MISCELLANEOUS NOTES

BERNOULLI'S THEOREM AND TSHEBYCHEFF'S ANALOGUE

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Bernoulli's theorem on the probability that in a series of n trials (where the chance of success is constant from trial to trial and equals p) the number of successes m should be within a certain range of the expectation value, can be stated as

$$P\{|m/n-p|<\epsilon\}>1-\eta \quad \text{if } n>n_0(\epsilon,\eta) \qquad (1\cdot1)$$

where ϵ and η are arbitrary small numbers and

$$n_0(\epsilon,\eta)=(1/\epsilon)+(1+\epsilon)\log(1/\eta)/\epsilon^2. \qquad (1\cdot2)$$

By introducing n independent stochastic variables $x_1, x_2, \ldots, x_n$ each of which can take values 1 or 0 with probabilities p or q we can deduce from Tshebycheff's law of large numbers a theorem which is of the same form as Bernoulli's theorem (1·1), except that $n_0(\epsilon,\eta)$ is a function of ϵ and η different from that given by (1.2). This theorem we shall call Tshebycheff's analogue of Bernoulli's theorem. In its general form Tshebycheff's law of large numbers can be stated as

$$P\{|(x_1+x_2+\ldots+x_n)/n-E(x_1+x_2+\ldots+x_n)/n|<\epsilon\}>1-\frac{V(x_1+x_2+\ldots+x_n)/\epsilon^2}{n} \qquad (2\cdot1)$$

where $x_1, x_2, \ldots, x_n$ are any general stochastic variables, $E(x_1+x_2+\ldots+x_n)$ the mathematical expectation and $V(x_1+x_2+\ldots+x_n)$ the variance of $x_1+x_2+\ldots+x_n$, both being supposed to exist. If now, instead of $x_1, x_2, \ldots, x_n$ being general stochastic variables, they are assumed to be mutually independent stochastic variables such that each can take on values 1 or 0 with probabilities p or q then

$$\left.\begin{aligned}(x_1+x_2+\ldots+x_n) &= m(\text{number of successes})\\ E(x_1+x_2+\ldots+x_n)/n &= p\\ V(x_1+x_2+\ldots+x_n)/n &= pq/n\end{aligned}\right\} \qquad (2\cdot2)$$

It will be thus seen that by utilising (2.2), (2.1) can be reduced to

$$P\{|m/n-p|<\epsilon\}>1-pq/n\epsilon^2 \qquad (2\cdot3)$$

Since p and q are both positive, and $p+q=1$, the maximum value of pq is reached when $p=q$ and that value is $\frac{1}{4}$. Hence (2.3) implies another inequality relation

$$P\{|m/n-p|<\epsilon\}>1-1/4n\epsilon^2 \qquad (2\cdot4)$$

It follows that given two arbitrary small numbers ϵ and η we have

$$P\{|m/n-p|<\epsilon\}>1-\eta \quad \text{if} \quad n\geqslant n'_0(\epsilon,\eta) \qquad (2\cdot5)$$

where $n'_0(\epsilon,\eta)=1/4\epsilon^2\eta$. Relation (2.5) we shall call Tshebycheff's analogue of Bernoulli's theorem, which latter would be defined by (1.1) and (1.2). Now both Bernoulli's theorem and Tschebycheff's analogue are approximate and fairly close but not exact inequality relations, that is, in neither case would the inequality relation (1.1) or (2.5) be necessarily reversed if $n\leqslant n_0(\epsilon,\eta)$ defined by (1.2) or $n\leqslant n'_0(\epsilon,\eta)$ defined by (2.5). There is, of course, a least number $n''_0(\epsilon,\eta)$ which is yet unknown such that the inequality relation (1.1) or (2.5) holds if $n>n''_0(\epsilon,\eta)$ and is reversed if $n<n''_0(\epsilon,\eta)$. The question now naturally arises as to which of $n_0(\epsilon,\eta)$ or $n'_0(\epsilon,\eta)$ is nearer to $n''_0(\epsilon,\eta)$ for different values of ϵ and η; both are, of course, greater than $n''_0(\epsilon,\eta)$. This point we have tried to settle by numerical-cum-graphical methods in the following manner.

Considering the (ϵ,η) plane we are generally interested in the square defined by ϵ varying from 0 to 1, and η varying from 0 to 1. Inside this square we are interested in knowing which of $n_0(\epsilon,\eta)$ or $n'_0(\epsilon,\eta)$ is greater. We have thus to find whether $1/\epsilon+(1+\epsilon)\log(1/\eta)/\epsilon^2$ or $1/4\epsilon^2\eta$ is greater.

If $n_\omega(\epsilon, \eta)$ is greater, then Bernoulli's theorem gives a wider limit and is thus worse, and if $n'_\omega(\epsilon, \eta)$ is greater it is the other way about. Equating the two we have

$$1/\epsilon+(1+\epsilon) \log (1/\eta)/\epsilon^2=1/4\epsilon^2\eta \qquad \cdots \quad (3\cdot1)$$

or

$$\epsilon=\{1/4\eta-\log (1/\eta)\}/\{1+\log (1/\eta)\} \qquad \cdots \quad (3\cdot2)$$

If this curve had cut the (ϵ, η) plane outside the square defined above, then within our region of interest either $n_\omega(\epsilon, \eta)$ or $n'_\omega(\epsilon, \eta)$ would have been throughout the greater of the two, that is, either Tshebycheff's analogue or Bernoulli's theorem would have been better. But, as the accompanying table and graph would show, this curve (3.2) cuts across the square and divides it into two portions in one of which $n_\omega(\epsilon,\eta)>n'_\omega(\epsilon, \eta)$ and in the other $n'_\omega(\epsilon, \eta)>n_\omega(\epsilon,\eta)$; that is, in one region Tshebycheff's analogue is closer and in the other Bernoulli's theorem is closer.

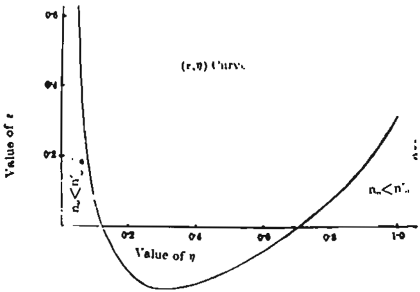

TABLE 1. VALUES OF ϵ FOR DIFFERENT VALUES OF η

η	ϵ	η	ϵ	η	ϵ	η	ϵ
0·01	3·6386	0·09	0·1085	0·17	−0·1087	0·50	−0·1141
0·02	1·7484	0·10	0·0598	0·18	−0·1200	0·55	−0·0897
0·03	1·0710	0·11	0·0204	0·19	−0·1296	0·60	−0·0623
0·04	0·7185	0·12	−0·0118	0·20	−0·1377	0·70	0·0003
0·05	0·5016	0·13	−0·0385	0·25	−0·1619	0·80	0·0731
0·06	0·3540	0·14	−0·0608	0·30	−0·1682	0·90	0·1560
0·07	0·2493	0·15	−0·0795	0·40	−0·1520	1·00	0·2500
0·08	0·1700	0·16	−0·0954	0·45	−0·1351		

It is easily seen that at the origin $n'_\omega(\epsilon, \eta) > n_\omega(\epsilon, \eta)$ and hence in the immediate neighbourhood of the origin and this is the region in which we are most interested, Bernoulli's theorem gives a closer limit than Tshebycheff's analogue.

It is well known that the proof of Bernoulli's theorem (even as modified by later mathematicians is long and complicated though extremely ingenious, while Tshebycheff's theorem and its analogue rest on a proof which is as simple as it is short. Hence Bernoulli's theorem (1.1) and (1.2) might have been discarded in favour of Tshebycheff's analogue (2.4) and (2.5) if only $n'_\omega(\epsilon,\eta)$ were less than $n_\omega(\epsilon, \eta)$ within the region of interest in the immediate neighbourhood of the point $\epsilon=0$, $\eta=0$, but our investigation shows that near the origin it is just the other way about, and hence the arduous mathematical processes involved in the proof of (1 1) and (1.2) are fully justified, as against the shorter and simpler reasoning behind (2.4) and (2.5).

Paper received : 1 *January*, 1943.