Quantum Stochastic Dilation of a Class of Quantum Dynamical Semigroups and Quantum Random Walks

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To

my parents

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Contents

Notations

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{R}_+$	Set of positive real numbers
$\mathbb{C}$	Set of complex numbers
$Re(z)$	real part of the complex number z
$\mathcal{C}_c^1(\mathbb{R}_+)$	Space of all complex valued once continuously differentiable function on $\mathbb{R}_+$ with compact support
$\mathcal{A}, \mathcal{B}$	$*$-algebras
$\mathcal{B}(X, Y)$	The space of all bounded linear operators from Banach space X to Banach space Y
$a(t)$	Annihilation operator
$a^\dagger(t)$	Creation operator
$\Lambda(t)$	Conservation operator
$\mathcal{H}, \mathbf{h}$ etc.	Hilbert spaces
$\Gamma(\mathcal{H})$	The symmetric Fock space over the Hilbert space $\mathcal{H}$
Ω	The vacuum vector in symmetric Fock space $\Gamma(\mathcal{H})$
$\mathcal{B}(\mathcal{H})$	The algebra of all bounded operators on $\mathcal{H}$
$\|u >< v\|$	Rank one operator $\mathcal{H} \ni w \mapsto \langle v, w \rangle u$
$\mathcal{D}(T)$	Domain of the operator T
$Ran(T)$	Range of the operator T
M_n	The algebra of $n \times n$ matrices with complex entries
$\Gamma(S_h)$	toy Fock space associated with the regular partition S_h of $\mathbb{R}_+$ with width h
P_h	Orthogonal projection of symmetric Fock space Γ onto toy Fock space $\Gamma(S_h)$

Introduction

In the theory of classical dynamical system, Markov processes, or equivalently, the associated expectation semigroups (Markov semigroups) are often used to model the irreversible time evolution of the system. Here, the Markov processes describe the evolution of the total (along with the environmental) system which is given by a stochastic differential flow equation and the evolution within the original system is obtained by taking conditional expectation with respect to the filtration of the above stochastic process. However, in accordance with theory of quantum mechanics, semigroups of completely positive (CP) maps acting on algebra of observables of the system make a natural appearance and the dynamics of an irreversible quantum dynamical system is modeled by a CP semigroup on some appropriate algebra. Here the algebra need not be commutative in contrast to the classical situation and such semigroups are of great interest from physical as well as mathematical point of view.

A linear map T between two $*$-algebras is said to be *completely positive* if $T \otimes id_{M_n(\mathbb{C})}$ is positive for each $n \geq 1$. Let $\mathcal{A}$ be a C^*-algebra. A one parameter C_0-semigroup $\{T_t\}$ of CP maps on $\mathcal{A}$ is called a *quantum dynamical semigroup (QDS)*. A QDS on a von Neumann algebra $\mathcal{A}$, is a one-parameter C_0-semigroup $\{T_t\}$ of normal (continuous with respect to ultraweak topology) CP maps on $\mathcal{A}$. A QDS is said to be conservative if it preserves the identity element. Given a QDS on a C^* or von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathbf{h}_0)$ ($\mathbf{h}_0$ is the initial Hilbert space) a natural question arises whether it can be dilated, that is, whether we can find a family of $*$-homomorphisms $j_t : \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is a $*$-algebra containing $\mathcal{A}$ with a conditional expectation $\mathbb{E}_0 : \mathcal{B} \to \mathcal{A}$, such that $\mathbb{E}_0 j_t = T_t, \forall t \geq 0$. Motivated by the classical case, it is natural to look for a quantum analogue of classical probability theory and then obtain a time indexed family j_t of $*$-homomorphisms from $\mathcal{A}$ to the larger algebra $\mathcal{B}$, modeling the total system, consisting of the original system and some "quantum noise", so that j_t satisfies a suitable differential equation. With the theory of quantum stochastic calculus as developed by the pioneering works of Hudson and Parthasarathy [19] and improved by a number of authors, a notion of quantum stochastic differential flow was formulated by Evans and Hudson [11], [10] and subsequently studied by many authors [31, 34, 12, 13]. In this formulation $\mathcal{B}$ is given by $\mathcal{A} \bigotimes \mathcal{B}((\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0))))$ for some separable Hilbert space $\mathbf{k}_0$ where $\Gamma(\mathbf{k})$ denotes the symmetric Fock space over the Hilbert space $\mathbf{k}$. The family of $*$-homomorphisms j_t is obtained as the solution of *Evans-Hudson (EH) type quantum stochastic differential equation (qsde)*

$$dj_t(a) = \sum_{\mu,\nu \geq 0} j_t(\theta^\mu_\nu(a)) \, d\Lambda^\mu_\nu(t), \; j_0(a) = a, \; \forall \; a \in \mathcal{A}^\infty,$$

where $\mathcal{A}^\infty$ is a dense $*$-subalgebra of $\mathcal{A}$, θ^μ_ν are linear maps (called structure maps) on $\mathcal{A}$ with θ^0_0 as the infinitesimal generator of the QDS T_t and $\{\Lambda^\mu_\nu\}$ is the family of fundamental processes associated with an orthonormal basis $\{e_i : i \geq 1\}$ of $\mathbf{k}_0$ [33, 29]. Here, $\mathbb{E}_0$ is the vacuum conditional expectation. Such a family $\{j_t\}$ of $*$-homomorphisms is called an *Evans-Hudson (EH) dilation* for the QDS $\{T_t\}$. In some situation one can obtain a $*$-homomorphic flow $\{j_t\}_{t\geq 0}$ implemented by a unitary valued process U_t on $\mathbf{h}_0 \otimes \Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0))$ (i.e., $j_t(x) = U_t^*(x \otimes 1_\Gamma)U_t$) satisfying Hudson-Parthasarathy type equation

$$dU_t = \sum_{\mu,\nu} U_t L^\mu_\nu \, d\Lambda^\mu_\nu(t), \; U_0 = 1,$$

for a suitable family of operators $\{L^\mu_\nu\}$ on the initial Hilbert space $\mathbf{h}_0$. Such a dilation is called *Hudson-Parthasarathy (HP) type dilation.*

In order to obtain a solution of EH or HP type flow equation, one may encounter all sorts of technical difficulties, arising due to possible unboundedness of the structure maps or the possible infinite dimension of noise space. There is a considerable amount of literature related to the existence, uniqueness and characterization of HP type flows [31, 32, 13, 14] and EH type flows [15, 24, 18, 25, 1, 26, 27] under various analytic assumptions on structure maps.

Given a QDS T_t on a C^* or von Neumann algebra the question of obtaining an EH or HP dilation is investigated by many authors and answered in some situations, for example when the QDS is uniformly continuous. The infinitesimal generator $\mathcal{L}$ of a uniformly continuous QDS is a bounded and *conditionally completely positive (CCP)* map, i.e. for $n \geq 1$,

$$\sum_{i,j=1}^{n} b_i^* \mathcal{L}(a_i^* a_j) b_j \geq 0, \quad \text{for any } a_i, b_i\text{'s in } \mathcal{A} \text{ such that } \sum_{i=1}^{n} a_i b_i = 0.$$

Such a map $\mathcal{L}$ admits unique structure given by Christensen and Evans [7]. Starting from this bounded generator one can define structure maps. In [15], a coordinate-free language of quantum stochastic calculus is developed and a canonical EH dilation for arbitrary uniformly continuous QDS on von Neumann algebras is obtained. Later in [18], construction of EH flow is extended to uniformly continuous QDS on separable C^*-algebras. This sums up the situation about QDS with bounded generators.

On the other hand, in case of strongly continuous QDS, the generator is unbounded and does not admit structure as in case of uniformly continuous QDS. The infinitesimal generator can be describe as a form [8]. There is no general method to to obtain an HP or EH type flows. Many authors have worked in this direction under suitable analytic assumptions and some partial success has been achieved [12, 14, 30, 32, 17].

In this thesis, we have investigated the possibility of constructing EH dilation for a larger class of QDS with unbounded generators. Restricting ourselves to a particular model [28] of *Uniformly Hyperfinite* C^*-algebra $\mathcal{A} = \otimes_{j\in\mathbb{Z}^d} M_N(\mathbb{C})$, where N and d be two fixed positive integers, we consider the strongly continuous QDS generated by formal Lindbladian associated with an element $r \in \mathcal{A}$. Let tr be the unique normalized trace on $\mathcal{A}$ and $\mathbf{h}_0 = L^2(\mathcal{A}, tr)$, the GNS space for the pair $(\mathcal{A}, tr)$.

For $x \in M_N(\mathbb{C})$ and $j \in \mathbb{Z}^d$, let $x^{(j)}$ denote an element in $\mathcal{A}$ whose j-th component is x and rest are identity of $M_N(\mathbb{C})$. For a simple tensor element $a \in \mathcal{A}$, let $a_{(j)}$ be the j-th component of a. The support of a, denoted by $supp(a)$ is defined to be the set $\{j \in \mathbb{Z}^d : a_{(j)} \neq 1\}$. For a general element $a \in \mathcal{A}$ such that $a = \sum_{n=1}^{\infty} c_n a_n$ with a_n's simple tensor elements in $\mathcal{A}$ and c_n's complex coefficients, we define $supp(a) := \bigcup_{n\geq 1} supp(a_n)$. Let $\mathcal{A}_{loc}$ be the $*$-subalgebra of $\mathcal{A}$ generated by elements $a \in \mathcal{A}$ with finite support.

We note that $M_N(\mathbb{C})$ is spanned by a pair of noncommutative representatives $\{U, V\}$ of $\mathbb{Z}_N = \{0, 1 \cdots N-1\}$ such that $U^N = V^N = 1$ and $UV = wVU$, where $w \in \mathbb{C}$ is the primitive N-th root of unity. Now let us consider the infinite group $\mathcal{G} := \prod_{j\in\mathbb{Z}^d} \mathbb{Z}_N \times \mathbb{Z}_N$ and the projective unitary representation of $\mathcal{G}$ given by $\mathcal{G} \ni g \mapsto U_g = \prod_{j\in\mathbb{Z}^d} U^{(j)^{\alpha_j}} V^{(j)^{\beta_j}} \in \mathcal{A}$ where j-th component of g, $g_j = (\alpha_j, \beta_j)$. It is clear that any element $a \in \mathcal{A}$ can be written as $x = \sum_{g\in\mathcal{G}} c_g U_g$ with coefficients from $\mathbb{C}$.

For a given $r \in \mathcal{A}$, formally we define the Lindbladian $\mathcal{L} = \sum_{k\in\mathbb{Z}^d} \mathcal{L}_k$, where $\mathcal{L}_k x = \frac{1}{2}\{[r_k^*, x]\, r_k + r_k^*\, [x, r_k]\}$. Here $r_k = \tau_k(r)$ with $\tau : \mathbb{Z}^d \to Auto(\mathcal{A})$ induced by the coordinate translation on the lattice $\mathbb{Z}^d$.

For a suitable class of element $r = \sum_{g\in\mathcal{G}} c_g U_g$ so that $\sum_{g\in\mathcal{G}} |c_g|\ |g|^2 < \infty$, the associated Lindbladian $\mathcal{L}$ is defined on a dense $*$- subalgebra containing the $*$-subalgebra $\mathcal{A}_{\rm loc}$ and its closure is the generator of a conservative contractive QDS T_t [28] on $\mathcal{A}$. We define a family of maps $\{\theta_\nu^\mu : \mu, \nu \in \mathbb{Z}^d \cup \{0\}\}$ given by

$$\begin{aligned} \theta_\nu^\mu &= \mathcal{L},\ \text{for } (\mu,\nu) = (0,0), \\ &= \delta_k,\ \text{for } (\mu,\nu) = (k,0), \\ &= \delta_k^\dagger,\ \text{for } (\mu,\nu) = (0,k), \\ &= 0,\ \text{otherwise}, \end{aligned}$$

where $\delta_k, \delta_k^\dagger$ are bounded derivation on $\mathcal{A}$ given by $\delta_k(x) = [x, r_k]$ and $\delta_i^\dagger(x) := (\delta_k(x^*))^* = [r_k^*, x], \forall x \in \mathcal{A}$. In order to construct an EH dilation for the QDS T_t, we would like to solve the following qsde in $\mathcal{B}(\mathbf{h}_0) \bigotimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, l^2(\mathbb{Z}^d))))$,

$$j_t(x) = x + \int_0^t \sum_{\mu,\nu} j_s(\theta_\nu^\mu(x)) d\Lambda_\nu^\mu(s), \forall x \in \mathcal{A}_{\rm loc}.$$

In full generality the problem of obtaining an EH flow satisfying the above qsde seem to be intractable. However, EH flows for a class of QDS are obtained by standard iteration method. In order to dilate (EH type) more general QDS we follow a different path using the idea of constructing EH flow as a limit of associated quantum random walk [3, 23, 35].

In order to define a quantum random walk we use toy Fock spaces and basic operators on them [3]. Let Γ be the symmetric Fock space $\Gamma(L^2(\mathcal{K}))$ where $\mathcal{K}$ is

$L^2(\mathbb{R}_+, \mathbf{k}_0)$ with $\mathbf{k}_0$ is a complex separable Hilbert space. Let $\Lambda^\mu_\nu : \mu, \nu \geq 0$ be the family of fundamental processes on Γ with respect a fixed orthonormal basis $\{e_i : i \geq 1\}$ of $\mathbf{k}_0$ and Ω be the vacuum vector in Γ. For any partition $S \equiv (0, t_1, t_2 \cdots)$ of $\mathbb{R}_+$ the symmetric Fock space Γ can be viewed as infinite tensor product $\bigotimes_{n\geq 1} \Gamma_n$ with respect to the stabilizing vector $\Omega = \otimes_{n\geq 1}\Omega_n$, where $\Omega_n = \Omega_{(t_{n-1},t_n]}$ is the vacuum vector in $\Gamma_n = \Gamma(\mathcal{K}_{(t_{n-1},t_n]})$. For any $0 \leq s \leq t$ and $i \geq 1$ we define a vector $\chi^i_{(s,t]} := \frac{1_{(s,t]}\otimes e_i}{\sqrt{t-s}} \in \mathcal{K}_{(s,t]}$. It is clear that $\{\chi^i_{(s,t]}\}_{i\geq 1}$ is an orthonormal family in $\mathcal{K}_{(s,t]}$ and hence in $\Gamma_{(s,t]}$. Here we note that the Hilbert subspace $\mathbf{k_{(s,t]}}$ of $\Gamma_{(s,t]}$ spanned by these orthonormal vectors is canonically isomorphic to $\mathbf{k}_0$. Let us consider the subspace $\hat{\mathbf{k}}_{\mathbf{(s,t]}} = \mathbb{C}\ \Omega_{(s,t]} \bigoplus \mathbf{k_{(s,t]}}$ of Γ which is isomorphic to $\hat{\mathbf{k}}_0 = \mathbb{C} \bigoplus \mathbf{k}_0$ and we denote the space $\hat{\mathbf{k}}_{\mathbf{(t_{n-1},t_n]}}$ by $\hat{\mathbf{k}}_{\mathbf{n}}$. Now we define the *toy Fock space* $\Gamma(S)$ associated with the partition S, to be the subspace, infinite tensor product $\bigotimes_{n\geq 1} \hat{\mathbf{k}}_{\mathbf{n}}$ with respect to the stabilizing vector $\Omega = \otimes_{n\geq 1}\Omega_n$. Let $P(S)$ be the orthogonal projection of Γ onto the toy Fock space $\Gamma(S)$. Without loss of generality let us consider toy Fock spaces $\Gamma(S_h)$ associated with regular partition $S_h \equiv (0, h, 2h \cdots)$ for some $h > 0$ and denote the orthogonal projection by P_h. As the width h of the partition tends to 0, the orthogonal projection P_h converges strongly to identity on Γ. Now we define a family of operators $\{N^\mu_\nu[k] : \mu, \nu \geq 0, k \geq 1\}$ on the Fock space Γ, given by

$$\begin{aligned}
N^\mu_\nu[k] &= P_0[k]\frac{\Lambda^0_0[k]}{h} = P_0[k] \text{ for } (\mu,\nu) = (0,0),\\
&= \frac{\Lambda^0_j[k]}{\sqrt{h}} P_1[k] \text{ for } (\mu,\nu) = (0,j),\\
&= P_1[k]\frac{\Lambda^i_0[k]}{\sqrt{h}} \text{ for } (\mu,\nu) = (i,0),\\
&= P_1[k](\Lambda^i_j[k])P_1[k]P_h[k] \text{ for } (\mu,\nu) = (i,j),
\end{aligned}$$

where $P_0[k]$ and $P_1[k]$ are the orthogonal projections from Γ_k onto the one dimensional subspace spanned by Ω_k and $L^2([(k-1)h, kh], \mathbf{k}_0)$ respectively. Here, we have used the notations $\Lambda^\mu_\nu[k]$ for $\Lambda^\mu_\nu((k-1)h, kh]$ and $P_h[k]$ for the associated toy Fock space orthogonal projection restricted to the interval $[(k-1)h, kh]$. Clearly these operators $N^\mu_\nu[k]$'s act non trivially on Γ_k and as identity on the other components and they leave the subspace $\Gamma(S_h)$ invariant.

Given a $*$-homomorphic family $\{\beta(h) : h > 0\} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$, for each h we define a family of $*$- homomorphism $j_t^{(h)} : \mathcal{A} \rightarrow \mathcal{A} \bigotimes \mathcal{B}(\Gamma)$, as follows. For a given $t \geq 0$, we subdivide the interval $[0, t]$ into $[k] \equiv ((k-1)h, kh]\,,\ 1 \leq k \leq n$ so that $t \in ((n-1)h, nh]$ and set

$$\begin{aligned} p_0^{(h)}(x) &= x \otimes 1_\Gamma, \\ p_{kh}^{(h)}(x) &= \sum_{\mu,\nu} p_{(k-1)h}^{(h)}(\beta_\nu^\mu(h, x)) \otimes N_\nu^\mu[k] \end{aligned}$$

and $p_t^{(h)} = p_{nh}^{(h)}$. This family $\{p_t^{(h)} : t \geq 0\}$ is called the *quantum random walk* associated with the $*$-homomorphism $\beta(h)$.
Let us summerize the main observations, made in this thesis:

1. We construct EH flow for QDS T_t associated with $r \in \mathcal{A}$ such that r_k commute for different k's [16]. Covariance of the EH flow with respect to τ is proved.

2. For QDS T^ϕ associated with partial states ϕ on $\mathcal{A}$, EH flows are constructed and ergodicity of such flows are discussed.

3. Various estimate on toy fock space $\Gamma(S_h)$ for the operators $h^{\epsilon_{\mu,\nu}} N_\mu^\nu[k] - \Lambda_\mu^\nu[k]$ with $\epsilon_{0,0} = 1, \epsilon_{0,i} = \epsilon_{j,0} = \frac{1}{2}, \epsilon_{i,j} = 0$ for $i, j \geq 1$ are observed.

4. In coordinate-free language of quantum stochastic calculus similar estimate for basic operators as in 3 are established and quantum random walks are defined.

5. Strong convergence of the quantum random walk, associated with bounded structure maps, is proved under certain assumption using coordinate-free language of quantum stochastic calculus.

6. For a larger class of QDS T_t associated with elements $r \in \mathcal{A}_{\mathrm{loc}}$, $*$-homomorphic family $\{\beta(h) : h > 0\} : \mathcal{A} \rightarrow \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ are constructed. Which satisfies, for any (μ, ν), $\lim_{h\rightarrow 0} \frac{\beta_\nu^\mu(h,x) - \delta_\nu^\mu x}{h^{\epsilon_{\mu,\nu}}} = \theta_\nu^\mu(x), \forall x \in \mathcal{A}_{\mathrm{loc}}$. By using basic operators N_ν^μ quantum random walks are defined.

7. Weak convergence of the above quantum random walk is proved and observed that the weak limit satisfies the qsde with structure maps θ_ν^μ.

Contents of the thesis are in following order:

In Chapter-1, background materials for the thesis are briefly recalled. We begin with most basic objects, namely C^* and von Neumann algebra. Some basic results and concepts from theory of operator algebras, including a short description of UHF C^*-algebras [9] are given. In the end of this section completely positive maps are introduced and Stinespring's dilation [36] theorem is mentioned. Next section is devoted to semigroup theory and evolution equation on Banach space [37]. The Hille-Yosida theorem and results on perturbation are stated without proof. From approximation theory of semigroups Chernoff's theorem [5] and Trotter-Kato theorem [20] are recalled. Finally, the characterization of the generator of uniformly continuous QDS due to Christensen and Evans [7] is given. In section 3 Quantum Stochastic Calculus on symmetric Fock space, including a coordinate free description [15], is briefly recalled.

In Chapter-2, the class of QDS [28] on UHF C^*-algebra $\mathcal{A} = \otimes_{j\in\mathbb{Z}^d} M_N(\mathbb{C})$ is discussed in detail. For an element r, formally we define the Lindbladian $\mathcal{L} = \sum_{k\in\mathbb{Z}^d} \mathcal{L}_k$, where $\mathcal{L}_k x = \frac{1}{2}\{[r_k^*, x]\, r_k + r_k^*\, [x, r_k]\}$. For $r = \sum_{g\in\mathcal{G}} c_g U_g : \sum_{g\in\mathcal{G}} |c_g|\ |g|^2 < \infty$ the associated Lindbladian $\mathcal{L}$ is defined on a dense $*$- subalgebra $\mathcal{C}^1(\mathcal{A})$ containing $\mathcal{A}_{\text{loc}}$ and by Hille-Yosida theorem it is shown that the closure $\mathcal{L}$ is the generator of a conservative contractive QDS T_t on $\mathcal{A}$. Moreover these QDS are covariant with respect to the action τ of discrete infinite group $\mathbb{Z}^d$. For a particular class of QDS, namely, the QDS T^ϕ associated with partial state ϕ on $\mathcal{A}$, ergodicity properties are established.

In Chapter-3, we construct EH flows for a class of QDS $\{T_t\}$ of the Chapter-2. Here we consider the QDS associated with $r \in \mathcal{A}$ such that translates r_k for different $k \in \mathbb{Z}^d$ are commuting. In this situation we control the growth of $\|\theta_{\nu_1}^{\mu_1} \cdots \theta_{\nu_n}^{\mu_n}(x)\|$ for $x \in \mathcal{A}_{\text{loc}}$ and employ iteration technique, to obtain a unique solution for EH flow equation. Exploiting the commuting properties of r_k's, we show that j_t is a weak $*$-homomorphism and then by standard method we conclude that j_t is a $*$-homomorphism. Thus we obtain an EH dilation for QDS $\{T_t\}$. We also observe that the EH flow j_t is covariant with respect to τ. In last section of the chapter, QDS $\{T_t^\phi\}$ associated with partial states ϕ are considered. These QDS are ergodic in the sense of [28]. Here, the element $r \in \mathcal{A}$ whose associated Lindbladian is the generator of the QDS $\{T_t^\phi\}$ is supported on only one lattice point. An EH dilation for such QDS is obtained with a simple argument. As for the QDS, ergodicity of the corresponding EH flows are discussed.

In Chapter-4, we begin with a brief description of the toy Fock space $\Gamma(S)$ as a subspace of the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0))$ associated with a partition S of $\mathbb{R}_+$ [3]. The *toy Fock space* $\Gamma(S)$ associated with the partition S, is defined to be the subspace, infinite tensor product $\bigotimes_{n\geq 1} \hat{\mathbf{k}}_\mathbf{n}$ with respect to the stabilizing vector $\Omega = \otimes_{n\geq 1}\Omega_n$. Without loss of generality we consider the toy Fock space $\Gamma(S_h)$ associated with a regular partition $S_h \equiv (0, h, 2h \cdots)$ for some $h > 0$ and denote the orthogonal projection by P_h. The approximate basic operators N_ν^μ are defined and various estimates on the Fock space for operators $h^{\epsilon_{\mu,\nu}} N_\mu^\nu[k] - \Lambda_\mu^\nu[k]$, where $\epsilon_{i,j} = 0, \epsilon_{0,i} = \epsilon_{j,0} = \frac{1}{2}$ for $i, j \geq 1$, are obtained.

For a given $*$-homomorphic family $\{\beta(h) : h > 0\} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$, for each h we define a family of $*$-homomorphism family *quantum random walk* $p_t^{(h)} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\Gamma)$, by setting

$$p_0^{(h)}(x) = x \otimes 1_\Gamma,$$

$$p_t^{(h)}(x) = p_{nh}^{(h)}(x) = \sum_{\mu,\nu} p_{(n-1)h}^{(h)}(\beta_\nu^\mu(h, x)) \otimes N_\nu^\mu[n]$$

for $t \in ((n-1)h, nh]$. We call this family $\{p_t^{(h)} : t \geq 0\}$ as *quantum random walk* associated with homomorphism $\beta(h)$.

In one dimensional noise case [35] EH flows are constructed by using quantum random walk model following [23, 3]. There an EH flow (with bounded structure maps) is obtained as a strong limit of associated quantum random walks. Here, we have discussed the strong convergence of quantum random walks in the situation with infinite dimensional noise. To handle the presence of infinitely many noise components we have used coordinate-free language of quantum stochastic calculus developed in [15]. For $S \in \mathcal{B}(\mathbf{h}_0), R \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0)$ and $T \in \mathcal{B}(\mathbf{h}_0 \bigotimes \mathbf{k}_0)$ we define the operators as follows, for $k \geq 1$.

$$N_S^1[k] = SP_0[k]\frac{\Lambda_S^1[k]}{h} = SP_0[k],$$
$$N_R^2[k] = \frac{\Lambda_R^2[k]}{\sqrt{h}}P_1[k],$$
$$N_R^3[k] = P_1[k]\frac{\Lambda_R^3[k]}{\sqrt{h}},$$
$$N_T^4[k] = P_1[k](\Lambda_T^4[k])P_1[k]P_h[k]$$

where

$$\Lambda_S^1[k] = \mathcal{I}_S((k-1)h, kh),$$
$$\Lambda_R^2[k] = a_R((k-1)h, kh),$$
$$\Lambda_R^3[k] = a_R^\dagger((k-1)h, kh),$$
$$\Lambda_T^4[k] = \Lambda_T((k-1)h, kh).$$

Similar estimates on Fock space for these operators $h^{\epsilon_l}N^l[k] - \Lambda^l[k]$ (where, $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = \frac{1}{2}$ and $\epsilon_4 = 0$) are obtained as earlier.
Let $\{\beta(h)\}$ be a $*$-homomorphic family from a von Neumann algebra $\mathcal{A}$ to $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$. For $x \in \mathcal{A}$, $\beta(h,x) = ((\beta_\nu^\mu(x))) = \begin{pmatrix} \beta_1(h,x) & (\beta_2(h,x))^* \\ \beta_3(h,x) & \beta_4(h,x) \end{pmatrix}$ with respect to direct sum decomposition $\hat{\mathbf{k}}_0 = \mathbb{C} \bigoplus \mathbf{k}_0$. Now using the basic operators N^l we define a quantum random walk $p_t^{(h)}$ associated with $\beta(h)$.

Let T_t be a uniformly continuous conservative QDS on von Neumann algebra $\mathcal{A}$ with the generator $\mathcal{L}$. Then by results in [15] there exists a Hilbert space $\mathbf{k}_0$ and structure maps $(\mathcal{L}, \delta, \sigma)$ where, $\mathcal{L} \in \mathcal{B}(\mathcal{A}), \delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \bigotimes \mathbf{k}_0)$ and $\sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \bigotimes \mathcal{B}(\mathbf{k}_0))$ so that the map
$\Theta = ((\theta_\nu^\mu)) = \begin{pmatrix} \theta_1 & (\theta_2(\cdot))^* \\ \theta_3 & \theta_4 \end{pmatrix} = \begin{pmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \sigma \end{pmatrix} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}_0})$ is a bounded CCP map with the structure

$$\theta(x) = V^*(x \otimes 1_{\hat{\mathbf{k}}_0})V + W(x \otimes 1_{\hat{\mathbf{k}}_0}) + (x \otimes 1_{\hat{\mathbf{k}}_0})W^*, \forall x \in \mathcal{A},$$

where $V, W \in \mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0)$. The qsde,

$$j_t(x) = x \otimes 1_\Gamma + \sum_{\mu,\nu \geq 0} \int_0^t j_s(\theta_\nu^\mu(x)) d\Lambda_\nu^\mu(s) \ , \forall x \in \mathcal{A}$$

admit a unique strongly continuous solution j_t.

To obtain $*$-homomrphic property of the family $\{j_t\}$ we shall make the following assumption. Let $\{\beta(h)\}$ be a $*$-homomorphic family from $\mathcal{A}$ to $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ satisfying for $l = 1, 2, 3$ and 4,

$$\|\beta_l(h, x) - b_l(x) - h^{\varepsilon_l}\theta_l(x)\| \leq C\|x\|h^{1+\varepsilon_l}, \forall x \in \mathcal{A},$$

where b_l's are linear maps given by $b_1(x) = x, b_4(x) = x \otimes 1_{\mathbf{k}_0}, b_2(x) = b_3(x) = 0 \in \mathcal{A} \bigotimes \mathbf{k}_0$. Moreover, this estimates extend uniformly for $m \geq 0$, ampliating Θ, b and β as maps from $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}})$ into $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}}) \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ i.e for any l

$$\|\beta_l(h, X) - b_l(X) - h^{\varepsilon_l}\theta_l(X)\| \leq C'\|X\|h^{1+\varepsilon_l}, \forall X \in \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}})$$

for some constant C' independent of $m \geq 0$.

Using various estimates on the Fock space and the estimates in the above assumption we have shown that $p_t^{(h)}$ converges strongly to j_t and it follows that j_t is a $*$-homomorphism.

In Chapter-5, we focus once again on the UHF model discussed earlier. Here we consider a larger class of QDS associated with elements $r \in \mathcal{A}_{\mathrm{loc}}$. Using local structure of the algebra we have constructed a $*$-homomorphic family $\{\beta(h) : h > 0\} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$, and we obtain a quantum random walk $p_t^{(h)}$ associated with $\beta(h)$ using basic operators N_ν^μ appear in previous Chapter. It is observed that for any (μ, ν), $\lim_{h \to 0} \frac{\beta_\nu^\mu(h,x) - \delta_\nu^\mu x}{h^{\epsilon_{\mu,\nu}}} = \theta_\nu^\mu(x), \forall x \in \mathcal{A}_{\mathrm{loc}}$. Next, using the above fact we prove that for any $x \in \mathcal{A}_{\mathrm{loc}}$, $p_t^{(h)}(x)$ converges weakly and the weak limit j_t satisfies the EH type qsde on $\mathcal{A}_{\mathrm{loc}}$ with structure maps θ_ν^μ, but we are not yet able to conclude wheather j_t is a $*$-homomorphism.

Chapter 1

Preliminaries

Here basic results and concepts from theory of operator algebras, including a short description of UHF C^*-algebras are given , for detail we refer to [9, 5, 20, 37, 36]. In the last section Quantum Stochastic Calculus on symmetric Fock space [33], including a coordinate-free description [15], is recalled. Let us begin with most basic objects, the C^* and von Neumann algebras.

1.1 C^* and von Neumann algebras

1.1.1 C^*-algebras

Definition 1.1.1. *A complex $*$-algebra $\mathcal{A}$, equipped with a C^*-norm, i.e. $\|x^*x\| = \|x\|^2$, is called a pre-C^*-algebra. Furthermore, if $\mathcal{A}$ is complete with respect to C^*-norm, then it is called a C^*-algebra.*

A C^*-algebra is called unital or nonunital depending upon the existence of identity element on it. For any Hilbert space $\mathcal{H}$, the space of all bounded linear operators on $\mathcal{H}$ with operator norm and its closed $*$-subalgebras are some concrete examples of C^*-algebra. For a locally compact Hausdorff space X, the space $C_0(X)$ of all complex valued continuous functions on X, vanishing at infinity, with supremum norm and with complex conjugation as the $*$-operation forms a commutative C^*-algebra under pointwise addition and multiplication. In particular for compact $X, C_0(X) = C(X)$, the space of all continuous functions on X, is a unital C^*-algebra. The following result completely characterizes the commutative C^*-algebras :

Theorem 1.1.2. *(Gelfand-Naimark) Let $\mathcal{A}$ be a commutative C^*-algebra. Then there exists a locally compact Hausdorff space X such that $\mathcal{A}$ is isometrically isomorphic to the C^*-algebra $C_0(X)$. Moreover if $\mathcal{A}$ is unital, then X is compact.*

Any nonunital C^*-algebra can always be isometrically embedded as a two sided ideal in a unital C^*-algebra canonically. So for rest of the section we consider $\mathcal{A}$ to be a unital C^*-algebra. For $a \in \mathcal{A}$, the resolvent of a, denoted by $\rho(a)$, is the subset $\{\lambda \in \mathbb{C} : (a-\lambda)^{-1} \in \mathcal{A}\}$; and its complement is called the spectrum of a, denoted by $\sigma(a)$. The spectral radius $spr(a)$ of a is defined to be, $spr(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$. It is a basic fact of theory of C^*-algebras that the norm on a C^*-algebra is uniquely determined by the algebraic structure. For any element a, $\|a\|^2 = spr(a^*a)$.
There is a rich functional calculus which enables one to form functions of elements of the C^*-algebra $\mathcal{A}$. For any function f which is holomorphic in some domain containing $\sigma(a)$, one obtains an element $f(a) \in \mathcal{A}$ by the holomorphic functional calculus. Furthermore, for any normal element x, there is a continuous functional calculus sending $f \in C(\sigma(a))$ to $f(a) \in \mathcal{A}$ where $f \mapsto f(a)$ is a $*$-isometric isomorphism from $C(\sigma(a))$ onto $C^*(a)$, the C^*-subalgebra of $\mathcal{A}$ generated by a. In particular, for any positive element $a \geq 0$, i.e. a can be written as $a = b^*b$ for some $b \in \mathcal{A}$, we can form a positive square root $\sqrt{a} \in \mathcal{A}$ satisfying $\sqrt{x}^2 = x$. A linear functional ϕ on $\mathcal{A}$ is said to be positive if $\phi(a^*a) \geq 0$ for all a. This is a remarkable and useful result that an element $a \in \mathcal{A}$ is positive if and only if $\phi(a) \geq 0$ for every positive functional ϕ on $\mathcal{A}$. It can be shown that the algebraic property of positivity implies the boundedness of ϕ, in particular $\|\phi\| = \phi(1)$. Any positive linear functional ϕ with $\phi(1) = 1$ is called a *state* on $\mathcal{A}$. It is said to be *faithful* if $\phi(a^*a) = 0$ implies $a = 0$, *pure* if any state ψ satisfies $0 \leq \psi \leq \phi$ must be of the form $\psi = \lambda\phi$ for some $\lambda \in [0,1]$ and *trace* if $\phi(ab) = \phi(ba)$ $\forall a, b \in \mathcal{A}$. . Here we state the celebrated theorem due to Gelfand, Naimark and Segal, known as the *GNS construction* for a state.

Theorem 1.1.3. *Let $\mathcal{A}$ be a C^*-algebra. Given a state ϕ on $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}_\phi$, a $*$-representation π_ϕ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\phi)$ and a vector $\xi_\phi \in \mathcal{H}_\phi$ which is cyclic* *in the sense that* $\{\pi_\phi(a)\xi_\phi : a \in \mathcal{A}\}$ *is total in* $\mathcal{H}_\phi$*, satisfying*

$$\phi(a) = \langle \xi_\phi, \pi_\phi(a)\xi_\phi \rangle, \ \forall a \in \mathcal{A}.$$

Moreover, ϕ *is pure if and only if* π_ϕ *is irreducible.*

This triple $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ is called the *GNS triple* for $(\mathcal{A}, \phi)$ and $\mathcal{H}_\phi$ is called *GNS Hilbert space* for the pair $(\mathcal{A}, \phi)$ and it is denoted by $L^2(\mathcal{A}, \phi)$.

1.1.2 UHF C^*-algebra

(Ref. [9]) A special class of C^*-algebras, namely approximately finite dimensional C^*-algebras (in short AF C^*-algebra) are built out of the class of finite dimensional matrix algebras. These algebras are well understood and arise in the study of quantum statistical mechanics. Before going to give the definition of AF algebra, let us note the following useful facts.

Theorem 1.1.4. *Any finite dimensional* C^**-algebra* $\mathcal{A}$ *is* $*$*-isomorphic to a direct sum of full matrix algebras, i,e.*

$$\mathcal{A} \simeq M_{n_1}(\mathbb{C}) \bigoplus \cdots M_{n_k}(\mathbb{C}) \textit{ for some } n_1, n_2, \cdots n_k \geq 1.$$

So, in particular, every finite dimensional C^*-algebra is unital.
The following result describe how one finite dimensional algebra fits into another.

Lemma 1.1.5. *Let* $\mathcal{A}, \mathcal{B}$ *be two finite dimensional* C^**-algebras such that*

$$\mathcal{A} \simeq M_{n_1}(\mathbb{C}) \bigoplus \cdots M_{n_k}(\mathbb{C}) \textit{and}$$

$$\mathcal{B} \simeq M_{m_1}(\mathbb{C}) \bigoplus \cdots M_{m_l}(\mathbb{C}).$$

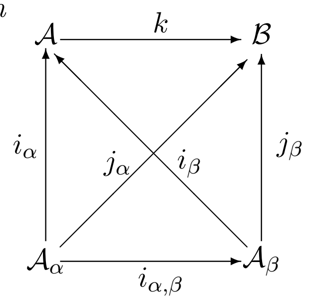
Then a $*$*-homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ *is uniquely determined upto unitary equivalence by an* $l \times k$*-matrix* $((\lambda_{ij}))$ *with entries from non negative integers such that*

$$\sum_{j=1}^{k} \lambda_{ij} n_j \leq m_i, \forall i = 1, 2, \cdots l. \tag{1.1. 1}$$

Moreover, in case of ϕ *is unital, equality holds in 1.1. 1.*

So, given any two finite dimensional C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, the lower dimensional one can be embedded isometrically in the higher dimensional algebra in various ways. Now let us define the Inductive Limit of C^*-algebras and the AF C^*-algebras we are interested in.

Definition 1.1.6. *Let $\{\mathcal{A}_\alpha\}_{\alpha\in I}$ be a directed family of C^*-algebras, i.e. for any $\alpha < \beta$ in the directed set I, there is an isometric isomorphism $i_{\alpha,\beta}$ from $\mathcal{A}_\alpha$ into $\mathcal{A}_\beta$ and $i_{\alpha,\beta} = i_{\gamma,\beta} \circ i_{\alpha,\gamma}$ when $\alpha < \gamma < \beta$. Then there exists a universal C^*-algebra $\mathcal{A}$, called Inductive Limit of the directed family $(\mathcal{A}_\alpha, i_{\alpha,\beta})$ and isometric isomorphism i_α from $\mathcal{A}_\alpha$ into $\mathcal{A}$ such that $i_\alpha = i_\beta \circ i_{\alpha,\beta}$ and such that $\mathcal{A} = \bigcup_{\alpha\in I} i_\alpha(\mathcal{A}_\alpha)$. The Inductive Limit has universal property that for any C^*-algebra $\mathcal{B}$ with isometric isomorphisms j_α from $\mathcal{A}_\alpha$ into $\mathcal{B}$ such that $j_\alpha = j_\beta \circ i_{\alpha,\beta}$, there exists an isometric isomorphism $k : \mathcal{A} \to \mathcal{B}$ and following diagram*



commutes.

Definition 1.1.7. *A C^*-algebra $\mathcal{A}$ is said to be an AF C^*-algebra if it is the Inductive Limit of a family of C^*-subalgebras $\{\mathcal{A}_n : n \geq 0\}$ with isometric imbeddings $i_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ for $n \geq 0$. Here $\mathcal{A}_0 = \mathbb{C}I$ in case of $\mathcal{A}$ is unital and $\mathcal{A} = \overline{\bigcup_{n\geq 0}\mathcal{A}_n}$, the norm closure.*

Example 1.1.8. *For any complex separable Hilbert space $\mathcal{H}$, let us consider the C^*-algebra $\mathcal{A} = \mathbb{C}\, I + \mathcal{B}_0(\mathcal{H})$, where $\mathcal{B}_0(\mathcal{H})$ is the space of all compact operators on $\mathcal{H}$. For an increasing sequence of orthogonal projections P_n with rank$(P_n) = n$, converging strongly to the identity, we set $\mathcal{A}_n = \mathbb{C}\ P_n^\perp + P_n\ \mathcal{B}_0(\mathcal{H})\ P_n \simeq \mathbb{C} + M_n(\mathbb{C}), \mathcal{A}$ is the closure of $\bigcup_{n\geq 0}\mathcal{A}_n$ with the canonical imbedding of $M_n(\mathbb{C})$ into $M_{n+1}(\mathbb{C})$, which sends $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$*

The union $\bigcup_{n\geq 0}\mathcal{A}_n$ will be not effected if we take a subsequence instead of this chain of subalgebra $\mathcal{A}_n$. In fact the union is unique upto unitary conjugation, more precisely:

Theorem 1.1.9. *Let $\mathcal{A}$ be an AF C^*-algebra, such that it can be written as the closure of the increasing union of two chains*

$$\mathcal{A} = \overline{\bigcup_{m\geq 0}\mathcal{A}_m} = \overline{\bigcup_{n\geq 0}\mathcal{B}_n}.$$

Then for any $\epsilon \geq 0$ there exists a unitary operator W in the unitization of $\mathcal{A}$ with $\|W - I\| < \epsilon$ such that

$$\bigcup_{m\geq 0}\mathcal{A}_m = W(\bigcup_{n\geq 0}\mathcal{B}_n)W^*.$$

In particular there are subsequences $\{m_i\}$ and $\{n_i\}$ of $\mathbb{N}$ so that $\mathcal{A}_{m_i} \subseteq W\mathcal{B}_{n_i}W^* \subseteq \mathcal{A}_{m_{i+1}}, \forall i \geq 1$, i.e. $\mathcal{A}_{m_i} \hookrightarrow \mathcal{B}_{n_i}$ and in $\mathcal{B}_{n_i} \hookrightarrow \mathcal{A}_{m_{i+1}}$ isometrically.

As an immediate consequence of this theorem we obtain that If $\mathcal{A} = \overline{\bigcup_{m\geq 0}\mathcal{A}_m}$ and $\mathcal{B} = \overline{\bigcup_{n\geq 0}\mathcal{B}_n}$ are two $*$-isomorphic AF C^*-algebras, then $\bigcup_{m\geq 0}\mathcal{A}_m$ and $\bigcup_{n\geq 0}\mathcal{B}_n$ are also $*$-isomorphic.

Next we discuss about a particular class of AF C^*-algebras called *Uniformly hyperfinite* C^*-algebras (in short UHF C^*-algebras).

Definition 1.1.10. *An AF C^*-algebra is said to be a UHF C^*-algebra if it is an increasing union of unital subalgebras which are isomorphic to full matrix algebras $\{M_{k_n}(\mathbb{C})\}$ for some sequence of positive integers $\{k_n\}$.*

Since a unital imbedding of $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$ requires m/n (m divides n), we have an increasing sequence $k_1/k_2 \cdots$. So for a prime number p there exists a unique number $\epsilon_p \in \{1, 2, \cdots \infty\}$, given by $\epsilon_p = \sup\{l : p^l/k_n \text{ as } n \to \infty\}$. Now we define a number $\delta(\mathcal{A})$ associated with the UHF C^*-algebra $\mathcal{A}$, known as *supernatural number*, by a formal product:

$$\delta(\mathcal{A}) = \prod_{p:prime} p^{\epsilon_p}.$$

This number gives a complete invariant for the class of UHF C^*-algebras by the following result of Glimm:

Theorem 1.1.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be two UHF C^*-algebras. Then $\mathcal{A}$ is isomorphic to $\mathcal{B}$ if and only if $\delta(\mathcal{A}) = \delta(\mathcal{B})$.*

In particular, we are interested in the class of N^∞ UHF C^*-algebras obtained as infinite tensor product of finite dimensional matrix algebra $M_N(\mathbb{C})$. For a fixed pair of positive integers d and N first let us consider the infinite lattice $\mathbb{Z}^d$. For $j = (j_1, j_1, \cdots j_d)$ in $\mathbb{Z}^d$ let $|j|$ denote $\max\{|j_i| : i = 1, 2, \cdots d\}$. For a finite subset Λ of $\mathbb{Z}^d$ we define $|\Lambda|$ to be the cardinality of Λ. Now, let us consider the infinite algebraic tensor product $\bigotimes_{j\in\mathbb{Z}^d} M_N(\mathbb{C})$, with respect to the stabilizing sequence of identities $I \in M_N(\mathbb{C})$. For an increasing sequence of finite subsets $\{\Lambda_n\}_{n\geq 1}, \Lambda_n = \{j : |j| \leq n\}$ of $\mathbb{Z}^d$, let $\mathcal{A}_n = \bigotimes_{j\in\Lambda_n} M_N(\mathbb{C})$ and $\mathcal{A}_0 = \mathbb{C}I$. It is clear that $\mathcal{A}_n = M_{k_n}(\mathbb{C})$, where $k_n = N^{|\Lambda_n|}, |\Lambda_n| = (2n + 1)^d$ and $\mathcal{A}_n$ is isometrically embedded in $\mathcal{A}_{n+1}$ by sending $\mathcal{A}_n \ni a \mapsto a\otimes I$, where I is the identity element in $M_{N^{2d}}(\mathbb{C})$. Thus $\mathcal{A}$ is an increasing union of full matrix algebra $\mathcal{A}_n$ and is called N^∞-UHF C^*-algebra.

1.1.3 von Neumann algebras

For a Hilbert space $\mathcal{H}$, the space of all bounded linear operators $\mathcal{B}(\mathcal{H})$, as a Banach space equipped with the operator-norm topology. There are many other useful topologies with respect to which $\mathcal{B}(\mathcal{H})$ is a locally convex topological vector space. The most useful ones are weak, strong, ultra-weak and ultra-strong topologies. However, although $\mathcal{B}(\mathcal{H})$ is complete in each of these topologies, a general C^*- subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ need not be so. It is easily provable that $\mathcal{A}$ is complete in all of the above four locally convex topologies if and only if it is complete in any one of them, and in such a case $\mathcal{A}$ is said to be a *von Neumann algebra.* Furthermore, the strong (respectively weak) and ultra-strong (respectively ultra-weak) topologies coincide on norm-bounded convex subsets of $\mathcal{A}$. For a σ-finite measure space $(X, \mathcal{F}, \mu)$, $L^\infty(X, \mathcal{F}, \mu)$, the space of all bounded measurable functions on X is a commutative von Neumann algebra. For any subset $M \subseteq \mathcal{B}(\mathcal{H})$, we denote M', the commutant of M in $\mathcal{B}(\mathcal{H})$, i.e. $\{a \in \mathcal{B}(\mathcal{H}) : am = ma, \forall m \in M\}$. The following fundamental result due to von Neumann, known as the *"Double commutant theorem"* is of fundamental importance in the study of von Neumann algebras.

Theorem 1.1.12. *Let $\mathcal{A}$ be a C^*-subalgebra in $\mathcal{B}(\mathcal{H})$ with trivial null space. Then $\mathcal{A}''(= (\mathcal{A}')') = \bar{\mathcal{A}}^w = \bar{\mathcal{A}}^s$, where $\bar{\mathcal{A}}^w$ and $\bar{\mathcal{A}}^s$ are closure of $\mathcal{A}$ in weak and strong operator topologies of $\mathcal{B}(\mathcal{H})$ respectively.*

Thus, in particular, any unital C^*-subalgebra $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.

A state ϕ on a von Neumann algebra $\mathcal{A}$ is said to be *normal* if whenever $\{a_\alpha\}$ is an increasing net in $\mathcal{A}$ such that $0 \leq a_\alpha \uparrow a$, one has $\phi(a_\alpha)$ increases to $\phi(a)$. More generally, we call a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ (where $\mathcal{B}$ is a von Neumann algebra) to be normal if whenever $0 \leq a_\alpha \uparrow a$ in $\mathcal{A}$, one has $\Phi(a_\alpha) \uparrow \Phi(a)$ weakly in $\mathcal{B}$. It is known that a positive linear map is normal if and only if it is continuous with respect to the ultra-weak topology mentioned earlier. In view of this fact, we shall say that a bounded linear map between two von Neumann algebras is normal if it is continuous with respect to the respective two ultra-weak topologies. Normal states, and more generally normal positive linear maps (in particular, normal $*$- homomorphisms) play a major role in the study of von Neumann algebras. The following result describes the structure of a normal state.

Theorem 1.1.13. *[5] ϕ is a normal state of a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ if and only if there is a positive trace-class operator ρ on $\mathcal{H}$ such that $\phi(a) = tr(\rho a)$ for all $a \in \mathcal{A}$.*

1.2 Hilbert modules

A Hilbert space is a complex vector space equipped with a complex valued inner product. A natural generalization of this is the concept of Hilbert module, which has become quite an important tool of analysis and mathematical physics in recent times. Let us briefly introduce the concept of Hilbert von Neumann modules. For a comprehensive study of such structures we refered to [22].

Hilbert C^*-modules

Definition 1.2.1. *Given a C^* algebra $\mathcal{A}$, a semi-Hilbert $\mathcal{A}$-module E is a right $\mathcal{A}$-module equipped with a sesquilinear map $\langle .,. \rangle : E \times E \to \mathcal{A}$ satisfying $\langle x, y\rangle^* = \langle y, x\rangle, \langle x, ya\rangle = \langle x, y\rangle a$ and $\langle x, x\rangle \geq 0$ for $x, y \in E$ and $a \in \mathcal{A}$. A semi-Hilbert module E is called a pre-Hilbert module if $\langle x, x\rangle = 0$ if and only if $x = 0$; and it is* *called a Hilbert C^*-module if furthermore E is complete in the norm $x \mapsto \|\langle x, x\rangle\|^{\frac{1}{2}}$ where $\|.\|$ the C^* norm of $\mathcal{A}$.*

It is clear that any semi-Hilbert $\mathcal{A}$-module can be made into a Hilbert module in a canonical way : first quotienting it by the ideal $\{x : \langle x, x\rangle = 0\}$ and then completing the quotient.

The $\mathcal{A}$-valued inner product $\langle ., .\rangle$ of a Hilbert module shares some of the important properties of usual complex valued inner product of a Hilbert space, such as the Cauchy-Schwarz inequality. However, some of the crucial properties of Hilbert spaces do not extend to general Hilbert modules : the most remarkable ones are the projection theorem and self-duality. Closed submodules of a Hilbert module need not be orthocomplemented, that is, given a closed submodule F of E, there need not exist any closed submodule G such that $E = F \bigoplus G$. Furthermore, the Banach space of all $\mathcal{A}$-valued, $\mathcal{A}$-linear, bounded maps on a Hilbert $\mathcal{A}$-module E may not be isometrically anti-isomorphic to E, in contrast to the Riesz's theorem for complex Hilbert space. For example, a bounded $\mathcal{A}$-linear map from one Hilbert $\mathcal{A}$-module to another may not have an adjoint. For this reason, the role played by the set of bounded linear maps between Hilbert spaces is taken over by the set of adjointable $\mathcal{A}$-linear maps.

Definition 1.2.2. *Let E and F be two Hilbert $\mathcal{A}$-modules. We say that an $\mathcal{A}$-linear map L from E to F is adjointable if there exists a bounded $\mathcal{A}$-linear map L^* from F to E such that $\langle L(x), y\rangle = \langle x, L^*(y)\rangle$ for all $x \in E,\ y \in F$. We call L^* the adjoint of L. The set of all adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$. In case $E = F$, we write $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$.*

It may be noted that an adjointable map is automatically bounded.

Let us fix two Hilbert $\mathcal{A}$-modules E and F. For $t \in \mathcal{L}(E, F)$ and $x \in E$, it is easy to prove that $\langle tx, tx\rangle \leq \|t\|^2\langle x, x\rangle$, where $\|t\|$ denotes the map-norm of t. The topology on $\mathcal{L}(E, F)$ given by the family of seminorms $\{\|.\|_x, \|.\|_y : x \in E, y \in F\}$ where $\|t\|_x = \langle tx, tx\rangle^{\frac{1}{2}}$ and $\|t\|_y = \langle t^*y, t^*y\rangle^{\frac{1}{2}}$, is known as the strict topology. For $x \in E$, $y \in F$, we denote by $\theta_{x,y}$ the element of $\mathcal{L}(E, F)$ defined by $\theta_{x,y}(z) = y\langle x, z\rangle$ $(z \in F)$. The norm-closed subset generated by $\mathcal{A}$-linear span of $\{\theta_{x,y} : x \in E, y \in F\}$ is called the set of compact operators and denoted by $\mathcal{K}(E, F)$. It should be noted that these objects need not be compact in the sense of compact operators between two Banach spaces, though this abuse of terminology has become standard. It is known that $\mathcal{K}(E,F)$ is dense in $\mathcal{L}(E,F)$ in the strict topology. In case $F=E$, we denote $\mathcal{K}(E,F)$ by $\mathcal{K}(E)$. Note that both $\mathcal{L}(E)$ and $\mathcal{K}(E)$ are C^* algebras.

Hilbert von Neumann modules

If $\mathcal{A}$ is a concrete C^* algebra in $\mathcal{B}(\mathbf{h})$ for some Hilbert space $\mathbf{h}$, then for any Hilbert space $\mathcal{H}$, the pre-Hilbert module $\mathcal{A}\bigotimes_{\text{alg}}\mathcal{H}$ may be viewed as a subset of $\mathcal{B}(\mathbf{h},\mathbf{h}\bigotimes\mathcal{H})$ and $\mathcal{A}\bigotimes_{C^*}\mathcal{H}$ is the closure of this subset under the operator-norm inherited from $\mathcal{B}(\mathbf{h},\mathbf{h}\bigotimes\mathcal{H})$. Instead, we may inherit one of the locally convex topologies from $\mathcal{B}(\mathbf{h},\mathbf{h}\bigotimes\mathcal{H})$, e.g., the topology of strong convergence, and close $\mathcal{A}\bigotimes_{\text{alg}}\mathcal{H}$ under that topology. This will lead to another topological module, in general bigger than $\mathcal{A}\bigotimes_{C^*}\mathcal{H}$. We denote the closure by $\mathcal{A}\bigotimes_s\mathcal{H}$ or simply by $\mathcal{A}\bigotimes\mathcal{H}$ when there is no possibility of confusion. $\mathcal{A}\bigotimes_s\mathcal{H}$ has a natural $\mathcal{A}''$ module action from both sides and has a natural $\mathcal{A}''$ -valued inner product. In view of this, we assume that $\mathcal{A}$ itself is a unital von Neumann algebra in $\mathcal{B}(\mathbf{h})$. We note a few simple but useful facts about the Hilbert von Neumann module $\mathcal{A}\bigotimes\mathcal{H}$. For this, let us first introduce some notations. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $A\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_1\bigotimes\mathcal{H}_2)$. For each $f\in\mathcal{H}_2$, we define a linear operator $\langle f,A\rangle$ on $\mathcal{H}_1$ such that,

$$\langle\langle f,A\rangle u,v\rangle=\langle Au,v\otimes f\rangle,\forall u,v\in\mathcal{H}_1. \tag{1.2. 1}$$

We shall denote by $\langle A,f\rangle$ the adjoint of $\langle f,A\rangle$, whenever it exists. Clearly, if A is bounded, then so is $\langle f,A\rangle$ and $\|\langle f,A\rangle\|\leq\|A\|\,\|f\|$. Similarly, for any $T\in\mathcal{B}(\mathcal{H}_1\bigotimes\mathcal{H}_2)$ and $f\in\mathcal{H}_2$, one can define $T_f\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_1\bigotimes\mathcal{H}_2)$ by setting

$$T_fu=T(u\otimes f),\forall u\in\mathcal{H}_1. \tag{1.2. 2}$$

With the above notations, let us recall some properties of $\mathcal{A}\otimes\mathcal{H}$.

Lemma 1.2.3. *Any element X of Hilbert von Neumann module $\mathcal{A}\bigotimes\mathcal{H}$ can be written as, $X=\sum_{\alpha\in J}x_\alpha\otimes\gamma_\alpha$, where $\{\gamma_\alpha\}_{\alpha\in J}$ is an orthonormal basis of $\mathcal{H}$ and $x_\alpha\in\mathcal{A}$. The above sum over a possibly uncountable index set J makes sense in the* *usual way: it is strongly convergent and* $\forall u \in \mathbf{h}$, *there exists an at most countable subset* J_u *of* J *such that* $Xu = \sum_{\alpha \in J_u} (x_\alpha u) \otimes \gamma_\alpha$. *Moreover, once* $\{\gamma_\alpha\}$ *is fixed,* x_α*'s are uniquely determined by* X.

Corollary 1.2.4. *Let* $X, Y \in \mathcal{A} \bigotimes \mathcal{H}$ *be given by* $X = \sum_{\alpha \in J} x_\alpha \otimes \gamma_\alpha$ *and* $Y = \sum_{\alpha \in J} y_\alpha \otimes \gamma_\alpha$ *as in the lemma above. For any finite subset* I *of* J, *if we denote by* X_I *and* Y_I *the elements* $\sum_{\alpha \in I} x_\alpha \otimes \gamma_\alpha$ *and* $\sum_{\alpha \in I} y_\alpha \otimes \gamma_\alpha$ *respectively, then* $\lim_I \langle X_I, Y_I \rangle = \langle X, Y \rangle$ *where the limit is taken over the directed family of finite subsets of* J *with usual partial ordering by inclusion.*

Proof. The proof is an easy adaptation of Lemma 27.7 in [33]. □

Here, we note a convenient necessary and sufficient criterion from [15] for verifying whether an element of $\mathcal{B}(\mathbf{h}, \mathbf{h} \bigotimes \mathcal{H})$ belongs to $\mathcal{A} \bigotimes \mathcal{H}$.

Lemma 1.2.5. *Let* $X \in \mathcal{B}(\mathbf{h}, \mathbf{h} \bigotimes \mathcal{H})$. *Then* X *belongs to* $\mathcal{A} \bigotimes \mathcal{H}$ *if and only if* $\langle \gamma, X \rangle \in \mathcal{A}$ *for all* γ *in some dense subset* $\mathcal{D}$ *of* $\mathcal{H}$.

In case $\mathcal{H} = \Gamma(\mathbf{k})$, we call the module $\mathcal{A} \bigotimes \Gamma(\mathbf{k})$ as the right Fock $\mathcal{A}$-module over $\Gamma(\mathbf{k})$, for short the *Fock module*, and denote it by $\mathcal{A} \bigotimes \Gamma$.

1.3 Some general theory of Semigroups on Banach spaces

Here, we recall some standard and useful results from the theory of semigroups of bounded linear operators on a Banach space [20, 37]. Let X be a Banach space. A semigroup on X is a one parameter family of bounded linear operators $\{T_t : t \in \mathbb{R}_+\}$ on X satisfying $T_s.T_t = T_{s+t}, \forall s, t \geq 0,\ T_0 = I$. If $\lim_{t \to 0} T_t a = a, \forall a \in X$, then the semigroup is called strongly continuous (or C_0-semigroup). For a C_0-semigroup $\{T_t\}$, we define a linear operator $\mathcal{L}$ on X, with domain

$$\mathcal{D}(\mathcal{L}) = \{x \in X : \lim_{t \to 0} \frac{T_t(x) - x}{t} \ \textit{exists}\}$$

given by

$$\mathcal{L}(x) = \lim_{t \to 0} \frac{T_t(x) - x}{t}.$$

This operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is called the infinitesimal generator of the semigroup $\{T_t\}$ and $\mathcal{D}(\mathcal{L})$ is dense. A semigroup $\{T_t\}$ is called uniformly continuous if $\lim_{t \to 0} \|T_t -$

$I\| = 0$. For such a semigroup the generator is bounded. Any C_0-semigroup $\{T_t\}$ on X is *quasi-bounded,* i.e. there exist constants $M \ge 0$ and $\beta \ge 0$ such that $\|T_t\| \le Me^{\beta t}$ $\forall t \ge 0$. Semigroup $\{T_t\}$ is called isometric or contractive according to each T_t is. For any contractive C_0-semigroup $\{T_t\}$, resolvent of the generator $\mathcal{L}$ is given by Laplace transform of the semigroup,

$$(\lambda 1 - \mathcal{L})^{-1} = \int_0^\infty e^{-\lambda t} T_t \, dt, \forall \lambda > 0.$$

The following useful theorem due to Hille and Yosida characterizes generators of C_0-semigroups.

Theorem 1.3.1. *(Hille-Yosida theorem) Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ be a densely defined closed linear operator on X. Then $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the generator of a quasi-bounded C_0-semigroup $\{T_t\}$ such that $\|T_t\| \le Me^{\beta t}$ $\forall t \ge 0$ for some constants $M \ge 0$ and $\beta \ge 0$ if and only if $\mathcal{L}$ satisfies*

$$\|(\mathcal{L} - \lambda 1)^{-1}\| \le \frac{M}{Re(\lambda) - \beta}, \text{ for some } \lambda \in \rho(\mathcal{L}) \text{ with } Re(\lambda) > \beta. \qquad (1.3.\ 1)$$

Let $\mathcal{G}(M, \beta)$ denote the class of all linear operators $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ on X satisfying 1.3. 1. Thus in particular the generator of a contractive C_0-semigroup is belong to $\mathcal{G}(1, 0)$.

Now we recall some useful results for analyzing perturbation, convergence and approximation of C_0-semigroup. First let us introduce the notion of relative boundedness.

Definition 1.3.2. *Let $\mathcal{L}$ and A be two operators with same domain space X with $\mathcal{D}(\mathcal{L}) \subseteq \mathcal{D}(A)$. Then the operator A is said to be relatively bounded with respect to $\mathcal{L}$ if there exist nonnegetive constants a and b such that*

$$\|Ax\| \le a\|x\| + b\|\mathcal{L}x\|, \forall x \in \mathcal{D}(\mathcal{L}). \qquad (1.3.\ 2)$$

The infimum of all possible constants b in 1.3. 2 is called relative bound of A with respect to $\mathcal{L}$.

The following theorem gives stability condition for perturbation of contraction semigroups.

Theorem 1.3.3. *[20] Let $\mathcal{L}, A \in \mathcal{G}(1,0)$ and let A be relatively bounded with respect to $\mathcal{L}$ with relative bound less than $\frac{1}{2}$. Then the perturbed operator $\mathcal{L}+A$ is also in the class $\mathcal{G}(1,0)$.*

Following results give the convergence of C_0-semigroups.

Theorem 1.3.4. *[20]*
Suppose $\{T_t^{(n)}\}_{n\geq 1}$ and $\{T_t\}$ are C_0-semigroups on a Banach space X with the generators $\mathcal{L}^{(n)}$ and $\mathcal{L}$ in $\mathcal{G}(M,\beta)$ respectively, for some fixed M and β. Then $T_t^{(n)}$ converges strongly to T_t if and only if $\mathcal{L}^{(n)}$ converges strongly to $\mathcal{L}$ in the generalized sense, i.e. $(\mathcal{L}^{(n)}-\lambda)^{-1}$ converges strongly to $(\mathcal{L}-\lambda)^{-1}$ for every $Re(\lambda) > \beta$.

Theorem 1.3.5. *(Chernoff's theorem [5]) Let P_t be a contractive C_0-semigroup on a Banach space X with generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$. Suppose $F : \mathbb{R}_+ \to \mathcal{B}(X)$ satisfies $F(0) = 1, \|F(t)\| \leq 1$ and $\lim_{n\to 0} \|\frac{F(t)-1}{t}x - \mathcal{L}(x)\| = 0, \forall x \in \mathcal{D}$, where $\mathcal{D}$ is a core of $\mathcal{L}$. Then for any $t \geq 0$,*
$\lim_{n\to 0} \|(F(\frac{t}{n}))^n(x) - P_t(x)\| = 0, \forall x \in \mathcal{A}$.

Evolution equations on Banach spaces

Let X be a Banach space. We look for a solution $x : \mathbb{R}_+ \to X$ of the differential equation

$$\frac{dx(t)}{dt} = A(t)x(t),\ x(0) = x_0, \tag{1.3. 3}$$

where $A(t)$ is a one parameter family of operators on X with domain $\mathcal{D}(A(t))$.

Theorem 1.3.6. *Let $A(t) = \mathcal{L} + B(t)$ such that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the generator of a contractive C_0-semigroup P_t on X and $t \mapsto B(t) \in \mathcal{B}(X)$ is locally bounded. Then the differential equation*

$$\frac{dT(t)(x)}{dt} = T(t)A(t)x, T(0)x = x,\ \forall x \in \mathcal{D}(\mathcal{L})$$

admits a unique solution $T(t)$.

Proof. Let $T > 0$ be fixed real number. Then for $0 \leq t \leq T$, we can find a constant $M < \infty$ such that

$$\sup_{0\leq t\leq T} \|B(t)\| \leq M.$$

For $n \geq 0$, we set bounded operators $T^{(n)}(t)$ as follows

$$
\begin{aligned}
T^{(0)}(t)(x) &= P_t(x) \\
T^{(1)}(t)(x) &= P_t(x) + \int_0^t B(s)P_s(x)ds \\
T^{(n+1)}(t) &= P_t(x) + \int_0^t T^{(n)}(s)B(s)P_{t-s}(x)ds.
\end{aligned}
$$

Now, for any $n \geq 1$

$$\|T^{(n+1)}(t)(x) - T^{(n)}(t)(x)\| \leq \int_0^t \|(T^{(n)}(s) - T^{(n-1)}(s))B(s)P_{t-s}(x)\|ds.$$

Taking supremum over $\{\|x\| \leq 1\}$ we get

$$\|T^{(n+1)}(t) - T^{(n)}(t)\| \leq M \int_0^t \|T^{(n)}(s) - T^{(n-1)}(s)\|ds. \tag{1.3. 4}$$

Repeatedly estimating right hand side, we obtained,

$$
\begin{aligned}
&\|T^{(n+1)}(t) - T^{(n)}(t)\| \\
&\leq M^{n+1} \int_0^t \int_0^{s_1} \cdots \int_0^{s_n} ds_{n+1}ds_n \cdots ds_1 \\
&= M^{n+1} \frac{t^{n+1}}{n+1!},
\end{aligned}
$$

which implies $\{T^{(n)}(t)\}_{n\geq 0}$ is Cauchy in $\mathcal{B}(X)$ and the limit, say $T(t)$, is given by

$$T(t) = P_t + \sum_{n\geq 0} [T^{(n+1)}(t) - T^{(n)}(t)]$$

and $\|T(t)\| \leq e^{tM}$. □

1.4 Completely positive maps and Quantum dynamical semigroups

Completely positive maps

Let $\mathcal{A}, \mathcal{B}$ be two unital $*$-algebra.

Definition 1.4.1. *A linear map $T : \mathcal{A} \to \mathcal{B}$ is said to be positive if $T(a^*a) \geq 0$ in $\mathcal{B}$ for all $a \in \mathcal{A}$. T is called completely positive (in short CP) if, for each $n \geq 1$, $T \otimes id_{M_n(\mathbb{C})} : \mathcal{A} \bigotimes M_n(\mathbb{C}) \to \mathcal{B} \bigotimes M_n(\mathbb{C})$ is positive, where $id_{M_n(\mathbb{C})}$ is the identity map from the finite dimensional matrix algebra $M_n(\mathbb{C})$ to it itself.*

Any $*$-homomorphism is a CP map but converse is not true in general. However, due to Stinespring's theorem we have the following characterization of CP map on $*$-algebras.

Theorem 1.4.2. *(Stinespring's Dilation Theorem) [36] Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{H}$ be a complex Hilbert space. Let $T : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map. Then there exists a Hilbert space $\mathcal{K}$, a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with the minimality condition that the span of $\{\pi(a)Vu : a \in \mathcal{A},\ u \in \mathcal{H}\}$ is total in $\mathcal{K}$, and the map T is given by:*
$T(a) = V^*\pi(a)V,\ \forall a \in \mathcal{A}$,
Such a triple $(\mathcal{K}, \pi, V)$ is called the 'Stinespring's triple' associated with T, is unique in the sense that if $(\mathcal{K}', \pi', V')$ is another such triple, and then there is a unitary operator $\Gamma : \mathcal{K} \to \mathcal{K}'$ such that $\pi'(a) = \Gamma\pi(a)\Gamma^*$ and $V' = \Gamma V$. Furthermore, if $\mathcal{A}$ is a von Neumann algebra and T is normal, π can be chosen to be normal.

The following result shows that the distinction between positivity and completely positivity appears only for noncommutative algebras.

Theorem 1.4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. Then any positive map $T : \mathcal{A} \mapsto \mathcal{B}$ is CP if either of $\mathcal{A}$ or $\mathcal{B}$ is abelian.*

Now let us introduce conditionally completely positive maps.

Definition 1.4.4. *A linear map $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ is said to be conditionally completely positive (in short CCP) if*

$$\sum_{i,j=1}^{n} b_i^* \mathcal{L}(a_i^* a_j) b_j \geq 0, \quad \text{for any } a_i, b_i \in \mathcal{A}, i = 1, 2, \cdots n, n \geq 1$$

such that $\sum_{i=1}^{n} a_i b_i = 0$.

The CCP maps play an important role in study of CP semigroups.

1.4.1 Quantum dynamical semigroup

Let $\mathcal{A}$ be a C^*-algebra.

Definition 1.4.5. *A one parameter C_0-semigroup $\{T_t\}$ of CP maps on $\mathcal{A}$ into itself is called a quantum dynamical semigroup (in short 'QDS'). On a von Neumann algebra $\mathcal{A}$, a QDS is a one parameter C_0-semigroup $\{T_t\}$ of normal CP maps.*

Any bounded CCP map $\mathcal{L}$ on a C^*-algebra $\mathcal{A}$ is the generator of uniformly continuous QDS $\{T_t = e^{t\mathcal{L}}\}$. Conversely, the generator of a uniformly continuous QDS is a bounded CCP map. The important and very useful structure theorem of Christensen and Evans [7] asserts that:

Theorem 1.4.6. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a C^* or von Neumann algebra, $\{T_t\}$ be a uniformly continuous QDS on $\mathcal{A}$ with the generator $\mathcal{L}$. Then there exists $l \in \mathcal{A}''$ and a Hilbert space $\mathcal{K}, R \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ with the minimality condition: that the span of $\{(Ra - \pi(a)R)u : a \in \mathcal{A}, u \in \mathcal{H}\}$ is total in $\mathcal{K}$ such that*

$$\mathcal{L}(a) = R^*\pi(a)R + la + al^*, \ \forall a \in \mathcal{A}.$$

In case of $\mathcal{A}$ is unital, $\mathcal{L}(1) = 0$ and $l = iH - \frac{1}{2}R^*R$ for some self adjoint element $H \in \mathcal{A}''$ and $\mathcal{L}$ takes the form,

$$\mathcal{L}(a) = R^*\pi(a)R - \frac{1}{2}R^*Ra - \frac{1}{2}aR^*R + i[H, a], \ \forall a \in \mathcal{A}.$$

1.5 Quantum stochastic calculus on symmetric Fock space

All the Hilbert spaces appearing here are assumed to be complex and separable with inner product $\langle \cdot, \cdot \rangle$ which is linear in the second variable. Let us first recall the definition of infinite tensor product of Hilbert spaces.

Infinite tensor product of Hilbert spaces

Let $\{\mathcal{H}_l, \langle ., . \rangle_l\}_{l \geq 1}$ be a family of Hilbert spaces with $\{e^{(l)}_{n_l}\}_{n_l \geq 1}$ be an orthonormal basis for $\mathcal{H}_l$. Let S be the set of all sequences $\underline{n} = \{n_l\}$ of positive integers. We define a vector space W spanned by finite linear combinations of elements from the set $W_0 = \{e_{\underline{n}} = e^{(1)}_{n_1} \otimes e^{(2)}_{n_2} \otimes \cdots : \underline{n} \in S\}$. A typical vector in $u \in W$ is given by

$$u = \sum_{\underline{n} \in S} c(\underline{n}) e_{\underline{n}},$$

for some function $c : S \to \mathbb{C}$ such that $c(\underline{n}) = 0$ for all but finitely many $\underline{n} \in S$ and the zero vector $0 \in W$ corresponds to c with $c(\underline{n}) = 0, \forall \underline{n} \in S$. We define an inner product $\langle ., . \rangle$ on W by setting, for two elements $u = \sum_{\underline{n} \in S} c(\underline{n}) e_{\underline{n}}$ and $v = \sum_{\underline{n} \in S} d(\underline{n}) e_{\underline{n}} \in W$,

$$\langle u, v \rangle = \sum_{\underline{n} \in S} \overline{c(\underline{n})} d(\underline{n}). \tag{1.5. 1}$$

It is clear that $\|v\| := \langle v, v \rangle = 0$ iff $v = 0$.

Definition 1.5.1. *The completion of the inner product space* $(W, \langle ., . \rangle)$ *is called the infinite tensor product of the family of Hilbert spaces* $\{\mathcal{H}_l\}$ *and it is denoted by* $\bigotimes_{l \geq 1} \mathcal{H}_l$ *and vector* $e_{\underline{n}} \in \bigotimes_{l \geq 1} \mathcal{H}_l$ *is denoted by* $\otimes_{l \geq 1} e^{(l)}_{n_l}$.

By definition $\{e_{\underline{n}} : \underline{n} \in S\}$ form an orthonormal basis for infinite tensor product $\bigotimes_{l \geq 1} \mathcal{H}_l$.

In order to define the infinite tensor product of the family of Hilbert spaces $\{\mathcal{H}_l\}$ with respect to a sequence of unit vectors $\{u^{(l)} : l \geq 1\}$, $u^{(l)} \in \mathcal{H}_l$, for each $l \geq 1$, let us consider an orthonormal basis $\{e^{(l)}_{n_l}\}_{n_l \geq 1}$ for $\mathcal{H}_l$ such that $e^{(l)}_1 = u^{(l)}$.

Definition 1.5.2. *Let us consider the closure of the subspace spanned by orthonormal vectors* $e_{\underline{n}} \in \bigotimes_{l \geq 1} \mathcal{H}_l$ *such that* $n_l = 1$ *i.e.* $e^{(l)}_{n_l} = u^{(l)}$ *for all but finitely many* $l \geq 1$. *This Hilbert subspace is called the infinite tensor product of the family of Hilbert spaces* $\{\mathcal{H}_l\}$ *with respect to the stabilizing vector* $\{u^{(l)}\}$.

1.5.1 Symmetric Fock space

For any Hilbert space $\mathcal{K}$ and $n \geq 1$, let $\mathcal{K}^{\textcircled{n}}$ stand for the n-fold symmetric tensor product of $\mathcal{K}$ and $\mathcal{K}^{\textcircled{0}}$ be the one dimensional complex Hilbert space $\mathbb{C}$. We denote the symmetric Fock space $\bigoplus_{n \geq 0} \mathcal{K}^{\textcircled{n}}$ over $\mathcal{K}$ by $\Gamma(\mathcal{K})$ or simply by Γ. For $f \in \mathcal{K}$, we denote by $\mathbf{e}(f)$ the exponential vector in Γ associated with f :

$$\mathbf{e}(f) = \oplus_{n \geq 0} \frac{1}{\sqrt{n!}} f^{(n)},$$

where $f^{(n)} = \underbrace{f \otimes f \otimes \cdots \otimes f}_{n-copies}$ for $n > 0$ with the convention $f^{(0)} = 1$. The exponential vector $\mathbf{e}(0) = 1 \oplus 0 \oplus \cdots$, associated with $f = 0$, is called the vacuum vector in Γ and it is denoted by Ω. For any subset $M \subseteq \mathcal{K}$, the family of exponential vectors $\{\mathbf{e}(f) : f \in M\}$ is a linearly independent set in $\Gamma(\mathcal{K})$. Let $\mathcal{E}(M)$ be the subspace spaned by above exponential vectors. For a dense set M, $\mathcal{E}(M)$ is dense in $\Gamma(\mathcal{K})$.

For $f \in \mathcal{K}$ and $U \in \mathcal{U}(\mathcal{K})$, the space of unitary operators on $\mathcal{K}$, the Weyl operator $W(f, U)$ associated with the pair f, U is defined by,

$$W(f, U)\mathbf{e}(g) = e^{-\frac{1}{2}\|f\|^2 - \langle f, Ug \rangle}\mathbf{e}(f + Ug), \forall g \in \mathcal{K}.$$

For any operator $H \in \mathcal{B}(\mathcal{K})$, the second quantization $\Gamma(H)$ of H is given by,

$$\Gamma(H)\mathbf{e}(g) = \mathbf{e}(Hg)$$

Now we consider the following operators $a(f), a^\dagger(f)$ and $\lambda(H)$, obtained from the Weyl operators. On the finite particle vectors, we have

- $a(f)g^{(n)} = \sqrt{n}\langle f, g \rangle g^{(n-1)}$
- $a^\dagger(f)g^{(n)} = \sum_{r=0}^{n} \frac{1}{\sqrt{n+1}} g^{(r)} f g^{(n-r)}$
- $\lambda(H)g^{(n)} = \sum_{n=0}^{n-1} g^{(r)} H g g^{(n-r)}$.

In view of these properties $a(f), a^\dagger(f)$ and $\lambda(H)$ are called the annihilation operator associated with f, the creation operator associated with f and the conservation operator associated with H respectively. The space $\mathcal{E}(\mathcal{K})$ is contained in the domain of all these operators and we have,

- $a(f)\mathbf{e}(g) = \langle f, g \rangle \mathbf{e}(g)$
- $a^\dagger(f)\mathbf{e}(g) = \frac{d}{dt}|_{t=0}\mathbf{e}(g + tf)$
- $\langle \mathbf{e}(g_1), \lambda(H)\mathbf{e}(g_2) \rangle = \langle f, Hg_2 \rangle \langle \mathbf{e}(g_1), \mathbf{e}(g_2) \rangle = \langle \mathbf{e}(g_1), a^\dagger(Hg_2)\mathbf{e}(g_2) \rangle$
- $\langle a^\dagger(f)\mathbf{e}(g_1), \mathbf{e}(g_2) \rangle = \langle \mathbf{e}(g_1), a(f)\mathbf{e}(g_2) \rangle = \langle f, g_2 \rangle \langle \mathbf{e}(g_1), \mathbf{e}(g_2) \rangle$.

Let $\mathbf{k}_0$ be a complex separable Hilbert space with an orthonormal basis $\{e_i\}_{i \geq 1}$. Let $\mathcal{K} = L^2(\mathbb{R}_+, \mathbf{k}_0) \simeq L^2(\mathbb{R}_+) \bigotimes \mathbf{k}_0$. So that any $f \in \mathcal{K}$ decomposes as $f = \sum_{k \geq 1} f_k e_k$

with $f_k \in L^2(\mathbb{R}_+)$. We take the freedom to use the same symbol f_k to denote the function in $L^2(\mathbb{R}_+, \mathbf{k}_0)$ as well, whenever it is clear from the context. For any $0 < s < t < \infty$, let $P_{s]}, P_{(s,t]}$ and $P_{[t}$ are the canonical orthogonal projections $1_{[0,s]}, 1_{(s,t]}$ and $1_{[t,\infty)}$ respectively. Denoting by $\mathcal{K}_{s]}, \mathcal{K}_{(s,t]}$ and $\mathcal{K}_{[t}$, the range of projections $P_{s]}, P_{(s,t]}$ and $P_{[t}$ respectively, we have $\mathcal{K} = \mathcal{K}_{s]} \bigoplus \mathcal{K}_{(s,t]} \bigoplus \mathcal{K}_{[t}$. Thus any function f in $\mathcal{K}$ decomposes as $f = f_{s]} \oplus f_{(s,t]} \oplus f_{[t}$, where $f_{s]} = 1_{[0,s]} f$, $f_{(s,t]} = 1_{(s,t]} f$ and $f_{[t} = 1_{[t,\infty)} f$. The symmetric Fock space $\Gamma(\mathcal{K})$ over $\mathcal{K}$ can be written as a tensor product $\Gamma(\mathcal{K}) = \Gamma_{s]} \bigotimes \Gamma_{(s,t]} \bigotimes \Gamma_{[t}$ and the vacuum vector in the Fock space $\Gamma(\mathcal{K})$ can be written as $\Omega = \Omega_{s]} \otimes \Omega_{(s,t]} \otimes \Omega_{[t}$, where $\Gamma_{s]} = \Gamma(\mathcal{K}_{s]}), \Gamma_{(s,t]} = \Gamma(\mathcal{K}_{(s,t]})$ and $\Gamma_{[t} = \Gamma(\mathcal{K}_{[t})$ with vacuum vectors $\Omega_{s]}, \Omega_{(s,t]}$ and $\Omega_{[t}$ respectively.

For the latter part of the thesis let us fix the convention that for two vector spaces V_1 and $V_2, V_1 \bigotimes V_2$ denotes the algebraic tensor product as long as at least one of the two spaces involved are not completed, but when both are complete spaces, $\bigotimes$ stands for topological tensor product and $\bigotimes_{\text{alg}}$ stands for algebraic tensor product.

1.5.2 Quantum stochastic integration

Let $\mathbf{h}_0$ be a Hilbert space and $\widetilde{\mathcal{H}} = \mathbf{h}_0 \bigotimes \Gamma(\mathcal{K})$. For $0 < s < t < \infty$, we write

$$\widetilde{\mathcal{H}}_{0]} = \mathbf{h}_0, \widetilde{\mathcal{H}}_{s]} = \mathbf{h}_0 \bigotimes \Gamma(\mathcal{K}_{s]})$$

$$\widetilde{\mathcal{H}}_{(s,t]} = \Gamma(\mathcal{K}_{(s,t]}), \widetilde{\mathcal{H}}_{[t} = \Gamma(\mathcal{K}_{[t}).$$

Let $\mathcal{D}_0$ and $\mathcal{M}$ be two dense subspaces of $\mathbf{h}_0$ and $\mathcal{K}$ respectively. The algebraic tensor product $\mathcal{D}_0 \bigotimes \mathcal{M}$ is a dense subspaces of $\widetilde{\mathcal{H}}$.

Definition 1.5.3. *A family of operators $\{L_t\}_{t\geq 0}$ on $\widetilde{\mathcal{H}}$ is said to be a $(\mathcal{D}_0, \mathcal{M})$-adapted process if,*

1. $\mathcal{D}_0(L_t) \supseteq \mathcal{D}_0 \bigotimes \mathcal{M}, \ \forall t \geq 0$,

2. For $t \geq 0, u \in \mathcal{D}_0$ and $f \in \mathcal{M}$,

$$L_t \; u\boldsymbol{e}(f_{t]}) \in \widetilde{\mathcal{H}}_{t]} \text{ and } L_t \; u\boldsymbol{e}(f) = L_t \; u\boldsymbol{e}(f_{t]}) \otimes \boldsymbol{e}(f_{[t}).$$

It is said to be *regular,* if in addition, for every $u \in \mathcal{D}_0$ and $f \in \mathcal{M}$, the map $t \mapsto L_t \; u\mathbf{e}(f)$ from $\mathbb{R}_+$ into $\widetilde{\mathcal{H}}$ is continuous. An adapted process is called bounded, contractive, isometric, co-isometric or unitary if the operator L_t's are so. Let $\mathcal{B} = \mathcal{B}(\widetilde{\mathcal{H}}) \simeq \mathcal{B}_0 \bigotimes \mathcal{B}(\Gamma)$, where $\mathcal{B}_0$ is stands for $\mathcal{B}(\mathbf{h}_0)$. For $0 < s < t < \infty$, $\mathcal{B}$ can be written as $\mathcal{B} = \mathcal{B}_{s]} \bigotimes \mathcal{B}_{(s,t]} \bigotimes \mathcal{B}_{[t}$, where $\mathcal{B}_{s]} = \mathcal{B}_0 \bigotimes \mathcal{B}(\Gamma(\mathcal{K}_{s]})), \mathcal{B}_{(s,t]} = \mathcal{B}(\Gamma(\mathcal{K}_{(s,t]}))$ and $\mathcal{B}_{[t} = \mathcal{B}(\Gamma(\mathcal{K}_{[t}))$. These von Neumann algebras are canonically embedded in $\mathcal{B}$. For any operators $L \in \mathcal{B}_0$ and $T \in \mathcal{B}_{(s,t]}$ can be identified with their ampliations $L \otimes 1_{\Gamma_{(s,t)}}$ and $1_{\mathbf{h}_0} \otimes T$. Further, any operator $L \in \mathcal{B}_{s]}$ can be identified with the process given by

$$\begin{aligned} L_t &= 1_{[0,t]} L \text{ if } t \leq s \\ &= L\ 1_{[s,t]} \text{ if } t \geq s. \end{aligned}$$

Let us introduce the vacuum conditional expectation $\mathbb{E}_0 : \mathcal{B}_0 \bigotimes \mathcal{B}(\Gamma) \to \mathcal{B}_0$, which is given by, for $X \in \mathcal{B}_0 \bigotimes \mathcal{B}(\Gamma)$

$$\langle u, \mathbb{E}_0(X) v \rangle = \langle u\Omega, Xv\Omega \rangle, \forall u, v \in \mathbf{h}_0.$$

The fundamental processes $\{\Lambda^\mu_\nu : \mu, \nu \geq 0\}$ associated with the orthonormal basis $\{e_j : j \geq 1\}$ are given by

$$\Lambda^\mu_\nu(t) = \begin{cases} t1 \text{ , for } (\mu,\nu) = (0,0), \\ a(1_{[0,t]} \otimes e_j) \text{ , for } (\mu,\nu) = (0,j) \\ a^\dagger(1_{[0,t]} \otimes e_i) \text{ , for } (\mu,\nu) = (i,0) \\ \Lambda(M_{1_{[0,t]}} \otimes |e_i >< e_j|) \text{ , for } (\mu,\nu) = (i,j) \end{cases} \tag{1.5. 2}$$

where $M_{1_{[0,t]}}$ is the multiplication operator on $L^2(\mathbb{R}_+)$ by the characteristic function of the interval $[0,t]$. All these processes $\{\Lambda^\mu_\nu(t)\}$ are defined on the exponential domain $\mathcal{E}(\mathcal{K})$ and as per our convention, $\Lambda^\mu_\nu(t)$'s are also identified with their ampliations $1_{\mathbf{h}_0} \otimes \Lambda^\mu_\nu(t)$. For details, the reader is referred to [33, 29].

The *quantum Ito formula* can be express as

$$d\Lambda^\mu_\nu d\Lambda^\xi_\eta = \hat{\delta^\xi_\nu} d\Lambda^\mu_\eta, \ \forall \mu, \nu, \xi, \eta \geq 0 \tag{1.5. 3}$$

where

$$\begin{aligned} \hat{\delta^\mu_\nu} =& \ 0 \text{ for } \mu = 0 \text{ or } \nu = 0, \\ =& \ \delta^\mu_\nu, \text{ otherwise} \end{aligned}$$

with δ^μ_ν is the Dirac delta function given by

$$\begin{aligned}\delta^\mu_\nu = & \ 1 \text{ for } \mu = \nu \\ = & \ 0, \text{ otherwise.}\end{aligned}$$

Now we shall discuss quantum stochastic integration with respect to the above basic integrator processes. First let us consider integration of simple adapted processes. An $(\mathbf{h}_0, \mathcal{K})$-adapted process L is said to be simple with respect to a partition $S \equiv (0 = t_0 < t_1 < \cdots)$ of $\mathbb{R}_+$, if

$$L(t) = L(t_k), \text{ for } t \in (t_{k-1}, t_k].$$

For any $\mu, \nu \geq 0$, we define a simple process X given by, for $t \in (t_{n-1}, t_n],\ n \geq 1$,

$$X_t = \sum_{k=1}^{n} L_{t_{k-1}}[\Lambda^\mu_\nu(t_k) - \Lambda^\mu_\nu(t_{k-1})] + L_{t_{n-1}}[\Lambda^\mu_\nu(t) - \Lambda^\mu_\nu(t_{n-1})].$$

The process X is called the quantum stochastic integral of L with respect to Λ^μ_ν and written as $\int_0^t L(s)d\Lambda^\mu_\nu(s)$.
For a simple adapted process L the following observations are immediate.

Proposition 1.5.4. *(First fundamental lemma) For any (μ, ν) and $t \geq 0, u, h \in \mathbf{h}_0$ and $f, g \in \mathcal{K}$ we have*

$$\langle u\boldsymbol{e}(f), X(t)v\boldsymbol{e}(g)\rangle = \int_0^t \overline{f_\mu(s)} g_\nu(s) \langle u\boldsymbol{e}(f), L(s)v\boldsymbol{e}(g)\rangle ds.$$

Let L and M be two simple adapted processes with respect to a common partition $0 = t_0 < t_1 \cdots$ of $\mathbb{R}_+$ and $t = t_n$ for some $n \geq 0$. Let

$$X(t) = \int_0^t L(s)d\Lambda^\mu_\nu(s) \text{ and} Y_t = \int_0^t M(s)d\Lambda^\xi_\eta(s).$$

Then for any $u, v \in \mathbf{h}_0; f, g \in L^2(\mathbb{R}_+, \mathbf{k}_0)$,

$$\begin{aligned}&\langle X(t)u\mathbf{e}(f), Y_t v\mathbf{e}(g)\rangle \\ = \ & \sum_{k=1}^{n} \langle L(t_{k-1})u\mathbf{e}(f_{t_{k-1}]}), M(t_{k-1})v\mathbf{e}(g_{t_{k-1}]})\rangle \langle \Lambda^\mu_\nu(k)\mathbf{e}(f_{[t_k}), \Lambda^\xi_\eta(k)\mathbf{e}(g_{[t_k})\rangle \\ + \ & \sum_{k=1}^{n} \langle X(t_{k-1})u\mathbf{e}(f_{t_{k-1}]}), M(t_{k-1})v\mathbf{e}(g_{t_{k-1}]})\rangle \langle \mathbf{e}(f_{[t_k}), \Lambda^\xi_\eta(k)\mathbf{e}(g_{[t_k})\rangle\end{aligned}$$

$$+\sum_{k=1}^{n}\langle L(t_{k-1})u\mathbf{e}(f_{t_{k-1}]}), Y(t_{k-1})v\mathbf{e}(g_{t_{k-1}]})\rangle\langle\Lambda^{\mu}_{\nu}(k)\mathbf{e}(f_{[t_k}), \mathbf{e}(g_{[t_k})\rangle \qquad (1.5.\ 4)$$

From (1.5. 4) and (1.5. 3) the following useful resulted, called the "Second fundamental lemma" follows (detail can be found in [33]) easily.

Proposition 1.5.5.

$$\begin{aligned}
&\langle X(t)u\boldsymbol{e}(f), Y_t v\boldsymbol{e}(g)\rangle \\
= &\int_0^t \delta^{\mu}_{\xi}\overline{f_\nu(s)}g_\eta(s)\langle L(s)u\boldsymbol{e}(f), M(s)v\boldsymbol{e}(g)\rangle ds \\
+ &\int_0^t \overline{f_\xi(s)}g_\eta(s)\langle X(s)u\boldsymbol{e}(f), M(s)v\boldsymbol{e}(g)\rangle ds \\
+ &\int_0^t \overline{f_\nu(s)}g_\mu(s)\langle L(s)u\boldsymbol{e}(f), Y(s)v\boldsymbol{e}(g)\rangle ds.
\end{aligned}$$

For more general processes, we have,

Proposition 1.5.6. *Let L be a $(\mathbf{h}_0, \mathcal{K})$-adapted process satisfying, for $u \in \mathbf{h}_0, f \in \mathcal{K}$*
1. $t \mapsto L(t)u\boldsymbol{e}(f)$ is left continuous.
2. $\sup_{0\leq s\leq t}\|L(s)u\boldsymbol{e}(f)\| < \infty,\ \forall t \geq 0$.
Then there exists a sequence of simple $(\mathbf{h}_0, \mathcal{K})$-adapted process $\{L_n\}$ such that

$$\lim_{n\to\infty} L_n(t)u\boldsymbol{e}(f) = L(t)u\boldsymbol{e}(f),\ \forall t \geq 0$$

and for any $\mu, \nu \geq 0$,

$$s\text{ - }\lim_{n\to\infty}\int_0^t L_n(s)d\Lambda^{\mu}_{\nu}(s) \text{ exists on the domain } \mathbf{h}_0 \bigotimes \mathcal{E}(\mathcal{K}).$$

The strong limit, say $X(t)$, is independent of the choice of approximating sequence.

We call the limit $X(t)$ to be the quantum stochastic integration of L with respect to Λ^{μ}_{ν}. For all such processes, the first and second fundamental lemma hold. Let us denote the space of all such integrable processes by $L(\mathbf{h}_0, \mathcal{K})$.

Proposition 1.5.7. *Let $\{L^{\mu}_{\nu}\}$ be a family in $L(\mathbf{h}_0, \mathcal{K})$ such that for any $t \geq 0, \nu \geq 0,\ u \in \mathbf{h}_0$ and $f \in \mathcal{K}$*

$$\int_0^t \sum_{\mu\geq 0}\|L^{\mu}_{\nu}(s)u\boldsymbol{e}(f)\|^2 d\gamma_f(s) < \infty$$

where $\gamma_f(t) = \int_0^t (1 + \|f(s)\|^2) ds$. *Then there exists a regular* $(\mathbf{h}_0, \mathcal{K})$*-adapted process* X *with*

1. $\lim_{n\to\infty} \sup_{0\leq s\leq t} \|X_n(t)u\boldsymbol{e}(f) - X(t)u\boldsymbol{e}(f)\| = 0$, *where*

$$X_n(t) = \sum_{0\leq\mu,\nu\leq n} \int_0^t L_\nu^\mu(s) d\Lambda_\nu^\mu(s).$$

2. $\|X(t)u\boldsymbol{e}(f)\|^2 \leq 2e^{\gamma_f(t)} \sum_{j\geq 0} \int_0^t \|L_\nu^\mu(s)u\boldsymbol{e}(f)\|^2 d\gamma_f(s)$.

Such a family $\{L_\nu^\mu\}$ is called stochastically integrable with respect to $\{\Lambda_\nu^\mu\}$ and its stochastic integral is given by,

$$X(t) = \sum_{\mu,\nu\geq 0} \int_0^t L_\nu^\mu(s) d\Lambda_\nu^\mu(s), \ \forall t \geq 0$$

and we write,

$$dX = \sum_{\mu,\nu\geq 0} L_\nu^\mu d\Lambda_\nu^\mu.$$

Let $\{L_\nu^\mu\}$ and $\{Y_\nu^\mu\}$ be two stochastically integrable families. Then we have

Proposition 1.5.8. *For any* $t \geq 0, u, h \in \mathbf{h}_0$ *and* $f, g \in \mathcal{K}$

1. First fundamental lemma:

$$\langle u\boldsymbol{e}(f), X(t)v\boldsymbol{e}(g)\rangle = \sum_{\mu,\nu\geq 0} \int_0^t f_\mu(s)\bar{g_\nu}(s)\langle u\boldsymbol{e}(f), L_\nu^\mu(s)v\boldsymbol{e}(g)\rangle ds.$$

2. Second fundamental lemma:

$$\langle X(t)u\boldsymbol{e}(f), Y_t v\boldsymbol{e}(g)\rangle = \sum_{\mu,\nu\geq 0} \int_0^t ds\overline{f_\mu(s)}g_\nu(s)\{\langle X(s)u\boldsymbol{e}(f), M_\nu^\mu(s)v\boldsymbol{e}(g)\rangle$$

$$+\langle L_\mu^\nu(s)u\boldsymbol{e}(f), Y(s)v\boldsymbol{e}(g)\rangle + \sum_{\xi\geq 0}\langle L_\mu^\xi(s)u\boldsymbol{e}(f), M_\nu^\xi(s)v\boldsymbol{e}(g)\rangle\}.$$

Proposition 1.5.9. *Let* $\{L_\nu^\mu\}$ *be a family in* $\mathcal{B}(\mathbf{h}_0)$ *such that for any* $\nu \geq 0$ *there exists a constant* $c_\nu \geq 0$ *satisfying*

$$\sum_{\mu\geq 1} \|L_\nu^\mu(u)\|^2 \leq c_\nu^2 \|u\|^2, \ \forall u \in \mathbf{h}_0.$$

Then there exists a unique regular $(\mathbf{h}_0, \mathcal{K})$*-adapted process* $X \equiv \{X(t) : t \geq 0\}$ *which satisfies the differential equation*

$$dX = \sum_{\mu,\nu \geq 0} X L^\mu_\nu d\Lambda^\mu_\nu \tag{1.5. 5}$$

with initial condition $X(0) = x_0 \otimes 1,$ *for some given* $x_0 \in \mathcal{B}(\mathbf{h}_0)$.

The next result tells us about the existence of unitary operator valued solution of

$$dU = \sum_{\mu,\nu \geq 0} U L^\mu_\nu d\Lambda^\mu_\nu,\ U(0) = 1. \tag{1.5. 6}$$

Proposition 1.5.10. *Let* $H \in \mathcal{B}(\mathbf{h}_0)$ *be self adjoint,* $\{r_i : i \geq 1\}, \{s^\mu_\nu : i, j \geq 1\}$ *be bounded operators in* $\mathbf{h}_0$ *such that* $S = \sum_{i,j \geq 1} s^\mu_\nu \otimes |e_\mu >< e_\nu|$ *is a unitary operator in* $\mathbf{h}_0 \bigotimes \mathbf{k}_0$ *and for some constant* $c \geq 0$,

$$\sum_{i \geq 1} \|r_i(u)\|^2 \leq c^2 \|u\|^2,\ \forall u \in \mathbf{h}_0.$$

If we set the coefficients L^μ_ν *in the equation (1.5. 6) as follows,*

$$\begin{aligned} L^\mu_\nu &= -(H + \frac{1}{2}\sum_{k \geq 1} r_k^* r_k),\ \textit{for } (\mu, \nu) = (0, 0) \\ &= -\sum_{k \geq 1} r_k^* s^k_j,\ \textit{for } (\mu, \nu) = (0, j) \\ &= r_i,\ \textit{for } (\mu, \nu) = (i, 0) \\ &= s^\mu_\nu - \delta^\mu_\nu,\ \textit{for } (\mu, \nu) = (i, j), \end{aligned}$$

there exists a unique unitary operator valued process $U(t)$ *satisfying (1.5. 6) .*

The equation (1.5. 6) can be interpreted as a Schrödinger equation in the presence of noise. Now let us look at the Heisenberg picture of this equation. For any $x \in \mathcal{B}(\mathbf{h}_0)$ let us define

$$j_t(x) = U(t)^*(x \otimes 1)U(t), \forall t \geq 0.$$

This defines a family $j_t : \mathcal{B}(\mathbf{h}_0) \to \mathcal{B}(\mathbf{h}_0 \bigotimes \Gamma(\mathcal{K}))$ of $*$-homomorphisms and for each $x, j_t(x)$ is a regular $(\mathbf{h}_0, \mathcal{K})$-adapted process satisfying

$$dj_t(x) = \sum_{\mu,\nu \geq 0} j_t(\theta^\mu_\nu(x)) d\Lambda^\mu_\nu(t) \tag{1.5. 7}$$

$$j_0(x) = x \otimes 1_\Gamma,$$

where $\{\theta^\mu_\nu\}$ be the family of bounded maps from $\mathcal{A}$ to itself, given by,

$$\theta^\mu_\nu(x) = \begin{cases} -[H,x] - \frac{1}{2}\sum_{k\geq 1} r_k^* r_k x + x r_k^* r_k - 2 r_k^* x r_k \text{ , for } (\mu,\nu) = (0,0) \\ \sum_{k\geq 1}[r_k^*, x] s_j^k, \text{ for } (\mu,\nu) = (0,j) \\ \sum_{k\geq 1}(s_i^k)^*[x, r_k], \text{ for } (\mu,\nu) = (i,0) \\ (\sum_{k\geq 1}(s_i^k)^* x s_j^k) - \delta^i_j x, \text{for } (\mu,\nu) = (i,j). \end{cases} \quad (1.5.\ 8)$$

The vacuum conditional expectation of j_t,

$$\mathbb{E}_0 j_t = P_t,$$

where $\{P_t\}$ is the QDS with generator θ^0_0.

Definition 1.5.11. *[33] (***Evans-Hudson (EH) flow***)*
Let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathcal{B}(\mathbf{h}_0)$. A family $\{j_t\}$ of unital $*$-homomorphisms from $\mathcal{A}$ into $\mathcal{B}(\mathbf{h}_0 \bigotimes \Gamma)$ is called an EH flow with the initial algebra $\mathcal{A}$ if the following conditions are satisfied,
1. $j_0(x) = x \otimes 1,\ \forall x \in \mathcal{A}$
2. $j_t(x) \in \mathcal{B}_{t]}$
3. There exists a family of maps $\{\theta^\mu_\nu : \mu,\nu \geq 0\}$ defined on a dense $*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ and taking values in $\mathcal{A}$, such that $j_t(x)$ for $x \in \mathcal{A}_0$ is a regular adapted process, obtained as a solution of (1.5. 7).

The family $\{\theta^\mu_\nu : \mu,\nu \geq 0\}$ is called the family for the *structure maps* of EH flow j_t and they satisfy the following properties

$$\begin{aligned} &1.\ \theta^\mu_\nu\text{'s are linear maps.} \\ &2.\ \theta^\mu_\nu(1) = 0,\ \forall \mu,\nu \geq 0,\ \text{if } 1 \in \mathcal{A}_0 \\ &3.\ \theta^\mu_\nu(x^*) = (\theta^\mu_\nu(x))^*, \forall x \in \mathcal{A}_0 \\ &4.\ \text{For any } x,y \in \mathcal{A}_0,\ \mu,\nu \geq 0 \\ &\quad \theta^\mu_\nu(xy) - x\theta^\mu_\nu(y) - \theta^\mu_\nu(x)y = \sum_{\xi\geq 1}\theta^\mu_\xi(x)\theta^\xi_\nu(y) \end{aligned} \quad (1.5.\ 9)$$

1.5.3 Coordinate-free Quantum stochastic calculus

Here, we shall briefly discuss the coordinate-free language of quantum stochastic calculus developed in [15].

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, $R \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \bigotimes \mathcal{H}_2)$ and $T \in \mathcal{B}(\mathcal{H}_1 \bigotimes \mathcal{H}_2)$. For any $f \in \mathcal{H}_2$, let $\langle f, R\rangle \in \mathcal{B}(\mathcal{H}_1)$ and $T_f \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \bigotimes \mathcal{H}_2)$ be defined as in (1.2. 1) and (1.2. 2) respectively. Let S be the symmetrization operator from free Fock space $\Gamma_f(\mathcal{H}_2)$ to symmetric Fock space $\Gamma_s(\mathcal{H}_2)$. Action of S on finite particle vector is given by,

$$S(f_1 \otimes f_2 \cdots f_m) = \frac{1}{(m-1)!} \sum_{\sigma \in S_m} f_{\sigma(1)} \otimes f_{\sigma(2)} \cdots f_{\sigma(m)},$$

where S_m is group of permutation on m points. For $R \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \bigotimes \mathcal{H}_2)$, we define the creation operator $a^\dagger(R)$ which will act on the linear span of vectors of the form $uf^{\otimes^n}$ and on $\mathcal{H}_1 \bigotimes_{alg} \mathcal{E}(\mathcal{H}_2)$ as follows,

$$a^\dagger(R)(uf^{\otimes^n}) = \frac{1}{\sqrt{n+1}} (1_{\mathcal{H}_1} \otimes S)((Ru) \otimes f^{\otimes^n}), \tag{1.5. 10}$$

It is easy to observe that $\sum_{n\geq 0} \frac{1}{n!}\|a^\dagger(R)(uf^{\otimes^n})\|^2 < \infty$, which allows us to define $a^\dagger(R)(u\mathbf{e}(f))$ as the direct sum $\bigoplus_{n\geq 0} \frac{1}{(n!)^{\frac{1}{2}}} a^\dagger(R)(uf^{\otimes^n})$. The annihilation operator a(R) is defined by,

$$a(R)u\mathbf{e}(f) := \langle R, f\rangle u\mathbf{e}(f). \tag{1.5. 11}$$

Now define the conservation operator $\Lambda(T)$ by,

$$\Lambda(T)u\mathbf{e}(f) := a^\dagger(T_f)(u\mathbf{e}(f)). \tag{1.5. 12}$$

Next, to define the fundamental processes, we need some more notations. Let $\mathbf{h}_0, \mathbf{k}_0$ be two Hilbert space and $\mathcal{K} = L^2(\mathbb{R}_+, \mathbf{k}_0)$. Let $R \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0)$, $T \in \mathcal{B}(\mathbf{h}_0 \bigotimes \mathbf{k}_0)$. For $t \geq 0$ and any bounded interval $\Delta \subseteq [t, \infty)$, we define $R_t^\Delta : \mathbf{h}_0 \bigotimes \Gamma_{t]} \to \mathbf{h}_0 \bigotimes \Gamma_{t]} \bigotimes \mathcal{K}_{[t}$ by,

$$R_t^\Delta(u\mathbf{e}(f_{t]})) = P((1_{\mathbf{h}_0} \otimes \chi_\Delta)(Ru) \otimes \mathbf{e}(f_{t]})$$

where $\chi_\Delta : \mathbf{k}_0 \to \mathcal{K}_{[t}$ is the operator which takes z to $1_\Delta(\cdot)z$ for $z \in \mathbf{k}_0$ and P is the canonical unitary isomorphism from $\mathbf{h}_0 \bigotimes \mathcal{K}_{[t} \bigotimes \Gamma_{t]}$ to $\mathbf{h}_0 \bigotimes \Gamma_{t]} \bigotimes \mathcal{K}_{[t}$. Now we define the creation process $a_R^\dagger(\Delta)$ as :

$$a_R^\dagger(\Delta) = a^\dagger(R_t^\Delta), \tag{1.5. 13}$$

where $a^\dagger(R_t^\Delta)$ carries the meaning as in (1.5. 10), with $\mathcal{H}_1 = \mathbf{h}_0 \bigotimes \Gamma_{t]}, \mathcal{H}_2 = \mathcal{K}_{[t}$. Let $T^\Delta_{f_{[t}}$ be the linear map from $\mathbf{h}_0 \bigotimes \Gamma_{t]}$ to $\mathbf{h}_0 \bigotimes \Gamma_{t]} \bigotimes \mathcal{K}_{[t}$. is given by

$$T^\Delta_{f_{[t}}(u\mathbf{e}(f_{t]})) = P(1 \otimes \hat{\chi}_\Delta)(\hat{T}(uf_{[t}) \otimes \mathbf{e}(f_{t]})). \tag{1.5. 14}$$

Here, $\hat{T}$ is the linear map on $\mathbf{h}_0 \bigotimes L^2((t,\infty),\mathbf{k}_0) \equiv L^2((t,\infty),\mathbf{h}_0 \bigotimes \mathbf{k}_0)$ given by $\hat{T}(uf_{[t})(s) = T(uf(s)), \forall s \geq t$ and $\hat{\chi}_\Delta$ is multiplication by characteristic function 1_Δ. With the above notation, we define the annihilation and conservation processes $a_R(\Delta)$ and $\Lambda_T(\Delta)$ by

$$a_R(\Delta)(u_t\mathbf{e}(f) = ((\int_\Delta \langle R, f(s)\rangle ds)u\mathbf{e}(f_{t]}))\mathbf{e}(f_{[t}), \tag{1.5. 15}$$

$$= \int_\Delta R^*(uf(s))\ ds\ \mathbf{e}(f).$$

and

$$\Lambda_T(\Delta)(u_t\mathbf{e}(f) = a^\dagger(T^\Delta_{f_{[t}})(u\mathbf{e}(f); \tag{1.5. 16}$$

Here let us recall some preliminary observations from [15] which will be needed later on.

Lemma 1.5.12. *Let $\Delta, \Delta' \subseteq (t,\infty)$ be intervals of finite length, H_t, H_t' be two adapted processes and $u, v \in \mathbf{h}_0$; $g, f \in \mathcal{K}$.*
1. For $R, S \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0)$ we have,

$$\begin{aligned}
&\langle H_t a_R^\dagger(\Delta)(v\boldsymbol{e}(g)),\ H_t' a_S^\dagger(\Delta')(u\boldsymbol{e}(f))\rangle \\
&= \langle \boldsymbol{e}(g_{t]}), \boldsymbol{e}(f_{t]})\rangle[\langle H_t R_t^\Delta(v\boldsymbol{e}(g_{t]})),\ H_t' S_t^{\Delta'}(u\boldsymbol{e}(f_{t]}))\rangle \\
&\quad + \langle\langle f_{[t}, H_t R_t^\Delta\rangle v\boldsymbol{e}(g_{t]}),\ \langle g_{[t}, H_t' S_t^{\Delta'}\rangle u\boldsymbol{e}(f_{t]})\rangle] \\
&= \int_{\Delta\cap\Delta'} \langle (H_t PR)(v\boldsymbol{e}(g)),\ (H_t' PS)(u\boldsymbol{e}(f))\rangle ds \\
&\quad + \int_\Delta \int_{\Delta'} \langle\langle f(s), H_t PR\rangle(v\boldsymbol{e}(g)), \langle g(s'),\ H_t' PS\rangle(u\boldsymbol{e}(f))\rangle ds\ ds'.
\end{aligned}$$

2. For $T, T' \in \mathcal{B}(\mathbf{h}_0 \bigotimes \mathbf{k}_0)$ *we have,*

$$\begin{aligned}&\langle (H_t T^{\Delta}_{g_{[t}})(v\boldsymbol{e}(g)), (H'_t T^{\Delta'}_{f_{[t}})(u\boldsymbol{e}(f))\rangle \\ &= \int_{\Delta\cap\Delta'} \langle H_t P T P^*(v\boldsymbol{e}(g)g(s)), H'_t P T' P^*(u\boldsymbol{e}(f)f(s))\rangle ds,\end{aligned}$$

and

$$\langle g_{[t}, H_t T^{\Delta}_{f_{[t}}\rangle = \int_{\Delta} \langle g(s), H_t T_{f(s)}\rangle ds.$$

As in the coordinatized version, integral with respect to above four basic processes can be defined. Here the fundamental lemmas take the following form. Let E, F, G, H and E', F', G', H' be integrable $(\mathbf{h}_0, \mathcal{K})$-adapted and

$$X_t = \int_0^t \left(E_s \Lambda_T(ds) + F_s a_R(ds) + G_s a^{\dagger}_S(ds) + H_s ds \right),$$

$$X'_t = \int_0^t \left(E'_s \Lambda_{T'}(ds) + F'_s a_{R'}(ds) + G'_s a^{\dagger}_{S'}(ds) + H' ds \right).$$

Then for $u, v \in \mathbf{h}_0; f, g \in L^2(\mathbb{R}_+, \mathbf{k}_0)$, we have :

Proposition 1.5.13. *(i) First fundamental formula*

$$\begin{aligned}&< X_t v\boldsymbol{e}(g), u\boldsymbol{e}(f) > \\ &= \int_0^t ds < \{< f(s), E_s P T_{g(s)} > + F_s < R, g(s) > + \\ &\qquad G_s < f(s), S > + H_s\}(v\boldsymbol{e}(g)), u\boldsymbol{e}(f) >\end{aligned}$$

(ii) Second fundamental formula (or Quantum Ito formula) can be put in a convenient symbolic form as follows. Let $\tilde{\pi_0}(x)$ *denote* $x \otimes 1_{\Gamma(k)}$ *and* $\pi_0(x)$ *denote* $x \otimes 1_{\mathbf{k}_0}$. *Then*

$a_R(dt)\tilde{\pi_0}(x)a^{\dagger}_S(dt) = R^*\pi_0(x)S dt,\ \Lambda_T(dt)\tilde{\pi_0}(x)\Lambda_{T'}(dt) = \Lambda_{T\pi_0(x)T'}(dt),$
$\Lambda_T(dt)\tilde{\pi_0}(x)a^{\dagger}_S(dt) = a^{\dagger}_{T\pi_0(x)S}(dt),\ a_S(dt)\tilde{\pi_0}(x)\Lambda_T(dt) = a_{T^*\pi_0(x)S}(dt),$
and the products of all other types are 0.

(iii) The $(\mathbf{h}_0, \mathcal{K})$*-adapted process* X_t *satisfies the estimate*

$$||X_t v\boldsymbol{e}(g)||^2 \leq e^t \int_0^t ds[||\{E_s PT_{g(s)} + G_s PS\}(v\boldsymbol{e}(g))||^2 \tag{1.5. 17}$$
$$+||\{< g(s), E_s PT_{g(s)} > +F_s < R, g(s) > + < g(s), G_s PS > +H_s\}(v\boldsymbol{e}(g))||^2].$$

In [15, 18], EH flows are constructed on C^* and von Neumann algebras when the structure maps are bounded. Let us briefly recall and state the main result obtained there. Let $\mathcal{A} \subseteq \mathcal{B}(\mathbf{h}_0)$ be a von Neumann algebra and $\mathbf{k}_0$ be a Hilbert space. Let us consider the Hilbert module $\mathcal{A} \bigotimes \mathbf{k}_0$ and define the fundamental processes in the Fock module $\mathcal{A} \bigotimes \Gamma$.

Assume that we are given the *structure maps,* that is, the triple of normal maps $(\mathcal{L}, \delta, \sigma)$, where $\mathcal{L} \in \mathcal{B}(\mathcal{A}), \delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \bigotimes \mathbf{k}_0)$ and $\sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \bigotimes \mathcal{B}(\mathbf{k}_0))$ satisfying:

(S1) $\sigma(x) = \pi(x) - x \otimes 1_{\mathbf{k}_0} \equiv \Sigma^*(x \otimes 1_{\mathbf{k}_0})\Sigma - x \otimes 1_{\mathbf{k}_0}$, where Σ is a partial isometry in $\mathbf{h}_0 \otimes \mathbf{k}_0$ such that $\pi : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\mathbf{k}_0)$ is a $*$-representation.

(S2) $\delta(x) = \pi(x)R - Rx$, where $R \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \otimes \mathbf{k}_0)$ so that δ is a π-derivation, i.e. $\delta(xy) = \delta(x)y + \pi(x)\delta(y)$.

(S3) $\mathcal{L}(x) = R^*\pi(x)R + lx + xl^*$, where $l \in \mathcal{A}$ with the condition $R^*\pi(x)R, \in \mathcal{A}$, $\mathcal{L}(1) = 0$ so that $\mathcal{L}$ satisfies the second order cocycle relation with δ as coboundary, i.e.

$$\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x)^*y = \delta(x)^*\delta(y) \;\forall x, y \in \mathcal{A}.$$

Here we note one important result from [15]

Theorem 1.5.14. *Given a uniformly continuous conservative QDS* T_t *on* $\mathcal{A}$ *with the generator* $\mathcal{L}$ *there exists a Hilbert space* $\mathbf{k}_0$*, a normal* $*$*-representation* $\pi : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\mathbf{k}_0)$ *and* $R \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0)$ *such that the above hypotheses (S1)-(S3) are satisfied.*

Let us define a map Θ associated with structure maps $(\mathcal{L}, \delta, \sigma)$ as follows $\Theta(x) := \begin{pmatrix} \theta_1(x) & (\theta_2((x)))^* \\ \theta_3(x) & \theta_4(x) \end{pmatrix} = \begin{pmatrix} \mathcal{L}(x) & \delta^\dagger(x) \\ \delta(x) & \sigma(x) \end{pmatrix}, \forall x \in \mathcal{A}$, where $\delta^\dagger(x) := (\delta(x^*))^*$, so that $\Theta : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ is a bounded linear map with respect to direct sum decomposition $\hat{\mathbf{k}}_0 = \mathbb{C} \bigoplus \mathbf{k}_0$. The following observation [15] sums up the important properties of the map Θ.

Lemma 1.5.15. *[15] Let Θ be as above. Then*
(i)There exists bounded operators $V, W \in \mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0)$ such that

$$\theta(x) = V^*(x \otimes 1_{\hat{\mathbf{k}}_0})V + W(x \otimes 1_{\hat{\mathbf{k}}_0}) + (x \otimes 1_{\hat{\mathbf{k}}_0})W^*. \tag{1.5. 18}$$

(ii) Θ is conditionally completely positive and satisfies structure relation:

$$\theta(xy) = \theta(x)(y \otimes 1_{\hat{\mathbf{k}}_0}) + (x \otimes 1_{\hat{\mathbf{k}}_0})\theta(y) + \theta(x)\hat{Q}\theta(y), \forall x, y \in \mathcal{A},$$

where $\hat{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathbf{h}_0 \bigotimes \mathbf{k}_0} \end{pmatrix} \in \mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0)$.
(iii) There exists Hilbert space $\mathcal{H}$, $D \in \mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0, \mathbf{h}_0 \bigotimes \mathcal{H})$ such that

$$\|\Theta(x)\xi\| \leq \|(x \otimes 1_{\mathcal{H}})D\xi\|, \forall x \in \mathcal{A}, \xi \in \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0. \tag{1.5. 19}$$

Proof. Define the following maps with respect to the direct sum decomposition $\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0 = \mathbf{h}_0 \bigoplus (\mathbf{h}_0 \bigotimes \mathbf{k}_0)$:
$\tilde{R} = \begin{pmatrix} 0 & 0 \\ R & -1_{\mathbf{h}_0 \bigotimes \mathbf{k}_0} \end{pmatrix}, \tilde{\pi}(x) = \begin{pmatrix} x & 0 \\ 0 & \pi(x) \end{pmatrix}, W = \begin{pmatrix} l & 0 \\ R & -\frac{1}{2}1_{\mathbf{h}_0 \otimes \mathbf{k}_0} \end{pmatrix},$
$\tilde{\Sigma} = \begin{pmatrix} 1_{\mathbf{h}_0} & 0 \\ 0 & \Sigma \end{pmatrix}.$
Then it is easy to see that (i) is verified with $V = \tilde{\Sigma}\tilde{R}$. That Θ is conditionally completely positive and satisfies the structure relation in (ii) is also an easy consequence of (i) and (S1)-(S3). To prove the estimate 1.5. 19 let us consider the following. From the structure of Θ given above, for any $\xi \in \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0$

$$\|\Theta(x)\xi\|^2 \leq 3 \left(\|V\|^2 \ \|(x \otimes 1_{\hat{\mathbf{k}}_0})V\xi\|^2 + ||W|| \ \|(x \otimes 1_{\hat{\mathbf{k}}_0})\xi\|^2 + \|(x \otimes 1_{\hat{\mathbf{k}}_0})W^*\xi\|^2 \right).$$

Thus required estimate follows with the choice of Hilbert space $\mathcal{H} = \hat{\mathbf{k}}_0 \bigoplus \hat{\mathbf{k}}_0 \bigoplus \hat{\mathbf{k}}_0$ and $D \in \mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0, \mathbf{h}_0 \bigotimes \mathcal{H},)$ given by

$$D\xi = \sqrt{3} \left(||V|| \ V\xi \oplus ||W|| \ 1_{\mathbf{h}_0 \otimes \hat{\mathbf{k}}_0}\xi \oplus W^*\xi \right).$$

□

Now we introduce the basic map-valued processes. Fix $t \geq 0$, a bounded interval $\Delta \subseteq [t, \infty)$, elements $x_1, x_2, \ldots, x_n \in \mathcal{A}$ and vectors $f_1, f_2, \ldots, f_n \in \mathcal{K}; u \in \mathbf{h}_0$. We define :

$$\left(\mathcal{I}_{\mathcal{L}}(\Delta)(\sum_{i=1}^{n} x_i \otimes \mathbf{e}(f_i))\right) u = \sum_{i=1}^{n} |\Delta|(\mathcal{L}(x_i)u) \otimes \mathbf{e}(f_i)),$$

$$\left(a_\delta(\Delta)(\sum_{i=1}^{n} x_i \otimes \mathbf{e}(f_i))\right) u = \sum_{i=1}^{n} a_{\delta(x_i^*)}(\Delta)(u\mathbf{e}(f_i)),$$

$$\left(a_\delta^\dagger(\Delta)(\sum_{i=1}^{n} x_i \otimes \mathbf{e}(f_i))\right) u = \sum_{i=1}^{n} a_{\delta(x_i)}^\dagger(\Delta)(u\mathbf{e}(f_i)),$$

$$\left(\Lambda_\sigma(\Delta)(\sum_{i=1}^{n} x_i \otimes \mathbf{e}(f_i))\right) u = \sum_{i=1}^{n} \Lambda_{\sigma(x_i)}(\Delta)(u\mathbf{e}(f_i)),$$

where $|\Delta|$ denotes the length of Δ. The above processes are well define on $\mathcal{A} \bigotimes_{\mathrm{alg}} \mathcal{E}(\mathcal{K})$ and they take values in $\mathcal{A} \bigotimes \Gamma$.

Definition 1.5.16. *A family of maps $\{Y_t\}_{t \geq 0}$ from $\mathcal{A} \bigotimes_{\mathrm{alg}} \mathcal{E}(\mathcal{K})$ to $\mathcal{A} \bigotimes \Gamma$ is said to be:*
(i) adapted, if there is a family of maps $Y_t' : \mathcal{A} \bigotimes_{\mathrm{alg}} \mathcal{E}(\mathcal{K}_{t]}) \to \mathcal{A} \bigotimes \Gamma_{t]}$ such that $Y_t(x \otimes \boldsymbol{e}(f)) = Y_t'(x \otimes \boldsymbol{e}(f_{t]})) \otimes \boldsymbol{e}(f_{[t})$ for all $x \in \mathcal{A}$, $f \in \mathcal{K}$ and $t \geq 0$;
(ii) regular, if $t \mapsto Y_t(x \otimes \boldsymbol{e}(f))u$ is continuous for every fixed $x \in \mathcal{A}$, $u \in h$, $f \in \mathcal{K}$.

For an adapted regular process Y_t satisfying

$$\sup_{0 \leq t \leq \tau} \|Y_t(x \otimes \mathbf{e}(f))u\| \leq C(f, \tau) ||(x \otimes 1_{\mathcal{H}''})ru||, \qquad (1.5.\ 20)$$

for $x \in \mathcal{A}, u \in \mathbf{h}_0, f \in \mathcal{C}$, where $\mathcal{C}$ is the space of all bounded continuous functions in $\mathcal{K}$ and $\mathcal{H}''$ is a Hilbert space, $r \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathcal{H}'')$, one can define the stochastic integral

$$\int_0^t Y_s \circ (a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_{\mathcal{L}})(ds) \qquad (1.5.\ 21)$$

as follows. First let us recall the following useful Lemma from [15].

Lemma 1.5.17. *[15]* **(The Lifting lemma)**
Let $\mathcal{H}$ be a Hilbert space and $\mathcal{V}$ be a vector space. Let $\beta : \mathcal{A} \bigotimes_{\mathrm{alg}} \mathcal{V} \to \mathcal{A} \bigotimes \mathcal{H}$ be a linear map satisfying the estimate

$$\|\beta(x \otimes \eta)u\| \leq c_\eta \|(x \otimes 1_{\mathcal{H}''})ru\| \qquad (1.5.\ 22)$$

for some Hilbert space $\mathcal{H}''$ and $r \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathcal{H}'')$ (both independent of η) and for some constant c_η depending on η. Then, for any Hilbert space $\mathcal{H}'$, we can define a *map* $\tilde{\beta} : (\mathcal{A} \bigotimes \mathcal{H}') \otimes_{\rm alg} \mathcal{V} \to \mathcal{A} \bigotimes (\mathcal{H} \bigotimes \mathcal{H}')$ *by setting* $\tilde{\beta}(x \otimes f \otimes \eta) = \beta(x \otimes \eta) \otimes f$ *for* $x \in \mathcal{A}, \eta \in \mathcal{V}, f \in \mathcal{H}'$. *Moreover,* $\tilde{\beta}$ *satisfies the estimate*

$$\|\tilde{\beta}(X \otimes \eta)u\| \leq c_\eta \|(X \otimes 1_{\mathcal{H}''})ru\|, \tag{1.5. 23}$$

where $X \in \mathcal{A} \bigotimes \mathcal{H}'$.

In (1.5. 21), the integrals corresponding to a_δ and $\mathcal{I}_{\mathcal{L}}$ belong to one class while the other two belong to another. In fact, we define $\int_0^t Y_s \circ (a_\delta + \mathcal{I}_{\mathcal{L}})(ds)(x \otimes \mathbf{e}(f))$ by setting it to be equal to $\int_0^t Y_s((\mathcal{L}(x) + \langle \delta(x^*), f(s) \rangle) \otimes \mathbf{e}(f))ds$. For the integral involving the other two processes, we need to consider $\widetilde{Y_s} : \mathcal{A} \bigotimes \mathbf{k}_0 \otimes \mathcal{E}(\mathcal{C}_{s]}) \to \mathcal{A} \bigotimes \Gamma_{s]} \bigotimes \mathbf{k}_0$ as is given by the lifting lemma 1.5.17, where $\mathcal{C}_{s]} = \mathcal{C} \bigcap \mathcal{K}_{s]}$. Defining two maps $S(s) : \mathbf{h}_0 \bigotimes_{\rm alg} \mathcal{E}(\mathcal{C}_{s]}) \to \mathbf{h}_0 \bigotimes \Gamma_{s]} \bigotimes \mathbf{k}_0$ and $T(s) : \mathbf{h}_0 \bigotimes_{\rm alg} \mathcal{E}(\mathcal{C}_{s]}) \bigotimes \mathbf{k}_0 \to \mathbf{h}_0 \bigotimes \Gamma_{s]} \bigotimes \mathbf{k}_0$ by

$$S(s)(u\mathbf{e}(f_{s]})) = \widetilde{Y_s}(\delta(x) \otimes \mathbf{e}(f_{s]}))u,$$

and

$$T(s)(ue(g_s) \otimes f(s)) = \widetilde{Y_s}(\sigma(x)_{f(s)} \otimes e(g_{s]}))u,$$

the integral $\int_0^t Y_s \circ (\Lambda_\sigma(ds) + a_\delta^\dagger(ds)(x \otimes \mathbf{e}(f))u$ is defined to be $\left(\int_0^t \Lambda_T(ds) + a_S^\dagger(ds)\right) u\mathbf{e}(f)$. For detail see [15], it is shown that

Proposition 1.5.18. *[15] The integral* $Z_t = \int_0^t Y_s \circ (a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_{\mathcal{L}})(ds)$, *where* Y_t *satisfies (1.5. 20), is well define on* $\mathcal{A} \bigotimes_{\rm alg} \mathcal{E}(\mathcal{C})$ *as a regular process. Moreover, it satisfies an estimate :*

$$\begin{aligned} &\|\{Z_t(x \otimes \mathbf{e}(f)\}u\|^2 \\ &\leq \; 2e^t \int_0^t exp(\|f_{[s}\|^2)\{\|\hat{Y}_s(\Theta(x)_{\hat{f}(s)} \otimes \mathbf{e}(f_{s]}))u\|^2 + \\ &\quad \|\langle f(s), \hat{Y}_s(\Theta(x)_{\hat{f}(s)} \otimes \mathbf{e}(f_{s]}))\rangle u\|^2\}ds, \end{aligned} \tag{1.5. 24}$$

$$\leq 2e^t(1 + \|f\|_\infty^2) \int_0^t exp(\|f_{[s}\|^2)\|\hat{Y}_s(\Theta(x)_{\hat{f}(s)} \otimes \mathbf{e}(f_{s]}))u\|^2 ds,$$

where Θ *is as defined earlier,* $\hat{Y}_s = Y_s \oplus \widetilde{Y_s} : \mathcal{A} \bigotimes \hat{\mathbf{k}}_0 \bigotimes_{\rm alg} \mathcal{E}(\mathcal{C}_{s]}) \to \mathcal{A} \bigotimes \Gamma_{s]} \bigotimes \hat{\mathbf{k}}_0$, $\hat{f}(s) = 1 \oplus f(s)$ *and* $f(s)$ *is identified with* $0 \oplus f(s)$ *in* $\hat{\mathbf{k}}_0$.

Here, we note that $\hat{Y}_s = (Y_s \otimes 1_{\hat{\mathbf{k}}_0})Q$ where, $Q : \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0 \bigotimes \Gamma \to \mathbf{h}_0 \bigotimes \Gamma \bigotimes \hat{\mathbf{k}}_0$ is the unitary operator which interchanges the second and third tensor components.

Now let us state the main result in [15] concerning the existence-uniqueness and homomorphism properties of EH flow

Theorem 1.5.19. *[15] (i) Let $\tau \geq 0$ be fixed. There exists a unique solution J_t of the equation ,*

$$J_t = id_{\mathcal{A} \bigotimes \Gamma} + \int_0^t J_s \circ (a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_{\mathcal{L}})(ds), \quad 0 \leq t \leq \tau \tag{1.5. 25}$$

as an adapted regular process mapping $\mathcal{A} \bigotimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \bigotimes \Gamma$ and satisfies the estimate

$$\sup_{0 \leq t \leq \tau} ||J_t(x \otimes \boldsymbol{e}(f))u|| \leq C'(f)||(x \otimes 1_{\Gamma_{\mathrm{fr}}(L^2([0,\tau],\mathcal{H}))})E_\tau u||,$$

where $f \in \mathcal{C}, E_\tau \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \Gamma_{\mathrm{fr}}(L^2([0,\tau],\mathcal{H})))$, $C'(f)$ is some constant and $\Gamma_{\mathrm{fr}}(L^2([0,\tau],\mathcal{H}))$ is the free Fock space over $L^2([0,\tau],\mathcal{H})$.
(ii) Setting $j_t(x)(u\boldsymbol{e}(g)) = J_t(x \otimes \boldsymbol{e}(g))u$, we have
(a) $\langle j_t(x)u\boldsymbol{e}(g), j_t(y)v\boldsymbol{e}(f)\rangle = \langle u\boldsymbol{e}(g), j_t(x^*y)v\boldsymbol{e}(f)\rangle$ $\forall g, f \in \mathcal{C}$, and
(b) j_t extends uniquely to a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{A} \bigotimes \mathcal{B}(\Gamma)$.

Proof. (i) Let us write for $\Delta \subseteq [0,\infty)$, $M(\Delta) \equiv a_\delta(\Delta) + a_\delta^\dagger(\Delta) + \Lambda_\sigma(\Delta) + \mathcal{I}_{\mathcal{L}}(\Delta)$, and set up an iteration by

$$J_t^{(n+1)}(x \otimes \mathbf{e}(f)) = \int_0^t J_s^{(n)} \circ M(ds)(x \otimes \mathbf{e}(f)), J_t^{(0)}(x \otimes \mathbf{e}(f)) = x \otimes \mathbf{e}(f),$$

with $x \in \mathcal{A}$ and $f \in \mathcal{C}$ fixed. Since $J_t^{(1)} = M([0,t])$, $J_t^{(1)}$ is a regular adapted process and by the definition of $M(\Delta)$, estimate (1.5. 17) and estimate (1.5. 19) in Lemma 1.5.15,

$$\begin{aligned} ||J_t^{(1)}(x \otimes \mathbf{e}(f))u||^2 &\leq 2e^\tau ||\mathbf{e}(f)||^2 \int_0^t ds||\Theta(x)(u \otimes \hat{f}(s))||^2||\hat{f}(s)||^2 \\ &\leq 2||\mathbf{e}(f)||^2 e^\tau \int_0^t ds||\hat{f}(s)||^2||(x \otimes 1_{\hat{\mathbf{k}}_0})D(u \otimes \hat{f}(s))||^2. \end{aligned}$$

For the given f, defining $E_t^{(1)} : \mathbf{h}_0 \to \mathbf{h}_0 \bigotimes L^2([0,\tau],\mathcal{H})$ by

$$(E_t^{(1)}u)(s) = D(u \otimes \hat{f}(s)||\hat{f}_{t]}(s)||),$$

where $\hat{f}_{t]}(s) = 1 \oplus f_{t]}(s)$, the above estimate reduces to

$$||J_t^{(1)}(x \otimes \mathbf{e}(f))u||^2 \leq 2||\mathbf{e}(f)||^2 e^\tau ||(x \otimes 1_{L^2([0,\tau],\mathcal{H})})E_t^{(1)}u||^2. \tag{1.5. 26}$$

It is also easy to see from the definition of $E_t^{(1)}$ that for $t \leq \tau$ $\|(x \otimes 1_{L^2([0,\tau],\mathcal{H})})E_t^{(1)}u\|^2 \leq \|(x \otimes 1_{L^2([0,\tau],\mathcal{H})})E_\tau^{(1)}u\|^2$. Which shows that $(J_t^{(1)})$ is indeed adapted regular process, so that $\int_0^t J_s^{(1)} \circ M(ds)$ is well-defined. Now, an application of the lifting lemma leads to

$$||\widehat{J_t^{(1)}}(X \otimes \mathbf{e}(f))u||^2 \leq 2||\mathbf{e}(f)||^2 e^\tau ||(X \otimes 1_{L^2([0,\tau],\mathcal{H})})E_\tau^{(1)}u||^2,$$

for $X \in \mathcal{A} \otimes \hat{\mathbf{k}}_0$, where as in the previous proposition, $\widehat{J_t^{(1)}} = J_t^{(1)} \oplus \widetilde{J_t^{(1)}}$. As an induction hypothesis, assume that $J_t^{(n)}$ is a regular adapted process having an estimate $||J_t^{(n)}(x \otimes \mathbf{e}(f))u||^2 \leq C^n ||\mathbf{e}(f)||^2 ||(x \otimes 1_{L^2([0,\tau],\mathcal{H})^{\otimes n}})E_\tau^{(n)}u||^2$, where $C = 2e^\tau$ and $E_\tau^{(n)} : \mathbf{h}_0 \to \mathbf{h}_0 \otimes L^2([0,\tau],\mathcal{H})^{\otimes^n}$ defined as :

$$(E_\tau^{(n)}u)(s_1, s_2, \ldots s_n) = (D \otimes 1_{L^2([0,\tau],\mathcal{H})^{\otimes n-1}})\mathcal{P}_n\{(E_\tau^{(n-1)}u)(s_2, \ldots s_n) \otimes \hat{f}(s_1)||\hat{f}_{t]}(s_1)||\}$$

where $\mathcal{P}_n : \mathbf{h}_0 \bigotimes L^2([0,\tau],\mathcal{H})^{\otimes^{(n-1)}} \bigotimes \hat{\mathbf{k}}_0 \to \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0 \bigotimes L^2([0,\tau],\mathcal{H})^{\otimes^{(n-1)}}$ is the operator which interchanges the second and third tensor components and $E_\tau^{(0)} = 1_h$. Then by an application of the proposition 1.5.18 one can verify that $J_t^{(n+1)}$ also satisfies a similar estimate, and is indeed an adapted regular process for each n. Thus, if we put $J_t = \sum_{n=0}^\infty J_t^{(n)}$, then

$$\begin{aligned} ||J_t(x \otimes \mathbf{e}(f))u|| &\leq \sum_{n=0}^\infty ||J_t^{(n)}(x \otimes \mathbf{e}(f))u|| \\ &\leq ||\mathbf{e}(f)|| \sum_{n=0}^\infty C^{\frac{n}{2}}(n!)^{-\frac{1}{4}}||(x \otimes 1_{L^2([0,\tau],\mathcal{H})^{\otimes n}})(n!)^{\frac{1}{4}}E_\tau^{(n)}u|| \\ &\leq ||\mathbf{e}(f)|| \left(\sum_{n=0}^\infty \frac{C^n}{\sqrt{n!}}\right)^{\frac{1}{2}} ||(x \otimes 1_{\Gamma_{\text{fr}}(L^2([0,\tau],\mathcal{H}))})E_\tau u||, \end{aligned} \quad (1.5.\ 27)$$

where we have set $E_\tau : \mathbf{h}_0 \to \mathbf{h}_0 \bigotimes \Gamma_{\text{fr}}(L^2([0,\tau],\mathcal{H}))$ by $E_\tau u = \oplus_{n=0}^\infty (n!)^{\frac{1}{4}} E_\tau^{(n)}u$. It is easy to see that

$$\begin{aligned} ||E_\tau u||^2 &= \sum_{n=0}^\infty (n!)^{\frac{1}{2}}||E_\tau^{(n)}u||^2 \\ &\leq ||u||^2 \sum_{n=0}^\infty (n!)^{\frac{1}{2}}||D||^{2n}\{\int_{0<s_n<s_{n-1}<\ldots s_1<\tau} ds_n \ldots ds_1 ||\hat{f}(s_n)||^4 \ldots ||\hat{f}(s_1)||^4\} \\ &= ||u||^2 \sum_{n=0}^\infty (n!)^{-\frac{1}{2}}||D||^{2n}\mu_f(t)^n, \end{aligned}$$

where $\mu_f(t) = \int_0^t ||\hat{f}(s)||^4 ds$. The estimate (1.5. 27) proves the existence of the solution of equation (1.5. 25), as well as its strong continuity. The uniqueness of the solution follows along standard lines of argument. □

Here we are omitting the proof that $\{j_t\}$ is a homomorphic family, in Chapter-4 we shall show that $\{j_t\}$ is a strong limit of a family of $*$-homomorphism $\{p_t^{(h)}\}_{h>0}$, called quantum random walks, and it follows that $\{j_t\}$ is a $*$-homomorphic flow.

Chapter 2

A Class of Quantum Dynamical Semigroups on UHF C*-algebras

In this chapter we shall discuss the class of strongly continuous QDS constructed in [28] on the UHF C^*-algebras of class N^∞.

Let $\mathcal{A}$ be the UHF C^*-algebra generated as the C^*-completion of the infinite tensor product $\bigotimes_{j\in\mathbb{Z}^d} M_N(\mathbb{C})$, where N and d are two fixed positive integers with the unique normalized trace, denoted by tr. For $x \in M_N(\mathbb{C})$ and $j \in \mathbb{Z}^d$, let $x^{(j)}$ denote an element in $\mathcal{A}$ whose j-th component is x and rest are identity of $M_N(\mathbb{C})$. For a simple tensor element $a \in \mathcal{A}$, let $a_{(j)}$ be the j-th component of a. The support of a, denoted by $supp(a)$, is defined to be the set $\{j \in \mathbb{Z}^d : a_{(j)} \neq 1\}$. For a general element $a \in \mathcal{A}$ such that $a = \sum_{n=1}^{\infty} c_n a_n$ with a_n's simple tensor elements in $\mathcal{A}$ and c_n's complex coefficients, we define $supp(a) := \bigcup_{n\geq 1} supp(a_n)$ and we set $|a| =$ cardinality of $supp(a)$. For any $\Lambda \subseteq \mathbb{Z}^d$, let $\mathcal{A}_\Lambda$ denote the $*$-subalgebra generated by elements of $\mathcal{A}$ with support Λ. When $\Lambda = \{k\}$, we write $\mathcal{A}_k$ instead of $\mathcal{A}_{\{k\}}$. Let $\mathcal{A}_{\rm loc}$ be the $*$-subalgebra of $\mathcal{A}$ generated by elements $a \in \mathcal{A}$ of finite support or equivalently by $\{x^{(j)} : x \in M_N(\mathbb{C}), j \in \mathbb{Z}^d\}$. Clearly $\mathcal{A}_{\rm loc}$ is dense in $\mathcal{A}$. For $k \in \mathbb{Z}^d$, the translation τ_k on $\mathcal{A}$ is an automorphism determined by $\tau_k(x^{(j)}) := x^{(j+k)}\ \forall x \in M_N(\mathbb{C})$ and $j \in \mathbb{Z}^d$. Thus, we get an action τ of the infinite discrete group $\mathbb{Z}^d$ on $\mathcal{A}$. For $x \in \mathcal{A}$ we denote $\tau_k(x)$ by x_k.

We also need another dense subset of $\mathcal{A}$, which is in a sense like the first Sobolev space in $\mathcal{A}$. For this, we need to note that $M_N(\mathbb{C})$ is spanned by a pair of noncommutative representatives $\{U, V\}$ of $\mathbb{Z}_N = \{0, 1 \cdots N-1\}$ such that $U^N = V^N = 1$ and $UV = wVU$, where $w \in \mathbb{C}$ is the primitive N-th root of unity. These U, V can be chosen to be the $N \times N$ circulant matrices. In particular for $N = 2$, a possible choice is given by $U = \sigma_x$ and $V = \sigma_z$, where σ_x and σ_z denote the Pauli-spin matrices. Let G be the cyclic group $\mathbb{Z}_N \times \mathbb{Z}_N$. For $g = (\alpha, \beta) \in G$, its inverse is $-g = (-\alpha, -\beta)$, where $-\alpha$ and $-\beta$ are inverse of α and β in G respectively. Now for $j \in \mathbb{Z}^d$ and $g = (\alpha, \beta) \in G$, we set $W_{j,g} = {U^{(j)}}^{\alpha} {V^{(j)}}^{\beta} \in \mathcal{A}$ and an automorphism $\pi_{j,g}$ of $\mathcal{A}$, given by $\pi_{j,g}(x) = W_{j,g}\ x\ W_{j,g}^*$. We define

$$\sigma_{j,g}(x) = \pi_{j,g}(x) - x, \ \forall x \in \mathcal{A}, \text{ and } \|x\|_1 = \sum_{j,g} \|\sigma_{j,g}(x)\|.$$

Let $\mathcal{C}^1(\mathcal{A}) = \{x \in \mathcal{A}\colon \|x\|_1 < \infty\}$. It is easy to see that $\|x^*\|_1 = \|\tau_j(x)\|_1 = \|x\|_1$ and since $\mathcal{C}^1(\mathcal{A})$ contains the dense $*$-subalgebra $\mathcal{A}_{\text{loc}}$, $\mathcal{C}^1(\mathcal{A})$ is a dense τ invariant $*$-subalgebra of $\mathcal{A}$.

Let $\mathcal{G} := \prod_{j \in \mathbb{Z}^d} G$ be the infinite direct product of the finite group G at each lattice site. Thus each $g \in \mathcal{G}$ has j-th component $g^{(j)} = (\alpha_j, \beta_j) \in G$. For $g \in \mathcal{G}$ we define its support by $supp(g) = \{j \in \mathbb{Z}^d : g^{(j)} \neq (0,0)\}$ and $|g|$ = cardinality of $supp(g)$. Let us consider the projective unitary representation of $\mathcal{G}$ given by $\mathcal{G} \ni g \mapsto U_g = \prod_{j \in \mathbb{Z}^d} W_{j,g^{(j)}} \in \mathcal{A}$.

2.1 QDS generated by formal Lindbladian

For a given completely positive map T on $\mathcal{A}$, we formally define a map $\mathcal{L}$ associated with T by setting $\mathcal{L} = \sum_{k \in \mathbb{Z}^d} \mathcal{L}_k$,
where

$$\mathcal{L}_k x = \tau_k \mathcal{L}_0(\tau_{-k} x), \ \forall x \in \mathcal{A}$$

with

$$\mathcal{L}_0(x) = -\frac{1}{2}\{T(1), x\} + T(x), \tag{2.1. 1}$$

and $\{A, B\} := AB + BA$.
In particular we consider the completely positive map

$$Tx = \sum_{l=0}^{\infty} a_l^* x a_l, \ \forall x \in \mathcal{A} \tag{2.1. 2}$$

with a sequence of elements $\{a_l\}$ in $\mathcal{A}$, such that $a_l = \sum_{g\in\mathcal{G}} c_{l,g} U_g$ and

$$\sum_{l=1}^{\infty} \sum_{g\in\mathcal{G}} |c_{l,g}|\,|g|^2 < \infty. \tag{2.1. 3}$$

It is clear that the map $\mathcal{L}$, associated with the above sequence $\{a_l\}$, i.e. associated with the completely positive map T given by (2.1. 2), takes the form

$$\mathcal{L}(x) = \sum_{k\in\mathbb{Z}^d} \mathcal{L}_k(x)$$

with

$$\mathcal{L}_k(x) = \frac{1}{2}\sum_{l=1}^{\infty}\{[\tau_k(a_l)^*, x]\tau_k(a_l) + \tau_k(a_l)^*[x, \tau_k(a_l)]\}, \forall k \in \mathbb{Z}^d.$$

Let us state and prove the main result obtained in [28].

Theorem 2.1.1. *[28] (i) The map $\mathcal{L}$ formally defined above is well define on the dense $*$-subalgebra $\mathcal{C}^1(\mathcal{A})$.*
(ii) The closure of $(\mathcal{L}, \mathcal{C}^1(\mathcal{A}))$ is the generator of a contractive conservative QDS $\{T_t : t \geq 0\}$ on $\mathcal{A}$,
(iii) The semigroup $\{T_t\}$ leaves $\mathcal{C}^1(\mathcal{A})$ invariant.

Proof. For simplicity let us prove the result for $\mathcal{L}$, associated with CP map $T(x) = r^*xr$, where $r = \sum_{g\in\mathcal{G}} c_g U_g \in \mathcal{A}$ such that $|r|_2 := \sum_{g\in\mathcal{G}} |c_g|\,|g|^2 < \infty$. For $\mathcal{L}$, associated with CP map T given by (2.1. 2), condition (2.1. 3) on the sequence $\{a_l\}$ will allow the proof to go through.
The map $\mathcal{L}$ associated with r, takes the form,

$$\mathcal{L}(x) = \sum_{k\in\mathbb{Z}^d} \mathcal{L}_k(x)$$

with

$$\mathcal{L}_k(x) = \frac{1}{2}\{[r_k^*, x]r_k + r_k^*[x, r_k]\}, \forall k \in \mathbb{Z}^d.$$

Denoting these two bounded derivations $[r_k^*, .]$ and $[., r_k]$ on $\mathcal{A}$ by $\delta_k^\dagger$ and δ_k respectively, $\mathcal{L}(x) = \frac{1}{2}\sum_{k\in\mathbb{Z}^d} \delta_k^\dagger(x) r_k + r_k^* \delta_k(x)$.

(i) For $x \in \mathcal{C}^1(\mathcal{A})$, let us estimate the norm of $\mathcal{L}(x)$,

$$\begin{aligned}\|\mathcal{L}(x)\| &\leq \frac{1}{2}\sum_{k\in\mathbb{Z}^d}\|\delta_k^{\dagger}(x)r_k + r_k^*\delta_k(x)\| \\ &\leq \frac{\|r\|}{2}\sum_{k\in\mathbb{Z}^d}\left(\|\delta_k^{\dagger}(x)\| + \|\delta_k(x)\|\right) \\ &\leq \frac{\|r\|}{2}\sum_{k\in\mathbb{Z}^d}\sum_{g\in\mathcal{G}}|c_g|\left(\|[\tau_k U_g, x]\| + \|[\tau_k U_g^*, x]\|\right).\end{aligned}$$

Since we have

$$\|[U_g, x]\| = \|[\textstyle\prod_{j\in\mathbb{Z}^d} W_{j,g^{(j)}}, x]\| \leq \displaystyle\sum_{j\in supp(g)}\|[W_{j,g^{(j)}}, x]\| = \sum_{j\in supp(g)}\|\sigma_{j,g^{(j)}}(x)\|,$$

it follows that

$$\begin{aligned}\|\mathcal{L}(x)\| &\leq \frac{\|r\|}{2}\sum_{k\in\mathbb{Z}^d}\sum_{g\in\mathcal{G}}|c_g|\sum_{j\in supp(g)+k}\left(\|\sigma_{j,g^{(j-k)}}(x)\| + \|\sigma_{j,-g^{(j-k)}}(x)\|\right) \\ &\leq \|r\|\sum_{k\in\mathbb{Z}^d}\sum_{g\in\mathcal{G}}|c_g|\sum_{j\in supp(g)+k}\sum_{g'\in G}\|\sigma_{j,g'}(x)\| \\ &\leq \|r\|\sum_{g\in\mathcal{G}}|c_g||g|\sum_{k\in\mathbb{Z}^d}\sum_{g'\in G}\|\sigma_{j,g'}(x)\| \\ &\leq \|r\|\sum_{g\in\mathcal{G}}|c_g|\ |g|\ ||x||_1 \\ &\leq |r|_2{}^2||x||_1\end{aligned}$$

(ii) Step-1. In order to apply Hille-Yosida theorem let us first observe the following. Let $\lambda > 0$ and let x be a self adjoint element in $\mathcal{C}^1(\mathcal{A})$. Then there exists a bounded operator Γ on $l^1(\mathbb{Z}^d \times G)$ such that

$$(\lambda - \Gamma)(\|\sigma_{\cdot}(x)\|)(j, g') \leq \|\sigma_{j,g'}((\lambda - \mathcal{L})x)\|. \qquad (2.1.\ 4)$$

In fact, Γ can be chosen to be an infinite positive matrix of the form,

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)},$$

with the action of $\Gamma^{(0)}$ and $\Gamma^{(1)}$ on $f \in l^1(\mathbb{Z}^d \times G)$ given by:

$$\Gamma^{(0)} f\ (j, g') = 2\sum_{g\in\mathcal{G}}|c_g|\sum_{k\in supp(g)}\sum_{l\in supp(g)}\sum_{g''\in G} f(j - k + l, g'')$$

and

$$\Gamma^{(1)}f\ (j,g') = 2\sum_{k\in\mathbb{Z}^d}\{\sum_{\mathcal{G}\ni g: j-k\in supp(g)}|c_g|\}\sum_{h\in\mathcal{G}}|c_h|\sum_{l\in supp(h)}\sum_{g''\in G}f(l+k,g'')$$

It may be noted that (2.1. 4) implies that for $\lambda > \|\Gamma\|_{l^1},\ \Gamma-\lambda$ is invertible and one has

$$\|\sigma_{j,g'}(x)\| \leq (\lambda-\Gamma)^{-1}(\|\sigma.((\lambda-\mathcal{L})x)\|)(j,g'). \tag{2.1. 5}$$

To prove (2.1. 4) let us consider the following. For a fixed $\lambda > 0$, a self adjoint element $x\in\mathcal{C}^1(\mathcal{A})$, setting $y := (\lambda-\mathcal{L})x$, we have for $(j,g')\in\mathbb{Z}^d\times G, g' = (\alpha,\beta)$

$$\sigma_{j,g'}(x) = \frac{1}{\lambda}\{\sigma_{j,g'}(y)+\sigma_{j,g'}(\mathcal{L}(x))\}.$$

Now we have, by definition

$$\begin{aligned}
&\sigma_{j,g'}(\mathcal{L}(x))\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}^d}\pi_{j,g'}([r_k^*,x]r_k)-[r_k^*,x]r_k+\pi_{j,g'}(r_k^*[x,r_k])-r_k^*[x,r_k]\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}^d}A_k(\sigma_{j,g'}(x))+[\sigma_{j,g'}(r_k^*),x]\pi_{j,g'}(r_k)+\pi_{j,g'}(r_k^*)[x,\sigma_{j,g'}(r_k)]
\end{aligned}$$

$$+[r_k^*,x]\sigma_{j,g'}(r_k)+\sigma_{j,g'}(r_k^*)[x,r_k]. \tag{2.1. 6}$$

Where,

$$A_k(x) = [\pi_{j,g'}(r_k^*),x]\pi_{j,g'}(r_k)+\pi_{j,g'}(r_k^*)[x,\pi_{j,g'}(r_k)].$$

It is clear that for each k, A_k is a conditionally completely positive bounded map and $A_k(1) = 0$. Thus A_k is the generator of a contractive CP semigroup, say $\{P_t^{(k)}\}$. As x is self adjoint, so is $\sigma_{j,g'}(x)$, we can find a state ψ on $\mathcal{A}$ such that

$$|\psi(\sigma_{j,g'}(x))| = \|\sigma_{j,g'}(x)\|$$

First let us assume,

$$\psi(\sigma_{j,g'}(x)) = \|\sigma_{j,g'}(x)\|. \tag{2.1. 7}$$

Since $\{P_t^{(k)}\}$ is positive and contractive and x is self adjoint, we have,

$$\psi(P_t^{(k)}(\sigma_{j,g'}(x)))$$

$$\leq |\psi(P_t^{(k)}(\sigma_{j,g'}(x)))|$$

$$\leq \|P_t^{(k)}(\sigma_{j,g'}(x))\|$$

$$\leq \|\sigma_{j,g'}(x)\| = \psi(P_0^{(k)}(\sigma_{j,g'}(x))).$$

Thus

$$\frac{d}{dt}|_{t=0}\psi(P_t^{(k)}(\sigma_{j,g'}(x))) = \psi(A_k(\sigma_{j,g'}(x))) \leq 0. \tag{2.1. 8}$$

Now evaluating the state ψ on $\sigma_{j,g'}(x)$ and using (2.1. 7), we get

$$\|\sigma_{j,g'}(x)\| = \frac{1}{\lambda}\{\psi(\sigma_{j,g'}(y)) + \psi(\sigma_{j,g'}(\mathcal{L}(x)))\}.$$

By (2.1. 6) and (2.1. 8), this gives

$$\|\sigma_{j,g'}(x)\| \leq \frac{1}{\lambda}\{\psi(\sigma_{j,g'}(y)) + \frac{1}{2}\sum_{k\in\mathbb{Z}^d}\{\psi([\sigma_{j,g'}(r_k^*), x]\pi_{j,g'}(r_k)) + \psi(\pi_{j,g'}(r_k^*)[x, \sigma_{j,g'}(r_k)])$$

$$+\psi([r_k^*, x]\sigma_{j,g'}(r_k)) + \psi(\sigma_{j,g'}(r_k^*)[x, r_k])\}$$

$$\leq \frac{1}{\lambda}\|\sigma_{j,g'}(y)\| + \frac{1}{2\lambda}\sum_{k\in\mathbb{Z}^d}\{\|r\| \ \|[\sigma_{j,g'}(r_k^*), x]\| + \|r\| \ \|[\sigma_{j,g'}(r_k), x]\|$$

$$+\|[r_k^*, x]\| \ \|\sigma_{j,g'}(r_k)\| + +\|\sigma_{j,g'}(r_k^*)\| \ \|[r_k, x]\|\}. \tag{2.1. 9}$$

If $\psi(\sigma_{j,g'}(x)) = -\|\sigma_{j,g'}(x)\|$, replacing x by $-x$, same argument as above gives the inequality (2.1. 9).

Now in order to estimate the second term of (2.1. 9), let us take $g \in \mathcal{G}$ with $j \in supp(g)$ and observe that

$$\|[\sigma_{j,g'}(U_g), x]\| = \|[{U^{(j)}}^{\alpha}{V^{(j)}}^{\beta}U_g{V^{(j)}}^{-\beta}{U^{(j)}}^{-\alpha} - U_g, x]\|$$

$$= \|[(w^{\alpha(\beta_j-\beta)-\beta(\alpha_j-\alpha)} - 1)U_g, x]\| \leq 2\|[U_g, x]\|$$

So we have,

$$
\begin{aligned}
&\|[\sigma_{j,g'}(r_k), x]\| \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{g:j-k\in supp(g)} |c_g| \|[\sigma_{j,g'}(\tau_k U_g), x]\| \\
&\leq 2\sum_{k\in\mathbb{Z}^d}\sum_{g:j-k\in supp(g)} |c_g| \|[\tau_k U_g, x]\| \\
&\leq 2\sum_{k\in\mathbb{Z}^d}\sum_{g:j-k\in supp(g)} |c_g| \sum_{l:l-k\in supp(g)} \|\sigma_{l,g^{(l-k)}}(x)\| \\
&\leq 2\sum_{k\in\mathbb{Z}^d}\sum_{g:j-k\in supp(g)} |c_g| \sum_{l:l-k\in supp(g)}\sum_{g''\in G} \|\sigma_{l,g''}(x)\| \\
&\leq 2\sum_{k\in\mathbb{Z}^d}\sum_{g:j+k\in supp(g)} |c_g| \sum_{l:l+k\in supp(g)}\sum_{g''\in G} \|\sigma_{l,g''}(x)\| \\
&\leq 2\sum_{g\in\mathcal{G}} |c_g| \sum_{k\in supp(g)-j}\sum_{l\in supp(g)-k}\sum_{g''\in G} \|\sigma_{l,g''}(x)\| \\
&\leq 2\sum_{g\in\mathcal{G}} |c_g| \sum_{k\in supp(g)}\sum_{l\in supp(g)}\sum_{g''\in G} \|\sigma_{j-k+l,g''}(x)\|
\end{aligned}
$$

Thus,

$$
\frac{1}{2}\sum_{k\in\mathbb{Z}^d}\{\|r\|\ \|[\sigma_{j,g'}(r_k^*), x]\| + \|r\|\ \|[\sigma_{j,g'}(r_k), x]\|\} \leq \Gamma^{(0)}(\|\sigma.(x)\|)(j,g')
$$

Similar estimate gives,

$$
\|\sigma_{j,g'}(r_k^*)\|\ \|[r_k, x]\| \leq \{2\sum_{g:j-k\in supp(g)} |c_g|\}\sum_{h\in\mathcal{G}}|c_h|\sum_{l\in supp(h)}\sum_{g''\in G}\|\sigma_{l+k,g''}(x)\|
$$

So, we have,

$$
\frac{1}{2}\sum_{k\in\mathbb{Z}^d}\left(\|\sigma_{j,g'}(r_k^*)\|\ \|[r_k, x]\| + \|[r_k^*, x]\|\ \|\sigma_{j,g'}(r_k)\|\right) \leq \Gamma^{(1)}\|\sigma.(x)\|(j,g')
$$

A simple estimate now gives,

$$
\|\Gamma(f)\|_{l_1} \leq N^2\{\sum_{g\in\mathcal{G}}|c_g|\ |g|^2\}\{4\|r\| + \sum_{g\in\mathcal{G}}|c_g|\ |g|^2\}\|f\|_{l_1}
$$

Step-2. For each $n\geq 1$, setting $\mathcal{L}^{(n)} = \sum_{|k|\leq n}\mathcal{L}_k$, it is clear that $\mathcal{L}^{(n)}$ is a bounded CCP map on $\mathcal{A}$. So $\mathcal{L}^{(n)}$ is the generator of QDS $\{T_t^{(n)}\}$ on $\mathcal{A}$ and for $\lambda > 0, \|(\mathcal{L}^{(n)} - \lambda)(x)\| \geq \lambda\|x\|,\ \forall x\in\mathcal{A}$ and hence $\|(\mathcal{L}^{(n)}-\lambda)^{-1}\| \leq \frac{1}{\lambda}$. For $\lambda > \|\Gamma\|_{l^1}$, in order to show that $Ran(\mathcal{L}-\lambda)$ is dense in $(\mathcal{A}, \|\cdot\|)$, we consider the following. Let y be a self adjoint element in $\mathcal{C}^1(\mathcal{A})$. Since $(\mathcal{L}^{(n)}-\lambda)$ is invertible for every n, we can choose $x_n \in \mathcal{A}$ (in fact self adjoint) so that $(\mathcal{L}^{(n)}-\lambda)(x_n) = y$. Note that (2.1. 5) also holds if $\mathcal{L}$ is replaced by $\mathcal{L}^{(n)}$ and thus we have

$$\|\sigma_{\alpha'}(x_n)\| \leq \sum_{\alpha}\{(\lambda-\Gamma)^{-1}\}_{\alpha',\alpha}\|\sigma_\alpha(y)\|. \tag{2.1. 10}$$

Summing over α' it follows that

$$\|(x_n)\|_1 \leq \|(\lambda-\Gamma)^{-1}\| \; \|y\|_1 < \infty$$

and so $x_n \in \mathcal{C}^1(\mathcal{A})$. Now setting $y_n = (\mathcal{L}-\lambda)(x_n)$,

$$\|y_n - y\| = \|(\mathcal{L}-\mathcal{L}^{(n)})x_n\| = \sum_{|k|>n}\mathcal{L}_k(x_n).$$

The above quantity is clearly dominated by

$$\begin{aligned} &\|r\| \sum_{|k|>n} \sum_{g\in\mathcal{G}} |c_g| \sum_{g_1\in G} \sum_{j\in supp(g)+k} \|\sigma_{j,g_1}(x_n)\| \\ &\leq \|r\| \sum_{|k|>n} \sum_{g\in\mathcal{G}} |c_g| \sum_{g_1\in G} \sum_{j\in supp(g)+k} (\lambda-\Gamma)^{-1}\{\|\sigma_\cdot(y)\|\}(j,g_1) \end{aligned} \tag{2.1. 11}$$

Since

$$\begin{aligned} &\sum_{|k|>1} \sum_{g\in\mathcal{G}} |c_g| \sum_{g_1\in G} \sum_{j\in supp(g)+k} (\lambda-\Gamma)^{-1}\{\|\sigma_\cdot(y)\|\}(j,g_1) \\ \leq\ &\sum_{g\in\mathcal{G}} |c_g| \; |g| \sum_{j\in\mathbb{Z}^d} \sum_{g_1\in G} (\lambda-\Gamma)^{-1}\{\|\sigma_\cdot(y)\|\}(j,g_1) \\ \leq\ &\sum_{g\in\mathcal{G}} |c_g| \; |g| \; \|(\lambda-\Gamma)^{-1}\| \; \|y\|_1 < \infty, \end{aligned}$$

(2.1. 11) goes to 0 as n tends to ∞ which shows that y_n converges to y. For a general elements $y \in \mathcal{C}^1(\mathcal{A})$, using the above argument for the real and imaginary parts of y we can find a sequence $y_n \in (\mathcal{L}-\lambda)(\mathcal{C}^1(\mathcal{A}))$ such that approximate y_n converges to y. Thus $Ran(\mathcal{L}-\lambda)$ as well as $(\mathcal{L}-\lambda)(\mathcal{C}^1(\mathcal{A}))$ are dense in $\mathcal{C}^1(\mathcal{A})$ and hence in $\mathcal{A}$ too.

Now for $y = (\mathcal{L} - \lambda)(x)$ in the dense $*$-subspace $(\mathcal{L} - \lambda)(\mathcal{C}^1(\mathcal{A}))$, we have

$$\begin{aligned} &\|(\mathcal{L}^{(n)} - \lambda)^{-1}(y) - (\mathcal{L} - \lambda)^{-1}(y)\| \\ &= \|(\mathcal{L}^{(n)} - \lambda)^{-1}(\mathcal{L} - \mathcal{L}^{(n)})(\mathcal{L} - \lambda)^{-1}(y)\| \\ &\leq \|(\mathcal{L}^{(n)} - \lambda)^{-1}\| \|\sum_{|k|>n} \mathcal{L}_k(x)\| \\ &\leq \frac{1}{\lambda}\|\sum_{|k|>n} \mathcal{L}_k(x)\|. \end{aligned}$$

So $\|(\mathcal{L}^{(n)} - \lambda)^{-1}(y) - (\mathcal{L} - \lambda)^{-1}(y)\|$ converges to 0 as n tends to ∞ and hence

$$\|(\mathcal{L} - \lambda)^{-1}(y)\| \leq \frac{1}{\lambda}\|y\|.$$

From the Hille-Yosida theorem it follows that $\mathcal{L}$ is the generator of a strongly continuous contractive semigroup. Now by theorem 1.3.4, the contractive semigroups $T_t^{(n)}$ converges to T_t strongly as n tends to ∞. Thus T_t is a contractive CP semigroup on $\mathcal{A}$ as each $T_t^{(n)}$ is so.
The semigroup T_t satisfies

$$T_t(x) = x + \int_0^t T_s(\mathcal{L}(x))ds, \ \forall x \in Dom(\mathcal{L}).$$

Since $1 \in \mathcal{C}^1(\mathcal{A})$ and $\mathcal{L}(1) = 0$, it follows that $T_t(1) = 1, \forall t \geq 0$.
(iii) By (2.1. 5) for $\lambda > \beta := \|\Gamma\|_{l^1}$ and self adjoint $x \in \mathcal{C}^1(\mathcal{A})$, we have

$$\|\sigma_{j,g}(\lambda - \mathcal{L})^{-1}(x)\| \leq (\lambda - \Gamma)^{-1}\|\sigma.(x)\|(j,g), \ \forall (j,g) \in \mathbb{Z}^d \times G.$$

Summing over all $(j,g) \in \mathbb{Z}^d \times G$, we get

$$\|(\lambda - \mathcal{L})^{-1}(x)\|_1 \leq (\lambda - \beta)^{-1}\|x\|_1.$$

Thus for $x \in \mathcal{C}^1(\mathcal{A})$,

$$\|T_t(x)\|_1 \leq 2\ e^{t\beta}\|x\|_1 < \infty.$$

□

2.2 Ergodicity of the QDS

Following [28], we say that a contractive CP semigroup T_t is *ergodic* if there exists an invariant state ψ satisfying

$$\|T_t(x) - \psi(x)1\| \to 0 \text{ as } t \to \infty, \ \forall x \in \mathcal{A}. \tag{2.2. 1}$$

In [28], the author has discussed some criteria for ergodicity of the QDS T_t. Some examples of such semigroups associated with partial states on the UHF algebra and their perturbation are given.

For a state ϕ on $M_N(\mathbb{C})$ and $k \in \mathbb{Z}^d$, the partial state ϕ_k on $\mathcal{A}$ is a CP map determined by $\phi_k(x) = \phi(x_{(k)})x_{\{k\}^c}$, for $x = x_{(k)}x_{\{k\}^c}$, where $x_{(k)} \in \mathcal{A}_k = \mathcal{A}_{\{k\}}$ and $x_{\{k\}^c} \in \mathcal{A}_{\{k\}^c}$ with $\{k\}^c$ stands for the complement set $\mathbb{Z}^d \setminus \{k\}$. Here, recall that for any set Λ in $\mathbb{Z}^d$, $\mathcal{A}_\Lambda$ denotes the sub-algebra of elements with support contain in Λ. By (2.1. 1) the Lindbladian $\mathcal{L}^\phi$ corresponding to the partial state ϕ_0 is formally given by

$$\mathcal{L}^\phi(x) = \sum_{k \in \mathbb{Z}^d} \mathcal{L}^\phi_k(x), \text{ where } \mathcal{L}^\phi_k(x) = \phi_k(x) - x. \tag{2.2. 2}$$

For the state ϕ we can find vectors $\{\xi_l : l = 1, 2 \cdots N\}$ in $\mathbb{C}^N$ such that

$$\phi(x) = \sum_{l=1}^{N} \langle \xi_l, x\xi_l \rangle, \ \forall x \in M_N(\mathbb{C}).$$

Let us reindex the set $\{(n,l) : n, l = 1, \cdots N\}$ by $\{m = 1, \cdots N^2\}$. Now for a fixed orthonormal basis $\{e_n : n = 1, \cdots N\}$ of $\mathbb{C}^N$, defining N^2 many rank one operators $L^{(m)} := |\xi_l >< e_n|$ where index m corresponds to (n,l), we have

$$\phi(x) = \sum_{m=1}^{N^2} L^{(m)^*} x L^{(m)}, \ \forall x \in M_N(\mathbb{C}), \text{ and } \sum_{m=1}^{N^2} L^{(m)^*} L^{(m)} = 1.$$

For $m = 1, \cdots N^2$, we consider the element $L_0^{(m)} \in \mathcal{A}_0$ with the zeroth component being $L^{(m)}$. Now for $k \in \mathbb{Z}^d$, writing $L_k^{(m)} = \tau_k(L_0^{(m)})$, the partial state ϕ_k is given by,

$$\phi_k(x) = \sum_{m=1}^{N^2} L_k^{(m)^*} x L_k^{(m)}, \ \forall x \in \mathcal{A}.$$

So the formal Lindbladian $\mathcal{L}^\phi$ takes the form

$$\mathcal{L}^\phi(x) = \sum_{k\in\mathbb{Z}^d} \mathcal{L}_k^\phi(x),$$

with

$$\mathcal{L}_k^\phi(x) = \frac{1}{2}\sum_{m=1}^{N^2}[L_k^{(m)^*}, x]L_k^{(m)} + L_k^{(m)^*}[x, L_k^{(m)}].$$

It follows from Theorem 2.1.1 that $\mathcal{L}^\phi$ is defined on $\mathcal{C}^1(\mathcal{A})$. Moreover, the closure of $(\mathcal{L}^\phi, \mathcal{C}^1(\mathcal{A}))$ (which we denote by same symbol $\mathcal{L}^\phi$) is the generator of a contractive CP semigroup T_t^ϕ on $\mathcal{A}$.
From (2.2. 2), for any element $x_k^{(k)} \in \mathcal{A}$ with support $\{k\}$, by induction we have $(\mathcal{L}^\phi)^n(x_k^{(k)}) = (-1)^n[x_k^{(k)} - \phi(x_k)]$ and hence

$$T_t^\phi(x_k) = \sum_{n\geq 0}\frac{t^n}{n!}(\mathcal{L}^\phi)^n(x_k^{(k)})$$

$$= \phi(x_k) + e^{-t}(x_k^{(k)} - \phi(x_k))\}.$$

For simple tensor element, in particular for $x = x_i^{(i)}x_j^{(j)}$,

$$(\mathcal{L}^\phi)^n(x) = (-1)^n\Big([x_i^{(i)} - \phi(x_i)]x_j^{(j)} + x_i^{(i)}[x_j^{(j)} - \phi(x_j)]$$
$$+ c(n)\ [x_i^{(i)} - \phi(x_i)][x_j^{(j)} - \phi(x_j)]\Big),$$

where

$$c(n) = \sum_{m=1}^{n-1}\frac{n!}{m!\ (n-m)!} = 2^n - 2.$$

Thus

$$\begin{aligned}T_t^\phi(x) &= \sum_{n\geq 0}\frac{t^n}{n!}(\mathcal{L}^\phi)^n(x)\\ &= \Big[\phi(x_i^{(i)}) + e^{-t}(x_i^{(i)} - \phi(x_i^{(i)}))\Big]\Big[\phi(x_j^{(j)}) + e^{-t}(x_j^{(j)} - \phi(x_j^{(j)}))\Big]\\ &= T_t^\phi(x_i^{(i)})T_t^\phi(x_j^{(j)}).\end{aligned}$$

On simple tensor element $x = \prod_{k\in\Lambda} x_k^{(k)}$ with support Λ,

$$T_t^\phi(\prod_{k\in\Lambda} x_k^{(k)}) = \prod_{k\in\Lambda}\ \{\phi(x_k) + e^{-t}(x_k^{(k)} - \phi(x_k))\}$$

and hence

$$\lim_{t\to\infty} \|T_t^{\phi}(\prod_{k\in\Lambda} x_k^{(k)}) - \prod_{k\in\Lambda}\phi(x_k)\| = 0.$$

Now setting $\Phi(\prod_{k\in\Lambda} x_k^{(k)}) = \prod_{k\in\Lambda}\phi(x_k)$ and for any $x = \sum_{g\in\mathcal{G}} c_g U_g \in \mathcal{A}_{\text{loc}}$ defining $\Phi(x) = \sum_{g\in\mathcal{G}} c_g \Phi(U_g)$, it follows that

$$\lim_{t\to\infty} \|T_t^{\phi}(x) - \Phi(x)\| = 0, \forall x \in \mathcal{A}_{\text{loc}}.$$

Since, $\{T_t^{\phi}\}$ is a CP contractive semigroup $\|\Phi(x)\| \le \|x\|, \forall x \in \mathcal{A}_{\text{loc}}$ and Φ extends as a state on $\mathcal{A}$ such that

$$\lim_{t\to\infty} T_t^{\phi}(x) = \Phi(x), \forall x \in \mathcal{A}.$$

The dilation problem for this T_t^{ϕ} will be addressed in the next Chapter.

Now we consider the perturbation of the contractive CP semigroup T_t^{ϕ}. Let T_t be the QDS with the generator $\mathcal{L}$ appearing in the Theorem 2.1.1. For any real number c, let us consider

$$\mathcal{L}^{(c)}(x) = \mathcal{L}^{(\phi)} + c\mathcal{L}$$

It is clear that $\mathcal{L}^{(c)}$ is the Lindbladian associated with the completely positive map

$$T(x) = \sum_{m=1}^{N^2} L_k^{(m)^*} x L_k^{(m)} + c\sum_{l=0}^{\infty} a_l^* x a_l, \forall x \in \mathcal{A},$$

and by Theorem 2.1.1 it follows that the closure of $(\mathcal{L}^{(c)}, \mathcal{C}^1(\mathcal{A}))$ is the generator of a QDS $T_t^{(c)}$. From [28], here we shall state a result concerning the ergodicity of CP semigroup $T_t^{(c)}$.

Theorem 2.2.1. *[28] There exists a constant c_0 such that for $0 \le c \le c_0$, the above contractive CP semigroup $T_t^{(c)}$ is ergodic with respect to the invariant state $\Phi^{(c)}$ satisfying*

$$\|T_t^{(c)}(x)\|_1 \le 2e^{-(1-\frac{c}{c_0})t}\|x\|_1, \tag{2.2. 3}$$

$$\|T_t^{(c)}(x) - \Phi^{(c)}(x)1\| \le \frac{4}{N^2} e^{-(1-\frac{c}{c_0})t}\|x\|_1, \ \forall x \in \mathcal{C}^1(\mathcal{A}).$$

The following result determines the invariant state $\Phi^{(c)}$.

Proposition 2.2.2. *The invariant state $\Phi^{(c)}$ corresponding to the ergodic QDS $T_t^{(c)}$ is given by*

$$\Phi^{(c)}(x) = \Phi(x) + c\int_0^\infty \Phi(\mathcal{L}(T_s^{(c)}(x)))ds, \forall x \in \mathcal{C}^1(\mathcal{A}).$$

Proof. Since for any $x \in \mathcal{A}$, $T_t^{\phi}(x)$ converges to $\Phi(x)$ as t tends to ∞ and for any $t \geq 0, x \in \mathcal{C}^1(\mathcal{A})$,

$$T_t^{(c)}(x) - T_t^{(\phi)}(x) = c\int_0^t T_{t-s}^{(\phi)}\mathcal{L}T_s^{(c)}(x)ds,$$

it is enough to show that for $x \in \mathcal{C}^1(\mathcal{A})$,

$$\lim_{t\to\infty} \| \int_0^t T_{t-s}^{(\phi)}\mathcal{L}T_s^{(c)}(x)ds - \int_0^\infty \Phi(\mathcal{L}(T_s^{(c)}(x)))ds\| = 0. \tag{2.2. 4}$$

To prove (2.2. 4), we observe that

$$\begin{aligned}&\| \int_0^t T_{t-s}^{(\phi)}\mathcal{L}T_s^{(c)}(x)ds - \int_0^\infty \Phi(\mathcal{L}(T_s^{(c)}(x)))ds\| \\ &\leq \| \int_0^t (T_{t-s}^{(\phi)} - \Phi)\mathcal{L}T_s^{(c)}(x)ds\| + \| \int_t^\infty \Phi(\mathcal{L}(T_s^{(c)}(x)))ds\|.\end{aligned}$$

Since $T_s^{(c)}(x) \in \mathcal{C}^1(\mathcal{A})$, we have by (2.2. 3) $\|\mathcal{L}(T_s^{(c)}(x))\| \leq Me^{-\mu s}\|x\|_1$ for some positive constants μ and M independent of s. Thus the integrands in the first and second terms are dominated by the integrable function $f(s) = 2Me^{-\mu s}\|x\|_1$. Clearly, the second term converges to 0 as t tends to ∞. Since for fixed $s \geq 0$ and $x \in \mathcal{C}^1(\mathcal{A})$,

$$\lim_{t\to 0} \|(T_{t-s}^{(\phi)} - \Phi)\mathcal{L}T_s^{(c)}(x)\| = 0, \forall s \geq 0,$$

by dominated convergence theorem the first term goes to 0 as t tends to ∞. □

Chapter 3

EH Dilation for a Class of QDS by Iteration Method

In this chapter we investigate the possibility of constructing EH flows for the QDS on UHF C^*-algebras, discussed in the previous section. Although the question is not answered in full generality, EH flows for a class of QDS are constructed.

Let $r = \sum_{g \in \mathcal{G}} c_g U_g \in \mathcal{A}$ such that $\sum_{g \in \mathcal{G}} |c_g||g|^2 < \infty$. The Lindbladian $\mathcal{L}$ associated with the element r, i.e. associated with the CP map T given by $T(x) = r^* x r$ takes the form

$$\mathcal{L}(x) = \sum_{k \in \mathbb{Z}^d} \delta_k^\dagger(x) r_k + r_k^* \delta_k(x), \tag{3.0. 1}$$

where $r_k := \tau_k(r)$ and $\delta_k, \delta_k^\dagger$ are bounded derivations on $\mathcal{A}$ defined by

$$\delta_k(x) = [x, r_k] \text{ and } \delta_k^\dagger(x) := (\delta_k(x^*))^* = [r_k^*, x], \forall x \in \mathcal{A}. \tag{3.0. 2}$$

It follows from [28] that the closure of $(\mathcal{L}, \mathcal{C}^1(\mathcal{A}))$ is the generator of a QDS T_t on $\mathcal{A}$. In order to construct an EH flow for the QDS T_t, we would like to solve the following qsde in $\mathcal{B}(L^2(\mathcal{A}, tr)) \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0)))$:

$$dj_t(x) = \sum_{k \in \mathbb{Z}^d} j_t(\delta_k^\dagger(x)) da_k(t) + \sum_{k \in \mathbb{Z}^d} j_t(\delta_k(x)) da_k^\dagger(t) + j_t(\mathcal{L}(x)) dt, \tag{3.0. 3}$$

$$j_0(x) = x \otimes 1_\Gamma \ , x \in \mathcal{A}_{\text{loc}}.$$

Let us first look at the corresponding Hudson-Parthasarathy equation in $L^2(\mathcal{A}, tr) \otimes \Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0))$ given by

$$dU_t = \{\sum_{k \in \mathbb{Z}^d} [r_k^* da_k(t) - r_k da_k^\dagger(t)] - \frac{1}{2} \sum_{k \in \mathbb{Z}^d} r_k^* r_k dt\} U_t, \qquad (3.0.\ 4)$$

$$U_0(x) = 1_{L^2 \otimes \Gamma}.$$

However, though each $r_k \in \mathcal{A}$ and hence is in $\mathcal{B}(L^2(\mathcal{A}, tr))$, the equation (3.0. 4) does not in general admit a solution since

$$\langle u, \sum_{k \in \mathbb{Z}^d} r_k^* r_k u \rangle = \sum_{k \in \mathbb{Z}^d} \|r_k u\|^2 \ \forall u \in L^2(\mathcal{A}, tr)$$

is not convergent in general and hence $\sum_{k \in \mathbb{Z}^d} r_k \otimes e_k$ does not define an element in $\mathcal{A} \otimes \mathbf{k}_0$. For example, let r be the single-supported unitary element $U^{(j)} \in \mathcal{A}$ for some $j \in \mathbb{Z}^d$ so that $r_k = U^{(k+j)}$ is a unitary for each $k \in \mathbb{Z}^d$ and hence

$$\sum_{k \in \mathbb{Z}^d} \|r_k u\|^2 = \sum_{k \in \mathbb{Z}^d} \|u\|^2 = \infty.$$

However, as we shall see, in many situations there exist Evans-Hudson flows, even though the corresponding Hudson-Parthasarathy equations (3.0. 4) do not admit solution.
There are some cases when an Evans-Hudson flow can be seen to be implemented by a solution of a Hudson-Parthasarathy equation.

Hudson-Parthasarathy type flow

Here we shall construct HP type flow for the QDS associated with self adjoint element r in $\mathcal{A}$, using the results in [30, 32]. This method is not applicable for CP semigroup associate with non self adjoint elements of $\mathcal{A}$. Let us recall some results from [30, 32] which will be needed in the sequel.
Let $\mathcal{I}$ be the collection of $L = \{L_\nu^\mu : \mu, \nu \geq 0\}$, where $L_\nu^\mu \in \mathcal{B}(\mathbf{h}_0)$ and for each $\nu, \exists$ a constant C_ν depending upon L and ν such that $\sum_{\mu \geq 0} \|L_\nu^\mu u\|^2 \leq C_\nu \|u\|^2, \ \forall u \in \mathbf{h}_0$, furthermore, for any $\mu, \nu \geq 0, L_\nu^\mu + (L_\mu^\nu)^* + \sum_{k \geq 1} (L_k^\mu)^* L_\nu^k = 0$. We define $\tilde{\mathcal{I}} = \{\tilde{L} = \{\tilde{L}_\mu^\nu = (L_\nu^\mu)^*\} : L \in \mathcal{I}\}$.

Theorem 3.0.1. *[30] Let $L = \{L^\mu_\nu : \mu, \nu \geq 0\}$ be a family of linear operator in $\mathbf{h}_0$ such that:*

1. L^0_0 is the generator of a strongly continuous contractive semigroup with $\mathcal{D}$ as a core and $\mathcal{D} \subseteq \mathcal{D}(L^\mu_\nu), \forall \mu, \nu$,

2. there exists a sequence $L(n) \in \mathcal{I} \cap \tilde{\mathcal{I}}, n \geq 1$, so that for any $u \in \mathcal{D}$

$$\lim_{n \to \infty} L^\mu_\nu(n) u = L^\mu_\nu u, \forall \mu, \nu.$$

Then there exists a unique strongly continuous contractive solution V_t of the qsde

$$dV_t = \sum_{\mu,\nu} V_t L^\mu_\nu d\Lambda^\mu_\nu; \ V_0 = 1. \tag{3.0. 5}$$

Moreover, if $\beta_\lambda = \tilde{\beta}_\lambda = 0$, for some $\lambda \geq 0$, where β_λ is

$$\{X \in \mathcal{B}(\mathbf{h}_0)^+ : \langle u, X L^0_0 v\rangle + \langle L^0_0 u, X v\rangle + \sum_{k \in \mathbb{Z}^d} \langle L^k_0 u, X L^0_k v\rangle = \lambda \langle u, X v\rangle, \forall u, v \in \mathcal{D}\}$$

and $\tilde{\beta}_\lambda$ is similarly defined corresponding to $\tilde{L}$, then V is a unitary process.

Next result give the sufficient condition for $\beta_\lambda = \tilde{\beta}_\lambda = 0$.

Theorem 3.0.2. *[32] Let $(Y, \mathcal{D})$ be the generator of a strongly continuous contractive semigroup on $\mathbf{h}_0$ and $S_k : k \geq 1$ be a family of densly define operator on $\mathbf{h}_0$ such that:*

1. $\mathcal{D} \subseteq \mathcal{D}(S_k), \forall k$ and

2. $\langle u, Yu\rangle + \langle Yu, u\rangle + \sum_{k \geq 1} \langle S_k u, S_k u\rangle \leq 0, \forall u \in \mathcal{D}$.

Then the map $\tilde{\mathcal{L}}$ on $\mathcal{B}(\mathbf{h}_0)$ formally define by

$$\tilde{\mathcal{L}}(x) := \frac{1}{2} \sum_{k \in \mathbb{Z}^d} (2 S_k x S_k - S_k S_k x - x S_k S_k)$$

is well define on a suitable domain and its closure generates a contractive QDS P_t on $\mathcal{B}(\mathbf{h}_0)$. Moreover, the following statements are equivalent:

(a). The semigroup P_t is conservative.

(b). For any fixed $\lambda > 0$, the set

$$\{x \in \mathcal{B}(\mathbf{h}_0)^+ : \langle u, xYv\rangle + \langle Yu, xv\rangle + \sum_{k \in \mathbb{Z}^d} \langle S_k u, x S_k v\rangle = \lambda \langle u, xv\rangle, \forall u, v \in \mathcal{D}(Y)\}$$

contains only trivial element $0 \in \mathcal{B}(\mathbf{h}_0)$.

Now let us consider QDS T_t on UHF C^*-algebra $\mathcal{A}$ associated with self adjoint element $r \in \mathcal{A}$. Here the generator $\mathcal{L}$ satisfies

$$\mathcal{L}(x) = -\frac{1}{2} \sum_{k \in \mathbb{Z}^d} [r_k, [r_k, x]] \; \forall x \in \mathcal{C}^1(A).$$

The UHF C^*-algebra $\mathcal{A}$ is isometrically embedded as a dense subspace in the GNS Hilbert space $\mathbf{h}_0 = L^2(\mathcal{A}, tr)$, with canonical embedding i. For any fixed a and $b \in \mathcal{A}$ it is clear that $\|i(axb)\|^2_{\mathbf{h}_0} \leq \|a\|^2 \|b\|^2 tr(x^*x), \forall x \in \mathcal{A}$. Thus for any $k \in \mathbb{Z}^d$, the map S_k on $\mathcal{A}$ defined by $S_k(i(x)) = i([r_k, x]), \forall x \in \mathcal{A}$, extends uniquely to a bounded self adjoint operator on $\mathbf{h}_0$. We define a family of operator $L = \{L^\mu_\nu\}$ given by

$$\begin{aligned} L^\mu_\nu &= -\frac{1}{2} \sum_{k \in \mathbb{Z}^d} S_k S_k, \quad \text{for} \quad (\mu, \nu) = (0,0) \\ &= -S_i, \quad \text{for } (\mu, \nu) = (i, 0) \\ &= S_j, \; \text{for } (\mu, \nu) = (0, j) \\ &= 0 \text{ otherwise }. \end{aligned}$$

Theorem 3.0.3. *Let L be describe as above, then there exists unique unitary valued process on* $\mathbf{h}_0 \otimes \Gamma$, *satisfying the HP type qsde*

$$dV_t = \sum_{\mu, \nu \in \mathbb{Z}^d \cup \{0\}} V_t L^\mu_\nu d\Lambda^\mu_\nu; \; V_0 = 1. \tag{3.0. 6}$$

Proof. From the definition of L we have $\mathcal{D} := i(\mathcal{A}_{loc}) \subseteq \mathcal{D}(L^\mu_\nu), \forall \mu, \nu$. Now let us consider the sequence $L(n)$ given by

$$\begin{aligned} L^\mu_\nu(n) &= -\frac{1}{2} \sum_{|k| \leq n} S_k S_k, \; \text{for } (\mu, \nu) = (0,0) \\ &= -S_i, \quad \text{for } (\mu, \nu) = (i, 0) : |i| \leq n \\ &= S_j, \text{for } (\mu, \nu) = (0, j) : |j| \leq n \\ &= 0 \text{ otherwise }. \end{aligned}$$

It is clear that $L(n) = \{L^\mu_\nu(n)\}$ and $\tilde{L}(n) = \{\tilde{L}^\mu_\nu(n)\}$ belong to the class $\mathcal{I}$, and for $x \in \mathcal{A}_{\text{loc}}$ one has $\lim_{n \to \infty} L^\mu_\nu(n)(i(x)) = L^\mu_\nu(i(x))$. Also, we note that

$$L^0_0(i(x)) = -\frac{1}{2} \sum_{0 \neq k \in \mathbb{Z}^d} S_k S_k(i(x)) = i(\mathcal{L}(x)) \; \forall x \in \mathcal{A}_{\text{loc}}.$$

From the proof of Theorem 2.1.1, it follows that $Ran(L_0^0-\mu)$ is dense in the subspace $i(\mathcal{A})$ and hence dense in $\mathbf{h}_0$, and $\|(L_0^0-\mu)^{-1}\| \leq \frac{1}{\mu}, \forall \mu > 0$. So L_0^0 is the generator of strongly continuous contractive semigroup on $\mathbf{h}_0$. Thus by Theorem 3.0.1, it follows that there exists unique contraction valued solution V_t for the qsde (3.0. 6).
For any $x \in \mathcal{C}^1(\mathcal{A})$, we define a map $\tilde{\mathcal{L}}(x)$ by

$$(\tilde{\mathcal{L}}(x))(i(y)) := \frac{1}{2}\sum_{k\in\mathbb{Z}^d}(2S_kxS_k - S_kS_kx - xS_kS_k)(i(y)), \forall y \in \mathcal{A}.$$

Clearly $\tilde{\mathcal{L}}(x)$ extends uniquely to a bounded linear operator on $\mathbf{h}_0$ and satisfies

$$(\tilde{\mathcal{L}}(x))(i(y)) = i(\mathcal{L}(x)y), \forall y \in \mathcal{A}.$$

Since $1 \in \mathcal{C}^1(\mathcal{A}) \subseteq \mathcal{D}(\tilde{\mathcal{L}})$, closure of $\tilde{\mathcal{L}}$ generates a contractive conservative CP semigroup Q_t on $\mathcal{B}(\mathbf{h}_0)$. Now applying Theorem 3.0.2, it follows that $\beta_\lambda = \{0\}$. By a similar argument it can be shown that $\tilde{\beta}_\lambda = \{0\}$ and hence the solution V_t for above qsde (3.0. 6) is a unitary value adapted process. □

Thus η_t given by $\eta_t(x) = V_t^*(x \otimes 1_\Gamma)V_t,$ for $x \in \mathcal{A}$ satisfies EH type flow equation,

$$\begin{aligned} d\eta_t(x) &= \sum_{j\in\mathbb{Z}^d}\eta_t([S_k,x])da_k(t) + \sum_{j\in\mathbb{Z}^d}\eta_t([x,S_k])da_k^\dagger(t) + \eta_t(\tilde{\mathcal{L}}(x))dt, \\ \eta_0(x) &= x \otimes 1_\Gamma. \end{aligned}$$

3.1 EH dilation

Here we restrict ourselves to QDS T_t associated with element $r \in \mathcal{A}$ such that translate r_k for different $k \in \mathbb{Z}^d$ are commuting. Let $a, b \in \mathbb{Z}_N$ be fixed and $W = U^aV^b \in \mathcal{M}_N(\mathbb{C})$. We consider the following representation of the infinite product group $\mathcal{G}' := \prod_{j\in\mathbb{Z}^d}\mathbb{Z}_N$, given by

$$\mathcal{G}' \ni g \mapsto W_g = \prod_{j\in\mathbb{Z}^d} W^{(j)^{\alpha_j}}, \text{ where } g = (\alpha_j).$$

For any $y \in \mathcal{A},\ y = \sum_{g\in\mathcal{G}} c_gU_g$ and for $n \geq 1$ we define

$$\vartheta_n(y) = \sum_{g\in\mathcal{G}} |c_g|\,|g|^n.$$

Now we consider $r \in \mathcal{A},\ r = \sum_{g \in \mathcal{G}'} c_g W_g$ such that $\sum_{g \in \mathcal{G}'} |c_g|\,|g|^2 < \infty$. It is clear that $\vartheta_1(r) = \sum_{g \in \mathcal{G}'} |c_g|\,|g| < \infty$. We note that any $x \in \mathcal{A}_{\rm loc}$ can be written as $x = \sum_{h \in \mathcal{G}} c_h U_h$, with complex coefficients c_h satisfying $c_h = 0$ for all h such that $supp(h) \bigcap supp(x)$ is empty. So

$$\vartheta_n(x) = \sum_{h \in \mathcal{G}} |c_h|\,|h|^n < \infty \text{ for } n \geq 1,$$

and it is clear that

$$\vartheta_n(x) \leq |x|^n \sum_{h \in \mathcal{G}} |c_h| \leq c_x^n$$

where $c_x = |x|(1 + \sum_{h \in \mathcal{G}} |c_h|)$. Let us consider the formal Lindbladian $\mathcal{L}$ associated with the element r,

$$\mathcal{L} = \sum_{k \in \mathbb{Z}^d} \mathcal{L}_k,$$

where $\mathcal{L}_k(x) = \frac{1}{2}\delta_k^\dagger(x) r_k + r_k^* \delta_k(x)$. Now consider the conservative CP semigroup T_t with generator $\mathcal{L}$. In order to obtain EH dilation for CP semigroup T_t we solve the qsde 3.0. 3 by employing iteration method. For this we need some estimate on product of structure maps. First let us fix some notations. For $n \geq 1$, we denote the set of integers $\{1, 2, \cdots n\}$ by I_n and for $1 \leq p \leq n,\ P = \{l_1, l_2 \cdots l_p\} \subseteq I_n$ with $l_1 < l_2 < \cdots < l_p$, we define a map from the n-fold Cartesian product of $\mathbb{Z}^d$ to that of p copies of $\mathbb{Z}^d$ by

$$\bar{k}(I_n) = (k_1, k_2 \cdots k_n) \mapsto \bar{k}(P) := (k_{l_1}, k_{l_2} \cdots k_{l_p})$$

and similarly, $\bar{\varepsilon}(P) := (\varepsilon_{l_1}, \varepsilon_{l_2}, \cdots \varepsilon_{l_p})$ for a vector $\bar{\varepsilon}(I_n) = (\varepsilon_1, \varepsilon_2, \cdots \varepsilon_n)$ in the n-fold Cartesian product of $\{-1, 0, 1\}$.

For brevity of notations, we write $\bar{\varepsilon}(P) \equiv c \in \{-1, 0, 1\}$ to mean that all $\varepsilon_{l_i} = c$ and denote $\bar{k}(I_n)$ and $\bar{\varepsilon}(I_n)$ by $\bar{k}(n)$ and $\bar{\varepsilon}(n)$ respectively. Setting $\delta_k^\varepsilon = \delta_k^\dagger, \mathcal{L}_k$ and δ_k depending upon $\varepsilon = -1, 0$ and 1 respectively, we write $R(\bar{k}) = r_{k_1} r_{k_2} \cdots r_{k_p}$ and $\delta(\bar{k}, \bar{\varepsilon}) = \delta_{k_p}^{\varepsilon_p} \cdots \delta_{k_1}^{\varepsilon_1}$ for any $\bar{k} = (k_1, k_2 \cdots k_p)$ and $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2 \cdots \varepsilon_p)$.

Now we have the following useful Lemma,

Lemma 3.1.1. *Let r, x and constant c_x be as above. Then*
(i) For any $n \geq 1$,

$$\sum_{\bar{k}(n)} \|\delta(\bar{k}(n), \bar{\varepsilon}(n))(x)\| \leq (2\vartheta_1(r) c_x)^n \ \forall x \in \mathcal{A}_{loc},$$

where $\bar{\varepsilon}(n)$ is such that $\varepsilon_l \neq 0, \ \forall \ l \in I_n$.
(ii) For any $n \geq 1$ and $\bar{k}(n)$,

$$\begin{aligned}
&\mathcal{L}_{k_n} \cdots \mathcal{L}_{k_1}(x) \\
&= \frac{1}{2^n} \sum_{p=0,1\cdots n} \sum_{P \subseteq I_n : |P|=p} R(\bar{k}(P^c))^* \delta(\bar{k}(n), \bar{\varepsilon}_{(P)}(n))(x) R(\bar{k}(P)),
\end{aligned}$$

where $\bar{\varepsilon}_{(P)}(n)$ is such that $\bar{\varepsilon}_{(P)}(P) \equiv -1$ and $\bar{\varepsilon}_{(P)}(P^c) \equiv 1$.
(iii) For any $n \geq 1, p \leq n, P \subseteq I_n$ and $\bar{\varepsilon}(n)$ such that $\bar{\varepsilon}(P)$ contains all those components equal to 0, we have,

$$\begin{aligned}
\sum_{\bar{k}(n)} \|\delta(\bar{k}(n), \bar{\varepsilon}(n))(x)\| &\leq \|r\|^p (2\vartheta_1(r) c_x)^n \\
&\leq (1 + \|r\|)^n (2\vartheta_1(r) c_x)^n.
\end{aligned}$$

(iv) Let $m_1, m_2 \geq 1$; $x, y \in \mathcal{A}_{\rm loc}$ and $\bar{\varepsilon}'(m_1), \bar{\varepsilon}''(m_2)$ be two fixed tuples. Then for $n \geq 1$ and $\bar{\varepsilon}(n)$ as in (iii), we have,

$$\begin{aligned}
&\sum_{\bar{k}(n), \bar{k}'(m_1), \bar{k}''(m_2)} \|\delta(\bar{k}(n), \bar{\varepsilon}(n))\{\delta(\bar{k}'(m_1), \bar{\varepsilon}'(m_1))(x) \cdot \delta(\bar{k}''(m_2), \bar{\varepsilon}''(m_2))(y)\}\| \\
&\leq 2^n (1 + \|r\|)^{2n+m_1+m_2} (2\vartheta_1(r) c_{x,y})^{n+m_1+m_2},
\end{aligned}$$

where $c_{x,y} = max\{c_x, c_y\}$.

Proof. (i) As r^* is again of the same form as r, it is enough to observe the following :

$$\sum_{k_n, \cdots k_1} \| \, [r_{k_n}, \cdots [r_{k_1}, x]] \cdots] \, \| \leq (2\vartheta_1(r) c_x)^n \ \forall x \in \mathcal{A}_{loc} \ .$$

In order to prove this let us consider

$$LHS = \sum_{k_n, \cdots k_1} \sum_{g_n, \cdots g_1 \in \mathcal{G}'; h \in \mathcal{G}} |c_{g_n}| \cdots |c_{g_1}| \ |c_h| \ \| \, [\tau_{k_n} W_{g_n}, \cdots [\tau_{k_1} W_{g_1}, U_h]] \cdots] \, \|.$$

We note that for any two commuting elements A, B in $\mathcal{A}$, $[A, \ [B, x]] = [B, \ [A, x]]$. Thus, for the commutator $[\tau_{k_n} W_{g_n}, \cdots [\tau_{k_1} W_{g_1}, U_h]] \cdots]$ to be nonzero, it is necessary to have $(supp(g_i)+k_i)\bigcap supp(h)\neq\phi$ for each $i=1,2,\cdots n$. Clearly the number of choices of such $k_i\in\mathbb{Z}^d$ is at most $|g_i|\cdot|h|$. Thus we get,

$$\begin{aligned}&\sum_{k_n,\cdots k_1}\|\,[r_{k_n},\cdots[r_{k_1},x]]\cdots]\,\|\\&\quad\leq\sum_{g_n,\cdots g_1\in\mathcal{G}';h\in\mathcal{G}}|c_{g_n}|\cdots|c_{g_1}||c_h||g_n|\cdots|g_1||h|^n2^n\\&\quad\leq(2\vartheta_1(r)c_x)^n.\end{aligned}$$

(ii) The proof is by induction. For any $k\in\mathbb{Z}^d$ we have,

$$\mathcal{L}_k(x)=\frac{1}{2}\sum_{k\in\mathbb{Z}^d}\delta_k^\dagger(x)r_k+r_k^*\delta_k(x),$$

so it is trivially true for $n=1$. Let us assume it to be true for some $m>1$ and for any $k_{m+1}\in\mathbb{Z}^d$ consider $\mathcal{L}_{k_{m+1}}\mathcal{L}_{k_m}\cdots\mathcal{L}_{k_1}(x)$. By applying the statement for $n=m$ we get,

$$\begin{aligned}&\mathcal{L}_{k_{m+1}}\mathcal{L}_{k_m}\cdots\mathcal{L}_{k_1}(x)\\&=\frac{1}{2^{m+1}}\sum_{p=0,1\cdots m}\ \sum_{P\subseteq I_m:|P|=p}[\delta^*_{k_{m+1}}\{R(\bar{k}(P^c))^*\delta(\bar{k}(m),\bar{\varepsilon}_{(P)}(m))(x)R(\bar{k}(P))\}r_{k_{m+1}}\\&\qquad+r^*_{k_{m+1}}\delta_{k_{m+1}}\{R(\bar{k}(P^c))^*\delta(\bar{k}(m),\bar{\varepsilon}_{(P)}(m))(x)R(\bar{k}(P))\}].\end{aligned}$$

Since r_k's are commuting with each other, the above expression becomes

$$\begin{aligned}&\frac{1}{2^{m+1}}\sum_{p=0,1\cdots m}\ \sum_{P\subseteq I_m:|P|=p}[R(\bar{k}(P^c))^*\delta^*_{k_{m+1}}\delta(\bar{k}(m),\bar{\varepsilon}_{(P)}(m))(x)R(\bar{k}(P))r_{k_{m+1}}\\&\qquad+r^*_{k_{m+1}}R(\bar{k}(P^c))^*\delta_{k_{m+1}}\delta(\bar{k}(m),\bar{\varepsilon}_{(P)}(m))(x)R(\bar{k}(P))]\\&=\frac{1}{2^{m+1}}\sum_{p=0,1\cdots m+1}\ \sum_{P\subseteq I_{m+1}:|P|=p}R(\bar{k}(P^c))^*\delta(\bar{k}(m+1),\bar{\varepsilon}_{(P)}(m+1))(x)R(\bar{k}(P)).\end{aligned}$$

(iii) By simple application of (ii),

$$\begin{aligned}&\delta(\bar{k}(n),\bar{\varepsilon}(n))(x)\\&=\frac{1}{2^p}\sum_{q=0,1\cdots p}\ \sum_{Q\subseteq P:|Q|=q}R(\bar{k}(P\setminus Q))^*\delta(\bar{k}(n),\bar{\varepsilon}_{(Q,P)}(n))(x)R(\bar{k}(Q)),\qquad(3.1.\ 1)\end{aligned}$$

where $\bar{\varepsilon}_{(Q,P)}(n)$ is defined to be the map from the n-fold Cartesian product of $\{-1,0,1\}$ to itself, given by $\bar{\varepsilon}(n) \mapsto \bar{\varepsilon}_{(Q,P)}(n)$ such that $\bar{\varepsilon}_{(Q,P)}(Q) \equiv -1, \bar{\varepsilon}_{(Q,P)}(P \setminus Q) \equiv 1$ and
$\bar{\varepsilon}_{(Q,P)}(I_n \setminus P) = \bar{\varepsilon}(I_n \setminus P)$. Now (iii) follows from (i).

(iv) By (3.1. 1) we have,

$$\begin{aligned}
&LHS \\
&= \frac{1}{2^p} \sum_{\bar{k}(n),\bar{k}'(m_1),\bar{k}''(m_2)} \sum_{q=0,1\cdots p} \sum_{Q\subseteq P:|Q|=q} \|R(\bar{k}(P\setminus Q))^* \\
&\quad \delta(\bar{k}(n),\bar{\varepsilon}_{(Q,P)}(n)) \ [\delta(\bar{k}'(m_1),\bar{\varepsilon}'(m_1))(x)\cdot\delta(\bar{k}''(m_2),\bar{\varepsilon}''(m_2))(y)] \ R(\bar{k}(Q))\|.
\end{aligned}$$

Now applying the Leibnitz rule, it can be seen to be less than or equal to

$$\begin{aligned}
&\frac{\|r\|^p}{2^p} \sum_{\bar{k}(n),\bar{k}'(m_1),\bar{k}''(m_2)} \sum_{q=0,1\cdots p} \sum_{Q\subseteq P:|Q|=q} \sum_{l=0,1\cdots n} \sum_{L\subseteq I_n:|L|=l} \\
&\quad \|\delta(\bar{k}(L),\bar{\varepsilon}_{(Q,P)}(L))\delta(\bar{k}'(m_1),\bar{\varepsilon}'(m_1))(x)\| \\
&\quad \|\delta(\bar{k}(L^c),\bar{\varepsilon}_{(Q,P)}(L^c))[\delta(\bar{k}''(m_2),\bar{\varepsilon}''(m_2))(y)]\|.
\end{aligned}$$

Using (iii), we obtain,

$$\begin{aligned}
&LHS \\
&\leq \frac{(1+\|r\|)^n}{2^p} \sum_{q=0,1\cdots p} \frac{p!}{(p-q)!\ q!} \sum_{l=0,1\cdots n} \frac{n!}{(n-l)!\ l!} (1+\|r\|)^{l+m_1} (2\vartheta_1(r)c_x)^{l+m_1} \\
&\qquad \cdot (1+\|r\|)^{n-l+m_2} (2\vartheta_1(r)c_y)^{n-l+m_2} \\
&\qquad \leq 2^n (1+\|r\|)^{2n+m_1+m_2} (2\vartheta_1(r)c_{x,y})^{n+m_1+m_2}.
\end{aligned}$$

□

Now we are in a position to prove the following result about existence of an Evans-Hudson flow for QDS T_t associated with the element $r \in \mathcal{A}$ discussed above.

Theorem 3.1.2. *(a) For $t \geq 0$, there exists a unique solution j_t of the qsde,*

$$dj_t(x) = \sum_{j\in\mathbb{Z}^d} j_t(\delta_j^\dagger x) da_j(t) + \sum_{j\in\mathbb{Z}^d} j_t(\delta_j x) da^\dagger{}_j(t) + j_t(\mathcal{L}x)dt, \tag{3.1. 2}$$

$$j_0(x) = x \otimes 1_\Gamma, \ \forall x \in \mathcal{A}_{\text{loc}},$$

such that $j_t(1) = 1, \ \forall t \geq 0.$

(b) For $x, y \in \mathcal{A}_{\rm loc}$ *and* $u, v \in \mathbf{h}_0, f, g \in \mathcal{C}$,

$$\langle u\boldsymbol{e}(f), j_t(xy)v\boldsymbol{e}(g)\rangle = \langle j_t(x^*)u\boldsymbol{e}(f), j_t(y)v\boldsymbol{e}(g)\rangle. \tag{3.1. 3}$$

(c) j_t *extends uniquely to a unital* C^**-homomorphism from* $\mathcal{A}$ *into* $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$.

Proof. We note first that $\mathcal{A}_{\rm loc}$ is a dense $*$-subalgebra of $\mathcal{A}$.

(a) As usual, we solve the qsde by iteration. For $t_0 \geq 0, t \leq t_0$ and $x \in \mathcal{A}_{\rm loc}$, we set $j_t^{(0)}(x) = x \otimes 1_\Gamma$ and for $n \geq 1$,

$$j_t^{(n)}(x) = x \otimes 1_\Gamma$$

$$+ \int_0^t \sum_{j \in \mathbb{Z}^d} j_s^{(n-1)}(\delta_j^\dagger(x))da_j(s) + \sum_{j \in \mathbb{Z}^d} j_s^{(n-1)}(\delta_j(x))da_j^\dagger(s) + j_s^{(n-1)}(\mathcal{L}(x))ds. \tag{3.1. 4}$$

Then for $u \in \mathbf{h}_0$ and $f \in \mathcal{C}$, we can show by induction, that

$$\|\{j_t^{(n)}(x) - j_t^{(n-1)}(x)\}u\mathbf{e}(f)\|$$

$$\leq \frac{(t_0 c_f)^{n/2}}{\sqrt{n!}}\|u\mathbf{e}(f)\| \sum_{\bar{k}(n)} \sum_{\bar{\varepsilon}(n)} \|\delta(\bar{k}(n), \bar{\varepsilon}(n))(x)\|, \tag{3.1. 5}$$

where $c_f = 2e^{\gamma_f(t_0)}(1 + \|f\|_\infty^2)$, with $\gamma_f(t_0) = \int_0^{t_0}(1 + \|f(s)\|^2)ds$. For $n = 1$, by the basic estimate of quantum stochastic integral [33],

$$\begin{aligned}
&\|\{j_t^{(1)}(x) - j_t^{(0)}(x)\}u\mathbf{e}(f)\|^2 \\
&\quad = \|\{\int_0^t \sum_{j \in \mathbb{Z}^d} \delta_j^\dagger(x)da_j(s) + \sum_{j \in \mathbb{Z}^d} \delta_j(x)da_j^\dagger(s) + \mathcal{L}(x)ds\}u\mathbf{e}(f)\|^2 \\
&\quad \leq 2e^{\gamma_f(t_0)}\|\mathbf{e}(f)\|^2 \int_0^t \{\sum_{j \in \mathbb{Z}^d} \|\delta_j^\dagger(x)u\|^2 + \sum_{j \in \mathbb{Z}^d} \|\delta_j(x)u\|^2 + \|\mathcal{L}(x)u\|^2\}(1 + \|f(s)\|)^2 ds \\
&\quad \leq c_f t_0 \|\mathbf{e}(f)\|^2 \{\sum_{j \in \mathbb{Z}^d} \|\delta_j^\dagger(x)u\| + \|\delta_j(x)u\| + \|\mathcal{L}_j(x)u\|\}^2.
\end{aligned}$$

Thus (3.1. 5) is true for $n = 1$. Inductively assuming the estimate for some $m > 1$, we have by the same argument as above,

$$\begin{aligned}
&\|\{j_t^{(m+1)}(x) - j_t^{(m)}(x)\}u\mathbf{e}(f)\|^2 \\
= \quad & \|\{\int_0^t \sum_{j\in\mathbb{Z}^d}[j_{s_m}^{(m)}(\delta_j^\dagger(x)) - j_{s_m}^{(m-1)}(\delta_j^\dagger(x))]da_j(s_m) \\
& + \sum_{j\in\mathbb{Z}^d}[j_{s_m}^{(m)}(\delta_j(x)) - j_{s_m}^{(m-1)}(\delta_j(x))]da_j^\dagger(s_m) \\
& + [j_{s_m}^{(m)}(\mathcal{L}(x)) - j_{s_m}^{(m-1)}(\mathcal{L}(x))]ds_m\}u\mathbf{e}(f)\|^2 \\
\leq \quad & 2e^{\gamma_f(t_0)}\int_0^t\{\sum_{j\in\mathbb{Z}^d}\|[j_{s_m}^{(m)}(\delta_j^\dagger(x)) - j_{s_m}^{(m-1)}(\delta_j^\dagger(x))]u\mathbf{e}(f)\|^2 \\
& + \sum_{j\in\mathbb{Z}^d}\|[j_{s_m}^{(m)}(\delta_j(x)) - j_{s_m}^{(m-1)}(\delta_j(x))]u\mathbf{e}(f)\|^2 \\
& + \|[j_{s_m}^{(m)}(\mathcal{L}(x)) - j_{s_m}^{(m-1)}(\mathcal{L}(x))]u\mathbf{e}(f)\|^2\}(1+\|f(s_m)\|^2)ds_m \\
\leq \quad & c_f\int_0^t[\sum_{j\in\mathbb{Z}^d}\|[j_{s_m}^{(m)}(\delta_j{}^\dagger(x)) - j_{s_m}^{(m-1)}(\delta_j^\dagger(x))]u\mathbf{e}(f)\| \\
& + \sum_{j\in\mathbb{Z}^d}\|[j_{s_m}^{(m)}(\delta_j(x)) - j_{s_m}^{(m-1)}(\delta_j(x))]u\mathbf{e}(f)\| \\
& + \|[j_{s_m}^{(m)}(\mathcal{L}(x)) - j_{s_m}^{(m-1)}(\mathcal{L}(x))]u\mathbf{e}(f)\|]^2 ds_m.
\end{aligned}$$

Now applying (3.1. 5) for $n = m$, we get the required estimate for $n = m + 1$ and furthermore by the estimate of Lemma 3.1.1 (iii),

$$\|\{j_t^{(n)}(x) - j_t^{(n-1)}(x)\}u\mathbf{e}(f)\| \leq 3^n\frac{(t_0c_f)^{n/2}}{\sqrt{n!}}\|u\mathbf{e}(f)\|(1+\|r\|)^n(1+2\vartheta_1(r)c_x)^n.$$

Thus it follows that the sequence $\{j_t^{(n)}(x)u\mathbf{e}(f)\}$ is Cauchy. We define $j_t(x)u\mathbf{e}(f)$ to be $\lim_{n\to\infty} j_t{}^{(n)}u\mathbf{e}(f)$, that is

$$j_t(x)u\mathbf{e}(f) = xu\otimes\mathbf{e}(f) + \sum_{n\geq 1}\{j_t^{(n)}(x) - j_t^{(n-1)}(x)\}u\mathbf{e}(f) \qquad (3.1.\ 6)$$

and one has

$$\|j_t(x)u\mathbf{e}(f)\| \leq \|u\mathbf{e}(f)\|\ [\|x\| + \sum_{n\geq 1}3^n\frac{(t_0c_f)^{n/2}}{\sqrt{n!}}(1+\|r\|)^n(1+2\vartheta_1(r)c_x)^n]. \quad (3.1.\ 7)$$

Uniqueness follows by setting,

$$q_t(x) = j_t(x) - j_t'(x)$$

and observing

$$dq_t(x) = \sum_{j\in\mathbb{Z}^d} q_t(\delta_j^\dagger(x))da_j(t) + \sum_{j\in\mathbb{Z}^d} q_t(\delta_j(x))da^\dagger{}_j(t) + q_t(\mathcal{L}(x))dt,\ q_0(x) = 0.$$

Exactly similar estimate as above shows that, for all $n \geq 1$,

$$\|q_t(x)u\mathbf{e}(f)\| \leq \frac{(t_0 c_f)^{n/2}}{\sqrt{n!}}\|u\mathbf{e}(f)\| \sum_{\bar{k}(n)}\sum_{\bar{\varepsilon}(n)} \|\delta(\bar{k}(n),\bar{\varepsilon}(n))(x)\|.$$

Since by Lemma 3.1.1(iii) the sum grows as n-th power, $q_t(x) = 0\ \forall x \in \mathcal{A}_{\rm loc}$, showing the uniqueness of the solution. As $1 \in \mathcal{A}_{\rm loc}$ with $\mathcal{L}_k(1) = \delta_k^\dagger(1) = \delta_k(1) = 0$ it follows from the qsde (3.1. 2) that $j_t(1) = 1$.

(b) For $u\mathbf{e}(f), v\mathbf{e}(g) \in h \otimes \mathcal{E}(\mathcal{C})$ and $x, y \in \mathcal{A}_{\rm loc}$, we have, by induction,

$$\langle j_t^{(n)}(x^*)u\mathbf{e}(f), v\mathbf{e}(g)\rangle = \langle u\mathbf{e}(f), j_t^{(n)}(x)v\mathbf{e}(g)\rangle.$$

Now as n tends to ∞, we get

$$\langle j_t(x^*)u\mathbf{e}(f), v\mathbf{e}(g)\rangle = \langle u\mathbf{e}(f), j_t(x)v\mathbf{e}(g)\rangle.$$

We define

$$\Phi_t(x,y) = \langle u\mathbf{e}(f), j_t(xy)v\mathbf{e}(g)\rangle - \langle j_t(x^*)u\mathbf{e}(f), j_t(y)v\mathbf{e}(g)\rangle.$$

Setting $(\zeta_k(l), \eta_k(l)) = (\delta_k, id),\ (id, \delta_k),\ (\delta_k^\dagger, id),\ (id, \delta_k^\dagger),\ (\mathcal{L}_k, id),\ (id, \mathcal{L}_k)$ and $(\delta_k^\dagger, \delta_k)$
for $l = 1, 2, \cdots 7$ respectively, one has

$$|\Phi_t(x,y)| \leq c_{f,g}^n \sum_{l_n,\cdots l_1} \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_1}$$

$$\sum_{k_n,\cdots k_1} |\Phi_{s_1}(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)x,\ \eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)y)|ds_0\cdots ds_{n-1}\ \forall n \geq 1, \quad (3.1.\ 8)$$

where $c_{f,g} = (1 + {t_0}^{1/2})(\|f\|_\infty + \|g\|_\infty)$. By the quantum Ito formula and cocyle properties of structure operators, i.e. $\mathcal{L}(xy) = x\mathcal{L}(y) + \mathcal{L}(x)y + \sum_{k\in\mathbb{Z}^d}\delta_k^\dagger(x)\delta_k(y)$, we have,

$$
\begin{aligned}
&\Phi_t(x,y)\\
&= \int_0^t \sum_k \{\Phi_s(\delta_k(x),y)+\Phi_s(x,\delta_k(y))\} f_k(s)ds\\
&+\int_0^t \sum_k \{\Phi_s(\delta_k^\dagger(x),y)+\Phi_s(x,\delta_k^\dagger(y))\} \bar{g}_k(s)ds\\
&+\int_0^t \sum_k \{\Phi_s(\mathcal{L}_k(x),y)+\Phi_s(x,\mathcal{L}_k(y))+\Phi_s(\delta_k^\dagger(x),\delta_k(y))\} ds,
\end{aligned}
$$

which gives the estimate for $n=1$:

$$|\Phi_t(x,y)| \le c_{f,g} \sum_{l=1\cdots 7} \int_0^t \sum_k |\Phi_s(\zeta_k(l)(x),\eta_k(l)(y))| ds \ . \tag{3.1. 9}$$

If we now assume (3.1. 8) for some $m>1$, an application of (3.1. 9) gives the required estimate for $n=m+1$.

At this point we note the following, which can be verified easily by (3.1. 6) , (3.1. 7) and Lemma 3.1.1 (iv).

(1) For any n-tuple $(l_1,l_2\cdots l_n)$ in $\{1,2\cdots 7\}$

$$
\begin{aligned}
&\sum_{k_n,\dots k_1} \|j_s(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)(x)\cdot\eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)(y))v\mathbf{e}(g)\|\\
&\qquad \le C_{g,x,y}\{(1+\|r\|)(1+2\vartheta_1(r)c_{x,y})\}^{2n}\|v\mathbf{e}(g)\|,
\end{aligned} \tag{3.1. 10}
$$

where for any $g\in\mathcal{C}$

$$C_{g,x,y} = 1+\sum_{m\ge 1} 3^m \frac{(t_0c_g)^{m/2}}{\sqrt{m!}}\{(1+\|r\|)(1+2\vartheta_1(r)c_{x,y})\}^{2m}.$$

(2) For any $s\le t_0,\ p\le n$ and $\bar{\varepsilon}(p)$,

$$
\begin{aligned}
&\sum_{\bar{k}(p)} \|j_s\{\delta(\bar{k}(p),\bar{\varepsilon}(p))(y)\}v\mathbf{e}(g)\|\\
&\le C_{g,x,y}\{(1+\|r\|)(1+2\vartheta_1(r)c_{x,y})\}^{n}\|v\mathbf{e}(g)\|.
\end{aligned} \tag{3.1. 11}
$$

(3) Since $\vartheta_p(x)=\vartheta_p(x^*)$ and $\{\delta(\bar{k}(p),\bar{\varepsilon}(p))(x)\}^*$ can also be written as $\delta(\bar{k}(p),\bar{\varepsilon}'(p))(x^*)$ for some $\bar{\varepsilon}'(p)$, we have

$$\sum_{\bar{k}(p)} \|j_s\{\delta(\bar{k}(p),\bar{\varepsilon}(p))(x)\}^* u\mathbf{e}(f)\|$$

$$\leq C_{f,x,y}\{(1+\|r\|)(1+2\vartheta_1(r)c_{x,y})\}^n\|u\mathbf{e}(f)\|. \qquad (3.1.\ 12)$$

For any fixed n-tuple $(l_1,\cdots l_n)$, it is easy to observe from the definition of Φ_s that

$$\begin{aligned}&\sum_{\bar{k}(n)}|\Phi_s(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)x,\eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)y)|\\&\quad\leq\sum_{k_n,\ldots k_1}\|u\mathbf{e}(f)\|\cdot\|j_s(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)x\cdot\eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)y)v\mathbf{e}(g)\|\\&\quad+\|j_s\{(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)(x))^*\}u\mathbf{e}(f)\|\cdot\|j_s(\eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)(y))v\mathbf{e}(g)\|.\end{aligned}$$

The estimates (3.1. 10), (3.1. 11) and (3.1. 12) yield :

$$\begin{aligned}&\sum_{\bar{k}(n)}|\Phi_s(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)x,\ \eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)y)|\\&\quad\leq\{(1+\|r\|)(1+2\vartheta_1(r)c_{x,y})\}^{2n}\|u\mathbf{e}(f)\|\cdot\|v\mathbf{e}(g)\|(C_{g,x,y}+C_{f,x,y}C_{g,x,y})\\&\quad=C\{(1+\|r\|)(1+2\vartheta_1(r)c_{x,y})\}^{2n},\end{aligned}$$

with $C=\|u\mathbf{e}(f)\|\cdot\|v\mathbf{e}(g)\|(C_{g,x,y}+C_{f,x,y}C_{g,x,y})$.
Now by (3.1. 8),

$$|\Phi_t(x,y)|\leq C\ \frac{(7\ t_0c_{f,g})^n}{n!}\{(1+\|r\|)(1+2\vartheta_1(r)c_{x,y})\}^{2n},\ \forall\ n\geq 1,$$

which implies $\Phi_t(x,y)=0$.
(c) Let $\xi=\sum c_ju_j\mathbf{e}(f_j)$ be a vector in the algebraic tensor product of $\mathbf{h}_0$ and $\mathcal{E}(\mathcal{C})$. If $y\in\mathcal{A}^+_{\text{loc}}$, y is actually an $N^{|y|}\times N^{|y|}$-dim positive matrix and hence it admits a unique square root $\sqrt{y}\in\mathcal{A}^+_{\text{loc}}$. For any $x\in\mathcal{A}^+_{loc}$, setting $y=\sqrt{\|x\|1-x}$ so that $y\in\mathcal{A}^+_{\text{loc}}$, we get

$$\begin{aligned}\|j_t(y)\xi\|^2&=\langle j_t(y)\xi,j_t(y)\xi\rangle\\&=\sum\bar{c}_ic_j\langle j_t(y)u_i\mathbf{e}(f_i),j_t(y)u_j\mathbf{e}(f_j)\rangle\\&=\sum\bar{c}_ic_j\langle u_i\mathbf{e}(f_i),j_t(\|x\|1-x)u_j\mathbf{e}(f_j)\rangle\ \text{(by (b))}\\&=\|x\|\cdot\|\xi\|^2-\langle\xi,j_t(x)\xi\rangle,\end{aligned}$$

where we have used the fact that $1\in\mathcal{A}_{\text{loc}}$ and $j_t(1)=1$. Now let $x\in\mathcal{A}_{loc}$ be arbitrary and applying the above for x^*x as well as (b) we get,

$$\begin{aligned}\|j_t(x)\xi\|^2 &= \langle j_t(x)\xi, j_t(x)\xi\rangle \\ &= \sum \bar{c}_i c_j \langle j_t(x)u_i\mathbf{e}(f_i), j_t(x)u_j\mathbf{e}(f_j)\rangle \\ &= \sum \bar{c}_i c_j \langle u_i\mathbf{e}(f_i), j_t(x^*x)u_j\mathbf{e}(f_j)\rangle \\ &= \langle \xi, j_t(x^*x)\xi\rangle \\ &\leq \|x^*x\| \cdot \|\xi\|^2 = \|x\|^2 \cdot \|\xi\|^2 \\ &\text{or } \|j_t(x)\xi\| \leq \|x\| \cdot \|\xi\|.\end{aligned}$$

This inequality obviously extends to all $\xi \in \mathbf{h}_0 \otimes \Gamma$. Noting that $j_t(1) = 1,\ \forall t$, we get

$$\|j_t(x)\| \leq \|x\| \text{ and } \|j_t\| = 1.$$

Thus j_t extends uniquely to a unital C^*-homomorphism satisfying the qsde (3.1. 2) and hence is an Evans-Hudson flow on $\mathcal{A}$ with T_t as its expectation semigroup. That the range of j_t is in $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$ is clear from the construction of j_t. □

Now let us recall the the ergodic QDS T_t^ϕ associated with a partial state ϕ_0 discussed in Chapter-2. It may be noted that the generator $\mathcal{L}^\phi$ of T_t^ϕ satisfies

$$\mathcal{L}^\phi(x) = \sum_{k\in\mathbb{Z}^d} \frac{1}{2} \sum_{m=1}^{N^2} [L_k^{(m)^*}, x]L_k^{(m)} + L_k^{(m)^*}[x, L_k^{(m)}], \forall x \in \mathcal{A}_{\text{loc}}.$$

We have also obtained an Evans-Hudson type dilation for these QDS T_t^ϕ.

Theorem 3.1.3. *Let T_t^ϕ be the QDS associated with a partial state ϕ_0. Then :*
(a) For each $k \in \mathbb{Z}^d$ and $t \geq 0$ there exists a unique solution $\eta_t^{(k)}$ for the qsde,

$$d\eta_t^{(k)}(x) = \eta_t^{(k)}(\sum_{m=1}^{N^2}[L_k^{(m)^*}, x_{(k)}])da_k(t) + \eta_t^{(k)}(\sum_{m=1}^{N^2}[x_{(k)}, L_k^{(m)}])da_k^\dagger(t) + \eta_t^{(k)}(\mathcal{L}_k^\phi x_{(k)})dt, \tag{3.1. 13}$$

$$\eta_0(x_{(k)}) = x_{(k)} \otimes 1_\Gamma,\ \forall x_{(k)} \in \mathcal{A}_k,$$

as a unital $*$-homomorphism from $\mathcal{A}_k$ into $\mathcal{A}_k \bigotimes \mathcal{B}(\Gamma)$. Moreover, for different k and k', $\eta_t^{(k)}$ and $\eta_t^{(k')}$ commute in the sense that, $\eta_t^{(k)}(x_{(k)})$ and $\eta_t^{(k')}(x_{k'})$ commute for every $x_{(k)} \in \mathcal{A}_k$ and $x_{k'} \in \mathcal{A}_{k'}$,

(b) There exists a unique unital $*$-homomorphism η_t from $\mathcal{A}_{\text{loc}}$ into $\mathcal{A}'' \bigotimes \mathcal{B}(\Gamma)$ such that it coincide with $\eta_t^{(k)}$ on $\mathcal{A}_k$,
(c) η_t extends uniquely as a unital C^*-homomorphism from $\mathcal{A}$ into $\mathcal{A}'' \bigotimes \mathcal{B}(\Gamma)$.

Proof. (a) For any $k \in \mathbb{Z}^d$ and $t \geq 0$ let us consider the qsde (3.1. 13). Here we have only finitely many nontrivial structure maps on the finite dimensional unital C^*-algebra $\mathcal{A}_k$, satisfying the structure equation. So there exists a unique solution $\eta_t^{(k)}$ as a unital $*$-homomorphism from $\mathcal{A}_k$ into $\mathcal{A}_k \bigotimes \mathcal{B}(\Gamma)$. Since for different k and k' the associated structure maps commute and for any $x_{(k)} \in \mathcal{A}_k$ and $x_{(k')} \in \mathcal{A}_{k'}$ Ito term absent in $d(\eta_t^{(k)}(x_{(k)})\eta_t^{(k')}(x_{(k')}))$, it follows that $\eta_t^{(k)}(x_{(k)})$ and $\eta_t^{(k')}(x_{(k')})$ commute.
(b) For any finite $\Lambda \subseteq \mathbb{Z}^d,\ t \geq 0$ and simple tensor element $x_\Lambda = \prod_{k\in\Lambda} x_{(k)} \in \mathcal{A}_\Lambda$, the map $\eta_t^{(\Lambda)}$ given by

$$\eta_t^{(\Lambda)}(x_\Lambda) := \prod_{k\in\Lambda}\eta_t^{(k)}(x_{(k)})$$

is well defined from $\mathcal{A}_\Lambda$ to $\mathcal{A}_\Lambda \bigotimes \mathcal{B}(\Gamma)$ as $\eta_t^{(k)}$'s commute. Differentiating $\eta_t^{(\Lambda)}(x_\Lambda)$ with respect to t, it follows that $\eta_t^{(\Lambda)}(x_\Lambda)$ satisfies the qsde,

$$d\eta_t^{(\Lambda)}(x_\Lambda) = \sum_{k\in\Lambda} \eta_t^{(\Lambda)}(\sum_{m=1}^{N^2}[L_k^{(m)^*}, x_\Lambda])da_k(t) + \sum_{k\in\Lambda} \eta_t^{(\Lambda)}(\sum_{m=1}^{N^2}[x_\Lambda, L_k^{(m)}])da_k^\dagger(t) \tag{3.1. 14}$$

$$+\eta_t^{(\Lambda)}(\sum_{k\in\Lambda}\mathcal{L}_k^\phi x_\Lambda)dt, \quad \eta_0^{(\Lambda)}(x_\Lambda) = x_\Lambda \otimes 1_\Gamma.$$

We now want to show

$$\eta_t^{(\Lambda)}(xy) = \eta_t^{(\Lambda)}(x)\cdot\eta_t^{(\Lambda)}(y), \text{ for simple tensor elements } x, y \in \mathcal{A}_{\text{loc}}. \tag{3.1. 15}$$

Since each $\eta_t^{(k)}$ is unital and $\eta_t^{(\Lambda')}$ agrees with $\eta_t^{(\Lambda)}$ for simple tensor elements in $\mathcal{A}_\Lambda$ whenever Λ is a finite subset of Λ', it is suffices to show (3.1. 15) for $x, y \in \mathcal{A}_\Lambda$, where $\Lambda \subseteq \mathbb{Z}^d$ is a finite set. For $x = \prod_{k\in\Lambda} x_{(k)}$ and $y = \prod_{k\in\Lambda} y_{(k)} \in \mathcal{A}_\Lambda$ we have,

$$\begin{aligned}\eta_t^{(\Lambda)}(xy) &= \eta_t^{(\Lambda)}\prod_{k\in\Lambda}(x_{(k)}y_{(k)}) = \prod_{k\in\Lambda}\eta_t^{(k)}(x_{(k)}y_{(k)}) \\ &= \prod_{k\in\Lambda}\eta_t^{(k)}(x_{(k)})\eta_t^{(k)}(y_{(k)}) = \prod_{k\in\Lambda}\eta_t^{(k)}(x_{(k)})\prod_{k\in\Lambda}\eta_t^{(k)}(y_{(k)}).\end{aligned}$$

Similarly

$$\eta_t^{(\Lambda)}(x^*) = (\eta_t^{(\Lambda)}(x))^*. \tag{3.1. 16}$$

Noting that any element $x \in \mathcal{A}_{\text{loc}}$ can be written as a linear combination of simple tensor elements $\{U_g : g \in \mathcal{G}\}$, say $x = \sum_{g\in\mathcal{G}} c_g U_g$ with $c_g = 0$ when $supp(g)$ is outside $supp(x) = \Lambda$, we define

$$\eta_t(x) = \sum_{g\in\mathcal{G}} c_g \eta_t^{(\Lambda)}(U_g).$$

For x and $y \in \mathcal{A}_{\text{loc}}$, with $x = \sum_{g\in\mathcal{G}} c_g U_g$ and $y = \sum_{h\in\mathcal{G}} c_h U_h$, such that $supp(x) = supp(y) = \Lambda$,

$$\begin{aligned}\eta_t(xy) &= \eta_t(\sum_{g,h\in\mathcal{G}} c_g c_h U_g U_h) \\ &= \sum_{g,h\in\mathcal{G}} c_g c_h \eta_t^{(\Lambda)}(U_g U_h) = \sum_{g,h\in\mathcal{G}} c_g c_h \eta_t^{(\Lambda)}(U_g)\eta_t^{(\Lambda)}(U_h) \text{ (by (3.1. 15))} \\ &= \eta_t(\sum_{g\in\mathcal{G}} c_g U_g)\eta_t(\sum_{h\in\mathcal{G}} c_h U_h) \\ &= \eta_t(x)\eta_t(y).\end{aligned}$$

It follows from (3.1. 16) that $\eta_t(x^*) = (\eta_t(x))^* \ \forall x \in \mathcal{A}_{\text{loc}}$. Thus η_t is a unital $*$-homomorphism from $\mathcal{A}_{\text{loc}}$ into $\mathcal{A}'' \bigotimes \mathcal{B}(\Gamma)$.

(c) We recall that $\mathcal{A}_{\text{loc}}^+$ is closed under taking square root, as already noted in the proof of Theorem 3.1.2(c). Thus for $x \in \mathcal{A}_{\text{loc}}$, $\sqrt{\|x\|^2 1 - x^* x} \in \mathcal{A}_{\text{loc}}^+$. Since η_t is a unital $*$-homomorphism on $\mathcal{A}_{\text{loc}}$,

$$\eta_t(\|x\|^2 1 - x^* x) \geq 0$$

$$\Rightarrow \eta_t(x^* x) \leq \|x\|^2 1$$

$$\Rightarrow \|\eta_t(x^* x)\| \leq \|x\|^2$$

$$\Rightarrow \|\eta_t(x)\| \leq \|x\|.$$

So η_t extends uniquely as a unital C^*-homomorphism from $\mathcal{A}$ into $\mathcal{A}'' \bigotimes \mathcal{B}(\Gamma)$.

□

3.2 Covariance of the EH flows

Let $\mathcal{B}$ be a C^* (or von Neumann) algebra, G be a locally compact group with an action α on $\mathcal{B}$.

Definition 3.2.1. *A QDS $\{T_t : t \geq 0\}$ on $\mathcal{B}$ is said to be covariant with respect to α, if*

$$\alpha_g \circ T_t(x) = T_t \circ \alpha_g(x), \forall t \geq 0, g \in G, x \in \mathcal{B}.$$

Given such a covariant QDS a natural question arises whether there exists a covariant Evans-Hudson dilation for $\{T_t\}$. The question is discussed in [6] for uniformly continuous QDS.

In this section we shall prove that the Evans-Hudson flows constructed in the previous section are covariant. It can be easily observed that

$$\delta_k \tau_j = \tau_j \delta_{k-j} \text{ and } {\delta^\dagger}_k \tau_j = \tau_j {\delta^\dagger}_{k-j}, \ \forall j, k \in \mathbb{Z}^d, \tag{3.2. 1}$$

and we have the following Lemma,

Lemma 3.2.2. *(i)$\mathcal{L}\tau_j(x) = \tau_j \mathcal{L}(x)$ $\forall x \in Dom(\mathcal{L})$,*
(ii)$T_t \tau_j = \tau_j T_t$, i.e. T_t is covariant.

Proof. (i) We note that $\mathcal{C}^1(\mathcal{A})$ is invariant under τ and thus for $x \in \mathcal{C}^1(\mathcal{A})$,

$$\begin{aligned}
\mathcal{L}(\tau_j(x)) &= \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \delta_k^\dagger(\tau_j(x)) r_k + r_k^* \delta_k(\tau_j(x)) \\
&= \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \tau_j \delta_{k-j}^\dagger(x) r_k + {r_k}^* \tau_j \delta_{k-j}(x) \quad \text{(by 3.2.2)} \\
&= \frac{1}{2} \tau_j \{ \sum_{k \in \mathbb{Z}^d} \delta_{k-j}^\dagger(x) r_{k-j} + r_{k-j}^* \delta_{k-j}(x) \} \\
&= \tau_j(\mathcal{L}(x)).
\end{aligned}$$

For $x \in Dom(\mathcal{L})$, we choose a sequence $\{x_n\}$ in $\mathcal{C}^1(\mathcal{A})$ and an element $y \in \mathcal{A}$ such that $y = \mathcal{L}(x),\ x_n$ converge to x and $\mathcal{L}(x_n)$ converge to y. As τ_j is an automorphism for any $j \in \mathbb{Z}^d,\ \tau_j(x_n)$ and $\tau_j \mathcal{L}(x_n)$ converge to $\tau_j(x)$ and $\tau_j(y)$ respectively. Since $x_n \in \mathcal{C}^1(\mathcal{A})$ and $\mathcal{L}(\tau_j(x_n)) = \tau_j \mathcal{L}(x_n)$, we get

$$\tau_j(x) \in Dom(\mathcal{L}) \text{ and } \ \mathcal{L}\tau_j(x) = \tau_j \mathcal{L}(x).$$

(ii) By (i), for $x \in Dom(\mathcal{L})$ and $\ 0 \leq s \leq t$ we have,

$$\frac{d}{ds} T_s \circ \tau_j \circ P_{t-s}(x) = T_s \circ \mathcal{L} \circ \tau_j \circ P_{t-s}(x) - T_s \circ \tau_j \circ \mathcal{L} \circ P_{t-s}(x) = 0.$$

This implies that $T_s \circ \tau_j \circ P_{t-s}(x)$ is independent of s for every j and $0 \leq s \leq t$. Setting $s = 0$ and t respectively and using the fact that T_t is bounded we get $T_t \tau_j = \tau_j T_t$. □

We note that $j_t : \mathcal{A} \to \mathcal{A}'' \bigotimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0)))$, where $\mathbf{k}_0 = l^2(\mathbb{Z}^d)$ with a canonical basis $\{e_k\}$, as mentioned earlier. We define the canonical bilateral shift s by $s_j e_k = e_{k+j}, \forall j, k \in \mathbb{Z}^d$ and let $\gamma_j = \Gamma(1 \otimes s_j)$ be the second quantization of $1 \otimes s_j$, i.e. $\gamma_j \mathbf{e}(\sum f_l(.)e_l) = \mathbf{e}(\sum f_l(.)e_{l+j})$. This defines a unitary representation of $\mathbb{Z}^d$ in Γ. We set an action $\sigma = \tau \otimes \lambda$ of $\mathbb{Z}^d$ on $\mathcal{A}'' \bigotimes \mathcal{B}(\Gamma)$, where $\lambda_j(y) = \gamma_j y \gamma_{-j}$ $\forall y \in \mathcal{B}(\Gamma)$.

By definition of fundamental processes $a_k(t)$ given by $a_k(t)\mathbf{e}(g) = \int_0^t g_k(s)ds\ \mathbf{e}(g)$, it can be observed that

$$\begin{aligned}
\lambda_j a_k(t)\mathbf{e}(g) &= \gamma_j a_k(t)\gamma_{-j}\mathbf{e}(g) = \gamma_j a_k(t)\mathbf{e}(\sum \langle g(\cdot), e_{l+j}\rangle e_l) \\
&= \int_0^t \langle g, e_{k+j}\rangle(s)ds\ \gamma_j \left(\mathbf{e}(\sum \langle g(\cdot), e_{l+j}\rangle e_l)\right) \\
&= \int_0^t \langle g, e_{k+j}\rangle(s)ds\ \mathbf{e}(\sum \langle g(\cdot), e_{l+j}\rangle e_{l+j}) \\
&= a_{k+j}(t)\mathbf{e}(g).
\end{aligned}$$

Since $\langle \mathbf{e}(f), \lambda_j a_k(t)\mathbf{e}(g)\rangle = \langle \lambda_j a_k^\dagger(t)\mathbf{e}(f), \mathbf{e}(g)\rangle$, it follows that

$$\lambda_j a_k(t) = a_{k+j}(t) \text{ and } \lambda_j a_k^\dagger(t) = a_{k+j}^\dagger(t). \qquad (3.2.\ 2)$$

Theorem 3.2.3. *The Evans-Hudson flow j_t of the QDS T_t is covariant with respect to the actions τ and σ, i.e.*

$$\sigma_j j_t \tau_{-j}(x) = j_t(x)\ \forall x \in \mathcal{A},\ t \geq 0 \textit{ and } k \in \mathbb{Z}^d.$$

Proof. For a fixed $j \in \mathbb{Z}^d$ we set $j'_t = \sigma_j j_t \tau_{-j}$, $\forall t \geq 0$. Using the qsde (3.1. 2) and Lemma 3.2.2, (3.2. 1), (3.2. 2) we have for $x \in \mathcal{A}_{\text{loc}}$,

$$\begin{aligned}
&j'_t(x) - j'_0(x)\\
&\quad = \int_0^t \sum_{k\in\mathbb{Z}^d} \sigma_j j_s(\delta_k^\dagger(\tau_{-j}(x)))da_k(s) + \int_0^t \sum_{k\in\mathbb{Z}^d} \sigma_j j_s(\delta_k(\tau_{-j}(x)))da_k^\dagger(s)\\
&\qquad + \int_0^t \sigma_j j_s(\mathcal{L}(\tau_{-j}(x)))ds\\
&\quad = \int_0^t \sum_{k\in\mathbb{Z}^d} \sigma_j j_s \tau_{-j}(\delta_{k+j}^\dagger(x))da_{k+j}(s) + \int_0^t \sum_{k\in\mathbb{Z}^d} \sigma_j j_s \tau_{-j}(\delta_{k+j}(x))da_{k+j}^\dagger(s)\\
&\qquad + \int_0^t \sigma_j j_s \tau_{-j}(\mathcal{L}(x))ds\\
&\quad = \int_0^t \sum_{k\in\mathbb{Z}^d} j'_s(\delta_k^\dagger(x))da_k(s) + \int_0^t \sum_{k\in\mathbb{Z}^d} j'_s(\delta_k(x))da_k^\dagger(s) + \int_0^t j'_s(\mathcal{L}x)ds.
\end{aligned}$$

Since $j'_0(x) = \sigma_j j_0 \tau_{-j}(x) = \sigma_j(\tau_{-j}(x) \otimes 1_\Gamma) = x \otimes 1_\Gamma = j_0(x)$, it follows from the uniqueness of solution of the qsde (3.1. 2) that $j'_t(x) = j_t(x)$ for all $t \geq 0$ and $x \in \mathcal{A}_{\text{loc}}$. As both j'_t and j_t are bounded maps, we have $j'_t = j_t$. □

Remark 3.2.4. *By similar arguments as above, the Evans-Hudson flow for the QDS T_t^ϕ associated with partial state ϕ_0 can be seen to be covariant with respect to the same actions τ and σ of $\mathbb{Z}^d$.*

3.3 Ergodicity of the EH flows

Let us recall the ergodic QDS T_t^ϕ associated with the partial state ϕ_0, for which we have constructed an Evans-Hudson flow η_t in section 3. It may be noted that T_t^ϕ has the unique invariant state Φ. We have the following result on ergodicity of η_t with respect to the weak operator topology.

Theorem 3.3.1. *The Evans-Hudson flow η_t of the ergodic QDS T_t^ϕ is ergodic with respect to the invariant state Φ, in the sense that*

$$\eta_t(x) \to \Phi(x) \otimes 1_\Gamma \text{ weakly } \forall x \in \mathcal{A}.$$

Proof. Since η_t and T_t^ϕ are norm contractive, $\mathcal{A}_{\text{loc}}$ is norm-dense in $\mathcal{A}$, and $T_t^\phi(x)$ converges to $\Phi(x)1$ for all $x \in \mathcal{A}$, it is enough to show that for $x \in \mathcal{A}_{\text{loc}}$, $\eta_t(x) -$

$T_t^\phi(x)\otimes 1_\Gamma \to 0$ weakly as $t\to\infty$. Furthermore, it suffices to show that $\langle \xi_1, (\eta_t(x) - T_t^\phi(x)\otimes 1_\Gamma)\xi_2\rangle \to 0$ as $t\to\infty$, where ξ_1,ξ_2 vary over the linear span of vectors of the form $ve(f)$, with $f=\sum_{|k|\leq n} f_k\otimes e_k$ for some n and f_k's are in $L^1(\mathbb{R}_+)\bigcap L^2(\mathbb{R}_+)$. For notational simplicity denoting the bounded derivations on $\mathcal{A}$,

$$x\mapsto \sum_{m=1}^{N^2}[x, L_k^{(m)}] \text{ and } x\mapsto \sum_{m=1}^{N^2}[L_k^{(m)^*}, x]$$

by ρ_k and $\rho_k^\dagger$ respectively, we note that η_t satisfies the qsde

$$d\eta_t(x) = \sum_{k\in\mathbb{Z}^d}\eta_t(\rho_k^\dagger(x))da_k(t) + \sum_{k\in\mathbb{Z}^d}\eta_t(\rho_k(x))da_k^\dagger(t) + \eta_t(\mathcal{L}^\phi(x))dt, \qquad (3.3.\ 1)$$

$$\eta_0(x) = x\otimes 1_\Gamma, \forall x\in\mathcal{A}_{\text{loc}}.$$

For $t\geq 0, u,v\in \mathbf{h}_0$ and $f,g\in L^2(\mathbb{R}_+,\mathbf{k}_0)\bigcap L^1(\mathbb{R}_+,\mathbf{k}_0)$ such that $f=\sum_{|k|\leq n} f_k\otimes e_k$ and $g=\sum_{|k|\leq n} g_k\otimes e_k$ and $x\in\mathcal{A}_{\text{loc}}$, we consider the following,

$$\begin{aligned}
&|\langle u\mathbf{e}(f), [\eta_t(x) - T_t^\phi(x)\otimes 1_\Gamma]v\mathbf{e}(g)\rangle|\\
&= |\langle u\mathbf{e}(f), [\int_0^t d(\eta_s T_{t-s}^\phi(x))]v\mathbf{e}(g)\rangle|\\
&= |\langle u\mathbf{e}(f), [\int_0^t \sum_{k\in\mathbb{Z}^d}\eta_q\{\rho_k(T_{t-q}^\phi(x))\}da_k^\dagger(q) + \eta_q\{\rho_k^\dagger(T_{t-q}^\phi(x))\}da_k(q)]v\mathbf{e}(g)\rangle|\\
&\leq \sum_{|k|\leq n}\int_0^t |\langle u\mathbf{e}(f), \eta_q\{\rho_k(T_{t-q}^\phi(x))\}v\mathbf{e}(g)\rangle|\ \|g(q)\|dq\\
&+ \sum_{|k|\leq n}\int_0^t |\langle u\mathbf{e}(f), \eta_q\{\rho_k^\dagger(T_{t-q}^\phi(x))\}v\mathbf{e}(g)\rangle|\ \|f(q)\|dq\\
&= \sum_{|k|\leq n}\int_0^t |\langle u\mathbf{e}(f), \eta_q\{\rho_k(T_{t-q}^\phi(x) - \Phi(x))\}v\mathbf{e}(g)\rangle|\ \|g(q)\|dq\\
&+ \sum_{|k|\leq n}\int_0^t |\langle u\mathbf{e}(f), \eta_q\{\rho_k^\dagger(T_{t-q}^\phi(x) - \Phi(x))\}v\mathbf{e}(g)\rangle|\ \|f(q)\|dq\\
&\leq \|u\mathbf{e}(f)\|\|v\mathbf{e}(g)\|\sum_{|k|\leq n}\left(\int_0^\infty \|\rho_k(T_{t-q}^\phi(x) - \Phi(x))\|\|g(q)\|dq\right.\\
&\qquad\qquad \left. + \int_0^\infty \|\rho_k^\dagger(T_{t-q}^\phi(x) - \Phi(x))\|\|f(q)\|dq\right).
\end{aligned}$$

Since ρ_k, $\rho_k^\dagger$ are bounded linear maps and since $f,g\in L^1(\mathbb{R}_+,\mathbf{k}_0)$, the integrands above are dominated by an integrable function $M(\|f(q)\|+\|g(q)\|)$ for some constant

M independent of q. Now, since $\|T_t^\phi(x) - \Phi(x)1\|$ converges to 0 as t tends to ∞, by dominated convergence theorem both the integrals in the above expression tend to 0 as t tends to ∞. This completes the proof. □

Remark 3.3.2. $\eta_t(x)$ *does not converge strongly, for if it did, then* $x \mapsto \Phi(x) \otimes 1_\Gamma$ *would be a homomorphism, i.e.* Φ *would be a multiplicative non zero functional on the UHF algebra* $\mathcal{A}$*, contradictory to the fact that* $\mathcal{A}$ *does not have any such functional.*

Now let us look at the perturbation of the semigroup T_t^ϕ by the semigroup T_t associated with some single-supported element $r \in \mathcal{A}_0$. Recall that the generator $\mathcal{L}$ of QDS T_t satisfies

$$\mathcal{L}(x) = \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \{[r_k^*, x] r_k + r_k^* [x, r_k]\}, \forall x \in \mathcal{C}^1(\mathcal{A}).$$

Setting $L^{(N^2+1)} = r$, for any real c, the generator $\mathcal{L}^{(c)}$ of the perturbed QDS $T_t^{(c)}$ satisfies,

$$\mathcal{L}^{(c)}(x) = \mathcal{L}^\phi(x) + c\mathcal{L}(x) = \sum_{k \in \mathbb{Z}^d} \mathcal{L}_k^{(c)},$$

with

$$\mathcal{L}_k^{(c)} = \frac{1}{2} \sum_{m=1}^{N^2+1} [L_k^{(m)^*}, x] L_k^{(m)} + L_k^{(m)^*} [x, L_k^{(m)}], \forall x \in \mathcal{C}^1(\mathcal{A}).$$

So by the same arguments used in the construction of the Evans-Hudson flow for the unperturbed semigroup T_t^ϕ one can obtain an Evans-Hudson flow for the perturbed QDS $T_t^{(c)}$. Moreover, for small perturbation parameter $c \geq 0$ for which $T_t^{(c)}$ is ergodic by Theorem 2.2.1, the associated Evans-Hudson flow is also ergodic with respect to the same invariant state in the sense of previous Theorem 3.3.1.

Chapter 4

Toy Fock Space and QRW Approach to the Construction of EH flow

In Chapter-1, quantum stochastic calculus on the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0))$ is discussed. Here following [2, 3] we shall describe a family of subspaces of $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0))$, indexed by some partition of $\mathbb{R}_+$. The subspace will be called *toy Fock space* associated with the corresponding partition. Next, using basic operators on toy Fock spaces, quantum random walks are defined as in [23], and then strong convergence of quantum random walks associated with bounded structure maps is proved under suitable assumptions, extendings the result obtained in [35] in case of one dimensional noise. To handle infinite dimensional noise we have used the coordinate-free language of quantum stochastic calculus developed in [15].

4.1 Toy Fock space and basic operators

First we note that, for any $n \geq 0$, the n-fold symmetric tensor product of $\mathcal{K} = L^2(\mathbb{R}_+, \mathbf{k}_0)$ and their direct sum can canonically be embedded in $\Gamma(\mathcal{K})$. We also have

Lemma 4.1.1. *For any partition $S \equiv (0 = t_0 < t_1 < t_2 \cdots)$ of $\mathbb{R}_+$, the Fock space $\Gamma(\mathcal{K})$ can be viewed as the infinite tensor product $\bigotimes_{n\geq 1} \Gamma_n$ of symmetric Fock spaces*

$\{\Gamma_n = \Gamma(\mathcal{K}_{(t_{n-1},t_n]})\}_{n\geq 1}$ *with respect to the stabilizing sequence* $\Omega = \{\Omega_n : n \geq 1\}$, *where* $\Omega_n = \Omega_{(t_{n-1},t_n]}$ *is the vacuum vector in* Γ_n.

Proof. The set of all exponential vectors $\mathcal{E}(\mathcal{K})$ is total in Γ. By definition we know that the set W of all vector $\xi = \otimes_{n\geq 1}\xi_n$ such that ξ_n is an exponential vector in Γ_n and $\xi_n = \Omega_n$ for all but finitely many $n \geq 1$, is total in $\bigotimes_{n\geq 1}\Gamma_n$. It is clear that any vector in W can be written as an exponential vector of the form $\mathbf{e}(f_{t_n]}) \otimes \Omega_{[t_n}$ for some $f \in \mathcal{K}$ and hence $W \subseteq \mathcal{E}(\mathcal{K})$. Thus it is enough to show that for any $f \in \mathcal{K}$ can be approximated in norm by a sequence $\{\eta_n\}$ in W. For a given f, let us consider the sequence vectors $\{\eta_n = \mathbf{e}(f_{t_n]}) \otimes \Omega_{[t_n}\}$. Then we have

$$\begin{aligned}
&\|\mathbf{e}(f) - \eta_n\|^2 \\
&= \|\mathbf{e}(f_{t_n]})\|^2 \ \|(\mathbf{e}(f_{[t_n}) - \Omega_{[t_n})\|^2 \\
&= \|\mathbf{e}(f_{t_n]})\|^2 \ (\|\mathbf{e}(f_{[t_n})\|^2 - 1) \\
&\leq \|\mathbf{e}(f_{t_n]})\|^2 \ \|f_{[t_n}\|^2 \|(\mathbf{e}(f_{[t_n})\|^2 \\
&= \|\mathbf{e}(f)\|^2 \ \|f_{[t_n}\|^2.
\end{aligned}$$

Since $f \in \mathcal{K}$, $\|f_{[t_n}\|^2 = \int_{t_n}^{\infty} \|f(s)\|^2 ds$ goes to 0 as n tends to ∞ and $\lim_{n\to\infty} \|\mathbf{e}(f) - \eta_n\| = 0$. □

Let $\{e_i\}$ is a fixed othonormal basis of $\mathbf{k}_0$ as mentioned earlier. For any $0 \leq s \leq t$ and $i \geq 1$ we define a vector $\chi^i_{(s,t]} := \frac{1_{(s,t]}\otimes e_i}{\sqrt{t-s}} \in \mathcal{K}_{(s,t]}$. It is clear that $\{\chi^i_{(s,t]}\}_{i\geq 1}$ is an orthonormal family in $\mathcal{K}_{(s,t]}$ and hence in $\Gamma_{(s,t]}$. Here we note that the Hilbert subspace $\mathbf{k}_{(\mathbf{s},\mathbf{t}]}$ of $\Gamma_{(s,t]}$ spanned by these orthonormal vectors is canonically isomorphic to $\mathbf{k}_0$. Let us consider the subspace $\hat{\mathbf{k}}_{(s,t]} = \mathbb{C}\ \Omega_{(s,t]} \bigoplus \mathbf{k}_{(\mathbf{s},\mathbf{t}]}$ of Γ and denote the space $\hat{\mathbf{k}}_{(t_{n-1},t_n]}$ by $\hat{\mathbf{k}}_n$, which is isomorphic to $\hat{\mathbf{k}}_0 := \mathbb{C} \bigoplus \mathbf{k}_0$. Now we are in a position to define the toy Fock spaces.

Definition 4.1.2. *The toy Fock space associated with the partition* S *of* $\mathbb{R}_+$ *is defined to be the subspace* $\Gamma(S) := \bigotimes_{n\geq 1}\hat{\mathbf{k}}_n$ *with respect to the stabilizing vector* $\Omega = \otimes_{n\geq 1}\Omega_n$.

For notational simplicity we write χ^i_n for the vector $\chi^i_{(t_{n-1},t_n]}$. Let $\sqcap$ be the set of all finite subsets of $\mathbb{N}\times\mathbb{N}$. Thus an element $A \in \sqcap$ is given by $A = \{m_1, i_1; m_2, i_2; \cdots m_n, i_n\}$

for some n with $1 \leq m_1 < m_2 \cdots m_n < \infty$. For $A \in \sqcap$, we associate a vector

$$\chi_A = \Omega_1 \otimes \Omega_2 \otimes \cdots \chi_{m_1}^{i_1} \otimes \cdots \chi_{m_2}^{i_2} \otimes \cdots \chi_{m_n}^{i_n} \otimes \Omega_{m_n+1} \cdots$$

in the toy Fock space $\Gamma(S)$. Clearly this family $\{\chi_A : A \in \sqcap\}$ forms an orthonormal basis for $\Gamma(S)$. Let $P(S)$ be the orthogonal projection of Γonto the toy Fock space $\Gamma(S)$. Without loss of generality now onwards let us consider toy Fock spaces $\Gamma(S_h)$ associated with regular partition $S_h \equiv (0, h, \cdots)$ for some $h > 0$ and denote the orthogonal projection by P_h. The projection P_h is given by

$$P_h = P_0 \oplus \bigoplus_{n\geq 1} \sum_{1\leq m_1 < m_2 \cdots < m_n} \sum_{i_1, i_2 \cdots i_n \geq 1} \bigotimes_{l=1}^{n} |\chi_{m_l}^{i_l} >< \chi_{m_l}^{i_l}|,$$

where P_0 is the orthogonal projection of the symmetric Fock space Γ onto the one dimensional Hilbert space $\mathbb{C}\Omega$. A simple computation shows that, for $f \in \mathcal{K}$, given by $f = \sum_{i\geq 1} f_i \otimes e_i$ with $f_i \in L^2(\mathbb{R}_+)$,

$$\begin{aligned}
&P_h(\Omega) = \Omega, \\
&P_h f = \sum_{m,i\geq 1} \frac{1}{\sqrt{h}} \int_{(m-1)h}^{mh} f_i(s) ds \; \chi_m^i, \\
&P_h \mathbf{e}(f) = \Omega \oplus \bigoplus_{n\geq 1} \frac{1}{\sqrt{n!}} \left[\sum_{1\leq m_1 < m_2 \cdots < m_n} \sum_{i_1, i_2 \cdots i_n \geq 1} \bigotimes_{l=1}^{n} \frac{1}{\sqrt{h}} \int_{(m_l-1)h}^{m_l h} f_{i_l}(s) ds \; \chi_{m_l}^{i_l} \right]
\end{aligned}$$

and furthermore,

$$\begin{aligned}
&P_h \mathbf{e}(f) = P_h \mathbf{e}(f_{(k-1)h]}) P_h \mathbf{e}(f_{[k]}) P_h \mathbf{e}(f_{[kh}) \text{ and} \\
&P_h \mathbf{e}(f_{[k]}) = \Omega_k \oplus \sum_{i\geq 1} \frac{1}{\sqrt{h}} \int_{(k-1)h}^{kh} f_i(s) ds \frac{1_{((k-1)h, kh]} \otimes e_i}{\sqrt{h}}.
\end{aligned}$$

Now we define a family of operators $\{N_\nu^\mu[k] : \mu, \nu \geq 0, k \geq 1\}$ on the Fock space Γ, given by

$$\begin{aligned}
N_\nu^\mu[k] &= P_0[k] \frac{\Lambda_0^0[k]}{h} = P_0[k] \text{ for } (\mu, \nu) = (0, 0), \\
&= \frac{\Lambda_j^0[k]}{\sqrt{h}} P_1[k] \text{ for } (\mu, \nu) = (0, j), \\
&= P_1[k] \frac{\Lambda_0^i[k]}{\sqrt{h}} \text{ for } (\mu, \nu) = (i, 0), \\
&= P_1[k] (\Lambda_j^i[k]) P_1[k] P_h[k] \text{ for } (\mu, \nu) = (i, j),
\end{aligned} \tag{4.1. 1}$$

where $P_0[k]$ and $P_1[k]$ are the orthogonal projections from Γ_k onto the one dimensional subspace spanned by Ω_k and $L^2([(k-1)h, kh], \mathbf{k}_0)$ respectively. We have used the notations $\Lambda^\mu_\nu[k]$ for $\Lambda^\mu_\nu((k-1)h, kh]$ and $P_h[k]$ for the associated toy Fock space orthogonal projection restricted to the interval $[(k-1)h, kh]$. These operators $N^\mu_\nu[k]$'s act nontrivially only on Γ_k and as identity on the other components and they leave the subspace $\Gamma(S_h)$ invariant. For simplicity let us denote the interval $((k-1)h, kh]$ by $[k]$ and write $f_{[k]}$ for $f_{((k-1)h,kh]}$. From the definitions we have, for any $f \in L^2(\mathbb{R}_+, \mathbf{k}_0)$,

$$\begin{aligned}
N^0_0[k]\mathbf{e}(f) &= \mathbf{e}(f_{(k-1)h]})\Omega_k \mathbf{e}(f_{[kh}),\\
N^0_j[k]\mathbf{e}(f) &= \frac{1}{\sqrt{h}}\int_{[k]} f_j(s)ds\ \mathbf{e}(f_{(k-1)h]})\Omega_k \mathbf{e}(f_{[kh}),\\
N^i_0[k]\mathbf{e}(f) &= \mathbf{e}(f_{(k-1)h]})\frac{1_{[k]}\otimes e_i}{\sqrt{h}}\mathbf{e}(f_{[kh}),\\
N^i_j[k]\mathbf{e}(f) &= \frac{1}{\sqrt{h}}\int_{[k]} f_j(s)\ ds\ \mathbf{e}(f_{(k-1)h]})\frac{1_{[k]}\otimes e_i}{\sqrt{h}}\mathbf{e}(f_{[kh}).
\end{aligned}$$

It can easily be observed that

$$\begin{aligned}
&(N^\mu_\nu[k])^* = N^\nu_\mu[k]\\
&N^\mu_\nu[k]N^\xi_\eta[k] = \delta^\xi_\nu N^\mu_\eta[k]\\
&\sum_{\mu,\nu}\delta^\mu_\nu N^\mu_\nu[k] = P_h[k].
\end{aligned} \tag{4.1. 2}$$

Here we also note that

$$\begin{aligned}
\|\Lambda^0_0[k]\mathbf{e}(f_{[k]})\| &= h\|\mathbf{e}(f_{[k]})\|\\
\|\Lambda^0_j[k]\mathbf{e}(f_{[k]})\| &= |\int_{[k]} f_j(s)ds|\|\mathbf{e}(f_{[k]})\|\\
\|\Lambda^i_0[k]\mathbf{e}(f_{[k]})\|^2 &= \left(h + |\int_{[k]} f_i(s)ds|^2\right)\|\mathbf{e}(f_{[k]})\|^2\\
\|\Lambda^i_j[k]\mathbf{e}(f_{[k]})\|^2 &= \left[\int_{[k]}|f_j(s)|^2ds + |\int_{[k]}\overline{f_i(s)}f_j(s)ds|^2\right]\|\mathbf{e}(f_{[k]})\|^2.
\end{aligned}$$

Let us consider the subspace $\mathcal{M}$ of $L^2(\mathbb{R}_+, \mathbf{k}_0)$, given by

$$\mathcal{M} = \{f \in L^2(\mathbb{R}_+, \mathbf{k}_0) : f_i \in \mathcal{C}^1_c(\mathbb{R}_+) \text{ and } f_i = 0 \text{ for all but finitely many } \ i\}.$$

Clearly $\mathcal{M}$ is a dense subspace, so the algebraic tensor product $\mathbf{h}_0 \bigotimes \mathcal{E}(M)$ is dense in $\mathbf{h}_0 \bigotimes \Gamma$. For $f \in \mathcal{M}$ we define a constant $c_f := \sum_{i\geq 1} \sup_\tau |f_i'(\tau)|$ where f_i' denotes the first derivative of the function f_i. Now we have the following estimates.

Lemma 4.1.3. *(a). For any* $f \in \mathcal{M}, k \geq 1$,
$\|(1 - P_h[k])\boldsymbol{e}(f_{[k]})\| \leq h(c_f + \|f\|_\infty)\|\boldsymbol{e}(f_{[k]})\|$.
(b). For any $k \geq 1$ *and* $f \in \mathcal{M}$,

1. $\|\{h\ N_0^0[k] - \Lambda_0^0[k]\}\boldsymbol{e}(f_{[k]})\| \leq h^{\frac{3}{2}}\|f\|_\infty\|\boldsymbol{e}(f_{[k]})\|$,

2. $\|\{\sqrt{h}\ N_j^0[k] - \Lambda_j^0[k]\}\boldsymbol{e}(f_{[k]})\| \leq h^{\frac{3}{2}}\|f\|_\infty^2\|\boldsymbol{e}(f_{[k]})\|$,

3. $\|\{\sqrt{h}\ N_0^i[k] - \Lambda_0^i[k]\}\boldsymbol{e}(f_{[k]})\| \leq 2h\|f\|_\infty\|\boldsymbol{e}(f_{[k]})\|$,

4. $\|\{N_j^i[k] - \Lambda_j^i[k]\}\boldsymbol{e}(f_{[k]})\| \leq hc_1(f)\|\boldsymbol{e}(f_{[k]})\|$,
   where $c_1(f)^2 = 2\|f\|_\infty^4 + c_f\|f\|_\infty$.

(c). For any $k \geq 1$ *and* $f, g \in \mathcal{M}$,

1. $|\langle \boldsymbol{e}(g_{[k]}), \{h\ N_0^0[k] - \Lambda_0^0[k]\}\boldsymbol{e}(f_{[k]})\rangle| \leq h^{\frac{3}{2}}\|f\|_\infty\|\boldsymbol{e}(f_{[k]})\|\|\boldsymbol{e}(g_{[k]})\|$,

2. $|\langle \boldsymbol{e}(g_{[k]}), \{\sqrt{h}\ N_j^0[k] - \Lambda_j^0[k]\}\boldsymbol{e}(f_{[k]})\rangle| \leq h^{\frac{3}{2}}\|f\|_\infty^2\|\boldsymbol{e}(f_{[k]})\|\|\boldsymbol{e}(g_{[k]})\|$,

3. $|\langle \boldsymbol{e}(g_{[k]}), \{\sqrt{h}\ N_0^i[k] - \Lambda_0^i[k]\}\boldsymbol{e}(f_{[k]})\rangle|$
   $\leq 2h^2\|f\|_\infty\|g\|_\infty\|^2\boldsymbol{e}(f_{[k]})\|^2\|\boldsymbol{e}(g_{[k]})\|^2$,

4. $|\langle \boldsymbol{e}(g_{[k]}), \{N_j^i[k] - \Lambda_j^i[k]\}\boldsymbol{e}(f_{[k]})\rangle|$
   $\leq h^2c_2(f,g)\|\boldsymbol{e}(f_{[k]})\|^2\|\boldsymbol{e}(g_{[k]})\|^2$, *where* $c_2(f,g) = (\|f\|_\infty\|g\|_\infty)^2 + c_f\|g\|_\infty$.

Proof. (a). We have

$$
\begin{aligned}
&\|(1 - P_h[k])\mathbf{e}(f_{[k]})\| \\
&= \|(P_0 + P_1 - P_h)\mathbf{e}(f_{[k]}) + [1 - P_0 - P_1]\mathbf{e}(f_{[k]})\| \\
&= \|f_{[k]} - P_h f_{[k]} + [1 - P_0 - P_1]\mathbf{e}(f_{[k]})\| \\
&\leq \|f_{[k]} - P_h f_{[k]}\| + \|[1 - P_0 - P_1]\mathbf{e}(f_{[k]})\|.
\end{aligned}
$$

It is clear that $\|[1-P_0-P_1]\mathbf{e}(f_{[k]})\| \leq h\|f\|_\infty^2\|\mathbf{e}(f_{[k]})\|$. Let us consider the first term,

$$\begin{aligned}
&\|f_{[k]} - P_h f_{[k]}\|^2 \\
&= \|\sum_{i\geq 1} 1_{[k]}(f_i - \frac{1}{h}\int_{[k]} f_i(s)ds)e_i\|^2 \\
&= \sum_{i\geq 1} \|1_{[k]}(f_i - \frac{1}{h}\int_{[k]} f_i(s)ds)\|^2 \\
&= \sum_{i\geq 1} \int_{[k]} dr|f_i(r) - \frac{1}{h}\int_{[k]} f_i(s)ds|^2 \\
&= \sum_{i\geq 1} \frac{1}{h^2}\int_{[k]} dr|\int_{[k]} (f_i(r) - f_i(s)ds|^2 \\
&\leq \sum_{i\geq 1} \frac{1}{h^2}\int_{[k]} dr[\int_{[k]} |f_i(r) - f_i(s|ds]^2 \\
&\leq \sum_{i\geq 1} \frac{1}{h^2}\int_{[k]} dr[\int_{[k]} h\sup|f_i'(\tau)|ds]^2 \\
&\leq c_f^2 h^3.
\end{aligned}$$

This completes the proof.

(b) (1). By definitions we have

$$\begin{aligned}
&\|\{h\ N_0^0[k] - \Lambda_0^0[k]\}\mathbf{e}(f_{[k]})\| \\
&= h\|\Omega_{[k]} - \mathbf{e}(f_{[k]})\|.
\end{aligned}$$

First let us estimate,

$$\begin{aligned}
&\|\Omega_{[k]} - \mathbf{e}(f_{[k]})\|^2 \\
&= 1 + \|\mathbf{e}(f_{[k]})\|^2 - 2 \\
&= e^{\|f_{[k]}\|^2} - 1 \leq \|f_{[k]}\|^2\ e^{\|f_{[k]}\|^2} \\
&\leq h\|f\|_\infty^2 \mathbf{e}(f_{[k]})\|^2.
\end{aligned}$$

So we get

$$\|\Omega_{[k]} - \mathbf{e}(f_{[k]})\| \leq \sqrt{h}\|f\|_\infty \mathbf{e}(f_{[k]})\| \tag{4.1. 3}$$

and the required estimate follows.

(2). By definitions,

$$\begin{aligned}
&\|\{\sqrt{h}\ N_j^0[k] - \Lambda_j^0[k]\}\mathbf{e}(f_{[k]})\| \\
&= \|\int_{[k]} f_j(s)ds(\Omega_{[k]} - \mathbf{e}(f_{[k]})\|.
\end{aligned}$$

Thus required estimate follows from (4.1. 3).

(3). We have

$$\begin{aligned}
&\|\{\sqrt{h}\ N_0^i[k] - \Lambda_0^i[k]\}\mathbf{e}(f_{[k]})\|^2 \\
&= \|1_{[k]} \otimes e_i - \Lambda_0^i[k]\mathbf{e}(f_{[k]})\|^2 \\
&= h + \|\Lambda_0^i[k]\mathbf{e}(f_{[k]})\|^2 - 2Re\langle 1_{[k]} \otimes e_i, \Lambda_0^i[k]\mathbf{e}(f_{[k]})\rangle \\
&= h + h\|\mathbf{e}(f_{[k]})\|^2 + |\int_{[k]} f_i(s)ds|^2\|\mathbf{e}(f_{[k]})\|^2 \\
&\quad -2Re\langle \Lambda_i^0[k](1_{[k]} \otimes e_i), \mathbf{e}(f_{[k]})\rangle \\
&= h + h\|\mathbf{e}(f_{[k]})\|^2 + |\int_{[k]} f_i(s)ds|^2\|\mathbf{e}(f_{[k]})\|^2 - 2h \\
&= h\{\|\mathbf{e}(f_{[k]})\|^2 - 1\} + |\int_{[k]} f_i(s)ds|^2\|\mathbf{e}(f_{[k]})\|^2 \\
&\leq 2h^2\|f\|_\infty^2\|\mathbf{e}(f_{[k]})\|^2.
\end{aligned}$$

(4). Let us consider the following,

$$\begin{aligned}
&\|\{N_j^i[k] - \Lambda_j^i[k]\}\mathbf{e}(f_{[k]})\|^2 \\
&= \|N_j^i[k]\mathbf{e}(f_{[k]})\|^2 + \|\Lambda_j^i[k]\mathbf{e}(f_{[k]})\|^2 - 2Re\langle N_j^i[k]\mathbf{e}(f_{[k]}), \Lambda_j^i[k]\mathbf{e}(f_{[k]})\rangle \\
&= \frac{1}{h}|\int_{[k]} f_j(s)ds|^2 + \left(\int_{[k]} |f_j(s)|^2 ds + |\int_{[k]} \overline{f_i(s)}f_j(s)ds|^2\right)\|\mathbf{e}(f_{[k]})\|^2 \\
&\quad - \frac{2}{h}Re\left(\int_{[k]} \overline{f_j(s)}ds\langle (1_{[k]} \otimes e_i), \Lambda_j^i[k]\mathbf{e}(f_{[k]})\rangle\right).
\end{aligned}$$

Since $\langle \Lambda_i^j[k](1_{[k]} \otimes e_i), \mathbf{e}(f_{[k]})\rangle = \langle (1_{[k]} \otimes e_j), \mathbf{e}(f_{[k]})\rangle = \int_{[k]} f_j(s)ds$, we get

$$\begin{aligned}
&\|\{N_j^i[k] - \Lambda_j^i[k]\}\mathbf{e}(f_{[k]})\|^2 \\
&= \{\int_{[k]} |f_j(s)|^2 ds + |\int_{[k]} \overline{f_i(s)}f_j(s)ds|^2\}\|\mathbf{e}(f_{[k]})\|^2 - \frac{1}{h}|\int_{[k]} f_j(s)ds|^2 \\
&= |\int_{[k]} \overline{f_i(s)}f_j(s)ds|^2\|\mathbf{e}(f_{[k]})\|^2 + \int_{[k]} |f_j(s)|^2 ds\{e^{\|(f_{[k]}\|^2} - 1\} \\
&\quad +\frac{1}{h}\int_{[k]} \overline{f_j(s)}ds\{\int_{[k]} (f_j(s) - f_j(q))dq\} \\
&\leq h^2(2\|f\|_\infty^4 + h^2 c_f\|f\|_\infty)\|\mathbf{e}(f_{[k]})\|^2,
\end{aligned}$$

(c). The estimates in (1) and (2) immediately follow from (a).

(3). From the definitions,

$$\begin{aligned}
&|\langle \mathbf{e}(g_{[k]}), \{\sqrt{h}\; N_0^i[k] - \Lambda_0^i[k]\}\mathbf{e}(f_{[k]})\rangle| \\
&= \; |\int_{[k]} g_i(s)ds||1 - \langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle| \\
&\leq \; \int_{[k]} |g_i(s)|ds|1 - e^{\langle g_{[k]}, f_{[k]}\rangle}|.
\end{aligned}$$

In order to estimate $|1-e^{\langle g_{[k]}, f_{[k]}\rangle}|$, we note that for any complex number z, $|1-e^z| \leq 2|z|e^{|z|}$. Thus

$$\begin{aligned}
&|1 - e^{\langle g_{[k]}, f_{[k]}\rangle}| \\
&\leq \; 2|\langle g_{[k]}, f_{[k]}\rangle| \; |e^{\langle g_{[k]}, f_{[k]}\rangle}| \\
&\leq \; 2\int_{[k]} |\langle g(s), f(s)\rangle|ds \; e^{\|g_{[k]}\| \; \|f_{[k]}\|} \\
&\leq \; 2\int_{[k]} \|g(s)\| \; \|f(s)\|ds \; e^{\|g_{[k]}\|^2 + \|f_{[k]}\|^2}.
\end{aligned}$$

Which gives

$$|1 - e^{\langle g_{[k]}, f_{[k]}\rangle}| \leq 2h\|g\|_\infty \; \|f\|_\infty \|\mathbf{e}(g_{[k]})\|^2 \; \|\mathbf{e}(f_{[k]})\|^2, \tag{4.1. 4}$$

and the required estimate follows.

(4). From the definitions,

$$\begin{aligned}
&\langle \mathbf{e}(g_{[k]}), \{N_j^i[k] - \Lambda_j^i[k]\}\mathbf{e}(f_{[k]})\rangle \\
&= \; \frac{1}{h}\int_{[k]} \overline{g_i(s)}ds \int_{[k]} f_j(s)ds - \int_{[k]} \overline{g_i(s)} f_j(s)ds\langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle \\
&= \; \frac{1}{h}\int_{[k]} \overline{g_i(s)}ds \int_{[k]} [f_j(q) - f_j(s)]\, dq \\
&\quad + \int_{[k]} \overline{g_i(s)} f_j(s)ds \; \left(1 - \langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle\right).
\end{aligned}$$

Thus we get

$$\begin{aligned}
&|\langle \mathbf{e}(g_{[k]}), \{N_j^i[k] - \Lambda_j^i[k]\}\mathbf{e}(f_{[k]})\rangle| \\
&\leq \frac{1}{h}\int_{[k]} |g_i(s)|ds \int_{[k]} |f_j(q) - f_j(s)|dq \\
&+ \int_{[k]} |g_i(s)| \; |f_j(s)|ds \; |1 - \langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle|.
\end{aligned}$$

Using the inequality (4.1. 4) the required estimate follows.

□

The toy Fock spaces $\Gamma(S_h)$ approximate the Fock space Γ in the following sense [3]:

Lemma 4.1.4. *(a). The family of othogonal projections P_h converges strongly to identity operator in Γ as h tends to* 0.
(b). For any $f \in \mathcal{M}$, $k \geq 1$ and for any $t \geq 0$, setting $n = [\frac{t}{h}] + 1$

1. $\lim_{h\to 0} \|[\sum_{k=1}^n hN_0^0[k] - \Lambda_0^0(t)]\boldsymbol{e}(f)\| = 0,$

2. $\lim_{h\to 0} \|[\sum_{k=1}^n \sqrt{h}N_j^0[k] - \Lambda_j^0(t)]\boldsymbol{e}(f)\| = 0,$

3. $\lim_{h\to 0} \|[\sum_{k=1}^n \sqrt{h}N_0^i[k] - \Lambda_0^i(t)]\boldsymbol{e}(f)\| = 0,$

4. $\lim_{h\to 0} \|[\sum_{k=1}^n N_j^i[k] - \Lambda_j^i(t)]\boldsymbol{e}(f)\| = 0.$

Proof. (a). It is enough to show that for any $f \in \mathcal{M}, t \geq 0$

$$\lim_{h\to 0} \|(1 - P_h)\mathbf{e}(f_{nh]})\| = 0,$$

where n is as in part (b) of the Lemma. We have

$$\begin{aligned}
&\|(1 - P_h)\mathbf{e}(f_{nh]})\|^2 \\
&= \|\sum_{k=1}^n \mathbf{e}(f_{(k-1)h]})(1 - P_h[k])\mathbf{e}(f_{[k]})P_h\mathbf{e}(f_{[kh,nh]})\|^2 \\
&= \sum_{k=1}^n \|\mathbf{e}(f_{(k-1)h]})\|^2\|(1 - P_h[k])\mathbf{e}(f_{[k]})\|^2\|P_h\mathbf{e}(f_{[kh,nh]})\|^2 \\
&\quad +2\ Re\ \sum_{k=1}^n \langle(1 - P_h)\mathbf{e}(f_{(k-1)h]}), \mathbf{e}(f_{(k-1)h]})\rangle \\
&\qquad \langle P_h\mathbf{e}(f_{[k]}), (1 - P_h[k])\mathbf{e}(f_{[k]})\rangle\|P_h\mathbf{e}(f_{[kh,nh]})\|^2.
\end{aligned}$$

Here $\langle P_h\mathbf{e}(f_{[k]}), (1 - P_h[k])\mathbf{e}(f_{[k]})\rangle = 0$ and using Lemma 4.1.3 (a) we obtain,

$$\begin{aligned}
&\|(1 - P_h)\mathbf{e}(f_{nh]})\|^2 \\
&\leq \sum_{k=1}^n h^2(c_f + \|f\|_\infty^2)\|\mathbf{e}(f)\|^2 \\
&\leq th(c_f + \|f\|_\infty^2)\|\mathbf{e}(f)\|^2.
\end{aligned}$$

This completes the proof.

(b). For $(\mu,\nu)=(0,0)$ and $(0,j)$, convergence follows directly from Lemma 4.1.3 (b) and for other processes it is necessary to give a better estimate. Let us write

$$\begin{aligned}&\sum_{k=1}^{n}\sqrt{h}N_0^i[k]-\Lambda_0^i(t)\\ &\quad=\sum_{k=1}^{n}[\sqrt{h}N_0^i[k]-\Lambda_0^i[k]]+\Lambda_0^i(nh,t).\end{aligned}$$

Now for $m\geq 1$, setting

$$X_{mh}=\sum_{k=1}^{m}[\sqrt{h}N_0^i[k]-\Lambda_0^i[k]],$$

we get

$$\begin{aligned}&\|[\sum_{k=1}^{n}\sqrt{h}N_0^i[k]-\Lambda_0^i(t)]\mathbf{e}(f)\|^2\\ &\quad\leq 2\{\|X_{nh}\mathbf{e}(f)\|^2+\|\Lambda_0^i(nh,t)\mathbf{e}(f)\|^2\}.\end{aligned}$$

It is clear that $\|\Lambda_0^i(nh,t)\mathbf{e}(f)\|$ tends to 0 as $h\to 0$ and by Lemma 4.1.3(a) $\|X_{(k-1)h}\mathbf{e}(f_{(k-1)h]})\|\leq C$, for some constant independent of h and k. Now we consider the first term,

$$\begin{aligned}&\|X_{nh}\mathbf{e}(f)\|^2\\ &=\sum_{k=1}^{n}\|[\sqrt{h}N_0^i[k]-\Lambda_0^i(t)]\mathbf{e}(f_{[k]})\|^2\|\mathbf{e}(f-f_{[k]})\|^2\\ &\quad-2Re\sum_{k=1}^{n}\langle X_{(k-1)h}\mathbf{e}(f_{(k-1)h]}),\mathbf{e}(f_{(k-1)h]})\rangle\\ &\qquad\langle\mathbf{e}(f_{[k]}),[\sqrt{h}N_0^i[k]-\Lambda_0^i[k]]\mathbf{e}(f_{[k]})\rangle\|\mathbf{e}(f_{[kh})\|^2.\end{aligned}$$

Since $\|X_{(k-1)h}\mathbf{e}(f_{(k-1)h]})\|$ is uniformly bounded in h and k, by (a.3) and (b.3) in Lemma 4.1.3 it follows that $\|[\sum_{k=1}^{n}\sqrt{h}N_0^i[k]-\Lambda_0^i(t)]\mathbf{e}(f)\|$ goes to 0 as $h\to 0$. By a very similar argument the convergence of (i,j)-th term can be proved. □

4.2 Quantum random walk

Let $\mathcal{A} \subseteq \mathcal{B}(\mathbf{h}_0)$ be a C^* or von Neumann algebra. Let $\{\beta(h) : h > 0\} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ be a family of $*$-homomorphisms. For any $x \in \mathcal{A}$, it can be written as

$$\beta(h,x) = \sum_{\mu,\nu} \beta^\mu_\nu(h,x) \otimes |e_\mu >< e_\nu|,$$

where the components $\beta^\mu_\nu(h)$ are contractive linear maps on $\mathcal{A}$ satisfying

- $\beta^\mu_\nu(h,x^*) = (\beta^\nu_\mu(h,x))^*$,
- $\beta^\mu_\nu(h,xy) = \sum_\xi \beta^\mu_\xi(h,x)\beta^\xi_\nu(h,y)$.

Now for any $h > 0$ and $k \geq 1$ we define a linear map $\rho_k(h)$ by

$$\rho_k(h)(x) = \rho_k(h,x) := \sum_{\mu,\nu} \beta^\mu_\nu(h,x) \otimes N^\mu_\nu[k], \forall x \in \mathcal{A}.$$

It follows from the $*$-homomorphic property of $\beta(h)$ and the relations (4.1. 2) among the basic operators $N^\mu_\nu[k]$'s that the map $\rho_k(h)$ is a $*$-homomorphism from $\mathcal{A}$ to $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_k)$. Here we note that the toy Fock space $\Gamma(S_h)$ is invariant under $\rho_k(h,x)$. Now we consider the family linear maps $p^{(h)}_t : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\Gamma)$, given as follows:

$$\left.\begin{array}{l} p^{(h)}_0(x) = x \otimes 1_\Gamma \\ p^{(h)}_t(x) = p^{(h)}_{nh}(x) = \sum_{\mu,\nu} p^{(h)}_{(n-1)h}(\beta^\mu_\nu(h,x)) \otimes N^\mu_\nu[n], \end{array}\right\} \tag{4.2. 1}$$

for $t \in ((n-1)h, nh]$.
It is clear from the definition that

$$p^{(h)}_{nh}(x) = \rho_1(h) \cdots \rho_n(h)(x)$$

and hence $p^{(h)}_t$ is a $*$-homomorphic family. Clearly, $p^{(h)}_t$ leaves the toy Fock space $\Gamma(S_h)$ invariant. We call this family a *quantum random walk* in short *QRW*.

It is an intersting question that when such a quantum random walk converges as h tends to 0. For any finite dimensional noise space $\mathbf{k}_0$, adapting the proof of strong convergence from [35], under bounded assumption on structure maps $\{\theta^\mu_\nu\}$ and β such that

$$\|\beta^\mu_\nu(h,x) - x\delta^\mu_\nu - h^{\varepsilon_{\mu,\nu}}\theta^\mu_\nu(x)\| \leq Ch^{1+\varepsilon_{\mu,\nu}}\|x\|, \forall x \in \mathcal{A}$$

for some constant C independent of x and h, it can be proved that the quantum random walk $p_t^{(h)}$ constructed above is strongly convergent. In Chapter-5, we shall explore the same question for the UHF model. There, we shall construct quantum random walk associated with a QDS and discuss the convergence issues related to dilation of the QDS.

Let us conclude this Chapter by showing, (in the next section) under suitable assumptions on the $*$-homomorphic family $\{\beta(h) : h > 0\} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$, but noise dimension is not necessarily finite, the associated quantum random walks converges strongly. Thus it follows in particular that the limit j_t is a family of $*$-homomorphism.

4.3 EH flow as a strong limit of Quantum random walk

Here, we shall use coordinate-free language of quantum stochastic calculus to handle infinite dimensional noise. We first recall the basic operators on the toy Fock space in the coordinate-free formalism [15] and then use them to prove the strong convergence of quantum random walks under the assumption of boundedness of the structure maps extending the result in [35]

4.3.1 Coordinate-free basic operators and Quantum random walk

Here we redefine basic operators associated with toy Fock space [3] $\Gamma(S_h)$ using the fundamental processes in coordinate-free language of quantum stochastic calculus, developed in [15] and obtained some estimate as in previous section. For $S \in \mathcal{B}(\mathbf{h}_0), R \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0)$ and $T \in \mathcal{B}(\mathbf{h}_0 \bigotimes \mathbf{k}_0)$ let us define four basic operators as follows, for $k \geq 1$,

$$\begin{aligned}
N_S^1[k] &= SP_0[k] = P_0[k]\frac{\Lambda_S^1[k]}{h}, \\
N_R^2[k] &= \frac{\Lambda_R^2[k]}{\sqrt{h}}P_1[k], \\
N_R^3[k] &= P_1[k]\frac{\Lambda_R^3[k]}{\sqrt{h}}, \\
N_T^4[k] &= P_1[k](\Lambda_T^4[k])P_1[k]P_h[k],
\end{aligned} \tag{4.3. 1}$$

where

$$\begin{aligned}
\Lambda^1_S[k] &= \mathcal{I}_S((k-1)h, kh),\\
\Lambda^2_R[k] &= a_R((k-1)h, kh),\\
\Lambda^3_R[k] &= a^\dagger_R((k-1)h, kh),\\
\Lambda^4_T[k] &= \Lambda_T((k-1)h, kh).
\end{aligned} \tag{4.3. 2}$$

All these maps $\mathcal{B}(\mathbf{h}_0) \ni S \mapsto \Lambda^1_S[k]$,
$\mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0) \ni R \mapsto \Lambda^2_R[k],\ \Lambda^3_R[k]$ and $\mathcal{B}(\mathbf{h}_0 \bigotimes \mathbf{k}_0) \ni T \mapsto \Lambda^4_T[k]$ are linear, and hence the maps $\mathcal{B}(\mathbf{h}_0) \ni S \mapsto N^1_S[k], \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0) \ni R \mapsto N^2_R[k],\ N^3_R[k]$ and $\mathcal{B}(\mathbf{h}_0 \bigotimes \mathbf{k}_0) \ni T \mapsto N^4_T[k]$ are so. It is clear that the subspace $\Gamma(S_h)$ is invariant under all these operators N^l and their action on $\mathbf{h}_0 \otimes \Gamma$: for $u \in \mathbf{h}_0, f \in L^2(\mathbb{R}_+, \mathbf{k}_0)$ are given by

$$\begin{aligned}
N^1_S[k] u\mathbf{e}(f_{[k]}) &= Su \otimes \Omega_{[k]},\\
N^2_R[k] u\mathbf{e}(f_{[k]}) &= \frac{\Lambda^2_R[k]}{\sqrt{h}} u \otimes f_{[k]}\\
&= \frac{1}{\sqrt{h}} \int_{[k]} R^*(uf(s)) ds \Omega_{[k]},\\
N^3_R[k] u\mathbf{e}(f_{[k]}) &= \frac{\Lambda^2_R[k]}{\sqrt{h}} u \otimes \Omega_{[k]}\\
&= \frac{1_{\mathbf{h}_0} \otimes 1_{[k]}}{\sqrt{h}}\ Ru\\
N^4_T[k] u\mathbf{e}(f_{[k]}) &= \Lambda^4_T[k]) P_h[k] f_{[k]}\\
&= (1_{\mathbf{h}_0} \otimes 1_{[k]}) Tu \otimes P_h f(\cdot).
\end{aligned} \tag{4.3. 3}$$

For any $S_1, S_2 \in \mathcal{B}(\mathbf{h}_0), R_1, R_2 \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0)$ and $T_1, T_2 \in \mathcal{B}(\mathbf{h}_0 \bigotimes \mathbf{k}_0)$ we observe the following simple but useful identities, which are easy to derive.

- $(N^2_R[k])^2 = (N^3_R[k])^2 = 0$,
- $N^1_{S_1}[k]\ N^1_{S_2}[k] = N^1_{S_1 S_2}[k]$,
- $N^2_{R_1}[k]\ N^3_{R_2}[k] = N^1_{R_1^* R_2}[k]$,
- $N^1_S[k]\ N^2_R[k] = N^2_{RS^*}[k]$,
- $N^2_R[k]\ N^4_T[k] = N^2_{T^* R}[k]$,

- $N_R^3[k]\ N_S^1[k] = N_{RS}^3[k]$,
- $N_T^4[k]\ N_R^3[k] = N_{TR}^3[k]$,
- $N_{R_1}^3[k]\ N_{R_2}^2[k] = N_{R_1 R_2^*}^4[k]$,
- $N_{T_1}^4[k]\ N_{T_2}^4[k] = N_{T_1 T_2}^4[k]$,
- $N_S^1[k] + N_{S\otimes 1_{\mathbf{k}_0}}^4[k] = S \otimes P_h[k]$

From (4.3. 3) we have

$$\begin{aligned} &\|N_S^1[k]u\mathbf{e}(f_{[k]})\| = \|Su\|, \\ &\|N_R^2[k]u\mathbf{e}(f_{[k]})\| \leq \sqrt{h}\|R\|\ \|u\|\ \|f\|_\infty, \\ &\|N_R^3[k]u\mathbf{e}(f_{[k]})\| \leq \|Ru\| \\ &\|N_T^4[k]u\mathbf{e}(f_{[k]})\| \leq \sqrt{h}\|T\|\|u\|\|f\|. \end{aligned} \tag{4.3. 4}$$

Here we also note the following which can be verified easily using Lemma 1.5.12

$$\begin{aligned} &\|\Lambda_R^3[k]u\mathbf{e}(f_{[k]})\|^2 \\ &= \|(1_{\mathbf{h}_0 \otimes 1_{[k]}})Ru\|^2\|\mathbf{e}(f_{[k]})\|^2 + \|\int_{[k]} R^*(uf(s))ds\|^2\|\mathbf{e}(f_{[k]})\|^2, \\ &\|\Lambda_T^4[k]u\mathbf{e}(f_{[k]})\|^2 \\ &= \int_{[k]} \|Tuf(s)\|^2 ds\|\mathbf{e}(f_{[k]})\|^2 + \|\int_{[k]} \langle f(s), T_{f(s)}\rangle\ ds\ u\mathbf{e}(f_{[k]})\|^2. \end{aligned} \tag{4.3. 5}$$

For the basic operators N^l's we have the following estimates:

Lemma 4.3.1. *(a). For any $k \geq 1$ and $u \in \mathbf{h}_0, f \in \mathcal{M}$,*

1. $\|\{h\ N_S^1[k] - \Lambda_S^1[k]\}u\boldsymbol{e}(f_{[k]})\| \leq h^{\frac{3}{2}}\|Su\|\|f\|_\infty\|\boldsymbol{e}(f_{[k]})\|$,

2. $\|\{\sqrt{h}\ N_R^2[k] - \Lambda_R^2[k]\}u\boldsymbol{e}(f_{[k]})\| \leq h^{\frac{3}{2}}\|R\|\ \|u\|\|f\|_\infty^2\|\boldsymbol{e}(f_{[k]})\|$,

3. $\|\{\sqrt{h}\ N_R^3[k]-\Lambda_R^3[k]\}u\boldsymbol{e}(f_{[k]})\| \leq 2h\|Ru\|\|f\|_\infty\|\boldsymbol{e}(f_{[k]})\|,$

4. $\|\{N_T^4[k]-\Lambda_T^4[k]\}u\boldsymbol{e}(f_{[k]})\| \leq 2h\|T\|(c_f+\|f\|_\infty^2)\|u\boldsymbol{e}(f_{[k]})\|.$

(b). For any $k \geq 1$ *and* $u,v \in \mathbf{h}_0, f,g \in \mathcal{M},$ *we have*

1. $|\langle v\boldsymbol{e}(g_{[k]}), \{h\ N_S^1[k]-\Lambda_S^1[k]\}u\boldsymbol{e}(f_{[k]})\rangle|$

   $\leq h^{\frac{3}{2}}\|Su\|\|f\|_\infty\|\boldsymbol{e}(f_{[k]})\|\|v\boldsymbol{e}(g_{[k]})\|,$

2. $|\langle v\boldsymbol{e}(g_{[k]}), \{\sqrt{h}\ N_R^2[k]-\Lambda_R^2[k]\}u\boldsymbol{e}(f_{[k]})\rangle|$

   $\leq h^{\frac{3}{2}}\|R\|\ \|u\|\|f\|_\infty^2\|g\|_\infty\|\boldsymbol{e}(f_{[k]})\|\|v\boldsymbol{e}(g_{[k]})\|,$

3. $|\langle v\boldsymbol{e}(g_{[k]}), \{\sqrt{h}\ N_R^3[k]-\Lambda_R^3[k]\}u\boldsymbol{e}(f_{[k]})\rangle|$

   $\leq 2h^2\|Ru\|\ \|v\|\ \|f\|_\infty\ \|g\|_\infty\ \|\boldsymbol{e}(f_{[k]})\|^2\ \|\boldsymbol{e}(g_{[k]})\|^2,$

4. $|\langle v\boldsymbol{e}(g_{[k]}), \{N_T^4[k]-\Lambda_T^4[k]\}u\boldsymbol{e}(f_{[k]})\rangle|$

   $\leq h^2\left[(\|f\|_\infty+c_f)\|g\|_\infty\right]^2\|T\|\ \|u\|\ \|v\|\ \|\boldsymbol{e}(f_{[k]})\|^2\ \|\boldsymbol{e}(g_{[k]})\|^2.$

Proof. a.(1) It is clear from the definition that

$$\begin{aligned}\|\{h\ N_S^1[k]-\Lambda_S^1[k]\}u\mathbf{e}(f_{[k]})\| &= h\|Su(\Omega_{[k]}-\mathbf{e}(f_{[k]}))\|\\ &= h\|Su\|\|\Omega_{[k]}-\mathbf{e}(f_{[k]})\|\\ &\leq h^{\frac{3}{2}}\|Su\|\|f\|_\infty\|\mathbf{e}(f_{[k]})\|.\end{aligned}$$

(2) From the definitions, we have

$$\begin{aligned}\|\{\sqrt{h}\ N_R^2[k]-\Lambda_R^2[k]\}u\mathbf{e}(f_{[k]})\| &= \|\int_{[k]}R^*(uf(s))ds\ (\Omega_{[k]}-\mathbf{e}(f_{[k]}))\|\\ &\leq \int_{[k]}\|R^*(uf(s))\|ds\ \|(\Omega_{[k]}-\mathbf{e}(f_{[k]}))\|\\ &\leq h^{\frac{3}{2}}\|R\|\ \|u\|\|f\|_\infty^2\|\mathbf{e}(f_{[k]})\|.\end{aligned}$$

(3) We have

$$\begin{aligned}
&\|\{\sqrt{h}\ N_R^3[k] - \Lambda_R^3[k]\}u\mathbf{e}(f_{[k]})\|^2 \\
&= \ \|(1_{\mathbf{h}_0} \otimes 1_{[k]})\ Ru - \Lambda_R^3[k]u\mathbf{e}(f_{[k]})\|^2 \\
&= \ \|(1_{\mathbf{h}_0} \otimes 1_{[k]})\ Ru\|^2 + \|\Lambda_R^3[k]u\mathbf{e}(f_{[k]})\|^2 \\
&\quad - 2Re\langle(1_{\mathbf{h}_0} \otimes 1_{[k]})\ Ru, \Lambda_R^3[k]u\mathbf{e}(f_{[k]})\rangle
\end{aligned}$$

Now using (4.3. 5) and the definition of Λ_R^3 the above quantity is equal to

$$\begin{aligned}
&\|(1_{\mathbf{h}_0} \otimes 1_{[k]})Ru\|^2 + \|(1_{\mathbf{h}_0} \otimes 1_{[k]})Ru\|^2\|\mathbf{e}(f_{[k]})\|^2 \\
&\quad + \|\int_{[k]} R^*(uf(s))ds\|^2\|\mathbf{e}(f_{[k]})\|^2 - 2\|(1_{\mathbf{h}_0} \otimes 1_{[k]})Ru\|^2 \\
&= \|(1_{\mathbf{h}_0} \otimes 1_{[k]})Ru\|^2[\|\mathbf{e}(f_{[k]})\|^2 - 1] + \|\int_{[k]} R^*(uf(s))ds\|^2\|\mathbf{e}(f_{[k]})\|^2 \\
&\leq 2h^2\|R\|^2\|u\|^2\|f\|_\infty^2\|\mathbf{e}(f_{[k]})\|^2.
\end{aligned}$$

(4) We have

$$\begin{aligned}
&\|\{N_T^4[k] - \Lambda_T^4[k]\}u\mathbf{e}(f_{[k]})\|^2 \\
&= \|(1_{\mathbf{h}_0} \otimes 1_{[k]})T(u \otimes P_h f(\cdot))\|^2 + \|\Lambda_T^4[k]u\mathbf{e}(f_{[k]})\|^2 \\
&\quad - 2Re\langle(1_{\mathbf{h}_0} \otimes 1_{[k]})T(u \otimes P_h f(\cdot)), \Lambda_T^4[k]u\mathbf{e}(f_{[k]})\rangle.
\end{aligned}$$

By the definition of Λ_T^4

$$\begin{aligned}
&\langle(1_{\mathbf{h}_0} \otimes 1_{[k]})T(u \otimes P_h f(\cdot)), \Lambda_T^4[k]u\mathbf{e}(f_{[k]})\rangle \\
&= \langle(1_{\mathbf{h}_0} \otimes 1_{[k]})T(u \otimes P_h f(\cdot)), a^\dagger(T_{f_{[k]}}^{[k]})u\mathbf{e}(f_{[k]})\rangle \\
&= \langle(1_{\mathbf{h}_0} \otimes 1_{[k]})T(u \otimes P_h f(\cdot)), T_{f_{[k]}}^{[k]}(u\Omega_{[k]})\rangle \\
&= \int_{[k]} \langle T(uP_h(f)(s)), T(uf(s))\rangle\ ds.
\end{aligned}$$

Thus using (4.3. 5) we obtained

$$
\begin{aligned}
&\|\{N^4_T[k]-\Lambda^4_T[k]\}u\mathbf{e}(f_{[k]})\|^2\\
&=\int_{[k]}\|T(uP_h(f)(s))\|^2ds\\
&\quad+\int_{[k]}\|T(uf(s))\|^2ds\|\mathbf{e}(f_{[k]})\|^2+\|\int_{[k]}\langle f(s),T_{f(s)}\rangle\ ds\ u\mathbf{e}(f_{[k]})\|^2\\
&\quad-2Re\int_{[k]}\langle T(uP_h(f)(s)),T(uf(s))\rangle ds\\
&=\int_{[k]}\|Tuf(s)\|^2ds(\|\mathbf{e}(f_{[k]})\|^2-1)+\|\int_{[k]}\langle f(s),T_{f(s)}\rangle\ ds\ u\mathbf{e}(f_{[k]})\|^2\\
&\quad+\int_{[k]}\|T(u\otimes(1-P_h)(f)(s))\|^2ds\\
&\leq 2h^2\|T\|^2\|f\|^4_\infty\|u\mathbf{e}(f_{[k]})\|^2+\|T\|^2\|u\|^2\int_{[k]}\|(1-P_h)(f)(s))\|^2ds\\
&\leq 2h^2\|T\|^2\|f\|^4_\infty\|u\mathbf{e}(f_{[k]})\|^2+\|T\|^2\|u\|^2\|(1-P_h)(f_{[k]})\|^2.
\end{aligned}
$$

Since $\|(1-P_h)\mathbf{e}(f_{[k]})\|^2\leq h^2c_f$, the required estimate follows.

(b). The estimates (1) and (2) follow directly from (a).

(3) From the definitions

$$
\begin{aligned}
&\langle v\mathbf{e}(g_{[k]}),\{\sqrt{h}\ N^3_R[k]-\Lambda^3_R[k]\}u\mathbf{e}(f_{[k]})\rangle\\
&\quad=\quad\langle v\mathbf{e}(g_{[k]}),\sqrt{h}\ N^3_R[k]u\mathbf{e}(f_{[k]})\rangle-\langle v\mathbf{e}(g_{[k]}),\Lambda^3_R[k]u\mathbf{e}(f_{[k]})\rangle\\
&\quad=\quad\langle v\mathbf{e}(g_{[k]}),(1_{\mathbf{h}_0}\otimes 1_{[k]})\ Ru\rangle-\langle\Lambda^2_R[k]v\mathbf{e}(g_{[k]}),u\mathbf{e}(f_{[k]})\rangle\\
&\quad=\quad\int_{[k]}\langle Ru,vg(s)\rangle ds(1-\langle\mathbf{e}(g_{[k]}),\mathbf{e}(f_{[k]})\rangle).
\end{aligned}
$$

Thus we have obtained the required estimate,

$$
\begin{aligned}
&|\langle v\mathbf{e}(g_{[k]}),\{\sqrt{h}\ N^3_R[k]-\Lambda^3_R[k]\}u\mathbf{e}(f_{[k]})\rangle|\\
&\quad\leq\quad h^2\|Ru\|\ \|v\|\ \|f\|_\infty\|g\|_\infty\|^2\mathbf{e}(f_{[k]})\|^2\|\mathbf{e}(g_{[k]})\|^2.
\end{aligned}
$$

4. By definition of N^4_T and Λ^4_T

$$
\begin{aligned}
&\langle v\mathbf{e}(g_{[k]}),\{N^4_T[k]-\Lambda^4_T[k]\}u\mathbf{e}(f_{[k]})\rangle\\
&\quad=\quad\langle v\mathbf{e}(g_{[k]}),N^4_T[k]u\mathbf{e}(f_{[k]})\rangle-\langle v\mathbf{e}(g_{[k]}),\Lambda^4_T[k]u\mathbf{e}(f_{[k]})\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle v\mathbf{e}(g_{[k]}), (1_{\mathbf{h}_0} \otimes 1_{[k]})T(uP_h f(\cdot))\rangle - \langle v\mathbf{e}(g_{[k]}), a^\dagger(T^{[k]}_{f_{[k]}})u\mathbf{e}(f_{[k]})\rangle \\
&= \int_{[k]} \langle vg(s), T(u(P_h f)(s))\rangle ds - \int_{[k]} \langle vg(s), T(uf(s))\rangle ds \langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle. \\
&= \int_{[k]} \langle vg(s), T[u\left((P_h - 1)f\right)(s)]\rangle ds \\
&\quad + \int_{[k]} \langle vg(s), T(uf(s))\rangle ds [1 - \langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle].
\end{aligned}
$$

So we get

$$
\begin{aligned}
&|\langle v\mathbf{e}(g_{[k]}), \{N^4_T[k] - \Lambda^4_T[k]\}u\mathbf{e}(f_{[k]})\rangle| \\
&\leq \left(\int_{[k]} \|vg(s)\|^2 ds\right)^{\frac{1}{2}} \left(\int_{[k]} \|T[u((P_h - 1)f_{[k]}(s))]\|^2 ds\right)^{\frac{1}{2}} \\
&\quad + \int_{[k]} \|vg(s)\| \|T(uf(s))\| ds \ \|1 - \langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle\| \\
&\leq h\|v\|\|g\|_\infty \|T\|\|u\|\|(P_h[k] - 1)f_{[k]}\| \\
&\quad + h\|v\|\|g\|_\infty \|T\|\|u\|\|f\|_\infty \ \|1 - \langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle\|.
\end{aligned}
$$

Using the estimates of $\|(P_h[k] - 1)f_{[k]}\|$ and $\|1 - \langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle\|$ the required estimate follows. □

Remark 4.3.2. *The estimates in the above Lemma will also hold if we replace the initial Hilbert space* $\mathbf{h}_0$ *by* $\mathbf{h}_0 \bigotimes \Gamma_{(k-1)h]}$ *and take* $S \in \mathcal{B}(\mathbf{h}_0 \bigotimes \Gamma_{(k-1)h]}), R \in \mathcal{B}(\mathbf{h}_0 \bigotimes \Gamma_{(k-1)h]}, \mathbf{h}_0 \bigotimes \Gamma_{(k-1)h]} \bigotimes \mathbf{k}_0)$ *and* $T \in \mathcal{B}(\mathbf{h}_0 \bigotimes \Gamma_{(k-1)h]} \bigotimes \mathbf{k}_0)$.

Quantum random walk

Let $\mathcal{A} \subseteq \mathcal{B}(\mathbf{h}_0)$ be a von Neumann algebra. Let us consider the Hilbert von Neumann module $\mathcal{A} \bigotimes \mathbf{k}_0$. Suppose we are given with a family of $*$-homomorphisms $\{\beta(h)\}_{h>0}$ from $\mathcal{A}$ to $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$. For $h > 0, \beta(h)$ can be written as $\beta(h,x) = \begin{pmatrix} \beta_1(h,x) & (\beta_2(h,x))^* \\ \beta_3(h,x) & \beta_4(h,x) \end{pmatrix}, \forall x \in \mathcal{A}$, where the components $\beta_l(h)$'s are contractive maps and $\beta_1(h) \in \mathcal{B}(\mathcal{A}), \beta_4(h) \in \mathcal{B}(\mathcal{A}, \mathcal{A} \bigotimes \mathcal{B}(\mathbf{k}_0))$ and $\beta_2(h), \beta_3(h) \in \mathcal{B}(\mathcal{A}, \mathcal{A} \bigotimes \mathbf{k}_0)$. The $*$-homomorphic properties of $\beta(h)$ can be translated into the following properties of $\beta_l(h)$'s.

- $\beta_1(h, x^*) = (\beta_1(h,x))^*$,

- $\beta_4(h, x^*) = (\beta_4(h, x))^*$,
- $\beta_3(h, x^*) = \beta_2(h, x)$,
- $\beta_1(h, xy) = \beta_1(h, x)\beta_1(h, y) + (\beta_2(h, x))^*\beta_3(h, y)$,
- $\beta_2(h, xy) = \beta_1(h, x)(\beta_2(h, y))^* + (\beta_2(h, x))^*\beta_4(h, y)$,
- $\beta_3(h, xy) = \beta_3(h, x)\beta_1(h, y) + \beta_4(h, x)\beta_3(h, y)$,
- $\beta_4(h, xy) = \beta_3(h, x)(\beta_2(h, y))^* + \beta_4(h, x)\beta_4(h, y)$.

We define a family of maps $\mathcal{P}_t^{(h)} : \mathcal{A} \bigotimes \mathcal{E}(\mathcal{K}) \to \mathcal{A} \bigotimes \Gamma$ as follows. We subdivide the interval $[0, t]$ into $[k] \equiv ((k-1)h, kh]$, $1 \le k \le n$ so that $t \in ((n-1)h, nh]$ as earlier and set for $x \in \mathcal{A}$, $f \in \mathcal{K}$

$$\left.\begin{aligned} &\mathcal{P}_0^{(h)}(x\mathbf{e}(f)) = x\mathbf{e}(f) \\ &\mathcal{P}_{kh}^{(h)}(x\mathbf{e}(f)) = \textstyle\sum_{l=1}^4 \mathcal{P}_{(k-1)h}^{(h)} N^l_{\beta_l(h,x)}[k]\mathbf{e}(f) \end{aligned}\right\} \tag{4.3. 6}$$

and $\mathcal{P}_t^{(h)} = \mathcal{P}_{nh}^{(h)}$.
Now setting a family of linear maps $p_t^{(h)} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\Gamma)$, by $p_t^{(h)}(x)u\mathbf{e}(f) := \mathcal{P}_t^{(h)}(x\mathbf{e}(f))u, \forall u \in \mathbf{h}_0$ we have

$$\left.\begin{aligned} &p_0^{(h)}(x)u\mathbf{e}(f) = xu\mathbf{e}(f) \\ &p_t^{(h)}(x)u\mathbf{e}(f) = p_{nh}^{(h)}(x)u\mathbf{e}(f) = \textstyle\sum_{l=1}^4 N^l_{p_{(n-1)h}^{(h)}(\beta_l(h,x))}[n]u\mathbf{e}(f). \end{aligned}\right\} \tag{4.3. 7}$$

As per our convention $p_{(n-1)h}^{(h)}$ appear above are identified with their ampliations $p_{(n-1)h}^{(h)} \otimes 1_{\mathbf{k}_0}$ as well as $p_{(n-1)h}^{(h)} \otimes 1_{\mathcal{B}(\mathbf{k}_0)}$. For $k \ge 1, l = 1, 2, 3$ and 4, $N^l_{p_{(k-1)h}^{(h)}(\beta_l(h,x))}[k]$ are defined in terms of

$$\Lambda^1_{(p_{(k-1)h}^{(h)}(\beta_1(h,x))}[k],\ \Lambda^2_{(p_{(k-1)h}^{(h)} \otimes 1_{\mathbf{k}_0})(\beta_2(h,x))}[k],\ \Lambda^3_{(p_{(k-1)h}^{(h)} \otimes 1_{\mathbf{k}_0})(\beta_3(h,x))}[k]$$

and $\Lambda^4_{(p_{(k-1)h}^{(h)} \otimes 1_{\mathcal{B}(\mathbf{k}_0)})(\beta_4(h,x))}[k]$ where, for example $\Lambda^2_{(p_{(k-1)h}^{(h)} \otimes 1_{\mathbf{k}_0})(\beta_2(h,x))}[k]$ carries the meaning of $a^\dagger_{(p_{(k-1)h}^{(h)} \otimes 1_{\mathbf{k}_0})(\beta_2(h,x))}[k]$ with initial Hilbert space $\mathbf{h}_0 \bigotimes \Gamma_{(k-1)h]}$.

For notational simplicity, for any bounded $*$-preserving map

$$\alpha : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0),\ \alpha(x) = \begin{pmatrix} \alpha_1(h,x) & (\alpha_2(h,x))^* \\ \alpha_3(h,x) & \alpha_4(h,x) \end{pmatrix},$$

we write $N_{\alpha(h,x)}[k]$ for $\sum_{l=1}^4 N^l_{\alpha_l(h,x)}[k]$ and $\Lambda_{\alpha(h,x)}[k]$ for $\sum_{l=1}^4 \Lambda^l_{\alpha_l(h,x)}[k]$. Now for each $k \geq 1$ defining a linear map $\rho_k(h,x) = N_{\beta(h,x)}[k], p^{(h)}_{nh}$ can be written as $p^{(h)}_{nh} = \rho_1(h) \cdots \rho_n(h)$. By the properties of the family $\{\beta_l(h)\}$ and $\{N^l[k]\}$, each $\rho_k(h)$ is a $*$-homomorphism and hence $p^{(h)}_t$ is so.

Lemma 4.3.3. *For any $t \geq 0, t \in ((n-1)h, nh]$ for some $n \geq 1$ and $x \in \mathcal{A}, u \in \mathbf{h}_0$ and $f \in \mathcal{K}$*

$$\mathcal{P}^{(h)}_t(x\boldsymbol{e}(f))u = xu\boldsymbol{e}(f) + \sum_{k=1}^n \mathcal{P}^{(h)}_{(k-1)h} N_{\beta(h,x)-b(x)}[k]\boldsymbol{e}(f)u + F(h,x,u,f), \quad (4.3.\ 8)$$

where $b(x) = \begin{pmatrix} b_1(x) & (b_2(x))^* \\ b_3(x) & b_4(x) \end{pmatrix} = x \otimes 1_{\hat{\mathbf{k}}_0}$ *and*
$F(h,x,u,f) = -\sum_{k=1}^n \mathcal{P}^{(h)}_{(k-1)h}(x(1_\Gamma - P_h[k])\boldsymbol{e}(f))u$. *Moreover, for any $f \in \mathcal{M}$*

$$\|F(h,x,u,f)\|^2 \leq h\ c(f,t)\|x\|^2\ \|u\|^2, \quad (4.3.\ 9)$$

where $c(f,t) = 2t(c_f + \|f\|_\infty)\|\boldsymbol{e}(f)\|$.

Proof. Since for any $k \geq 1$,

$$N_{b(x)}[k] = \sum_{l=1}^4 N^l_{b_l(x)}[k] = N^1_x[k] + N^4_{x\otimes 1_{\mathbf{k}_0}}[k] = x \otimes P_h[k],$$

We get

$$\begin{aligned} \mathcal{P}^{(h)}_t(x\mathbf{e}(f))u &= \mathcal{P}^{(h)}_{nh}(x\mathbf{e}(f))u \\ &= xu\mathbf{e}(f) + \sum_{k=1}^n (\mathcal{P}^{(h)}_{kh} - \mathcal{P}^{(h)}_{(k-1)h})(x\mathbf{e}(f))u \\ &= xu\mathbf{e}(f) + \sum_{k=1}^n \mathcal{P}^{(h)}_{(k-1)h} N_{\beta(h,x)-b(x)}[k]\mathbf{e}(f)u \\ &\quad - \sum_{k=1}^n \mathcal{P}^{(h)}_{(k-1)h}(x \otimes 1_\Gamma - N_{b(x)}[k])\mathbf{e}(f)u \\ &= xu\mathbf{e}(f) + \sum_{k=1}^n \mathcal{P}^{(h)}_{(k-1)h} N_{\beta(h,x)-b(x)}[k]\mathbf{e}(f)u + F(h,x,u,f). \end{aligned}$$

In order to obtained (4.3. 9) let us consider the following. For any $1 \leq m \leq n$ setting $Z_m = \sum_{k=1}^m p^{(h)}_{(k-1)h}(x)(1 - P_h[k])$, we have
$\|Z_m u\mathbf{e}(f_{mh]})\| \leq \sum_{k=1}^m \|p^{(h)}_{(k-1)h}(x)u\mathbf{e}(f_{(k-1)h]})\| \ \|(1 - P_h[k])\mathbf{e}(f_{[k]})\|\|\mathbf{e}(f_{(kh,mh]})\|$.
Now using Lemma 4.1.3(a) and the fact that $p^{(h)}_{kh}$'s are homomorphisms,

$$\begin{aligned}
&\|Z_m u\mathbf{e}(f_{mh]})\| \\
&\leq \sum_{k=1}^m h(c_f + \|f\|_\infty)\|x\|\|u\mathbf{e}(f_{mh]})\| \\
&\leq t(c_f + \|f\|_\infty)\|x\|\|u\mathbf{e}(f_{mh]})\|.
\end{aligned}$$

By (1.5. 4) we have

$$\begin{aligned}
&\|F(h, x, u, f)\|^2 \\
&= \sum_{k=1}^n \|p^{(h)}_{(k-1)h}(x)u\mathbf{e}(f_{(k-1)h]})\|^2\|(1 - P_h[k])\mathbf{e}(f_{[k]})\|^2 \ \|\mathbf{e}(f_{[kh})\|^2 \\
&\quad + 2Re\sum_{k=1}^n \langle Z_{k-1}u\mathbf{e}(f_{(k-1)h]}), p^{(h)}_{(k-1)h}(x)u\mathbf{e}(f_{(k-1)h]})\rangle \\
&\qquad\qquad \langle \mathbf{e}(f_{[k]}), (1 - P_h[k])\mathbf{e}(f_{[k]})\rangle \ \|\mathbf{e}(f_{[kh})\|^2 \\
&\leq \sum_{k=1}^n \|x\|^2\|u\mathbf{e}(f_{(k-1)h]})\|^2\|(1 - P_h[k])\mathbf{e}(f_{[k]})\|^2 \ \|\mathbf{e}(f_{[kh})\|^2 \\
&\quad + 2\sum_{k=1}^n \|Z_{k-1}u\mathbf{e}(f_{(k-1)h]})\| \ \|x\|\|u\mathbf{e}(f_{(k-1)h]})\| \\
&\qquad\qquad \|(1 - P_h[k])\mathbf{e}(f_{[k]})\|^2 \ \|\mathbf{e}(f_{[kh})\|^2.
\end{aligned}$$

Using the uniform bound for $\|Z_{k-1}u\mathbf{e}(f_{(k-1)h]})\|$ and Lemma 4.1.3(a) the required estimate follows. □

By above Lemma and the definition $p^{(h)}_t$ we have

$$\begin{aligned}
\mathcal{P}^{(h)}_t(x\mathbf{e}(f))u = p^{(h)}_t(x)u\mathbf{e}(f) &= xu\mathbf{e}(f) \\
&+ \sum_{k=1}^n N_{p^{(h)}_{(k-1)h}(\beta(h,x)-b(x))}[k]u\mathbf{e}(f) + F(h, x, u, f) \qquad (4.3.\ 10)
\end{aligned}$$

4.3.2 Strong convergence of Quantum random walk: with bounded structure maps

Here, we shall prove the strong convergence of quantum random walk $p^{(h)}_t$ extending the ideas in [35], where the strong convergence was obtained under bounded assumption on structure maps in one dimensional noise situation.

Let T_t be a uniformly continuous conservative QDS on von Neumann algebra $\mathcal{A}$ with the generator $\mathcal{L}$. Then by Theorem 1.5.14 and Lemma 1.5.15 (for detail see [15]):

(i) There exists a Hilbert space $\mathbf{k}_0$ and structure maps $(\mathcal{L}, \delta, \sigma)$ satisfying the hypotheses (S1), (S2) and (S3).

(ii)The map $\Theta = \begin{pmatrix} \theta_1 & (\theta_2(\cdot))^* \\ \theta_3 & \theta_4 \end{pmatrix} = \begin{pmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \sigma \end{pmatrix} : \mathcal{A} \rightarrow \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}_0})$ is a bounded CCP map with the structure (1.5. 18)

$$\theta(x) = V^*(x \otimes 1_{\hat{\mathbf{k}}_0})V + W(x \otimes 1_{\hat{\mathbf{k}}_0}) + (x \otimes 1_{\hat{\mathbf{k}}_0})W^*, \forall x \in \mathcal{A},$$

where $V, W \in \mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0)$, and the estimate (1.5. 19).

(iii) Let $\tau \geq 0$ be fixed. There exists a unique solution J_t of the equation,

$$J_t = id_{\mathcal{A} \bigotimes \Gamma} + \int_0^t J_s \Lambda_\Theta(ds), \; 0 \leq t \leq \tau \tag{4.3. 11}$$

(here we have written $\Lambda_\Theta(ds)$ for $\Lambda^1_{\theta_1}(ds) + \Lambda^2_{\theta_3}(ds) + \Lambda^3_{\theta_3}(ds) + \Lambda^4_{\theta_4}(ds)$)

as a regular adapted process mapping $\mathcal{A} \bigotimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \bigotimes \Gamma$ and satisfies

$$\sup_{0 \leq t \leq \tau} ||J_t(x \otimes \mathbf{e}(f))u|| \leq C'(f)||(x \otimes 1_{\Gamma_{\mathrm{fr}}(L^2([0,\tau],\mathcal{H}))})E_\tau u||,$$

where $f \in \mathcal{C}, E_t \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \Gamma_{\mathrm{fr}}(L^2([0,\tau],\mathcal{H})))$, $C'(f)$ is some constant and $\Gamma_{\mathrm{fr}}(L^2([0,\tau],\mathcal{H}))$ is the free Fock space over $L^2([0,\tau],\mathcal{H})$.

For $m \geq 0$, let us consider the ampliation

$\Theta_{(m)} : \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}) \rightarrow \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}) \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ of the map Θ given by

$$\Theta_{(m)}(X) = Q_m^* \left(\Theta \otimes id_{\mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ})}(X) \right) Q_m \tag{4.3. 12}$$

where $Q_m : \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ} \bigotimes \hat{\mathbf{k}}_0 \rightarrow \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ}$ is the unitary operator which interchanges the second and third tensor components. From the structure (1.5. 18) of the map Θ,

$$\Theta_{(m)}(X) = Q_m^*(V^* \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})Q_m(X \otimes 1_{\hat{\mathbf{k}}_0})Q_m^*(V \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})Q_m$$
$$+Q_m^*(W \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})Q_m(X \otimes 1_{\hat{\mathbf{k}}_0}) + (X \otimes 1_{\hat{\mathbf{k}}_0})Q_m^*(W^* \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})Q_m.$$

For $\xi \in \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ} \bigotimes \hat{\mathbf{k}}_0$,

$$\|\Theta_{(m)}(X)\xi\|^2 \leq 3 \Big[\|V\|^2 \|(X \otimes 1_{\hat{\mathbf{k}}_0}) Q_m^* (V \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) Q_m \xi\|^2$$
$$+ \|W\|^2 \|(X \otimes 1_{\hat{\mathbf{k}}_0})\xi\|^2 + \|(X \otimes 1_{\hat{\mathbf{k}}_0}) Q_m^* (W^* \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) Q_m \xi\|^2 \Big].$$

Setting

$$D_m \xi = \sqrt{3} \Big[\|V\| Q_m^* (V \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) Q_m \xi \oplus \|W\| \xi \oplus Q_m^* (W^* \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) Q_m \xi \Big],$$

$D_m \in \mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ} \bigotimes \hat{\mathbf{k}}_0, \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ} \bigotimes \mathcal{H})$ (where $\mathcal{H} = \hat{\mathbf{k}}_0 \oplus \hat{\mathbf{k}}_0 \oplus \hat{\mathbf{k}}_0$ as earlier) and

$$\|\Theta_{(m)}(X)\xi\| \leq \|(X \otimes 1_{\mathcal{H}}) D_m \xi\|, \forall X \in \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}). \quad (4.3.\ 13)$$

Thus $\|\Theta_{(m)}\| \leq \|D_m\|$, by definition $\|D_m\|^2 \leq 3(\|V\|^4 + \|W\|^2), \forall m \geq 0$ and hence Θ can be extend as a map $\bigoplus_{m \geq 0} \Theta_{(m)}$ from $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\text{fr}}(\hat{\mathbf{k}}_0))$ into itself with $\| \bigoplus_{m \geq 0} \Theta_{(m)}\| \leq 3(\|V\|^2 + \|W\|)$, we denote this map by same symbol Θ.
For any fixed $m \geq 0$ let us look at the following qsde on $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}) \bigotimes \Gamma$

$$\eta_{m,t} = id_{\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}) \bigotimes \Gamma} + \int_0^t \eta_{m,s} \Lambda_\Theta(ds), 0 \leq t \leq \tau. \quad (4.3.\ 14)$$

Since we have the estimate, for any $X \in \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ})$, $\xi \in \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ} \bigotimes \hat{\mathbf{k}}_0$

$$\|\Theta(X)\xi\| = \|\Theta_{(m)}(X)\xi\| \leq \|(X \otimes 1_{\mathcal{H}}) D_m \xi\|,$$

by a simple adaptation of the proof of Theorem 1.5.19, it can be shown that
(i) the qsde (4.3. 14) admit a unique solution $\eta_{m,t}$ as an adapted regular process mapping $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}) \bigotimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}) \bigotimes \Gamma$.
(ii) $\eta_{m,t}$ satisfies the estimate

$$\sup_{0 \leq t \leq \tau} ||\eta_{m,t}(X \otimes \mathbf{e}(f))\xi|| \leq C'(f) ||(X \otimes 1_{\Gamma_{\text{fr}}(L^2([0,\tau],\mathcal{H}))}) E_\tau \xi||, \quad (4.3.\ 15)$$

where $f \in \mathcal{C}$ and $C'(f)$ is some constant. The operator E_τ appears above is an element of
$\mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ}, \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ} \bigotimes \Gamma_{\text{fr}}(L^2([0,\tau],\mathcal{H})))$, define as follows:

$$E_\tau \xi = \bigoplus_{n \geq 0} (n!)^{\frac{1}{4}} E_\tau^{(n)} \xi,$$

where $E_\tau^{(n)} \in \mathcal{B}(\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{\textcircled{m}}, \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{\textcircled{m}} \bigotimes (L^2([0,\tau],\mathcal{H}))^{\textcircled{n}})$ given by, for $\xi \in \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{\textcircled{m}}$

$$\begin{aligned} &E_\tau^{(0)}\xi = \xi, \\ &(E_\tau^{(1)}\xi)(s) = D(\xi \otimes \hat{f}(s)||\hat{f}_{t]}(s)||) \text{ and iteratively} \\ &(E_\tau^{(n)}\xi)(s_1, s_2, \dots s_n) = (D_m \otimes 1_{L^2([0,\tau],\mathcal{H})^{\otimes n-1}})\mathcal{Q}_n \\ &\qquad\qquad \{(E_\tau^{(n-1)}u)(s_2, \dots s_n) \otimes \hat{f}(s_1)||\hat{f}_{t]}(s_1)||\} \end{aligned}$$

($\mathcal{Q}_n : \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{\textcircled{m}} \bigotimes L^2([0,\tau],\mathcal{H})^{\otimes(n-1)} \bigotimes \hat{\mathbf{k}}_0 \to \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{\textcircled{m}} \bigotimes \hat{\mathbf{k}}_0 \bigotimes L^2([0,\tau],\mathcal{H})^{\otimes(n-1)}$ is the unitary operator which interchanges the third and fourth tensor components).

It is clear that $J_t \otimes id_{\mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}})} (\equiv \Upsilon_m^*(J_t \otimes id_{\mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}})})\Upsilon_m : \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}}) \bigotimes \mathcal{E}(\mathcal{C}) \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}}) \bigotimes \Gamma$, where $\Upsilon_m : \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{\textcircled{m}} \bigotimes \Gamma \to \mathbf{h}_0 \bigotimes \Gamma \bigotimes \hat{\mathbf{k}}_0^{\textcircled{m}})$ satisfies the qsde (4.3. 14) and hence $\eta_{m,t} = J_t \otimes id_{\mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}})}$. By definition of E_τ, it can be easily seen that $\|E_\tau\|$ uniformly bounded for $m \geq 0$ and hence the estimate (4.3. 15) allow us to extend $\{J_t\}$ as a regular adapted process $\{\bigoplus_{m\geq 0} J_t \otimes id_{\mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}})}\}$ mapping $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\text{fr}}(\hat{\mathbf{k}}_0)) \bigotimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\text{fr}}(\hat{\mathbf{k}}_0)) \bigotimes \Gamma$, we denote this family by same symbol J_t. For a given $f \in \mathcal{C}$ this J_t satisfies

$$\|J_t(X\mathbf{e}(f)\xi\| \leq D'\|X\|\|\xi\|, \forall X \in \mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\text{fr}}(\hat{\mathbf{k}}_0)) \text{ and } \xi \in \mathbf{h}_0 \bigotimes \Gamma_{\text{fr}}(\hat{\mathbf{k}}_0)), \tag{4.3. 16}$$

for some constant D' independent of X and ξ.

To obtain the $*$-homomorphic property of the family $j_t : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\Gamma)$ $(j_t(x)u\mathbf{e}(f) := J_t(x\mathbf{e}(f))u)$ we shall prove that the $\{j_t\}$ is a strong limit of a family of quantum random walks $\{p_t^{(h)} : h > 0\}$ associated with a family $\beta(h) : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ of $*$-homomorphism under the following assumption on $\{\beta(h) : h \geq 0\}$. Here first we note that since each $\beta(h)$ is a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$, as Θ it can be extend as a bounded map from $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\text{fr}}(\hat{\mathbf{k}}_0))$ into itself.

Assumption:

- A1. The family of linear maps $E(h) : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ given by, for $x \in \mathcal{A}$
$$E(h,x) = \begin{pmatrix} h^{-2}\left[\beta_1(h,x) - x - h\theta_1(x)\right] & h^{-\frac{3}{2}}(\beta_2(h,x) - \sqrt{h}\theta_2(x))^* \\ h^{-\frac{3}{2}}[\beta_3(h,x) - \sqrt{h}\theta_3(x)] & h^{-1}\left[\beta_4(h,x) - x \otimes 1_{\mathbf{k}_0} - \theta_4(x)\right] \end{pmatrix},$$
is uniformly norm bounded, also as maps from $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\text{fr}}(\hat{\mathbf{k}}_0))$ into itself, have uniform norm bound i.e. $\|E(h)\| \leq M$, for some constant M independent of h.

In particular, it follows from assumption (A1) that for any l

$$\|\beta_l(h,X) - b_l(X) - h^{\varepsilon_l}\theta_l(X)\| \leq M\|X\|h^{1+\varepsilon_l}, \forall X \in \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{\textcircled{m}}) \tag{4.3. 17}$$

where $\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = \frac{1}{2}$ and $\varepsilon_4 = 0$.

Here it may be noted that one can find a $*$-homomorphic family $\{\beta(h)\}_{h>0}$ with assumption A1 starting from the generator of a uniformly continuous QDS T_t on a von Neumann algebra $\mathcal{A}$, for example with generator $\mathcal{L}$ satisfying

$$\mathcal{L}(x) = R^*(x \otimes 1_{\mathbf{k}_0})R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R, \forall x \in \mathcal{A} \tag{4.3. 18}$$

for some Hilbert space $\mathbf{k}_0$ and $R \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \bigotimes \mathbf{k}_0)$.

Theorem 4.3.4. *Let $\mathcal{L}$ be given by (4.3. 18). Then there exists a $*$-homomorphic family $\{\beta(h)\}_{h>0}$ with assumption A1.*

Proof. Here the map Θ is given by $\Theta(x) = \begin{pmatrix} \theta_1(x) & (\theta_2(x))^* \\ \theta_3(x) & \theta_4(x) \end{pmatrix} = \begin{pmatrix} \mathcal{L}(x) & \delta^\dagger(x) \\ \delta(x) & \sigma(x) \end{pmatrix}$, $\forall x \in \mathcal{A}$, where $\delta(x) = (x \otimes 1_{\mathbf{k}_0})R - Rx, \delta^\dagger(x) = (\delta(x^*))^* = R^*(x \otimes 1_{\mathbf{k}_0}) - xR^*$ and $\sigma = 0$. Setting $\widetilde{R} = \begin{pmatrix} 0 & -R^* \\ R & 0 \end{pmatrix}$ from $\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0$ to itself. It is clear that $\widetilde{R}$ is a bounded skew symmetric operator thus it generate a one parameter unitary group $\{e^{t\widetilde{R}}\}$. For $h > 0$, we consider the unitary operator $U(h) = e^{\sqrt{h}\widetilde{R}}$ which can be written as $\begin{pmatrix} \cos(\sqrt{h}|R|) & -\sqrt{h}D(h)R^* \\ \sqrt{h}RD(h) & \cos(\sqrt{h}|R^*|) \end{pmatrix}$ where $D(h) = \sin(\sqrt{h}|R|)(\sqrt{h}|R|)^{-1}$ and $|R|, |R^*|$ denote the positive square root of R^*R and RR^* respectively. It can easily be observed that

$$\begin{aligned}
&\|\cos(\sqrt{h}|R|) - 1_{\mathbf{h}_0} + \frac{h}{2}|R|^2\| \leq h^2\|R\|^4, \\
&\|\cos(\sqrt{h}|R|) - 1_{\mathbf{h}_0}\| \leq h\|R\|^2, \\
&\|\cos(\sqrt{h}|R^*|) - 1_{\mathbf{h}_0 \otimes \mathbf{k}_0}\| \leq h\|R\|^2, \\
&\|D(h) - 1_{\mathbf{h}_0}\| \leq h\|R\|^2, \\
&\|\cos(\sqrt{h}|R|)\| \leq 1, \\
&\|D(h)\| \leq 1.
\end{aligned} \tag{4.3. 19}$$

Now we define a $*$-homomorphism $\beta(h)$ from $\mathcal{A}$ to $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ implemented by the unitary $U(h)$, i.e. for $x \in \mathcal{A}, \beta(h,x) := \beta(h)(x) = (U(h))^*(x \otimes 1_{\hat{\mathbf{k}}_0})U(h)$. So for any

$$x \in \mathcal{A}, \beta(h,x) = \begin{pmatrix} \beta_1(h,x) & (\beta_2(h,x))^* \\ \beta_3(h,x) & \beta_4(h,x) \end{pmatrix}$$

$$= \begin{pmatrix} \{\cos(\sqrt{h}|R|)x\cos(\sqrt{h}|R|) & \{-\sqrt{h}\cos(\sqrt{h}|R|)xD(h)R^* \\ +hD(h)R^*(x\otimes 1_{\mathbf{k}_0})RD(h)\} & +\sqrt{h}D(h)R^*(x\otimes 1_{\mathbf{k}_0})\cos(\sqrt{h}|R^*|)\} \\ & \\ \{-\sqrt{h}RD(h)x\cos(\sqrt{h}|R|) & \{hRD(h)xD(h)R^* \\ +\sqrt{h}\cos(\sqrt{h}|R^*|)(x\otimes 1_{\mathbf{k}_0})RD(h)\} & +\cos(\sqrt{h}|R^*|)(x\otimes 1_{\mathbf{k}_0})\cos(\sqrt{h}|R^*|)\} \end{pmatrix}.$$

We have

$$\begin{aligned}
&\beta_1(h,x) - x - h\theta_1(x) \\
&= \cos(\sqrt{h}|R|)x\cos(\sqrt{h}|R|) + hD(h)R^*(x\otimes 1_{\mathbf{k}_0})RD(h) \\
&\quad - x - h\left(R^*(x\otimes 1_{\mathbf{k}_0})R - \frac{1}{2}|R|^2x - \frac{1}{2}x|R|^2\right) \\
&= \left[\cos(\sqrt{h}|R|) - 1_{\mathbf{h}_0} + \frac{1}{2}|R|^2\right] x\cos(\sqrt{h}|R|) \\
&\quad + x\left[\cos(\sqrt{h}|R|) - 1_{\mathbf{h}_0} + \frac{1}{2}|R|^2\right] + \frac{1}{2}|R|^2x\left[1_{\mathbf{h}_0} - \cos(\sqrt{h}|R|)\right] \\
&\quad + h[D(h) - 1_{\mathbf{h}_0}]R^*(x\otimes 1_{\mathbf{k}_0})RD(h) + hR^*(x\otimes 1_{\mathbf{k}_0})RD(h).
\end{aligned}$$

By (4.3. 19) we get

$$\|\beta_1(h,x) - x - h\theta_1(x)\| \leq 5h^2\|R\|^4\|x\|. \tag{4.3. 20}$$

By definition we have

$$\begin{aligned}
&\beta_2(x^*) - \sqrt{h}\theta_2(x^*) = \beta_3(x) - \sqrt{h}\theta_3(x) \\
&= \sqrt{h}\left[-RD(h)x\cos(\sqrt{h}|R|) + \cos(\sqrt{h}|R^*|)(x\otimes 1_{\mathbf{k}_0})RD(h) - (x\otimes 1_{\mathbf{k}_0})R + Rx\right] \\
&= \sqrt{h}\Big[-RD(h)x[\cos(\sqrt{h}|R|) - 1_{\mathbf{h}_0}] - R[D(h) - 1_{\mathbf{h}_0}]x \\
&\quad + \cos(\sqrt{h}|R^*|)(x\otimes 1_{\mathbf{k}_0})R[D(h) - 1_{\mathbf{h}_0}] + [\cos(\sqrt{h}|R^*|) - 1_{\mathbf{h}_0}](x\otimes 1_{\mathbf{k}_0})R\Big].
\end{aligned}$$

Using (4.3. 19) we get

$$\begin{aligned}
&\|\beta_2(x^*) - \sqrt{h}\theta_2(x^*)\| = \|\beta_3(x) - \sqrt{h}\theta_3(x)\| \\
&\leq \sqrt{h}\Big[\|RD(h)x[\cos(\sqrt{h}|R|) - 1_{\mathbf{h}_0}]\| + \|R[D(h) - 1_{\mathbf{h}_0}]x\| \\
&\quad + \|\cos(\sqrt{h}|R^*|)(x\otimes 1_{\mathbf{k}_0})R[D(h) - 1_{\mathbf{h}_0}]\| + \|[\cos(\sqrt{h}|R^*|) - 1_{\mathbf{h}_0\otimes\mathbf{k}_0}](x\otimes 1_{\mathbf{k}_0})R\|\Big] \\
&\leq 4h^{\frac{3}{2}}\|R\|^3\|x\|.
\end{aligned}$$

Now let us consider $\beta_4(x) - \theta_4(x)$, we have

$$\begin{aligned}
&\|\beta_4(x) - \theta_4(x)\| \\
&= \|hRD(h)xD(h)R^* + \cos(\sqrt{h}|R^*|)(x \otimes 1_{\mathbf{k}_0}) \cos(\sqrt{h}|R^*|) - (x \otimes 1_{\mathbf{k}_0})\| \\
&\leq h\|RD(h)xD(h)R^*\| + \|[\cos(\sqrt{h}|R^*|) - 1_{\mathbf{h}_0 \otimes \mathbf{k}_0}](x \otimes 1_{\mathbf{k}_0}) \cos(\sqrt{h}|R^*|)\| \\
&\quad + \|(x \otimes 1_{\mathbf{k}_0})[\cos(\sqrt{h}|R^*|) - 1_{\mathbf{h}_0 \otimes \mathbf{k}_0}]\| \\
&\leq 3h\|R\|^2\|x\|.
\end{aligned}$$

Thus for $l = 1, 2, 3$ and 4,

$$\|\beta_l(h, x) - b_l(x) - h^{\varepsilon_l}\theta_l(x)\| \leq M\|x\|h^{1+\varepsilon_l}, \forall x \in \mathcal{A}, \tag{4.3. 21}$$

where constant $M = 5(\|R\|^2 + \|R\|^3 + \|R\|^4)$.

For $m \geq 0$, let us consider the ampliation of the maps Θ, b and β as maps from $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\mathrm{fr}}(\hat{\mathbf{k}}_0))$ into itself. For $X \in \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ})$

$$\Theta(X) = \Theta_{(m)}(X) = Q_m^* \left(\Theta \otimes id_{\mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ})}(X) \right) Q_m,$$

where $Q_m : \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ} \bigotimes \hat{\mathbf{k}}_0 \to \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ}$ is the unitary operator which interchanges the second and third tensor components. This operator $Q_m = 1_{\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ}} \oplus P_m$ where $q_m : \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ} \bigotimes \mathbf{k}_0 \to \mathbf{h}_0 \bigotimes \mathbf{k}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ}$ is define as Q_m. By definition we have

$$\begin{aligned}
\theta_1(X) &= (R^* \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) q_m (X \otimes 1_{\mathbf{k}_0}) q_m^* (R \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) - \frac{1}{2}(|R| \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})X - \frac{1}{2}X(|R| \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) \\
\theta_2(X)^* &= \left[X(R^* \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) q_m - (R^* \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) q_m (X \otimes 1_{\mathbf{k}_0}) \right] q_m^* \\
\theta_3(X) &= q_m^* \left[(R \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) X - q_m (X \otimes 1_{\mathbf{k}_0}) q_m^* (R \otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) \right] \\
\theta_4(X) &= 0
\end{aligned}$$

and components of $\beta(h, X)$ are

$$
\begin{aligned}
\beta_1(h,X) &= (\cos(\sqrt{h}|R|)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})X(\cos(\sqrt{h}|R|)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) \\
&\quad + h(D(h)R^*\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})q_m(X\otimes 1_{\mathbf{k}_0})q_m^*(RD(h)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) \\
\beta_2(h,X)^* &= \Big[-\sqrt{h}(\cos(\sqrt{h}|R|)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})X(D(h)R^*\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) \\
&\quad +\sqrt{h}(D(h)R^*\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})q_m(X\otimes 1_{\mathbf{k}_0})q_m^*(\cos(\sqrt{h}|R^*|)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})\Big]\, q_m \\
\beta_3(h,X) &= q_m^*\Big[-\sqrt{h}(RD(h)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})X(\cos(\sqrt{h}|R|)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) \\
&\quad +\sqrt{h}(\cos(\sqrt{h}|R^*|)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})q_m(X\otimes 1_{\mathbf{k}_0})q_m^*(RD(h)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})\Big] \\
\beta_4(h,X) &= q_m^*\Big[h(RD(h)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})X(D(h)R^*\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}}) \\
&\quad +(\cos(\sqrt{h}|R^*|)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})q_m(X\otimes 1_{\mathbf{k}_0})q_m^*(\cos(\sqrt{h}|R^*|)\otimes 1_{\hat{\mathbf{k}}_0^{ⓜ}})\Big]\, q_m.
\end{aligned}
$$

By same argument as for (4.3. 21) one has

$$\|\beta_1(h,X)-b_l(X)-h\theta_1(X)\|\leq C'h^{1+\varepsilon_l}\|X\|, \forall X\in\mathcal{A}\bigotimes\mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}),$$

for some constant C' independent of $h>0, m\geq 0$. Thus $\|E(h)(X)\|\leq M\|X\|, \forall X\in \mathcal{A}\bigotimes\mathcal{B}(\Gamma_{\mathrm{fr}}(\hat{\mathbf{k}}_0))$, for some constant M independent of h. □

Let $\mathcal{B}=\mathcal{A}\bigotimes\mathcal{B}(\Gamma_{\mathrm{fr}}(\hat{\mathbf{k}}_0))\bigotimes\mathcal{B}(\Gamma)$, which can be decomposed as $\mathcal{B}=\left(\mathcal{A}\bigotimes\bigoplus_{m\geq 0}\mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ})\right)\bigoplus\mathcal{B}_c$ for some subspace $\mathcal{B}_c$. Now let us consider the extensions of all these maps $\Theta, \beta(h), b, p_t^{(h)}$ and $\mathcal{P}_t^{(h)}$ as bounded linear maps from $\mathcal{B}$ into itself, given by, for example extention of $p_t^{(h)}$ is $\bigoplus_{m\geq 0} p_t^{(h)}\otimes id_{\mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ})}\oplus 0_{\mathcal{B}_c}$. We denote these extentions by same symbols as the original maps. From the assumption A1 it follows that

$$\|\beta_l(h,X)-b_l(X)-h^{\varepsilon_l}\theta_l(X)\|\leq C\|X\|h^{1+\varepsilon_l}, \forall X\in\mathcal{A}\bigotimes\mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}). \qquad (4.3.\ 22)$$

For $h\geq 0$ we define a map $\Theta(h):=\begin{pmatrix} h\theta_1 & h^{\frac{1}{2}}(\theta_2(\cdot))^* \\ h^{\frac{1}{2}}\theta_3 & \theta_4\end{pmatrix}$ from $\mathcal{A}$ to $\mathcal{A}\bigotimes\mathcal{B}(\hat{\mathbf{k}}_0)$, as the map Θ, $\Theta(h)$ also extend a bounded map from $\mathcal{B}$ into itself. Here we have the following observations which will be needed later for proving the convergence of quantum random walk $p_t^{(h)}$.

Lemma 4.3.5. *For any $l, X \in \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ}), \xi \in \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ}$ and $f \in \mathcal{M}$ we have*

1. $\| \sum_{k=1}^n N_{p^{(h)}_{(k-1)h}[\beta(h,X)-b(X)-\Theta(h,X)]}[k]\xi \boldsymbol{e}(f)\| \leq \sqrt{h} C_1(f,t)\|X\|\|\xi\|$

2. $\| \sum_{k=1}^n \left[N_{p^{(h)}_{(k-1)h}(\Theta(h,X))}[k] - \Lambda_{p^{(h)}_{(k-1)h}(\Theta(X))}[k] \right] \xi \boldsymbol{e}(f)\|^2$

   $\leq hC_2(f,t)\|X\|^2\|\xi\|^2,$

3. $\| \sum_{k=1}^n p^{(h)}_{(k-1)h}(X)(1-P_h[k])\xi \boldsymbol{e}(f))\|^2 \leq h\ c(f,t)\|X\|^2\|\xi\|^2.$

where constants $c(f,t)$ is as in Lemma 4.3.3, $C_1(f,t) = t(1+\|f\|_\infty)\|\boldsymbol{e}(f)\|$ and $C_2(f,t) = (1+t)^2(\|f\|_\infty + \|f\|^2_\infty)^2(1+\|\Theta\|)^2\|\boldsymbol{e}(f)\|^2$.

Proof. (1). For any l we have

$$\begin{aligned} &\| \sum_{k=1}^n N^l_{p^{(h)}_{(k-1)h}[\beta_l(h,X)-b_l(X)-h^{\varepsilon_l}\theta_l(X)]}[k]\xi \mathbf{e}(f)\| \\ &\leq \sum_{k=1}^n \|N^l_{p^{(h)}_{(k-1)h}[\beta_l(h,X)-b_l(X)-h^{\varepsilon_l}\theta_l(X)]}[k]\xi_{k-1}\mathbf{e}(f_{[k]})\|\|\mathbf{e}(f_{[kh})\| \end{aligned}$$

where $\xi_{k-1} = \xi \mathbf{e}(f_{(k-1)h]})$ is a vector in the initial Hilbert space $\mathbf{h}_0 \bigotimes \Gamma_{\text{fr}}(\hat{\mathbf{k}}_0) \bigotimes \Gamma_{(k-1)h]}$. For any l, from (4.3. 22) and contractivity of $p^{(h)}_t$, we get

$$\|p^{(h)}_{(k-1)h}[\beta_l(h,X) - b_l(X) - h^{\varepsilon_l}\theta_l(X)]\| \leq Ch^{1+\varepsilon_l}\|X\|,$$

hence by (4.3. 4) the above quantity is dominated by $\sum_{k=1}^n h^{\frac{3}{2}} C(1+\|f\|_\infty)\|X\|\ \|\xi \mathbf{e}(f)\|$ and required estimate follows.
(2). By Lemma 4.3.1 the terms correspond to $l = 1, 2$ can be estimated as,

$$\begin{aligned} &\| \sum_{k=1}^n \left[h^{\varepsilon_l} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] \right] \xi \mathbf{e}(f)\| \\ \leq &\sum_{k=1}^n \| \left[h^{\varepsilon_l} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] \right] \xi_{k-1}\mathbf{e}(f_{[k]})\|\|\mathbf{e}(f_{[kh})\| \\ \leq &\sum_{k=1}^n h^{\frac{3}{2}} \|p^{(h)}_{(k-1)h}(\theta_l(X))\|\|\xi_{k-1}\|(\|f\|_\infty + \|f\|^2_\infty)\|\mathbf{e}(f_{[(k-1)h})\| \\ \leq &(\|f\|_\infty + \|f\|^2_\infty)\|\Theta\| \sum_{k=1}^n h^{\frac{3}{2}}\|X\|\|\xi \mathbf{e}(f)\|. \end{aligned}$$

Thus the required estimate follows. Now consider other two terms correspond to $l = 3$ and 4. Setting for $1 \leq m \leq n$

$$Z_m = \sum_{k=1}^{m} \left[\sqrt{h} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] \right],$$

by Lemma 4.3.1 (a), we have

$$\begin{aligned} &\|Z_m u \mathbf{e}(f_{mh]})\| \\ \leq & \sum_{k=1}^{m} \| \left[\sqrt{h} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] \right] \xi_{(k-1)} \mathbf{e}(f_{[k]}) \| \|\mathbf{e}(f_{(kh,mh]})\| \\ \leq & \sum_{k=1}^{m} h \|p^{(h)}_{(k-1)h}(\theta_l(X)) \xi_{k-1}\| \ \|f\|_\infty \ \|\mathbf{e}(f_{[k]})\| \|\mathbf{e}(f_{(kh,mh]})\|. \end{aligned}$$

Thus

$$\|Z_m u \mathbf{e}(f_{mh]})\| \leq t \|\Theta\| \ \|f\|_\infty \|X\| \|\xi \mathbf{e}(f_{mh]})\|. \tag{4.3. 23}$$

We have the following equality,

$$\begin{aligned} &\|Z_n \xi \mathbf{e}(f)\|^2 \\ = & \sum_{k=1}^{n} \| \left[\sqrt{h} N^l_{(k-1)h^{(h)}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] \right] \xi_{(k-1)} \mathbf{e}(f_{[k]}) \|^2 \mathbf{e}(f_{[kh})\|^2 \\ & +2 \ Re \sum_{k=1}^{n} \langle Z_{k-1} \xi_{k-1} \mathbf{e}(f_{[k]}), \ \left[\sqrt{h} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] \right] \\ & \qquad \xi_{k-1} \mathbf{e}(f_{[k]}) \rangle \|\mathbf{e}(f_{[kh})\|^2. \end{aligned}$$

By the estimate in Lemma 4.3.1,

$$\begin{aligned} &\| \sum_{k=1}^{n} \left[\sqrt{h} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] \right] \xi \mathbf{e}(f) \|^2 \\ & \leq \sum_{k=1}^{n} h^2 \|p^{(h)}_{(k-1)h}(\theta_l(X)) \xi_{k-1}\|^2 \|f\|^2_\infty \|\mathbf{e}(f_{[(k-1)h})\|^2 \\ & \quad +2 \sum_{k=1}^{n} h^2 \|Z_{k-1} \xi_{k-1}\| \|p^{(h)}_{(k-1)h}(\theta_l(X))\| \ \|\xi_{k-1}\| \ \|f\|^2_\infty \|\mathbf{e}(f_{[(k-1)h})\|^2. \end{aligned}$$

Now using (4.3. 23), above quantity is less than or equal to

$$\begin{aligned} & \sum_{k=1}^{n} h^2 \|f\|^2_\infty \ \|X\|^2 \|\xi \mathbf{e}(f)\|^2 \\ & +2 \sum_{k=1}^{n} h^2 t \|\Theta\|^2 \ \|f\|^3_\infty \|X\|^2 \|\xi \mathbf{e}(f)\|^2 \end{aligned}$$

and required estimate follows.

(3.) The proof is same as for estimate (4.3. 9) in Lemma 4.3.3. □

Now we shall prove the strong convergence of the quantum random walks $p_t^{(h)}$. Note that $J_t : \mathcal{A} \bigotimes \mathcal{E}(\mathcal{K}) \to \mathcal{A} \bigotimes \Gamma$ is the unique solution of the qsde

$$J_t = id_{\mathcal{A} \bigotimes \Gamma} + \int_0^t J_s \Lambda_\Theta(ds). \tag{4.3. 24}$$

We define a family of maps $J_t^{(h)}$ by

$$J_0^{(h)}(x\mathbf{e}(f))u = xu\mathbf{e}(f)$$
$$J_t^{(h)}(x\mathbf{e}(f))u = J_{nh}^{(h)}(x\mathbf{e}(f))u = xu\mathbf{e}(f) + \sum_{k=1}^{n} J_{(k-1)h}(\Lambda_{\Theta(x)}[k]\mathbf{e}(f))u$$

for $t \in ((n-1)h, nh]$. Thus by definition

$$J_t^{(h)}(x\mathbf{e}(f))u = J_{nh}^{(h)}(x\mathbf{e}(f))u = xu\mathbf{e}(f) + \sum_{k=1}^{n} \Lambda_{j_{(k-1)h}(\Theta(x))}[k]u\mathbf{e}(f). \tag{4.3. 25}$$

For $u \in \mathbf{h}_0, f \in \mathcal{M}$ the adapted process J_t satisfies

$$J_t(x\mathbf{e}(f))u = xu\mathbf{e}(f) + \int_0^t J_s \Lambda_\Theta(ds)(x\mathbf{e}(f))u$$

and the map $t \mapsto J_t(x\mathbf{e}(f))u$ is continuous. Thus by definition of this integral

$$\lim_{h\to 0} \|J_t(x\mathbf{e}(f))u - J_t^{(h)}(x\mathbf{e}(f))u\| = 0$$

and hence

$$\lim_{h\to 0} \|j_t(x)u\mathbf{e}(f) - j_t^{(h)}(x)u\mathbf{e}(f)\| = 0. \tag{4.3. 26}$$

Under the assumptions A1 we have the following result:

Theorem 4.3.6. *Let $p_t^{(h)}$ be the quantum random walk associated with $\beta(h)$. Then for each $x \in \mathcal{A}$ and $t \geq 0, p_t^{(h)}(x)$ converges strongly to $j_t(x)$. Thus $j_t : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\Gamma)$ is a $*$-homomorphic flow.*

Proof. In order to prove

$$\lim_{h\to 0} \|p_t^{(h)}(x)u\mathbf{e}(f)) - j_t(x)u\mathbf{e}(f)\| = 0, \ \forall u \in \mathbf{h}_0, f \in \mathcal{M}, \tag{4.3. 27}$$

by (4.3. 26) it is sufficient to show that

$$\lim_{h\to 0} \|p_t^{(h)}(x)u\mathbf{e}(f)) - j_t^{(h)}(x)u\mathbf{e}(f)\| = 0 \forall u \in \mathbf{h}_0, f \in \mathcal{M}. \quad (4.3.\ 28)$$

For any fixed $h > 0, f \in \mathcal{M}$ let us define a family of bounded linear maps

$$W_t^{(h)} : \mathcal{A} \to \mathcal{A} \bigotimes \Gamma$$

given by, for $x \in \mathcal{A}$ and $u \in \mathbf{h}_0$,

$$W_t^{(h)}(x)u = p_t^{(h)}(x)u\mathbf{e}(f)) - j_t^{(h)}(x)u\mathbf{e}(f)$$

$$= [\mathcal{P}_t^{(h)}(x\mathbf{e}(f)) - J_t^{(h)}(x\mathbf{e}(f)]u =: Y_t^{(h)}(x\mathbf{e}(f))u.$$

Here, recall that $\{J_t\}$ extend as a regular adapted process mapping $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\rm fr}(\hat{\mathbf{k}}_0)) \bigotimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\rm fr}(\hat{\mathbf{k}}_0)) \bigotimes \Gamma$ and hence for each $X \in \mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\rm fr}(\hat{\mathbf{k}}_0))$ the family $\{j_t(X)\}$ define by $j_t(X)\xi\mathbf{e}(f) = J_t(X \otimes \mathbf{e}(f))\xi, \forall \xi \in \mathbf{h}_0 \bigotimes \Gamma_{\rm fr}(\hat{\mathbf{k}}_0), f \in \mathcal{C}$, is a regular $(\mathbf{h}_0 \bigotimes \Gamma_{\rm fr}, \mathcal{K})$-adapted process. For a given $f \in \mathcal{M} \subseteq \mathcal{C}$ by estimate (4.3. 16), $W_t^{(h)}$ extend as a bounded linear map from $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\rm fr}(\hat{\mathbf{k}}_0))$ into $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\rm fr}(\hat{\mathbf{k}}_0)) \bigotimes \Gamma$.

Viewing $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\rm fr}(\hat{\mathbf{k}}_0))$ and $\mathcal{A} \bigotimes \mathcal{B}(\Gamma_{\rm fr}(\hat{\mathbf{k}}_0)) \bigotimes \Gamma$ as subspaces of $\mathcal{B}$, let us denote by same symbol $W_t^{(h)}$ to the canonical extentions of $W_t^{(h)}$ as linear maps from $\mathcal{B}$ into itself preserving the norm.

In order to prove (4.3. 28) we shall show that $\|W_t^{(h)}\|$ (as maps from $\mathcal{B}$ into itself) converges to 0 as h tends to 0. For any $X \in \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ})$ and $\xi \in \mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{ⓜ}$

by (4.3. 25) and (4.3. 10), we have

$$
\begin{aligned}
& W_t^{(h)}(X)\xi \\
&= \sum_{k=1}^{n} \left[N_{p^{(h)}_{(k-1)h}(\beta(h,X)-b(X))}[k] - \Lambda_{j_{(k-1)h}(\Theta(X))}[k] \right] \xi \mathbf{e}(f) \\
&\qquad - \sum_{k=1}^{n} p^{(h)}_{(k-1)h}(X)(1-P_h[k])\xi\mathbf{e}(f) \\
&= \sum_{k=1}^{n} \Bigg[\left(N_{p^{(h)}_{(k-1)h}(\beta(h,X)-b(X))}[k] - N_{p^{(h)}_{(k-1)h}(\Theta(h,X))}[k] \right) \xi \mathbf{e}(f) \\
&\qquad + \left(N_{p^{(h)}_{(k-1)h}(\Theta(h,X))}[k] - \Lambda_{p^{(h)}_{(k-1)h}(\Theta(X))}[k] \right) \xi \mathbf{e}(f) \\
&\qquad + \left(\Lambda_{p^{(h)}_{(k-1)h}(\Theta(X))}[k] - \Lambda_{j_{(k-1)h}(\Theta(X))}[k] \right) \xi \mathbf{e}(f) \Bigg] \\
&\quad - \sum_{k=1}^{n} p^{(h)}_{(k-1)h}(X)(1-P_h[k])\xi\mathbf{e}(f)
\end{aligned}
$$

Using linearity of $N_{(\cdot)}[k]$ and $\Lambda_{(\cdot)}[k]$,

$$
\begin{aligned}
& \|W_t^{(h)}(X)\xi\|^2 \\
&\le 4 \Bigg(\| \sum_{k=1}^{n} N_{p^{(h)}_{(k-1)h}(\beta(h,X)-b(X)-\Theta(h,X))}[k] \xi_{k-1} \mathbf{e}(f_{[(k-1)h})\|^2 \\
&\quad + \| \sum_{k=1}^{n} \left[N_{p^{(h)}_{(k-1)h}(\Theta(h,X))}[k] - \Lambda_{p^{(h)}_{(k-1)h}(\Theta(X))}[k] \right] \xi_{k-1} \mathbf{e}(f_{[(k-1)h})\|^2 \\
&\quad + \| \sum_{k=1}^{n} p^{(h)}_{(k-1)h}(X)(1-P_h[k])\xi_{k-1}\mathbf{e}(f_{[(k-1)h})\|^2 \\
&\quad + \| \sum_{k=1}^{n} \Lambda_{\left[p^{(h)}_{(k-1)h} - j_{(k-1)h}\right](\Theta(X))}[k] \xi_{k-1} \mathbf{e}(f_{[(k-1)h})\|^2 \Bigg) \\
&= 4(I_1 + I_2 + I_3 + I_4).
\end{aligned}
$$

By Lemma 4.3.5 we have

$$I_1 + I_2 + I_3 \le const(f,t)\|X\|^2\|\xi\|^2 h.$$

Now let us consider the terms in I_4. We have by estimate (1.5. 24) in Proposition

1.5.18

$$\begin{aligned}
&\|\sum_{k=1}^{n}\Lambda_{[p^{(h)}_{(k-1)h}-j_{(k-1)h}](\Theta(X))}[k]\xi\mathbf{e}(f)\|^2\\
&=\|\sum_{k=1}^{n}Y^{(h)}_{(k-1)h}\Lambda_{\Theta}[k]X\mathbf{e}(f)\xi\|^2\\
&=\|\int_0^{nh}Y^{(h)}_s\Lambda_{\Theta}(ds)X\mathbf{e}(f)\xi\|^2\\
&\le 2e^t(1+\|f\|^2_\infty)\int_0^{nh}\|[(Y^{(h)}_s\otimes 1_{\hat{\mathbf{k}}_0})(\Theta(X)_{\hat{f}(s)}\mathbf{e}(f))]\xi\|^2ds.
\end{aligned}$$

It can be easily seen that

$$(Y^{(h)}_s\otimes 1_{\hat{\mathbf{k}}_0})(\Theta(X)_{\hat{f}(s)}\mathbf{e}(f))=[(Y^{(h)}_s\otimes id_{\mathcal{B}(\hat{\mathbf{k}}_0)})(\Theta(X)\mathbf{e}(f))]_{\hat{f}(s)}=[Y^{(h)}_s(\Theta(X)\mathbf{e}(f))]_{\hat{f}(s)},$$

so the above quantity is equal to

$$\begin{aligned}
&2e^t(1+\|f\|^2_\infty)\int_0^{nh}\|[Y^{(h)}_s(\Theta(X)\mathbf{e}(f))]_{\hat{f}(s)}\xi\|^2ds,\\
&=2e^t(1+\|f\|^2_\infty)\int_0^{nh}\|W^{(h)}_s(\Theta(X))\xi\otimes\hat{f}(s)\|^2ds\\
&\le 2e^t(1+\|f\|^2_\infty)^2\sum_{k=1}^{n}h\|W^{(h)}_{(k-1)h}\|^2\|\Theta(X))\|^2\|\xi\|^2\\
&\le c_f\sum_{k=1}^{n}h\|W^{(h)}_{(k-1)h}\|^2\|\Theta\|^2\|X\|^2\|\xi\|^2.
\end{aligned}$$

Combining all the above estimates, we obtained

$$\begin{aligned}
&\|W^{(h)}_t(X)\xi\|^2 \qquad (4.3.\ 29)\\
&\le\ hC\|X\|^2\|\xi\|^2+D\sum_{k=1}^{n}h\|W^{(h)}_{(k-1)h}\|^2\|X\|^2\|\xi\|^2
\end{aligned}$$

for some constant C and D independent of h. For any $X\in\mathcal{B}$ and $\xi\in\mathbf{h}_0\bigotimes\Gamma_{\mathrm{fr}}(\hat{\mathbf{k}}_0)\bigotimes\Gamma$ we can write $X=\bigoplus_{m\ge 0}X_m\oplus X'$ with $X_m\in\mathcal{A}\bigotimes\mathcal{B}(\hat{\mathbf{k}}_0^{ⓜ})$, $X'\in\mathcal{B}_c$ and $\xi=\bigoplus_{m\ge 0}\xi_m\oplus\xi'$ with $\xi_m\in\mathbf{h}_0\bigotimes\hat{\mathbf{k}}_0^{ⓜ}$ and ξ' belong to orthogonal complement of

$\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}_0^{(m)}$ for all $m \geq 0$. Using the estimate (4.3. 29) we have

$$\begin{aligned} &\|W_t^{(h)}(X)\xi\|^2 \\ = &\sum_{m\geq 0} \|W_t^{(h)}(X_m)\xi_m\|^2 \\ \leq &\; hC \sum_{m\geq 0} \|X_m\|^2\|\xi_m\|^2 + D\sum_{k=1}^{n} h\|W_{(k-1)h}^{(h)}\|^2 \sum_{m\geq 0} \|X_m\|^2\|\xi_m\|^2 \\ \leq &\; hC\|X\|^2\|\xi\|^2 + D\sum_{k=1}^{n} h\|W_{(k-1)h}^{(h)}\|^2\|X\|^2\|\xi\|^2 \end{aligned}$$

Taking supremum over all $\xi \in \mathbf{h}_0 \bigotimes \Gamma_{\text{fr}}(\hat{\mathbf{k}}_0) \bigotimes \Gamma, X \in \mathcal{B}$ such that $\|\xi\| \leq 1, \|X\| \leq 1$ we get

$$\|W_t^{(h)}\|^2 = \|W_{nh}^{(h)}\|^2 \leq hC + hD\sum_{k=1}^{n} \|W_{(k-1)h}^{(h)}\|^2. \qquad (4.3.\ 30)$$

By definition $\|W_0^{(h)}\|^2 = 0$ so (4.3. 30) gives $\|W_h^{(h)}\|^2 \leq hc$ and

$$\|W_{2h}^{(h)}\|^2 \leq hc + hD\|W_h^{(h)}\|^2 \leq ch(1+hD).$$

Then by induction it follows that

$$\|W_t^{(h)}\|^2 = \|W_{nh}^{(h)}\|^2 \leq hC(1+hD)^{n-1} \leq hCe^{Dt}$$

and hence

$$\lim_{h\to 0} \|W_t^{(h)}\|^2 = 0, \quad \text{inparticular} \quad \lim_{h\to 0} \|p_t^{(h)}(x)u\mathbf{e}(f)) - j_t^{(h)}(x)u\mathbf{e}(f)\| = 0.$$

Which says that for any $u \in \mathbf{h}_0$ and $f \in \mathcal{M}$, $\{p_t^{(h)}(x)u\mathbf{e}(f)) : h > 0\}$ is Cauchy in $\mathbf{h}_0 \bigotimes \Gamma$. Since $\|p_t^{(h)}(x)\| \leq \|x\|$ and algebraic tensor product $\mathbf{h}_0 \bigotimes \mathcal{E}(\mathcal{M})$ is dense in $\mathbf{h}_0 \bigotimes \Gamma$ it follows that $\{p_t^{(h)}(x)\xi : h > 0\}$ is Cauchy for all $\xi \in \mathbf{h}_0 \bigotimes \Gamma$ and hence for each $x \in \mathcal{A}$, $\{p_t^{(h)}(x)\}$ converges strongly to $j_t(x)$. Thus $j_t : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\Gamma)$ is a contractive $*$-homomorphic flow. □

Remark 4.3.7. *(i) It may be observed that in the above quantum stochastic dilation $\{j_t\}$ of the dynamical semigroup $\{T_t\}$ there is no "Poisson" term since $\theta_4(x) = 0$ for all $x \in \mathcal{A}$. This is only to be expected since the choice of representation of $\mathcal{A}$ is $x \otimes 1_{\mathbf{k}_0}$ for all $x \in \mathcal{A}$. The more general case of dilation using the convergence*

of quantum random walks where the representation is non trivial (and therefore will have non zero "Poisson" component) is being investigated.
(ii) The method of proof employed above does not seem to be amenable to adaptation for a dynamical semigroup with unbounded generator. On the other hand, one has example of the convergence of random walks to diffusion processes (which of course, has unbounded generators) in the classical case. For the handling of these cases, one may have to find different method to replace the proof of Theorem 4.3.6.

Chapter 5

Weak Stochastic Dilation of UHF Dynamics by QRW Model

In Chapter-3, EH flows are constructed for a class of QDS associated with relatively simpler elements of $\mathcal{A}$. There, we have followed the standard iteration method to obtain the EH flow as a solution of the qsde (3.0. 3). In this chapter, we shall discuss the same question in a greater generality and consider a larger class of QDS T_t associated with $r \in \mathcal{A}_{\text{loc}}$. Here, our approach is different form the standard method. Following [23], we adopt the idea of constructing the EH flow as a limit of the associated quantum random walks. Starting from an $r \in \mathcal{A}_{\text{loc}}$, we construct a $*$-homomorphic family $\{\beta(h) : h > 0\} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$ and then prove that the family of quantum random walks $\{p_t^{(h)}\}$ associated with β converges weakly as h tends to 0. From which it follows that the weak limit $\{j_t\}$ is a CP flow and satisfies the required qsde.

5.1 Quantum random walk on UHF Model

For a fixed $r \in \mathcal{A}_{\text{loc}}$, the closure of the associated Lindbladian $(\mathcal{L}, \mathcal{C}^1(\mathcal{A}))$, generates a QDS $\{T_t : t \geq 0\}$ on $\mathcal{A}$. Here $\mathcal{L}$ takes the form:

$$\mathcal{L}(x) = \sum_{k \in \mathbb{Z}^d} \mathcal{L}_k(x), \forall x \in \mathcal{C}^1(\mathcal{A}) \ \text{ with } \mathcal{L}_k(x) = \frac{1}{2}\{[r_k^*, x]r_k + r_k^*[x, r_k]\}, \forall k \in \mathbb{Z}^d.$$

For $n \geq 0$, let $R_{(n)}$ be the bounded operator from $\mathbf{h}_0$ to $\mathbf{h}_0 \bigotimes \mathbf{k}_0$, given by $R_{(n)} = \sum_{|k|=n} r_k \otimes e_k$. Now we define,

$$\delta(x) = \sum_{n \geq 0} \delta_{(n)}(x), \text{ where } \delta_{(n)}(x) = (x \otimes 1_{\mathbf{k}_0}) R_{(n)} - R_{(n)} x = \sum_{|k|=n} \delta_k(x) \otimes e_k$$

and

$$\delta^\dagger(x) = \sum_{n \geq 0} \delta^\dagger_{(n)}(x), \text{ where } \delta^\dagger_{(n)}(x) = R^*_{(n)}(x \otimes 1_{\mathbf{k}_0}) - x R^*_{(n)} = \sum_{|k|=n} \delta^\dagger_k(x).$$

The Lindbladian $\mathcal{L}$ can be written as

$$\mathcal{L} = \sum_{n \geq 0} \mathcal{L}_{(n)}, \text{ where } \mathcal{L}_{(n)}(x) = R^*_{(n)}(x \otimes 1_{\mathbf{k}_0} R_{(n)} - \frac{1}{2} R^*_{(n)} R_{(n)} x - \frac{1}{2} x R^*_{(n)} R_{(n)} = \sum_{|k|=n} \mathcal{L}_k.$$

It is clear that all these maps $\mathcal{L}, \delta$ and $\delta^\dagger$ are well define on $\mathcal{A}_{\text{loc}}$ and for a fixed $x \in \mathcal{A}_{\text{loc}}$ there exists an $n_x \geq 1$ such that

$$\mathcal{L}_{(n)}(x) = \delta_{(n)}(x) = \delta^\dagger_{(n)}(x) = 0, \forall n \geq n_x.$$

Let us recall the family of maps $\{\theta^\mu_\nu\}$ given by

$$\begin{aligned} \theta^\mu_\nu &= \mathcal{L}, \text{ for } (\mu, \nu) = (0, 0), \\ &= \delta_i, \text{ for } (\mu, \nu) = (i, 0), \\ &= \delta^\dagger_j, \text{ for } (\mu, \nu) = (0, j), \\ &= 0, \text{ otherwise.} \end{aligned}$$

We are looking for a solution of the qsde

$$dj_t(x) = \sum_{\mu, \nu \geq 0} j_t(\theta^\mu_\nu(x)) d\Lambda^\mu_\nu(t), \tag{5.1. 1}$$

$$j_0(x) = x \otimes 1_\Gamma, \ \forall x \in \mathcal{A}_{\text{loc}}.$$

In order to construct QRW let us consider the following. For each $n \geq 0$ we define an operator $\widetilde{R}_n = \begin{pmatrix} 0 & -R^*_n \\ R_n & 0 \end{pmatrix}$ from $\mathbf{h}_0 \bigotimes \hat{\mathbf{k}}$ to itself. It is clear that $\widetilde{R}_n$ is a skew symmetric bounded operator and hence is the generator of a one parameter unitary group

$\{e^{t\widetilde{R}_n} : t \in \mathbb{R}\}$. For $h > 0$, let us consider the unitary operator $U_n(h) := e^{\sqrt{h}\widetilde{R}_n}$ which can be written as $\begin{pmatrix} \cos(\sqrt{h}|R_n|) & -\sqrt{h}D_n(h)R_n^* \\ \sqrt{h}R_nD_n(h) & \cos(\sqrt{h}|R_n^*|) \end{pmatrix}$ where $|R_n|, |R_n^*|$ are the square root of $R_n^*R_n$, $R_nR_n^*$ respectively and $D_n(h)$ is the self adjoint element $\sin(\sqrt{h}|R_n|)(\sqrt{h}|R_n|)^{-1} \in \mathcal{A}$. We can rewrite $U_n(h)$ as
$U_n(h) = \begin{pmatrix} 1 - \frac{h}{2}|R_n|^2 + h^2E_n(h) & -\sqrt{h}R_n^* + h^{\frac{3}{2}}F_n(h) \\ \sqrt{h}R_n + h^{\frac{3}{2}}G_n(h) & 1 + hH_n(h) \end{pmatrix}$, where $E_n(h), F_n(h), G_n(h)$ and $H_n(h)$ are bounded operators given by,

- $E_n(h) = \frac{\cos(\sqrt{h}|R_n|) - 1 + \frac{h}{2}|R_n|^2}{h^2}$

- $H_n(h) = \frac{\cos(\sqrt{h}|R_n^*|) - 1}{h}$

- $G_n(h) = -(F_n(h))^* = R_n\frac{D_n(h) - 1}{h}$.

By 4.3. 19) all these operators are uniformly bounded in h (but not in n). Now we have the following.

Lemma 5.1.1. *For each $n \geq 0$, $U_n(h) \cdots U_1(h)U_0(h)$*
$$= \begin{pmatrix} 1 - \frac{h}{2}\sum_{k=0}^{n}|R_k|^2 + h^2E^{(n)}(h) & -\sqrt{h}\sum_{k=0}^{n}R_k^* + h^{\frac{3}{2}}F^{(n)}(h) \\ \sqrt{h}\sum_{k=0}^{n}R_k + h^{\frac{3}{2}}G^{(n)}(h) & 1 + hH^{(n)}(h) \end{pmatrix},$$
where $E^{(n)}(h), F^{(n)}(h), G^{(n)}(h)$ and $H^{(n)}(h)$ are bounded operators with uniform norm bound in h.

Proof. We prove the result by induction. For $n = 0$, the statement is valid. Now suppose that it is true for some $n \geq 1$. We have $U_{(n+1)}(h)U_n(h) \cdots U_1(h)U_0(h)$
$$= U_{(n+1)}(h)\begin{pmatrix} 1 - \frac{h}{2}\sum_{k=0}^{n}|R_k|^2 + h^2E^{(n)}(h) & -\sqrt{h}\sum_{k=0}^{n}R_k^* + h^{\frac{3}{2}}F^{(n)}(h) \\ \sqrt{h}\sum_{k=0}^{n}R_k + h^{\frac{3}{2}}G^{(n)}(h) & 1 + hH^{(n)}(h) \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \frac{h}{2}\sum_{k=0}^{n+1}|R_k|^2 + h^2E^{(n+1)}(h) & -\sqrt{h}\sum_{k=0}^{n+1}R_k^* + h^{\frac{3}{2}}F^{(n+1)}(h) \\ \sqrt{h}\sum_{k=0}^{n+1}R_k + h^{\frac{3}{2}}G^{(n+1)}(h) & 1 + hH^{(n+1)}(h) \end{pmatrix},$$ where

$E^{(n+1)}(h)$, $F^{(n+1)}(h)$, $G^{(n+1)}(h)$ and $H^{(n+1)}(h)$ are bounded operators, given by:

$$\begin{aligned}
&E^{(n+1)}(h)\\
= \quad & E^{(n)}(h) + E_{n+1}(h) + h^2 E_{(n+1)}(h)E^{(n)}(h) - \frac{h}{2}E_{n+1}(h)\sum_{k=0}^{n}|R_k|^2\\
&-\frac{h}{2}|R_{n+1}|^2E^{(n)}(h) + \frac{1}{4}|R_{n+1}|^2\sum_{k=0}^{n}|R_k|^2 - R_{n+1}^*G^{(n)}(h)\\
&+F_{n+1}(h)\sum_{k=0}^{n}R_k + hF_{n+1}(h)G^{(n)}(h),
\end{aligned}$$

$$\begin{aligned}
&F^{(n+1)}(h)\\
= \quad & F^{(n)}(h) + \frac{1}{2}|R_{n+1}|^2\sum_{k=0}^{n}R_k^* - \frac{h^2}{2}|R_{n+1}|^2F^{(n)}\\
&-h^2E_{n+1}(h)\sum_{k=0}^{n}R_k^* + h^2E_{n+1}(h)F^{(n)}(h) - R_{n+1}^*H^{(n)}(h)\\
&+F^{(n+1)}(h) + hF^{(n+1)}(h)H^{(n)}(h),
\end{aligned}$$

$$\begin{aligned}
&G^{(n+1)}(h)\\
= \quad & -\frac{1}{2}R_{n+1}\sum_{k=0}^{n}|R_k|^2 + hR_{n+1}E^{(n)}(h) + G_{n+1}(h) - hG_{n+1}(h)\sum_{k=0}^{n}|R_k|^2\\
&+h^2G_{n+1}(h)E^{(n)}(h) + G^{(n)}(h) + H_{n+1}(h)\sum_{k=0}^{n}R_k + hH_{n+1}(h)G^{(n)}(h),
\end{aligned}$$

and

$$\begin{aligned}
&H^{(n+1)}(h)\\
= \quad & -R_{n+1}\sum_{k=0}^{n}R_k^* + hR_{n+1}F^{(n)}(h)\\
&-hG_{n+1}(h)\sum_{k=0}^{n}R_k^* + h^2G_{n+1}(h)F^{(n)}(h)\\
&+H_{n+1}(h) + H^{(n)}(h) + hH_{n+1}(h)H^{(n)}(h).
\end{aligned}$$

This enable us to obtain the required identity for $n+1$, thereby completing the proof. $\square$

Now we shall prove the following result which will be needed for proving the weak convergence of the QRW $p_t^{(h)}$.

Proposition 5.1.2. *There exists a family of* $*$*-homomorphism* $\beta(h) : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$, *satisfying, for any* $\mu, \nu \in \mathbb{Z}^d \cup \{0\}$

$$\|\beta_\nu^\mu(h,x) - \delta_\nu^\mu x - h^{\epsilon_{\mu,\nu}} \theta_\nu^\mu(x)\| \le C_x h^{1+\epsilon_{\mu,\nu}}, \forall x \in \mathcal{A}_{\text{loc}}. \tag{5.1. 2}$$

Proof. For any $n \ge 0$ we consider the $*$-homomorphism $\beta^{(n)}(h) : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$, implemented by the unitary $U_n(h) \cdots U_0(h)$, i.e. for $x \in \mathcal{A}$,

$$\beta^{(n)}(h,x) = (U_0(h))^* \cdots (U_n(h))^* (x \otimes 1_{\hat{\mathbf{k}}_0}) U_n(h) \cdots U_0(h).$$

A simple computation shows that

$$\beta^{(n)}(h,x) = \begin{pmatrix} x + h\sum_{k=0}^n \mathcal{L}_{(k)}(x) + h^2 A_n(h,x) & \sqrt{h}\sum_{k=0}^n \delta_{(k)}^\dagger(x) + h^{\frac{3}{2}} B_n(h,x) \\ \sqrt{h}\sum_{k=0}^n \delta_{(k)}(x) + h^{\frac{3}{2}} C_n(h,x) & x \otimes 1_{\mathbf{k}_0} + hD_n(h,x) \end{pmatrix}. \tag{5.1. 3}$$

Here $A_n(h,.), B_n(h,.), C_n(h,.)$ and $D_n(h,.)$ are bounded maps, given by

$$\begin{aligned} A_n(h,x) = {} & xE^{(n)}(h) + \frac{1}{4}\sum_{k=0}^n |R_k|^2 x \sum_{k=0}^n |R_k|^2 \\ & + E^{(n)}(h)x + h^2 E^{(n)}(h) x E^{(n)}(h) - \frac{h}{2}\sum_{k=0}^n |R_k|^2 x E^{(n)}(h) \\ & - \frac{h}{2} E^{(n)}(h) x \sum_{k=0}^n |R_k|^2 + (G^{(n)}(h))^* (x \otimes 1_{\mathbf{k}_0}) \sum_{k=0}^n R_k \\ & + \sum_{k=0}^n R_k (x \otimes 1_{\mathbf{k}_0}) G^{(n)}(h) + h(G^{(n)}(h))^* (x \otimes 1_{\mathbf{k}_0}) G^{(n)}(h), \end{aligned}$$

$$\begin{aligned}
B_n(h,x) &= xF^{(n)}(h) + \frac{1}{2}\sum_{k=0}^{n}|R_k|^2 x \sum_{k=0}^{n} R_k^* \\
&-\frac{h^2}{2}\sum_{k=0}^{n}|R_k|^2 x F^{(n)}(h) - hE^{(n)}(h)x\sum_{k=0}^{n} R_k^* \\
&+h^2E^{(n)}(h)xF^{(n)}(h) + (G^{(n)}(h))^*(x\otimes 1_{\mathbf{k}_0}) + \sum_{k=0}^{n} R_k^*(x\otimes 1_{\mathbf{k}_0})H^{(n)}(h) \\
&+h(G^{(n)}(h))^*(x\otimes 1_{\mathbf{k}_0})H^{(n)}(h),
\end{aligned}$$

$$\begin{aligned}
C_n(h,x) &= \sum_{k=0}^{n} R_k x \sum_{k=0}^{n}|R_k|^2 - h\sum_{k=0}^{n} R_k x E^{(n)}(h) \\
&+(F^{(n)}(h))^* x - \frac{h}{2}(F^{(n)}(h))^* x \sum_{k=0}^{n}|R_k|^2 \\
&+h^2(F^{(n)}(h))^* x E^{(n)}(h) + (x\otimes 1_{\mathbf{k}_0})G^{(n)}(h) \\
&+H^{(n)}(h)(x\otimes 1_{\mathbf{k}_0})\sum_{k=0}^{n} R_k + H^{(n)}(h)(x\otimes 1_{\mathbf{k}_0})G^{(n)}(h)
\end{aligned}$$

and

$$\begin{aligned}
D_n(h,x) &= \sum_{k=0}^{n} R_k x \sum_{k=0}^{n} R_k^* - h\sum_{k=0}^{n} R_k x F^{(n)}(h) \\
&+(F^{(n)}(h))^* x \sum_{k=0}^{n} R_k^* + h^2 H^{(n)}(h)(x\otimes 1_{\mathbf{k}_0}) \\
&+(x\otimes 1_{\mathbf{k}_0})H^{(n)}(h + H^{(n)}(h)(x\otimes 1_{\mathbf{k}_0})H^{(n)}(h).
\end{aligned}$$

In order to define a $*$-homomorphism $\beta(h) : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$, we first note that for any fixed $x \in \mathcal{A}_{\text{loc}}$, there exists an integer n_x such that

$$(U_{(n)}(h))^*(x\otimes 1_{\hat{\mathbf{k}}_0})U_{(n)}(h) = x\otimes 1_{\hat{\mathbf{k}}_0},$$

so $\beta^{(m)}(h,x) = \beta^{(n)}(h,x), \forall m,n \geq n_x$. Now setting $\beta(h)(x) \equiv \beta(h,x) := \beta^{(n_x)}(h,x)$, we get a $*$-homomorphism $\beta(h)$ from $\mathcal{A}_{\text{loc}}$ into $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$. Since $\mathcal{A}_{\text{loc}}$ is norm dense in $\mathcal{A}$, the map $\beta(h)$ extends as a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{A} \bigotimes \mathcal{B}(\hat{\mathbf{k}}_0)$. By 5.1. 3 for any $x \in \mathcal{A}_{\text{loc}}$ we have,

$$\beta(h,x) = \left(\begin{array}{cc} x + h\mathcal{L}(x) + h^2 A_{n_x}(h,x) & \sqrt{h}\delta^\dagger(x) + h^{\frac{3}{2}} B_{n_x}(h,x) \\ \sqrt{h}\delta(x) + h^{\frac{3}{2}} C_{n_x}(h,x) & x \otimes 1_{\mathbf{k}} + hD_{n_x}(h,x) \end{array} \right)$$

where $A_{n_x}(h,.)$, $B_{n_x}(h,.)$, $C_{n_x}(h,.)$ and $D_{n_x}(h,.)$ are bounded maps with uniform norm bound in h. Thus we obtain the required estimates 5.1. 2.

□

Now we define QRW

$$p_t^{(h)} : \mathcal{A} \to \mathcal{A} \bigotimes \mathcal{B}(\Gamma)$$

associated with $\beta(h)$ as in previous Chapter. First subdivide the interval $[0,t]$ into $[k] \equiv ((k-1)h, kh]$, $1 \leq k \leq n$ so that $t \in ((n-1)h, nh]$ and set

$$\left. \begin{array}{l} p_0^{(h)}(x) = x \otimes 1, \\ p_{kh}^{(h)}(x) = \sum_{\mu,\nu} p_{(k-1)h}^{(h)}(\beta_\nu^\mu(h,x)) \otimes N_\nu^\mu[k] \end{array} \right\} \quad (5.1.\ 4)$$

and $p_t^{(h)} = p_{nh}^{(h)}$.

5.2 Weak convergence of the QRW

Here we shall discuss the weak convergence of QRW constructed in the previous section. Let $\mathcal{S}$ be the collection of all simple function $f \in L^2(\mathbb{R}_+, \mathbf{k}_0))$ such that

$$f = \sum_{q=1}^{n} 1_{[a_q,b_q]} \otimes e_{k_q}$$

for some $n \geq 1$ and partition $(0 \leq a_1 < b_1 < a_2 < b_2 \cdots a_n < b_n < \infty)$ of $\mathbb{R}_+$. It is clear that $\mathcal{S}$ is total in $L^2(\mathbb{R}_+, \mathbf{k}_0)$ and hence $\mathbf{h}_0 \bigotimes \mathcal{E}(\mathcal{S})$ is dense in $\mathbf{h}_0 \bigotimes \Gamma$.
We have the following approximation result.

Theorem 5.2.1. *Let $p_t^{(h)}$ be the QRW associated with $\beta(h)$. Then*
(i). For each $x \in \mathcal{A}_{\text{loc}}$

$$\lim_{h \to 0} \langle \xi_1, p_t^{(h)}(x) \xi_2 \rangle \text{ exists}, \ \forall \xi_1, \xi_2 \in \mathbf{h}_0 \bigotimes \mathcal{E}(\mathcal{S}).$$

(ii). For $x \in \mathcal{A}_{\text{loc}}$ setting a map $j_t(x)$ by

$$\langle \xi_1, j_t(x)\xi_2 \rangle := \lim_{h \to 0} \langle \xi_1, p_t^{(h)}(x)\xi_2 \rangle, \forall \xi_1, \xi_2 \in \mathbf{h}_0 \bigotimes \mathcal{E}(\mathcal{S}),$$

$j_t(x)$ extends uniquely as a bounded operator from $\mathbf{h}_0 \bigotimes \Gamma$ to itself. For each $t \geq 0$ the map j_t extends to a unique bounded CP map from $\mathcal{A}$ to $\mathcal{A}'' \bigotimes \mathcal{B}(\Gamma)$ satisfying $\|j_t(x)\| \leq \|x\|, \forall x \in \mathcal{A}$.

(iii). The CP flow j_t satisfies the required qsde 5.1. 1.

Proof. Note that except the generator θ_0^0 of the contractive QDS P_t all other structure maps θ_ν^μ's are bounded and in particular $\theta_j^i = 0, \forall i, j \in \mathbb{Z}^d$. Thus by Theorem 1.3.3 the closure of each of the operators $\theta_0^0 + \theta_j^0, \theta_0^0 + \theta_0^i$ and $\theta_0^0 + \theta_j^0 + \theta_0^i + \theta_j^i$ generates a C_0-semigroup of contraction on $\mathcal{A}$. In fact, for any locally bounded functions $f, g \in \mathcal{K}$ by Theorem 1.3.6 the evolution equation

$$\frac{dT_t^{f,g}(x)}{dt} = T_t^{f,g}(\sum_{\mu,\nu=0}^{q} \bar{f}_\mu(t) g_\nu(t) \theta_\nu^\mu(x)), T_0^{f,g}(x) = e^{\langle f,g \rangle} x,$$

admits a unique solution.

(i). Let $\{T_t^{\mu,\nu} : \mu, \nu \geq 0\}$, be the family of contractive C_0-semigroup on $\mathcal{A}$ given by

$$\begin{aligned} T_t^{\mu,\nu} = T_t = e^{t\theta_0^0}, & \text{ for } (\mu,\nu) = (0,0) \\ = e^{t(\theta_0^0 + \theta_j^0)}, & \text{ for } (\mu,\nu) = (0,j) \\ = e^{t(\theta_0^0 + \theta_0^i)}, & \text{ for } (\mu,\nu) = (i,0) \\ = e^{t(\theta_0^0 + \theta_j^0 + \theta_0^i + \theta_j^i)}, & \text{ for } (\mu,\nu) = (i,j). \end{aligned}$$

Let $\tau \geq 0$ be fixed, $0 \leq t \leq \tau$ and $f, g \in L^2(\mathbb{R}_+, \mathbf{k}_0)$. For $x \in \mathcal{A}$ and $u, v \in \mathbf{h}_0$ we have

$$\begin{aligned} & \langle u \otimes \mathbf{e}(f), p_t^{(h)}(x)\ v \otimes \mathbf{e}(g) \rangle \qquad (5.2.\ 1) \\ = & \langle u \otimes \mathbf{e}(f), \rho_1(h) \cdots \rho_n(h,x)\ v \otimes \mathbf{e}(g) \rangle \\ = & \langle u, A_1 \cdots A_n(x) v \rangle \langle \mathbf{e}(f_{[nh}), \mathbf{e}(g_{[nh})\rangle, \end{aligned}$$

where A_k's are bounded linear maps from $\mathcal{A}$ to itself given by,

$$A_k(x) = \sum_{\mu,\nu \geq 0} \langle \mathbf{e}(f_{[k]}), N_\nu^\mu[k] \mathbf{e}(g_{[k]}) \rangle \beta_\nu^\mu(h,x).$$

For $f = g = 0$ 5.2. 1 gives

$$\langle u \otimes \Omega, p_t^{(h)}(x)\ v \otimes \Omega \rangle = \langle u \otimes, (\beta_0^0(h))^n(x) v \rangle.$$

Since we have by (5.1. 2)

$$\lim_{h\to 0}\frac{\beta_0^0(h,x)-x}{h}=\theta_0^0(x),\forall x\in\mathcal{A}_{\rm loc},$$

theorem (1.3.5) gives,

$$\lim_{h\to 0}(\beta_0^0(h))^n(x)=T_t(x)\forall x\in\mathcal{A}.$$

Thus we obtain

$$\lim_{h\to 0}\langle u\otimes\Omega,p_t^{(h)}(x)\ v\otimes\Omega\rangle=\langle u,T_t(x)v\rangle,\forall x\in\mathcal{A}.$$

Now for $f=1_{[0,\tau]}\otimes e_i, g=1_{[0,\tau]}\otimes e_j\in\mathcal{S}$ and $u,v\in\mathbf{h}_0$, we have,

$$\langle u\otimes\mathbf{e}(f),p_t^{(h)}(x)\ v\otimes\mathbf{e}(g)\rangle$$

$$=\langle u,A_1\cdots A_n(x)v\rangle\langle\mathbf{e}(f_{[nh}),\mathbf{e}(g_{[nh})\rangle,$$

where

$$\begin{aligned}A_k(x)=&\sum_{\mu,\nu\geq 0}\langle\mathbf{e}(f_{[k]}),N_\nu^\mu[k]\mathbf{e}(g_{[k]})\rangle\beta_\nu^\mu(h,x)\\
=&\quad\langle\Omega_{[k]}+1_{[k]}\otimes e_i,\Omega_{[k]}\rangle\beta_0^0(h,x)\\
&\quad+\langle\Omega_{[k]}+1_{[k]}\otimes e_i,\sqrt{h}\Omega_{[k]}\rangle\beta_j^0(h,x)\\
&\quad+\sum_{l\geq 1}\langle\Omega_{[k]}+1_{[k]}\otimes e_i,\frac{1_{[k]}\otimes e_l}{\sqrt{h}}\rangle\beta_0^l(h,x)\\
&\quad+\sum_{l\geq 1}\langle\Omega_{[k]}+1_{[k]}\otimes e_i,1_{[k]}\otimes e_l\rangle\beta_j^l(h,x)\\
=&\quad\beta_0^0(h,x)+\sqrt{h}\beta_j^0(h,x)+\sqrt{h}\beta_0^i(h,x)+h\beta_j^i(h,x).\end{aligned}$$

So we get

$$\begin{aligned}&\langle u\otimes\mathbf{e}(f),p_t^{(h)}(x)\ v\otimes\mathbf{e}(g)\rangle\\
&=\langle u,(\beta_0^0(h)+\sqrt{h}\beta_j^0(h)+\sqrt{h}\beta_0^i(h)+h\beta_j^i(h))^n(x)v\rangle\langle\mathbf{e}(f_{[nh}),\mathbf{e}(g_{[nh})\rangle.\end{aligned}$$

Since $p_t^{(h)}$ is a homomorphic family, we have

$$\|(\beta_0^0(h)+\sqrt{h}\beta_j^0(h)+\sqrt{h}\beta_0^i(h)+h\beta_j^i(h))^n\|\leq e^{\langle f,g\rangle}.$$

It follows from 5.1. 2 that for all $x \in \mathcal{A}_{\text{loc}}$,

$$\lim_{h\to 0} \|\frac{1}{h}[\beta_0^0(h) + \sqrt{h}\beta_j^0(h) + \sqrt{h}\beta_0^i(h) + h\beta_j^i(h) - 1]x - [\theta_0^0 + \theta_j^0 + \theta_0^i + \theta_j^i + \delta_j^i 1]x\| = 0$$

hence by theorem (1.3.5),

$$\lim_{h\to 0}(\beta_0^0(h) + \sqrt{h}\beta_j^0(h) + \sqrt{h}\beta_0^i(h) + h\beta_j^i(h))^n(x) = e^{t\delta_j^i}T_t^{i,j}(x) = e^{\langle f,g\rangle}T_t(x), \forall x \in \mathcal{A}.$$

Thus

$$\lim_{h\to 0}\langle u \otimes \mathbf{e}(f), p_t^{(h)}(x)\ v \otimes \mathbf{e}(g)\rangle = \langle u, e^{\langle f,g\rangle}T_t(x)v\rangle. \tag{5.2. 2}$$

Similarly, one can see (5.2. 2) for the cases $f \equiv 0, g = 1_{[0,\tau]} \otimes e_j$
and $f = 1_{[0,\tau]} \otimes e_i, g \equiv 0$.
Now for $f = (1_{[0,s]} \otimes e_{i_1}) \oplus (1_{[s,\tau]} \otimes e_{i_2})$ and $g = (1_{[0,s]} \otimes e_{j_1}) \oplus (1_{[s,\tau]} \otimes e_{j_2})$ such that $s \le t \le \tau$ let $n_1 = [\frac{s}{h}]$ and $n_2 = [\frac{t-s}{h}]$, let us compute the inner product,

$$\begin{aligned}&\langle u \otimes \mathbf{e}(f), p_t^{(h)}(x)\ v \otimes \mathbf{e}(g)\rangle \\ &= \langle u, (\beta_0^0(h) + \sqrt{h}\beta_{j_1}^0(h) + \sqrt{h}\beta_0^{i_1}(h) + h\beta_{j_1}^{i_1}(h))^{n_1}(\beta_0^0(h) \\ &\quad + \sqrt{h}\beta_{j_2}^0(h) + \sqrt{h}\beta_0^{i_2}(h) + h\beta_{j_2}^{i_2}(h))^{n_2}(x)v\rangle.\end{aligned}$$

Since $\|(\beta_0^0(h) + \sqrt{h}\beta_{j_1}^0(h) + \sqrt{h}\beta_0^{i_1}(h) + h\beta_{j_1}^{i_1}(h))^{n_1}\| \le |e^{\langle f,g\rangle}|$ we get

$$\lim_{h\to 0}\langle u \otimes \mathbf{e}(f), p_t^{(h)}(x)\ v \otimes \mathbf{e}(g)\rangle = \langle u, e^{\langle f,g\rangle}T_s^{i_1,j_1}T_{t-s}^{i_2,j_2}(x)v\rangle.$$

Now let us consider arbitrary $f, g \in \mathcal{S}$. For any $f, g \in \mathcal{S}$ we can choose a partition $(0 = t_0 < t_1 \cdots < t_n = t)$ of $\mathbb{R}_+$ such that on the interval $[0, t]$,

- $f = \sum_{q=1}^n 1_{[t_{q-1},t_q]} \otimes e_{\mu_q}$
- $g = \sum_{q=1}^n 1_{[t_{q-1},t_q]} \otimes e_{\nu_q}$.

Here μ_q, ν_q include the index 0 with the convention (strictly restricted to here only) that $e_0 = 0 \in \mathbf{k}_0$. Now we set contractive maps,

$$T_t^{f,g} = T_{t_1}^{\mu_1,\nu_1}T_{t_2-t_1}^{\mu_2,\nu_2}\cdots T_{t-t_{n-1}}^{\mu_n,\nu_n}$$

as in [1], then it can be easily shown from the above observations that

$$\lim_{h\to 0}\langle u \otimes \mathbf{e}(f), p_t^{(h)}(x)\ v \otimes \mathbf{e}(g)\rangle = \langle u, e^{\langle f,g\rangle}T_t^{f,g}(x)v\rangle.$$

Now let us consider arbitrary vector in the algebraic tensor product $\mathbf{h}_0 \bigotimes_{\text{alg}} \mathcal{E}(\mathcal{S})$. For $\xi_1 = \sum_{k=1}^{p} u_k \otimes \mathbf{e}(f_k), \xi_2 = \sum_{l=1}^{q} v_l \otimes \mathbf{e}(g_l) \in \mathbf{h}_0 \bigotimes_{\text{alg}} \mathcal{E}(\mathcal{S})$,

$$\langle \xi_1, p_t^{(h)}(x)\xi_2 \rangle = \sum_{k,l} \langle u_k \otimes \mathbf{e}(f_k), p_t^{(h)}(x) v_l \otimes \mathbf{e}(g_l) \rangle.$$

This implies that $\lim_{h\to 0} \langle \xi_1, p_t^{(h)}(x)\xi_2 \rangle$ exists.

(ii). For $x \in \mathcal{A}_{\text{loc}}, 0 \le t \le \tau$, let us define $j_t(x)$ by

$$\langle \xi_1, j_t(x)\xi_2 \rangle = \lim_{h\to 0} \langle \xi_1, p_t^{(h)}(x)\xi_2 \rangle, \forall \xi_1, \xi_2 \in \mathbf{h}_0 \bigotimes \mathcal{E}(\mathcal{S}).$$

Since $p_t^{(h)}$ is a contractive family we obtain

$$|\langle \xi_1, j_t(x)\xi_2 \rangle| \le \|x\| \|\xi_1\|.\|\xi_2\|.$$

Thus $j_t(x)$ extends uniquely to a bounded operator on $\mathbf{h}_0 \bigotimes \Gamma$, with $\|j_t(x)\| \le \|x\|, \forall x \in \mathcal{A}_{\text{loc}}$. Since $\mathcal{A}_{\text{loc}}$ is norm dense in $\mathcal{A}$ for each $t \le \tau$, j_t extends uniquely to a contractive map from $\mathcal{A}$ to $\mathcal{A}'' \bigotimes \mathcal{B}(\Gamma)$ satisfying $\|j_t\| \le 1$. As the weak limit of a $*$-homomorphic family $p_t^{(h)}$, j_t is a family of completely positive contractions.

(iii). For $0 \le t \le \tau$, $x \in \mathcal{A}_{\text{loc}}$, we have,

$$\langle u \otimes \mathbf{e}(f), j_t(x) v \otimes \mathbf{e}(g) \rangle = \langle u, e^{\langle f,g \rangle} T_t^{f,g}(x) v \rangle, \forall u \otimes \mathbf{e}(f), v \otimes \mathbf{e}(g) \in \mathbf{h}_0 \bigotimes \mathcal{E}(\mathcal{S})$$

and $T_t^{f,g}(x)$ satisfies the evolution equation

$$\frac{dT_t^{f,g}(x)}{dt} = T_t^{f,g}\Big(\sum_{\mu,\nu=0}^{q} \bar{f}_\mu(t) g_\nu(t) \theta_\nu^\mu(x)\Big), T_0^{f,g}(x) = e^{\langle f,g \rangle} x.$$

Thus the family j_t of completely positive contractions satisfies the qsde (5.1. 1). □

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