \exp\left(-\int W(x)dx\right) = \exp\{i[\frac{\alpha x^3}{3} + \beta x]\}\frac{|1+igx|}{|1+fx^2|}. \quad (15)$$

As a final comment to the above construction, we give a particular explicit solution of the constraints (12), (13): $f = 3g^2 = -\beta^{-1} = 2^{4/3}\cdot 3^{-1/3}$. It is possible to generalize the quartic potential taking into account additional terms of more complicate structure of the type $P'_M(x)/P_M(x)$ with P_M a complex PT-invariant polynomials.

One can also try to include an inverse square barrier by considering a PT-odd superpotential:

$$W(x) = ix_\epsilon^2 - 1/x_\epsilon \quad (16)$$

leading to the pair of isospectral potentials:

$$V_-(x) = -x_\epsilon^4 - 4ix_\epsilon \quad (17)$$

$$V_+(x) = -x_\epsilon^4 + \frac{2}{x_\epsilon^2} \quad (18)$$

The potential $V_+(x)$ is physically well defined for $\epsilon \to 0$ on the half line where it is real. The most straightforward application of this isospectrality between complex and real superpartners applies therefore for the so called radial problem [22].

3. Sextic potential

While the sextic potential has been studied thoroughly both from algebraic and analytic points of view, including PT-symmetry [5], [19], [12], no systematic study in an algebraic framework included an inverse square barrier. Here an attempt is made to accomplish this. We discover that a rather general complex sextic potential with a barrier of the form $2/x_\epsilon^2$ has fully real spectrum though only $(2j+1)$ levels are known analytically. In order to match the discussion of the quartic potential of the previous Section, let us start considering a complex PT-symmetric potential:

$$V_6(x) = \rho x^6 + 3iax^5 + bx^4 + icx^3 + dx^2 + iex, \quad (19)$$

[7] It is relevant to remark that this zero energy bound state of the real potential V_- has a double degeneracy, corresponding to $\Psi_0^{(-)}$, $\Psi_0^{(-)*}$, which are not proportional to each other because of the x-dependent phase.

where one can take $\rho = \pm$ and a, b, c, d, e are again, in general, arbitrary real parameters. The restrictions, sufficient to ensure the real character of the spectrum, read:

$$V_6(x) = \rho x_\epsilon^6 + A x_\epsilon^4 + B x_\epsilon^2 + C \tag{20}$$

with A, B, C - real. Eq.(20) leads to the determination $\epsilon = \frac{1}{2}\rho a$ and to two constraints in Eq.(19) :

$$c = 5a^3 + 2\rho ab$$

$$e = a^5 + \rho a^3 b + \rho ad.$$

The additional constraint (see [23])

$$d = -\frac{75}{64}a^4 + \frac{3}{8}a^2 b + \frac{1}{4}b^2 - 8j - 3 \tag{21}$$

leads to quasi-solvability of the model (19) for $(2j+1)$ levels with real energies.

While in the quartic case (3), discussed in the previous Section, we could only ensure that the QES levels have real energies, for the sextic case (19) we can build a QES model, which not only has $(2j+1)$ levels with real energy, but has the full real spectrum. The reason is that for the sextic potential the limit $\epsilon \to 0$ does not lead to any problem.

For $j = 0$ supersymmetrization is straightforward:

$$W(x) = \rho^{1/2} x_\epsilon^3 + \frac{A}{2\rho^{1/2}} x_\epsilon \tag{22}$$

and the ground state wave function of the corresponding supersymmetrical hamiltonian takes the form:

$$\Psi_0(x) = exp(-\int W(x)dx) = exp(-\frac{1}{4}\rho^{1/2} x_\epsilon^4 - \frac{A}{4\rho^{1/2}} x_\epsilon^2). \tag{23}$$

This wave function is obviously normalizable for positive values of ρ, taking the arithmetic determination of the root. For larger values of j quasi-solvability amounts to add additional terms of the form P_M'/P_M in a superpotential (22).

Methods based on Lie algebra [5], [6], [7] to construct QES complex polynomial potentials have been developed in [24]. Here we apply this method to a sextic (in terms of x_ϵ) potential which can accomodate a $2/(x_\epsilon^2)$ barrier[8] [14].

Let us start with the potential

$$V(x) = x_\epsilon^6 + a_4 x_\epsilon^4 + a_2 x_\epsilon^2 + \frac{a_{-2}}{x_\epsilon^2} \tag{24}$$

where a_k are real. It is convenient to define the superpotential

$$W(x_\epsilon) = x_\epsilon^3 + a x_\epsilon + \frac{\gamma}{x_\epsilon} \tag{25}$$

[8] This construction can not be directly generalized to a generic strength of the barrier with the same structure for W as in (25).

with real values for the constants a, γ. After the "gauge" transformation

$$\Psi(x_\epsilon) \equiv \exp(-\int W(x_\epsilon)dx)\phi(x_\epsilon) \tag{26}$$

the Schroedinger equation takes the form:

$$[-\partial^2 + 2W(x_\epsilon)\partial + (V(x_\epsilon) - W^2(x_\epsilon) + W'(x_\epsilon) - E)]\phi(x_\epsilon) = 0. \tag{27}$$

By the substitutions $z \equiv x_\epsilon^2$ and $\chi(z) \equiv \phi(x_\epsilon)$ and by the requirement:

$$a = a_4/2; \quad \gamma(\gamma+1) = a_{-2}, \tag{28}$$

it can be further transformed to:

$$\{-4z\frac{d^2}{dz^2} + (4z^2 + 4az + 4\gamma - 2)\frac{d}{dz} + (a_2 + 3 - a^2 - 2\gamma)z + a - 2a\gamma - E\}\chi(z) = 0. \tag{29}$$

Introducing the same $sl(2)$ generators as in the previous Section

$$J^- = \frac{d}{dz}; \quad J^+ = z^2\frac{d}{dz} - 2jz; \quad J^0 = z\frac{d}{dz} - j, \tag{30}$$

which satisfy (8), the Schroedinger equation (29) finally can be cast in algebraic form as:

$$(-4J^0J^- + 4J^+ + (4\gamma - 2 - 4j)J^- + 4aJ^0 + \bar{E})\chi(z) = 0, \tag{31}$$

where $\bar{E} = a - E - 2a\gamma + 4aj$ and

$$a_2 = a_4^2/4 - 8j - 3 + 2\gamma.$$

Thus the potential of (24) becomes:

$$V(x) = x_\epsilon^6 + 2ax_\epsilon^4 + (a^2 - 8j - 3 + 2\gamma)x_\epsilon^2 + \frac{\gamma(\gamma+1)}{x_\epsilon^2} \tag{32}$$

and is QES for $j = 0, 1/2, 1, ...$ The corresponding wave functions can be written as:

$$\Psi(x_\epsilon) = P_{2j+1}(x_\epsilon)\exp(-\int W(x_\epsilon)dx). \tag{33}$$

Notice that for $j = 0$ the potential

$$V = V_- + a(1 - 2\gamma) = W^2 - W' + a(1 - 2\gamma)$$

has the ground state

$$\Psi_0(x_\epsilon) = \exp(-\int W(x_\epsilon)dx). \tag{34}$$

with energy given by $E_0 = a(1-2\gamma)$. Taking $\gamma = +1$, the partner potential $V_+ = W^2 + W'$, has no barrier term and is a polynomial with real coefficients. Thus the spectrum of (32) for $j = 0$ is real though only one level is known analytically.

For $j = 1/2$ one can also obtain the reality of the full spectrum using SUSY techniques [21] as follows. Starting from

$$W(x) = x_\epsilon^3 + ax_\epsilon + \frac{\gamma}{x_\epsilon} - \frac{2fx_\epsilon}{1 + fx_\epsilon^2}. \tag{35}$$

it is easy to show that $W^2 - W'$ coincides with potential (32) for $j = 1/2$ and $\gamma = 1$ apart from a constant for $f = a + \sqrt{a^2 - 2}$. If $a > \sqrt{2}$, the partner potential $W^2 + W'$ has no barrier term and, having a well defined [9] behaviour for $\epsilon \to 0$, has real spectrum.

[9] f is positive.

4. Conclusions.

In this paper we have discovered a new class of PT-symmetric quartic and sextic potentials with real energy spectrum. We have found a class of quartic potentials which have a real superpartner, thus ensuring real spectrum for them. A PT-symmetric sextic potential with a $2/x_\epsilon^2$ barrier term can be accomodated in our scheme of quasi-exactly solvable PT-symmetric potentials with a full real spectrum. As an outlook on future perspectives we would like to point out that it is possible to formulate an innovated [13] SUSY framework where the two superpartners live in two different half planes of the complex $x-$plane achieving thereby normalizability of both superpartner wave functions. This approach should be particularly interesting for the quartic type Eq.(15)) because it would enlarge the class of PT-symmetric potentials with real spectrum. An interesting feature of this approach will be the violation of the Witten criterion similarly to higher order SUSY QM [22].

Acknowledgments.

Two of the authors (M.I. and R.R.) thank INFN and University of Bologna for warm hospitality. The work of M.I. was partially supported by RFBR (No. 99-01-00736). P.R. would like to thank the Abdus Salam ICTP for hospitality.

References

1. A. Andrianov, F. Cannata, J.-P. Dedonder, M. Ioffe, *Int. J. of Mod. Phys.* **A14** (1999) 2675;
2. F. Cannata, G. Junker, J. Trost, *Phys. Lett.* **A246** (1998) 219;
3. A. Khare, B. P. Mandal, arXiv:quant-ph/0004019;
4. C. M. Bender, S. Boettler, P. N. Meisinger, *J. of Math. Phys.* **40** (1999) 2201;
5. A. Turbiner, *Commun. Math. Phys.* **118** (1988) 467;
6. A. Ushveridze, *Sov. J. Part. Nucl.* **20** (1989) 504 [Transl. from *Fiz. Elem. Chast. Atom. Yad.*, **20** (1989) 1185];
7. M. Shifman, *Contemp. Math*, **160** (1994) 237;
8. G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics*, Springer, Berlin, 1996;
9. F.Cooper, A.Khare, U.Sukhatme, *Phys. Rep.* **25** (1995) 268;
10. B. Bagchi, C. Quesne, *Phys. Lett.* **A273** (2000) 285;
11. G. Levai, M. Znojil, *J. of Phys. A: Math.Gen.* **33** (2000) 7165;

12. B. Bagchi, F. Cannata, C. Quesne, *Phys. Lett.* **A269** (2000) 79;

13. M. Znojil, F. Cannata, B. Bagchi, R. Roychoudhury, *Phys. Lett.* **B483** (2000) 284;

14. M. Znojil, *J. of Phys. A: Math.Gen.* **33** (2000) 6825;

15. M. Znojil, *Phys. Lett.* **A259** (1999) 220;

16. A. Andrianov, F. Cannata, M. Ioffe, D. Nishnianidze, *Phys.Lett.* **A266** (2000) 341;

17. B. Bagchi, R. Roychoudhury, *J. of Phys. A: Math.Gen.* **33** (2000) L1;

18. C. Bender, S. Boettler, *J. of Phys. A: Math.Gen.* **31** (1998) L273;

19. C. M. Bender, G. V. Dunne, *J. of Math. Phys.* **37** (1996) 6;

20. F. Finkel, A.Gonzalez-Lopez, N. Kamran, P. J. Olver, M. A. Rodriguez, arXiv:hep-th/9603139;

21. B. Roy, P. Roy, R. Roychoudhury, *Fortschr. Phys.* **39** (1991) 3, 211;

22. A. Andrianov, F. Cannata, J.-P. Dedonder, M. Ioffe, *Int. J. of Mod. Phys.* **A10** (1995) 2683;

23. P. Roy, R. Roychoudhury, *P ys. Lett.* **A214** (1996) 266;

24. P. Roy, R. Roychoudhu