

OPTIMAL CLONING AND NO SIGNALING

Sibasish Ghosh¹, Guruprasad Kar, Anirban Roy²

Physics and Applied Mathematics Unit

Indian Statistical Institute

203 B. T. Road, Calcutta-700 035

India.

ABSTRACT

It is shown that no signaling constraint generates the whole class of $1 \rightarrow 2$ optimal quantum cloning machines of single qubits

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Perfect cloning (*i.e.*, copying) of an unknown quantum mechanical state is known to be impossible, as shown by Wootters and Zurek. This is valid for pure quantum mechanical states [1], and well as for mixed states [2]. Bužek and Hillery have provided a universal $1 \rightarrow 2$ cloning machine, which produces two identical but imperfect copies of an arbitrary single qubit state [3]. Bruß et. al. [4] have shown that this symmetric (as the two copies produced are identical) universal cloning machine of Bužek and Hillery is optimal. Recently Cerf [5] has provided a concept of asymmetric quantum cloning when the two output states of the cloner are not identical, but at the same time, these two output states are specifically related to the input. The cloning operation presented in [5] is universal for qubits, *i.e.*, the fidelity of cloning does not depend on the input qubit state. Bužek et. al. [6] have provided a universal $1 \rightarrow 2$ cloning network for asymmetric cloning using local unitary operations and controlled NOT (C-NOT) operations, where the input state is a single qubit, and the optimal symmetric cloning machine of Bužek and Hillery [3] is reproduced.

In a very interesting way, Gisin [7] has connected the $1 \rightarrow 2$ symmetric cloning operation of qubits with no signaling property (which states that superluminal signaling is impossible in quantum mechanics). In this letter, we shall reproduce the result of Bužek et. al. [6] using no signaling condition. And our derivation shows that the universal $1 \rightarrow 2$ asymmetric cloning machine of Bužek et. al. [6] is optimal (described below).

a_0 corresponds to the original single qubit, a_1 corresponds to the blank copy (which is also in a single qubit state), and b_1 corresponds to the machine of the cloning process. Let $\rho_{a_0}^{\text{in}}(\vec{m}) = (1/2)(I + \vec{m} \cdot \vec{\sigma})$ be the density matrix of the input single qubit state (which is unknown, as the Bloch vector $\vec{m} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is unknown) entering into the asymmetric quantum cloning machine (AQCM). We want to clone (asymmetrically) this qubit universally, *i.e.*, input-state independently (*i.e.*, independent of the Bloch vector \vec{m}), in such a way that the density matrices of the two clones $\rho_{a_j}^{\text{out}}(\vec{m})$ ($j = 0, 1$) at the output of the AQCM are of the forms

$$\rho_{a_j}^{\text{out}}(\vec{m}) = s_j \rho_{a_j}^{\text{in}}(\vec{m}) + \frac{1 - s_j}{2} I, \quad (1)$$

(for $j = 0, 1$) where I is the 2×2 identity matrix. Equation (1) is referred as the isotropy condition. Obviously here $0 \leq s_0, s_1 \leq 1$. For symmetric QCM, $s_0 = s_1$. Let

$\rho_{a_0 a_1}^{\text{out}}(\vec{m})$ be the two qubit output density matrix of the AQCM, obtained after employing the trace operation on the machine states in the output pure state $|\psi\rangle_{a_0 a_1 \text{ machine}}^{\text{out}}(\vec{m})$, obtained by applying the asymmetric cloning operation on $\rho_{a_0}^{\text{in}}(\vec{m})$. In full generality, $\rho_{a_0 a_1}^{\text{out}}(\vec{m})$ can be written as

$$\rho_{a_0 a_1}^{\text{out}}(\vec{m}) = \frac{1}{4} \left(I \times I + s_0 \vec{m} \cdot \vec{\sigma} \otimes I + s_1 I \otimes \vec{m} \cdot \vec{\sigma} + \sum_{j,k=x,y,z} t_{jk} \sigma_j \otimes \sigma_k \right). \quad (2)$$

The AQCM will be universal if it acts similarly on all input states, *i.e.*, if

$$\rho_{a_0 a_1}^{\text{out}}(\mathbf{R}\vec{m}) = U(\mathbf{R}) \otimes U(\mathbf{R}) \rho_{a_0 a_1}^{\text{out}}(\vec{m}) U(\mathbf{R})^\dagger \otimes U(\mathbf{R})^\dagger, \quad (3)$$

where $\mathbf{R} \equiv \mathbf{R}(\vec{n}, \alpha)$ represents an arbitrary rotation (in $SO(3)$) about an axis along the unit vector \vec{n} through an angle α of the Bloch vector \vec{m} , and $U(\mathbf{R}) \equiv e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}}$ is the corresponding 2×2 unitary operation (it is in $SU(2)$) acting on the two 2-dimensional Hilbert spaces corresponding to the two qubits a_0 and a_1 . As a consequence of this property (given by equation (3)), we see that (see (7)) $\rho_{a_0 a_1}^{\text{out}}(\vec{m})$ is invariant under rotation of \vec{m} , *i.e.*,

$$\left[e^{i\alpha\vec{m}\cdot\vec{\sigma}} \otimes e^{i\alpha\vec{m}\cdot\vec{\sigma}}, \rho_{a_0 a_1}^{\text{out}}(\vec{m}) \right] = 0 \text{ for all real } \alpha. \quad (4)$$

Equation (4) imposes the following conditions on the parameters t_{jk} :

$$\left. \begin{aligned} -m_z t_{xy} + m_y t_{xz} - m_z t_{yx} + m_y t_{zx} &= 0, \\ m_z t_{xx} - m_x t_{xz} - m_z t_{yy} + m_y t_{zy} &= 0, \\ -m_y t_{xx} + m_x t_{xy} - m_z t_{yz} + m_y t_{zz} &= 0, \\ m_z t_{xx} - m_z t_{yy} + m_y t_{yz} - m_x t_{zx} &= 0, \\ m_z t_{xy} + m_z t_{yx} - m_x t_{yz} - m_x t_{zy} &= 0, \\ m_z t_{xz} - m_y t_{yx} + m_x t_{yy} - m_x t_{zz} &= 0, \\ -m_y t_{xx} + m_x t_{yx} - m_z t_{zy} + m_y t_{zz} &= 0, \\ -m_y t_{xy} + m_x t_{yy} + m_z t_{zx} - m_x t_{zz} &= 0, \\ -m_y t_{xz} + m_x t_{yz} - m_y t_{zx} + m_x t_{zy} &= 0. \end{aligned} \right\} \quad (5)$$

Particularly, for $\vec{m} = (0, 0, 1) \equiv \uparrow$, we have $t_{xx}^\uparrow = t_{yy}^\uparrow$, $t_{xy}^\uparrow = -t_{yx}^\uparrow$ and

$$t_{yz}^\uparrow = t_{zy}^\uparrow = t_{zx}^\uparrow = t_{xz}^\uparrow = 0.$$

For $\vec{m} = (0, 0, -1) \equiv \downarrow$, we have $t_{xx}^\downarrow = t_{yy}^\downarrow$, $t_{xy}^\downarrow = -t_{yx}^\downarrow$ and $t_{yz}^\downarrow = t_{zy}^\downarrow = t_{zx}^\downarrow = t_{xz}^\downarrow = 0$.

For $\vec{m} = (1, 0, 0) \equiv \rightarrow$, we have $t_{yy}^- = t_{zz}^-$, $t_{yz}^- = -t_{zy}^-$ and $t_{xx}^- = t_{xz}^- = t_{xy}^- = t_{yx}^- = 0$.

And for $\vec{m} = (-1, 0, 0) \equiv \leftarrow$, we have $t_{yy}^- = t_{zz}^-$, $t_{yz}^- = -t_{zy}^-$ and $t_{xx}^- = t_{xz}^- = t_{xy}^- = t_{yx}^- = 0$.

Our motivation is to find out bounds on s_0 and s_1 (or some sort of relation between them), using the conditions imposed on t_{jk} 's by equation (5), the no signaling condition (to be described bellow), and the positive semi-definiteness of the density matrices $\rho_{a_0 a_1}^{\text{out}}(\vec{m})$ for each Bloch vector \vec{m} .

If it would have been possible to distinguish between different mixtures that can be prepared at a distance (*e.g.*, between $\rho_{a_0 a_1}^{\text{out}}(\vec{m}) + \rho_{a_0 a_1}^{\text{out}}(-\vec{m})$ and $\rho_{a_0 a_1}^{\text{out}}(\vec{m}') + \rho_{a_0 a_1}^{\text{out}}(-\vec{m}')$), then non-locality in quantum mechanics could be used for signaling (*i.e.*, superluminal signaling) through that distance, and hence we would reach a contradiction between quantum mechanics and relativity (8). Thus we have to maintain no signality, which imposes that the mixtures $\rho_{a_0 a_1}^{\text{out}}(\vec{m}) + \rho_{a_0 a_1}^{\text{out}}(-\vec{m})$ and $\rho_{a_0 a_1}^{\text{out}}(\vec{m}') + \rho_{a_0 a_1}^{\text{out}}(-\vec{m}')$ (of the output states), corresponding to the indistinguishable mixtures $(1/2)(I + \vec{m} \cdot \vec{\sigma}) + (1/2)(I - \vec{m} \cdot \vec{\sigma})$ and $(1/2)(I + \vec{m}' \cdot \vec{\sigma}) + (1/2)(I - \vec{m}' \cdot \vec{\sigma})$ respectively (of the input states), are themselves indistinguishable (7). So, without loss of generality, we can write

$$\rho_{a_0 a_1}^{\text{out}}(0, 0, 1) + \rho_{a_0 a_1}^{\text{out}}(0, 0, -1) = \rho_{a_0 a_1}^{\text{out}}(1, 0, 0) + \rho_{a_0 a_1}^{\text{out}}(-1, 0, 0). \quad (6)$$

Using equations (5) and (6), we get the following expression for $\rho_{a_0 a_1}^{\text{out}}(\uparrow)$, where $\uparrow = (0, 0, 1)$:

$$\begin{aligned} \rho_{a_0 a_1}^{\text{out}}(\uparrow) = \frac{1}{4} [& I \otimes I + s_0 \sigma_z \otimes I + s_1 I \otimes \sigma_z + t(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \\ & + t_{xy}(\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x)], \end{aligned} \quad (7)$$

where $t = t_{xx}^\uparrow = t_{yy}^\uparrow = t_{zz}^\uparrow$ and $t_{xy} = t_{xy}^\uparrow$ are both real quantities. The (real) eigen values of $\rho_{a_0 a_1}^{\text{out}}(\uparrow)$ are given by

$$\frac{1}{4} \{ 1 + t \pm (s_0 + s_1) \}, \quad \frac{1}{4} [1 - t \pm \{ 4t^2 + 4t_{xy}^2 + (s_0 - s_1)^2 \}^{1/2}]. \quad (8)$$

All these eigen values must be non-negative, and so we must have

$$s_0 + s_1 \leq 1 + t, \quad (9)$$

$$(s_0 - s_1)^2 + 4t_{xy}^2 \leq (1 + t)(1 - 3t), \quad (10)$$

$$-1 \leq t \leq \frac{1}{3}. \quad (11)$$

From equation (9) we see that maximum values of both s_0 and s_1 will occur when

$$s_0 + s_1 = 1 + t. \quad (12)$$

So, using equation (12), we get from equation (10) that

$$s_0^2 + s_1^2 + s_0s_1 - s_0 - s_1 + t_{xy}^2 \leq 0. \quad (13)$$

The optimal symmetric cloning machine of Bužek and Hillery [3] (where $s_0 = s_1 = 2/3$) will be reproduced here if we take $t_{xy} = 0$, and then condition (13) is exactly equation (11) of [6]. And from equation (12) we see that the relation (13) has to be satisfied by the reduction factors s_0, s_1 of an optimal AQCM, which implies that the AQCM of Bužek et. al. [6] is optimal.

No signaling constraint was used [7] to derive the optimality of the (universal) symmetric cloning machine of Bužek and Hillery [3], and in this paper the same constraint has been used to find out the optimality of the (universal) asymmetric cloning machine of Bužek et. al. [6].

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