Bell's inequality violation and symmetry

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Abstract

Bell's inequality violation is related to the breakdown of symmetry of photonic field states. The states allowing the violation are characterized by a parameter γ associated with the interaction of the nonlinear medium and radiation. The violation is shown for small values of γ, where the particle aspect of light dominates. The degrading of the entanglement of the beam with increasing γ is discussed. The essential local noncommutativity of the operators involved is obvious.

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1. Introduction

It is well known that group theory is particularly suited for quantum mechanics. The deep connection of group theory with symmetry suggests that casting counterintuitive quantum mechanical results in group theoretical format should be helpful to its comprehension. Violation of Bell's inequality [1] is, thus, a natural candidate. In the present Letter we outline our version of this approach which underscores the crucial role of operators noncommutativity [2], which is a local effect, in Bell's inequality violation (BIQV, henceforth). The quantum states of the radiation field that are often involved in the studies of BIQV (e.g., "entangled states", "phase shifted states", etc.) were analysed within Lie group terminology in [3,4]. In the next section we use their terminology to formulate our problem. That section provides the mathematical background for our study and relates it to the generation of the states involved. The state providing the entanglement of distant parts of the field is generated via a down converter. The vacuum state we consider is the limit wherein no photon pairs are being generated by the down converter. The symmetry of the whole system (i.e., of all parts inclusive of those that may be widely separated in space) is realized by a polarization rotator acting in unison on the distinct parts of our system—denoted by A and B henceforth (cf. Fig. 1).

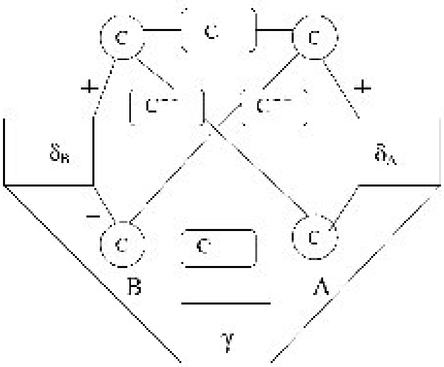

Fig. 1. Schematics of the apparatus: γ is the down converter, δ_A, δ_B are polarization phase vectors, C are the counters, $C^{\alpha\beta}$ are counters' correlators.

The symmetry operation in our study is this symmetry with respect to the rotation of the polarization. (It corresponds to the spherical symmetry enjoyed by the S state of the spins in standard discussions (e.g., [5,6]) of BIQV.) It is the breaking of this symmetry that will be shown to be associated with BIQV. Our approach is closely related to that of Ralph et al. [7]. However, we consider the whole range of the parameter and, since our emphasis is on symmetry rather than on the realization, we allow both polarizations to be included equally in the emerging squeezed light. The time duration for each operation and the strength of the nonlinearity of the medium are parametrized by a parameter γ. Thus γ is a measure of the entanglement of the beam under study which, in agreement with other studies [6], degrades with increasing γ and can be associated with the "fuzziness" parameter of reference [8]: only low values of γ allow BIQV. The conclusions and some discussion are contained in the last section. Appendix A contains the proof for the particularly simple dependence of the expectation values considered on the crucial symmetry breaking parameter—a dependence which underpins the symmetry aspect of our formulation.

2. Group theoretical formulation of the Bell problem

Crucial to any discussion of "nonlocality" in quantum mechanics (and Bell's inequality violation is no exception) is the notion of "entangled states", whose spatially separated parts, A and B, can provide counterintuitive correlations [1]. For the case at hand, this state is taken to be

$$|\Psi\rangle = \exp[i\gamma K_x]|0\rangle, \tag{1}$$

$$K_x = \frac{1}{2}\left[a_+^+ b_+^+ + a_- b_- + a_-^+ b_-^+ + a_+ b_+\right]. \tag{2}$$

Here the $+$ and $-$ subscripts denote the polarization relative to some chosen axis common to A and B (cf. Fig. 1). The state $|\Psi\rangle$ (Eq. (1)) will be recognized as the state generated by an appropriate down converter [3].

Our symmetry operator is taken to be

$$K_0 = \frac{i}{2}\left[a_+^+ a_- - a_-^+ a_+ + b_+^+ b_- - b_-^+ b_+\right]. \tag{3}$$

Since

$$[K_0, K_x] = 0 \tag{4}$$

and

$$K_0|0\rangle = 0, \tag{5}$$

we have

$$|\Psi'\rangle \equiv \exp(-i\delta K_0)|\Psi\rangle = |\Psi\rangle, \tag{6}$$

i.e., $\exp(-i\delta K_0)$ is a symmetry operator—the state is invariant under this operation.

Note that K_0 is made up, additively, of two parts

$$K_0^A = \frac{i}{2}\left[a_+^+ a_- - a_-^+ a_+\right],$$

$$K_0^B = \frac{i}{2}\left[b_+^+ b_- - b_-^+ b_+\right], \tag{7}$$

and each of these parts acts on a distinct local part. While

$$K_0 = K_0^A + K_0^B \tag{8}$$

is a symmetry operator for the system as a whole—the operator $\exp(i\delta_A K_0^A)$ is a symmetry *breaking* operator for our state, Eq. (1)—it breaks the K_0 symmetry. This operator does not commute with K_x.

The correlations under scrutiny are

$$C^{\alpha\beta}(\delta_A, \delta_B) = \left\langle a_\alpha^+(\delta_A) a_\alpha(\delta_A) b_\beta^+(\delta_B) b_\beta(\delta_B)\right\rangle. \tag{9}$$

Here α, β are polarization indices, and

$$a_\alpha^+(\delta_A) = \exp^{i\delta_A K_0^A} a_\alpha^+ \exp^{-i\delta_A K_0^A}, \quad \alpha = \pm,$$

$$b_\beta(\delta_B) = \exp^{i\delta_B K_0^B} b_\beta \exp^{-i\delta_B K_0^B}, \quad \beta = \pm, \tag{10}$$

and their Hermitian adjoints. The expectation value (e.g., in Eq. (9)) is with respect to $|\Psi\rangle$. It is shown in Appendix A that

$$C^{\alpha\beta}(\delta_A,\delta_B)=C^{\alpha\beta}(\delta_A-\delta_B,0). \tag{11}$$

We will, henceforth, denote $C^{\alpha\beta}(\delta_A-\delta_B,0)$ by $C^{\alpha\beta}(\delta)$ with $\delta=\delta_A-\delta_B$.

Next we evaluate $C^{\alpha\beta}(0)$. To this end we use the readily verifiable formulae

$$\begin{aligned} a_\alpha^+[\gamma] &\equiv \exp^{i\gamma K_x} a_\alpha^+ \exp^{-i\gamma K_x} \\ &= a_\alpha^+\cosh\gamma + ib_\alpha \sinh\gamma, \end{aligned} \tag{12}$$

$$b_\beta[\gamma]=b_\beta\cosh\gamma - ia_\beta^+\sinh\gamma, \tag{13}$$

and their Hermitian adjoints.

Substituting these into

$$C^{\alpha\beta}(0)=\langle\psi|a_\alpha^+a_\alpha b_\beta^+b_\beta|\Psi\rangle, \tag{14}$$

we get

$$C^{\alpha\beta}(0)=\sinh^4\gamma, \quad \alpha\neq\beta, \tag{15}$$

$$C^{\alpha\beta}(0)=\cosh^2\gamma\sinh^2\gamma+\sinh^4\gamma, \quad \alpha=\beta. \tag{16}$$

Noting that (cf. Appendix A)

$$C^{\alpha\beta}(0)=C^{\beta\alpha}(\delta), \tag{17}$$

we evaluate $C^{\alpha\beta}(\delta)$ to be

$$\begin{aligned} C^{-+}(\delta) &= C^{+-}(\delta) \\ &= C^{+-}(0)\cos^2\delta + C^{++}(0)\sin^2\delta, \\ C^{--}(\delta) &= C^{++}(\delta) \\ &= C^{++}(0)\cos^2\delta + C^{+-}(0)\sin^2\delta. \end{aligned} \tag{18}$$

Defining

$$C(\delta)=\sum_{\alpha,\beta}C^{\alpha\beta}(\delta)=C(0)=C^+ + C^-, \tag{19}$$

with

$$\begin{aligned} C^+ &\equiv C^{++}(0)+C^{--}(0) \\ &= 2\big(\cosh^2\gamma\sinh^2\gamma+\sinh^4\gamma\big), \\ C^- &\equiv C^{-+}(0)+C^{+-}(0)=2\sinh^4\gamma, \end{aligned} \tag{20}$$

we note that $C(\delta)$ is independent of δ.

Consider now

$$P^{\alpha\beta}(\delta)\equiv\frac{C^{\alpha\beta}(\delta)}{C(\delta)}=\frac{C^{\alpha\beta}(\delta)}{C(0)}, \tag{21}$$

where we have used the equality $C^{\alpha\beta}(0)=C^{\beta\alpha}(0)$, Eq. (17), and Eq. (19). $P^{\alpha\beta}(\delta)$ is the normalized probability for realizing the correlation $C^{\alpha\beta}(\delta)$.

We now define the expectation value [6], $E(\delta)$, by

$$E(\delta)=P^{++}(\delta)+P^{--}(\delta)-P^{+-}(\delta)-P^{-+}(\delta). \tag{22}$$

It gives the expectation value of obtaining equally polarized correlation. Utilizing our previous definitions we obtain for the expectation value:

$$\begin{aligned} E(\delta) &= \frac{C^+ - C^-}{C(0)}\cos(2\delta) \\ &= \frac{1}{1+2\tanh^2\gamma}\cos(2\delta). \end{aligned} \tag{23}$$

This expression reduces to

$$\begin{aligned} &E(\delta)\approx\cos(2\delta), \quad \gamma\ll 1, \\ &E(\delta)\approx\frac{1}{3}\cos(2\delta), \quad \gamma\gg 1. \end{aligned} \tag{24}$$

Thus, in agreement with [6,7], large values of γ reduce the allowed values of $E(\delta)$ and, as will be shown below, preclude BIQV. This can be expressed differently as follows: state (1) has an ill-defined particle number; larger values of γ signify that larger admixtures are involved, thereby degrading the entanglement in its simplest form (where only two particles are involved). This introduces "haziness" into the entanglement which can be parametrized by λ [8] ($\lambda=1$ signifies "sharp" observables [8]). An appropriate measure for this is the mean square number deviation:

$$\lambda=\frac{\langle n_\alpha^2\rangle-\langle n_\alpha\rangle^2}{\langle n_\alpha^2\rangle}=\frac{1}{1+\tanh^2\gamma}. \tag{25}$$

Note that this expression for the mean square deviation coincides with the thermal one if we identify $\tanh^2\equiv\exp(-1/T)$, T being a dimensionless temperature [9].

A direct substitution of $E(\delta)$, Eq. (23), into the expression for Bell's inequality [6,7],

$$\begin{aligned} &\big|E(\delta_A,\delta_B)+E\big(\delta_A,\delta'_B\big)+E\big(\delta'_A,\delta_B\big) \\ &\quad - E\big(\delta'_A,\delta'_B\big)\big|\leqslant 2, \end{aligned} \tag{26}$$

yields

$$\begin{aligned} &\frac{1}{1+2\tanh^2\gamma}\Big[\big|\cos 2(\delta_A-\delta_B)+\cos 2\big(\delta_A-\delta'_B\big) \\ &\quad +\cos 2\big(\delta'_A-\delta_B\big)-\cos 2\big(\delta'_A-\delta'_B\big)\big|\Big]\leqslant 2. \end{aligned} \tag{27}$$

The quantity in the square brackets is maximized at

$$\delta_A = 0, \qquad \delta_B = \frac{\pi}{8},$$
$$\delta'_A = \frac{\pi}{4}, \qquad \delta'_B = -\frac{\pi}{8}, \tag{28}$$

for which the square bracket term is $2\sqrt{2}$. Thus, for γ such that $1 + \tanh^2\gamma < \sqrt{2}$, inequality (26) can be violated. The limitation to small values of γ can also be interpreted to mean that only when $|\Psi\rangle$ involves a few photons is the violation possible—or, put in another way, only when the granulated facet of light dominates, the violation is attainable. Increasing γ degrades the entanglement in that it allows admixtures of various particle number states; this is expressed in by "haziness" parameter, Eq. (25). The experimental realization is seen to be difficult, as the expression for the probability of correlated photons involves a normalised expression, i.e., we require reference readings. These are given by Eqs. (15), (16), which give vanishing readings for $\gamma \to 0$. Thus while for γ small but finite observation of BIQV is possible, this becomes increasingly more difficult as $\gamma \to 0$.

3. Concluding remarks

The violation of Bell's inequality observed within quantum optics was discussed in terms of symmetry breaking. It was shown that when the wavefunction as a whole possesses a global symmetry—the rotation of the polarization in the case studied—the breaking of the symmetry retains the strong quantum correlation that allows the violation of Bell's inequality. This presentation relates directly to the (non)commutativity of operators: the complete symmetry operation commutes with all the relevant quantities—the breaking of the symmetry is connected to noncommutativity of local operators whose expectation values allow the violation of the inequality.

The parameter γ is, in effect, a measure for an effective temperature ($\tanh^2\gamma = \exp(-1/T)$) of each separate beam when its reduced density matrix is considered [9,10] (cf. Eq. (23) above). Thus the degrading of the entanglement of the state with increasing γ—in the sense of its increasing reluctance to allow the violation of Bell's inequality—can also be viewed as due to the increase in the effective temperature of each beam.

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Appendix A. Proof of $C^{\alpha\beta}(\delta_A, \delta_B) = C^{\alpha\beta}(\delta_A - \delta_B, 0)$

By definition

$$C^{\alpha\beta}(\delta_A, \delta_B) = \langle\Psi| \exp^{i\delta_A K_0^A + i\delta_B K_0^B} a_\alpha^+ a_\alpha b_\beta^+ b_\beta \times \exp^{-i\delta_A K_0^A - i\delta_B K_0^B} |\Psi\rangle.$$

Defining

$$\delta = \delta_A - \delta_B \quad \text{and} \quad \Delta = \delta_A + \delta_B$$

allows the rewriting of the correlation function as

$$C^{\alpha\beta}(\delta_A, \delta_B) = \langle\Psi| \exp^{i(\delta/2)(K_0^A - K_0^B)} a_\alpha^+ a_\alpha b_\beta^+ b_\beta \times \exp^{-i(\delta/2)(K_0^A - K_0^B)} |\Psi\rangle.$$

We have used the attribute that $K_0 = K_0^A + K_0^B$ is a symmetry operation, i.e., it commutes with K_x and $K_0|0\rangle = 0$, and that it commutes with K_0^A and K_0^B. By the same reasoning we add/subtract $i(\delta/2)K_0$ to remove the explicit appearance of K_0^B to get

$$C^{\alpha\beta}(\delta_A, \delta_B) = C^{\alpha\beta}(\delta, 0) \equiv C^{\alpha\beta}(\delta),$$

and that it is symmetric in δ:

$$C^{\alpha\beta}(\delta) = C^{\alpha\beta}(-\delta) = C^{\beta\alpha}(-\delta).$$

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