# Uncertainty Principles on Nilpotent Lie Groups

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A thesis submitted to Indian Statistical Institute
in partial fulfilment of requirements
for the degree of Doctor of Philosophy

**Bangalore**
**July 2005**

ACKNOWLEDGEMENTS

A long exhausting journey becomes easier when we travel with acquaintances. I owe to all of them who have provided support, encouragement and assistance along the way in writing this thesis. It is a great pleasure for me to get an opportunity to express my gratitude to all of them.

My deepest gratitude is to my supervisor Professor S. Thangavelu for his continuous guidance and stimulating discussions on various aspects of Harmonic analysis.

I am grateful to the Stat-Math faculty at the Indian Statistical Institute, especially to Professor S.C. Bagchi, Professor G. Misra and Professor A. Sitaram for their excellent teaching, guidance and endless support. Their ideas and concepts have had a remarkable influence on my entire research career.

No words can express my indebtedness to Dr. Rudra P. Sarkar for the extensive discussions that I had with him, and for his brotherly advice that helped me to stay in the right track all along.

A great amount of gratitude is deserved by Dr. E. K. Narayanan for reading my thesis with infinite patience and providing suggestions for its improvement.

I would like to thank all of my friends for their inspiration from the beginning until this lovely moment. My special thanks go to Mr. Binod Kumar Sahoo who not only shared many of his experiences and thoughts with me during the last four years but also showed himself to be a very kind person, and a most helpful and trusted friend. I will miss you all.

I also extend my sincere thanks to Professor N.S.N. Sastry, Head of the

Stat-Math Unit, ISI Bangalore Center, for providing me everything, I have asked for throughout the development of this research.

I owe thanks beyond measure to all other members of the ISI Bangalore Center, for their cheerful assistance. However, Ms. Asha Lata and Ms. Mohana Devi, who extended their helping hands throughout, deserve a more personal note of gratitude. I shall cherish their friendship forever.

I wish to thank the Indian Statistical Institute for providing me financial assistances and research facilities.

This acknowledgements is surely incomplete without expressing my indebtedness to my parents and my brothers for their unconditional love and support. Without their encouragement and understanding it would have been impossible for me to finish this work.

CONTENTS

# 0. PREFACE

Roughly speaking the *Uncertainty Principle* says that " A nonzero function $f$ and its Fourier transform $\hat{f}$ cannot be sharply localized simultaneously". There are several ways of measuring localization of a function and depending on it one can formulate different versions of uncertainty principle. A classical theorem of Hardy [14] proved way back in 1933 states that $f$ and $\hat{f}$ both cannot have arbitrary Gaussian decay. Here the localization of $f$ and $\hat{f}$ are measured in the sense of rapid decay at infinity. More precisely, for a measurable function $f$ on $\mathbb{R}$ if

$$|f(x)| \leq Ce^{-ax^2}, \ |\hat{f}(\xi)| \leq Ce^{-b\xi^2}$$

for some $a, b > 0$, then $f = 0$ for $ab > 1/4$ and $f(x) = Ce^{-ax^2}$ for $ab = 1/4$. Also there are various ways of measuring decay of $f$ and $\hat{f}$. Cowling and Price [6], Beurling [17] measured the decay in terms of integral estimates of $f$ and $\hat{f}$. Recently Bonami et al [5] generalized the result of Beurling and characterized Hermite functions. The theorems of Hardy and Cowling–Price follow from the stronger result of Bonami et al. Narayanan and Ray observed in [23] that Hardy's theorem can be viewed as a characterization of the Heat kernel associated with the Laplacian on $\mathbb{R}$.

Analogues of Hardy and Cowling–Price theorems for various Lie groups have received considerable attention during the last decade. Heisenberg group is the most well known example from realm of nilpotent Lie groups. Uncertainty principles on Heisenberg group was first considered by Sitaram, Sundari, Thangavelu in [30]. After that Bagchi and Ray [2] proved Cowling–Price theorem for Heisenberg group. In the last few years Thangavelu proved analogues of Paley–Wiener theorem and Hardy's theorem for Heisenberg groups, see the monograph [34] and the references there. He formulated the Hardy's theorem in terms of the Heat kernel associated with the sublaplacian on the Heisenberg group (see [32]). We call it the Heat kernel version of Hardy's theorem.

An analogue of Hardy's theorem for all simply connected nilpotent Lie groups was proved by Kaniuth and Kumar [19]. A slightly different version of Hardy's theorem was also proved in [1] for connected simply connected step two nilpotent Lie groups. Cowling–Price theorem has been considered by Ray [28] for connected simply connected step two nilpotent Lie groups. All these authors measure the decay of the group Fourier transform $\hat{f}(\lambda)$ in terms of the Hilbert–Schmidt norm and the results are essentially for the central variable.

However the research in this direction is still incomplete since in most of the results proved in the above mentioned papers the case $ab = 1/4$ has been left open. Once we have the result for $ab = 1/4$ we can always deduce the case $ab > 1/4$. Thangavelu conjectured in his book [34] that the heat kernel version of Hardy's theorem is true for all stratified groups.

The main purpose of this thesis is to prove the cases $ab = 1/4$ and get complete analogues theorem of Hardy, Cowling–Price and Beurling for connected simply connected step two groups. The conjecture of Thangavelu will be proved for all step two stratified groups. We prove a different version of Cowling–Price theorem for Euclidean Fourier transform and as an application of it, formulate and prove a version of Cowling–Price theorem for general nilpotent Lie groups.

The structure of the thesis is as follows:

In **Chapter 1** we introduce nonisotropic Hesienberg group $H_d^n$ and describe its representations. We prove vector valued Beurling's theorem and as an application of it we get theorem of Hardy and a version of Cowling–Price for $H_d^n$. A complete analogue of Cowling–Price and Morgan's theorem has been obtained after proving a version of Beurling's theorem for $H_d^n$. We define Fourier–Weyl transform of a measurable function on $H_d^n$ and formulate a version of Beurling's theorem. Finally we prove heat kernel versions of Hardy and Cowling–Price theorems.

In **Chapter 2** we describe the irreducible unitary representations, the Plancherel formula and other relevant aspects of step two connected simply connected nilpotent Lie groups. We prove the heat kernel versions of Hardy and Cowling–Price theorems for all step two stratified groups. Also an alternative condition on $f$ is obtained to formulate and prove heat kernel versions of Hardy and Cowling–Price theorems for all connected simply connected step two groups. Analogue of Beurling's theorem for all step two groups is also proved. All other results proved in Chapter 1 has been extended for step two groups.

In **Chapter 3** we obtain a new version of Cowling–Price theorem for Euclidean Fourier transform where the decay has been measured only on $\hat{f}$ and its derivatives. We also give a comparative study of Cowling–Price theorem and its new version. We use this new version to obtain an uncertainty principle for operators and Cowling–Price theorem for Laguerre expansions of polyradial functions. Finally we get an uncertainty principle for all connected simply connected nilpotent Lie groups.

The thesis is based on [25, 26] and [24]. The paper [25] will appear in J. Austral. Math. Soc. (series A).

# 1. UNCERTAINTY PRINCIPLES ON HEISENBERG GROUPS

Our aim in this chapter is to prove some uncertainty principles for non–isotropic Heisenberg groups. Some of these results are known in the case of isotropic Heisenberg groups but the proofs do not extend to the non–isotropic case. We provide different proofs which work for both cases.

## *1.1 Heisenberg groups and their representations*

Given $d = (d_1, d_2, \cdots, d_n), d_j > 0$ the *non-isotropic Heisenberg group* $H_d^n$ is $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,t)(w,s) = (z + w, t + s + \frac{1}{2}\sum_{j=1}^{n} d_j \mathrm{Im}(z_j \bar{w}_j)).$$

For each $\lambda \in \mathbb{R} \setminus \{0\}$ there exists an irreducible unitary representation $\pi_\lambda$ realized on $L^2(\mathbb{R}^n)$ given by

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(\sum\limits_{j=1}^{n} d_j(x_j\xi_j + \frac{1}{2}x_j y_j))} \phi(\xi + y),$$

where $\phi \in L^2(\mathbb{R}^n)$ and $z = x + iy$. These are all the infinite dimensional irreducible unitary representations of $H_d^n$ up to unitary equivalence. For $f \in L^1(H_d^n)$, its group Fourier transform $\hat{f}(\lambda)$ is defined by

$$\hat{f}(\lambda) = \int_{H_d^n} f(z,t)\pi_\lambda(z,t)\, dz\, dt. \tag{1.1.1}$$

We define $\pi_\lambda(z) = \pi_\lambda(z, 0)$ so that $\pi_\lambda(z,t) = e^{i\lambda t}\pi_\lambda(z, 0)$. For $f \in L^1(\mathbb{C}^n)$, we define the bounded operator $T_\lambda(f)$ on $L^2(\mathbb{R}^n)$ by

$$T_\lambda(f)\phi = \int_{\mathbb{C}^n} f(z)\pi_\lambda(z)\phi\, dz\ . \tag{1.1.2}$$

It is clear that $\|T_\lambda(f)\| \leq \|f\|_1$ and for $f \in L^1(\mathbb{C}^n)\bigcap L^2(\mathbb{C}^n)$, it can be shown that $T_\lambda(f)$ is a Hilbert-Schmidt operator and we have the Plancherel theorem

$$\|T_\lambda(f)\|_{\text{HS}}^2 = (2\pi)^n|\lambda|^{-n}\prod_{j=1}^n d_j^{-1}\int_{\mathbb{C}^n}|f(z)|^2\ dz. \tag{1.1.3}$$

Thus $T_\lambda$ is an isometric isomorphism between $L^2(\mathbb{C}^n)$ and $\mathcal{S}_2$, the Hilbert space of all Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. For $f \in L^1(H_d^n)$, let

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z,t)\ dt$$

be the inverse Fourier transform of $f$ in the $t$–variable. Then from the definition of $\hat{f}(\lambda)$, it follows that $\hat{f}(\lambda) = T_\lambda(f^\lambda)$.

If $d_j = 1$ for all $j$ then $H_d^n$ is denoted by $H^n$, called the *Heisenberg group*. For $H^n$ the representations corresponding to $\pi_\lambda$ will be denoted by $\rho_\lambda$. Thus $\rho_\lambda(z,t)\phi(\xi) = e^{i\lambda t}e^{i\lambda(x\cdot\xi+\frac{1}{2}x\cdot y)}\phi(\xi+y)$, for all $\phi \in L^2(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}\setminus\{0\}$. In this case we will denote $T_\lambda(f)$ by $W_\lambda(f)$ which will be called the Weyl transform of $f$. For $\lambda = 1$, we define $W(z) = \rho_1(z)$.

For $x \in \mathbb{R}$ and $k \in \mathbb{N}$ the polynomial $H_k(x)$ of degree $k$ is defined by the formula

$$H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}(e^{-x^2}). \tag{1.1.4}$$

We define the Hermite function $h_k(x)$ by

$$h_k(x) = (2^k k!\sqrt{\pi})^{-1/2}H_k(x)e^{-\frac{x^2}{2}}.$$

For $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{N}^n$, the normalized Hermite function $\Phi_\mu(x)$ on $\mathbb{R}^n$ is defined by

$$\Phi_\mu(x) = h_{\mu_1}(x_1)\cdots h_{\mu_n}(x_n). \tag{1.1.5}$$

Hermite functions are eigenfunctions of the Hermite operator $H = -\triangle + |x|^2$ and they form an orthonormal basis for $L^2(\mathbb{R}^n)$. For $\mu,\nu \in \mathbb{N}^n$, the special Hermite function $\Phi_{\mu\nu}$ is defined by

$$\Phi_{\mu\nu}(z) = (2\pi)^{-\frac{n}{2}}\left(W(z)\Phi_\mu, \Phi_\nu\right). \tag{1.1.6}$$

These functions form an orthonormal basis for $L^2(\mathbb{C}^n)$ and they are expressible in terms of Laguerre functions. For our purposes, we only require the formula

$$\Phi_{\mu,0}(z) = (2\pi)^{-\frac{n}{2}} \left(\frac{1}{\mu!}\right)^{\frac{1}{2}} \left(\frac{i\bar{z}}{\sqrt{2}}\right)^{\mu} e^{-\frac{1}{4}|z|^2}. \tag{1.1.7}$$

For a detailed account of Hermite and special Hermite functions we refer to [36].

A basis for the Lie algebra of $H_d^n$ is given by the left invariant vector fields

$$X_j(d) = \frac{\partial}{\partial x_j} + \frac{1}{2}d_j y_j \frac{\partial}{\partial t}, \quad Y_j(d) = \frac{\partial}{\partial y_j} - \frac{1}{2}d_j x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n.$$

and $T$. We define the sublaplacian $\mathcal{L}_d$ by

$$\mathcal{L}_d = -\sum_{j=1}^{n}(X_j(d)^2 + Y_j(d)^2). \tag{1.1.8}$$

The sublaplacian is a subelliptic operator which generates a heat diffusion semigroup. Let $q_{a,d}$ be the heat kernel corresponding to this sublaplacian which is given by

$$q_{a,d}(z,t) = \int_{-\infty}^{\infty} e^{i\lambda t} q_{a,d}^{\lambda}(z,t)\ dt, \tag{1.1.9}$$

where

$$q_{a,d}^{\lambda}(z) = C_n \prod_{j=1}^{n} \left(\frac{d_j\lambda}{\sinh d_j\lambda a}\right) e^{-\frac{1}{4}d_j\lambda(\coth d_j\lambda a)|z_j|^2}. \tag{1.1.10}$$

Then using the formula ( 1.1.10 ), it can be proved as in the case of $H^n$ (see [34]) that it satisfies the estimate

$$|q_{a,d}(z,t)| \leq Ce^{-\frac{A}{a}(|z|^2+|t|)}, \tag{1.1.11}$$

for some $C, A > 0$. We define

$$H(\lambda d) = \sum_{j=1}^{n} \left(-\frac{\partial^2}{\partial \xi_j^2} + \lambda^2 d_j^2 \xi_j^2\right). \tag{1.1.12}$$

Using the explicit formula for the representations $\pi_\lambda$ we can show that

$$\pi_\lambda(\mathcal{L}_d) = H(\lambda d),\ \pi_\lambda(q_{a,d}) = e^{-aH(\lambda d)}. \tag{1.1.13}$$

Given $r = (r_1, r_2, \cdots, r_n),\ r_j > 0$, we define $U(r) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ by

$$U(r)\phi(\xi) = \prod_{j=1}^{n} r^{\frac{1}{4}}\phi(\sqrt{r}_1\xi_1, \sqrt{r}_2\xi_2 \cdots, \sqrt{r}_n\xi_n).$$

Then $U(r)$ is a unitary operator on $L^2(\mathbb{R}^n)$ and

$$H(\lambda d)U(|\lambda|d)\Phi_\mu = \left(\sum_{j=1}^{n}(|\lambda|(2\mu_j + 1)d_j)\right)U(|\lambda|d)\Phi_\mu.$$

If $d_j = 1$ for all $j$, $H(\lambda d)$ reduces to the scaled Hermite operator

$$H(\lambda) = -\triangle + \lambda^2|x|^2.$$

## 1.2 Uncertainty Principles on $H_d^n$

Roughly speaking the Uncertainty Principle says that "A non zero function $f$ and its Fourier transform $\hat{f}$ cannot be localized simultaneously." The simplest example of this phenomenon is the Paley–Wiener theorem: the Euclidean Fourier transform of a smooth compactly supported function on $\mathbb{R}^n$ can be extended as an entire function on $\mathbb{C}^n$ and hence cannot be compactly supported. Consider a function $f$ such that for some $a, b > 0$

$$f(x) = O(e^{-a|x|^2}) \text{ and } \hat{f}(\xi) = O(e^{-b|\xi|^2})$$

where $\hat{f}$ is the Euclidean Fourier transform of $f$ defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

For $\delta > 0$, let $f_\delta(x) = f(\delta x)$ so that $f_\delta(x) = O(e^{-a\delta^2|x|^2})$ and $\widehat{f_\delta}(\xi) = O(e^{-\frac{b}{\delta^2}|x|^2})$. As $\delta$ increases, $f$ has faster decay. On the other hand decay of $\hat{f}$ becomes slower. So $f$ and $\hat{f}$ both cannot have arbitrary Gaussian decay, as was proved by Hardy [14] in 1933 for $n = 1$.

**Theorem 1.2.1. (Hardy)** *Let $f$ be a measurable function on $\mathbb{R}^n$ such that*

$$|f(x)| \leq Ce^{-a|x|^2}, \quad |\hat{f}(x)| \leq Ce^{-b|x|^2}.$$

*If $ab > 1/4$ then $f = 0$ almost everywhere and $f(x) = Ce^{-a|x|^2}$ for $ab = 1/4$.*

Since $ab > 1/4$ implies $f = 0$ the result of Hardy is an example of uncertainty principle for the Fourier transform. The case $ab = 1/4$ is considered as a characterisation of the Gaussian. The Hermite functions $\Phi_\mu(x)$ satisfy the conditions of Hardy's theorem for any $a = b < 1/2$ and hence in the case $ab < 1/4$ there are infinitely many linearly independent functions satisfying the hypotheses of the theorem.

In 1983, Cowling and Price [6] replaced the $L^\infty$ estimates on $f$ and $\hat{f}$ by $L^p$ estimates and obtained a generalization of Hardy's theorem.

**Theorem 1.2.2. (Cowling–Price)** *For $1 \leq p, q \leq \infty$ let $f\phi_a^{-1} \in L^p(\mathbb{R}^n)$ and $\hat{f}\phi_b^{-1} \in L^q(\mathbb{R}^n)$, where $\phi_a(x) = e^{-a|x|^2}$. Then $f = 0$ whenever $ab > 1/4$.*

Note that the case $p = q = \infty$ in the above theorem is Hardy's theorem with $ab > 1/4$. The above result is true even if $ab = 1/4$ under the added assumption that $\min(p, q) < \infty$. In fact we have the following results which are stronger than theorems of Hardy and Cowling–Price.

**Theorem 1.2.3.** *Suppose $f$ is a measurable function on $\mathbb{R}^n$ such that it satisfies the estimates*

$$|f(x)| \leq C(1 + |x|)^m e^{-a|x|^2} \text{ and } |\hat{f}(\xi)| \leq C(1 + |\xi|)^m e^{-b|\xi|^2}.$$

*Then for $ab > 1/4$, $f = 0$ and whenever $ab = 1/4$, $f(x) = P(x)e^{-a|x|^2}$, where $P$ is a polynomial with $\deg\ P \leq m$.*

**Theorem 1.2.4.** *Let $N \geq 0$ and $1 \leq p, q \leq \infty$. Assume $f \in L^2(\mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}^n} \left( \frac{f(x)e^{a|x|^2}}{(1 + |x|)^N} \right)^p dx < \infty \text{ and } \int_{\mathbb{R}^n} \left( \frac{\hat{f}(\xi)e^{a|\xi|^2}}{(1 + |\xi|)^N} \right)^q d\xi < \infty.$$

*If $ab > 1/4$ then $f = 0$ and for $ab = 1/4$ $f(x) = P(x)e^{-a|x|^2}$, where $P$ is a polynomial with* $\deg\ P < \inf\{N - \frac{n}{p}, N - \frac{n}{q}\}$.

The above results follow from the following stronger result known as Beurling's theorem.

**Theorem 1.2.5.** *Let $f \in L^2(\mathbb{R}^n)$ be such that for some $N \geq 0$ it satisfies the condition*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\hat{f}(y)|e^{|x||y|}}{(1+|x|+|y|)^N}\ dx\ dy < \infty.$$

*Then $f(x) = P(x)e^{-\alpha|x|^2}$ where $P$ is a polynomial and $\alpha > 0$.*

This result is an immediate corollary of the following theorem due to Bonami et al [5].

**Theorem 1.2.6.** *Let $f \in L^2(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\hat{f}(y)|e^{|\langle x,y\rangle|}}{(1+|x|+|y|)^N} dx\ dy < \infty$$

*for some $N \geq 0$. Then $f = 0$ whenever $N \leq n$. If $N > n$, then the above holds if and only if $f$ can be written as*

$$f(x) = P(x)e^{-\frac{1}{2}\langle x,Ax\rangle}$$

*where $A$ is a real, positive definite, symmetric matrix and $P$ is a polynomial with* $\deg P < \frac{(N-n)}{2}$.

Also the inequality case of the following theorem known as Morgan's theorem discussed in [2] can be proved using Theorem 1.2.5.

**Theorem 1.2.7.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that*

*(i)* $|f(x)| \leq Ce^{-a|x|^p}$

*(ii)* $|\hat{f}(x)| \leq Ce^{-b|x|^q}$

*where $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $(ap)^{1/p}\,(bq)^{1/q} \geq 1$. Then $f = 0$ almost everywhere unless $p = q = 2$ and $ab = 1/4$, in which case $f(x) = Ce^{-a|x|^2}$.*

### 1.2.1 Vector valued uncertainty principles and their applications

In this section we formulate and prove analogues of the above theorems for $H^n_d$. In order to do that we need vector valued versions of the above theorems. We first state and prove a vector valued version of Theorem 1.2.6. In what follows $\mathcal{H}$ denotes a separable Hilbert space and $L^2(\mathbb{R}^n, \mathcal{H})$ stands for all $\mathcal{H}$–valued function $f$ on $\mathbb{R}^n$ such that $\|f(x)\|$ is square integrable on $\mathbb{R}^n$.

**Theorem 1.2.8.** *Suppose $f \in L^2(\mathbb{R}^n, \mathcal{H})$ be such that for some $N \geq 0$ it satisfies*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\|f(x)\|\|\hat{f}(y)\|e^{|(x,y)|}}{(1+|x|+|y|)^N} dx\ dy < \infty.$$

*If $N \leq n$, then $f = 0$. If $N > n$, then the above holds if and only if $f$ can be written as*

$$f(x) = P(x)e^{-\frac{1}{2}\langle x, Ax\rangle}$$

*where $A$ is a real, positive definite, symmetric matrix and $P(x) = \sum_{|\alpha|\leq m} x^\alpha \psi_\alpha$, $\psi_\alpha \in \mathcal{H}$ and $m < \frac{(N-n)}{2}$.*

*Proof.* Take $\phi \in \mathcal{H}$ and consider the function $F_\phi(x) = \langle \phi, f(x)\rangle$. Since $\widehat{F_\phi}(\xi) = \langle \phi, \hat{f}(\xi)\rangle$, using Cauchy–Schwarz and the hypothesis of the theorem we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|F_\phi(x)||\widehat{F_\phi}(\xi)|e^{|(x,\xi)|}}{(1+|x|+|\xi|)^N}\ dx\ d\xi < \infty.$$

Applying Theorem 1.2.6 to the function $F_\phi$ we have $F_\phi(x) = e^{-\langle A(\phi)x,x\rangle}P_\phi(x)$, where $A(\phi)$ is a real, positive definite, symmetric matrix and $P_\phi(x)$ is a polynomial of degree $< \frac{(N-n)}{2}$. We will write $P_\phi(x) = \sum_{|\alpha|\leq m} C_\alpha(\phi)x^\alpha$. We want to show that $A(\phi) = A$ is independent of $\phi$. To show this we need to prove the following:

If for each $x \in \mathbb{R}$

$$\phi \to e^{-(a(\phi)x^2+b(\phi)x)}P_\phi(x), \tag{1.2.1}$$

where $P_\phi(x)$ is a polynomial in $x$, is a bounded linear functional on $\mathcal{H}$ then $a(\phi),\ b(\phi)$ are independent of $\phi$.

From the linearity of the above map ( 1.2.1 ) we have

$$\begin{aligned} & e^{-(a(\phi)x^2+b(\phi)x)}P_\phi(x) + e^{-(a(\psi)x^2+b(\psi)x)}P_\psi(x) \\ = \ & e^{-(a(\phi+\psi)x^2+b(\phi+\psi)x)}P_{\phi+\psi}(x). \end{aligned} \tag{1.2.2}$$

Without loss of generality we assume $a(\phi) < a(\psi)$.

*Case*$(i)$: If $a(\phi) < a(\psi) < a(\phi+\psi)$, we have from the above equation

$$\begin{aligned} & e^{((a(\phi+\psi)-a(\phi))x^2-b(\phi)x)}P_\phi(x) + e^{((a(\phi+\psi)-a(\psi))x^2-b(\psi)x)}P_\psi(x) \\ = \ & e^{-b(\phi+\psi)x}P_{\phi+\psi}(x). \end{aligned} \tag{1.2.3}$$

This shows that left hand side of the above equation ( 1.2.3 ) grows faster than the right hand side of the above equation and hence $a(\phi) = a(\psi) = a(\phi+\psi)$.

*Case*$(ii)$: If $a(\phi+\psi) < a(\phi) < a(\psi)$, the left hand side of ( 1.2.3 ) decays faster than the right hand side of ( 1.2.3 ) and hence $a(\phi) = a(\psi) = a(\phi+\psi)$.

*Case*$(iii)$: If $a(\phi) < a(\phi+\psi) < a(\psi)$, we rewrite the equation ( 1.2.2 ) as

$$\begin{aligned} & e^{((a(\psi)-a(\phi))x^2-b(\phi)x)} - e^{((a(\psi)-a(\phi+\psi))x^2-b(\phi+\psi)x)}P_{\phi+\psi}(x) \\ = \ & -e^{-b(\psi)x}P_\psi(x). \end{aligned}$$

Arguing as before we conclude $a(\phi) = a(\psi) = a(\phi+\psi)$.

Similar argument will show $b(\phi) = b(\psi)$ for all $\phi,\psi \in \mathcal{H}$. Let us prove that $A(\phi)$ is independent of $\phi$ using the above result. Since $A(\phi)$ is a symmetric matrix,

$$\begin{aligned} \langle A(\phi)x, x\rangle &= \sum_{1\le j,k\le n} C_{jk}(\phi)x_jx_k \\ &= \sum_{j,k\ne 1} C_{jk}(\phi)x_jx_k + 2\left(\sum_{j\ne 1} C_{j1}(\phi)x_j\right)x_1 + C_{11}(\phi)x_1^2. \end{aligned}$$

We recall that for each $x \in \mathbb{R}^n$, $\phi \mapsto e^{-\langle A(\phi)x,x\rangle}P_\phi(x)$ is a bounded linear functional on $\mathcal{H}$. If we vary the variable $x_1$ keeping other $(n-1)$ variables fixed and use the above result we get

$$C_{11}(\phi)x_1^2 + \sum_{j=1}^{n} C_{1j}(\phi)x_j = C_{11}(\psi)x_1^2 + \sum_{j=1}^{n} C_{1j}(\psi)x_j$$

for all $\phi, \psi \in \mathcal{H}$. This is true for all $(x_1, \cdots, x_n) \in \mathbb{R}^n$ and hence $C_{1j}(\phi) = C_{1j}(\psi)$ for all $\phi, \psi \in \mathcal{H}$ and for all $1 \leq j \leq n$ . Similarly if we vary the variable $x_j$ keeping other variables fixed we will get $C_{kj}(\phi) = C_{kj}(\psi)$ for all $\phi, \psi \in \mathcal{H}$ and for all $1 \leq k \leq n$. Finally we conclude $A(\phi) = A$ is independent of $\phi$. Therefore, $\phi \mapsto P_\phi(x)$ is also a bounded linear functional on $\mathcal{H}$ for all $x \in \mathbb{R}^n$. We write $P_\phi(x) = \sum_{|\alpha| \leq m} C_\alpha(\phi)x^\alpha$. We claim that $\phi \mapsto C_\alpha(\phi)$ is a bounded linear functional on $\mathcal{H}$ for each $\alpha$. Let us consider the case $n = 1$. In this case $P_\phi(x) = \sum_{k=0}^{m} C_k(\phi)x^k$ for all $x \in \mathbb{R}$. Choose $x_i \in \mathbb{R}$ such that $x_i \neq x_j$, for all $0 \leq i, j \leq m$. We consider a system of linear equations given by:

$$\begin{pmatrix} 1 & x_0 & \cdots & x_0^m \\ 1 & x_1 & \cdots & x_1^m \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & \cdots & x_m^m \end{pmatrix} \begin{pmatrix} C_0(\phi) \\ C_1(\phi) \\ \vdots \\ C_m(\phi) \end{pmatrix} = \begin{pmatrix} P_\phi(x_0) \\ P_\phi(x_1) \\ \vdots \\ P_\phi(x_m) \end{pmatrix}.$$

Since $x_i \neq x_j$ for all $0 \leq i, j \leq m$ the determinant of the $(m+1) \times (m+1)$ Vandermonde matrix is nonzero. Therefore, for each $j$, $C_j(\phi)$ is a linear combination of $\{P_\phi(x_0), \cdots, P_\phi(x_m)\}$. It follows that $\phi \mapsto C_j(\phi)$ is a bounded linear functional on $\mathcal{H}$. Now consider the case $n > 1$. Suppose that our claim is true for all $n' \leq (n-1)$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ write $x = (x_1, \tilde{x})$, $\alpha = (\alpha_1, \tilde{\alpha})$, where $\tilde{x} = (x_2, \cdots, x_n)$ and $\tilde{\alpha} = (\alpha_2, \cdots, \alpha_n)$. Therefore we can write

$$P_\phi(x) = \sum_{\alpha_1=0}^{m} \left( \sum_{|\tilde{\alpha}| \leq m} C_{\alpha_1, \alpha'}(\phi)\tilde{x}^{\tilde{\alpha}} \right) x_1^{\alpha_1}.$$

From the case $n = 1$, we get for each $\tilde{x} \in \mathbb{R}^{n-1}$, $\phi \mapsto \sum_{|\tilde{\alpha}| \leq m} C_{\alpha_1, \alpha'}(\phi)\tilde{x}^{\tilde{\alpha}}$ is a bounded linear functional on $\mathcal{H}$ for each $(\alpha_1, \tilde{\alpha})$. Now using the induction hypothesis it follows that for each $\alpha = (\alpha_1, \tilde{\alpha})$, $\phi \mapsto C_\alpha(\phi)$ is a bounded linear functional on $\mathcal{H}$. Therefore, for each $\alpha$ there exists $\psi_\alpha \in \mathcal{H}$ such that $C_\alpha(\phi) = \langle \phi, \psi_\alpha \rangle$ and hence the theorem is proved. □

We state below a vector valued version of Theorem 1.2.5 which is a consequence of Theorem 1.2.8.

**Theorem 1.2.9.** *Let $f \in L^2(\mathbb{R}^n, \mathcal{H})$ be such that for some $N \geq 0$*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\|f(x)\| \|f(y)\| e^{|x||y|}}{(1+|x|+|y|)^N} < \infty.$$

*Then $f(x) = e^{-a|x|^2} P(x)$ where $P(x) = \sum_{|\alpha| \leq m} x^\alpha \psi_\alpha$, $\psi_\alpha \in \mathcal{H}$,$a > 0$ and $m < \frac{(N-n)}{2}$ .*

In view of Theorem 1.2.9, all the theorems 1.2.3, 1.2.4 and 1.2.7 remain true for vector valued functions. If $f \in L^2(H^n_d)$ then the function $F$ defined by $F(t) = f(\cdot, t)$ is an $L^2(C^n)$ valued measurable function of $t$. Using the Plancherel formula and Theorem 1.2.8 we have the following theorem:

**Theorem 1.2.10.** *Suppose $f \in L^2(H^n_d)$ and for some $N \geq 0$, it satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|f(\cdot, t)\|_2 |\lambda|^{n/2} \|\hat{f}(\lambda)\|_{HS} e^{|t\lambda|}}{(1+|\lambda|+|t|)^N} \, dt \, d\lambda < \infty.$$

*Then $f(z,t) = e^{-at^2} P(z,t)$ for some $a > 0$ and $P(z,t) = \left( \sum_{j=0}^{m} \psi_j(z) t^j \right)$, where $\psi_j \in L^2(\mathbb{C}^n)$ and $m < \frac{N-1}{2}$.*

An analogue of Cowling–Price and Theorem 1.2.7 for $H^n$ has been proved in [2] but the equality case has not been treated there. We prove equality cases of theorems of Hardy and Cowling–Price. Immediate consequences of the above theorem are the following three corollaries.

**Corollary 1.2.11.** *Let $f$ be a measurable function on $H^n_d$ such that it satisfies*

*(i)* $|f(z,t)| \leq g(z)(1+|t|)^m e^{-at^2}$, *where* $g \in L^2(\mathbb{C}^n)$,

*(ii)* $|\lambda|^{n/2}\|\hat{f}(\lambda)\|_{HS} \le C(1+|\lambda|)^m e^{-b\lambda^2}$.

*Then* $f = 0$ *for* $ab > 1/4$ *and if* $ab = 1/4$ *then* $f(z,t) = e^{-at^2}P(z,t)$, *where* $P(z,t) = \sum_{j=0}^{m} t^j \psi_j(z)$, $\psi_j \in L^2(\mathbb{C}^n)$.

**Corollary 1.2.12.** *Suppose* $f \in L^2(H_d^n)$ *and it satisfies the estimates*

*(i)* $|f(z,t)| \le g(z)e^{-a|t|^p}$, *where* $g \in L^2(\mathbb{C}^n)$,

*(ii)* $|\lambda|^{n/2}\|\hat{f}(\lambda)\|_{HS} \le e^{-b|\lambda|^q}$,

*where* $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ *and* $(ap)^{1/p}(bq)^{1/q} \ge 1$. *Then* $f = 0$ *unless* $p = q = 2$ *and* $ab = 1/4$ *in which case* $f(z,t) = e^{-at^2}\psi(z)$ *for some* $\psi \in L^2(\mathbb{C}^n)$.

**Corollary 1.2.13.** *Suppose* $f \in L^2(H_d^n)$ *and for some* $N \ge 0$, *assume that*

$$\int_{\mathbb{R}} \left( \frac{\|f(\cdot,t)\|_2 e^{at^2}}{(1+|t|)^N} \right)^p dt < \infty \text{ and } \int_{\mathbb{R}} \left( \frac{|\lambda|^{n/2}\|\hat{f}(\lambda)\|_{HS} e^{b\lambda^2}}{(1+|\lambda|)^N} \right)^q d\lambda < \infty.$$

*If* $ab = 1/4$ *then* $f(z,t) = e^{-at^2}\left(\sum_{j=0}^{m}\psi_j(z)t^j\right)$, *where each* $\psi_j \in L^2(\mathbb{C}^n)$ *and* $m < \min\{N - \frac{n}{p}, N - \frac{n}{q}\}$.

We remark that Corollary 1.2.12 has been proved in [2] with an extra condition $p \ge 2$ and the equality case has not been treated. To get Cowling–Price theorem for $H_d^n$, we need a modified version of Theorem 1.2.5 for $n = 1$. We record it as the following theorem.

**Theorem 1.2.14.** *Let* $h \in L^2(\mathbb{R})$. *Assume for some* $N \ge 0$ *and* $\delta > 0$ *it satisfies*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|h(t)||\hat{h}(\lambda)|e^{|t||\lambda|}|Q(\lambda)|^{\delta}}{(1+|t|+|\lambda|)^N}\, dt\, d\lambda < \infty, \tag{1.2.4}$$

*where* $Q$ *is a polynomial of degree* $m$. *Then* $h(t) = P(t)e^{-at^2}$, *where* $P$ *is a polynomial of* $\deg < \frac{N-1-m\delta}{2}$.

*Proof.* The condition ( 1.2.4 ) of the theorem is equivalent to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|h(t)||\hat{h}(\lambda)|e^{|t||\lambda|}|Q(\lambda)|^{\delta}}{(1+|t|)^{N/2}\,(1+|\lambda|)^{N/2}}\;dt\;d\lambda < \infty. \tag{1.2.5}$$

Therefore,

$$\int_{\mathbb{R}^n} \frac{|h(t)|e^{|t||\lambda'|}}{(1+|t|)^{N/2}}\;dt < \infty \tag{1.2.6}$$

for sufficiently large $|\lambda'| > 1$ for which $|Q(\lambda)| > 1$ holds for all $|\lambda| > \lambda'$. Hence it is easy to see that for any $0 < \lambda_0 < |\lambda'|$

$$\int_{\mathbb{R}} |h(t)|e^{\lambda_0|t|}\;dt < \infty. \tag{1.2.7}$$

This shows that $h \in L^1(\mathbb{R})$ and $\hat{h}$ is analytic in the open strip $|\Im\lambda| < \lambda_0$. Using this fact together with ( 1.2.5 ) and ( 1.2.6 ), we have the integral

$$\begin{aligned}
& \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|h(t)||\hat{h}(\lambda)|e^{|t||\lambda|}}{(1+|t|)^{N/2}\,(1+|\lambda|)^{N/2}}\;dt\;d\lambda \\
= & \int_{\mathbb{R}}\int_{|\lambda|<|\lambda'|} \frac{|h(t)||\hat{h}(\lambda)|e^{|t||\lambda|}}{(1+|t|)^{N/2}\,(1+|\lambda|)^{N/2}}\;dt\;d\lambda \\
& + \int_{\mathbb{R}}\int_{|\lambda|\geq|\lambda'|} \frac{|h(t)||\hat{h}(\lambda)|e^{|t||\lambda|}}{(1+|t|)^{N/2}\,(1+|\lambda|)^{N/2}}\;dt\;d\lambda \\
\leq & \; C\int_{\mathbb{R}} \frac{|h(t)|e^{|t||\lambda'|}}{(1+|t|)^{N/2}}\;dt + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|h(t)||\hat{h}(\lambda)|e^{|t||\lambda|}|Q(\lambda)|^{\delta}}{(1+|t|)^{N/2}\,(1+|\lambda|)^{N/2}}\;dt\;d\lambda \\
< & \; \infty
\end{aligned}$$

If we apply Theorem 1.2.5 for $n = 1$ we will get $h(t) = P(t)e^{-at^2}$ for some $a > 0$ and $P$ is a polynomial of deg $< \frac{N-1}{2}$. But the hypothesis ( 1.2.4 ) will force $\deg P < \frac{N-1-m\delta}{2}$. □

With this preparation we establish a version of Theorem 1.2.5 for $H_d^n$.

**Theorem 1.2.15.** *Suppose $f \in L^1 \cap L^2(H_d^n)$ and for some $M, N \geq 0$, it satisfies*

$$\int_{H_d^n}\int_{\mathbb{R}} \frac{|f(z,t)|\|\hat{f}(\lambda)\|_{HS}e^{|t||\lambda|}}{(1+|z|)^{M}\,(1+|t|)^{N/2}\,(1+|\lambda|)^{N/2}}\;|\lambda|^n\;d\lambda\;dz\;dt < \infty.$$

*Then* $f(z,t)=e^{-at^2}(1+|z|)^M\left(\sum\limits_{j=0}^{m}\psi_j(z)t^j\right)$, *where* $\psi_j\in L^2(\mathbb{C}^n)$ *and* $m<\frac{N-n/2-1}{2}$.

*Proof.* For each pair $(\phi,\psi)$, where $\phi,\psi\in L^2(\mathbb{R}^n)$ we consider the function

$$F_{(\phi,\psi)}(t)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{C}^n}f(z,t)(1+|z|)^{-M}\overline{(W(z)\phi,\psi)}\,dz.$$

Then it follows that

$$\begin{aligned}\widehat{F_{(\phi,\psi)}}(\lambda) &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{C}^n}f^{-\lambda}(z)(1+|z|)^{-M}\overline{(W(z)\phi,\psi)}\,dz \quad (1.2.8)\\ &\leq C\left(\int_{\mathbb{C}^n}|f^{-\lambda}(z)|^2\,dz\right)^{1/2}\\ &= C|\lambda|^{n/2}\|\hat{f}(-\lambda)\|_{\mathrm{HS}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|F_{(\phi,\psi)}(t)||\widehat{F_{(\phi,\psi)}}(\lambda)|e^{|t||\lambda|}|\lambda|^{n/2}}{(1+|t|)^{N/2}\,(1+|\lambda|)^{N/2}}\,dt\,d\lambda\\ \leq\;& C\int_{H_d^n}\int_{\mathbb{R}}\frac{|f(z,t)|\|\hat{f}(\lambda)\|_{\mathrm{HS}}e^{|t||\lambda|}}{(1+|z|)^M\,(1+|t|)^{N/2}\,(1+|\lambda|)^{N/2}}|\lambda|^n\,d\lambda\,dz\,dt<\infty,\end{aligned}$$

from our hypothesis. Now applying Theorem (1.2.14) to the function $F_{(\phi,\psi)}$ with $\delta=n/2$ we have $F_{(\phi,\psi)}(t)=P_{(\phi,\psi)}(t)e^{-a(\phi,\psi)t^2}$, where $P_{(\phi,\psi)}$ is a polynomial with deg $<\frac{N-n/2-1}{2}$. As in the proof of Theorem 1.2.8 keeping $\psi$ fixed, it can be shown that $a(\phi,\psi)=a(\psi)$ is independent of $\phi$. Similarly keeping $\phi$ fixed, we can show that $a(\phi,\psi)=a(\psi)=a$ is independent of $(\phi,\psi)$. We recall that $\{\Phi_{\alpha,\beta}:\alpha,\beta\in\mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. Now we take $\phi=\Phi_\alpha$ and $\psi=\Phi_\beta$. Let $F_{\alpha,\beta}=F_{(\Phi_\alpha,\Phi_\beta)}$ and $P_{\alpha,\beta}=P_{(\Phi_\alpha,\Phi_\beta)}$. Since for each $t\in\mathbb{R}$, $(1+|\cdot|)^{-M}f(\cdot,t)\in L^2(C^n)$ the sequence $\{P_{\alpha,\beta}(t)\}\in l^2$ for all $t$. We write $P_{\alpha,\beta}(t)=\sum\limits_{j=0}^{m}a_j(\alpha,\beta)t^j$, $m<\frac{N-n/2-1}{2}$. Choose $t_i\in\mathbb{R}$ such that $t_i\neq t_j$, for all $0\leq i,j\leq m$. We consider a system of linear equations given by:

$$\begin{pmatrix} 1 & t_0 & \cdots & t_0^m \\ 1 & t_1 & \cdots & t_1^m \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_m & \cdots & t_m^m \end{pmatrix} \begin{pmatrix} \{a_0(\alpha,\beta)\} \\ \{a_1(\alpha,\beta)\} \\ \vdots \\ \{a_m(\alpha,\beta)\} \end{pmatrix} = \begin{pmatrix} \{P_{\alpha,\beta}(t_0)\} \\ \{P_{\alpha,\beta}(t_1)\} \\ \vdots \\ \{P_{\alpha,\beta}(t_m)\} \end{pmatrix}.$$

Since $t_i \neq t_j$ for all $i \neq j$, the determinant of the $(m+1)\times(m+1)$ Vandermonde matrix is nonzero. Therefore, $\{a_j(\alpha,\beta)\}$ will be a linear combination of members from $\{\{P_{\alpha,\beta}(t_j)\} : 0 \leq j \leq m\}$ and hence $\{a_j(\alpha,\beta)\} \in l^2$ for each $0 \leq j \leq m$. With this observation we can write

$$\begin{aligned} (1+|z|)^{-M} f(z,t) &= \left(\sum_{\alpha,\beta} P_{\alpha,\beta}(t)\Phi_{\alpha,\beta}(z)\right) e^{-at^2} \\ &= \left(\sum_{\alpha,\beta}\left(\sum_{j=0}^{m} a_j(\alpha,\beta)t^j\right)\Phi_{\alpha,\beta}(z)\right) e^{-at^2} \\ &= \left(\sum_{j=0}^{m}\left(\sum_{\alpha,\beta} a_j(\alpha,\beta)\Phi_{\alpha,\beta}(z)\right) t^j\right) e^{-at^2} \\ &= \left(\sum_{j=0}^{m} \psi_j(z)t^j\right) e^{-at^2}, \end{aligned}$$

where $\psi_j(\cdot) = \sum_{\alpha,\beta} a_j(\alpha,\beta)\Phi_{\alpha,\beta}(\cdot) \in L^2(\mathbb{C}^n)$ . □

Applying the above theorem, we get a complete analogue of Cowling–Price theorem for $H_d^n$ and we record it in the following corollary.

**Corollary 1.2.16.** *Suppose $f \in L^1 \cap L^2(H_d^n)$ satisfies the conditions*

*(i)* $\int_{H_d^n} e^{pa(|z|^2+t^2)}|f(z,t)|^p \, dz \, dt < \infty$,

*(ii)* $\int_{\mathbb{R}} e^{qb\lambda^2}\|\hat{f}(\lambda)\|_{HS}^q |\lambda|^n \, d\lambda < \infty$.

*Then for* $ab \geq 1/4$, *and* $\min\{p,q\} < \infty$, $f = 0$.

*Proof.* Using Hölder's inequality we can find $N, M > 0$ such that

$(i)'$ $\int_{H_d^n} \frac{|f(z,t)|e^{at^2}}{(1+|z|)^M(1+|t|)^N}\ dz\ dt < \infty$,

$(ii)'$ $\int_{\mathbb{R}} \frac{e^{b\lambda^2}\|\hat{f}(\lambda)\|_{\mathrm{HS}}}{(1+|\lambda|)^N}\ |\lambda|^n\ d\lambda < \infty$.

These two conditions together with $ab \geq 1/4$ give us

$$\int_{H_d^n} \int_{\mathbb{R}} \frac{|f(z,t)| \|\hat{f}(\lambda)\|_{\mathrm{HS}} e^{|t||\lambda|}}{(1+|z|)^M (1+|t|)^N (1+|\lambda|)^N}\ |\lambda|^n\ d\lambda\ dz\ dt < \infty.$$

Therefore, using Theorem 1.2.15 we get for some $m > 0$,

$$f(z,t) = e^{-at^2}(1+|z|)^M \left( \sum_{j=0}^{m} \psi_j(z) t^j \right),$$

where $\psi_j \in L^2(\mathbb{C}^n)$. Since $\min\{p,q\} < \infty$ the conditions $(i)$ and $(ii)$ will force $f$ to be zero almost everywhere. □

Let us assume that $f \in L^1 \cap L^2(H_d^n)$ is such that

$$\int_{\mathbb{C}^n} \left( \int_{\mathbb{R}} e^{pat^2} |f(z,t)|^p\ dt \right)^{1/p}\ dz < \infty.$$

Then using Hölder's inequality followed by Minkowski's integral inequality we get $N > 0$, for which

$$\begin{aligned} & \int_{H_d^n} \frac{e^{at^2}|f(z,t)|}{(1+|t|)^N}\ dz\ dt \\ \leq\ & C \left( \int_{\mathbb{R}} e^{pat^2} \left| \int_{\mathbb{C}^n} f(z,t)\ dz \right|^p dt \right)^{1/p} \\ \leq\ & \int_{\mathbb{C}^n} \left( \int_{\mathbb{R}} e^{pat^2} |f(z,t)|^p\ dt \right)^{1/p}\ dz < \infty. \end{aligned}$$

This observation gives us the following corollary which is also a version of Cowling–Price theorem.

**Corollary 1.2.17.** *Let $f \in L^1 \cap L^2(H_d^n)$ be such that* (*i*) $\int_{\mathbb{C}^n} \left( \int_{\mathbb{R}} e^{pat^2} |f(z,t)|^p \, dt \right)^{1/p} \, dz < \infty,$

(*ii*) $\int_{\mathbb{R}} \frac{e^{qb\lambda^2} \|\hat{f}(\lambda)\|_{HS}^q}{(1+|\lambda|)^N} \, |\lambda|^n d\lambda < \infty.$

*If* $ab \geq 1/4$ *and* $\min\{p, q\} < \infty$ *then* $f = 0$ *almost everywhere.*

The second condition of all the theorems and corollaries proved for $H_d^n$ is in terms of the Hilbert–Schmidt norm of $\hat{f}(\lambda)$ and these theorems are in some sense theorems for the $t$–variable. This can be easily justified if we consider functions of the form $f(z,t) = g(z)h(t)$. We are interested in formulating uncertainty principles in which both the variables $z, t$ are respected.

### *1.2.2 Heat kernel version of uncertainty principles for $H_d^n$*

The equality case of the Hardy's theorem on $\mathbb{R}^n$ can be viewed as a characterization of the heat kernel associated with the Laplacian $\triangle$. If $p_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$, $t > 0$ denotes the heat kernel associated to $\triangle$ we can rewrite Hardy's theorem as follows. (We call it the heat kernel version of Hardy's theorem.)

**Theorem 1.2.18.** *Suppose* $f$ *is a measurable function on* $\mathbb{R}^n$ *such that*

$$|f(x)| \leq C p_t(x) \text{ and } |\hat{f}(\xi)| \leq C \widehat{p_s}(\xi).$$

*Then for* $t < s$, $f = 0$ *and* $f(x) = C p_t(x)$ *for* $t = s$.

We remark that the heat kernel $q_{a,d}$ satisfies neither of the two conditions in Corollary 1.2.11. In fact, if $q_{a,d}(z,t) \leq C(z) e^{-\alpha t^2}$ for some $\alpha > 0$ then $q_{a,d}^{\lambda}$ extends to an entire function of $\lambda$ in the complex plane. But the explicit formula of $q_{a,d}^{\lambda}$ shows that it has singularity at $\pm i\pi k / a d_j$ for $k \in \mathbb{N} \setminus \{0\}$. Hence the estimate $q_{a,d}(z,t) \leq C(z) e^{-\alpha t^2}$ is not possible. On the other hand using the relation $\hat{q}_{a,d}(\lambda) = T_\lambda(q^\lambda_{a,d})$ and the formula ( 1.1.3 ) we have

$$\begin{aligned} |\lambda|^n \|\hat{q}_{a,d}(\lambda)\|^2_{\mathrm{HS}} &= (2\pi)^n \prod_{j=1}^n d_j^{-1} \int_{\mathbb{C}^n} |q^\lambda_{a,d}(z)|^2 \, dz \\ &= C_n \left( \prod_{j=1}^n \frac{\lambda d_j}{\sinh a\lambda d_j} \right)^2 \prod_{j=1}^n \int_{\mathbb{C}} e^{-\frac{1}{2}\lambda \coth a\lambda d_j |z_j|^2} \, dz_j \\ &= C_n \left( \prod_{j=1}^n \frac{\lambda d_j}{\sinh a\lambda d_j} \right)^2 \left( \prod_{j=1}^n \frac{2\pi}{\lambda \coth a\lambda d_j} \right) \\ &= C'_n \left( \prod_{j=1}^n \frac{\lambda}{\sinh 2a\lambda d_j} \right) \\ &\approx C \prod_{j=1}^n e^{-2a|\lambda| d_j} \end{aligned}$$

for sufficiently large $|\lambda|$. This shows that $q_{a,d}$ cannot satisfy the hypotheses of Corollary 1.2.11. So it is not possible to characterize $q_{a,d}$ by Corollary 1.2.11. Let $\mathcal{L}$ be the sublaplacian on $H^n$ with associated heat kernel $q_a(z,t)$. Then the following theorem has been proved in [32].

**Theorem 1.2.19.** *Suppose $f$ is a measurable function on $H^n$ that satisfies*

*(i)* $|f(z,t)| \leq C q_a(z,t)$,

*(ii)* $\hat{f}(\lambda)^* \hat{f}(\lambda) \leq C \hat{q}_{2b}(\lambda)$ *for all* $\lambda \neq 0$,

*for some $a, b > 0$. Then $f = 0$ almost everywhere whenever $a < b$.*

In [34], it has been conjectured that such a theorem is true for all stratified nilpotent Lie groups. The proof given in [32] uses Gelfand pairs associated to $H^n$ and properties of the metaplectic representations. Therefore, this proof is not suitable for generalizing to other nilpotent Lie groups. Here we give a proof which works for all non-isotropic Heisenberg groups. In order to do it, first we prove a version of Theorem 1.2.6 for $H^n_d$. This theorem will be used to get heat kernel versions of Hardy and Cowling–Price theorems for $H^n_d$. Thangavelu has defined Fourier–Weyl transform for a function on $H^n$ and used it to prove Paley–Wiener theorem for $H^n$ (see [33]). Before stating our theorems we need the following two definitions.

**Definition 1.2.20.** *For $\xi = (\xi', \xi'') \in \mathbb{R}^n \times \mathbb{R}^n$ and $f \in L^1(H^n_d)$, we define*

$$\hat{f}(\lambda, \xi) = \pi_\lambda(\xi' + i\xi'', 0)\hat{f}(\lambda)\pi_\lambda(\xi' + i\xi'', 0)^*$$

*and call it the Fourier–Weyl transform of $f$ on $H^n_d$.*

It can be easily checked that

$$\hat{f}(\lambda, \xi) = \int_{\mathbb{R}^{2n}} e^{i\lambda \sum\limits_{j=1}^{n} d_j(x_j\xi''_j - y_j\xi'_j)} f^\lambda(x, y)\pi_\lambda(x + iy, 0)dx\ dy,$$

where we have written $z = x + iy$ and $f^\lambda(x, y)$ stands for $f^\lambda(z)$.

**Definition 1.2.21.** *A function $\phi \in L^2(\mathbb{R}^n)$ is said to be an analytic vector for the representation $\pi_\lambda$ if for all $\psi \in L^2(\mathbb{R}^n)$ the function $(z, t) \to \langle \pi_\lambda(z, t)\phi, \psi\rangle$ is real analytic.*

Since $U(|\lambda|d)^{-1}\pi_\lambda(z, t)U(|\lambda|d) = \rho_1(\sqrt{|\lambda|d}\ z, t)$, it is sufficient to consider analytic vectors for the representation $\rho_1$ of $H^n$. In the following theorem we give a sufficient condition for a function $\phi \in L^2(\mathbb{R}^n)$ to be an analytic vector for $\rho_1$.

**Theorem 1.2.22.** *Let $\phi \in L^2(\mathbb{R}^n)$ be such that $|\langle \phi, \Phi_\alpha\rangle| \leq Ce^{-b(2|\alpha|+n)}$ for all $\alpha \in \mathbb{N}$ and for some $b > 0$. Then $\phi$ is an analytic vector for the representation $\rho_1$ of $H^n$.*

*Proof.* Since $\rho_1(z, t) = e^{it}\rho_1(z, 0)$ it is enough to show $\langle \rho_1(z, 0)\phi, \psi\rangle$ is real analytic function on $\mathbb{R}^{2n}$ for any $\psi \in L^2(\mathbb{R}^n)$. The following formula

$$\overline{\langle \rho_1(z, 0)\phi_1, \psi_1\rangle} \times \overline{\langle \rho_1(z, 0)\phi_2, \psi_2\rangle} = \langle \psi_2, \phi_1\rangle \overline{\langle \rho_1(z, 0)\phi_2, \psi_1\rangle}$$

has been proved in the proposition 1.47, page–32 of [10]. Therefore, for any $\psi \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\|\langle \rho_1(z, 0)\phi, \psi\rangle \times \Phi_{\alpha\alpha}\|_2 &= |\langle \Phi_\alpha, \phi\rangle|\ \|\langle \rho_1(z, 0)\Phi_\alpha, \psi\rangle\|_2 \\
&= |\langle \phi, \Phi_\alpha\rangle|\ \|\psi\|_2\|\Phi_\alpha\|_2 \\
&\leq Ce^{-b(2|\alpha|+n)}.
\end{aligned}$$

Let $\phi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$. It follows from the identity (see [34],page 58),

$$\phi_k(z) = (2\pi)^{n/2} \sum_{|\alpha|=k} \Phi_{\alpha\alpha}(z)$$

that

$$\begin{aligned} \|\langle \rho_1(z,0)\phi, \psi\rangle \times \phi_k\|_2 &\leq C \sum_{|\alpha|=k} e^{-b(2|\alpha|+n)} \\ &= C\frac{(k+n-1)!}{k!(n-1)!} e^{-b(2k+n)} \\ &\leq Ce^{-b'(2k+n)} \end{aligned}$$

for some $0 < b' < b$. We conclude $\langle \rho_1(z,0)\phi, \psi\rangle$ is real analytic using the following proposition proved in [30]. □

**Proposition 1.2.23.** *Suppose $g \in L^2(\mathbb{C}^n)$ is such that $\|g\times\phi_k\|_2 \leq Ce^{-b(2k+n)}$, for some $b > 0$. Then $g$ is real analytic.*

The above theorem shows that for each $\lambda \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{N}^n$, $\Phi_\alpha$ is an analytic vector for $\rho_1$ and hence is also an analytic vector for $\pi_\lambda$. So analytic vectors for $\pi_\lambda$ are dense in $L^2(\mathbb{R}^n)$.

**Theorem 1.2.24.** *Suppose $f \in L^1 \cap L^2(H_d^n)$. Assume that for each $\lambda \in \mathbb{R} \setminus \{0\}$ there exists an analytic vector $\phi_\lambda$ for $\pi_\lambda$ and $\psi_\lambda \in L^2(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^{2n}} \int_{H_d^n} |f(x+iy,t)||\langle \hat{f}(\lambda,(\xi',\xi''))\phi_\lambda, \psi_\lambda\rangle| \times e^{|\lambda|| \sum_{j=1}^{n} d_j(x_j\xi''_j - y_j\xi'_j)|} \, dx\, dy\, dt\, d\xi'\, d\xi'' < \infty.$$

*Then $f = 0$ almost everywhere .*

*Proof.* Let $g_\lambda(x,y) = f^\lambda(x,y)\langle \pi_\lambda(x+iy,0)\phi_\lambda, \psi_\lambda\rangle$. Then $\langle \hat{f}(\lambda,(\xi',\xi''))\phi_\lambda, \psi_\lambda\rangle$

is the Fourier transform of $g_\lambda$ at $|\lambda|(-d_1\xi_1'', \cdots, -d_n\xi_n'', d_1\xi_1', \cdots, d_n\xi_n')$. Now,

$$\begin{aligned} & \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} |g_\lambda(x,y)|\ |\widehat{g}_\lambda(\xi',\xi'')| e^{|x\cdot\xi'+y\cdot\xi''|}\ dx\ dy\ d\xi'\ d\xi'' \\ \leq & \int_{\mathbb{R}^{2n}} \int_{H_d^n} |f(x+iy,t)||\langle \hat{f}(\lambda,(\xi',\xi''))\phi_\lambda,\psi_\lambda\rangle| \\ & \times e^{|\lambda||\sum_{j=1}^{n} d_j(x_j\xi_j''-y_j\xi_j')|}\ dx\ dy\ dt\ d\xi'\ d\xi'' \\ < & \ \infty \end{aligned}$$

by our hypothesis. Applying Theorem 1.2.6 on $\mathbb{R}^{2n}$ to the function $g_\lambda$ we get $g_\lambda = 0$ almost everywhere. Therefore, the support of $f^\lambda$ is contained in $\{(x,y) : \langle \pi_\lambda(x+iy)\phi_\lambda, \psi_\lambda\rangle = 0\}$ which is a set of measure zero as $\phi_\lambda$ is an analytic vector. Since $f^\lambda = 0$ almost everywhere for each $\lambda \in \mathbb{R}\setminus\{0\}$ we have $f = 0$ almost everywhere. □

The following theorem is analogue of Theorem 1.2.6 for $H_d^n$.

**Theorem 1.2.25.** *Suppose $f \in L^1 \cap L^2(H_d^n)$. Assume that for each $\lambda \in \mathbb{R}\setminus\{0\}$ and for every pair $(\phi_\lambda, \psi_\lambda)$ , where $\phi_\lambda, \psi_\lambda \in L^2(\mathbb{R}^n)$*

$$\begin{aligned} & \int_{\mathbb{R}^{2n}} \int_{H_d^n} \frac{|f(x+iy,t)||\langle \hat{f}(\lambda,(\xi',\xi''))\phi_\lambda,\psi_\lambda\rangle|}{(1+|(x,y)|+|(\xi',\xi'')|)^N} \\ & \times e^{|\lambda||\sum_{j=1}^{n} d_j(x_j\xi_j''-y_j\xi_j')|}\ dx\ dy\ dt\ d\xi'\ d\xi'' < \infty, \end{aligned}$$

*for some $N \geq 0$. Then*

$$f^\lambda(x,y) = P_\lambda(x,y)\ exp(\frac{1}{4}\sum_{j=1}^{n}|\lambda| d_j(x_j^2+y_j^2)) e^{-\langle A(x,y),(x,y)\rangle},$$

*where $P_\lambda(x,y)$ is a polynomial in $(x,y)$ but depends on $\lambda$ and $A$ is a positive definite symmetric matrix.*

*Proof.* Consider the function $g_\lambda(x,y) = f^\lambda(x,y)\langle \pi_\lambda(x+iy)\phi_\lambda, \psi_\lambda\rangle$. From the hypothesis of the theorem we see that

$$\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \frac{|g_\lambda(x,y)|\ |\widehat{g}_\lambda(\xi',\xi'')| e^{|x\cdot\xi'+y\cdot\xi''|}}{(1+|(x,y)|+|(\xi',\xi'')|)^N}\ dx\ dy\ d\xi'\ d\xi'' < \infty.$$

Therefore, using Theorem 1.2.6 we get

$$f^\lambda(x,y)\langle \pi_\lambda(x+iy,0)\phi_\lambda,\psi_\lambda\rangle = P_{(\phi_\lambda,\psi_\lambda)}(x,y)e^{-\langle A(\phi_\lambda,\psi_\lambda)(x,y),(x,y)\rangle},$$

where $P_{(\phi_\lambda,\psi_\lambda)}$ is a polynomial in $(x,y)$. But as in the proof of Theorem 1.2.8 we can show that $A(\phi_\lambda,\psi_\lambda)=A$ is independent of the choice of $(\phi_\lambda,\psi_\lambda)$. In particular take $\phi_\lambda(x)=\psi_\lambda(x)=U(\lambda d)\Phi_0(x)$ which yields

$$f^\lambda(x,y) = P_\lambda(x,y)\ \exp(\frac{1}{4}\sum_{j=1}^{n}(|\lambda|d_j(x_j^2+y_j^2)))\ e^{-\langle A(x,y),(x,y)\rangle}$$

and $\deg P_\lambda < \frac{N-2n}{2}$. □

Assume $f\in L^1(H^n)\cap L^2(H^n)$ and $f$ is radial that is $f(\sigma\cdot z,t)=f(z,t)$ for all $\sigma\in U(n)$, the unitary group. In this case $\hat f(\lambda)$ is diagonalizable with respect to the orthonormal basis given by Hermite functions (see [34],page 62). So we can write

$$\hat f(\lambda)\phi = \sum_{\alpha\in\mathbb{N}^n} R_\alpha(\lambda)\langle\phi,\Phi_\alpha^\lambda\rangle\Phi_\alpha^\lambda,$$

where $\Phi_\alpha^\lambda(\xi)=|\lambda|^{n/4}\Phi_\alpha(\sqrt{|\lambda|}\xi)$. Suppose for each $\lambda\in\mathbb{R}\setminus\{0\}$ there exists an operator $S(\lambda)\in\mathcal{S}_2$, the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$ such that $\hat f(\lambda)=S(\lambda)e^{-b(H(\lambda))}$. Then it follows that for some $C>0$

$$|R_\alpha(\lambda)|\le Ce^{-(2|\alpha|+n)|\lambda|b},$$

for all $\alpha\in\mathbb{N}^n$. This shows the exponential decay of singular numbers of $\hat f(\lambda)$. It is to be noted that $\widehat{q_a}(\lambda)=e^{-aH(\lambda)}$. The condition that for each $\lambda\in\mathbb{R}\setminus\{0\}$ there exists $S(\lambda)\in\mathcal{S}_q$ such that $\hat f(\lambda)=S(\lambda)\widehat{q_a}(\lambda)$ is the analogue of the condition $\hat f(\widehat{p_s})^{-1}\in L^q(\mathbb{R}^n)$ in the case of Heisenberg group. Here $\mathcal{S}_q$ denotes the set of Schatten $q$–class operators. Let us see the effect of such an assumption on $\hat f(\lambda)$ on the Fourier–Weyl transform of $f$ on $H_d^n$. Assume for each $\lambda\in\mathbb{R}\setminus\{0\}$ there exists a bounded operator $S(\lambda)$ on $L^2(\mathbb{R}^n)$ such that $\hat f(\lambda)=S(\lambda)\,(H(\lambda d))^m\,e^{-bH(\lambda d)}$ for some $m>0$. Therefore, there exists

$C > 0$ such that for all $\phi \in L^2(\mathbb{R}^n)$, $\|S(\lambda)\phi\|_2 \leq C\|\phi\|_2$. As

$$H(|\lambda|d)U(|\lambda|d)\Phi_\alpha = \left(|\lambda|\sum_{j=1}^{n}(2\alpha_j+1)d_j\right)U(|\lambda|d)\Phi_\alpha$$

for all $\alpha \in \mathbb{N}^n$, we have

$$\begin{aligned}\|\hat{f}(\lambda)U(|\lambda|d)\Phi_\alpha\|_2 &\leq C\left(\sum_{j=1}^{n}(2\alpha_j+1)|\lambda|d_j\right)^m \\ &\quad\times\left(\prod_{j=1}^{n}e^{-b|\lambda|(2\alpha_j+1)d_j}\right)\end{aligned}$$

for all $\alpha \in \mathbb{N}^n$. Therefore, for any $0 < b' < b$, there exists a constant $C$ such that

$$\|\hat{f}(\lambda)U(|\lambda|d)\Phi_\alpha\|_2 \leq C\left(\prod_{j=1}^{n}e^{-b'|\lambda|(2\alpha_j+1)d_j}\right) \tag{1.2.9}$$

for all $\alpha \in \mathbb{N}$.

For $\xi = (\xi', \xi'') \in \mathbb{R}^n \times \mathbb{R}^n$ and $r = (r_1, \cdots, r_n) \in \mathbb{R}^n$ let us denote the point $(r_1\xi_1', \cdots r_n\xi_n')$ by $r\xi$ and $(r_1\xi_1', \cdots r_n\xi_n', r_1\xi'', \cdots, r_n\xi_n'')$ by $(r\xi', r\xi'')$ and write $r(\xi' + i\xi'')$ for the point $r\xi' + ir\xi'' \in \mathbb{C}^n$. Using the fact that $\{U(|\lambda|d)\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$, we compute

$$\begin{aligned}&\pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0 \\ = &\sum_\alpha \langle \pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_\alpha\rangle U(|\lambda|d)\Phi_\alpha \\ = &\sum_\alpha \langle U(|\lambda|d)^* \pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0, \Phi_\alpha\rangle U(|\lambda|d)\Phi_\alpha \\ = &\sum_\alpha \langle W(-\sqrt{|\lambda|d}(\xi' + i\xi''))\Phi_0, \Phi_\alpha\rangle U(|\lambda|d)\Phi_\alpha \\ = &\sum_\alpha \bar{\Phi}_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi''))U(|\lambda|d)\Phi_\alpha.\end{aligned}$$

Using ( 1.2.9 ) and the above expression for $\pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0$ we have

$$\begin{aligned}
& |\langle \hat{f}(\lambda, (\xi', \xi''))U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle| \\
= \; & |\langle \pi_\lambda(\xi' + i\xi'', 0)\hat{f}(\lambda)\pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle| \\
\leq \; & \|\hat{f}(\lambda)\pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0\|_2 \\
\leq \; & C\sum_\alpha |\Phi_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi'')| \; \|\hat{f}(\lambda)U(|\lambda|d)\Phi_\alpha\|_2 \\
\leq \; & C\sum_\alpha |\Phi_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi'')| \; e^{-b'|\lambda|(\sum_{j=1}^n (2\alpha_j+1)d_j)}.
\end{aligned}$$

We choose $b''$ with $a < b'' < b' < b$ and apply Cauchy–Schwarz inequality to get

$$\begin{aligned}
& |\langle \hat{f}(\lambda, (\xi', \xi''))U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle| \\
\leq \; & C\sum_\alpha |\Phi_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi'')| e^{-(b'-b'')|\lambda|(\sum_{j=1}^n (2\alpha_j+1)d_j)} e^{-b''|\lambda|(\sum_{j=1}^n (2\alpha_j+1)d_j)} \\
\leq \; & C\left(\sum_\alpha |\Phi_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi'')|^2 e^{-2b''|\lambda|(\sum_{j=1}^n (2\alpha_j+1)d_j)}\right)^{\frac{1}{2}} \\
\leq \; & C\left(\prod_{j=1}^n \sum_{\alpha_j=0}^\infty \left(\frac{1}{\alpha_j!}\; e^{-2b''|\lambda|(2\alpha_j+1)d_j} \left(\frac{1}{2}|\lambda|d_j({\xi'_j}^2 + {\xi''_j}^2)\right)^{\alpha_j} e^{-\frac{1}{2}|\lambda|d_j({\xi'_j}^2+{\xi''_j}^2)}\right)\right)^{\frac{1}{2}} \\
\leq \; & C\prod_{j=1}^n \left(e^{-\frac{1}{4}|\lambda|d_j({\xi'_j}^2+{\xi''_j}^2)} \left(\sum_{\alpha_j=0}^\infty \frac{1}{\alpha_j!}\left(\frac{1}{2}|\lambda|d_j({\xi'_j}^2 + {\xi''_j}^2)e^{-4b''|\lambda|d_j}\right)^{\alpha_j}\right)^{\frac{1}{2}}\right) \\
\leq \; & C\prod_{j=1}^n e^{-\frac{1}{4}|\lambda|d_j\left(1-e^{-4b''|\lambda|d_j}\right)({\xi'_j}^2+{\xi''_j}^2)}.
\end{aligned}$$

In the fourth step of the above calculation we have used the explicit formula ( 1.1.7 ) of $\Phi_{\alpha,0}$ . Thus the function $\langle \hat{f}(\lambda, (\xi', \xi''))U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle$ has a Gaussian decay.

**Theorem 1.2.26.** *Let $f$ be a function on $H_d^n$ such that for some $a, b > 0$*

*(i)* $(1+|z|)^{-m} f(q_{a,d})^{-1} \in L^p(H_d^n)$,

*(ii)* $\widehat{f}(\lambda) = S(\lambda)\,(\pi_\lambda(\mathcal{L}_d))^m\,\hat{q}_{b,d}(\lambda)$, $S(\lambda) \in \mathcal{S}_q$ *for all* $\lambda$,

*where* $1 \leq p, q \leq \infty$. *Then* $f = 0$ *almost everywhere whenever* $a < b$.

*Proof.* Define $g(z,t) = (1+|z|)^{-m}(q_{a,d}(z,t))^{-1} f(z,t)$. From the hypothesis $g \in L^p(H_d^n)$. We have the estimate $|q_{a,d}(z,t)| \leq Ce^{-\frac{A}{a}(|z|^2+|t|)}$, for some $A > 0$. Using the condition $g \in L^p(H_d^n)$, the following integral

$$\begin{aligned} |f^{\lambda+i\mu}(z)| &= |\int_{-\infty}^{\infty} e^{(i\lambda-\mu)t} q_{a,d}(z,t) g(z,t)\; dt| \\ &\leq C\int_{-\infty}^{\infty} e^{\mu t} e^{-\frac{A}{a}(|z|^2+|t|)} |g(z,t)|\; dt \end{aligned}$$

is finite for $|\mu| < \frac{A}{a}$. Now applying Morera's theorem it is easy to see that $f^\lambda$ can be extended as a holomorphic function in the strip $|\Im\lambda| < \frac{A}{a}$ of the complex plane.
For $p = \infty$,

$$\begin{aligned} (1+|z|)^{-m}|f^\lambda(z)| &\leq \|g(z,\cdot)\|_\infty \int_{-\infty}^{\infty} |q_{a,d}(z,t)|\; dt \\ &\leq \|g(z,\cdot)\|_\infty e^{-\frac{1}{4a}|z|^2}. \end{aligned}$$

For $1 \leq p \leq 2$,

$$\begin{aligned} (1+|z|)^{-m}|f^\lambda(z)| &\leq \int_{-\infty}^{\infty} |q_{a,d}(z,t)||g(z,t)|\; dt \\ &\leq \left(\int_{-\infty}^{\infty} |q_{a,d}(z,t)|^{p'}\; dt\right)^{\frac{1}{p'}} \left(\int_{-\infty}^{\infty} |g(z,t)|^p\; dt\right)^{\frac{1}{p}}. \end{aligned}$$

Now applying Hausdorff-Young inequality to the first integral

$$(1+|z|)^{-m}|f^\lambda(z)| \leq \left(\int_{-\infty}^{\infty} |q_{a,d}^\lambda(z)|^p\; d\lambda\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |g(z,t)|^p\; dt\right)^{\frac{1}{p}}$$

$$\begin{aligned}
&\leq (4\pi)^{-n}\left(\int_{-\infty}^{\infty}\prod_{j=1}^{n}\left(\frac{d_j\lambda}{\sinh d_j\lambda a}\right)^p e^{-\frac{p}{4}(d_j\lambda\coth d_j\lambda a)|z_j|^2}d\lambda\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_{-\infty}^{\infty}|g(z,t)|^p\,dt\right)^{\frac{1}{p}} \\
&\leq Ce^{-\frac{1}{4a}|z|^2}\left(\int_{-\infty}^{\infty}|g(z,t)|^p\,dt\right)^{\frac{1}{p}}.
\end{aligned}$$

When $2 < p < \infty$, $1 < p' < 2$ write $\frac{1}{p'} = \frac{\nu}{1} + \frac{1-\nu}{2}$ for some $0 < \nu < 1$. Since $\|q_{a,d}(z,\cdot)\|_1 \leq e^{-\frac{1}{4a}|z|^2}$ and $\|q_{a,d}(z,\cdot)\|_2 \leq e^{-\frac{1}{4a}|z|^2}$, applying Hölder's inequality with the pair of conjugate exponents $\frac{1}{\nu p'}$ and $\frac{2}{(1-\nu)p'}$, we get

$$\begin{aligned}
\int_{-\infty}^{\infty}|q_{a,d}(z,t)|^{p'}\,dt &= \int_{-\infty}^{\infty}|q_{a,d}(z,t)|^{\nu p'}|q_a(z,t)|^{(1-\nu)p'}\,dt \\
&\leq \|q_{a,d}(z,\cdot)\|_1^{\nu p'}\|q_{a,d}(z,\cdot)\|_2^{(1-\nu)p'}
\end{aligned}$$

which gives $\|q_{a,d}(\cdot,t)\|_{p'} \leq e^{-\frac{1}{4a}|z|^2}$. Therefore, $f^\lambda(\cdot)(1+|z|)^m e^{\frac{1}{4a}|z|^2}$ belongs to $L^p(\mathbb{C}^n)$ and hence for any $a' > a$, $f^\lambda(\cdot)e^{\frac{1}{4a'}|z|^2} \in L^p(\mathbb{C}^n)$ . Since every member of $\mathcal{S}_q$ is a bounded operator there exits $C > 0$ such that $\|S(\lambda)\phi\|_2 \leq C\|\phi\|_2$ for all $\phi \in L^2(\mathbb{R}^n)$. Since $\pi_\lambda(\mathcal{L}_d) = H(\lambda d)$, from the previous observation we have for any $0 < b' < b$

$$\begin{aligned}
&|\langle\hat{f}(\lambda,(\xi',\xi''))U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle| \\
&\leq C\prod_{j=1}^{n} e^{-\frac{1}{4}|\lambda|d_j\left(1-e^{-4b'|\lambda|d_j}\right)\left({\xi'_j}^2+{\xi''_j}^2\right)}. \qquad (1.2.10)
\end{aligned}$$

Let $g_\lambda(x,y) = f^\lambda(x,y)\langle\pi_\lambda(x+iy,0)\phi_\lambda,\phi_\lambda\rangle$ where $\phi_\lambda(x) = U(|\lambda|d)\Phi_0(x)$. Note that $\phi_\lambda$ is an analytic vector. Then

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}|g_\lambda(x,y)|\;|\widehat{g}_\lambda(\xi',\xi'')|e^{|x\cdot\xi'+y\cdot\xi''|}dx\;dy\;d\xi'\;d\xi'' \\
&\leq \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}|f^\lambda(x,y)|\;|\langle\hat{f}(\lambda,(\xi',\xi''))\phi_\lambda,\phi_\lambda\rangle|e^{|\lambda|\sum\limits_{j=1}^{n}d_j(|x_j\xi''_j|+|y_j\xi'_j|)}dx\;dy\;d\xi'\;d\xi'' \\
&\leq C\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}|f^\lambda(x,y)|\;e^{\frac{1}{4a'}(|x|^2+|y|^2)}e^{-\frac{1}{4a'}|((x,y)-2a|\lambda|d(\xi'',\xi'))|^2}\times \\
&\quad\prod_{j=1}^{n}e^{-|\lambda|d_j\left(\frac{1}{4}\left(1-e^{-4b'|\lambda|d_j}\right)-a'|\lambda|d_j\right)({\xi'_j}^2+\xi''_j)}dx\;dy\;d\xi'\;d\xi''
\end{aligned}$$

Since $f^\lambda e^{\frac{1}{4a'}(|x|^2+|y|^2)} \in L^p(\mathbb{R}^{2n})$, the above integral will be finite if $a'|\lambda|d_j < \frac{1}{4}\left(1-e^{-4b'|\lambda|d_j}\right)$. Therefore, applying Theorem 1.2.6 to the function $g_\lambda$, we conclude $g_\lambda = 0$ whenever $a'|\lambda|d_j < \frac{1}{4}\left(1-e^{-4b'|\lambda|d_j}\right)$. As the function $\langle \pi_\lambda(x+iy)\phi_\lambda, \phi_\lambda\rangle = \prod_{j=1}^{n} e^{-\frac{1}{4}|\lambda|\ d_j(x_j^2+y_j^2)}$ is non vanishing everywhere for all $\lambda$, we can conclude $f^\lambda = 0$ whenever $a'|\lambda|d_j < \frac{1}{4}(1-e^{-4b'|\lambda|d_j})$. Since $a < a' < b' < b$ we can choose $\delta > 0$ such that $a' < b'e^{-2b'|\lambda|d_j} < b'$ for all $j$ with $0 < |\lambda| < \delta$. Now

$$\begin{aligned} 1-e^{-4b'|\lambda|d_j} &= e^{-2b'|\lambda|d_j}(e^{2b'|\lambda|d_j} - e^{-2b'|\lambda|d_j}) \\ &> 4b'|\lambda|d_j e^{-2b'|\lambda|d_j} \\ &> 4a'|\lambda|d_j \end{aligned}$$

for all j and $\lambda$ with $0 < |\lambda| < \delta$. Since $f^\lambda$ can be extended as a holomorphic function in a strip of the complex plane we conclude $f^\lambda = 0$ for all $\lambda$ whenever $a < b$ and hence $f = 0$ almost everywhere. □

We also have the following version of Hardy's theorem.

**Theorem 1.2.27.** *If*

(*i*) $|f(z,t)| \le C(1+|z|)^m q_{a,d}(z,t)$,

(*ii*) $\hat{f}(\lambda)^*\hat{f}(\lambda) \le C(\pi_\lambda(\mathcal{L}_d))^m \hat{q}_{2b,d}(\lambda)$

*for some* $a, b > 0$, *then* $f = 0$ *whenever* $a < b$.

*Proof.* If $\hat{f}(\lambda)^*\hat{f}(\lambda) \le Ce^{-2bH(|\lambda|d)}$ we have the estimate

$$\|\hat{f}(\lambda)U(|\lambda|d)\Phi_\alpha\|_2 \le C\prod_{j=1}^{n} e^{-b|\lambda|(2\alpha_j+1)d_j}$$

for all $\alpha \in \mathbb{N}^n$ and hence $\langle \hat{f}(\lambda,(\xi',\xi''))U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle$ can be estimated as before. Thus the proof will be completed. □

# 2. UNCERTAINTY PRINCIPLES FOR STEP TWO NILPOTENT LIE GROUPS

The aim of this chapter is to prove various uncertainty principles for connected simply connected step two nilpotent lie groups. We have organised the chapter as follows: We describe the irreducible unitary representations, Plancherel formula, relevant aspects of step two nilpotent Lie groups. Then we extend the results proved for Heisenberg groups in earlier chapter, in the context of connected simply connected step two nilpotent Lie groups.

## *2.1 Preliminaries on step two nilpotent Lie groups*

Let $G$ be a step two connected simply connected nilpotent Lie group so that its Lie algebra $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{v}$ is any subspace of $\mathfrak{g}$ complementary to $\mathfrak{z}$. We choose an inner product on $\mathfrak{g}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal. Fix an orthonormal basis $\mathcal{B} = \{e_1, e_2 \cdots, e_m, T_1, \cdots, T_k\}$ so that $\mathfrak{v} = \mathbb{R}$ span$\{e_1, e_2 \cdots, e_m\}$ and $\mathfrak{z} = \mathbb{R}$ span$\{T_1, \cdots, T_k\}$. Since $\mathfrak{g}$ is nilpotent the exponential map is an analytic diffeomorphism . We can identify $G$ with $\mathfrak{v} \oplus \mathfrak{z}$ and write $(X + T)$ for $\exp(X+T)$ and denote it by $(X, T)$ where $X \in \mathfrak{v}$ and $T \in \mathfrak{z}$. The product law on $G$ is given by the Baker-Campbell-Hausdorff formula :

$$(X, T)(X', T') = (X + X', T + T' + \frac{1}{2}[X, X'])$$

for all $X, X' \in \mathfrak{v}$ and $T, T' \in \mathfrak{z}$. For any orthonormal basis $\{X_j : 1 \leq j \leq m\}$ of $\mathfrak{v}$ define the sublaplacian

$$\mathcal{L} = -\sum_{j=1}^{m} X_j^2.$$

### 2.1.1 Representations of step two nilpotent Lie groups

A complete account of representation theory for general connected simply connected nilpotent Lie groups can be found in [8]. Representations of step two connected simply connected nilpotent groups are easy to describe. Ray [28] has described their representations and proved the Plancherel theorem following the orbit method of Kirillov. Let $\mathfrak{g}^*$, $\mathfrak{z}^*$ be the real dual of $\mathfrak{g}$ and $\mathfrak{z}$ respectively. For each $\nu \in \mathfrak{z}^*$ consider the bilinear form $B'_\nu$ on $\mathfrak{v}$ defined by

$$B'_\nu(X,Y) = \nu([X,Y]) \text{ for all } X, Y \in \mathfrak{g}.$$

The radical $\tilde{\mathfrak{r}}_\nu$ of the bilinear form $B'_\nu$ is given by

$$\tilde{\mathfrak{r}}_\nu = \{X \in \mathfrak{g} : \nu([X,Y]) = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

Let $B_\nu$ be the restriction of $B'_\nu$ on $\mathfrak{v}$ and

$$\mathfrak{r}_\nu = \{X \in \mathfrak{v} : \nu([X,Y]) = 0 \text{ for all } Y \in \mathfrak{v}\}.$$

Let $X_i = e_i$ for all $1 \leq i \leq m$ and $X_{m+i} = T_i$ for all $1 \leq i \leq k$. Then $\mathcal{B} = \{X_1, \cdots, X_m, X_{m+1}, \cdots, X_{m+k}\}$. Let $\mathcal{B}^* = \{X_1^*, \cdots, X_m^*, X_{m+1}^*, \cdots, X_{m+k}^*\}$ be the dual basis of $\mathcal{B}$. We consider the matrix $(B_\nu(i,j))$ given by the bilinear form $B_\nu$ that is $(i,j)$ th entry of the matrix is $B_\nu(X_i, X_j)$. Let $B_\nu^i$ denotes the submatrix of $(B_\nu(i,j))$ consisting of first $i$ rows. If rank $B_\nu^i$ is strictly greater than rank $B_\nu^{(i-1)}$ then $i$ is called a jump index for $\nu$. Since $B_\nu$ is an alternating bilinear form $\nu$ has an even number of jump indices. The set of jump indices is denoted by $S = \{j_1, j_2 \cdots, j_{2n}\}$. These indices depend on $\nu$ as well as the order of the basis. But they are all same if we choose $\nu \in \mathcal{U} = \{\nu : \text{ rank } (B_\nu^i) \text{ is maximal for all } i.\}$, a Zariski open subset of $\mathfrak{z}^*$. Let $T = \{n_1, n_2, \cdots, n_r, m+1, \cdots, m+k\}$ be the complement of $S$ in $\{1, 2, \cdots, m, m+1, \cdots, m+k\}$. Let

$$V_T^* = \operatorname{Span}_{\mathbb{R}}\{X_{m+1}^*, \cdots, X_{m+k}^*, X_{n_i}^* : n_i \in T\}$$

and $\widetilde{V}_T^* = \{X_{n_i}^* : n_i \in T\}$. The irreducible unitary representations relevant to Plancherel measure of $\hat{G}$ are parametrized by the set $\Lambda = \widetilde{V}_T^* \times \mathcal{U}$.

If there exist $\nu \in \mathfrak{z}^*$ such that $B_\nu$ is nondegenerate then the Lie algebra is called an MW algebra after Moore and Wolf and the corresponding group is called an MW group. In this case $T = \{m+1, \cdots, m+k\}$ and $\mathcal{U} = \{\nu \in \mathfrak{z}^* : B_\nu \text{ is nondegeneate}\}$. The irreducible unitary representations relevant to Plancherel measure of $\hat{G}$ will be parametrized by $\Lambda = \{\nu \in \mathfrak{z}^* : B_\nu$ is nondegenerate.$\}$

For

$$(X,T) = \exp(\sum_{j=1}^{m} x_j X_j + \sum_{j=1}^{k} t_j X_{j+m}), \quad x_j, t_j \in \mathbb{R},$$

we define its norm by

$$|(X,T)| = (x_1^2 + \cdots + x_m^2 + t_1^2 + \cdots + t_k^2)^{1/2}.$$

The map

$$\begin{aligned}(x_1, \cdots, x_m, t_1 \cdots, t_k) \quad &\longrightarrow \quad \sum_{j=1}^{m} x_j X_j + \sum_{j=1}^{k} t_j X_{j+m} \\ &\longrightarrow \quad \exp\left(\sum_{j=1}^{m} x_j X_j + \sum_{j=1}^{k} t_j X_{j+m}\right)\end{aligned}$$

takes Lebesgue measure $dx_1 \cdots dx_m dt_1 \cdots dt_k$ of $\mathbb{R}^{m+k}$ to Haar measure on $G$. Any measurable function $f$ on $G$ will be identified with a function on $\mathbb{R}^{m+k}$. We identify $\mathfrak{g}^*$ with $\mathbb{R}^{m+k}$ with respect to the basis $\mathcal{B}^*$ and introduce the Euclidean norm relative to this basis.

*Step two groups without MW condition*

In this case $\mathfrak{r}_\nu \neq \{0\}$ for each $\nu \in \mathcal{U}$. Let $\mathfrak{m}_\nu$ be the orthogonal complement of $\mathfrak{r}_\nu$ in $\mathfrak{v}$. Then $B_\nu|_{\mathfrak{m}_\nu}$ is nondegenerate and hence $\dim \mathfrak{m}_\nu$ is $2n$. From the properties of an alternating bilinear form there exists an orthonormal basis

$$\{X_1(\nu), Y_1(\nu), \cdots, X_n(\nu), Y_n(\nu), Z_1(\nu), \cdots, Z_r(\nu)\}$$

of $\mathfrak{v}$ and positive numbers $d_i(\nu) > 0$ such that

(i) $\mathfrak{r}_\nu = \mathbb{R}$ span $\{Z_1(\nu), \cdots, Z_r(\nu)\}$,

(ii) $\nu([X_i(\nu), Y_j(\nu)]) = \delta_{i,j} d_j(\nu), 1 \leq i, j \leq n$ and
$\nu([X_i(\nu), X_j(\nu)]) = 0$, $\nu([Y_i(\nu), Y_j(\nu)]) = 0$ for $1 \leq i, j \leq n$,

(iii) $\mathbb{R}$ span $\{X_1(\nu) \cdots, X_n(\nu), Z_1(\nu), \cdots, Z_r(\nu), T_1, \cdots, T_k\} = \mathfrak{h}_\nu$ is a polarization for $\nu$.

This means the subalgebra $\mathfrak{h}_\nu$ is maximal with respect to the property $\nu([\mathfrak{h}_\nu, \mathfrak{h}_\nu]) = 0$. We call the basis

$$\{X_1(\nu), \cdots, X_n(\nu), Y_1(\nu), \cdots, Y_n(\nu), Z_1(\nu), \cdots, Z_r(\nu), T_1, \cdots, T_k\}$$

almost symplectic basis. Let $\xi_\nu = \mathbb{R}$ span$\{X_1(\nu) \cdots, X_n(\nu)\}$ and $\eta_\nu = \mathbb{R}$ span $\{Y_1(\nu), \cdots, Y_n(\nu)\}$. Then we have the decomposition $\mathfrak{g} = \xi_\nu \oplus \eta_\nu \oplus \mathfrak{r}_\nu \oplus \mathfrak{z}$. We denote the element $\exp(X + Y + Z + T)$ of $G$ by $(X, Y, Z, T)$ for $X \in \xi_\nu, Y \in \eta_\nu, Z \in \mathfrak{r}_\nu, T \in \mathfrak{z}$. Further we can write

$$(X, Y, Z, T) = \sum_{j=1}^{n} x_j(\nu) X_j(\nu) + \sum_{j=1}^{n} y_j(\nu) Y_j(\nu) + \sum_{j=1}^{r} z_j(\nu) Z_j(\nu) + \sum_{j=1}^{k} t_j T_j$$

and denote it by $(x, y, z, t)$ suppressing the dependence of $\nu$ which will be understood from the context. If we take $\lambda \in \Lambda$ then it can be written as $\lambda = (\mu, \nu)$, where $\mu \in \tilde{V}_T^* = \mathbb{R}$ span $\{X_{n_i}^* : 1 \leq i \leq r\}$ and $\nu \in \mathcal{U}$. Therefore, $\lambda = (\mu, \nu) \equiv \sum\limits_{i=1}^{r} \mu_i X_{n_i}^* + \sum\limits_{i=1}^{m} \nu_i T_i^*$. Let $\lambda' \in \mathfrak{g}^*$ such that $\lambda'(X_{j_i}) = 0$ for $1 \leq i \leq 2n$ and the restriction of $\lambda'$ to $V_T^*$ is $\lambda = (\mu, \nu)$. Let $\tilde{\mu}_i = \lambda'(Z_i(\nu))$ and consider the map

$$\phi : \tilde{V}_T^* \to \mathbb{R} \text{ span } \{Z_1(\nu)^*, \cdots, Z_r(\nu)^*\} \tag{2.1.1}$$

given by $\phi(\mu_1, \cdots, \mu_r) = (\tilde{\mu}_1, \cdots, \tilde{\mu}_r)$. Then it has been shown in [28] that $|\det J_\phi| = \frac{|Pf(\nu)|}{d_1(\nu) \cdots d_n(\nu)}$, where $J_\phi$ is the Jacobian matrix of $\phi$ and Pf $(\nu) = \sqrt{\det(B_\nu(j_i, j_s))}$ is called the Pfaffian of $\nu$. Now we want to study the behavior of $d_j(\nu)$ as $\nu \to 0$. We show that $d_j(\nu) \to 0$ as $\nu \to 0$. It is to be noted that $(B_{s\omega}(i,j)) = s(B_\omega(i,j))$ for $s \in (0,\infty)$ and $\omega \in S^{k-1}$. Then it follows that $X_j(s\omega) = X_j(\omega), Y_j(s\omega) = Y_j(\omega)$, $Z_j(s\omega) = Z_j(\omega)$ and $d_j(s\omega) = sd_j(\omega)$ for all $j$. The entries of the matrix $(B_\nu(i,j))$ are continuous functions of $\nu$ and $\pm id_j(\nu)$ being eigenvalues of the matrix, $d_j(\cdot)$ are also continuous in $\nu$. Using the fact that $d_j(s\omega) = sd_j(\omega)$ we conclude $d_j(\nu) \to 0$ as $\nu \to 0$.

We take $\lambda = (\mu,\nu) \in \Lambda$. Since $\lambda|[\mathfrak{h}_\nu, \mathfrak{h}_\nu] = 0$ we define character $\sigma_{\mu,\nu}$ of $H_\nu = \exp\,(h_\nu)$ by

$$\sigma_{\mu,\nu}(X,Z,T) = e^{i\tilde{\mu}(Z)+i\nu(T)}$$

for all $(X,Z,T) \in H_\nu$. For each $\lambda = (\mu,\nu) \in \Lambda$ we construct a new Hilbert space $\mathcal{H}_\lambda$ consisting of $\mathbb{C}$ valued measurable functions $f$ on $G$ such that for all $k \in H_\nu$

$$f(kg) = \sigma_{\mu,\nu}(k)f(g) \tag{2.1.2}$$

and

$$f(0,Y,0,0) \in L^2(\eta_\nu).$$

As any element of $G$ can be written uniquely as $h(0,Y,0,0)$ for some $h \in H_\nu$, we can identify $\mathcal{H}_\lambda$ with $L^2(\eta_\nu)$ . We define an irreducible unitary representation $\pi_{\mu,\nu}$ of $G$ on $\mathcal{H}_\lambda$ by

$$[\pi_{\mu,\nu}(g)f](g') = f(g'g) \tag{2.1.3}$$

for all $f \in \mathcal{H}_\lambda$ and $g, g' \in G$. Since we can write the following product uniquely

$$\begin{aligned} & (0,Y',0,0)(X,Y,Z,T) \\ = \quad & (X,0,Z,(T+[Y'+\frac{1}{2}Y, X-Y'+Z]))(0,Y+Y',0,0,0) \end{aligned}$$

using ( 2.1.3 ) and the identification of $\mathcal{H}_\lambda$ with $L^2(\eta_\nu)$ we get an irreducible unitary representations $\pi_{\mu,\nu}$ of $G$ realized on $L^2(\eta_\nu)$. It is the representation induced by $\sigma_{\mu,\nu}$ and can be described as follows :

$$\left(\pi_{\mu,\nu}(X,Y,Z,T)\phi\right)(Y') = e^{i\nu(T+[Y'+\frac{1}{2}Y,X-Y'+Z])}e^{i\tilde{\mu}(Z)}\phi(Y+Y')$$

for all $\phi \in L^2(\eta_\nu)$. Using the almost symplectic basis we have the following description :

$$\begin{aligned} & \left(\pi_{\mu,\nu}(x,y,z,t)\phi\right)(\xi) \\ = \quad & \exp(i\sum_{j=1}^{k}\nu_j t_j + i\sum_{j=1}^{r}\tilde{\mu}_j z_j + i\sum_{j=1}^{n} d_j(\nu)(x_j\xi_j + \frac{1}{2}x_j y_j))\phi(\xi + y) \end{aligned}$$

for all $\phi \in L^2(\eta_\nu)$.

Define the Fourier transform of $f \in L^1(G)$ by

$$\hat{f}(\mu,\nu) = \int_{\mathfrak{z}}\int_{\mathfrak{r}_\nu}\int_{\eta_\nu}\int_{\xi_\nu} f(x,y,z,t)\pi_{\mu,\nu}(x,y,z,t)\ dx\ dy\ dz\ dt$$

for $\lambda = (\mu,\nu) \in \Lambda$. We let

$$f^\nu(x,y,z) = \int_{\mathfrak{z}} \exp(i\sum_{j=1}^{k}\nu_j t_j) f(x,y,z,t)\ dt,$$

$$f^{\tilde{\mu},\nu}(x,y) = \int_{\mathfrak{r}_\nu}\int_{\mathfrak{z}} \exp(i\sum_{j=1}^{k}\nu_j t_j + i\sum_{j=1}^{r}\tilde{\mu}_j z_j) f(x,y,z,t)\ dt\ dz$$

for all $\tilde{\mu} \in \mathfrak{r}_\nu^*, \nu \in \mathfrak{z}^*$. If $f \in L^1 \cap L^2(G)$ then $\hat{f}(\mu,\nu)$ is an Hilbert–Schmidt operator. For all $\tilde{\mu} \in \mathfrak{r}_\nu^*$ we have

$$(2\pi)^{-n}\prod_{j=1}^{n} d_j(\nu)\|\hat{f}(\tilde{\mu},\nu)\|_{\mathrm{HS}}^2 = \int_{\eta_\nu}\int_{\xi_\nu} |f^{\tilde{\mu},\nu}(x,y)|^2\ dx\ dy. \quad (2.1.4)$$

Now applying Plancherel formula in the variable $\tilde{\mu}$ we get

$$\begin{aligned} & (2\pi)^{-(n+r)}\prod_{j=1}^{n} d_j(\nu)\int_{\mathfrak{r}_\nu^*}\|\hat{f}(\tilde{\mu},\nu)\|_{\mathrm{HS}}^2 d\tilde{\mu} \quad (2.1.5) \\ = \quad & \int_{\mathfrak{r}_\nu}\int_{\eta_\nu}\int_{\xi_\nu} |f^\nu(x,y,z)|^2\ dx\ dy\ dz \\ = \quad & \int_{\mathfrak{v}} |f^\nu(x,y,z)|^2\ dx\ dy\ dz. \end{aligned}$$

Polarizing this identity

$$\begin{aligned} & (2\pi)^{-(n+r)}\prod_{j=1}^{n} d_j(\nu)\int_{\mathfrak{r}_\nu^*} \mathrm{tr}\left(\hat{f}(\tilde{\mu},\nu)\hat{g}(\tilde{\mu},\nu)^*\right) d\tilde{\mu} \quad (2.1.6) \\ = \quad & \int_{\mathfrak{v}} f^\nu(x,y,z)\overline{g^\nu(x,y,z)}\ dx\ dy\ dz. \end{aligned}$$

Using the change of variables defined by the function $\phi$ in ( 2.1.1 ) we get

$$(2\pi)^{-(n+r)}\mathrm{Pf}(\nu)\int_{\tilde{V}_T^*}\|\hat{f}(\mu,\nu)\|_{\mathrm{HS}}^2\,d\mu \tag{2.1.7}$$
$$= \int_{\mathfrak{v}}|f^\nu(x,y,z)|^2\,dx\,dy\,dz$$

and

$$(2\pi)^{-(n+r)}\mathrm{Pf}(\nu)\int_{\tilde{V}_T^*}\mathrm{tr}\left(\hat{f}(\mu,\nu)\hat{g}(\mu,\nu)^*\right)d\mu \tag{2.1.8}$$
$$= \int_{\mathfrak{v}}f^\nu(x,y,z)\overline{g^\nu(x,y,z)}\,dx\,dy\,dz.$$

The Plancherel formula takes the following form:

$$\int_\Lambda\|\hat{f}(\mu,\nu)\|_{\mathrm{HS}}^2\,\mathrm{Pf}(\nu)d\nu\,d\mu = \int_G|f(x,y,z,t)|^2\,dx\,dy\,dz\,dt.$$

For $g\in L^2(\mathfrak{v})$, define the Hilbert–Schmidt operator $W_{\mu,\nu}(g)$ by

$$W_{\mu,\nu}(g) = \int_{\mathfrak{v}}g(x,y,z)\pi_{\mu,\nu}(x,y,z,0)\,dx\,dy\,dz. \tag{2.1.9}$$

With this notation, for all $g,h\in L^2(\mathfrak{v})$ we have

$$(2\pi)^{-(n+r)}\mathrm{Pf}(\nu)\int_{\tilde{V}_T^*}\mathrm{tr}(W_{\mu,\nu}(g)W_{\mu,\nu}(h)^*)\,d\mu \tag{2.1.10}$$
$$= \int_{\mathfrak{v}}g(x,y,z)\overline{h(x,y,z)}\,dx\,dy\,dz.$$

If $g$ is a Schwartz function then it has been shown in [1] that $|W_{\mu,\nu}(g)|$ is a trace class operator and $\mathrm{tr}(|W_{\mu,\nu}(g)|)$ can be estimated in terms of Schwartz semi norms of $g$. In fact $(1+|\mu|)^k\mathrm{tr}(|W_{\mu,\nu}(g)|)\le Cr(\nu)^l\|g\|_*$ for some $l,k>0$, where $r(\nu)=\sum_{j=1}^{n}(d_j(\nu)^2+d_j(\nu)^{-2})$ and $\|g\|_*$ is a suitable Schwartz seminorm.

*Step two groups with MW condition*

In this case the representations are parameterized by the Zariski open set $\Lambda = \{\nu \in \mathfrak{z}^* : B_\nu \text{ is nondegenerate}\}$ and is given by:

$$\begin{aligned} & (\pi_\nu(x,y,t)\phi)(\xi) \\ = \; & \exp(i\sum_{j=1}^{k}\nu_j t_j + i\sum_{j=1}^{n} d_j(\nu)(x_j\xi_j + \frac{1}{2}x_j y_j))\phi(\xi+y) \end{aligned} \quad (2.1.11)$$

for all $\phi \in L^2(\eta_\nu)$. Since $B_\nu$ is nondegenerate it is clear that $\text{Pf}(\nu) = \prod_{j=1}^{n} d_j(\nu)$. Define the Fourier transform of $f \in L^1(G)$ by

$$\hat{f}(\nu) = \int_{\mathfrak{z}}\int_{\eta_\nu}\int_{\xi_\nu} f(x,y,t)\pi_\nu(x,y,t)\ dx\ dy\ dt$$

for all $\nu \in \Lambda$. Also define

$$f^\nu(x,y) = \int_{\mathfrak{z}} \exp(i\sum_{j=1}^{k}\nu_j t_j) f(x,y,t)\ dx\ dy\ dt$$

for all $\nu \in \Lambda$. For each $\nu \in \Lambda$ and $g \in L^1 \bigcap L^2(\xi_\nu \oplus \eta_\nu)$, define the operator

$$W'_\nu(g) = \int_{\eta_\nu}\int_{\xi_\nu} g(x,y)\pi_\nu(x,y,0)\ dx\ dy. \quad (2.1.12)$$

Then $W'_\nu(g)$ is an integral operator with kernel

$$K_\nu(\xi,y) = \int_{\xi_\nu} \exp(i\frac{1}{2}\sum_{j=1}^{n} d_j(\nu)\,(\xi_j+y_j)\,x_j) g(x,y-\xi)\ dx$$

which is in $L^2(\eta_\nu)\otimes L^2(\eta_\nu)$. Moreover if $g \in L^2(\xi_\nu \oplus \eta_\nu)$ then $W'_\nu(g)$ is a Hilbert-Schmidt operator and we have the Plancherel theorem

$$\text{Pf}\ (\nu)\ \|W'_\nu(g)\|^2_{\text{HS}} = (2\pi)^n \int_{\eta_\nu}\int_{\xi_\nu} |g(x,y)|^2\ dx\ dy.$$

Polarizing this identity, we obtain

$$\begin{aligned} & \text{Pf}\ (\nu)\ \text{tr}\left(W'_\nu(g)^* W'_\nu(h)\right) \\ = \; & (2\pi)^n \int_{\eta_\nu}\int_{\xi_\nu} g(x,y)\bar{h}(x,y)\ dx\ dy. \end{aligned} \quad (2.1.13)$$

If $f \in L^1 \cap L^2(G)$ then from the definition of $\hat{f}(\nu)$ we have $\hat{f}(\nu) = W'_\nu(f^\nu)$ and hence

$$\mathrm{Pf}(\nu)\|\hat{f}(\nu)\|^2_{\mathrm{HS}} = (2\pi)^n \int_{\eta_\nu}\int_{\xi_\nu} |f^\nu(x,y)|^2 \, dx \, dy \tag{2.1.14}$$

and

$$\begin{aligned} &\mathrm{Pf}(\nu)\left(\mathrm{tr}\hat{f}(\nu)\hat{g}(\nu)^*\right) \qquad (2.1.15)\\ = \;& (2\pi)^n \int_{\eta_\nu}\int_{\xi_\nu} f^\nu(x,y)\overline{g^\nu(x,y)} \, dx \, dy. \end{aligned}$$

## 2.2 Step two stratified groups

The two step Lie algebra $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ is called stratified if $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$ and the corresponding group is called stratified group. If the group $G$ is stratified then it admits a natural family of dilations so it is a homogeneous group. Then there exists a smooth function $p_a(v,t)$ on $G \times (0,\infty)$ such that $f * p_a(v,t)$ solves the heat equation associated with the sublaplacian $\mathcal{L}$ with initial condition $f$, see Folland-Stein [13]. This $p_a$ is called the heat kernel associated with $\mathcal{L}$. In this section we will prove heat kernel versions of Hardy and Cowling-Price theorems for all step two stratified groups.

### 2.2.1 Uncertainty principles on step two stratified groups without MW-condition

If we write the sublaplacian with respect to an almost symplectic basis it takes the normal form

$$\mathcal{L} = -\sum_{j=1}^{n}\left(X_j^2(\nu) + Y_j^2(\nu)\right) - \sum_{j=1}^{r} Z_j^2(\nu). \tag{2.2.1}$$

We define

$$H(\tilde{\mu}, d(\nu)) = \sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial \xi_j^2} + d_j^2(\nu)\xi_j^2\right) + |\tilde{\mu}|^2. \tag{2.2.2}$$

Using the explicit form of the representations we can calculate that

$$\pi_{\mu,\nu}(\mathcal{L}) = H(\tilde{\mu}, d(\nu))$$

where $d(\nu) = (d_1(\nu), \cdots, d_n(\nu))$. Also we have

$$H(\tilde{\mu}, d(\nu))U(d(\nu))\Phi_\alpha = \left( |\tilde{\mu}|^2 + \sum_{j=1}^{n} (2\alpha_j + 1) d_j(\nu) \right) U(d(\nu))\Phi_\alpha$$

for all $\alpha \in \mathbb{N}^n$. Let $\Phi_\alpha^\nu = U(d(\nu))\Phi_\alpha$ and

$$\Phi_{\alpha,\beta}^\nu(x,y) = (2\pi)^{-\frac{n}{2}} (\prod_{j=1}^{n} d_j(\nu))^{\frac{1}{2}} (\pi_{0,\nu}(x,y,0,0)\Phi_\alpha^\nu, \Phi_\beta^\nu). \tag{2.2.3}$$

Then $\{\Phi_{\alpha,\beta}^\nu : \alpha, \beta \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\xi_\nu \oplus \eta_\nu)$. For a detailed account we refer to [28]. It can be proved as in the case of $H_d^n$ that

$$\hat{p}_a(\mu, \nu) = e^{-aH(\tilde{\mu}, d(\nu))}. \tag{2.2.4}$$

The heat kernel is explicitly given by

$$\begin{aligned} p_a^\nu(v) &= (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} e^{-\frac{1}{4a}\sum\limits_{j=1}^{r} \langle v, Z_j(\nu)\rangle^2} \\ &\quad \times \prod_{j=1}^{n} \left( \frac{d_j(\nu)}{\sinh a d_j(\nu)} \right) e^{-\frac{1}{4} d_j(\nu) \coth a d_j(\nu) \left( \langle v, X_j(\nu)\rangle^2 + \langle v, Y_j(\nu)\rangle^2 \right)}. \end{aligned}$$

Writing $\nu = s\omega$ for $s \in (0, \infty)$ and $\omega \in S^{k-1}$ we can compute $\lim_{s\to 0} p_a^{s\omega}(v)$ using the fact that $X_j(s\omega) = X_j(\omega), Y_j(s\omega) = Y_j(\omega), Z_j(s\omega) = Z_j(\omega)$ and $d_j(s\omega) = s d_j(\omega)$ for all $j$. We get

$$\begin{aligned} p_a^0(v) &= (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} a^{-n} e^{-\frac{1}{4a}\left( \sum\limits_{i=1}^{r} \langle v, Z_j(\omega)\rangle^2 + \sum\limits_{j=1}^{n} \langle v, X_j(\omega)\rangle^2 + \sum\limits_{j=1}^{n} \langle v, Y_j(\omega)\rangle^2 \right)} \\ &= (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} a^{-n} e^{-\frac{1}{4a}(\sum\limits_{j=1}^{m} \langle v, e_j\rangle^2)} e^{-\frac{1}{4a}(\sum\limits_{j=1}^{k} \langle v, T_j\rangle^2)} \\ &= (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} a^{-n} e^{-\frac{1}{4a}|v|^2}. \end{aligned} \tag{2.2.5}$$

Writing $v = (x(\nu), y(\nu), z(\nu)) \in \xi_\nu \oplus \eta_\nu \oplus r_\nu = \mathfrak{v}$,

$$p_a^0(x(\nu), y(\nu), z(\nu)) = (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} a^{-n} e^{-\frac{1}{4a}(|x(\nu)|^2 + |y(\nu)|^2 + |z(\nu)|^2)}.$$

With this preparation, we have the following versions of Hardy and Cowling–Price theorems.

**Theorem 2.2.1.** *Let $G$ be a step two stratified group without MW condition. Let $f$ be a function on $G$ such that*

*(i)* $|f(v,t)| \leq C(1+|v|)^l p_a(v,t)$,

*(ii)* $\hat{f}(\mu,\nu)^* \hat{f}(\mu,\nu) \leq C\,(\pi_{\mu,\nu}(\mathcal{L}))^l\,\hat{p}_{2b}(\mu,\nu)$ *for every* $(\mu,\nu) \in \Lambda$,

*where $l \geq 0$. Then $f = 0$ almost everywhere whenever $a < b$.*

*Proof.* Let $f^*(x,y,z,t) = \bar{f}(-x,-y,-z,-t)$ and consider the function

$$h_\nu(z) = \int_{\xi_\nu}\int_{\eta_\nu} f^\nu *_3 f^{*^\nu}(x,y,z)\ dx\ dy,$$

where $*_3$ means convolution is taken in the third variable. Since $|f^\nu(x,y,z)| \leq C(1+|(x,y,z)|)^l p_a^0(x,y,z)$ we have the following estimate on $h$ :

$$|h_\nu(z)| \leq C e^{-\frac{1}{8a'}|z|^2}$$

for any $a < a' < b$. Also

$$\begin{aligned}\hat{f}(\mu,\nu)^*\hat{f}(\mu,\nu) &\leq C(\pi_{\mu,\nu}(\mathcal{L}))^l \hat{p}_{2b}(\mu,\nu)\\ &= (H(\tilde{\mu},d(\nu)))^l\, e^{-2bH(\tilde{\mu},d(\nu))}\end{aligned}$$

for every $\mu \in \tilde{V}_T^*$ and $\nu \in \mathcal{U}$ from which we get

$$\begin{aligned}\|\hat{f}(\mu,\nu)\|_{\mathrm{HS}}^2 &\leq C\ e^{-2b|\tilde{\mu}|^2}\left(\sum_{j=1}^n(2\alpha_j+1)d_j(\nu)+|\tilde{\mu}|^2\right)^l \prod_{j=1}^n\sum_{\alpha_j\in\mathbb{N}} e^{-2b(2\alpha_j+1)d_j(\nu)}\\ &\leq C_\nu\ e^{-2b'|\tilde{\mu}|^2}\end{aligned}$$

for any $a < a' < b' < b$. Therefore,

$$\begin{aligned}\hat{h}_\nu(\tilde{\mu}) &= \int_{\xi_\nu}\int_{\eta_\nu} |f^{\tilde{\mu},\nu}(x,y)|^2\ dx\ dy\\ &= (2\pi)^{-n}\prod_{j=1}^n d_j(\nu)^{-1}\|\hat{f}(\mu,\nu)\|_{\mathrm{HS}}^2\\ &\leq C_\nu\ e^{-2b'|\tilde{\mu}|^2}.\end{aligned}$$

Now applying Hardy's theorem on $\mathfrak{r}_\nu$ we conclude that $h_\nu = 0$ and hence $\hat{f}(\mu,\nu) = 0$ for all $(\mu,\nu) \in \Lambda$, whenever $a < b$. Therefore, $f = 0$ almost everywhere for $a < b$. □

**Theorem 2.2.2.** *Let $G$ be a step two stratified group without MW condition. Let $f$ be a function on $G$ such that*

*(i)* $(1+|v|)^{-l} f p_a^{-1} \in L^p(G)$,

*(ii)* $\hat{f}(\mu,\nu) = S(\mu,\nu)\,(\pi_{\mu,\nu}(\mathcal{L}))^l\, \hat{p}_b(\mu,\nu)$

*with $S(\mu,\nu) \in \mathcal{S}_q$ for every $(\mu,\nu) \in \Lambda$ , where $l > 0$, $1 \le p,q \le \infty$. Then $f = 0$ almost everywhere whenever $a < b$.*

*Proof.* Using the explicit formula for $p_a^\nu(v)$ it can be proved as in the case of $H_d^n$ that $e^{\frac{1}{4a'}|v|^2} f^\nu \in L^p(\mathfrak{v})$ for any $a < a' < b$. Let $g(v,t) = e^{-\alpha|t|^2} h(v)$, where $\alpha > 0$ and $h$ is a smooth function with supp $h \subset \{v : |v| < \delta\}$. Choose $a''$ such that $a' < a'' < b$. Then for all $v \in \mathfrak{v}$ with $|v| > \frac{\delta\sqrt{a''}}{\sqrt{a''}-\sqrt{a'}}$ and $v' \in$ supp $h$ we have $|v-v'| \ge |v|-|v'| > |v|-\delta > |v|\sqrt{\frac{a'}{a''}}$. Since $g^\nu \in L^p(\mathfrak{v})$ for all $p$ we get by Hölder's inequality a constant $C > 0$ such that

$$\begin{aligned} C &\ge \int_{\mathfrak{v}} e^{\frac{1}{4a'}|v-v'|^2} |f^\nu(v-v')||g^\nu(v')|dv' \\ &\ge e^{\frac{1}{4a''}|v|^2} \int_{\mathfrak{v}} |f^\nu(v-v')|\;|g^\nu(v')|dv' \end{aligned}$$

for all $v$ with $|v| > \frac{\delta\sqrt{a''}}{\sqrt{a''}-\sqrt{a'}}$ . From the continuity of the function $(f*g)^\nu$ it follows that

$$\begin{aligned} |(f*g)^\nu(v)| &= |\int_{\mathfrak{v}} f^\nu(v-v') g(v') e^{\frac{i}{2}\nu([v,v'])}\, dv'| \\ &\le \int_{\mathfrak{v}} |f^\nu(v-v')||g^\nu(v')|dv' \\ &\le Ce^{-\frac{1}{4a''}|v|^2} \end{aligned}$$

for all $v \in \mathfrak{v}$. Since

$$H(\tilde{\mu}, d(\nu))U(d(\nu))\Phi_\alpha = \left( |\tilde{\mu}|^2 + \sum_{j=1}^n (2\alpha_j+1)d_j(\nu) \right) \Phi_\alpha$$

for all $\alpha \in \mathbb{N}$, from the hypothesis on $\hat{f}(\mu,\nu)$ it follows that for some constant $C > 0$ and $a'' < b' < b$,

$$\|\hat{f}(\mu,\nu)U(d(\nu))\Phi_\alpha\|_2 \leq C_\nu e^{-b'|\tilde{\mu}|^2}$$

for all $\alpha \in \mathbb{N}$. This shows that $\|\hat{f}(\mu,\nu)\|^2_{\mathrm{HS}} \leq Ce^{-2b'|\tilde{\mu}|^2}$. Therefore,

$$\begin{aligned} \|\widehat{f*g}(\mu,\nu)\|_{\mathrm{HS}} &\leq \|\hat{g}(\mu,\nu)\|_{\mathrm{op}}\|\hat{f}(\mu,\nu)\|_{\mathrm{HS}} \\ &\leq Ce^{-b'|\tilde{\mu}|^2}. \end{aligned}$$

From the proof of the previous theorem we conclude that $f * g = 0$ as $a < a'' < b$. Let $g(v,t) = (2\pi)^{-\frac{k}{2}}h(v)e^{-\frac{|t|^2}{2}}$ where $h$ is a compactly supported smooth function and $\int_{\mathfrak{v}} h(v)\ dv = 1$. Let $g_\epsilon(v,t) = \epsilon^{-(2n+k+r)}g(\frac{v}{\epsilon},\frac{t}{\epsilon})$ for $\epsilon > 0$. Then $\{g_\epsilon\}_{\epsilon>0}$ form an approximate identity. Since $f * g_\epsilon(v,t) = 0$ for all $\epsilon > 0$ whenever $a < b$, it follows that $f = 0$ for $a < b$. □

### 2.2.2 Uncertainty principles on step two stratified groups with MW condition

In this subsection we assume $G$ to be step two stratified group with MW condition. If we write the sublaplacian with respect to the symplectic basis it takes the normal form

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j(\nu)^2 + Y_j(\nu)^2)$$

and

$$\pi_\nu(\mathcal{L}) = \sum_{j=1}^{n}\left(-\frac{d^2}{d\xi_j^2} + d_j(\nu)^2\xi_j^2\right) = H(d(\nu)). \tag{2.2.6}$$

If $p_a$ denotes the heat kernel associated to $\mathcal{L}$ then

$$\hat{p}_a(\nu) = e^{-aH(d(\nu))} \tag{2.2.7}$$

and

$$\begin{aligned} &p_a^\nu(x(\nu),y(\nu)) \qquad\qquad (2.2.8)\\ = \quad &(4\pi)^{-n}\prod_{j=1}^{n}\left(\frac{d_j(\nu)}{\sinh\ ad_j(\nu)}\right)e^{-\frac{1}{4}d_j(\nu)(\coth\ ad_j(\nu))(x_j(\nu)^2+y_j(\nu)^2)}. \end{aligned}$$

From the work of Jerison and Sánchez-Calle [18] it is known that $p_a(v,t) \leq Ce^{-\frac{A}{a}|(v,t)|^2}$ for some $A > 0$, where $|\cdot|$ denotes a homogeneous norm on $G$. As in the case if $H_d^n$ we define the Fourier–Weyl transform of a function $f$ by

$$\hat{f}(\nu, (\xi', \xi'')) = \pi_\nu(\xi', \xi'', 0)\hat{f}(\nu)\pi_\nu(\xi', \xi'', 0)^*,$$

for $(\xi', \xi'') \in \xi_\nu \oplus \eta_\nu$.

**Theorem 2.2.3.** *Let $G$ be a step two stratified group with MW–condition and $f$ be a function on $G$ such that*

*(i)* $(1 + |v|)^{-l} f p_a^{-1} \in L^p(G)$,

*(ii)* $\hat{f}(\nu) = S(\nu)\,(\pi_\nu(\mathcal{L}))^l\,\hat{p}_b(\nu)$, $S(\nu) \in \mathcal{S}_q$ *for every* $\nu \in \Lambda$,

*where $l \geq 0$ and $1 \leq p, q \leq \infty$. Then $f = 0$ almost everywhere whenever $a < b$.*

*Proof.* Since any two homogeneous norms are equivalent, using the estimate of $p_a$ in terms of the homogeneous norm obtained from natural dilation, it can be shown that $f^\nu$ can be extended as a holomorphic function of $\nu$ in a strip of $\mathbb{C}^k$ and also using the explicit expression of $p_a^\nu$ we can conclude that $f^\nu(x,y)e^{\frac{1}{4a'}(|x|^2+|y|^2)} \in L^p(\xi_\nu \oplus \eta_\nu)$ for any $a' > a$. Let us compute $\pi_\nu(\xi', \xi'', 0)^* U(d(\nu))\Phi_0$ using the fact that $\{U(d(\nu))\Phi_\alpha : \alpha \in \mathbb{N}\}$ forms an orthonormal basis for $L^2(\eta_\nu)$. Therefore, we have

$$\begin{aligned}
& \pi_\nu(\xi', \xi'', 0)^* U(d(\nu))\Phi_0(x) \\
= & \sum_\alpha \langle \pi_\nu(\xi', \xi'', 0)^* U(d(\nu))\Phi_0, U(d(\nu))\Phi_\alpha \rangle U(d(\nu))\Phi_\alpha(x) \\
= & \sum_\alpha \langle U(d(\nu))^* \pi_\nu(\xi', \xi'', 0)^* U(d(\nu))\Phi_0, \Phi_\alpha \rangle U(d(\nu))\Phi_\alpha(x) \\
= & \sum_\alpha \langle W(-\sqrt{d(\nu)}(\xi' + i\xi''))\Phi_0, \Phi_\alpha \rangle U(d(\nu))\Phi_\alpha(x) \\
= & \sum_\alpha \bar{\Phi}_{\alpha,0}\left(\sqrt{d(\nu)}(\xi' + i\xi'')\right) U(d(\nu))\Phi_\alpha(x).
\end{aligned}$$

Let $\phi_\nu = U(d(\nu))\Phi_0$, then $\langle \pi_\nu(\xi', \xi'', 0)\phi_\nu, \phi_\nu\rangle = \prod_{j=1}^{n} e^{-\frac{1}{4}d_j(\nu)({\xi'_j}^2+{\xi''_j}^2)}$. Now

$$\begin{aligned} & |\langle \hat{f}(\nu, (\xi', \xi''))\phi_\nu, \phi_\nu\rangle| \qquad (2.2.9)\\ = \ & |\langle \pi_\nu(\xi' + i\xi'', 0)\hat{f}(\nu)\pi_\nu(\xi' + i\xi'', 0)^* U(d(\nu))\Phi_0, U(d(\nu))\Phi_0\rangle| \\ \leq \ & \|\hat{f}(\nu)\pi_\nu(\xi' + i\xi'', 0)^* U(d(\nu))\Phi_0\|_2 \\ \leq \ & C\sum_\alpha |\Phi_{\alpha,0}(\sqrt{d(\nu)}(\xi' + i\xi'')| \ \|\hat{f}(\nu)U(d(\nu))\Phi_\alpha\|_2. \end{aligned}$$

From the hypothesis on $\hat{f}(\nu)$, we get $C > 0$, such that for all $\alpha \in \mathbb{N}$

$$\begin{aligned} & \|\hat{f}(\nu)U(d(\nu))\Phi_\alpha\|_2 \qquad (2.2.10)\\ \leq \ & C\left(\sum_{j=1}^{n}(2\alpha_j + 1)d_j(\nu)\right)^l \prod_{j=1}^{n} e^{-b(2\alpha_j+1)d_j(\nu)}. \end{aligned}$$

Using the above estimate for $\|\hat{f}(\nu)U(d(\nu))\Phi_\alpha\|_2$ in (2.2.2) and proceeding as in the case of $H_d^n$ we see that for any $b'$ with $a' < b' < b$

$$\begin{aligned} & |\langle \hat{f}(\nu, (\xi', \xi''))\phi_\nu, \phi_\nu\rangle| \qquad (2.2.11)\\ \leq \ & C\prod_{j=1}^{n} e^{-\frac{1}{4}d_j(\nu)\left(1-e^{-4b'd_j(\nu)}\right)({\xi'_j}^2+{\xi''_j}^2)}. \end{aligned}$$

For each $\nu \in \Lambda$ we define the function $g_\nu$ on $\mathfrak{v} = \xi_\nu \oplus \eta_\nu$ by

$$g_\nu(x, y) = f^\nu(x, y)\langle \pi_\nu(x, y, 0)\phi_\nu, \phi_\nu\rangle.$$

Then

$$\begin{aligned} & \int_{\xi_\nu\oplus\eta_\nu}\int_{\xi_\nu\oplus\eta_\nu} |g_\nu(x, y)| \ |\widehat{g}_\nu(\xi', \xi'')| e^{|x\cdot\xi' + y\cdot\xi''|} dx\ dy\ d\xi'\ d\xi'' \\ \leq \ & \int_{\xi_\nu\oplus\eta_\nu}\int_{\xi_\nu\oplus\eta_\nu} |f^\nu(x, y)| \ |\langle \hat{f}(\nu, (\xi', \xi''))\phi_\nu, \phi_\nu\rangle| e^{\sum_{j=1}^{n} d_j(\nu)(|x_j\xi''_j| + |y_j\xi'_j|)} dx\ dy\ d\xi'\ d\xi'' \\ \leq \ & C\int_{\xi_\nu\oplus\eta_\nu}\int_{\xi_\nu\oplus\eta_\nu} |f^\nu(x, y)| \ e^{\frac{1}{4a'}(|x|^2+|y|^2)} e^{-\frac{1}{4a}|((x,y)-2a'd(\nu)(\xi'',\xi'))|^2} \times \\ & \prod_{j=1}^{n} e^{-d_j(\nu)\left(\frac{1}{4}\left(1-e^{-4b'd_j(\nu)}\right)-ad_j(\nu)\right)({\xi'_j}^2+{\xi''_j}^2)} dx\ dy\ d\xi'\ d\xi''. \end{aligned}$$

Since $f^\nu e^{\frac{1}{4a'}(|x|^2+|y|^2)} \in L^p(\mathfrak{v})$ the above integral will be finite if $a'd_j(\nu) < \frac{1}{4}\left(1-e^{-4b'd_j(\nu)}\right)$ for all $j$. If we apply Theorem 1.2.6 to the function it yields $f^\nu = 0$ almost everywhere if $a'd_j(\nu) < \frac{1}{4}\left(1-e^{-4b'd_j(\nu)}\right)$ as the function $\langle \pi_\nu(x,y,0)\phi_\nu, \phi_\nu\rangle$ is non vanishing. We recall that $d_j(\nu) \to 0$ as $\nu \to 0$ for all $j$. Since $a < a' < b' < b$, using the above fact we can choose $\delta > 0$ such that $a'd_j(\nu) < \frac{1}{4}\left(1-e^{-4b'd_j(\nu)}\right)$ for all $j$ and for all $|\nu| < \delta$ as in the proof of Theorem 1.2.26. This means $f^\nu = 0$ almost everywhere for $|\nu| < \delta$. We have already observed that good estimate of the heat kernel allowed to extend $f^\nu$ as a holomorphic function of $\nu$ in a strip of $\mathbb{C}^k$. Finally we have $f = 0$ almost everywhere since for all $\nu \in \Lambda$, $f^\nu(x,y) = 0$ almost everywhere in $(x,y)$. $\square$

In the above proof we have used the estimate $\|\hat{f}(\nu)U(d(\nu))\Phi_\alpha\|_2^2 \leq C\left(\sum_{j=1}^{n}(2d_j(\nu)+1)\right)^l \prod_{j=1}^{n} e^{-2b(2\alpha_j+1)d_j(\nu)}$ which is also true if we assume that $\hat{f}(\nu)^*\hat{f}(\nu) \leq C\pi_\nu(\mathcal{L}))^l\hat{p}_{2b}(\nu)$. Therefore, we have the following version of Hardy's theorem.

**Theorem 2.2.4.** *Let $G$ be a stratified step two group satisfying MW - condition and $f$ be a function on $G$ satisfying*

(*i*) $|f(v,t)| \leq C(1+|v|)^l p_a(v,t)$,

(*ii*) $\hat{f}(\nu)^* f(\nu) \leq C(\pi_\nu(\mathcal{L}))^l \hat{p}_{2b}(\nu)$, *for some* $l > 0$.

*Then $f = 0$ almost everywhere whenever $a < b$.*

## 2.3 Uncertainty principles on general step two groups

The main purpose of this section is to extend the results, in the context of step two groups, proved in the previous chapter . For general step two groups we find an alternative condition on $f$ and prove Hardy and Cowling–Price theorems. Also an analogue of the result of Bonami et al [5] will be proved. Ray in [28] has proved Hardy, Cowling–Price and an analogue of

Theorem 1.2.7 . The equality case has been left open. Another version of Hardy's theorem (inequality case) was proved in [1]. We treat here the equality case.

### 2.3.1 Uncertainty Principles on step two groups without MW condition

For a stratified group the associated sublaplacian is a positive Rockland operator (see [13], 4.20, page–130,). Then Theorem 4.25 of [13] asserts that this operator generates a diffusion semigroup with kernel $p_a(v,t)$. Such results are not available for general step two groups. We are looking for an alternative condition on $f$ and $\hat{f}(\mu,\nu)$ to formulate heat kernel versions of Hardy and Cowling–Price theorems for general step two groups. Let us first introduce Radon transform of functions on $\mathbb{R}^k$. For suitable function $g$ on $\mathbb{R}^k$ the Radon transform $Rg$ is a function on $\mathbb{R}\times S^{k-1}$, defined by

$$Rg(\omega,r) = R_\omega g(r) = \int_{x\cdot\omega=r} g(x)\ d\sigma, \tag{2.3.12}$$

where $d\sigma$ denotes the $(k-1)$–dimensional Lebesgue measure on the hyperplane $\{x : x\cdot\omega = r\}$. It is known that

$$R_\omega g(r) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-isr}\int_{\mathbb{R}^k} e^{is\langle\omega,t\rangle} g(t)\ dt\ ds. \tag{2.3.13}$$

For stratified step two group the condition $|f(v,t)| \leq Cp_a(v,t)$ implies $|R_\omega f(v,r)| \leq CR_\omega p_a(v,r)$. Writing $v = (x(\omega), y(\omega), z(\omega))$, using the above notation and the formula ( 2.3.13 ) we have

$$\begin{aligned} & R_\omega p_a(x(\omega),y(\omega),z(\omega),r) \\ = \ & C_n\int_{-\infty}^{\infty} e^{-irs}p_a^{s\omega}(x(\omega),y(\omega),z(\omega))\ ds \\ = \ & C_n e^{-\frac{1}{4a}|z(\omega)|^2}\int_{-\infty}^{\infty} e^{-irs}q^s_{a,d(\omega)}(x(\omega),y(\omega))\ ds \\ = \ & C_n e^{-\frac{1}{4a}|z(\omega)|^2} q_{a,d(\omega)}(x(\omega),y(\omega),s) \end{aligned} \tag{2.3.14}$$

So it is natural to formulate Hardy's theorem as follows:

**Theorem 2.3.1.** *Let $G$ be a step two group without MW condition. Let $f$ be a function on $G$ such that for every $\omega \in S^{k-1}$*

(*i*) $|R_\omega f(x(\omega), y(\omega), z(\omega), r)| \leq C(1 + |(x(\omega), y(\omega), z(\omega))|)^l \times e^{-\frac{1}{4a}|z(\omega)|^2} q_{a,d(\omega)}\left(x(\omega) + iy(\omega), r\right),$

(*ii*) $\hat{f}(\mu, \nu)^* \hat{f}(\mu, \nu) \leq C\left(\pi_{\mu,\nu}(\mathcal{L})\right)^l e^{-2b\pi_{\mu,\nu}(\mathcal{L})}$ *for every* $(\mu, \nu) \in \Lambda$*, where* $l \geq 0$.

*Then $f = 0$ almost everywhere whenever $a < b$.*

*Proof.* We have the formula

$$f^{r\omega}(x(\omega), y(\omega), z(\omega)) = \int_{-\infty}^{\infty} e^{irs} R_\omega f((x(\omega), y(\omega), z(\omega)), s))\, ds$$

for all $r \in (0, \infty), \omega \in S^{k-1}$. Then from the first hypothesis of the theorem we have

$$|f^{r\omega}(x(\omega), y(\omega), z(\omega))| \leq Ce^{-\frac{1}{4a'}(|x(\omega)^2| + |y(\omega)|^2 + |z(\omega)|^2)}$$

for any $a < a' < b$. Now proceeding as in the proof of Theorem 2.2.1 we get the desired result. □

Also we have the following version of Cowling–Price theorem:

**Theorem 2.3.2.** *Let $G$ be a step two group without MW–condition. Let $f$ be a function on $G$. Let us define the function $g_\omega$ by*

$$\begin{aligned} g_\omega(x(\omega), y(\omega), z(\omega), r) &= R_\omega f(x(\omega), y(\omega), z(\omega), r) e^{\frac{1}{4a}|z(\omega)|^2} \\ &\quad \times \left(q_{a,d(\omega)}\right)^{-1} (x(\omega) + iy(\omega), r)\,. \end{aligned}$$

*Suppose*

(*i*) $(1 + |(x(\omega), y(\omega), z(\omega))|)^{-l} g_\omega$ *belongs to* $L^p(\mathfrak{v} \times \mathbb{R})$,

(*ii*) $\hat{f}(\mu, \nu) = S(\mu, \nu)\left(\pi_{\mu,\nu}(\mathcal{L})\right)^l e^{-b\pi_{\mu,\nu}(\mathcal{L})}$,

*$S(\mu,\nu) \in \mathcal{S}_q$ for every $(\mu,\nu) \in \Lambda$, where $l > 0$ and $1 \leq p, q \leq \infty$. Then $f = 0$ almost everywhere whenever $a < b$.*

*Proof.* It is easy to see that for each $e^{\frac{1}{4a'}|\cdot|^2} f^{r\omega}(\cdot) \in L^p(\mathfrak{v})$ for any $a < a'$, for all $r \in (0,\infty)$ and $\omega \in S^{k-1}$ as in the case of $H_d^n$. Now the rest of the proof will be same as Theorem 2.2.2. □

We define Fourier–Weyl transform $\hat{f}((\mu,\nu),\xi)$ of a function $f$ on $G$ by

$$\hat{f}((\mu,\nu),\xi) = \pi_{\mu,\nu}(\xi',\xi'',0,0)\hat{f}(\mu,\nu)\pi_{\mu,\nu}(\xi',\xi'',0,0)^*$$

for all $\xi = (\xi',\xi'') \in \xi_\nu \oplus \eta_\nu$. A simple calculation shows that

$$\begin{aligned} &\hat{f}((\mu,\nu),(\xi',\xi'')) \\ = &\int_{\eta_\nu}\int_{\xi_\nu} \exp(i\sum_{j=1}^n d_j(\nu)(x_j\xi_j'' - y_j\xi_j'))f^{\tilde{\mu},\nu}(x,y)\pi_{0,\nu}(x,y,0,0)\,dx\,dy. \end{aligned}$$

We have the following theorem which is analogue of Theorem 1.2.24.

**Theorem 2.3.3.** *Suppose $f \in L^1 \cap L^2(G)$. Assume that for each $\nu \in \mathcal{U}$ there exits an analytic vector $\phi_\nu$ for $\pi_{0,\nu}$ and $\psi_\nu \in L^2(\eta_\nu)$ such that*

$$\begin{aligned} &\int_{\xi_\nu\oplus\eta_\nu\oplus\mathfrak{r}_\nu\oplus\mathfrak{z}}\int_{\eta_\nu}\int_{\xi_\nu} |f(x,y,z,t)|\,|\langle(\hat{f}(\mu,\nu),(\xi',\xi''))\phi_\nu,\psi_\nu\rangle| \times \\ &exp(|\sum_{j=1}^n d_j(\nu)(x_j\xi_j'' - y_j\xi_j')|)\,d\xi'\,d\xi''\,dx\,dy\,dz\,dt < \infty \end{aligned}$$

*for all $\mu \in \tilde{V}_T^*$. Then $f = 0$ almost everywhere.*

*Proof.* We define $g_{\tilde{\mu},\nu}(x,y) = f^{\tilde{\mu},\nu}(x,y)\langle\pi_{0,\nu}(x,y,0,0)\phi_\nu,\psi_\nu\rangle$ for all $\nu \in \mathcal{U}$ and $\tilde{\mu} \in \mathfrak{r}_\nu^*$. It follows from the explicit expression of $\hat{f}((\mu,\nu),(\xi',\xi''))$ that

$$\widehat{g_{\tilde{\mu},\nu}}(d(\nu)(-\eta,\xi)) = (2\pi)^n\langle\hat{f}((\tilde{\mu},\nu),(\xi',\xi''))\phi_\nu,\psi_\nu\rangle.$$

Let us consider the following integral:

$$\begin{aligned} &\int_{\eta_\nu}\int_{\xi_\nu}\int_{\eta_\nu}\int_{\xi_\nu} |g_{\tilde{\mu},\nu}(x,y)||\widehat{g_{\tilde{\mu},\nu}}(\xi',\xi'')|\exp(|\sum_{j=1}^{n} x\cdot\xi + y\cdot\xi''|)\ dx\ dy\ d\xi'\ d\xi'' \\ \leq\ & \int_{\eta_\nu}\int_{\xi_\nu}\int_{\xi_\nu\oplus\eta_\nu\oplus\mathfrak{r}_\nu\oplus\mathfrak{z}} |f(x,y,z,t)||\langle \hat{f}((\tilde{\mu},\nu),(\xi',\xi'')\phi_\nu,\psi_\nu\rangle| \\ & \times\exp(|\sum_{j=1}^{n} d_j(\nu)(x_j\xi_j'' - y_j\xi_j')|)\ dx\ dy\ dz\ dt\ d\xi'\ d\xi'' \\ <\ & \infty. \end{aligned}$$

If we apply Theorem 1.2.6 to the function $g_{\tilde{\mu},\nu}$, we get $g_{\tilde{\mu},\nu} = 0$ almost everywhere in $(\xi',\xi'')$. Since $\phi_\nu$ is an analytic vector, $f^{\tilde{\mu},\nu} = 0$ almost everywhere. But this is true for all $(\tilde{\mu},\nu) \in \mathfrak{r}_\nu^* \times \mathcal{U}$. Therefore, finally we have $f = 0$ almost everywhere. □

In the case of $H_d^n$ we have proved Hardy's theorem and Theorem 1.2.12 as corollaries of Theorem 1.2.10. Our plan is to prove analogue of Theorem 1.2.10 in this context. Then we will deduce theorem of Hardy and an analogue of Theorem 1.2.12 from it.

**Theorem 2.3.4.** *For $f \in L^2(G)$, let $g(\mu,\nu) = \left(Pf(\nu)\int_{\tilde{V}_T^*}\|\hat{f}(\mu,\nu)\|_{HS}^2\ d\mu\right)^{1/2}$. Suppose for some $N \geq 0$,*

$$\int_{\mathfrak{z}^*}\int_{\mathfrak{z}} \frac{\|f(\cdot,t)\|_2 g(\mu,\nu)e^{|\nu||t|}}{(1+|\nu|+|t|)^N}\ dt\ d\nu < \infty.$$

*Then $f(v,t) = e^{-a|t|^2}P(v,t)$ for some $a > 0$ and $P(v,t) = \left(\sum_{|\alpha|\leq m} t^\alpha\psi_\alpha(v)\right)$, where $\psi_\alpha \in L^2(\mathfrak{v})$ and $m < \frac{N-1}{2}$.*

*Proof.* Since $f \in L^2(G)$, the function $F$ defined by $F(t) = f(\cdot,t)$ is $L^2(\mathfrak{v})$ measurable function. Now using the formula ( 2.1.14 ) and Theorem 1.2.9 the proof will be finished. □

Immediate consequences of the above theorem are the following two corollaries.

**Corollary 2.3.5.** *Let $f$ be a measurable function on $G$ such that it satisfies*

*(i)* $|f(v,t)| \leq g(v)(1+|t|)^m e^{-a|t|^2}$, *where* $g \in L^2(\mathfrak{v})$,

*(ii)* $Pf(\nu)^{1/2} \left(\int_{V_T^*} \|\hat{f}(\mu,\nu)\|_{HS}^2 \, d\mu\right)^{1/2} \leq e^{-b|\nu|^2}$.

*Then* $f = 0$ *for* $ab > 1/4$ *and if* $ab = 1/4$ *then* $f(v,t) = P(v,t)e^{-a|t|^2}$, *where* $P(v,t) = \left(\sum_{|\alpha|\leq k} \psi_\alpha(v) t^\alpha\right) e^{-a|t|^2}$, *where* $\psi_\alpha \in L^2(\mathfrak{v})$ *and* $k \leq m$.

**Corollary 2.3.6.** *Suppose $f$ is a measurable function on $G$ such that it satisfies the estimates*

*(i)* $|f(v,t)| \leq g(v)e^{-a|t|^p}$, *where* $g \in L^2(\mathfrak{v})$,

*(ii)* $Pf(\nu)^{1/2} \left(\int_{\tilde{V}_T^*} \|\hat{f}(\mu,\nu)\|_{HS}^2 \, d\mu\right)^{1/2} \leq Ce^{-b|\nu|^q}$,

*where* $1 \leq p \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $(ap)^{1/p}(bq)^{1/q} \geq 1$. *Then* $f = 0$ *unless* $p = q = 2$ *and* $ab = 1/4$, *in which case* $f(v,t) = \psi(v)e^{-a|t|^2}$ *for some* $\psi \in L^2(\mathfrak{v})$.

Cowling–Price theorem for any nilpotent Lie group has been proved in [3]. It has been assumed that $2 \leq p, q \leq \infty$ and $ab > 1/4$. Ray assumed $1 \leq p \leq \infty, q \geq 2$ and $ab > 1/4$ in his proof of Cowling–Price theorem for step two nilpotent Lie groups which are not MW (see [28]). We prove the same theorem with assumptions $1 \leq p, q \leq \infty$ and $ab \geq 1/4$. We make use of the following theorem which is a modified version of Theorem 1.2.5. The proof (see [24]) is based on the use of Theorem 1.2.14 and Radon transform.

**Theorem 2.3.7.** *Suppose* $f \in L^2(\mathbb{R}^n)$. *Let for some* $\delta > 0$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\widehat{f}(y)|e^{|x||y|}|Q(y)|^\delta}{(1+|x|+|y|)^d} dxdy < \infty, \tag{2.3.15}$$

*where $Q$ is a polynomial of degree $m$. Then* $f(x) = P(x)e^{-a|x|^2}$ *for some* $a > 0$ *and polynomial $P$ with* $\deg P < \frac{d-n-m\delta}{2}$.

*Proof.* In Theorem 1.2.14 we have seen that the Theorem is true for $n = 1$. Let us assume $n > 1$. Since $A = \{y : \hat{f}(y) = 0 \text{ and } Q(y) = 0\}$ is a set of measure of zero there exists $y' \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)| e^{|x||y'|} \, dx < \infty.$$

This shows that $f$ is $L^1(\mathbb{R}^n)$ and $\hat{f}$ can be extended as a holomorphic function in a strip of $\mathbb{C}^n$. In particular $\hat{f}$ is real analytic on $\mathbb{R}^n$. In ( 2.3.15 ) we use polar coordinates for $y$, to see that there exists a subset $S$ of $S^{n-1}$ with full surface measure such that for every $\omega_2 \in S$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{|f(x)||\widehat{f}(s\omega_2)||s|^{n-1}|Q(s\omega_2)|e^{|x||s|}}{(1+|x|+|s|)^d} \, ds \, dx < \infty. \tag{2.3.16}$$

In view of ( 2.3.13 ) this is the same as for every $\omega_2 \in S$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{|f(x)||\widehat{R_{\omega_2} f}(s)||s|^{n-1}|Q(s\omega_2)|e^{|x||s|}}{(1+|x|+|s|)^d} \, ds \, dx < \infty. \tag{2.3.17}$$

**Step 1:** In this step we will show that for any $\omega_1 \in S^{n-1}$ and $\omega_2 \in S$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{R_{\omega_1}|f|(r)|\widehat{R_{\omega_2} f}(s)||s|^{n-1}|Q(s\omega_2)|e^{|r||s|}}{(1+|r|+|s|)^d} \, ds \, dr < \infty. \tag{2.3.18}$$

We will break the above integral into the following 3 parts and show that each part is finite. That is we will show:

(i)

$$\int_{\mathbb{R}} \int_{|s|>L} \frac{R(|f|)(\omega_1, r)|\widehat{R_{\omega_2} f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|}{(1+|r|+|s|)^d} \, ds \, dr < \infty$$

for $L > 0$ such that $L^2 + L > d$.

(ii)

$$\int_{|r|>M} \int_{|s|\le L} \frac{R(|f|)(\omega_1, r)|\widehat{R_{\omega_2} f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|}{(1+|r|+|s|)^d} \, ds \, dr < \infty$$

for $M = 2(L+1)$ and $L$ as in (i).

(iii)

$$\int_{|r|\leq M}\int_{|s|\leq L}\frac{R(|f|)(\omega_1,r)|\widehat{R_{\omega_2}f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|}{(1+|r|+|s|)^d}\,ds\,dr<\infty$$

for $M, L$ used in (i) and (ii).

*Proof of* (i): It is given that $L + L^2 > d$. We will show that for any $s$ such that $|s| \geq L$,

$$\frac{e^{|s||x|}}{(1+|x|+|s|)^d}\geq\frac{e^{|s||\langle x,\omega_1\rangle|}}{(1+|\langle x,\omega_1\rangle|+|s|)^d}. \tag{2.3.19}$$

Let $F(z) = \frac{e^{\alpha z}}{(1+\alpha+z)^d}$ for $\alpha > 0$ and $\alpha + \alpha^2 > d$. Then $F'(z) > 0$ for any $z \geq 0$. Therefore, if $z_1 \geq z_2 \geq 0$, then

$$\frac{e^{\alpha z_1}}{(1+\alpha+z_1)^d}\geq\frac{e^{\alpha z_2}}{(1+\alpha+z_2)^d}. \tag{2.3.20}$$

Note that $|x| \geq |\langle x, \omega_1\rangle|$ for all $x \in \mathbb{R}^n$ and $\omega_1 \in S^{n-1}$. Now take $z_1 = |x|$ and $z_2 = |\langle x, \omega_1\rangle|$. Then $z_1 \geq z_2 \geq 0$. We take $\alpha = |s| \geq L$ to get ( 2.3.19 ).

We start now from ( 2.3.17 ) and break it up as:

$$\int_{\mathbb{R}}\int_{x\cdot\omega_1=r}\int_{\mathbb{R}}\frac{|f(x)||\widehat{R_{\omega_2}f}(s)|e^{|x||s|}|s|^{n-1}|Q(s\omega_2)|}{(1+|x|+|s|)^d}ds\,d\sigma_1\,dr<\infty, \tag{2.3.21}$$

where $d\sigma_1$ denotes the Lebesgue measure on the hyper plane $\{x : x\cdot\omega_1 = r\}$. We use the inequality ( 2.3.19 ) to obtain:

$$\int_{\mathbb{R}}\int_{x\cdot\omega_1=r}\int_{|s|>L}\frac{|f(x)||\widehat{R_{\omega_2}f}(s)|e^{|\langle x,\omega_1\rangle||s|}|s|^{n-1}|Q(s\omega_2)|}{(1+|\langle x,\omega_1\rangle|+|s|)^d}\,ds\,d\sigma_1\,dr<\infty. \tag{2.3.22}$$

Now we put $\langle x, \omega_1\rangle = r$ in the above integral and use the definition of Radon transform to obtain,

$$\int_{\mathbb{R}}\int_{|s|>L}\frac{R(|f|)(\omega_1,r)|\widehat{R_{\omega_2}f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|}{(1+|r|+|s|)^d}\,ds\,dr<\infty. \tag{2.3.23}$$

This proves (i).

*Proof of* (ii): Let

$$I_2 = \int_{|r|>M} \int_{|s|\leq L} \frac{R(|f|)(\omega_1, r)|\widehat{R_{\omega_2} f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|}{(1+|r|+|s|)^d} drds.$$

It is clear that,

$$\begin{aligned} I_2 &\leq C \int_{|r|>M} \frac{R(|f|)(\omega_1, r)|e^{L|r|}}{(1+|r|)^d} dr \\ &= C \int_{|r|>M} \int_{x\cdot\omega_1 = r} \frac{|f(x)|e^{L|r|}}{(1+|r|)^d}\ d\sigma_1\ dr \\ &= CI_3, \end{aligned}$$

say. We will show that $I_3$ is finite for $M = 2(L+1)$.

We have already observed that $\hat{f}$ is real analytic on $\mathbb{R}^n$ and hence $\hat{f}(y) \neq 0$ for almost every $y \in \mathbb{R}^n$. Therefore, from ( 2.3.17 ) we can get a $s_0 \in \mathbb{R}$ with $|s_0| > 2L$ such that:

$$\int_{\mathbb{R}^n} \frac{|f(x)|e^{|x||s_0|}}{(1+|x|+|s_0|)^d} dx < \infty.$$

That is

$$\int_{\mathbb{R}} \int_{x\cdot\omega_1 = r} \frac{|f(x)|e^{|x||s_0|}}{(1+|x|+|s_0|)^d}\ d\sigma_1\ dr < \infty.$$

Notice that $|s_0| + |s_0|^2 > d$, since $|s_0| > 2L$ and $L + L^2 > d$. Now applying the argument of case (i) (see ( 2.3.20 )) to $|s_0|$ we get:

$$\frac{e^{|x||s_0|}}{(1+|x|+|s_0|)^d} \geq \frac{e^{|\langle x,\omega_1\rangle||s_0|}}{(1+|\langle x,\omega_1\rangle|+|s_0|)^d}$$

as $|\langle x, \omega_1\rangle| \leq |x|$. Therefore,

$$\begin{aligned} &\int_{|r|>M} \int_{x\cdot\omega_1 = r} \frac{|f(x)|e^{|r||s_0|}}{(1+|r|+|s_0|)^d}\ d\sigma_1\ dr \\ \leq\ &\int_{|r|>M} \int_{x\cdot\omega_1 = r} \frac{|f(x)|e^{|\langle x,\omega_1\rangle||s_0|}}{(1+|\langle x,\omega_1\rangle|+|s_0|)^d}\ d\sigma_1\ dr < \infty \end{aligned}$$

from the above observation. Note that $M + M^2 > d$ as $M = 2(L+1)$ and $L + L^2 > d$. Applying the argument of case (i) again (see ( 2.3.20 )) this time with $\alpha = |r| > M$ and $z_1 = |s_0|, z_2 = 2L$ we get,

$$\frac{e^{|s_0||r|}}{(1+|s_0|+|r|)^d} \geq \frac{e^{2L|r|}}{(1+2L+|r|)^d}.$$

Therefore,

$$\int_{|r|>M}\int_{x\cdot\omega_1=r} \frac{|f(x)|e^{2L|r|}}{(1+|r|+2L)^d}\, d\sigma_1\, dr < \infty.$$

From this it is easy to see that

$$\int_{|r|>M}\int_{x\cdot\omega_1=r} \frac{|f(x)|e^{L|r|}}{(1+|r|)^d}\, d\sigma_1\, dr < \infty$$

and hence, $I_3 < \infty$. This completes the proof of (ii).

*Proof of* (iii): As the domain $[-M, M] \times [-L, L]$ is compact and as

$$\frac{|\widehat{R_{\omega_2} f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|}{(1+|r|+|s|)^d}$$

is continuous in this domain, the integral is bounded by $C\int_{-M}^{M} R|f|(\omega_1, r)dr$. Now recall that $f \in L^1(\mathbb{R}^n)$. Therefore,

$$\begin{aligned}\int_{-M}^{M} R|f|(\omega_1, r)dr &\leq \int_{\mathbb{R}} R|f|(\omega_1, r)dr \\ &= \int_{\mathbb{R}}\int_{x\cdot\omega_1=r} |f(x)|d\sigma dr \\ &= \int_{\mathbb{R}^n} |f(x)|dx < \infty. \qquad (2.3.24)\end{aligned}$$

Thus from (i), (ii) and (iii) we obtain ( 2.3.18 ). This completes step 1.

**Step 2:** From ( 2.3.18 ) we see that for almost every $\omega \in S^{n-1}$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{R_\omega|f|(r)|\widehat{R_\omega f}(s)||s|^{n-1}|Q(s\omega)|^\delta e^{|r||s|}}{(1+r|+|s|)^d}\, dr\, ds < \infty. \qquad (2.3.25)$$

Since $|R_\omega f(r)| \leq R_\omega|f|(r)$ we have,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|R_\omega f(r)||\widehat{R_\omega f}(s)||s|^{n-1}|Q(s\omega)|^\delta e^{|r||s|}}{(1+|r|+|s|)^d}\, dr\, ds < \infty. \qquad (2.3.26)$$

Using the result for $n = 1$ we conclude that $R_\omega f(r) = A_\omega(r)e^{-\alpha r^2}$, for some polynomial $A_\omega$ which depends on $\omega$, $\deg A_\omega < \frac{d-m\delta-n}{2}$ and $\alpha$ is a positive constant. A priori, $\alpha$ also should depend on $\omega$. But we will see below that $\alpha$ is actually independent of $\omega$. It is clear that $\widehat{R_\omega f}(s) = P_\omega(s)e^{-\frac{1}{4\alpha}s^2}$, where $\deg P_\omega$ is same as $A_\omega$. Consider $\omega_1, \omega_2 \in S$ with $\omega_1 \neq \omega_2$ for which $R_{\omega_1}$, $R_{\omega_2}$ satisfy ( 2.3.18 ), that is

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|R_{\omega_1} f(r)||\widehat{R_{\omega_2} f}(s)||s|^{n-1}|Q(s\omega_2)|e^{|r||s|}}{(1+r|+|s|)^d}\, dr\; ds < \infty. \tag{2.3.27}$$

From the above argument it follows that $R_{\omega_1} f(r) = A_{\omega_1}(r)e^{-\alpha_1 r^2}$ and $\widehat{R_{\omega_2} f}(s) = P_{\omega_2}(s)e^{-\frac{1}{4\alpha_2}s^2}$ for some positive constants $\alpha_1, \alpha_2$. Suppose if possible $\alpha_1 \neq \alpha_2$. Without loss of generality suppose $\alpha_1 < \alpha_2$, otherwise we will change the role of $\omega_1$ and $\omega_2$. Substituting them back in ( 2.3.27 ) we see that

$$\begin{aligned} I &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{-(\sqrt{\alpha_1}|r|-\frac{1}{2\sqrt{\alpha_2}}|s|)^2} e^{|r||s|(1-\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}})}|s|^{n-1}|Q(s\omega_2)||A_{\omega_1}(r)||P_{\omega_2}(s)|}{(1+|r|+|s|)^d} drds \\ &< \infty. \end{aligned}$$

Fix $\epsilon > 0$, consider the set $A_\epsilon = \{(r,s) : r, s \geq 0 \text{ and } |\sqrt{\alpha_1}r - \frac{1}{\sqrt{\alpha_2}}s| \leq \epsilon\}$, which is a set of infinite measure. Since $\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}} < 1$, it is easy to see that there exists $C > 0$ such that the integrand of $I$ is greater than $C$ on the strip $A_\epsilon$ . Hence, $I \geq Cm(A_\epsilon) = \infty$. Thus we get $\alpha_1 = \alpha_2 = \alpha$ and $\widehat{R_\omega f}(s) = P_\omega(s)e^{-\frac{1}{4\alpha}s^2}$.

**Step 3:** We will show that $P_\omega(s) = P(s\omega)$ is a polynomial in $s\omega$, that is $P$ is a polynomial in $\mathbb{R}^n$. Recall that $\widehat{R_\omega f}(s) = \widehat{f}(s\omega)$ is a holomorphic function in a neighbourhood around 0. We can write $P_\omega(s) = \widehat{f}(s\omega)e^{\frac{1}{4\alpha}s^2} = \widehat{f}(s\omega)e^{\frac{1}{4\alpha}|s\omega|^2} = F(s\omega)$, say.

We write $F(s\omega) = \sum_{j=0}^k a_j(\omega)s^j$, where $k = \max\limits_{\omega\in S^{n-1}} \deg P_\omega < \frac{d-m\delta-n}{2}$. Then for $j = 0, 1, \ldots, k$

$$\frac{1}{j!}\frac{d^j}{ds^j}F(s\omega)\bigg|_{s=0} = a_j(\omega).$$

The left hand side is the restriction of a homogenous polynomial of degree $j$ to $S^{n-1}$. Therefore $F(s\omega)$ is a polynomial of degree $\leq k$ in $\mathbb{R}^n$. Therefore $\widehat{f}(x) = P(x)e^{-\frac{1}{4\alpha}|x|^2}$, where $\deg P < \frac{d-m\delta-n}{2}$. □

First we want to prove an analogue of Theorem 1.2.15 for step two groups and then as a corollary we deduce the Cowling–Price theorem.

**Theorem 2.3.8.** *Suppose $f \in L^1 \cap L^2(G)$ and for some $M, N \geq 0$, it satisfies*

$$\int_\Lambda \int_{\xi_\nu \oplus \eta_\nu \oplus \mathfrak{r}_\nu \oplus \mathfrak{z}} \frac{|f(x,y,z,t)| \|\hat{f}(\mu,\nu)\|_{HS} e^{|z||\mu|+|t||\nu|}}{(1+|(x,y)|)^M (1+|(z,t)|)^{N/2}(1+|(\mu,\nu)|)^{N/2}}$$
$$\times Pf(\nu)\ dx\ dy\ dz\ dt d\mu\ d\nu < \infty.$$

*Then for each $\nu \in \mathcal{U}$*

$$\begin{aligned} & f(x,y,z,t) \\ = \ & (1+|(x,y)|)^M \left( \sum_{|\gamma|+|\delta| \leq l} \Psi_{\gamma,\delta}(x,y) z^\gamma t^\delta \right) e^{-a(|z|^2+|t|^2)}, \end{aligned}$$

*where $\Psi_{\gamma,\delta} \in L^2(\xi_\nu \oplus \eta_\nu)$ and $l$ is an nonnegative integer.*

*Proof.* For each pair $(\phi, \psi)$, where $\phi, \psi \in L^2(\eta_\nu)$ let us consider the function $F_{(\phi,\psi)}$ defined by

$$\begin{aligned} & F_{(\phi,\psi)}(z,t) \\ = \ & \int_{\eta_\nu} \int_{\xi_\nu} f(x,y,z,t)(1+|(x,y)|)^{-M} \overline{(\pi_{0,\nu}(x,y,0,0)\phi, \psi)}\ dx\ dy. \end{aligned}$$

It follows that for all $(\tilde{\mu}, \nu) \in \mathfrak{r}_\nu^* \times \mathfrak{z}^*$

$$\begin{aligned} & \widehat{F_{(\phi,\psi)}}(\tilde{\mu},\nu) \\ = \ & \int_{\eta_\nu} \int_{\xi_\nu} f^{\tilde{\mu},\nu}(x,y)(1+|(x,y)|)^{-M} \overline{(\pi_{0,\nu}(x,y,0,0)\phi, \psi)}\ dx\ dy. \end{aligned}$$

Using Cauchy–Schwarz, we get

$$\begin{aligned}
&|\widehat{F_{(\phi,\psi)}}(\tilde{\mu},\nu)| \\
\leq\ & C\left(\int_{\eta_\nu}\int_{\xi_\nu}|f^{\tilde{\mu},\nu}(x,y)|^2\ dx\ dy\right)^{1/2} \\
=\ & \left(\prod_{j=1}^{n} d_j(\nu)\right)^{1/2} \|\hat{f}(\tilde{\mu},\nu)\|_{\mathrm{HS}}. \qquad (2.3.28)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathfrak{z}^*}\int_{\mathfrak{r}_\nu^*}\int_{\mathfrak{z}}\int_{\mathfrak{r}_\nu}\frac{|F_{(\phi,\psi)}(z,t)||\widehat{F_{(\phi,\psi)}}(\tilde{\mu},\nu)|e^{|\tilde{\mu}||z|+|\nu||t|}}{(1+|(z,t)|)^{N/2}(1+|(\tilde{\mu},\nu)|)^{N/2}} \\
&\times\left(\prod_{j=1}^{n} d_j(\nu)\right)^{1/2} dt\ dz\ d\tilde{\mu}\ d\nu \\
\leq\ & \int_{\mathfrak{z}^*}\int_{\mathfrak{r}_\nu^*}\int_{\xi_\nu\oplus\eta_\nu\oplus\mathfrak{r}_\nu\oplus\mathfrak{z}}\frac{|f(x,y,z,t)|\|\hat{f}(\tilde{\mu},\nu)\|_{\mathrm{HS}}e^{|z||\mu|+|t||\nu|}}{(1+|(x,y)|)^{M}(1+|(z,t)|)^{N/2}(1+|(\mu,\nu)|)^{N/2}} \\
&\times\left(\prod_{j=1}^{n} d_j(\nu)\right) dx\ dy\ dz\ dt\ d\tilde{\mu}\ d\nu \\
=\ & \int_{\Lambda}\int_{\xi_\nu\oplus\eta_\nu\oplus\mathfrak{r}_\nu\oplus\mathfrak{z}}\frac{|f(x,y,z,t)|\|\hat{f}(\mu,\nu)\|_{\mathrm{HS}}e^{|z||\mu|+|t||\nu|}}{(1+|(x,y)|)^{M}(1+|(z,t)|)^{N/2}(1+|(\mu,\nu)|)^{N/2}} \\
&\times \mathrm{Pf}(\nu)\ dx\ dy\ dz\ dt\ d\mu\ d\nu \\
<\ & \infty.
\end{aligned}$$

In the last step we have used the change of variables by the map $\phi$ defined in ( 2.1.1 ). Notice that $\prod_{j=1}^{n} d_j(\nu)$ is a polynomial in $\nu$. Now using Theorem 2.3.7 we have for each pair $(\phi,\psi)$

$$F_{(\phi,\psi)}(z,t) = P_{(\phi,\psi)}(z,t)e^{-a(\phi,\psi)(|z|^2+|t|^2)},$$

where $a(\phi,\psi) > 0$ and

$$P_{(\phi,\psi)}(z,t) = \sum_{|\gamma|+|\delta|\leq m} a_{(\gamma,\delta)}(\phi,\psi)z^\gamma t^\delta$$

and $m$ is independent of $(\phi, \psi)$. As in the proof of Theorem 1.2.15 we can show that $a(\phi, \psi) = a$ is independent of $(\phi, \psi)$. Finally taking $\phi = \Phi^\nu_\alpha$ and $\psi = \Phi^\nu_\beta$ and using the fact that $\{\Phi^\nu_{\alpha,\beta}(x, y) : \alpha, \beta \in \mathbb{N}^n\}$ (see 2.2.3 )forms an orthonormal basis for $L^2(\xi_\nu \oplus \eta_\nu)$ we can show as in the proof of Theorem 1.2.15

$$f(x, y, z, t) = (1 + |(x, y)|)^M \left( \sum_{|\gamma|+|\delta|\leq m} \Psi_{\gamma,\delta}(x, y) z^\gamma t^\delta \right) e^{-a(|z|^2+|t|^2)},$$

where $\Psi_{\gamma,\delta} \in L^2(\xi_\nu \oplus \eta_\nu)$.

□

Now we are ready to prove the following version of Cowling–Price theorem.

**Theorem 2.3.9.** *Suppose $f \in L^1 \cap L^2(G)$ and it satisfies the following conditions.*

(i) $\int_G e^{pa|(v,t)|^2} |f(v, t)|^p \, dv \, dt < \infty$,

(ii) $\int_\Lambda e^{bq|(\mu,\nu)|^2} \|\hat{f}(\mu, \nu)\|^q_{HS} \, Pf(\nu) d\mu \, d\nu < \infty$.

*Then for $ab \geq 1/4$ and $\min\{p, q\} < \infty$, $f = 0$ almost everywhere.*

*Proof.* Using Hölder's inequality we can find $M, N, C > 0$ such that for each $\nu \in \mathcal{U}$

(i)′ $\int_{\xi_\nu \oplus \eta_\nu \oplus \mathfrak{r}_\nu \oplus \mathfrak{z}} \frac{e^{a|(z,t)|^2} |f(x,y,z,t)|}{(1+|(x,y)|)^M (1+|(z,t)|)^N} \, dx \, dy \, dz \, dt < C$,

(ii)′ $\int_\Lambda \frac{e^{b|(\mu,\nu)|^2} \|\hat{f}(\mu,\nu)\|_{\mathrm{HS}}}{(1+|(\mu,\nu)|)^N} \, \mathrm{Pf}(\nu) d\mu \, d\nu < \infty$.

With this observation and using Theorem 2.3.8 we can conclude that $f = 0$ almost everywhere under the assumption $ab \geq 1/4$ and $\min\{p, q\} < \infty$. □

As in the case of $H^n_d$ we can get the following following analogue of Corollary 1.2.12.

**Theorem 2.3.10.** *Suppose $f \in L^2(G)$ satisfies the estimates for each $\nu \in \mathcal{U}$*

(i) $|f(x,y,z,t)| \leq g(x,y)e^{-a|(z,t)|^p}$, *where* $g \in L^2(\xi_\nu \oplus \eta_\nu)$,

(ii) $(\prod_{j=1}^{n} d_j(\nu))^{1/2}\|\hat{f}(\tilde{\mu},\nu)\|_{HS} \leq Ce^{-b|(\tilde{\mu},\gamma)|^q}$ *for all* $\tilde{\mu} \in \mathfrak{r}_\nu^*$,

*where* $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $(ap)^{1/p}(bq)^{1/q} \geq 1$. *Then* $f=0$ *almost everywhere unless* $p=q=2$ *and* $ab=1/4$ *in which case* $f(x,y,z,t) = \psi(x,y)e^{-a|(z,t)|^2}$ *for some* $\psi \in L^2(\xi_\nu \oplus \eta_\nu)$.

*Proof.* For each $(\alpha,\beta)$, consider the function

$$F_{\alpha,\beta}(z,t) = \int_{\eta_\nu}\int_{\xi_\nu} f(x,y,z,t)\overline{\Phi^\nu_{\alpha,\beta}(x,y)}\,dx\,dy. \tag{2.3.29}$$

Now proceeding as before and using the hypotheses we get

(i) $|F_{\alpha,\beta}(z,t)| \leq Ce^{-a|(z,t)|^p}$

(ii) $|\hat{F}_{\alpha,\beta}(\tilde{\mu},\nu)| \leq Ce^{-b|(\tilde{\mu},\nu)|^q}$.

Now applying Theorem 1.2.7 to the function $F_{\alpha,\beta}$ we get $F_{\alpha,\beta}=0$ almost everywhere unless $p=q=2$ and $ab=1/4$ in which case $F_{\alpha,\beta}(z,t) = C_{\alpha,\beta}e^{-a|(z,t)|^2}$. Therefore, $f=0$ almost everywhere for $(ap)^{1/p}(bq)^{1/q} \geq 1$ and $p \neq 2$. Let us consider the case $p=q=2$ and $ab=1/4$. Since $f \in L^2(G)$ we can express it as

$$\begin{aligned} f(x,y,z,t) &= e^{-a|(z,t)|^2}\sum_{(\alpha,\beta)} C_{\alpha,\beta}\Phi^\nu_{\alpha,\beta}(x,y) \\ &= e^{-a|(z,t)|^2}\psi(x,y), \end{aligned}$$

where $\psi \in L^2(\xi_\nu \oplus \eta_\nu)$. □

For all step two nilpotent Lie groups Astengo et al [1] have proved a version of Hardy's theorem. The equality case has been left open. Our method will give the equality case which implies the inequality case. We now prove the following version of Hardy's theorem.

**Theorem 2.3.11.** *Let $f \in L^1(G)$ satisfy the following conditions:*

*(i)* $\int_{\mathfrak{v}} |f(v,t)|(1+|v|^2)^{k'}\ dv \leq Ce^{-a|t|^2}$,

*(ii)* $Pf(\nu) \int_{\tilde{V}_T^*} \|\hat{f}(\mu,\nu)\|_{op} (1+|\mu|)^{-k'}\ d\mu \leq Cr(\nu)^l e^{-b|\nu|^2}$,

*where* $r(\nu) = \sum_{j=1}^{n} (d_j(\nu)^2 + d_j(\nu)^{-2})$, $k', l \in \mathbb{N}$. *If* $ab > 1/4$ *then* $f = 0$ *almost everywhere and for* $ab = 1/4$, $f(v,t) = f(v,0)e^{-a|t|^2}$.

*Proof.* Let $h(v,t) = f(v,t)(1+|v|^2)^{-k'}$ and $g$ be a Schwartz function. Consider the function

$$\begin{aligned} F(v,t) &= h(\cdot,t) * \bar{g}(v) \\ &= \int_{\mathfrak{v}} h(w,t)\bar{g}(v-w)\ dw. \end{aligned}$$

Taking the Fourier transform in the $t$ variable

$$\begin{aligned} F^\nu(v) &= \int_{\mathfrak{v}} \frac{f^\nu(v)}{(1+|v|^2)^{k'}} \bar{g}(v-w)\ dv\ dw \\ &= (2\pi)^{-(n+r)} \mathrm{Pf}(\nu) \int_{\tilde{V}_T^*} \mathrm{tr}\left(\hat{f}(\mu,\nu) W_{\mu,\nu}\left(h'_v\right)^*\right) d\mu, \end{aligned}$$

where $h'_v(w) = g(v-w)(1+|w|^2)^{-k'}$. Note that $h'_v$ is from Schwartz class and hence

$$\begin{aligned} |F^\nu(v)| &\leq (2\pi)^{-(n+r)} |\mathrm{Pf}(\nu)| \int_{r_\nu^*} \left|\mathrm{tr}\left(\hat{f}(\mu,\nu) W_{\mu,\nu}(h'_{v)})^*\right)\right|\ d\mu \\ &\leq C\ |\mathrm{Pf}(\nu)| \int_{r_\nu^*} \frac{\|\hat{f}(\mu,\nu)\|_{\mathrm{op}}}{(1+|\mu|)^{k'}} (1+|\mu|)^{k'}\ \mathrm{tr}\left(|W_{\mu,\nu}(h'_v)|\right)\ d\mu \\ &\leq C(v)\ r(\nu)^{k''} e^{-b|\nu|^2}, \end{aligned}$$

for some positive integer $k''$ using the second hypothesis of the theorem and estimate on $(1+|\mu|)^k \mathrm{tr}\,(|W_{\mu,\nu}(h'_v)|)$ as mentioned earlier . Let $D = \{\omega \in S^{k-1} : d_j(\omega) \neq 0$ for all $j\}$. Note that we have $d_j(\nu) > 0$ for all $j$ whenever $\nu \in \mathcal{U}$. Also $d_j(s\omega) = sd_j(\omega)$. If $\gamma$ denotes the surface measure on $S^{k-1}$ then $\gamma(S^{k-1} \setminus D) = 0$ as $\mathcal{U}$ is a set of full measure on $\mathbb{R}^k$. Let

$d(\omega) = \max_{1\leq j\leq n}\{d_j(\omega), d_j^{-1}(\omega)\}$. for fixed $\omega \in D$. Since $d_j(s\omega) = sd_j(\omega)$ for sufficiently large $|s|$, $\sum_{j=1}^{n} d_j(s\omega)^2 + d_j(s\omega)^{-2} \leq nd(\omega)(1+s^2)$. Therefore,

$$\begin{aligned} r(s\omega)^{k''} &= \left(\sum_{j=1}^{n} d_j(s\omega)^2 + d_j(s\omega)^{-2}\right)^{k''} \\ &\leq C'(\omega)(1+s^2)^{k''} \end{aligned}$$

for sufficiently large $s$. For each $\omega \in D$ consider the Radon transform $R_\omega F(v,s)$ of the function $F(v,t)$ in the last variable $t$. So we can conclude that

$$|\widehat{R_\omega F}(v,s)| \leq C(v)C'(\omega)(1+s^2)^{k''}e^{-bs^2}$$

for sufficiently large positive integer $m$ . Also we have $|R_\omega F(v,s)| \leq Ce^{-as^2}$ as $|F(v,t)| \leq Ce^{-a|t|^2}$ using the first hypothesis of the theorem. Now applying Hardy's theorem to the function $R_\omega F(v,\cdot)$ we conclude that for all $\omega \in D$, $F^{s\omega}(v) = C(v)C'(\omega)e^{-bs^2}$ whenever $ab = 1/4$ and $F^{s\omega}(v) = 0$ for $ab > 1/4$. Since $F^\nu$ is a continuous function of $\nu$ we conclude that $F^\nu(v) = C(v)e^{-b|\nu|^2}$ for $ab = 1/4$ and $F^\nu(v) = 0$ whenever $ab > 1/4$ for all $\nu \in \mathfrak{z}^*$. Finally we get $F(v,t) = h(\cdot,t) * \bar{g}(v) = C(v)e^{-a|t|^2}$ for $ab = 1/4$ and $F(v,t) = h(\cdot,t) * \bar{g}(v) = 0$ for $ab > 1/4$. Choosing $g$ from an approximate identity $\{g_m\}_m$ where each $g_m$ is of Schwartz class on $\mathfrak{v}$ we conclude that $h(\cdot,t)*\bar{g}_m(v) = 0$ for $ab > 1/4$ and $h(\cdot,t)*\bar{g}_m(v) = C_m(v)e^{-a|t|^2}$ for $ab = 1/4$. But $h(\cdot,t) * \bar{g}_m(v)$ converges to $h(v,t)$ as $m \to \infty$. Hence $C_m(v) \to C(v)$ as $m \to \infty$.
Therefore, $f(v,t) = C(v)(1 + |(v)|^2)^{k'}e^{-a|t|^2} = f(v,0)e^{-a|t|^2}$ for $ab = 1/4$ and $f(v,t) = 0$, whenever $ab > 1/4$. □

### 2.3.2 *Uncertainty principles on step two groups with MW condition*

All the theorems except Theorems 2.2.4, 2.2.2 proved for step two groups without MW condition can be formulated and proved for MW groups with obvious modifications. So we discuss here analogues of Theorems 2.2.4 and 2.2.2 for MW groups.

**Theorem 2.3.12.** *Let $G$ be a step two group with MW- condition. Let $f$ be a function on $G$ such that for every $\omega \in S^{k-1}$*

(i) $|R_\omega f(x(\omega), y(\omega), r)| \leq C(1 + |(x(\omega), y(\omega))|)^l q_{a,d(\omega)}\,(x(\omega) + iy(\omega), r)$,

(ii) $\hat{f}(\nu)^* \hat{f}(\nu) \leq C(\pi_\nu(\mathcal{L}))^l e^{-2b\pi_\nu(\mathcal{L})}$ *for every* $\nu \in \Lambda$,

*where $l > 0$. Then $f = 0$ whenever $a < b$.*

*Proof.* We recall the formula

$$f^{r\omega}(x(\omega), y(\omega)) = \int_{-\infty}^{\infty} e^{irs} R_\omega f((x(\omega), y(\omega)), s)\; ds$$

for all $r \in (0, \infty)$ and $\omega \in S^{k-1}$. Then from first hypothesis of the theorem we show that for any $b > a' > a$

$$|f^{r\omega}(x(\omega), y(\omega))| \leq Ce^{-\frac{1}{4a'}(|x(\omega)|^2 + |y(\omega)|^2)}.$$

Also using the above formula and the given estimate on $R_\omega f$ it can be shown that for each $\omega \in S^{k-1}$, $f^{r\omega}$ can be extended as a holomorphic function of $r \in \mathbb{C}$ in some strip $|\Im(r)| < \frac{A}{a}$. Now the rest of the proof will be same as Theorem 2.2.4. □

**Theorem 2.3.13.** *Let $G$ be a step two group with MW condition. Let $f$ be a function on $G$. Let*

$$g_\omega(x(\omega), y(\omega), r) = R_\omega f(x(\omega), y(\omega), r) \left( q_{a,d(\omega)}\,(x(\omega) + iy(\omega), r) \right)^{-1}.$$

*Suppose for every $\omega \in S^{k-1}$*

(i) $(1 + |(x(\omega), y(\omega))|)^{-l} g_\omega$ *belongs to* $L^p(\mathfrak{v} \times \mathbb{R})$, $1 \leq p \leq \infty$ *and*

(ii) $\hat{f}(\nu) = S(\nu)\,(\pi_\nu(\mathcal{L}))^l\, e^{-b\pi_\nu(\mathcal{L})}$,

*where $l > 0$, $1 \leq q \leq \infty$ and $S(\nu) \in \mathcal{S}_q$, for every $\nu \in \Lambda$. Then $f = 0$ almost everywhere whenever $a < b$.*

*Proof.* As before it can be shown that $f^{r\omega}$ can be extended as a holomorphic function of $r \in \mathbb{C}$ in the strip $|\Im(r)| < \frac{A}{a}$ for some $A > 0$ and $e^{\frac{1}{4a'}|.|^2} f^{r\omega} \in L^p(\mathfrak{v})$ for any $a < a' < b$. This is true for all $\omega \in S^{k-1}$. The rest will follow from the proof of Theorem 2.2.2. $\square$

# 3. UNCERTAINTY PRINCIPLES FOR GENERAL NILPOTENT LIE GROUPS

In the previous chapters we have proved various kinds of uncertainty principles for Heisenberg groups and step two nilpotent Lie groups. It is natural to ask up to what extent those results can be generalized for general nilpotent Lie groups. Recently a version of Hardy's theorem and Cowling–Price theorem for connected simply connected nilpotent Lie groups have been proved in [19] and [3] respectively. In the proof of heat kernel versions of Hardy and Cowling–Price theorems for step two stratified group we have used good estimates of the heat kernel as well as the explicit expression for the partial Fourier transform of the heat kernel in the central variable. For a general stratified group a good estimate of the heat kernel is available from the work of [18]. However we do not have an explicit expression for the partial Fourier transform of the heat kernel in the central variable. So there are technical problems in proving such a version of uncertainty principle even in the case of a stratified group. Due to the lack of such information in the general case we look for alternative versions of Hardy and Cowling–Price theorems.

## *3.1 A new version of Cowling–Price theorem for $\mathbb{R}^n$ and its application to nilpotent groups*

We replace the first condition of Theorem 1.2.2 by estimates on derivatives of $\hat{f}$ and get a new version of Cowling–Price theorem. We also give a comparative study of these two versions.

### 3.1.1 A new version of Cowling–Price

Consider the Gaussian $\phi_a(x)$ for which $\hat{\phi_a} = (4\pi a)^{-n/2}\phi_b$ with $b = 1/4a$. In view of the Plancherel theorem we have

$$\|\partial^\alpha \hat{\phi_a}\|_2^2 = \int_{\mathbb{R}^n} |x^\alpha \phi_a(x)|^2 \, dx = 2^n \prod_{j=1}^{n} \int_0^\infty t^{2\alpha_j} e^{-2at^2} dt$$

which gives the estimate

$$\|\partial^\alpha \hat{\phi_a}\|_2^2 = C \prod_{j=1}^{n} \Gamma\left(\alpha_j + \frac{1}{2}\right)(2a)^{-|\alpha|} \leq C \ \alpha!(2b)^{|\alpha|}. \tag{3.1.1}$$

If a function $f$ satisfies $|f(x)| \leq C \ \phi_a(x)$ then the derivatives of $\hat{f}$ satisfy the estimates

$$\|\partial^\alpha \hat{f}\|_2^2 \leq C \ \alpha!(2a)^{-|\alpha|}. \tag{3.1.2}$$

Replacing the pointwise estimate $|f(x)| \leq C \ \phi_a(x)$ by the slightly weaker estimate ( 3.1.1 ) we get the following uncertainty principle.

**Theorem 3.1.1.** *Let $f$ be a function on $\mathbb{R}^n$ such that $|\hat{f}(\xi)| \leq C \ e^{-b|\xi|^2}$, and for every $\alpha \in \mathbb{N}^n$, $\|\partial^\alpha \hat{f}\|_2^2 \leq C\alpha!(2a)^{-|\alpha|}$. Then $f = 0$ whenever $ab > \frac{1}{4}$ and when $ab = \frac{1}{4}$, $\hat{f}(\xi) = \phi(\xi)e^{-b|\xi|^2}$ where $\phi$ is an entire function on $\mathbb{C}^n$.*

We start with the following lemma which allows us to get pointwise estimates on $\partial^\alpha \hat{f}$ when we have estimates on $\|\partial^\alpha \hat{f}\|_2$.

**Lemma 3.1.2.** *Suppose we have*

$$\|\partial^\alpha \hat{f}\|_2^2 \leq C \ \alpha!(2a)^{-|\alpha|}$$

*for every $\alpha \in \mathbb{N}^n$. Then we also have*

$$|\partial^\alpha \hat{f}(\xi)|^2 \leq C \ \prod_{j=1}^{n}(\alpha_j + n)!(2a)^{-|\alpha|}$$

*for every $\alpha \in \mathbb{N}^n$.*

*Proof.* In view of Sobolev embedding theorem (see [11], page 270),

$$|\partial^\alpha \hat{f}(\xi)|^2 \leq C \sum_{|\beta|\leq n} \|\partial^{\alpha+\beta}\hat{f}\|_2^2$$

which gives the estimate

$$\begin{aligned} |\partial^\alpha \hat{f}(\xi)|^2 &\leq C \sum_{|\beta|\leq n} \|\partial^{\alpha+\beta}\hat{f}\|_2^2 \\ &\leq C \sum_{|\beta|\leq n} (2a)^{-(|\alpha|+|\beta|)}(\alpha+\beta)! \\ &\leq C \prod_{j=1}^{n}(\alpha_j+n)!(2a)^{-|\alpha|}. \end{aligned}$$

□

In view of this lemma, we only need to prove the following version of Theorem 3.1.1.

**Theorem 3.1.3.** *The conclusions of Theorem 3.1.1 are valid if we replace the estimates on* $\|\partial^\alpha \hat{f}\|_2$ *by*

$$|\partial^\alpha \hat{f}(\xi)|^2 \leq C \prod_{j=1}^{n}(\alpha_j+n)!(2a)^{-|\alpha|}$$

*for every* $\alpha \in \mathbb{N}^n$*.*

Let us complete the proof of it. We first consider the case $ab > 1/4$. We make use of the following lemma.

**Lemma 3.1.4.** *Let* $F(\xi)$ *be a smooth function on* $\mathbb{R}^n$ *which satisfies*

$$|\partial^\alpha F(\xi)|^2 \leq C \prod_{j=1}^{n}(\alpha_j+n)!(2a)^{-|\alpha|}$$

*for all* $\alpha \in \mathbb{N}^n$*. Then* $F$ *extends to* $\mathbb{C}^n$ *as an entire function which satisfies* $|F(\zeta)| \leq C\, e^{b|\Im\zeta|^2}$ *for every* $b > 1/4a$*.*

*Proof.* For $b > 1/4a$,

$$\begin{aligned}
|F(\xi+\eta)| &= |\sum_\alpha \frac{\partial^\alpha F(\xi)}{\alpha!}\eta^\alpha| \\
&\leq \sum_\alpha \left|\frac{\partial^\alpha F(\xi)}{\alpha!}\right| |\eta|^{|\alpha|} \\
&= \sum_\alpha \frac{|\partial^\alpha F(\xi)|}{\alpha!^{\frac{1}{2}}}(2b)^{-\frac{|\alpha|}{2}} . \frac{(2b)^{\frac{|\alpha|}{2}}}{\alpha!^{\frac{1}{2}}}|\eta|^{|\alpha|} \\
&\leq \left(\sum_\alpha \frac{|\partial^\alpha F(\xi)|^2}{\alpha!}(2b)^{-|\alpha|}\right)^{\frac{1}{2}} \left(\sum_\alpha \frac{\left(|\eta|^2 2b\right)^{|\alpha|}}{\alpha!}\right)^{\frac{1}{2}} \\
&\leq C\left(\sum_\alpha \frac{(\alpha+n)!}{\alpha!}\left(\frac{1}{4ab}\right)^{|\alpha|}\right)^{\frac{1}{2}} e^{b|\eta|^2} \\
&= C(n,b)e^{b|\eta|^2}.
\end{aligned}$$

This shows that $F$ can be extended as an entire function on $\mathbb{C}^n$ and it satisfies

$$|F(\xi+i\eta)| \leq \ C(n,b)e^{b|\eta|^2}.$$

Coming to the proof of the case $ab > 1/4$, choose $b'$ such that $b > b' > 1/4a$. By Lemma 3.1.4 we have

$$|\hat{f}(\zeta)| \leq C\ e^{b'|\Im\zeta|^2}.$$

As we have $|\hat{f}(\xi)| \leq C\ e^{-b|\xi|^2}$ and $b' < b$ we appeal to the following lemma to conclude that $\hat{f} = 0$. □

**Lemma 3.1.5.** *Let $F(\zeta)$ be an entire function on $\mathbb{C}^n$ which satisfies*

$$|F(\zeta)| \leq C\ e^{a|\Im\zeta|^2},\quad |F(\xi)| \leq Ce^{-b|\xi|^2}$$

*for $\zeta \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^n$. Then $F = 0$ whenever $a < b$ and $F(\zeta) = Ce^{-a\zeta^2}$ for $a = b$.*

Now take up the equality case. Clearly it is enough to prove it when $n = 1$. Indeed if we have the result in the one dimensional case then by considering the function

$$F(\xi_n) = \hat{f}(\xi', \xi_n), \quad \xi' = (\xi_1, \xi_2, \cdots \xi_{n-1})$$

which satisfies the estimates

$$|F(\xi_n)| \leq C(\xi')e^{-b\xi_n^2} \quad |\partial^k F(\xi_n)|^2 \leq C\ (k+n)!(2a)^{-k}$$

we obtain

$$F(\xi_n) = C(\xi', \xi_n)e^{-b\xi_n^2}.$$

But now the function $C(\xi', \xi_n)$ satisfies the same estimates as $\hat{f}$ on $\mathbb{R}^{n-1}$. By using induction we can obtain $\hat{f}(\xi) = \phi(\xi)e^{-b|\xi|^2}$ with $\phi$ bounded. For the one dimensional case using Lemma 3.1.4 $\hat{f}$ can be extended to $\mathbb{C}$ as an entire function of order at most 2. Since $\hat{f}$ cannot decay on $\mathbb{R}$ faster than its order its order is 2. Since we have the estimate

$$|\hat{f}(\zeta)| \leq C\ e^{b'|\Im\zeta|^2} \text{for all} \quad b' > \frac{1}{4a}$$

its type is $\frac{1}{4a}$. Now we apply the following result of Pfannschmidt [27] to the entire function $\hat{f}(\zeta)$.

**Theorem 3.1.6.** *Let $F$ be an entire function of one variable $\zeta$ of order $\rho$ ($\rho$ integer) and type $b$. Let*

$$h(\theta) = \limsup_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r^\rho}, \quad \theta \in [0, \infty)$$

*be its indicator and assume that*

$$h\left(\frac{2\pi j}{\rho}\right) \leq -b, j = 0, 1, 2, \cdots, \rho - 1.$$

*Then $F(\zeta) = P(\zeta)e^{-b\zeta^\rho}$ where $P(\zeta)$ is an entire function at most of minimal type of order $\rho$.*

The following remark is in order. In Theorem 3.1.3 with $ab = 1/4$ we have concluded that $\hat{f}(\xi) = \phi(\xi)e^{-b|\xi|^2}$. It would be nice to say something about $f$ itself. As $\phi(\zeta)$ is an entire function we have

$$\phi(\xi) = \sum_{|\alpha|\leq N} a_\alpha \xi^\alpha + \phi_N(\xi)$$

where $|\phi_N(\xi)| \leq C\ (1+|\xi|)^N$. This shows that, in view of the inversion formula

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\phi(\xi)e^{-b|\xi|^2}d\xi \tag{3.1.3}$$

$f$ can be written as

$$f(x) = \left(\sum_{|\alpha|\leq N} c_\alpha x^\alpha\right) e^{-a|x|^2} + f_N(x)$$

where

$$\hat{f}_N(\xi) = \phi_N(\xi)e^{-b|\xi|^2}.$$

We also have $(L^p, L^q)$ version, the case $ab > 1/4$ of Theorem 3.1.1.

**Theorem 3.1.7.** *Let $\hat{f}$ be a smooth function such that*

$$||\partial^\alpha \hat{f}||_p^2 \leq C\alpha!(2a)^{-|\alpha|}, \hat{f}e^{b|\cdot|^2} \in L^q(\mathbb{R}^n)$$

*where $1 \leq p, q \leq \infty$ then $f = 0$ for $ab > 1/4$.*

*Proof.* We reduce the conditions of the Theorem to the corresponding conditions when $p = q = \infty$ using a trick from [2]. Let $h$ be a smooth function with supp $\subset \{x : |x| < \delta\}$. Choose $b'$ such that $b > b' > \frac{1}{4a}$. Then for all $x \in \mathbb{R}^n$ with $|x| > \frac{\delta\sqrt{b}}{\sqrt{b}-\sqrt{b'}}$ and $y \in$ sup $h$ we have $|x-y| \geq |x|-|y| > |x|-\delta > |x|\sqrt{\frac{b'}{b}}$. Since $h \in L^p$ for all $p$ we get by Holder's inequality a constant $C > 0$ such that

$$\begin{aligned} C &\geq \int_{\mathbb{R}^n} e^{b|x-y|^2}|\hat{f}(x-y)||h(y)|dy \\ &\geq e^{b'|x|^2}\int_{\mathbb{R}^n} |\hat{f}(x-y)||h(y)|dy \\ &\geq e^{b'|x|^2}|\hat{f} * h(x)| \end{aligned}$$

for all $|x| > \frac{\delta\sqrt{b}}{\sqrt{b}-\sqrt{b'}}$. From the continuity of the function $\hat{f} * h$ it follows that

$$|\hat{f} * h(x)| \leq Ce^{-b'|x|^2}$$

for all $x \in \mathbb{R}^n$. We also have the following estimate

$$\begin{aligned} |\partial^\alpha(\hat{f} * h)(x)|^2 &\leq ||\partial^\alpha \hat{f}||_p^2 ||h||_{p'}^2 \\ &\leq C\alpha!(2a)^{-|\alpha|} \end{aligned}$$

Since $ab' > 1/4$ we can apply Theorem 3.1.3 to conclude $\hat{f} * h = 0$. As this is true for all $h \in C_0^\infty(\mathbb{R}^n)$ we get $\hat{f} = 0$ almost everywhere. □

### 3.1.2 Cowling–Price theorem versus its new version

To investigate the relation between Theorem 1.2.2 and Theorem 3.1.7 we need to compute the following integral.

$$\int_{\mathbb{R}^n} |x^\alpha e^{-a|x^2|}|^p dx = \prod_{j=1}^n 2\int_0^\infty r^{p\alpha_j} e^{-apr^2} dr. \tag{3.1.1}$$

It is enough to consider the case $n = 1$. Now

$$\begin{aligned} 2\int_0^\infty e^{-apx^2} x^{pk}\, dx &= (pa)^{-\frac{pk+1}{2}} \int_0^\infty e^{-x} x^{\frac{pk-1}{2}} dx \\ &= C(ap)^{-\frac{pk}{2}} \Gamma\left(\frac{pk+1}{2}\right) \end{aligned} \tag{3.1.2}$$

Using the Sterling's formula $\Gamma(t+1) \sim t^t e^{-t}\sqrt{2\pi t}$

$$\begin{aligned} (ap)^{-\frac{pk}{2}} \Gamma\left(\frac{pk+1}{2}\right) &= (ap)^{-\frac{pk}{2}} \Gamma\left(\frac{pk-1}{2} + 1\right) \\ &\leq C(ap)^{-\frac{pk}{2}} e^{-\frac{pk-1}{2}} \left(\frac{pk-1}{2}\right)^{\frac{pk}{2}} \\ &\leq Ce^{-\frac{pk}{2}} (2a)^{-\frac{pk}{2}} k^{\frac{pk}{2}} \\ &\leq C\left((2a)^{-k/2} k^{k/2} e^{-k/2}\right)^p \\ &\leq C\left((2a)^{-k} k^{k+\frac{1}{2}} e^{-k}\right)^{p/2} \\ &\leq C\left((2a)^{-k} k!\right)^{p/2}. \end{aligned} \tag{3.1.3}$$

Therefore, from ( 3.1.1 ) and ( 3.1.2 ) we have

$$\left(\int_{\mathbb{R}^n} |x^\alpha e^{-a|x^2|}|^p \ dx\right)^{1/p} \leq C\left((2a)^{-|\alpha|}\alpha!\right)^{1/2}.$$

Now if $f$ satisfies the conditions of Theorem 1.2.2, then using the above estimate and Hölder's inequality we have

$$\begin{aligned} |\partial^\alpha \hat{f}(\xi)|^2 &= |\widehat{x^\alpha f}(\xi)|^2 \\ &\leq \|x^\alpha f\|_1 \\ &\leq C||f\phi_a^{-1}||_p^2||x^\alpha \phi_a||_{p'}^2 \\ &\leq C\alpha!(2a)^{-|\alpha|}. \end{aligned}$$

Therefore, the hypotheses of Theorem 3.1.7 are satisfied and hence Theorem 1.2.2 can be deduced from Theorem 3.1.7.

We show below that the case $p = 2, q \in [1, \infty]$ of Theorem 1.2.2 is equivalent to the case $p = 2, q \in [1, \infty]$ of Theorem 3.1.7.

Suppose $f$ satisfies the hypothesis of Theorem 3.1.7 with $p = 2,\ q \in [1, \infty]$. Choose $a' < a$ but satisfying $a'b > \frac{1}{4}$ and consider $f\phi_{a'}^{-1}$. Expanding the Gaussian we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x)\phi_{a'}(x)^{-1}|^2 \ dx \\ =& \sum_{k=0}^{\infty}\left(\int_{\mathbb{R}^n} |f(x)|^2 \frac{(2a')^k}{k!}|x|^{2k} \ dx\right) \\ =& \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k}\frac{1}{\alpha!}\int_{\mathbb{R}^n}|f(x)|^2 x^{2\alpha} \ dx\right)(2a')^k. \end{aligned}$$

Under the hypothesis on $\|\partial^\alpha \hat{f}\|_2^2$ we get

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x)|^2 x^{2\alpha} \ dx \\ =& \int_{\mathbb{R}^n} |\partial^\alpha \hat{f}(x)|^2 \ dx \\ \leq& \ C\alpha!(2a)^{-|\alpha|}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_{\mathbb{R}^n} |f(x)\phi_{a'}(x)^{-1}|^2 \, dx \\
\leq \; & C\sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k}\frac{1}{\alpha!}\right)\left(\frac{a'}{a}\right)^k \\
\leq \; & C\sum_{k=0}^{\infty} k^{n-1}\left(\frac{a'}{a}\right)^k \\
< \; & \infty.
\end{aligned}$$

Hence the hypotheses of Theorem 1.2.2 (case $p = 2, q \in [q, \infty]$) are satisfied. On the other hand suppose $f$ satisfies hypothesis of Theorem 1.2.2 with $p = 2$. The above calculation shows that

$$||\partial^\alpha \hat{f}||_2^2 \leq C\alpha!(2a)^{-|\alpha|}$$

for every $\alpha \in \mathbb{N}^n$. Thus the hypothesis of Theorem 3.1.7 (case $p = 2, q \in [1, \infty]$) are satisfied.

### *3.1.3 An uncertainty principle for operators and Cowling–Price theorem for nilpotent Lie groups*

The group Fourier transform on a nilpotent Lie group $G$ is operator valued. Given an irreducible unitary representation $\pi$ of $G$ and a function $f$ on $G$ the operator $\hat{f}(\pi) = \pi(f)$ is realized on $L^2(\mathbb{R}^n)$ for a suitable $n$. In order to formulate an analogue of Theorem 3.1.1 we need such a result for operators. We require the notion of noncommutative derivatives.

Given a bounded linear operator $T$ on $L^2(\mathbb{R}^n)$, we define certain noncommutative derivatives of $T$ by

$$\delta_j T = [A_j, T], \;\; \bar{\delta}_j T = [T, A_j^*], \tag{3.1.1}$$

where $[T, S] = TS - ST$ is the commutator and $A_j = \frac{\partial}{\partial \xi_j} + \xi_j$, $A_j^* = -\frac{\partial}{\partial \xi_j} + \xi_j$ are the annihilation and creation operators. The above derivations were introduced by Mauceri [21] and Thangavelu has used them on several occasions, see [33, 34]. For multiindices $\alpha$, $\beta$ we define $\delta^\alpha T$ and $\bar{\delta}^\beta T$ iteratively.

Since for any Hilbert–Schmidt operator $T$ there exists $f \in L^2(\mathbb{C}^n)$ such that $T = W_\lambda(f)$ it is sufficient to consider an analogue of Theorem 3.1.1 for Weyl transform. The following Cowling–Price theorem for Weyl transform is a generalization of Theorem 1.6.5, page 43 of [35].

**Theorem 3.1.8.** *Let $fe^{a|z|^2} \in L^p(\mathbb{C}^n)$ and $W_\lambda(f) = S(\lambda)e^{-bH(\lambda)}$ for some $S(\lambda) \in \mathcal{S}_q$, where $1 \le p, q \le \infty$. Then $f = 0$ almost everywhere. whenever $a\frac{\tanh b\lambda}{\lambda} \ge \frac{1}{4}$ and min$(p,q) < \infty$. If $p = q = \infty$ then $f = 0$ for $a\frac{\tanh b\lambda}{\lambda} > 1/4$ and $f(z) = Ce^{-a|z|^2}$ for $a\frac{\tanh b\lambda}{\lambda} = 1/4$.*

*Proof.* We give a sketch of the proof as it is a modification of the proof of Theorem 2.9.4 of [34]. Let $\mathbb{T}^n$ denotes the subgroup of the unitary group $U(n)$ consisting of diagonal matrices. Then each element of $\mathbb{T}^n$ can be written in the form $e^{i\theta} = (e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n})$ so that $\mathbb{T}^n$ can be identified with $n$-torus. Let $m$ be an $n$-tuple of integers. Since $fe^{a|z|^2}$ is in $L^p(\mathbb{C}^n)$ it follows that $f_m e^{a|z|^2} \in L^p(\mathbb{C}^n)$ where

$$f_m(z) = \int_{\mathbb{T}^n} f(e^{i\theta} \cdot z)e^{-im.\theta}\, d\theta. \tag{3.1.2}$$

Now $W_\lambda(f) = S(\lambda)e^{-bH(\lambda)}$ for some $S(\lambda) \in \mathcal{S}_q$. Therefore, $S(\lambda)$ is a bounded operator and hence there exists $C > 0$ such that $\|W_\lambda(f)\Phi_\alpha^\lambda\|_2 \le Ce^{-b(2|\alpha|+n)|\lambda|}$ for all $\alpha \in \mathbb{N}^n$. Using this estimate it has been shown in page 91 of [34] that

$$|\langle f_m, \Phi^\lambda_{\alpha,\alpha+m}\rangle| \le C_n \prod_{i=1}^n \left((2\alpha_i + 1)^n e^{-b(2\alpha_i+1)|\lambda|}\right). \tag{3.1.3}$$

Let $\mathcal{F}_\lambda f$ stand for the symplectic Fourier transform of a function $f$ in $\mathbb{C}^n$ defined by

$$\mathcal{F}_\lambda f(z) = (2\pi)^{-n} \int_{\mathbb{C}^n} f(z - w)e^{i\frac{\lambda}{2}\Im(z\cdot\bar{w})}\, dw. \tag{3.1.4}$$

$\mathcal{F}_\lambda$ is related to the ordinary Fourier transform by $\mathcal{F}_\lambda f(z) = \hat{f}(-i\frac{\lambda}{2}z)$. We also have

$$\mathcal{F}_\lambda f_m(z) = \sum_\alpha \langle f_m, \Phi^\lambda_{\alpha,\alpha+m}\Phi^\lambda_{\alpha,\alpha+m}\rangle$$

in the sense of distribution. Using the explicit expression of $\Phi^\lambda_{\alpha,\alpha+m}$ it can be shown

$$|\mathcal{F}_\lambda f_m(z)| \quad \leq \quad C_{n,m}(\lambda)(1+|\lambda||z|^2)^l e^{-(\frac{1}{4}\lambda\tanh b\lambda)|z|^2}$$

for some positive integer $l > 0$, (see [34], page 93 ). Therefore, we have

$$|\hat{f}_m(z)| \leq C_{n,m}(\lambda)(1+|z|^2)^l e^{-\frac{\tanh b\lambda}{\lambda}|z|^2}.$$

So by the Cowling–Price theorem for the Euclidean Fourier transform we conclude the following:

**Case 1.** If $1 \leq p < \infty, q \leq \infty$, then for $a\frac{\tanh b\lambda}{\lambda} \geq 1/4$, $f_m = 0$ for all $m$ and hence $f = 0$ almost everywhere.

**Case 2.** If $p = \infty$, $1 \leq q < \infty$, then for $a\frac{\tanh b\lambda}{\lambda} > 1/4$, $f_m = 0$ and for $a\frac{\tanh b\lambda}{\lambda} = 1/4$ $f_m(z) = C_m e^{-a|z|^2}$ for all $m$. Since $f_m$ is $m$–homogeneous $C_m = 0$ except $m = 0$ and hence $f(z) = f_0(z) = Ce^{-a|z|^2}$ which yields $W_\lambda(f) = Ce^{-bH(\lambda)}$. But this is not compatible with the condition $W_\lambda(f) = S(\lambda)e^{-bH(\lambda)}$ for some $S(\lambda) \in \mathcal{S}_q$ and hence $f = 0$ almost everywhere.

**Case 3.** If $p = q = \infty$, then $f_m = 0$ for all $m$ whenever $a\frac{\tanh b\lambda}{\lambda} > 1/4$ and for $a\frac{\tanh b\lambda}{\lambda} = 1/4$ arguing as before $f(z) = f_0(z) = Ce^{-a|z|^2}$. □

Let $T$ be an Hilbert–Schmidt operator. Then there exists $f \in L^2(\mathbb{C}^n)$ such that $T = W_1(f)$. A simple calculation using the definition shows that $\delta_j W_1(f) = W_1(\bar{M}_j f)$ and $\bar{\delta}_j W_1(f) = W_1(M_j f)$ where $M_j f(z) = z_j f(z)$ and $\overline{M_j} f(z) = \bar{z}_j f(z)$. By iteration we get $\delta^\alpha W_1(f) = W_1(\bar{z}^\alpha f)$ and $\bar{\delta}^\beta W_1(f) = W_1(z^\beta f)$. With this observation we are ready to prove our operator analogue of Theorem 3.1.1.

**Theorem 3.1.9.** *Let $T \in \mathcal{S}_2$ satisfy the estimates*

(*i*) $T^*T \leq C\ e^{-2bH}$

(*ii*) $\|\delta^\alpha \overline{\delta}^\beta (T^*T)\|^2_{HS} \leq C\ (\alpha+\beta)! a^{|\alpha|+|\beta|}$ *for all* $\alpha, \beta \in N^n$,

*for some $a, b > 0$ . Then $T = 0$ whenever $a < 2\tanh 2b$.*

*Proof.* Let $S = T^*T$. Since $S$ is a Hilbert–Schmidt operator there exists $f \in L^2(\mathbb{C}^n)$ such that $S = W_1(f)$. We define the operator valued function $\tilde{f}(u,v)$ on $\mathbb{R}^{2n}$ given by

$$\tilde{f}(u,v) = W(u+iv)W_1(f)W(u+iv)^*. \tag{3.1.5}$$

As $W(z)$ is a projective representation of $\mathbb{C}^n$ it is easily seen that

$$\tilde{f}(u,v) = \int_{\mathbb{C}^n} e^{i(x.v-y.u)} f(x+iy)W(x+iy)\; dx\; dy. \tag{3.1.6}$$

Taking derivatives in $u, v$ and using the relations $\delta^\alpha W_1(f) = W_1(\bar{z}^\alpha f)$, $\bar{\delta}^\beta W_1(f) = W_1(z^\beta f)$ we get

$$\partial_u^\alpha \partial_v^\beta \tilde{f}(u,v) = 2^{-(|\alpha|+|\beta|)} W(u+iv)(\delta+\bar{\delta})^\beta(\delta-\bar{\delta})^\alpha W_1(f)W(u+iv)^*.$$

This identity shows that

$$\|\partial_u^\alpha \partial_v^\beta \tilde{f}(u,v)\|_{\mathrm{HS}}^2 \leq C\ (\alpha+\beta)!a^{|\alpha|+|\beta|} \tag{3.1.7}$$

whenever we have

$$\|\delta^\alpha \bar{\delta}^\beta W_1(f)\|_{\mathrm{HS}}^2 \leq C\ (\alpha+\beta)!a^{|\alpha|+|\beta|}. \tag{3.1.8}$$

Let $F$ be the function on $\mathbb{R}^{2n}$ defined by

$$F(u,v) = \langle \tilde{f}(u,v)\Phi_0, \Phi_0\rangle.$$

then $F(u,v)$ satisfies the following two properties:

$(i)$ $F(u,v)$ extends to $\mathbb{C}^{2n}$ as an entire function which satisfies the estimate $|F(\zeta)| \leq Ce^{a_1|\Im\zeta|^2}$ for some $a_1 < \frac{1}{2}(1-e^{-4b})$,

$(ii)$ $F(u,v) \leq C\ e^{-\frac{1}{2}(1-e^{-4b})(|u|^2+|v|^2)}$.

Assuming this claim for a moment we appeal to the following lemma.

**Lemma 3.1.10.** *Let $G(\zeta)$ be an entire function on $\mathbb{C}^n$ which satisfies $|G(\zeta)| \leq C\ e^{a|\Im\zeta|^2}$, $\zeta \in \mathbb{C}^n$ and $|G(\xi)| \leq C\ (1+|\xi|^2)^m e^{-b|\xi|^2}$, $\xi \in \mathbb{R}^n$. Then $G = 0$ whenever $a < b$.*

The lemma shows that $F = 0$ whenever $a_1 < \frac{1}{2}(1 - e^{-4b})$. Since $a < 2\tanh 2b$ we have $\coth 2b < \frac{2}{a}$ and so we can choose $b_1$ and $b_2$ such that $\coth 2b < 4b_1 < 4b_2 < \frac{2}{a}$. This gives $b_2 < \frac{1}{2a}$ and $1 + 4b_1 > 1 + \coth 2b = \frac{2}{1-e^{-4b}}$ or $\frac{2}{1+4b_1} < (1 - e^{-4b})$. In our claim we can take $a_1 = \frac{1}{1+4b_1}$ so that $F = 0$. Since

$$\begin{aligned} F(u,v) &= \langle W(u+iv)W_1(f)W(u+iv)^*\Phi_0, \Phi_0\rangle \\ &= \int_{\mathbb{R}^{2n}} e^{i(x.v-y.u)} f(x,y) e^{-\frac{1}{4}(|x|^2+|y|^2)}\, dx\, dy \end{aligned}$$

by Fourier inversion formula $f = 0$ almost everywhere proving the theorem as $S = T^*T = W_1(f) = 0$. It remains to prove the claim with $a_1 = \frac{1}{1+4b_1}$ where $b_1$ is chosen as above. As we have indicated earlier the estimates on $\delta^\alpha \bar{\delta}^\beta S$ give the estimates

$$\|\partial_u^\alpha \partial_v^\beta \tilde{f}(u,v)\|_{\mathrm{HS}}^2 \leq C\ (\alpha+\beta)! a^{|\alpha|+|\beta|} \tag{3.1.9}$$

for all $\alpha, \beta \in \mathbb{N}^n$ and hence

$$\|\partial_u^\alpha \partial_v^\beta F(u,v)\|_2^2 \leq C\ (\alpha+\beta)! a^{|\alpha|+|\beta|} \tag{3.1.10}$$

for all $\alpha, \beta \in \mathbb{N}^n$.

Since $S = W_1(f)$ using the Plancherel theorem we have the estimates

$$\int_{\mathbb{R}^{2n}} |x^\alpha y^\beta f(x,y)|^2 dx\ dy \leq C(\alpha+\beta)! a^{(|\alpha|+|\beta|)}.$$

We claim that

$$\int_{\mathbb{R}^{2n}} |f(x,y)|^2 e^{2b_2(|x|^2+|y|^2)}\, dx\ dy < \infty \tag{3.1.11}$$

for any $b_2 < \frac{1}{2a}$. To see this consider the series

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^{2n}} |f(x,y)|^2 (2b_2)^k (|x|^2+|y|^2)^k\, dx\ dy \tag{3.1.12}$$

which converges as long as $b_2 < \frac{1}{2a}$.

Now consider $F(u,v)$ which is given by

$$F(u,v) = \int_{\mathbb{R}^{2n}} f(x,y) e^{i(x\cdot v - y\cdot u)} e^{-\frac{1}{4}(|x|^2+|y|^2)}\, dx\, dy.$$

By Hölder's inequality

$$|\partial_u^\alpha \partial_v^\beta F(u,v)|^2 \leq C \int_{\mathbb{R}^{2n}} |x^\alpha y^\beta|^2 e^{-2(b_2+\frac{1}{4})(|x|^2+|y|^2)}\, dx\, dy \tag{3.1.13}$$

which gives the estimate

$$|\partial_u^\alpha \partial_v^\beta F(u,v)|^2 \leq C\ (\alpha+\beta)! \left(2\left(b_2+\frac{1}{4}\right)\right)^{-(|\alpha|+|\beta|)}. \tag{3.1.14}$$

Appealing to Lemma 3.1.10 we see that $F(u,v)$ extends to an entire function of type $a_2$ where $a_2 = \frac{1}{1+4b_2}$. Since $a_2 < a_1$ we get $|F(\zeta)| \leq C\ e^{a_1|\Im\zeta|^2}$ which proves the claim $(i)$. The second claim is proved using the bound $T^*T \leq C\ e^{-2bH}$. We have

$$\begin{aligned} F(u,v) &= (W(u+iv)T^*TW(u+iv)^*\Phi_0, \Phi_0) \\ &\leq C\ \left(W(u+iv)e^{-2bH}W(u+iv)^*\Phi_0, \Phi_0\right). \end{aligned} \tag{3.1.15}$$

We now expand $W(u+iv)^*\Phi_0 = W(-u-iv)\Phi_0$ in terms of $\Phi_\mu$:

$$W(u+iv)^*\Phi_0 = (2\pi)^{\frac{n}{2}} \sum_\mu \Phi_{0,\mu}(-u-iv)\Phi_\mu. \tag{3.1.16}$$

Since

$$e^{-2bH}\Phi_\mu = e^{-2b(2|\mu|+n)}\Phi_\mu$$

we have, using Parseval's formula for Hermite expansions,

$$F(u,v) \leq C \sum_\mu e^{-2b(2|\mu|+n)} |\Phi_{0,\mu}(-u-iv)|^2. \tag{3.1.17}$$

Now using the explicit formula ( 1.1.7 ) for $\Phi_{0,\mu}$, we get

$$F(u,v) \leq C\ \sum_\mu e^{-2b(2|\mu|+n)} \frac{1}{\mu!} \left(\frac{1}{2}(|u|^2+|v|^2)\right)^{|\mu|} e^{-\frac{1}{2}(|u|^2+|v|^2)}$$

which gives

$$F(u,v) \leq C\ e^{-\frac{1}{2}\left(1-e^{-4b}\right)(|u|^2+|v|^2)} \tag{3.1.18}$$

as desired. □

We now give an application of the above theorem for multiple Laguerre expansions. Let $f$ be a function in $L^2(\mathbb{C}^n)$ which is invariant under the action on $\mathbb{T}^n$. Then $f$ is called polyradial and it has expansion

$$f(z) = \sum_{\mu} (f, \Phi_{\mu\mu}) \Phi_{\mu\mu}(z).$$

Let $d$ be a function on $\mathbb{N}^n$. For each $j = 1, 2, \ldots, n$ we define the difference operators $\triangle_j^+$ and $\triangle_j^-$ by

$$(\triangle_j^+ d)(\mu) = d(\mu + e_j) - d(\mu), (\triangle_j^- d)(\mu) = d(\mu) - d(\mu - e_j),$$

where $e_j$ are the coordinate vectors. For multi-indices $\alpha$, $\beta \in \mathbb{N}^n$ we define

$$\triangle_+^\alpha = (\triangle_1^+)^{\alpha_1} (\triangle_2^+)^{\alpha_2} \cdots (\triangle_n^+)^{\alpha_n}, \triangle_-^\beta = (\triangle_1^-)^{\beta_1} (\triangle_2^-)^{\beta_2} \cdots (\triangle_n^-)^{\beta_n}.$$

With these notations we will prove the following corollary of the above theorem, which can be considered as Cowling–Price theorem for multiple Laguerre expansions of polyradial functions.

**Corollary 3.1.11.** *Let $f$ be a polyradial function which is in $L^2(\mathbb{C}^n)$, $C(\mu) = (f, \Phi_{\mu\mu})$ and $d(\mu) = |C(\mu)|^2$, $\mu \in \mathbb{N}^n$. If $C(\mu)$ satisfy the following conditions:*

(i) $|C(\mu)| \le C e^{-b(2|\mu|+n)}$

(ii) $\sum_{\mu} \frac{(\mu+\alpha)!}{(\mu-\beta)!} \left| \left( \triangle_-^\beta \triangle_+^\alpha d \right) (\mu) \right|^2 \le C(\alpha+\beta)! 2^{-(|\alpha|+|\beta|)} a^{(|\alpha|+|\beta|)}$

*where $a, b > 0$, then $f = 0$ whenever $a < 2 \tanh 2b$.*

*Proof.* Since $f$ is polyradial

$$W_1(f)\phi = \sum_{\mu} (f, \Phi_{\mu\mu})(\phi, \Phi_\mu)\Phi_\mu.$$

Using the formulae

$$A_j \Phi_\mu = (2\mu_j + 2)^{\frac{1}{2}} \Phi_{\mu+e_j}, \;\; A_j^* \Phi_\mu = (2\mu_j)^{\frac{1}{2}} \Phi_{\mu-e_j}$$

it is easy to see that

$$\begin{aligned}&\left(\delta^{\alpha}\overline{\delta}^{\beta}W_1(f)^*W_1(f)\right)\Phi_\mu\\ =\;& (-1)^{|\alpha|+|\beta|}2^{\frac{|\alpha|+|\beta|}{2}}\left(\frac{(\mu+\alpha)!}{(\mu-\beta)!}\right)^{\frac{1}{2}}\left(\triangle_-^{\beta}\triangle_+^{\alpha}d\right)(\mu)\Phi_{\mu+\alpha-\beta}.\end{aligned}$$

Then using the above conditions we will get

$$W_1(f)^*W_1(f)\le Ce^{-2bH},\ \|\delta^{\alpha}\overline{\delta}^{\beta}\left(W_1(f)^*W_1(f)\right)\|_{\mathrm{HS}}^2\le C(\alpha+\beta)!a^{|\alpha|+|\beta|}.$$

So by Theorem 3.1.9, $W_1(f)=0$ for $a<2\tanh 2b$ and hence $f=0$. □

As an immediate corollary of Theorem 3.1.9 we have the following theorem for general nilpotent Lie groups.

**Theorem 3.1.12.** *Let $G$ be a connected, simply connected nilpotent Lie group and let $\Lambda$ be a cross section for the generic coadjoint orbits parametrising the elements of $\hat{G}$ which are relevant for the Plancherel theorem. For each $\lambda\in\Lambda$ let $\pi_\lambda$ be the associated element of $\hat{G}$. Let $f\in L^1\bigcap L^2(G)$ satisfy the following conditions:*

(*i*) $\pi_\lambda(f)^*\pi_\lambda(f)\le C\ e^{-2b(\lambda)H}$

(*ii*) $\|\delta^{\alpha}\bar{\delta}^{\beta}\left(\pi_\lambda(f)^*\pi_\lambda(f)\right)\|_{HS}^2\le C\ (\alpha+\beta)!a(\lambda)^{|\alpha|+|\beta|}$

*where $a(\lambda),b(\lambda)>0$. Then $f=0$ whenever $a(\lambda)<2\tanh 2b(\lambda)$ for all $\lambda\in\Lambda$.*

For the case of of the Heisenberg group it can be easily checked using the explicit formula for the heat kernel that $|f^\lambda(z)|\le C\ q_a^\lambda(z)$ leads to the estimates

$$\|\delta^{\alpha}\bar{\delta}^{\beta}(\hat{f}(\lambda)^*\hat{f}(\lambda))\|_{\mathrm{HS}}^2\le C\ (\alpha+\beta)!(a|\lambda|)^{|\alpha|+|\beta|}. \tag{3.1.19}$$

Thus condition (*ii*) in the above theorem is a suitable alternative which compensates for the absence of a good formula for the heat kernel. In the case of the Heisenberg group we can replace the condition (*i*) by $\hat{f}(\lambda)^*\hat{f}(\lambda)\le$

$Ce^{-2bH(\lambda)}$. Note that $e^{-bH(\lambda)} = \hat{q}_b(\lambda)$ and so it is a natural candidate for measuring the decay of $\hat{f}(\lambda)$. As $H(\lambda)$ is unitarily equivalent to $|\lambda|H$ the condition $(i)$ is natural. The same comment applies to the case of all step two groups as the scaled Hermite operator is related to the sublaplacian even in that case. In the case of general nilpotent groups, there is no canonical way of measuring the decay of $\pi_\lambda(f)$. Therefore, we have used $e^{-b(\lambda)H}$ to measure the decay of the Fourier transform since we do not have any other choice.

# BIBLIOGRAPHY


[1] F. Astengo, M Cowling, B, Di Blasio and M. Sundari, '*Hardy's Uncertainty principle on certain Lie groups*', J. London Math. Soc. 62 (2000), 461–472.

[2] S. C. Bagchi and S. K. Ray, '*Uncertainty principles like Hardy's theorem on some Lie groups*', J. Austral. Math. Soc. (series A) 65 (1999), 289–302.

[3] A. Baklouti and N. B. Salah, '*The Cowling–Price theorem for nilpotent Lie groups*' Preprint.

[4] A. Baklouti, N. B. Salah and K. Smaoui, '*Some uncertainty principles on nilpotent Lie groups, Banach algebras and their applications*', Contemp. Math., 363, Amer. Math. Soc., Providence, RI, 2004, 39–52.

[5] A. Bonami, B. Demange and P. Jaming, '*Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms*', Revist. Math. Ibero. 19 (2003), no.1,23–55.

[6] M. Cowling and J. Price, '*Generalisations of Heisenberg's inequality, in Harmonic analysis (G. Mauceri, F. Ricci and G. Weisss, eds)*', Lecture notes in Math. 992, Springer, Berlin (1983).

[7] M. Cowling ('*Private communication*').

[8] L. J. Corwin and F. P. Greenleaf, '*Representations of nilpotent Lie groups and their applications, Part 1 : Basic theory and examples*', Cambridge University Press (1990).

[9] J.Dziubanski, W. Hebisch and J. Zienkiewicz, '*Note on semigroups generated by positive Rockland operators on graded homogeneous groups*' Studia Math. 110 (1994), 115–126.

[10] G. B. Folland, '*Harmonic analysis in phase space*', Ann. Math. Stud. 122, Princeton University Press, Princeton, NJ, (1989).

[11] G. B. Folland, '*Real analysis, Modern techniques and their applications*', John Willey and Sons, Inc, (1984).

[12] G. B. Folland and A. Sitaram, '*The uncertainty principles : A mathematical survey*', J. Fourier Anal. Apl. 3 (1997), 207–238.

[13] G. B. Folland and E.M. Stein, '*Hardy Spaces on Homogeneous groups*', Princeton University Press, Princeton, NJ, (1982).

[14] G. H. Hardy, '*A theorem concerning Fourier transforms*', J. London Math. Soc. 8 (1933), 227–231.

[15] V. Havin and B. Joricke, '*The uncertainty principle in Harmonic analysis*', Springer, Berlin, (1994).

[16] W. Hebisch, '*Sharp pointwise estimates for the kernels of the semigroups generated by sums of even powers of vector fields on homogeneous groups*', Studia Math. 95 (1989), 93–106.

[17] L. Hörmander, '*A uniqueness theorem of Beurling for Fourier transform pairs'*, Ark. Mat. 29 (1991), no. 2, 237–240.

[18] D.S Jerison, A. Sánchez-Calle, '*Estimates for the heat kernel for a sum of squares of vector fields*', Indiana Univ. Math. J. 35 (1986), 835–854.

[19] E. Kaniuth and A. Kumar, '*Hardy's theorem for simply connected nilpotent Lie groups*', Proc. Cambridge Philos. Soc. 131 (2001), 487–494.

[20] A. Kumar and C. R. Bhata, '*An uncertainty principle like Hrdy's theorem for nilpotent Lie groups*', J. Aust. Math. Soc. 77 (2004), 47–53.

[21] G. W. Morgan, '*A note on Fourier transform*', J. London Math. Soc. 9 (1934) , 187–192.

[22] D. Muller and F. Ricci, '*Solvability for a class of doubly characteristic differential operators on two step nilpotent Lie groups*', Ann. Math. 143 (1996), 1–46.

[23] E. K. Narayanan and S. K. Ray, '*The heat kernel and Hardy's theorem on symmetric spaces of noncompact type*', Proc. Indian Acad. Sci. Math. Sci. 112 (2002), no. 2, 321–330.

[24] S. Parui and R. P. Sarkar, '*Modified Beurling's theorem for* $\mathbb{R}^n$', Preprint.

[25] S. Parui and S. Thangavelu, '*Variations on a theorem of Cowling–Price with applications to nilpotent Lie groups*', J. Austral. Math. Soc. (series A) (to appear).

[26] S. Parui and S. Thangavelu, '*On theorems of Beurling and Hardy for certain step two nilpotent groups*', submitted.

[27] C. Pfannschmidt, '*A generalization of the theorem of Hardy: A most general version of the uncertainty principle for Fourier integrals*', Math. Nachr. 182(1996), 317–327.

[28] S. K. Ray, '*Uncertainty principles on two step nilpotent Lie groups*', Proc. Indian Acad. Sci. 111 (2001), 1–26.

[29] S. K. Ray and Rudra P. Sarkar, '*Cowling-Price theorem and characterization of heat kernel on symmetric spaces*', Proc. Indian Acad. Sci. 114 (2004), no. 2, 159–180.

[30] A. Sitaram, M. Sundari and S. Thangavelu, '*Uncertainty principles on certain Lie groups*', Proc. Indian Acad. Sci. 105 (1995), 135–151.

[31] L.M. Sun and S.L.Wang, '*Uncertainty principles on nilpotent Lie groups*', Acta Math. Sinica 42 (1999), no. 4, 597–604.

[32] S. Thangavelu, '*Hardy's theorem on the Heisenberg group revisited*', Math. Z. 242 (2002), 761–779.

[33] S. Thangavelu, '*On Paley-Wiener theorems for the Heisenberg group*', J. Funct. Anal. 115 (1993), no. 1, 24–44.

[34] S. Thangavelu, '*An introduction to the uncertainty Principle : Hardy's theorem on Lie groups*', Progr. Math. 217, Birkhäuser, Boston, (2004).

[35] S. Thangavelu, '*Harmonic analysis on the Heisenberg group*', Progr. Math. 159, Birkhauser, (1998).

[36] S. Thangavelu, '*Lectures on Hermite and Laguerre expansions*', Math. Notes. 42, Princeton University Press, Princeton, NJ, (1993).