MINIMAX THEOREMS AND PRODUCT SOLUTIONS

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PREFACE

This thesis is submitted to the Indian Statistical Institute in support of the author's application for the degree of Doctor of Philosophy. The thesis embodies the research carried out by the author during the period 1963-1966 under the supervision of Professor C.R.Rao at the Indian Statistical Institute, Calcutta.

This thesis consists of three chapters. The first chapter deals with general minimax theorems while the second deals with minimax theorem for certain types of kernels on the unit square and the last chapter is concerned with product solutions for simple games.

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C O N T E N T S

INTRODUCTION

It was John von-Neumann who laid the foundations of a mathematical theory of 'games of strategy'. The climax of the pioneering period of development came in 1944 with the publication of 'The Theory Of Games And Economic Behaviour' by John von-Neumann and Oscar Morgenstern. The field of game theory is now well established and widely diffused through the mathematical world - thanks to the success of the volumes entitled 'Contributions to the theory of games' and 'Advances in game theory'. These volumes comprise a collection of contributions to the theory of games and answer some questions raised explicitly or implicitly by von-Neumann.

My interest in the theory of games received great stimulus from the inspiring articles of Dr. L.S.Shapley and a number of others. These papers are listed at the end of this thesis.

Detailed introduction and summary will be given at the beginning of each chapter. I shall now describe in an outline the contents of this thesis. In chapter I, various sufficient conditions are given under which an infinite game with unbounded Kernel $K(x, y)$ possesses a solution. In some cases these sufficient conditions have been supplemented by effective necessary conditions. Finally an application to a minimax theorem, in probability theory is given.

In chapter II, we consider the bounded pay-off $K(x,y)$ defined on the unit square and whose discontinuities lie on a finite number of curves of the form $y = \varphi_k(x)$; $K = 1, 2, \ldots, n$. In general such games need not possess the min-max value - but it is shown that such games have min-max value provided the second player's mixed strategies are restricted to absolutely continuous distributions on the unit interval.

In the last chapter a new class of product solutions is obtained for the product game $J \otimes K$ where $J = M_3 \otimes B_4$; M_3 being 3-person majority game and B_4 being 1 - person pure bargaining game and K is an arbitrary simple game. These solutions need not have the property of full monotonicity in the sense of Shapley and theorem 5 or 6 of Shapley [Solutions of compound simple games - By L. S. Shapley - in 'Advances in game theory'] cannot predict these solutions.

CHAPTER I

Minimax Theorems

Introduction and summary 1: It was John von Neumann who proved the well known minimax theorem for finite games. His theorem can be stated mathematically as follows. If X and Y are compact convex subsets of R^m and R^n respectively and if K is a bilinear function on $X \times Y$ then K has a saddle point. That is $\underset{X}{\text{Max}}\ \underset{Y}{\text{Min}}\ K(x, y) = \underset{Y}{\text{Min}}\ \underset{X}{\text{Max}}\ K(x, y)$. Ville and Wald [21] have generalized von-Neuman's result when X and Y are allowed to be certain infinite dimensional linear spaces and K is assumed to be bilinear. It was Kneser [5], Ky Fan [6] and Berge [1] who proved minimax theorems for Concave Convex functions and one of the spaces is assumed to be compact or conditionally compact in a suitable topology. M. Sion [16] has proved minimax theorems for quasi concave - convex functions that are appropriately semi-continuous in the variables. Very rec recently Teh-Tjoe Tie [18] has proved a general minimax theorem for functions that are Concave-Convex like.

In this chapter we prove minimax theorems - of which one will be a generalization of Wald's theorem [22] while the rest will be general minimax theorems, that are extentions of T.T.Tie and Ky Fan. In all the theorems compactness or conditional compactness of one of the spaces has played a crucial role. More precisely, the assumption of compactness or conditional compactness enables one to reduce the problem to finite dimensional case under appropriate continuity assumptions on the Kernel $K(x, y)$ which in general will be unbounded. We will now state some of the theorems that are proved in this chapter.

<u>Theorem 1.3:</u> Let $K(x, y)$ be a real valued function (not necessarily bounded) defined on $X \times Y$ such that X is conditionally compact in the semi-intrinsic topology. Further suppose $K(x, y)$ is bounded in y for every fixed x and

$$K(\mu, \lambda) = \int \int K(x, y) d\mu(x) d\lambda(y)$$

$$= \int \int K(x, y) d\lambda(y) d\mu(x)$$

for all $\mu \in m_X^S$ and $\lambda \in \overset{*}{m}{}_Y^S$. Then the game is strictly determined i.e. $\operatorname{Inf}_{m_Y^S} \operatorname{Sup}_{m_X^S} K(\mu, \lambda) = \operatorname{Sup}_{m_X^S} \operatorname{Inf}_{m_Y^S} K(\mu, \lambda)$.

<u>Theorem 1.6</u>: If X is conditionally compact in the (S) topology and $K(x, y)$ is concave-convex like then

$$\operatorname*{Inf}_{Y} \operatorname*{Sup}_{X} K(x, y) = \operatorname*{Sup}_{X} \operatorname*{Inf}_{Y} K(x, y).$$

<u>Theorem 1.7</u>: Let K be defined over $X \times Y$. Suppose X is (S) conditionally compact. Then the following conditions are equivalent.

1) Given any $\varepsilon > 0$, any finite subsets A and B of X and Y respectively there exists $x' \in X$, $y' \in Y$ such that

$$K(x_i, y') \leq K(x', y_j) + \varepsilon \quad \text{for all } x_i \in A,\ y_j \in B.$$

2) $\operatorname*{Sup}_{X} \operatorname*{Inf}_{Y} K = \operatorname*{Inf}_{Y} \operatorname*{Sup}_{X} K$

3) $\operatorname*{Inf}_{Y} \operatorname*{sup}_{A} K \leq \operatorname*{sup}_{X} \operatorname*{Inf}_{Y} K$ for every finite set $A \subseteq X$

4) $\operatorname*{Sup}_{X} \operatorname*{Inf}_{B} K \geq \operatorname*{Inf}_{Y} \operatorname*{sup}_{X} K$ for every finite set $B \subseteq Y$

<u>Theorem 1.9</u>: Let X be a compact space. $K(x, y)$ is upper-semicontinuous in x for every fixed y. Then the following statements are equivalent.

1) $\inf_Y \sup_X K = \sup_X \inf_Y K$

2a) Given any $\epsilon > 0$, any finite subsets A, B of X, Y there exists x', y' such that $K(x_i, y') \leq K(x', y_j) + \epsilon$ for all $x_i \in A$, $y_j \in B$ and

2b) For any $C < \inf_Y \sup_X K$ there exists a finite set A of X such that for every y there exists an $x \in A$ with $K(x, y) \geq C$.

In the last section a simple general minimax theorem is proved and from that theorem as an application a result due to Ranga Rao [11] is deduced.

Sec. 2: Games of Strategy

If X and Y are non-empty sets and if $K(x, y)$ is a finite real valued function on the Cartesian product $X \times Y$ then the triple (X, Y, K) is called a game. In a game (X, Y, K) the elements of X are called pure strategies of player 1 and the elements of Y are called pure strategies of player 2. The pairs (x, y) where $x \in X$, $y \in Y$ will be called plays. For $x \in X$, $y \in Y$ the number $K(x, y)$ is the pay-off in the play (x, y).

Let $K(x, y)$ be bounded. We will now define the neighbourhood systems for every point x belonging to X.

$$I_{x_0, \epsilon} = \left\{ x \mid \sup_Y |K(x,y) - K(x_0,y)| < \epsilon \right\}$$

where $\epsilon > 0$ and $x_0 \in X$. This class of neighbourhoods will form a base which will induce a topology which we shall call the intrinsic topology or simply I-topology for X. Actually this topology is induced by the following pseudo metric

$$d(x_1, x_2) = \sup_Y |K(x_1, y) - K(x_2, y)|$$

But one can actually convert this into a metric by defining X' as the class of all sets of the form

$$\bigcap_{y \in Y} \left\{ x \mid K(x, y) = K(x_0, y) \right\} , x_0 \in X$$

and K' as the function on $X' \times Y$ satisfying $K'(x', y) = K(x, y)$ for $x \in x'$.

Let $\mathcal{B}_X^I$ and m_X^I denote the smallest σ-field containing I-open sets of X and the class of all probability distributions on X respectively. $\mathcal{B}_Y^I$ and m_Y^I can be defined in a similar way for Y. We will assume without loss of generality $\mathcal{B}_X^I$ and $\mathcal{B}_Y^I$ includes pure strategies. Elements of m_X (m_Y) are called mixed strategies.

<u>Definition</u>: X is said to be I-conditionally compact if and only if for every $\epsilon > 0$ there exists a finite $(x_1, x_2, \ldots, x_n)$ of X such that $\bigcup_{i=1}^{n} I_{x_i, \epsilon} = X$.

<u>Remark</u>: X is I-conditionally compact if and only if Y is I-conditionally compact. For a proof refer Wald [23]. We will now state the theorem of Wald [23] without proof.

<u>Theorem 1.1</u>: Let X be I-conditionally compact and let $K(\mu, \lambda) = \int \int K(x, y)\, d\mu(x) d\lambda(y)$ where $\mu \in m_X^I$, $\lambda \in m_Y^I$ then

$$\sup_{m_X^I} \operatorname{Inf}_{m_Y^I} K(\mu, \lambda) = \operatorname{Inf}_{m_Y^I} \sup_{m_X^I} K(\mu, \lambda)$$

In the next section we will prove a generalization of theorem 1.1 by weakening the topological assumptions in Theorem 1.1.

Sec. 3: Generalization of Theorem 1.1

We are now going to introduce topologies for X and Y which will lead to more general theorems than theorem 1.1. Let $K(x, y)$ be a (not necessarily bounded) real valued function defined on $X \times Y$. Then the class S of the subsets

$$S(x_o, \varepsilon) = \left\{ x \mid \sup_{Y} [K(x, y) - K(x^o, y)] < \varepsilon \right\}$$

$$\varepsilon > 0, \quad x_o \in X$$

is a base for a topology for X.

Similarly,

$$S(y_o, \varepsilon) = \left\{ y \mid \operatorname{Sup}_{X} [K(x, y_o) - K(x, y)] < \varepsilon \right\}$$

$$\varepsilon > 0, \quad y_o \in Y$$

is a base for a topology for Y.

We shall refer to these topologies as the semi-intrinsic topologies or simply the S-topology. This was first considered by Teh Tjoe-Tie [18].

Definition: X is said to be S-conditionally compact iff given any $\epsilon > 0$ there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ such that $\overline{\bigcup_{i=1}^{n} S(x_i, \epsilon)} = X$.

Remarks: Every S-open set of X is I-open. Also every I-conditionally compact set is S-conditionally compact. $K(x, y)$ is (S) upper semicontinuous in x for every fixed $y \in Y$ and (S) lower semicontinuous in y for every fixed $x \in X$. If X is (S) conditionally compact it need not imply that Y is S-conditionally compact. This is evident from the following example: Let X be the set of positive integers and Y the class of all subsets of X. Define $K(x, y) = -1$ if $x \in y$ and $= 1$ if $x \notin y$. It is readily seen that the Y-space is S-conditionally compact but the space X is not.

Let $\mathcal{B}_X^S$ and m_X^S denote the smallest σ-field containing S-open sets of X and the pure strategies and the class of all probability distributions on X. $\mathcal{B}_Y^S$ and m_Y^S are defined similarly. Let $\mathcal{B}$ be the smallest σ-algebra containing the rectangles $C \times D$; $C \in \mathcal{B}_X^S$, $D \in \mathcal{B}_Y^S$.

If X is I-separable then $\mathcal{B}_X^I = \mathcal{B}_X^S$ we will now prove the following known lemma [18] which is needed in the sequel.

<u>Lemma 1.2:</u> If X is (I) separable then $K(x, y)$ is (S) measurable.

<u>Proof</u>: We define:

$$Z = \{(x, y) \mid K(x, y) > a\} \quad a \text{ real}$$

$$C(x_1, r) = \{x \mid d(x_1, x) < r\} \quad r\text{-positive rational}$$

$$X_o = \{x \mid K(x, y) > a \text{ for at least one } y \in Y\}$$

Since $K(x, y)$ is (I) continuous for each fixed y, the set X_o is I-open. Let X_1 be any fixed countable dense subset in X_o and (x_o, y_1) be a point of Z Then there exists a point $x_1 \in X_1$ and a positive rational r such that

$$d(x_o, x_1) < r \quad \text{and} \quad r < K(x_1, y_1) - a$$

Let ϵ be a positive number such that $0 < r + \epsilon < K(x_1, y_1) - a$. Then, if (x, y) is a point of the rectangle $C(x_1, r) \times S(y_1, \epsilon)$ we have

$$K(x_1,y_1) - K(x,y) \leq |K(x_1,y_1) - K(x,y_1)| + [K(x,y_1) - K(x,y)] < r + \epsilon < K(x_1, y_1) - a$$

It follows that $K(x, y) > a$ or $C(x_1, r) \times S(y_1, \epsilon)$ is a subset of Z.

If $Y(x_1, r)$ is defined to be the set

$$Y(x_1, r) = \bigcup \left\{ S(y,\epsilon) : y \in Y,\ \epsilon > 0 \text{ and } 0 < r + \epsilon < K(x_1, y) - a \right\}$$

then it has been proved that for any point $(x, y) \in Z$ there exists a point $x_1 \in X_1$ and a rational $r > 0$ such that

$$(x, y) \in C(x_1, r) \times Y(x_1, r) \subset Z.$$

Since $C(x_1, r) \in \mathfrak{S}^{I}_{X}$ and the class of rectangles $C(x_1, r) \times Y(x_1, r)$ is countable, the set Z is S-measurable. We will now state and prove our theorem.

Theorem 1.3: Let $K(x, y)$ be a real-valued function (not necessarily bounded) defined on $X \times Y$ such that X is (S) conditionally compact. Further suppose $K(x, y)$ is bounded in y for every fixed x and

$$K(\mu, \lambda) = \iint K(x,y)d\mu(x)d\lambda(y) = \iint K(x,y)d\lambda(y)d\mu(x)$$

for all $\mu \in m_X^S$, $\lambda \in m_Y^S$ then

$$\operatorname*{Sup}_{m_X^S} \operatorname*{Inf}_{m_Y^S} K(\mu, \lambda) = \operatorname*{Inf}_{m_Y^S} \operatorname*{Sup}_{m_X^S} K(\mu, \lambda).$$

Proof of theorem 1.3: Given any $\epsilon > 0$, since X is S-conditionally compact there is a finite set A_ϵ such that for every x there exists an $x_i \in A_\epsilon$ with

$$K(x, y) \leq K(x_i, y) + \epsilon \qquad \text{for all } y.$$

Hence it follows that,

$$\operatorname*{Inf}_{m_Y^S} \operatorname*{sup}_{m_X^S} K(\mu, \lambda) \overset{*}{\leq} \operatorname*{Inf}_{m_Y^S} \operatorname*{Max}_{A_\epsilon} K(x_i, \lambda) + \epsilon .$$

Write $V = \underset{m_Y^S}{\mathrm{Inf}}\ \underset{A_C}{\mathrm{Max}}\ K(x_i, \lambda)$

Now we shall establish that there exists a probability vector

$$\xi = (\xi_1 \ldots \xi_n) \text{ such that}$$

$$\sum_1^n \xi_i K(x_i, \lambda) \geq V \quad \text{for all } \lambda.$$

Suppose, if it is not true, it means that for every ξ there exists a λ such that $\Sigma\, \xi_i K(x_i, \lambda) < V$. Since the set of probability vectors is compact, we can find a finite number of $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that for every ξ there exists one λ_j with $\Sigma\, \xi_i K(x_i, \lambda_j) < V$. In other words for every ξ there exists some probability vector $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ with $\Sigma\, \xi_i \eta_j K(x_i, \lambda_j) < V$.

Let $A(\xi, \eta) = \Sigma\, \xi_i \eta_j K(x_i, \lambda_j)$

It follows that $\underset{\xi}{\mathrm{Max}}\ \underset{\eta}{\mathrm{Min}}\ A(\xi, \eta) < V$.

By von-Neumann's minimax theorem we have $\underset{\eta}{\mathrm{Min}}\ \underset{\xi}{\mathrm{Max}}\ A(\xi,\eta) < V$ or $\Sigma\, \eta_j^o \xi_i K(x_i\ \lambda_j) < V$ for all ξ and for some η^o. This means $\Sigma\, \xi_i K(x_i, \Sigma\, \eta_j^o \lambda_j) < V$ for all ξ. That is

$K(x_i, \lambda') < V$ for $i = 1, 2, \ldots, n$

where $\lambda' = \Sigma \eta_j^o \lambda_j$. It follows that

$$\operatorname*{Inf}_{m_Y^S} \operatorname*{Max}_{A_\varepsilon} K(x_i, \lambda) < V = \operatorname*{Inf}_{m_Y^S} \operatorname*{Max}_{A_\varepsilon} K(x_i, \lambda)$$

which is impossible. Hence, for some ξ

$$\Sigma \xi_i K(x_i, \lambda) \geq V \quad \text{for all } \lambda$$

From this it is immediate that, $\sup_{m_X^S} \operatorname*{Inf}_{m_Y^S} K(\mu, \lambda) \geq V$

which in turn implies the conclusion of theorem 1.3. We will now quote the result due to T.T.Tie [18].

<u>Theorem 1.4</u>: Let $K(x, y)$ be a feal valued function on $X \times Y$ such that X and Y are both (S) conditionally. Compact. Further suppose

$$K(\mu,\lambda) = \iint K(x,y)d\mu(x)d\lambda(y) = \iint K(x,y)d\lambda(y)d\mu(x)$$

for all $\mu \in m_X^S$, $\lambda \in m_Y^S$ then $\operatorname*{Sup}_{m_X^S} \operatorname*{Inf}_{m_Y^S} K = \operatorname*{Inf}_{m_Y^S} \operatorname*{sup}_{m_X^S} K$.

We will now give an example to show that theorem 1.3 is not included in theorem 1.4. In other words theorem 1.3 is logically independent of theorem 1.4.

Example: Let X be the space of positive integers and Y the class of all subsets of X. Define $K(x,y) = -1$ if $x \in y$ and $= 1$ if $x \notin y$. It is readily seen that Y is (S) conditionally compact while X is not. As X is trivially I separable, from lemma 1.2 it follows $K(x, y)$ is S-measurable. Further $K(x, y)$ is bounded and hence $\iint K(x,y)d\mu(x)d\lambda(y) = \iint K(x,y)d\lambda(y)d\mu(x)$. All the conditions of theorem 1.3 are satisfied and it is quickly seen that $\underset{m_X^S}{\text{Sup}}\ \underset{m_Y^S}{\text{Inf}}\ K = \underset{m_Y^S}{\text{Inf}}\ \underset{m_X^S}{\text{Sup}}\ K = -1$. Theorem 1.4 cannot be applied applied here as X is not (S) conditionally compact.

Remark: It is not known whether the conclusion of theorem 1.3 remains true if one omits the assumption namely $K(x, y)$ is bounded in y for every fixed x.

Sec. 4: General Minimax Theorems:

Definition: $K(x,y)$ is upper semicontinuous in x for every fixed y iff $\{x \mid K(x,y) < r\}$ is open for every real number r. $K(x, y)$ is lower semicontinuous in y if and only if for every real number r, the set $\{y \mid K(x, y) > r\}$ is open.

Definition: $K(x, y)$ is said to be concave in x for fixed y iff for any $t \in [0, 1]$ and any $x_1, x_2 \in X$ there exists an element $tx_1 + (1-t)x_2 \in X$ such that

$$K(tx_1 + (1-t)x_2, y) \geq t\, K(x_1, y) + (1-t)K(x_2, y)$$

$K(x, y)$ is said to be convex in y for fixed x iff for any $t \in [0, 1]$ and any $y_1, y_2 \in Y$ there exists an element $ty_1 + (1-t)y_2 \in Y$ such that

$$K(x, ty_1 + (1-t)y_2) \leq tK(x, y_1) + (1-t)K(x, y_2)$$

This definition is meaningful iff the spaces X and Y have linear structures. We will now generalize this concept so that it may be valid for spaces which do not possess any linear structure. This was first done by Ky Fan [6].

<u>Definition</u>: A real valued function $K(x, y)$ is said to be concave like in x (convex-like in y) if and only if for any t in $[0, 1]$ and any two $x_1, x_2 \in X$ $(y_1, y_2 \in Y)$ there exists an $x_0 \in X$ $(y_0 \in Y)$ such that

$$tK(x_1, y) + (1-t)K(x_2, y) \leq K(x_0, y) \quad \text{for all } y$$

$$[t\, K(x, y_1) + (1-t)\, K(x, y_2) \geq K(x, y_0) \quad \text{for all } x\,]$$

$K(x, y)$ is concave-convex like if it is concave-like in x and convex-like in y.

A concept which is equivalent to I-conditionally compactness is that of almost periodic functions defined as follows [6]. A real valued bounded function $K(x, y)$ defined on the product of $X \times Y$ is left almost periodic iff given any $\epsilon > 0$ there exists a finite subset $[x_1, x_2, \ldots, x_n]$ of X such that for any $x \in X$ there is some x_i for which $|K(x, y) - K(x_i, y)| < \epsilon$ for all y. It is not hard to check that left almost periodicity of K implies and is implied by right almost periodicity (which can be defined in an obvious way) Ky Fan has proved the following theorem.

<u>Theorem 1.5</u>: If $K(x, y)$ is left almost periodic and concave-convex like then $\operatorname{Sup}_X \operatorname{Inf}_Y K = \operatorname{Inf}_Y \operatorname{Sup}_X K$. We will now prove the following theorem [7].

<u>Theorem 1.6</u>: Let $K(x, y)$ be a real valued (not necessarily bounded) function defined over arbitrary product space $X \times Y$. Suppose X is (S) conditionally compact and K is concave-convex like. Then

$$\operatorname{Sup}_X \operatorname{Inf}_Y K(x, y) = \operatorname{Inf}_Y \operatorname{Sup}_X K(x, y).$$

Remark: Theorem 1.6 includes theorem 1.5 as a special case. It is possible to give a direct proof of theorem 1.6 without resorting to von-Neumann's Minimax Theorem, but we will not be doing it. The following lemma is required to in sequel.

Lemma 1.7: Let A and B be any two finite subsets of X and Y respectively and K be concave-convex-like. Then there exists $x_0 \in X$, $y_0 \in Y$ such that

$$K(x_i, y_0) \leq K(x_0, y_j) \quad \text{for all} \quad x_i \in A, \quad y_j \in B.$$

Proof: The conclusion of the lemma is an immediate consequence of von-Neumann's theorem. From the lemma 1.7 it follows that for any two finite subsets A, B of X, Y respectively, $\underset{Y}{\text{Inf}}\ \underset{A}{\text{Max}}\ K(x, y) \leq \underset{X}{\text{Sup}}\ \underset{B}{\text{Min}}\ K(x, y)$.

Proof of Theorem 1.6: Since X is (S) conditionally compact, given any $\epsilon > 0$ there exists a finite set A_ϵ such that for any x one can find an $x_i \in A_\epsilon$ with

$$K(x, y) \leq K(x_i, y) + \epsilon \quad \text{for all} \quad y$$

Therefore we have,

$$\operatorname*{Inf}_{Y} \sup_{X} K \leq \operatorname*{Inf}_{Y} \operatorname*{Max}_{A_\epsilon} K + \epsilon$$

From lemma 1.7 it follows that

$$\operatorname*{Inf}_{Y} \sup_{X} K \leq \operatorname*{Inf}_{\mathfrak{B}} \sup_{X} \operatorname*{Inf}_{B} K + \epsilon$$

where $\mathfrak{B}$ denotes the class of all finite subsets of Y. Through out we fix the ϵ which we have chosen already. Now we shall prove

$$\operatorname*{Inf}_{\mathfrak{B}} \sup_{X} \operatorname*{Inf}_{B} K \leq \sup_{X} \operatorname*{Inf}_{Y} K + 2\epsilon$$

Let us write $V = \operatorname*{Inf}_{\mathfrak{B}} \sup_{X} \operatorname*{inf}_{B} K$

If $V = -\infty$ then we are through. V can never be $+\infty$ because of the assumption that X is S-conditionally compact. Hence we will assume V to be a finite real number. Suppose the above inequality is not true, then we have

$$\operatorname*{Sup}_{X} \operatorname*{Inf}_{Y} K < V - 2\epsilon \qquad \text{or}$$

$$\operatorname*{Inf}_{Y} K(x, y) < V - 2\epsilon \quad \text{for all } x$$

Therefore,

$$\operatorname*{Inf}_{Y} K(x_i, y) < V - 2\epsilon \quad \text{for all } x_i \in A_\epsilon.$$

It follows that there exists at least one y_i for every x_i such that

$$K(x_i, y_i) < V - 2\epsilon .$$

Since X is (S) conditionally compact, for any x there exists one index i such that

$$K(x, y_i) \leq K(x_i, y_i) + \epsilon < V - 2\epsilon + \epsilon = V - \epsilon .$$

In other words,

$$\sup_X \inf_B K(x, y_i) \leq V - \epsilon$$

where B is the finite set $[y_1, y_2, \ldots]$ that is $V \leq V - \epsilon$ which is impossible.

Therefore we have,

$$\inf_Y \sup_X K \leq \sup_X \inf_Y K + 3\epsilon$$

Since ϵ is arbitrary and $\inf_Y \sup_X K(x, y) \geq \sup_X \inf_Y K(x,y)$ holds good always, our theorem follows and thus the proof is complete.

Now one can prove the following more general theorem which also includes theorem 1.6.

Theorem 1.7: Let K be a real valued function defined on $X \times Y$. Further assume X to be (S) conditionally compact. Then the following conditions are equivalent.

1) Given any $\varepsilon > 0$, any finite subsets A and B of X and Y respectively there exists $x_0 \in X$, $y_0 \in Y$ such that

$$K(x_i, y_0) \leq K(x_0, y_j) + \varepsilon \quad \text{for all } x_i \in A, y_j \in B.$$

2) $\sup_X \inf_Y K = \inf_Y \sup_X K.$

3) $\inf_Y \sup_A K \leq \sup_X \inf_Y K$ for every finite set A of X.

4) $\sup_X \inf_B K \geq \inf_Y \sup_X K$ for every finite set B of Y.

Proof: One can prove as before [proof of theorem 1.6] that (1) => (2) and it is trivial to check that (2) => (1), (3) and (4). Using the (S) conditional compactness of X it can be shown that (3) as well as (4) implies (2). Hence the theorem follows. Theorem 1.7 includes the following theorem due to T. T. Tie.

Theorem 1.8: Let K^* be defined over $X \times Y$ and X be (S) conditionally compact. Suppose K is concave-convex like and $K(x, y)$ is bounded below in y for every fixed x. Then $\operatorname{Sup}_X \operatorname{Inf}_Y K = \operatorname{Inf}_Y \operatorname{Sup}_X K$.

Remark: Any function that is concave-convex like will have the property (1) of theorem 1.7. We will now state a minimax theorem that is valid when one of the spaces is compact.

Theorem 1.9: Let X be a compact space $K(x, y)$ is upper semicontinuous in x for every fixed y. Then the following two statements are equivalent.

(1) $\operatorname{Inf}_Y \operatorname{Sup}_X K = \operatorname{Sup}_X \operatorname{Inf}_Y K$

(2) (a) For any $\epsilon > 0$, any finite subsets A, B of X, Y there exists x_o, y_o such that $K(x_i, y_o) \leq K(x_o, y_j) + \epsilon$ for all $x_i \in A$, $y_j \in B$.

(b) For any $C < \operatorname{Inf}_Y \sup_X K$, there exists a finite set A of X such that for every y there exists an $x \in A$ with $K(x, y) \geq C$.

Proof: We will prove (2) => (1) as it can be easily checked that (1) => (2).
From condition (2) (a) if follows that

$$\operatorname{Inf}_Y \operatorname{Sup}_A K \leq \sup_X \operatorname{Inf}_B K$$

where A and B are any two finite sets of X and Y. Since X is compact and $K(x, y)$ is upper semicontinuous in x it is not hard to prove the following: Given any $C > \sup_X \operatorname{Inf}_Y K$, there exists a finite set B of Y such that for every x there exists a $y \in B$ with $K(x, y) < C$. We will denote this condition as (2)(b') we will now prove $\sup_X \operatorname{Inf}_Y K \geq \operatorname{Inf}_Y \sup_X K$. Suppose, if it is not true, then $\sup_X \operatorname{Inf}_Y K < \operatorname{Inf}_Y \operatorname{Sup}_X K$. So we can find two real numbers r_1 and r_2 such that

$$\sup_X \operatorname{Inf}_Y K < r_1 < r_2 < \operatorname{Inf}_Y \sup_X K$$

From (2)(b) and (2)(b') we have

$$\operatorname{Inf}_Y \sup_A K \geq r_2 \text{ and}$$

$$\operatorname{Sup}_X \operatorname{Inf}_B K \leq r_1$$

Since $r_1 < r_2$, $\sup_X \operatorname{Inf}_B K \leq r_1 < r_2 \leq \operatorname{Inf}_Y \sup_A K$ which contradicts our assumption (2a). Hence the theorem follows.

In fact one can prove the following for concave-convex like functions. If K is concave-convex like and further if either 2b or 2b' is satisfied (compactness assumption as well as upper semicontinuous assumption is omitted) then

$\sup_X \inf_Y K = \inf_Y \sup_X K$. This observation is due to M. Sion [14]. One can easily construct examples to show that theorem 1.9 is logically independent of the above fact. If we omit the assumptions of compactness and upper semi-continuity in theorem 1.9 then the following question arises. Will (2a) and (2b) alone imply (1)? We suspect that the answer is 'no', but we do not have any counter examples at present at our disposal.

For the sake of completeness we will state without proof the most general minimax theorem for quasi-concave-convex functions [15]. We will assume X and Y to be convex. K is said to be quasi-concave in x iff the set $\{x \mid K(x, y) \geq r\}$ is convex for every real number r. K is quasi-convex in y iff the set $\{y \mid K(x, y) \leq r\}$ is convex for every real number r. K is quasi-concave-convex iff it is quasi-concave in x and quasi-convex on y. Theorem of Sion can be stated as follows.

Theorem 1.10: Let X and Y be convex and suppose one of the spaces is compact. Further assume K is quasi-concave-convex and $K(x, y)$ is upper semicontinuous in and lower semicontinuous in y. Then

Then

$$\operatorname*{Sup}_{X} \operatorname*{Inf}_{Y} K' = \operatorname*{Inf}_{Y} \operatorname*{Sup}_{X}$$

Sec. 5: A minimax theorem and an application

In this section we will prove a very simple general minimax theorem that is valid for compact spaces. From this theorem Dini's theorem will follow immediately. We will also deduce a known theorem [12] in probability theory.

Theorem 1.11: Let $\{f_\alpha(x) : \alpha \in D\}$ be a collection of upper semi-conditions functions on a compact space X where D is a directed set. Further assume $\alpha, \beta \in D$ $\alpha \geq \beta$ implies $f_\alpha(x) \geq f_\beta(x)$ for all x. Then

$$\operatorname*{Sup}_{X} \operatorname*{Inf}_{D} f_\alpha(x) = \operatorname*{Inf}_{D} \operatorname*{sup}_{X} f_\alpha(x)$$

Theorem 1.11': $\alpha \geq \beta$ implies $f_\alpha(x) \leq f_\beta(x)$ for all x. $\operatorname*{Sup}_{X} \operatorname*{Inf}_{D} f_\alpha(x) = \operatorname*{Inf}_{D} \operatorname*{Sup}_{X} f_\alpha(x)$. We will give a simple direct proof of theorem 1.11 (as it may be possible to deduce it from the theorems proved in the previous section).

Proof of Theorem 1.11: Let if possible $\sup_X \operatorname{Inf}_D f_\alpha(x) < \operatorname{Inf}_D \operatorname{Sup}_X f_\alpha(x)$. Write $V = \operatorname{Inf}_D \sup_X f_\alpha(x)$

Consider $A_\alpha = \{x \mid f_\alpha(x) < V\}$

Then $\bigcup A_\alpha = X$ and each A_α is open in X.

Since X is compact, $\bigcup_{i=1}^{n} A_{\alpha_i} = X$

Let $\alpha_0 = \operatorname{Min} [\alpha_1, \alpha_2, \ldots, \alpha_n]$. Then it follows that for every x, $f_{\alpha_0}(x) < V$. That is $\operatorname{Inf}_D \sup_X f_\alpha(x) < V$ or $V < V$ which leads to a contradiction. We will now state the theorem of DINI.

Theorem 1.12: If a monotonically increasing net $f_n : n \in D$ of continuous real valued functions on a topological space X converges pointwise to a continuous function f then the net converges to uniformly on compacta.

Remark: This is essentially the same as the minimax theorem 1.11. Theorem 1.12 can be written as

$\operatorname{Inf}_{\alpha \in D} \sup_K (\phi_\alpha)(x) = \sup_K \operatorname{Inf}_\alpha (\phi_\alpha)(x)$ where $(\phi_\alpha)(x) = f - f_\alpha(x)$

and K is any compact subset of X. Since $\operatorname{Inf}_\alpha (\phi_\alpha)(x) = \lim (\phi_\alpha)(x) = 0$ it follows $\operatorname{Inf}_D \operatorname{Sup}_K (\phi_\alpha)(x) = 0$.

As an application of theorem 1.11 we will prove the following theorem due to Ranga Rao [13].

<u>Theorem 1.13</u>: Let X be any complete separable metric space. Let $\mu_n \Rightarrow \mu$ where μ_n and μ are probability measures on X. Let $\mathfrak{a} = \{f_t(x) : t \in T\}$ be any family of continuous functions on X with the following properties (i) a is equicontinuous and (ii) there exists a constant C such that $|f_t(x)| \leq C$ for all x and for all t. Then

$$\lim_{n \to \infty} \sup_{a} \left| \int f \, d\mu_n - \int f \, d\mu \right| = 0$$

Before converting this theorem into a minimax theorem we will start with some preliminaries.

<u>Definition:</u> We say $\mu_n \Rightarrow \mu$ i.e. μ_n converges weakly to μ iff $\lim_{n \to \infty} \int f \, d\mu_n = \int f d\mu$ for every continuous and bounded function on X.

<u>Definition:</u> $\{a\}$ is said to be equicontinuous at a point $x_0 \in X$ iff given any $\epsilon > 0$ there exists a δ depending only on x_0 and ϵ such that

$$|f(x) - f(x_0)| \leq \epsilon \quad \text{whenever} \quad d(x, x_0) \leq \delta \quad \text{and for all}$$

$f \in a$.

We say that the family is equincontinuous if it is equicontinuous at every point belonging to X.

From the theorem of Ascoli [4] it follows that the family $\{a\}$ of equicontinuous and uniformly bounded functions becomes conditionally compact in the uniform topology.

$f_n \to f$ in the uniform topology iff

$$\lim_{n \to \infty} \sup_K |f_n(x) - f(x)| = 0 \quad \text{for every compact subset } K \text{ of } X.$$

We will assume without loss of generality that the given family a is compact (otherwise we have to take its closure with respect to the uniform topology on compacta).
We need the following theorem due to Yu. V. Prokhorov [11] in the sequel.

<u>Theorem 1.14:</u> Let X be any complete separable metric space. Let Γ be any family of probability measures on X. Then Γ is conditionally compact iff given $\epsilon > 0$ there exists a compact set $K_\epsilon \subseteq X$ such that

$$\mu(K_\epsilon) \geq 1 - \epsilon \quad \text{for all } \mu \in \Gamma.$$

Define $(\phi_n)(f) = |\int f d\mu_n - \int f d\mu|$

and $b_n(f) = \sup_{m \geq n} (\phi_m)(f)$.

The proof of theorem 1.13 is complete if we show the following namely,

$$\sup_{\bar{a}} \inf_n b_n(f) = \inf_n \sup_{\bar{a}} b_n(f)$$

where $\bar{a}$ denotes the closure of a. But this will follow from theorem 1.11 if it is shown that $b_n(f)$ is upper semi-continuous in f for every fixed n. In fact we will now show that each $b_n(f)$ is continuous in f. Let $f_m \to f$ in the uniform topology. Since $\mu_n \Rightarrow \mu$, $\{\mu_n, \mu\}$ is compact in the weak topology. By theorem 1.14 it follows that given any $\epsilon > 0$ there exists a compact set K_ϵ such that $\mu_n(K_\epsilon) \geq 1 - \epsilon$ and $\mu(K_\epsilon) \geq 1 - \epsilon$. Now it follows that

$$|\int f_m d\mu_n - \int f_m d\mu - (\int f d\mu_n - \int f d\mu)|$$

$$\leq \int |f_m - f| d\mu_n + \int |f_m - f| d\mu$$

$$\leq \int_{K_\epsilon} |f_m - f| d\mu_n + \int_{K_\epsilon} |f_m - f| d\mu + \int_{K_\epsilon'} |f_m - f| d\mu_n + \int_{K_\epsilon'} |f_m - f| d\mu$$

$\leq (2 + 4c)\epsilon$ whenever $m \geq m_0$ and for all n.

This step proves the assertion namely that each $b_n'(f)$ is continuous in f. Therefore,

$$\sup_{\bar{a}} \inf_{n} b_n(f) = \inf_{n} \sup_{\bar{a}} b_n(f).$$

But $\inf_{n} b_n(f) = \lim_{n \to \infty} \sup \phi_n(f) = \lim \phi_n(f) = 0$

Hence, $$\sup_{\bar{a}} \inf_{n} b_n(f) = 0 = \inf_{n} \sup_{\bar{a}} b_n(f)$$

$$= \lim_{n \to \infty} \sup_{\bar{a}} b_n(f)$$

$$= \lim_{n \to \infty} \sup_{\bar{a}} \phi_n(f).$$

Thus the proof is complete.

CHAPTER II

On games played over the unit square.

Introduction and Summary 1: In attempts to find methods of solving a fairly wide class of (2-person zero-sum) games over the unit square, real success has been attained in only two cases: (i) games in which the problem is essentially a finite dimensional one, notably those with polynomial or polynomial like pay-offs and (ii) games having solutions which are absolutely continuous and can be solved via differential or integral equations. These latter games have discontinuities in the pay-offs or their derivatives. Besides, Karlin [3] has dealt with bell-shaped and Polya type Kernels successfully. Sion and Wolfe have given an example [17] thereby showing that the minimax theorem due to Glicksberg [2] cannot be extended in certain directions. Of particular interest in applications to problems of tactical games is the case where the pay-off $K(x, y)$ is bounded and has discontinuities along the diagonal $x = y$. In general the value of such games need not exist - Sion and Wolfe's example is one such. Our object is to show that such kernels have mixed value provided one of the players' mixed strategies are restricted to absolutely continuous distributions with respect to the Lebesgue measure. We hope that this theorém will be of some theoretical interest.

<u>Preliminaries</u>: Before proceeding to state and prove the theorem we begin with some notations and terminology. X and Y will stand for the unit interval [0, 1]. m_X, m_Y will stand for the probability measures on X and Y respectively. A_X or A_Y will denote the class of all absolutely continuous distributions in [0, 1].

<u>Definition</u>: A sequence of measures $\mu_n \in m_X$ is said to converge weakly to $\mu \in m_X$ if and only if for each bounded continuous function on X,

$$\lim_{n \to \infty} \int_0^1 f \, d\mu_n = \int_0^1 f \, d\mu .$$

We endow m_X and m_Y with weak topology. It is well known that m_X is metrizable and compact in the weak topology. [19].

Sec. 2: Statement and proof of theorem 2.1

Theorem 2.1: Let $K(x, y)$ be bounded on the unit square $0 \leq x, y \leq 1$ and all points of discontinuity lie on a finite number of curves of the form $y = \varphi_k(x)$ $k = 1,2,\ldots, n$ where $\varphi_k(x)$ are continuous functions. Further suppose,

$$K(\mu,\lambda) = \int_0^1\int_0^1 K(x,y)d\mu(x)d\lambda(y) = \int_0^1\int_0^1 K(x,y)d\lambda(y)d\mu(x)$$

Then

$$\max_{m_X} \inf_{A_Y} K(\mu, \lambda) = \inf_{A_Y} \max_{m_X} K(\mu, \lambda)$$

Proof of theorem 2.1: First we will prove for every $\lambda \in A_Y$, $\int K(x, y)d\lambda(y)$ is continuous in x. Let $\psi(x) = \int_0^1 K(x,y)d\lambda(y)$. Let $M = \operatorname{Sup} |K(x,y)|$ on the square $0 \leq x, y \leq 1$.

$$G = \{(x, y) \mid |y - \varphi_k(x)| < C/12Mn \text{ holds for some } K, \text{ where } K = 1,2,\ldots, n\}$$

and $F = G' \cap (X \times Y)$ where G' denotes the complement of G. Obviously F is closed and $K(x, y)^*$ is continuous on F. Since F is compact, K is uniformly continuous on F. That is, there exists a ∂ such that for points (x', y), (x'', y) in F for which $|x'-x''| < \partial$, the inequality $|K(x',y) - K(x'', y)| < \epsilon/3$ holds good. Now let x' and x'' be such that $|x'-x''| < \partial$.

Then

$$|(\phi)(x') - (\phi)(x'')| \leq \int_0^1 |K(x',y) - K(x'',y)|\lambda'(y)dy$$

where $\lambda'(y)$ is the Radon-Nykodym derivative of λ with respect to y.

R.H.S. of the above can be evaluated by integrating over the sum of the intervals

$$\begin{aligned} |y - \phi_k(x')| &< \epsilon/12Mn \\ |y - \phi_k(x'')| &< \epsilon/12Mn \end{aligned} \qquad K = 1, 2, \ldots, n$$

(denote the sum of these intervals by S_1) and over the complement S_1' of S_1 with respect to the closed interval $[0, 1]$. It is clear that the length of S_1 does not exceed $\epsilon/3M$.

Hence,

$$\int_{S_1'} |K(x',y) - K(x'',y)| \ \lambda'(y)dy \ \leq C/3$$

Since λ is absolutely continuous w.r.t. Lebesgue measure it follows that,

$$\int_{S_1} |K(x',y) - K(x'',y)|\lambda'(y)dy \leq 2Mr$$

where r is a small positive quantity $(\lambda(S_1) \leq r)$ with the property that $r \to 0$ as $\epsilon \to 0$. Therefore it follows that,

$$\int_0^1 |K(x',y) - K(x'',y)| \ \lambda'(y)dy \leq \frac{\epsilon}{3} + 2Mr$$

or $|(\perp)(x') - (\perp)(x'')| \leq C/3 + 2Mr$

which in turn implies that $(\perp)(x)$ is continuous in x. Since $(\perp)(x)$ is continuous in $[0, 1]$ it is bounded and hence we can conclude that $K(\mu,\lambda)$ is continuous in μ for every fixed $\lambda \in A_Y$. Further A_Y is convex, m_X is convex and compact in the weak topology. The conclusion of the theorem follows from the general minimax theorem that is given in Chapter 1.

We will now quote certain result in this direction without proof that are proved by E.B.Yanovskaya [24].

Theorem 2.2: Let $n_c = \left\{ F \mid \overline{\lim_{\epsilon \to 0}} \left| \frac{F(x+\epsilon)-F(x)}{\epsilon} \right| \leq C \right.$ with $C > 1 \Big\}$ where F is a distribution function on $[0,1]$. Suppose $K(x, y)$ is measurable and $\int_0^1 K(x,y)dx$ and $\int_0^1 K(x,y)dy$ are uniformly convergent then $\underset{n_c}{\text{Max}}\ \underset{n_D}{\text{Min}}\ K(\mu,\lambda) = \underset{n_D}{\text{Min}}\ \underset{n_c}{\text{Max}}\ K(\mu, \lambda)$.

Theorem 2.3: Let $K(x, y)$ be measurable and suppose $\int_S K(x,y)dy$ and $\int_S K(x,y)dx$ are continuous in x and y for any Lebesgue measurable set $S \subseteq [0, 1]$ then $\underset{n_D}{\text{Min}}\ \underset{m_X}{\text{Max}}\ K = \underset{m_X}{\text{Max}}\ \underset{n_D}{\text{Min}}\ K$.

Theorem 2.1 and 2.3 are similar in nature. A special case of theorem 2.2 is considered by Wald [22].

Sec. 3: Example of Sion and Wolfe: [17]

$$\text{Let } K(x, y) = \begin{cases} -1 & \text{if } x < y < x + \frac{1}{2} \\ 0 & \text{if } x = y \text{ or } y = x + \frac{1}{2} \\ +1 & \text{otherwise.} \end{cases}$$

This function K has its discontinuities along $y = x$ and $y = x + \frac{1}{2}$. We shall now show that,

$$\operatorname*{Sup}_{m_X} \operatorname*{Inf}_{m_Y} K(\mu, \lambda) = \frac{1}{3} \quad \text{and}$$

$$\operatorname*{Inf}_{m_Y} \operatorname*{Sup}_{m_X} K(\mu, \lambda) = \frac{3}{7}$$

Let μ be any probability measure on $[0, 1]$. If $\mu[0, \frac{1}{2})] \leq \frac{1}{3}$ let $y_\mu = 1$.

If $\mu[0, \frac{1}{2}) > \frac{1}{3}$, choose $\partial > 0$ such that

$$\mu[0, \tfrac{1}{2} - \partial) \geq \tfrac{1}{3} \text{ and let } y_\mu = \tfrac{1}{2} - \partial.$$

In either case, it is quickly checked that,

$$\operatorname*{Inf}_{m_Y} \int_0^1 \int_0^1 K(x,y)d\mu(x)d\lambda(y) \leq \int_0^1 K(x,y_\mu)d\mu(x) \leq \frac{1}{3}$$

On the other hand, if μ is chosen so that

$$\mu(0) = \mu(\tfrac{1}{2}) = \mu(1) = \tfrac{1}{3}$$

then for all y,

$$\int_0^1 K(x,y)d\mu(x) = \frac{1}{3}[K(0,y) + K(\tfrac{1}{2},y) + K(1, y)] \geq \frac{1}{3}.$$

Hence $$\sup_{m_X} \inf_{m_Y} K(\mu, \lambda) = \frac{1}{3} .$$

Similarly it can be shown that $\inf_{m_Y} \sup_{m_X} K(\mu,\lambda) = \frac{3}{7}$.

As the conditions of theorem 2.1 are satisfied in this example it follows that,

$$\max_{m_X} \inf_{A_Y} K = \inf_{A_Y} \max_{m_X} K = \frac{1}{3}$$

and

$$\sup_{A_X} \min_{m_Y} K = \min_{m_Y} \sup_{A_X} K = \frac{3}{7} .$$

This game can be considered as a continuous Blotto game as follows: Player I must assign a force x to the attack of one of two mountain passes, and $1 - x$ to the other. Player II must assign a force y to the defence of the first pass, and $1 - y$ to the other, at which is also located an extra stationary defence force of $\frac{1}{2}$. A player receives from the other a payment of 1 at each pass if his force at that exceeds his opponent's and receives nothing if they are equal there.

The pay-off is thus,

$$B(x, y) = \text{Sgn } (x - y) + \text{Sgn } ((1 - x) - (\tfrac{3}{2} - y))$$

It is easily checked that $1 + B(x, y) = K(x, y)$ so that this game has the value $-\frac{2}{3}$ (and $-\frac{4}{7}$) if player II (player I) restricts his attention only to absolutely continuous distributions.

CHAPTER III

Product solutions for simple games.

Introduction and summary 1: This is an investigation of the solutions of the games that are formed by combining two or more simple games played by separate groups of individuals. A game is called simple if every coalition of players either wins or loses. Given any two simple games - players are assumed to be distinct in the two games, we define the product as one in which every winning coalition must include winning contingents from both the components. A typical example of a product game is any organization in which some member has veto power. L.S. Shapley has obtained the following theorem on product solutions. [15], [14].

Theorem 3.1. For $i = 1,2$ let $[Y_i(\alpha) : 0 \leq \alpha \leq 1]$ be a monotonic family of solutions of $\Gamma (P_i, W_i)$ except that $Y_i(1)$ need not be externally stable and let $X_i (\alpha) = A_{P_i} - \text{dom}_i\ Y_i (\alpha)$. Then

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1-\alpha)$$

is a solution of $\Gamma(P, W) = \Gamma(P_1, W_1) \otimes \Gamma(P_2, W_2)$

In this connection Shapley has raised the following questions:

(1) Can the requirement of full monotonicity be relaxed outside a neighbourhood of $\alpha_1 = 1$ for the validity of theorem 3.1 ?

(2) If the answer to (1) is 'yes' then is it sufficient to assume monotonicity condition to hold good in any arbitrarily small neighbourhood of $\alpha_1 = 1$ for the validity of the above theorem ?

(3) If the answer to (1) is 'yes' then does there exist a solution to product simple games which need not have the property of full monotonicity ?

We answer the first question in the affirmative. We are unable to prove (2) in the general case. However we have succeeded in proving (2) in a special case. More precisely the following theorems are proved. [9]

<u>Theorem 3.2:</u> Let $[X_1(\alpha) \mid 0 \leq \alpha \leq 1]$ be any ∂-*monotonic family of compound or product solutions except* that $X_1(1)$ need not be externally stable to the game $J = M_3 \boxtimes B_4$ where M_3 denotes the simple majority game on $\overline{123}$ and B_4 the 1- person pure bargaining game. Then

$$X = \bigcup_{0 \leq \alpha \leq 1} Z_1(\alpha) \underset{\alpha}{\times} Z_2(1 - \alpha)$$

is a solution for $J \boxtimes K$ where K is any arbitrary simple game and $Z_1(\alpha) = A_4 - \text{dom}_1 X_1(\alpha)$ and $Z_2(1-\alpha) \equiv Z_2$ being any solution of K.

<u>Theorem 3.2[1]:</u> Let $X_1(\alpha)$ be C- solutions to J and $X_1(\alpha)$ be ∂-monotonic. Then X as defined above is a solution for $J \boxtimes K$ where K is any arbitrary simple game.

An example of a product solution is given in order to show that every product solution need not have the property of full monotonicity - in the sense of Shapley. The author is extremely grateful to L.S. Shapley for having raised questions 2 and 3 in a private communication to the author and for his interest in this work.

Sec. 2 Definitions and Notations:

Simple games: We shall denote a simple game by the symbol $\Gamma(P, W)$ where P is a finite set (players) and W is a collection of subsets of P (the winning coalitions). We demand that $P \in W$ and the empty set not an element of W.

Let $\Gamma(P_1, W_1)$ and $\Gamma(P_2, W_2)$ be two simple games with $P_1 \cap P_2 = \emptyset$ and let $P = P_1 \cup P_2$. Then the product

$$\Gamma(P_1, W_1) \otimes \Gamma(P_2, W_2) \text{ (for simplicity we will write } P_1 \otimes P_2\text{)}$$

is defined as the game $\Gamma(P, W)$ where W consists of all $S \subseteq P$ such that $S \cap P_1 \in W_1$ and $S \cap P_2 \in W_2$.

Imputations: Let A_P denote the simplex of real non-negative vectors x, such that $\sum_i^P x_i = 1$. These vectors are traditionally called 'imputations' where P is the set of players in a simple game. Let us write $x(S)$ for $\sum_{i \in S} x_i$. Let $R_S\, x$ be the restriction of x to S thus:

$$R_S\, x = \begin{cases} x_i & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

The barycentric projection of x on A_S is given by

$$B_S\, x = \frac{1}{x(S)}\, R_S\, x$$

This is well defined provided $x(S) > 0$. Let P_1 and P_2 be fixed disjoint sets. Let $P = P_1 \cup P_2$ and $A_{P_i} = [\, x \mid x \in A_P \text{ and } x(P_i) = 1 \,]$ If X and Y are subsets of A_{P_1} and A_{P_2} respectively we define an operation $\underset{\alpha}{\times}$ by $(0 \leq \alpha \leq 1)$

$$X \underset{\alpha}{\times} Y = [\, Z \mid Z = \alpha x + (1 - \alpha) y \ \text{ for some } x \in X, y \in Y \,]$$

We recall that a solution of the game $\Gamma(P, W)$ is a set X of imputations such that $X = A_P - \text{dom}\, X$ where dom X denotes the set of all $y \in A_P$ such that for some $x \in X$, the set $\{ i \mid x_i > y_i \}$ is an element of W. The notations dom_1 and dom_2 will be used for domination with respect to special classes W_1, W_2. J will always stand for the game $M_3 \otimes B_4$ where M_3 stands for 3-person majority simple game and B_4 - 1 person pure bargaining game.

<u>Definition:</u> A parametrized family of sets of imputations

$$[\ Y(\alpha) : 0 \leq \alpha \leq 1\ ]$$

will be called semimonotonic if for every α, β, x such that $0 \leq \alpha \leq \beta \leq 1$ and $x \in Y(\beta)$, there exists $y \in Y(\alpha)$ with

$$\alpha y \leq \beta x$$

<u>Definition:</u> A semimonotonic family $Y(\alpha)$ is called monotonic if for every α, β, y such that $0 \leq \alpha \leq \beta \leq 1$ and $y \in Y(\alpha)$ there exists $x \in Y(\beta)$ with

$$\alpha y \leq \beta x$$

We will now generalize this concept of monotonicity.

<u>Definition:</u> A semimonotonic family $Y(\alpha)$ is called ∂- monotonic $(0 \leq \partial \leq 1)$ if for every α, β, y such that $\partial \leq \alpha \leq \beta \leq 1$ and $y \in Y(\alpha)$ there exists $x \in Y(\beta)$ with

$$\alpha y \leq \beta x$$

<u>Remark:</u> Any ∂- monotonic family is necessarily ∂_1-monotonic if $\partial_1 \geq \partial$. [∂ - in general will stand for any positive number with $0 < \partial < 1$ unless otherwise it is stated].

Sec. 3: On two theorems of Shapley [10]

Shapley has proved theorem 3.1 and the following theorem [13].

Theorem 3.3: Let $X_i(\alpha)$ be semimonotonic families of solutions of $\Gamma(P_i, W_i)$. Then

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1-\alpha)$$

is a solution of $P_1 \otimes P_2$.

The purpose of this section is to prove a theorem which includes both the theorems.

Let $Y_i(\alpha) \subseteq A_{P_i}$ for $i = 1,2$. Further suppose the families to be semimonotonic. Let
$X_i(\alpha) = A_{P_i} - \mathrm{dom}_i\, Y_i(\alpha)$. For every $S_j \in W_1$. Consider the following sets:

$$\Lambda_1(S_j) = [\alpha \mid 1 > \alpha > 0 \text{ there exists } x_1 \in Y_1(\alpha) \text{ and } y_1 \in X_1(1) - Y_1(1) \text{ such that } \alpha x_1 > y_1 \text{ on } S_j]$$

Now choose one α_j from each $\Lambda_1(S_j)$ which are not empty. Let $\alpha_0 = \text{Min}\ \alpha_j$. We fix these α_j's and it is clear that $\alpha_0 > 0$. We wish to remark that there is certain amount of arbitrariness in choosing α_j's. Now we shall prove the following simple but useful lemma.

Lemma 3.4: If $Y_1(\alpha)$ is α_0-monotonic then there exists vectors $Z_j \in Y_1(1)$ for those j for which $\Lambda_1(S_j) \neq \emptyset$ and a ∂_1 with $0 < \partial_1 \leq \alpha_0$ such that

$$Z_j \geq \partial_1 \quad \text{on} \quad S_j$$

Further if we take any $\alpha \in \Lambda_1(S_j)$, $\alpha \geq \partial_1$.

Proof: Since $\alpha_j \in \Lambda_1(S_j)$ there exists $x_j \in Y_1(\alpha_j)$ such that $\alpha_j x_j > 0$ on S_j. Since $\alpha_0 \leq \alpha_j$ and the family is α_0-monotonic there exists $Z_j \in Y_1(1)$ such that

$$Z_j \geq \alpha_j x_j$$

Let e_j denote the least component among the s_j components of x_j. Obviously $e_j > 0$ and it follows that

$$Z_j \geq \alpha_j e_j \quad \text{on} \quad S_j.$$

Define $\partial_1 = \text{Min}\ \alpha_j\ _j$. Then we have

$$Z_j \geq \partial_1 \quad \text{on} \quad S_j$$

Also it is trivial to check that $0 < \partial_1 \leq \alpha_0$. Hence the first part of the proof of the lemma is complete. We shall now prove that any $\alpha \in \Lambda_1(S_j)$ will satisfy the inequality $\alpha \geq \partial_1$. Take any $\alpha \in \Lambda_1(S_j)$. This means, there exists an $x_1 \in Y_1(\alpha_j)$ and $y_1 \in X_1(1) - Y_1(1)$ such that

$\alpha x_1 > y_1$ on S_j. We claim that at least one of the S_j components of y_1 must be greater than or equal to ∂_1; otherwise $Z_j > y_1$ on S_j and therefore $y_1 \in \text{dom}\, z_j$ $\text{dom}_1 Y_1(1)$ which contradicts the assumption that $y_1 \in X_1(1) = A_{P_1} - \text{dom}_1 Y_1(1)$. Therefore we have $\alpha > y_1(S_j) \geq \partial_1$. Hence the proof of the lemma is complete.

Similarly we can get hold of a ∂_2 by constructing $\Lambda_2(S_j')$ for $S_j' \in W_2$. If all the $\Lambda_1(S_j)$ and $\Lambda_2(S_j')$ are empty we define ∂_0 to be any positive number in the interval $(0, 1]$ - in fact we can take $\partial_0 = 1$. If $\Lambda_1(S_j) \neq \emptyset$ for some j and $\Lambda_2(S_j') \neq \emptyset$ for some j we define $\partial_0 = \text{Min}\,[\partial_1, \partial_2]$. If $\Lambda_1(S_j) \neq \emptyset$ for some j $\Lambda_2(S_j') = \emptyset$ for all j we define $\partial_0 = \partial_1$ and in the other case we take $\partial = \partial_2$. In all these cases we see that $\partial_0 > 0$. Now we are in a position to state and prove our theorem:

<u>Theorem 3.5</u>: Let $Y_1(\alpha)$ and $Y_2(\alpha)$ be semimonotonic family of solutions of $\Gamma'(P_1, W_1)$ and $\Gamma'(P_2, W_2)$ respectively except that $Y_1(1)$ and $Y_2(1)$ need not be externally stable. Further suppose $Y_i(\alpha)$ is ∂_0 monotonic Then

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1-\alpha)$$

is a solution of $P_1 \boxed{\times} P_2$, where $X_i(\alpha) = A_{P_i} - \mathrm{dom}_i\, Y_i(\alpha)$.

<u>Remark</u>: Since $Y_i(\alpha)$ is semimonotonic it follows that $Y_i(1)$ is internally stable. If $Y_i(1)$ for $i = 1,2$ is also externally stable, that is, if $Y_i(1)$ is a solution then it is not hard to check that $\Lambda_1(S_j)$ and $\Lambda_2(S_j')$ are all empty and we can take $\partial_0 = 1$. In other words theorem 3.3 is included in our theorem. One can also prove that full monotonicity implies that the Λ-sets are all empty.

We will show by giving an example that theorem 3.5 is actually a generalization of theorems 3.1' and 3.3.

<u>Proof of theorem 3.5</u>: Our proof follows along the same line as the one given for theorem 3.3. Note that

$X_i(\alpha) = Y_i(\alpha)$ for all α except $\alpha = 1$.

<u>External Stability:</u> Take any $y \in A_P$ and define $\beta_i = y(P_i)$ and let y_i be the barycentric projection of y on A_{P_i}.

<u>Case 1:</u> $0 < \beta_1 < 1$.

<u>Case 1a:</u> $y_1 \in X_1(\beta_1)$, $y_2 \in X_2(\beta_2)$. Then

$$y = \beta_1 y_1 + \beta_2 y_2 \in X$$

<u>Case 1b:</u> $y_1 \notin X_1(\beta_1)$ $y_2 \notin X_2(\beta_2)$. Then

there exists $x_1 \in Y_1(\beta_1)$, $x_2 \in Y_2(\beta_2)$ such that $y_i \in \mathrm{dom}_i\, x_i$ for $i = 1,2$. Then the imputation defined by

$$x = \beta_1\, x_1 + \beta_2\, x_2$$

which is in X, clearly dominates y or $y \in \mathrm{dom}\, X$.

<u>Case 1c:</u> $y_1 \in X_1(\beta_1)$, $y_2 \notin X_2(\beta_2)$.

Find $x_2 \in Y_2(\beta_2)$ such that $y_2 \in \mathrm{dom}_2\, x_2$. Let $x_2 > y_2$ on $S_2' \in W_2$ choose $e > 0$ so that $\beta_2\, x_2 - \beta_2\, y_2 > e$ on S_2'. Using the semimonotonic property of $Y_2(\alpha)$, find $x_2' \in Y_2(\beta_2 - e)$ such that $\beta_2 x_2 - (\beta_2 - e)\, x_2'$ is non-negative. This vector must be less than or equal to e in all components, since no component of a non-negative vector can exceed the sum of all components. It follows that we have

$$(\beta_2 - e)\, x_2' > \beta_2\, y_2 \quad \text{on} \quad S_2' .$$

Let u_1 be any interior point of A_{P_1}. Then the imputation x defined by

$$x = \beta_1 y_1 + e u_1 + (\beta_2 - e)\, x_2'$$

dominates y on $P_1 \cup S_2'$. If $x \in X$ then $y \in \text{dom } X$ and we are through. Suppose $x \notin X$. Then the barycentric projection of x on A_{P_1} namely

$$x_0 = \frac{\beta_1}{\beta_1 + e}\, y_1 + \frac{e}{\beta_1 + e}\, u_1$$

must not be an element of $X_1(\beta_1 + e) = Y_1(\beta_1 + e)$. Hence we can find $x_1 \in Y_1(\beta_1 + e)$ such that $x_0 \in \text{dom}_1\, x_1$. Then it is clear that the imputation Z defined by

$$Z = (\beta_1 + e)\, x_1 + (\beta_2 - e)\, x_2'$$

dominates y. Since $Z \in X$ it follows that $y \in \text{dom } X$.

<u>Case 1d:</u> $y_1 \notin X_1(\beta_1)$ $y_2 \in X_2(\beta_2)$. Like case 1c.

Case 2: $\beta_1 = 0$

Case 2a: $y_2 \in X_2(1)$. Then $y \in X$

Case 2b: $y_2 \notin X_2(1)$.

This means $y_2 \in \mathrm{dom}_2\, Y_2(1)$ and hence argument of case 1c can be repeated with the understanding that $\beta_1 y_1 = 0$.

Case 3: $\beta_1 = 1$. Like case 2.

This completes the proof of the external stability of X.

Internal Stability of X:

Suppose there exists $x, y \in X$ such that $x \succ y$ on $S \subset W$. Let $S_1 = S \cap P_1$, $S_2 = S \cap P_2$, and $x = \alpha x_1 + (1-\alpha)x_2$, $y = \beta y_1 + (1-\beta)y_2$. Since $x \succ y$ on $S_1 \cup S_2 = S$ it follows that $0 < \alpha < 1$.

Case 1a: $1 > \beta \geq \alpha$.

Since $\alpha > 0$, β is also strictly positive. Also $\alpha x_1 > \beta y_1$ on S_1. Since $(Y_1(\alpha))$ is semimonotonic, there exists $x_1' \in Y_1(\alpha)$ such that $\beta y_1 \geq \alpha x_1'$. Hence, $\alpha x_1 > \alpha x_1'$ on S_1 or $x_1 > x_1'$ on S_1 contradicting the internal stability of $Y_1(\alpha)$.

Case 1b: $\alpha \geq \beta > 0$.

This means $1 > 1 - \beta \geq 1 - \alpha$. Since $\alpha \neq 1$ this case is similar to the previous case except that we have to utilise the semimonotonic property of $Y_2(\alpha)$.

<u>Case 2a:</u> $\beta_1 = 1$ that is $\alpha x_1 > y = y_1$ on S_1. If $y_1 \in Y_1(1)$, then using the semimonotonic property of $Y_1(\alpha)$ we will arrive at a contradiction. If $y_1 \in X_1(1) - Y_1(1)$ then $\alpha \in \Lambda_1(S_1)$ and hence by lemma 3.4, $\alpha \geq \delta_1 \geq \delta_0$. Since $Y_1(\alpha)$ is δ_0-monotonic one can find $y_1' \in Y_1(1)$ such that $y_1' \geq \alpha x_1$ that is $y_1' \geq \alpha x_1 > y_1$ on S_1. This implies $y_1 \in \text{dom}_1 Y_1(1)$ or $y_1 \notin X_1(1)$ which contradicts our assumption regarding y_1.

<u>Case 2b:</u> $\beta = 0$ or $1 - \beta = 1$ and the argument can be carried over as in case 2a:
Hence the proof of the theorem is complete.

Sec. 4: An example of a solution of product simple games

The following example shows that solutions to product simple games can be found that satisfy the conditions of theorem 3.5 but not that of 3.1 or 3.3. In other words every solution that satisfies the conditions of theorem 3.5

need not have the property of full monotonicity. My sincere thanks are due to L.S. Shapley for having raised this relevant point.

Example: The four person game J is defined by

$$J = \Gamma(\overline{1234}, \{\overline{124}, \overline{134}, \overline{234}, \overline{1234}\})$$

where members in the curly brackets denote the winning coalitions of the game J.

$$J = M_3 \boxtimes B_4$$

M_3 denotes the 3-person simple majority game and B_4 the 1- person pure bargaining game.

Now we shall write down the solutions of M_3 which are well known [19].

(a) the finite set

$$(\tfrac{1}{2}, \tfrac{1}{2}, 0), (\tfrac{1}{2}, 0, \tfrac{1}{2}), (0, \tfrac{1}{2}, \tfrac{1}{2})$$

(b_∂) the line-segment

$$\text{Set } \{(\partial, t, 1-\partial-t) \mid 0 \leq t \leq 1-\partial\}$$

for each $0 \leq \partial < \frac{1}{2}$

(c_∂) the line segment obtained by permuting players 1 and 2 in (b_∂)

(d_∂) the line segment obtained by permuting players 1 and 3 in (b_∂).

No semimonotonic family drawn from this list include representatives from more than one of the four groups (a) - (d); hence the only possible variation within such a family is in the value of ∂.

Define, for $0 \leq \alpha \leq \frac{3}{4}$

$$X_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta) \quad \text{where}$$

$$Y(\beta) = \{(\beta^2/2, \beta t, \beta(1-t) - \beta^2/2, 1-\beta) | 0 \leq t \leq 1-\beta/2\}$$

for $0 \leq \beta < 1$ and

$$Y(1) = \{(\tfrac{1}{2}, t, \tfrac{1}{2} - t, 0) | 0 \leq t \leq \tfrac{1}{2}\} \bigcup (0, \tfrac{1}{2}, \tfrac{1}{2}, 0)$$

Define for $\frac{3}{4} < \alpha \leq \alpha_0$ where α_0 is so chosen such that

$$\alpha_0 \left(1 - \frac{7}{8(1+\alpha_0)}\right) = \frac{1}{2}$$

$$X_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta) \quad \text{where}$$

$Y(\beta) = [(\frac{7}{8}\frac{\beta^2}{1+\alpha},\ \beta t,\ \beta(1-t) - \frac{7}{8}\ \frac{\beta^2}{1+\alpha},\ 1-\beta)$

where $0 \leq t \leq 1 - \frac{7\beta}{8(1+\alpha)}]$ for $0 \leq \beta \leq 1$.

For $\alpha_o < \alpha < 1$ define

$$X_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta) \quad \text{where}$$

$Y(\beta) = [((1 - \frac{1}{2\alpha})\beta^2,\ \beta t,\ \beta(1-t) - (1-\frac{1}{2\alpha})\beta^2,\ 1-\beta)$

where $0 \leq t \leq 1 - (1 - \frac{1}{2\alpha})\beta]$ for $0 \leq \beta \leq 1$

For $\alpha = 1$ define

$$X_1(1) = \bigcup_{0 \leq \beta \leq 1} Y(\beta) \quad \text{where for } 0 \leq \beta \leq 1,$$

$Y(\beta) = [(\beta^2/2,\ \beta t,\ \beta(1-t) - \beta^2/2,\ 1-\beta)$ for $0 \leq t \leq 1-\beta/2]$.

Now it is not hard to check that the family $X_1(\alpha)$ is semimonotonic and that each $X_1(\alpha)$ is a solution to the game J except that $X_1(1)$ is not externally stable because

$$(0, \tfrac{1}{2}, \tfrac{1}{2}, 0) \notin X_1(1) \cup \text{dom } X_1(1)$$

If $X_1'(1) = X_1(1) \cup (0, \frac{1}{2}, \frac{1}{2}, 0)$ then $X_1'(1)$ is a solution of J. But $\{X_1(\alpha): 0 \leq \alpha < 1\}$ together with $X_1'(1)$ is not semimonotonic, for, corresponding to $(0, \frac{1}{2}, \frac{1}{2}, 0) \in X_1'(1)$, there exists no element $x \in X_1(\alpha)$ for any $\alpha > \frac{3}{4}$ with $\alpha x \leq (0, \frac{1}{2}, \frac{1}{2}, 0)$. Further the family $[X_1(\alpha) : 0 \leq \alpha \leq 1]$ is not fully monotonic because corresponding to the element $(0, \frac{1}{2}, \frac{1}{2}, 0) \in X_1(\frac{3}{4})$ there exists no element $y \in X_1(1)$ with the property that $y \geq \frac{3}{4} (0, \frac{1}{2}, \frac{1}{2}, 0)$. Hence it is clear that the conditions of theorem 3.1 as well as theorem 3.3 are violated. Moreover it is not difficult to check that $\wedge(S_j) = \emptyset$ for all j and as such we can take $\delta_1 = 1$. Hence this family $X_1(\alpha)$ can be used to produce product solutions to arbitrary games of the form $J \otimes K$ - solutions which theorem 3.1 or theorem 3.3. Cannot predict.

Sec. 5: Proof of theorem 3.2

We will start with some preliminaries that are essential for the proof of theorem 3.2.

Definition: Let X be a solution to the product of simple games $P_1 \otimes P_2$. We say X is a product solution if and only if

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1 - \alpha)$$

where $[\,Y_i(\alpha) : 0 \leq \alpha \leq 1]$ are semimonotonic family of solutions to P_i except that $Y_i(1)$ need not be externally stable with $X_i(\alpha) = A_{P_i} - \mathrm{dom}_i\ Y_i(\alpha)$.

If $X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1 - \alpha)$ and $X_i(\alpha) \equiv X_i$ for $i = 1,2$ then we call X a compound set. We require the following theorem due to Shapley.

<u>Theorem 3.6</u>: A compound set X is a solution to $P_1 \; (\times) \; P_2$ iff X_i is a solution for P_i $(i = 1,2)$. For a proof see [15]. We call such solutions as compound solutions. In fact every compound solution is a product solution Now we are in a position to prove theorem 3.2.

<u>Proof:</u> Let $X = \bigcup_{0 \leq \alpha \leq 1} Z_1(\alpha) \underset{\alpha}{\times} Z_2\ (1 - \alpha)$ where

$$Z_1(\alpha) = A_J - \mathrm{dom}_1\ X_1(\alpha) \quad \text{and}$$

$$Z_2(1 - \alpha) \equiv Z_2 \quad \text{where } Z_2 \text{ is any solution of } K.$$

Observe that $Z_1(\alpha) = X_1(\alpha)$ for every α except $\alpha = 1$.

External stability can be established as in the case of theorem of 3.5 as the proof depends only on the semimonotonic property of $X_1(\alpha)$ and the external stability of $Z_1(\alpha)$. Hence it is enough to establish internal stability of X.

Internal stability of X:

Case 1: $X_1(\alpha)$ are all compound solutions except $X_1(1)$. Now one can write,

$$X_1(\alpha) = \left[(\beta(c, t, 1-t-c), 1-\beta) \middle| \begin{matrix} 0 \le t \le 1-c \\ 0 \le \beta \le 1 \end{matrix} \right]$$

This re-presentation is possible because of theorem 3.6 and the fact that explicit solution to M_3 is known completely.

Let α^n be a sequence of positive numbers increasing to 1 with

$$X_1(\alpha^n) = [\beta(c_n, t, 1-t-c_n), \; 1-\beta]$$

[For simplicity we will not be writing the possible values of t and β].

With loss of generality let $C_n \to C_0$ Consider the set

$$N = \{ x \mid x = (\beta(c_0, t, 1 - t - c_0), 1 - \beta) \text{ and there exists } x_{n_k} \in X_1(\alpha^{n_k}) \text{ such that } \alpha^{n_k} x_{n_k} \uparrow x \}$$

First we shall establish that the closure of $X_1(1)$ contains N. It is not difficult to see that N is nonempty.

Let $x \in N$. This means there exists $x_{n_k} \in X_1(\alpha^{n_k})$ with

$$\alpha^{n_k} x_{n_k} \uparrow x.$$

Except a finite number of α^{n_k} the remaining α^{n_k} will be greater than or equal to ∂. Since by assumption the family $X_1(\alpha)$ is ∂-monotonic it follows that there exists a sequence y^{n_k} belonging to $X_1(1)$ with $y^{n_k} \geq \alpha^{n_k} x_{n_k}$. Hence it follows that the closure of $X_1(1)$ - written as $\bar{X}_1(1)$ contains x. In other words $N \subseteq \bar{X}_1(1)$.

At this juncture we would like to make another observation namely $\bar{X}_1(1)$ to-gether with $[X_1(\alpha) : 0 \leq \alpha < 1]$ is a semimonotonic family and hence $\bar{X}_1(1)$ is also internally stable.

If $C_0 < \frac{1}{2}$, then $\bar{X}_1(1) = N$ and further N is a solution for J. Hence by theorem 3.3 it follows that X is internally stable.

If $C_0 = \frac{1}{2}$ then $\bar{X}_1(1) = N$

$$= [\beta(\tfrac{1}{2}, t, \tfrac{1}{2} - t), 1 - \beta]$$

It is clear that the set

$$[(0, \beta/2, \beta/2, 1-\beta)/ 0 \leq \beta \leq 1]$$

is not dominated by $\bar{X}_1(1)$.

To prove internal stability of X in this case, it is sufficient to establish that there does not exist any element $x \in X_1(\alpha)$ with αx dominating $(0, \beta/2, \beta/2, 1-\beta)$ for some β. In other words we have to show $\wedge_j(S_j) = \emptyset$ for all j.

Let if possible

$$\alpha\ (\beta'(x_1, x_2, x_3)\ 1-\beta') > (0, \beta^0/2, \beta^0/2, 1-\beta^0)$$

Via say $\overline{124}$. Hence $\alpha(1 - \beta') > 1 - \beta^0$ or $\beta^0 > \beta'$.

Let $X_1(\alpha) = [\beta(c, t, 1 - t - c), 1 - \beta]$.

Now it is clear that

$$\alpha\, \beta^{1}(1 - c) > \beta^{0}/2 > \beta'/2$$

$$\text{or} \quad \alpha(1 - c) > 1/2$$

Further $(1/2, 1/2, 0, 0) \in \bar{X}_1(1)$ and $X_1(\alpha)$ is semimonotonic It follows that

$$\alpha\ (c, 1-c, 0, 0) \leq (1/2, 1/2, 0, 0)$$

$$\text{or} \quad \alpha(1 - c) \leq 1/2 \quad \text{and}$$

hence there is a contradiction.

Let if possible $X_1(\alpha) = [\beta(t, 1-t, 0), 1-\beta]$.

It follows that $\alpha > 1/2$. We also have

$$(1/2, 0, 1/2, 0) \geq \alpha\ (1, 0, 0, 0)$$

$$\text{or} \quad \alpha \leq 1/2$$

which is impossible. Thus we have established internal stability of X in this case also. If for infinitely many n,

$$X_1(\alpha^n) = (\beta/2, 0, \beta/2, 1-\beta) \cup (\beta/2, \beta/2, 0, 1-\beta) \cup (0, \beta/2, \beta/2, 1-\beta)$$

then $\bar{X}_1(1)$ will also be the same set and this case is similar to the case when $\theta_o < \frac{1}{2}$ and internal stability of X can be shown via theorem 3.3.

<u>Case 2:</u> $X_1(\alpha)$ are **product** solutions.

Let α^n be an increasing sequence of positive numbers with $\alpha^n \to 1$

$$X_1(\alpha^n) = \left\{ (\beta(c_\beta^n, t, 1-c_\beta^n - t) \; 1-\beta) \quad \begin{matrix} 0 \leq \beta < 1 \\ 0 \leq t \leq 1-c_\beta^n \end{matrix} \right\} \cup Y_1^{\alpha^n}(1)$$

where $Y_1^{\alpha^n}(1)$ need not be an internally stable set for M_3. Consider as before the set,

$$N_\beta = \{x \mid x = (\beta(c_\beta^o, t, 1-t-c_\beta^o), 1-\beta) \text{ and there exists}$$

a sequence $x_{n_k} \in X_1(\alpha^{n_k})$ such that $\alpha^{n_k} x_{n_k} \uparrow x \}$

If $c_\beta < \frac{1}{2}$ for every β then,

$$\bar{X}_1(1) = \bigcup_{0 \leq \beta \leq 1} N_\beta$$

and $\bigcup N_\beta$ is a solution for J and internal stability follows by the theorem of Shapley.

Let $c_\beta^o = \frac{1}{2}$ for some β. To complete the proof of theorem 3.2 it is sufficient to establish that there does not exist any vector $x \in X_1(\alpha)$ with αx dominating y with

$y \in Z_1(1) - \bar{X}_1(1)$. Let if possible,

$$\alpha x > y \quad \text{via} \quad \overline{124} \quad \text{with} \quad y \in Z_1(1) - \bar{X}_1(1)$$

Let $y = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ and

$X(\alpha) = [\beta(c_\beta, t, 1 - t - c_\beta), 1 - \beta]$

$\alpha(\beta(c_\beta, t', 1 - t' - c_\beta), 1 - \beta) > y$ on $\overline{124}$

Choose any β' with $\alpha(1 - \beta) > 1 - \beta' > \epsilon_4$.

We will choose and fix β'.

Let $N_{\beta'} = [\beta'(c^o_{\beta'}, t, 1 - t - c^o_{\beta'}), 1 - \beta']$

If $\beta' c^o_{\beta'} \leq \epsilon_1$, then

$$\alpha x > (\beta'(c^o_{\beta'}, 0, 1 - c^o_{\beta'}), 1 - \beta') \quad \text{via} \quad 124$$

This will mean $\alpha x > \alpha w$ where $w \in X_1(\alpha)$ thereby. Contradicting the internal stability of $X_1(\alpha)$. If $\beta' c^o_{\beta'} > \epsilon_1$ then $\beta'(1 - c^o_{\beta'}) \leq \epsilon_2$ otherwise there will be an element in $N_{\beta'}$ dominating y thereby contradicting the assumption that $y \in Z_1(1)$. So we have $\beta'(1 - c^o_{\beta'}) \leq \epsilon_2$. Now a suitable positive t^o can be found such that

$$\alpha[\beta(c_\beta,\ t^o,\ 1-t^o-c_\beta),\ 1-\beta] > [\beta'(c^o_{\beta'},\ 1-c^o_{\beta'},\ 0), 1-\beta']$$

Via 234 which will once again contradict the internal stability of $X_1(\alpha)$. If $X_1(\alpha) = [\beta(t,\ 1-t,\ 0),\ 1-\beta]$ or contains elements of the form $(\beta/2,\ \beta/2,\ 0,\ 1-\beta)$ and whatever be the form of $N_{\beta'}$ then also one can show the impossibility of αx dominating $y \in Z_1(1) - \bar{X}_1(1)$. Thus the internal stability follows and the proof of theorem 3.2 is complete.

<u>Remark:</u> The example given in the previous section will serve as an example for theorem 3.2 as each $X_1(\alpha)$ constructed there are product solutions to J except $X_1(1)$ and the family is α_o-monotonic where $0 < \alpha_o < 1$. Theorem 3.2 has its own limitations because every solution to J need not be a product solution. We will now construct a solution which is not a product solution to J. Let $s = (0,\ 1/4,\ 1/4,\ 1/2)$ and L be the set defined as

$$L = L_1 \cup L_2 \cup L_3 \quad \text{where}$$

$$L_1 = \bigcup_{0 \leq t \leq 1/2} \{(\tfrac{1}{4}, x, y, t) \mid x, y \geq 0 \text{ and } x+y = \tfrac{3}{4} - t\}$$

$$L_2 = \bigcup_{\frac{1}{2} < t \leq \frac{3}{4}} \{(\tfrac{3}{4}-t, x, y, t) \mid x, y \geq 0,\ x+y = \tfrac{1}{4}\}$$

$$L_3 = \bigcup_{\frac{3}{4} < t \leq 1} \{(0, x, y, t) \mid x, y \geq 0,\ x + y = 1 - t\}$$

Define $H = L \cap \text{dom } s$

Then it can be shown that,

$$\{s\} \cup (L - H)$$

is a solution for J. For a proof of this assertion one can refer to the well known paper of Shapley [13]. Further it is easy to verify that the solution constructed above is not a product solution. But this is not a serious limitation as can be seen from the next section - where in we are going to show that theorem 3.2 is valid if one includes solutions of the type constructed by Shapley [13] containing an arbitrary closed component for the game J.

We are unable to settle the following question: Suppose $X_1(\alpha)$ is any ∂-monotonic family of solutions (not necessarily product-solutions or C-solutions) to J except that $X_1(1)$ need not be externally stable. Then will this family yield product solutions to games of the form $J \; (\boxtimes) \; K$? The author firmly believes that this problem can be settled provided one knows all possible solutions to the game J. However one can prove the following partial results which give product solutions to games of the form $M_3 \; (\boxtimes) \; K$ and $J \; (\boxtimes) \; K$.

<u>Theorem 3.7:</u> Let $Y_1(\alpha)$ be any ∂ monotonic family of solutions to $M_3 = \Gamma[\overline{123}, (\overline{12}, \overline{13}, \overline{23}, \overline{123})]$ except that $Y_1(1)$ need not be externally stable. Let $X_1(\alpha) = A_3 - \mathrm{dom}_1 \; Y_1(\alpha)$. Then

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2 \; (1 - \alpha)$$

is a solution for $M_3 \; (\boxtimes) \; K$ where $X_2(1-\beta) \equiv X_2$ is any solution of the simple game K.

<u>Theorem 3.8:</u> Let $X_1(\alpha)$ be $\frac{1}{2}$ monotonic family of solutions to J except that $X_1(1)$ need not be externally stable. Further suppose $[X_1(\alpha) \quad \alpha \geq \frac{1}{2}]$ are product solutions. Define $Z_1(\alpha) = A_J - \mathrm{dom} \; X_1(\alpha)$. Then

$$X = \bigcup_{0 \leq \alpha \leq 1} Z_1(\alpha) \underset{\alpha}{\times} Z_2(1 - \alpha)$$

is a solution for $J \; (\boxtimes) \; K$ where $Z_2(1 - \alpha) \equiv Z_2$, any solution for the simple game K.

<u>Remark:</u> We suspect that theorem 3.8 will be true if we replace 1/2 by any positive δ sufficiently near one. Proof of theorem 3.7 is simple and we will be proving only the internal stability of theorem 3.8.

<u>Internal stability of theorem 3.8</u>: We need only show that $\Lambda(S_j) = \emptyset$ for $j = 1,2,3$ where $S_1 = 124$, $S_2 = 134$ and $S_3 = 234$. Let if possible

$$\alpha(x_1, x_2, x_3, x_4) > y = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \text{ via say}$$

S_1 where $y \in Z_1(1) - \bar{X}_1(1)$.

Let $N_{\beta'} = (\beta'(c_{\beta'}, t, 1-t-c_{\beta'}), 1-\beta')$ where β' is so chosen such that $\alpha x_4 > 1 - \beta' > \varepsilon_4$.

$\beta' c_{\beta'}$ cannot be less than or equal to ε_1 as this contradicts the internal stability of $X_1(c)$. Hence $\beta' c_{\beta'} > \varepsilon_1$. Since $y \in Z_1(1)$ it followe, $\beta'(1 - c_{\beta'}) \leq \varepsilon_2, \varepsilon_3$. So we have the following inequalities

$$\alpha x_4 > 1 - \beta'$$

$$\alpha x_2 > \varepsilon_2 \geq \beta'(1 - c_{\beta'}) \geq \beta'/2.$$

Therefore $\alpha > 1 - \beta'/2 \geq 1/2$.

Since $X_1(\alpha)$ is $\frac{1}{2}$-monotonic, there exists a $w \in X_1(1)$ such that $w \geq \alpha x > y$ via S_1 which contradicts the assumption that $y \in Z_1(1)$.

Let if possible $N_{\beta'} = (\beta'/2, \beta'/2, 0, 1-\beta')(\_)$

$$(\beta'/2, 0, \beta'/2, 1-\beta')(\_)$$

$$(0, \beta'/2, \beta'/2, 1-\beta')$$

As before we have

$$\beta'/2 > e_1 \quad \text{and} \quad \beta'/2 \leq e_2$$

$$\alpha x_4 > 1 - \beta'$$

$$\alpha x_2 > \beta'/2$$

$$\text{or} \quad \alpha > 1/2$$

This once again contradicts the hypothesis that $y \in Z_1(1)$ Similarly other cases can be disposed of and thus the proof is complete.

In fact theorems (3.2) and 3.8 can be written in a slightly more general form as follows.

<u>Theorem 3.8'</u>: Let $X_1(\alpha)$ be any 1/2 - monotonic family of solutions to J. Further suppose $[X_1(\alpha): \alpha \geq \frac{1}{2}]$ are product solutions. Let $Z_1(\alpha) = A_J - \text{dom}_1\, X_1(\alpha)$ and

$Z_2(\alpha)$ be any semimonotonic family of solutions to K'. Then

$$X = \bigcup_{0 \leq \alpha \leq 1} Z_1(\alpha) \underset{\alpha}{\times} Z_2(1-\alpha)$$

is a solution for the product game $J \circledast K$. In a similar fashion theorem 3.2 can be stated.

Sec. 6: A class of product solutions

In the previous section we have constructed a solution that is not a product solution for J. In this section we will prove a theorem corresponding to 3.2 that will include solutions of the type constructed in section 5.

Let C be an arbitrary closed set in $[0, 1)$ and let S be the image of C under the one-one map

$$u_4 \in C \longrightarrow \langle 0, \frac{1-u_4}{2}, \frac{1-u_4}{2}, u_4 \rangle$$

consider the set L which consists of all imputations of the form

$$\left(\frac{1 - u_4 - \rho(u_4, C^0)}{2}, x, y, u_4\right), \; u_4 \in [0, 1]$$

where $C^0 = C \bigcup 1$, $\rho(u_4, C^0) = \inf_{u \in C^0} |u_4 - u|$

$x \geq 0 \quad y \geq 0$ and $x + y = \dfrac{1 - u_4 + \rho(u_4, C^0)}{2}$. Let

$$H = L \bigcap \text{dom } S$$

Then it follows that $S \bigcup (L - H)$ is a solution for J. For a proof refer [12]. We will call such solutions as C - solutions.

<u>Theorem 3.2'</u>: Let $X_1(\alpha)$ be C-solutions to J and $X_1(\alpha)$ be ∂-monotonic. Then

$$X = \bigcup_{0 \leq \alpha \leq 1} Z_1(\alpha) \underset{\alpha}{\times} Z_2(1-\alpha) \text{ is a}$$

solution for $J \otimes K$ where $Z_2(1 - \alpha) \equiv Z_2$ is any solution for K.

<u>Proof</u>: We will only indicate the proof of internal stability. Let if possible $\alpha(x_1, x_2, x_3, x_4) >$ $y = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ via 124, $y \in Z_1(1) - \bar{X}_1(1)$.

Using the θ-monotonic property, it can be shown that

$$w_1 = \left(\frac{1-u_4 - \rho^o}{2}, \frac{1 - u_4 + \rho^o}{2}, 0, u_4\right) \in \bar{X}_1(1) \quad \text{and}$$

$$w_2 = \left(\frac{1 - u_4 - \rho^o}{2}, 0, \frac{1 - u_4 + \rho^o}{2}, u_4\right) \in \bar{X}_1(1)$$

where $\alpha x_4 > u_4 > \epsilon_4$ and ρ^o is a non-negative real number.

If $\frac{1 - u_4 - \rho^o}{2} \leq \epsilon_1$ we are through. We will suppose $\frac{1 - u_4 - \rho^o}{2} > \epsilon_1$ and hence it follows that $\frac{1 - u_4 + \rho^o}{2} \leq \epsilon_2$.
If $x_3 > 0$ then $\alpha x > w_1$ via 234 which will in turn contradict the internal stability of $X_1(\alpha)$. So we will assume $x_3 = 0$.

Let C_α be the closed subset of $[0, 1)$ corresponding to $X(\alpha)$. If $x_4 \in C_\alpha$ then

$$\left(0, \frac{1 - x_4}{2}, \frac{1 - x_4}{2}, x_4\right) \in X(\alpha)$$

Since $x_3 = 0$ it follows that $x_1 = \frac{1 - x_4}{2}$ and $x_2 = \frac{1- x_4}{2}$
Hence,

$\alpha x > w_1$ via 124 which is impossible.

Let $x_4 \notin \underline{C_\alpha}$. This means $x_1 = \dfrac{1 - x_4 - \rho(x_4, C_\alpha^0)}{2}$ and

$x_2 = \dfrac{1 - x_4 + \rho(x_4, C_\alpha^0)}{2}$

where $\underline{C_\alpha^0} = \underline{C_\alpha} \cup \{1\}$

$\rho(x_4, C_\alpha^0) > 0$ for $x_4 \notin \underline{C_\alpha^0}$ and $\underline{C_\alpha^0}$ is compact.

Let $0 < \epsilon < \rho(x_4, C_\alpha^0)$ Then

$$\left(\frac{1 - x_4 - \rho(x_4, C_\alpha^0)}{2}, \frac{1 - x_4 + \rho(x_4, C_\alpha^0) - \epsilon}{2}, \frac{\epsilon}{2}, x_4\right) \in X_1(\alpha)$$

That is,

$$\alpha\left(\frac{1 - x_4 - \rho(x_4, C_\alpha^0)}{2}, \frac{1 - x_4 + \rho(x_4, C_\alpha^0) - \epsilon}{2}, \frac{\epsilon}{2}, x_4\right) > w_1$$

via 234 which will contradict the internal stability of $X(\alpha)$.

Consider the case when $x_3 = \dfrac{1 - x_4 - \rho(x_4, C_\alpha^0)}{2} = 0$ i.e. when $\rho(x_4, C_\alpha^0) = 1 - x_4$.

From the way in which every C-solution is constructed it follows that the first two coordinates can run between 0 to $1 - x_4$.

Hence we have,

$$\alpha(x_2, x_1, 0, x_4) > w_2 \quad \text{via } 124$$

which leads to a contradiction. Other cases can be disposed of similarly. Thus the proof is complete.

Remark: It is possible to construct a product solution but not a C-solution. In fact solutions constructed for J in sec. 4 are product solutions and not C-solutions.

Sec. 7: Concluding remarks

The pursuit of the complexities in the solutions of product games is justified in part by the insight this activity provides into the still more difficult problems surrounding the solutions of general n-person games. The results so far obtained on product games are primarily of a constructive nature. It may still be possible to construct a solution for J which is neither a product solution nor a C-solution thereby reflecting the conspicuous limitations of our present technique. Definitive results on composition of product simple games may be achieved provided one can characterise all possible solutions for simple games. At last we wish to remark that it will be of some interest if one can give a lower bound for ∂_0 in theorem 3.5. We have not yet succeeded in constructing a ∂- (where of course $0 < \partial < 1$) monotonic family of solutions to J wherein $\Lambda_1(S_j) \neq \emptyset$ for at least one j. If there exists no such family then it means that Shapley's conjecture is true in this special case. In fact no example is known to the author so far for any game whatsoever, where the Λ-sets are nonvaccuous, while the family of solutions is ∂- monotonic.

REFERENCES

1) Berge, C. [1954]: 'Sur une convexite regulier et ses a pplications a'la theorie des jeux'. Bulletin Society France 82.

2) Glicksberg, I.L. [1950]: 'Minimax theorem for upper and lower semicontinuous pay-offs', Rand Corporation Research Memorandum No.478.

3) Karlin, S. [1959]: Mathematical methods and the theory of games, programming and economics Vol. 2.

4) Kelley, J.L. [1955]: General topology - Von Nostrand and Company.

5) Kneser, H. [1952]: 'Sur Un theorem fundamental de la theorie des jeux', C.R. Academy of Sciences Paris.

6) Ky Fan [1953]: 'Minimax Theorems'. Proceedings of the National Academy of Sciences U.S.A. Vol. 39.

7) Parthasarathy, T. [1965]: 'A note on a minimax theorem of T.T.Tie'. To appear in Sankhya.

8) ------ [1965]: 'On games played over the unit square'. Submitted for publication in Sankhya.

9) ---------- [1965]: 'Solutions for product simple games'. Submitted for publication in SIAM Journal.

10) ----------- [1965]: 'A note on compound simple games' - To appear in the Proceedings of the American Mathematical Society.

11) Prohorov, Yu. V. [1956]: 'Convergence of random processes and limit theorems in the theory of probability'. Theory of probability and its applications [English Translation].

12) Ranga Rao, R. [1960]: 'Some problems in probability theory'. Thesis submitted to the Calcutta University.

13) Shapley, L.S. [1959]: 'A solution containing an arbitrary closed component'. Contributions to the theory of games Vol. 4.

14) ----------- [1962]: 'Compound simple games I', Rand Corporation Research Memorandum 3192.

15) ----------- [1964]: 'Solutions of compound simple games'. Advances in game theory.

16) Sion, M. [1958]: 'On general minimax theorem', Pacific Journal of Mathematics.

17) Sion, M. and Wolfe, P. [1957]: 'On a game without a value'. Contributions to the theory of games Vol. 3.

18) Teh Tjoe-Tie [1963]: 'Minimax theorems on conditionally compact sets'. Annals of Mathematical Statistics Vol. 34.

19) Varadarajan, V. S. [1958]: 'Weak convergence of measures on separable metric spaces', Sankhya.

20) von-Neumann, J. and Morgenstern, O. [1944]: Theory of games and Economic behaviour.

21) Wald, A. [1945]: 'Generalization of a theorem by von-Neumann Concerning Zerosum two person game', Annals of Mathematics.

22) Wald, A. [1950]: 'A note on zero-sum two person game', Annals of Mathematics.

23) ------- [1950]: Statistical Decision Functions, John Wiley publications.

24) Yanovskaya, E. B. [1964]: 'Minimax theorems for games on the unit square'. Theory of probability and its applications, Vol. 9. English Translation.