HARDY'S UNCERTAINTY PRINCIPLE ON SEMISIMPLE GROUPS

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A theorem of Hardy states that, if f is a function on $\mathbb{R}$ such that $|f(x)| \leq C\, e^{-\alpha|x|^2}$ for all x in $\mathbb{R}$ and $|\hat{f}(\xi)| \leq C\, e^{-\beta|\xi|^2}$ for all ξ in $\mathbb{R}$, where $\alpha > 0$, $\beta > 0$, and $\alpha\beta > 1/4$, then $f = 0$. Sitaram and Sundari generalised this theorem to semisimple groups with one conjugacy class of Cartan subgroups and to the K-invariant case for general semisimple groups. We extend the theorem to all semisimple groups.

1. Introduction.

The Uncertainty Principle states, roughly speaking, that a nonzero function f and its Fourier transform $\hat{f}$ cannot both be sharply localised. This fact may be manifested in different ways. The version of this phenomenon described in the abstract is due to Hardy [3]; we call it Hardy's Uncertainty Principle. Considerable attention has been devoted recently to discovering new forms of and new contexts for the Uncertainty Principle (see [2] for a recent comprehensive survey). In particular, Sitaram and Sundari [4] generalised Hardy's Uncertainty Principle to connected semisimple Lie groups with one conjugacy class of Cartan subgroups and to the K-invariant case for general connected semisimple Lie groups. We extend the theorem of Sitaram and Sundari [4], and establish a form of Hardy's Uncertainty Principle for all connected semisimple Lie groups with finite centre.

2. The theorem.

Let G be a connected real semisimple Lie group with finite centre. Let KAN be an Iwasawa decomposition of G, and let MAN be the associated minimal parabolic subgroup of G. The Lie algebras of G and A are denoted by $\mathfrak{g}$ and $\mathfrak{a}$. The Killing form of $\mathfrak{g}$ induces an inner product on $\mathfrak{a}$ and hence on the dual $\mathfrak{a}^*$; in both cases the corresponding norms are denoted by $|\cdot|$. Haar measures on K and G are fixed; that on K is normalised so that the total mass of K is 1. Integrals over G and K are relative to these Haar measures.

Any irreducible unitary representation μ of M may be realised as the left-translation representation on a finite-dimensional subspace $\mathcal{H}_\mu$ of $C(M)$, the space of continuous complex-valued functions on M. For such a μ, and λ in the complexification $\mathfrak{a}^*_{\mathbb{C}}$ of $\mathfrak{a}^*$, we define the space $\mathcal{H}^0_{\mu,\lambda}$ to be the subspace of $C(G)$ of all functions ξ with the properties that

$$\xi(gan) = \xi(g)\exp((i\lambda - \rho)\log a) \qquad \forall g \in G \quad \forall a \in A \quad \forall n \in N$$

and

$$m \mapsto \xi(gm) \in \mathcal{H}_\mu \qquad \forall g \in G.$$

Note that such functions are determined by their restrictions to K, i.e., effectively we are dealing with a subspace of $C(K)$. The representation $\pi^0_{\mu,\lambda}$ of G is the left-translation representation of G on this space. We define the inner product $\langle \xi, \eta \rangle$ of ξ and η in $\mathcal{H}^0_{\mu,\lambda}$ to be

$$\int_K \xi(k)\,\overline{\eta}(k)\,dk;$$

$\|\cdot\|$ denotes the associated norm.

Denote by $\mathcal{H}_{\mu,\lambda}$ the completion of $\mathcal{H}^0_{\mu,\lambda}$ with this norm, and by $\pi_{\mu,\lambda}$ the extension of $\pi^0_{\mu,\lambda}$ to $\mathcal{H}_{\mu,\lambda}$. The space $\mathcal{H}_{\mu,\lambda}$ may be identified with a subspace of $L^2(K)$, and $\mathcal{H}^0_{\mu,\lambda}$ with the space of continuous functions in $\mathcal{H}_{\mu,\lambda}$.

For μ in $\widehat{M}$ and λ in $\mathfrak{a}^*$, the representation $\pi_{\mu,\lambda}$ is unitary. This representation lifts to a representation of $L^1(G)$ by integration, as follows. First, for f in $L^1(G)$ and ξ and η in $\mathcal{H}_{\mu,\lambda}$, the integral

$$\int_G f(g)\,\langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle\,dg$$

converges, to $B_f(\xi,\eta)$ say. Next, B_f is a sesquilinear form on $\mathcal{H}_{\mu,\lambda}$. Thus there exists a unique bounded operator, denoted $\pi_{\mu,\lambda}(f)$, such that

$$\langle \pi_{\mu,\lambda}(f)\xi, \eta \rangle = \int_G f(g)\,\langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle\,dg \qquad \forall \xi, \eta \in \mathcal{H}_{\mu,\lambda}.$$

We denote by $\|\cdot\|$ the operator norm of such operators, relative to the given norm on $\mathcal{H}_{\mu,\lambda}$. If $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \setminus \mathfrak{a}^*$, then the matrix coefficients $g \mapsto \langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle$ need not be bounded, and for general f in $L^1(G)$ it may not be possible to define $\pi_{\mu,\lambda}(f)$. However, for f which decays sufficiently rapidly at infinity in G, in particular for f in the theorem below, $\pi_{\mu,\lambda}(f)$ may still be defined by the procedure above.

Theorem. *Suppose that C, α, C_μ, β_μ are positive constants and $\alpha\beta_\mu > 1/4$ for all μ in $\widehat{M}$, and that f is a measurable function on G such that*

$$|f(kak')| \le C\exp(-\alpha|\log a|^2) \qquad \forall k, k' \in K \quad \forall a \in A$$

and

$$\|\pi_{\mu,\lambda}(f)\| \le C_\mu \exp(-\beta_\mu|\lambda|^2) \qquad \forall \mu \in \widehat{M} \quad \forall \lambda \in \mathfrak{a}^*.$$

Then $f = 0$.

Proof. Let σ and τ be irreducible representations of K, with characters χ_σ and χ_τ. Define $f_{\sigma,\tau}$ by the formula

$$f_{\sigma,\tau}(g) = \dim\sigma \dim\tau \int_K \int_K \overline{\chi}_\sigma(k)\, \overline{\chi}_\tau(k')\, f(kgk')\, dk\, dk'.$$

By a straightforward estimate,

$$|f_{\sigma,\tau}(kak')| \leq C\,(\dim\sigma\dim\tau)^2 \exp(-\alpha|\log a|^2) \qquad \forall k,k' \in K \quad \forall a \in A.$$

Further, $\pi_{\mu,\lambda}(f_{\sigma,\tau})$ is the composition $P_\sigma \pi_{\mu,\lambda}(f) P_\tau$, where P_σ and P_τ are the projections of $L^2(K)$ onto the σ-isotypic and τ-isotypic subspaces, so that

$$\|\pi_{\mu,\lambda}(f_{\sigma,\tau})\| \leq C_\mu \exp(-\beta_\mu|\lambda|^2) \qquad \forall \mu \in \widehat{M} \quad \forall \lambda \in \mathfrak{a}^*.$$

Now the arguments of Section 3 of [4] show that, if α_μ is chosen such that $0 < \alpha_\mu < \alpha$ and $\alpha_\mu\beta_\mu > 1/4$, then

$$\begin{aligned}\|\pi_{\mu,\lambda}(f_{\sigma,\tau})\| &\leq C_{\sigma,\tau,\mu} \int_G \Phi_{i\,\mathrm{Re}(\lambda)}(x)\, |f(x)|\, dx \\ &\leq C'_{\sigma,\tau,\mu} \exp\left(\frac{|\lambda|^2}{4\alpha_\mu}\right) \qquad \forall \mu \in \widehat{M} \quad \forall \lambda \in a^*_{\mathbb{C}},\end{aligned}$$

where $\Phi_{i\,\mathrm{Re}(\lambda)}$ denotes the usual elementary spherical function, and hence that

$$\pi_{\mu,\lambda}(f_{\sigma,\tau}) = 0 \qquad \forall \mu \in \widehat{M} \quad \forall \lambda \in a^*_{\mathbb{C}}.$$

By Harish-Chandra's subquotient theorem (see G. Warner [5, p. 452]), if π is any irreducible unitary representation of G on a Hilbert space $\mathcal{H}_\pi$, then there exist μ in $\widehat{M}$ and λ in $\mathfrak{a}^*_{\mathbb{C}}$ and closed subspaces S_0 and S_1 of $\mathcal{H}_{\mu,\lambda}$ such that π is Naĭmark equivalent to the quotient representation $\dot{\pi}_{\mu,\lambda}$ of $\pi_{\mu,\lambda}$ on S_1/S_0. This means that there is an intertwining operator A_π with dense domain and range between $(\pi, \mathcal{H}_\pi)$ and $(\dot{\pi}_{\mu,\lambda}, S_1/S_0)$. Consequently $\pi(f_{\sigma,\tau}) = 0$, first on the domain of A_π by the intertwining property, and then on all $\mathcal{H}_\pi$ by continuity. In summary,

$$\langle \pi(f_{\sigma,\tau})\xi, \eta\rangle = 0 \qquad \forall \xi, \eta \in \mathcal{H}_\pi,$$

and therefore, summing over σ and τ, we see that

$$\langle \pi(f)\xi, \eta\rangle = 0 \qquad \forall \xi, \eta \in \mathcal{H}_\pi.$$

It follows that $\pi(f) = 0$ for all π in $\widehat{G}$, the unitary dual of G, whence $f = 0$ by the Plancherel theorem. □

The argument of this paper may also be applied in other contexts. For instance, we may show the following: if f is a measurable function on G, rapidly decreasing in the sense that for any B in $\mathbb{R}^+$ there exists A in $\mathbb{R}^+$ such that

$$|f(kak')| \leq A \exp(-\alpha B|\log a|) \qquad \forall k, k' \in K \quad \forall a \in A,$$

and if on each principal series induced from the minimal parabolic subgroup, the group-theoretic Fourier transform vanishes on a set of positive Plancherel measure, then f is zero. This is a qualitative uncertainty principle related to [**1**].

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