Points of weak*-norm continuity in the dual unit ball of injective tensor product spaces

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Abstract

In this note we exhibit points of weak*-norm continuity in the dual unit ball of the injective tensor product of two Banach spaces when one of them is a G-space.

Introduction

Let X be a Banach space. A point x^* of the dual unit ball $B(X^*)$ is said to be a point of weak*-norm continuity (w^*-PC for short) if whenever $\{x^*_\alpha\}$ is a net in $B(X^*)$ such that $x^*_\alpha \to x^*$ in the weak* topology then $x^*_\alpha \to x^*$ in the norm topology. In a recent work Hu and Smith [4] have given a complete description of w^*-PC's of the dual unit ball of $C(K, X)$, when K is a compact Hausdorff space (see [1] for a short proof of this result).

Since $C(K, X)$ is the injective tensor product of $C(K)$ and X (see [2], page 224), it is natural to ask if a description of w^*-PC's, similar to the ones given by Hu and Smith is valid if one replaces the space $C(K)$ by any Banach space Y such that Y^* is isometric to a $L^1(\mu)(\mu$ is a positive measure). In this short note we show that if $C(K)$ is replaced by a subspace of the form

$$Y = \{f \in C(K) : f(x_a) = \lambda_a f(y_a) \quad \forall a \in I\}$$

where $\{x_a, y_a, \lambda_a\}_{a \in I} \subset K \times K \times [-1, 1]$ is any family (these are the so called G-spaces, see [3], page 83), then an abstract form of the description of the w^*-PC's given by Hu and Smith ([4], Theorem 6) is still valid. We also obtain an analogue of a result of Hu and Smith (Theorem 4) about the strongly extreme points of the dual unit ball of $C(K,X)$ for general injective tensor products.

We will be using concepts and results from M-structure theory and the theory of L^1-preduals. Chapter I and Section II.5 from the monograph by Harmand, Werner and Werner contain all the material we need. For results relating to injective tensor products we refer the reader to the monograph by Diestel and Uhl.

All the Banach spaces are considered over the real scalar field.

Main result

We first recall the description of w^*-PC's of the dual unit ball of $C(K,X)$ (the space of X-valued continuous functions on a compact Hausdorff space K, equipped with the supremum norm).

Theorem ([4, Theorem 6])

An element μ of $C(K,X)^*$ is a weak point of continuity of the unit ball of $C(K,X)^*$ if and only if it has the form $\mu = \sum_{k\in I} L_{k,x_k^*}$ where $I = \{k \in K : k$ is an isolated point of $K\}$, $\sum_{k\in I} \|L_{k,x_k^*}\| = 1$ and for each k in I either $x_k^* = 0$ or $x_k^*/\|x_k^*\|$ is a weak* point of continuity of $B(X^*)$.*

Here L_{k,x_k^*} denotes the functional

$$(L_{k,x_k^*})(f) = x_k^*(f(k)) \quad \text{for any} \quad f \in C(K,X).$$

Before describing some w^*-PC's of the unit ball of $(Y \otimes_\epsilon X)^*$, when Y is a G-space, we analyze the description given by Hu and Smith.

Recall from ([3] Chapter I) that a projection $P : Y \to Y$ is said to be an M-projection if $\|y\| = \max\{\|P(y)\|, \|y - P(y)\|\}$ for all $y \in Y$. If $k \in K$ is an isolated point, clearly $f \to \chi_{\{k\}} f$ is an M-projection in the space $C(K)$ (or $C(K,X)$). The range of an M-projection is called an M-summand.

A closed subspace $M \subset X$ is said to be an M-ideal if there is a projection $P : X^* \to X^*$ such that $\|P(x)\| + \|x - P(x)\| = \|x\|$ for all $x \in X$ and Ker $P = M^\perp$ (such a projection is called a L-projection). An M-summand is in particular an M-ideal. We need a lemma that relates the w^*-PC's of $B(X^*)$ with those of $B(M^*)$.

Lemma 1

Let X be a Banach space and $M \subset X$ be an M-ideal and let P denote the projection as in the above definition. Let $x^* \in B(X^*)$ be such that $P(x^*) = x^*$ and $\|x^*|M\| = 1$. If x^* is a w^*-PC of $B(X^*)$ then it is a w^*-PC of $B(M^*)$.

If $y^* \in B(M^*)$ is a w^*-PC of $B(M^*)$ then it is a w^*-PC of $B(X^*)$.

Proof. Let us recall from ([3], page 11) that $P(x^*)$ is the unique norm-preserving extension of $x^*|M$. Thus M^* is isometrically realized as the range of P. Suppose $\{x^*_\alpha\}$ is a net in $B(M^*)$ such that $x^*_\alpha \to x^*$ in the weak* topology of M^*. We claim that $x^*_\alpha \to x^*$ in the weak* topology of X^* as well. To see this, suppose $\{x^*_{\alpha_\beta}\}$ is any subnet of $\{x^*_\alpha\}$ such that $x^*_{\alpha_\beta} \to y^*, y^* \in B(X^*)$ in the weak* topology. Since $\|y^*\| = 1$ and $y^* = x^*$ on M, by the uniqueness of the extension we get that $y^* = x^*$. Therefore $x^*_\alpha \to x^*$ in the weak* topology and hence in the norm. So $\|x^*_\alpha - x^*\|_M \to 0$.

The converse part is Lemma 3.1 in [1], for the sake of completeness we reproduce the details of the proof here.

Suppose $\{x^*_\alpha\}$ is a net in $B(X^*)$ such that $x^*_\alpha \to y^*$ in the weak* topology. Then by hypothesis $x^*_\alpha|M \to y^*$ in the norm topology. Therefore by the nature of P, $\|P(x^*_\alpha) - y^*\| \to 0$. Since $1 = \lim \|x^*_\alpha\| = \lim \|P(x^*_\alpha)\|$ and since P is a L-projection, we have that $\|x^*_\alpha - P(x^*_\alpha)\| \to 0$. Therefore $\|x^*_\alpha - y^*\| \to 0$. Hence y^* is a w^*-PC of $B(X^*)$. $\square$

We are now ready to describe some w^*-PC's of the unit ball of $(X \otimes_\epsilon Y)^*$.

We first fix some notation.

Let $E(Y)$ denote the extreme points of $B(Y^*)$.

Let $A \subset E(Y)$ be such that $E(Y) = A \cup -A, A \cap -A = \phi$.

Let $B = \{y^* \in A : \mathrm{Ker}(y^*) \text{ is an M-summand in } Y\}$.

Note that in the case $Y = C(K)$, if $k \in K$ is an isolated point then the evaluation functional (or the Dirac measure) $\delta(k)$ is in B.

For $y^* \in Y^*$ and $x^* \in X^*, y^* \otimes x^*$ denotes the functional $(y^* \otimes x^*)(y \otimes x) = y^*(y)x^*(x)$.

Lemma 2

Let $y^*_1, \cdots y^*_n$ be any distinct points of B. Let $x^*_1, \cdots x^*_n$ be w^*-PC's of $B(X^*)$. Then for any scalars $\alpha_1 \cdots \alpha_n$ such that $\sum_{i=1}^n |\alpha_i| = 1$, $F = \sum_{i=1}^n \alpha_i y^*_i \otimes x^*_i$ is a w^*-PC of the unit ball of $(Y \otimes_\epsilon X)^*$.

Proof. If P is an M-projection in Y then $P \otimes I$ is an M-projection in $Y \otimes_\epsilon X$ (see [3], page 264). Also $\bigcap_{i=1}^n \operatorname{Ker}(y_i^*)$ is an M-summand in Y (see [3], Theorem 1.10).

Let N denote the complementary summand. Note that $N^* =$ span $\{y_1^* \cdots y_n^*\}$. Also it is easy to see that for any scalars $\beta_1, \cdots \beta_n$, $\| \sum_{i=1}^n \beta_i y_i^* \| = \sum_{i=1}^n |\beta_i|$. Now since $N \otimes_\epsilon X$ is an M-ideal in $Y \otimes_\epsilon X$ in view of the above lemma, it is enough to show that F is a w^*-PC of the unit ball of $(N \otimes_\epsilon X)^* = N^* \otimes_\pi X^*$. (see [2] Chapter 8). Since $N^* = \bigoplus_{i=1}^n$ span $\{y_i^*\}$ (ℓ^1 - direct sum), this follows from the arguments given in the scalar case in ([1], Theorem 3.9). Thus F is indeed a w^*-PC of the unit ball of $(Y \otimes_\epsilon X)^*$. □

If one starts with a (infinite) sequence $\{y_n^*\}$ in B, it is not clear to us if in-general a functional of the form $F = \sum_{i=1}^\infty \alpha_i y_i^* \otimes x_i^*$ where $\{x_i^*\}$ is a sequence of w^*-PC's and $\sum_{i=1}^\infty |\alpha_i| = 1$ will be a w^*-PC again.

However if one considers G-spaces then we have the following.

Theorem

Let Y be a G-space and X a Banach space. Then for any sequence $\{y_n^*\}$ of distinct points in B and for any sequence $\{x_n^*\}$ of w^*-PC's of $B(X^*)$, $F = \sum_{i=1}^\infty \alpha_i y_i^* \otimes x_i^*$ is a w^*-PC of the unit ball of $(Y \otimes_\epsilon X)^*$.

Proof. Let $M = \{y \in Y : y(A \backslash B) = 0\}$.

Since Y is a G-space, for any $f \in A$ Ker (f) is an M-ideal. Also since in a G-space the intersection of any family of M-ideals is an M-ideal (see [3], Corollary 5.3), we have that M is an M-ideal.

Thus $M \otimes_\epsilon X$ is an M-ideal in $Y \otimes_\epsilon X$. As before we shall show that F is a w^*-PC of the unit ball of $(M \otimes_\epsilon X)^*$.

This is established by first showing that M is isometric to $c_0(\Gamma)$ where Γ is a discrete space. Then note that $(M \otimes_\epsilon X)^* = \ell^1(\Gamma) \otimes_\pi X^*$ (see [2] Chapter 8) and the conclusion follows from arguments similar to the ones indicated in [1].

Since M is an M-ideal, it is easy to see that $E(M) = B \cup -B$. Since for any $y^* \in B, \operatorname{Ker}(y^*)$ is an M-summand in Y and since an M-ideal is invariant under M-projections we get that Ker (y^*) is an M-summand when considered in the space M. It therefore follows that $E(M)$ when equipped with the "structure topology" (see [3], page 38) is a discrete space. Now applying Theorem 4.6 of [5] we conclude that M is isometric to $c_0(B)$ where B has the "structure topology". Therefore the conclusion follows. □

Remark 1. That the intersection of any family of M-ideals in Y is an M-ideal was only used to show that M is an M-ideal. Since the intersection of any finite number of M-ideals is always an M-ideal (see [3], Chapter I), when $A\backslash B$ is a finite set, the set M considered above is an M-ideal. The following example illustrates such a situation.

EXAMPLE: Let $\mathbb{N}\cup\{\infty\}$ denote the one point compactification of natural numbers.

Let $\mathcal{A} = \left\{ f \in C(\mathbb{N}\cup\{\infty\}) : f(\infty) = \frac{f(1)+f(2)}{2} \right\}$.

Let $K = \{\Lambda \in B(\mathcal{A}^*) : \Lambda(1) = 1\}$.

It is easy to see that extreme points of $K, \partial_e K = \{\delta(i) : i \in \mathbb{N}\}$ where $\delta(i)$ is the evaluation functional at i. Also $E = \partial_e K \cup -\partial_e K$.

Since the constant function $1 \in \mathcal{A}$ and $\partial_e K$ is not weak* closed, it follows easily that $\mathcal{A}$ is not a G-space. It is fairly easy to see that

$$B = \left\{ \delta(i) : i \in \mathbb{N}, i \neq 1, 2 \right\}.$$

Thus $M = \{f \in \mathcal{A} : f(1) = 0 = f(2)\}$ and is clearly an M-ideal in $\mathcal{A}$.

Therefore $\sum_{i=1}^{\infty} \alpha_i \delta(i+2)$ is a w^*-PC of the unit ball for any $\sum_{i=1}^{\infty} |\alpha_i| = 1$. It may be worth noting that $\mathcal{A}^* = \left\{ \sum_{i=1}^{\infty} \alpha_i \delta(i) : \alpha \in \ell^1 \right\}$.

Remark 2. Now consider the subspace

$$\mathcal{A}^1 = \left\{ f \in C(\mathbb{N}\cup\{\infty\}) : f(\infty) = \sum_{i=1}^{\infty} \frac{1}{2^i} f(i) \right\}$$

of $C(\mathbb{N}\cup\{\infty\})$. It can be shown that for this space $B = \emptyset$, though being an Asplund space there are plenty of w^*-PC's in the dual unit ball. Thus an L^1-predual can be rich in w^*-PC's, without having any w^*-PC's of the kind we have considered above.

We conclude by giving the analogue of a result of Hu and Smith (see [4], Theorem 4) that characterizes the strongly extreme points of the dual unit ball of $C(K, X)$ to the case of general injective tensor products. Our proof is easier than that of the one given by Hu and Smith and follows from our lemma about strongly extreme points of ℓ^1-direct sums.

We first recall the definition of a strongly extreme point. We choose the form of the definition that is easy to make use of in the rest of the work.

DEFINITION. $x \in B(X)$ is a strongly extreme point if and only if for any sequence $\{x_n\}$ and $\{y_n\}$ in $B(X)$ such that $\frac{x_n+y_n}{2} \to x$, then $\|x_n - y_n\| \to 0$.

Lemma 3

Let P be a L-projection in X with $M =$ Range P and $N = \text{Ker}P$. $x \in B(X)$ is a strongly extreme point if and only if $x \in B(M)$ or $x \in B(N)$ and is a strongly extreme point of that set.

Proof. Let $x \in B(X)$ be a strongly extreme point. Since x is in particular an extreme point, clearly $x \in B(M)$ or $x \in B(N)$. It is easy to see that x is a strongly extreme point of $B(M)$ or $B(N)$.

Conversely suppose $x \in B(M)$ is a strongly extreme point. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $B(X)$ such that $\frac{x_n+y_n}{2} \to x$.

Clearly $\frac{P(x_n)+P(y_n)}{2} \to x$ and since x is a strongly extreme point of $B(M)$, $\|P(x_n) - P(y_n)\| \to 0$. Also

$$\lim_n \|x_n\| = \lim_n \|y_n\| = \lim_n \|P(x_n)\| = \lim_n \|P(y_n)\| = 1.$$

Since P is a L-projection, we get that $\|x_n - P(x_n)\| \to 0$ and $\|y_n - P(y_n)\| \to 0$.

Now $\|x_n - y_n\| \leq \|x_n - P(x_n)\| + \|P(x_n) - P(y_n)\| + \|P(y_n) - y_n\|$.

Therefore $\|x_n - y_n\| \to 0$. Hence x is a strongly extreme point of $B(X)$. A similar argument holds when $x \in B(N)$ is a strongly extreme point. □

Corollary 1

Let $x^* \in B(X^*)$ be such that $\text{Ker}(x^*)$ is an M-ideal in X and let $y^* \in B(Y^*)$ be a strongly extreme point. Then $x^* \otimes y^*$ is a strongly extreme point of $B((X \otimes_\epsilon Y)^*)$.

Proof. As noted before $\text{Ker}(x^*) \otimes_\epsilon Y$ is an M-ideal in $X \otimes_\epsilon Y$ and hence its annihilator which can be identified as $\text{span}\{x^*\} \otimes_\pi Y^*$, is the range of a L-projection in $(X \otimes_\epsilon Y)^*$. Clearly $x^* \otimes y^*$ is a strongly extreme point of the unit ball of $\text{span}\{x^*\} \otimes_\pi Y^*$. Therefore it follows from the above Lemma that $x^* \otimes y^*$ is a strongly extreme point of $B((X \otimes_\epsilon Y)^*)$. □

Corollary 2 ([4], Theorem 4)

$\mu \in B(C(K, X)^*)$ is a strongly extreme point if and only if $\mu = \delta(k) \otimes x^*$ for some $k \in K$ and strongly extreme point $x^* \in B(X^*)$.

Proof. We use the identification of the dual of $C(K, X)$ as the space $M(K, X^*)$ of X^*-valued regular Borel measures. Here for any Borel set A, the mapping $F \to F/A$ (F/A is the Borel measure $(F/A)(B) = F(A \cap B)$) is a L-projection. When x^* is a strongly extreme point of $B(X^*)$ by taking $A = \{k\}$, and applying the above Lemma we conclude that $\delta(k) \otimes x^*$ is a strongly extreme point of $B(C(K, X)^*)$. That any strongly extreme point of $B(C(K, X)^*)$ has the above form follows from the arguments given by Hu and Smith. □

Remark 3. Considering $C(K,X)$ as a subspace of $WC(K,X)$ (functions that are continuous when X has the weak topology), it is natural to consider the validity of Theorem 6 of [4] for $WC(K,X)$. In a recent work [6] this author has established that a similar description of points of weak*-norm continuity of $B(WC(K,X)^*)$ is valid under the assumption that the set of isolated points is dense in K. Our arguments also cover the case of the space of operators, $\mathcal{L}(X, C(K))$.

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