Conditionally exactly solvable potentials and supersymmetric transformations

Géza Lévai [a,*], Pinaki Roy [b,1]

[a] Institute of Nuclear Research of the Hungarian Academy of Sciences, Debrecen, Pf. 51, 4001 Hungary
[b] Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta, 700035, India

Received 17 September 1999; accepted 25 October 1999
Communicated by P.R. Holland

Abstract

A general procedure is presented to construct conditionally exactly solvable (CES) potentials using the techniques of supersymmetric quantum mechanics. The method is illustrated with potentials related to the harmonic oscillator problem. Besides recovering known results, new CES potentials are also obtained within the framework of this general approach. The conditions under which this method leads to CES potentials are also discussed.

1. Introduction

Solvable potential problems have have played a dual role since the beginnings of quantum mechanics. First, they represented useful aids in modelling realistic physical problems, and second, they offered an interesting field of investigation in their own right. Related to this latter area, the concept of solvability has changed to some extent in recent years. Besides exactly solvable problems, for which the bound-state energy spectrum and solutions could be given in general analytical form, quasi-exactly solvable (QES) (see e.g. [1]) and conditionally exactly solvable (CES) [2] potential classes have also been identified recently.

In the first case only a finite number of eigenstates can be obtained exactly, while in the latter one analytical solutions are available only if some (or all) of the potential parameters are fine tuned to specific numerical values.

There are different types of CES potentials, which is also reflected in the way they can be most naturally constructed. Some of them, including the first CES potentials [2,3] (see also Ref. [4] for some interesting comments on the construction of CES potentials) have their bound-state solutions in terms of a single special function (polynomial); a structure characteristic of Natanzon class potentials [5]. For another class, these solutions have more complex structure, but generally they can be expressed in terms of the linear combination of two special functions. Typically these CES potentials are constructed as supersymmetric partners of some simple potentials [6,7]. Their CES nature hinges on the fact

* Corresponding author. E-mail: Levai@atomki.hu
[1] E-mail: Pinaki@isical.ac.in

whether the parameters of their partners can be chosen such that they can be reduced to some simple potential with known solutions and energy eigenvalues. According to the techniques of supersymmetric quantum mechanics, or SUSYQM (for reviews see e.g. [8–11]), the CES potentials constructed in this way are then essentially isospectral with their partners, i.e. the two spectra are identical or differ only in their ground state. The bound-state solutions of CES potentials are obtained from those of their simple (Natanzon-type) partner potentials by acting on these latter ones with linear differential operators. This explains why their bound-state solutions possess the specific structure described above.

In Refs. [6,7] some CES potentials have been constructed by SUSYQM. The aim of this publication is to show that this procedure can be made more systematic by making use of various types of SUSYQM transformations. The rather general nature of this treatment allows the recovery of known results and also the derivation of new CES potentials in the same framework. Our examples concern CES potentials related to the harmonic oscillator potential in three or one dimension (the standard examples of Refs. [6,7]), but the formalism is equally applicable to other types of potentials as well.

2. The conventional SUSYQM approach to CES potentials

Let us assume that there is a pair of SUSYQM partner potentials $V_{\pm}^{(0)}(r)$, which can be constructed from a superpotential $W_0(r)$ in the usual way:

$$V_{\pm}^{(0)}(r) = \tfrac{1}{2}\left[W_0^2(r) \pm W_0'(r)\right]. \tag{1}$$

Consider now a superpotential of the form

$$W(r) = W_0(r) + w(r). \tag{2}$$

The partner potentials generated from $W(r)$ are then

$$V_+(r) = V_+^{(0)}(r) + W_0(r)w(r) + \tfrac{1}{2}\left[w^2(r) + w'(r)\right] \tag{3}$$

$$V_-(r) = V_-^{(0)}(r) + W_0(r)w(r) + \tfrac{1}{2}\left[w^2(r) - w'(r)\right]. \tag{4}$$

Let us now insist on that one of these potentials, say $V_+(r)$ is related to some known potential up to an energy shift. In the simplest case this could be $V_+^{(0)}(r)$ in Eq. (1):

$$V_+(r) = V_+^{(0)}(r) + \Delta. \tag{5}$$

Combined with (3), this requirement immediately introduces a Riccati-type differential equation for $w(r)$:

$$\tfrac{1}{2}\left[w^2(r) + w'(r)\right] + W_0(r)w(r) = \Delta. \tag{6}$$

If this equation is solved, then a pair of SUSYQM potentials is obtained, from which one of the partner potentials, $V_+(r)$, corresponds to a known potential (up to an energy shift). Therefore, both the spectrum and the wavefunctions of the partner potential $V_-(r)$ can be obtained in the usual way.

In the examples in [6] $V_+^{(0)}(r)$ was the harmonic oscillator potential in 1 and 3 dimensions, with $W_0(r)$ being the corresponding superpotential. In both cases the structure of $w(r)$ was of the type

$$w(r) = \sum_{i=1}^{N} \frac{2g_i r}{1 + g_i r^2}. \tag{7}$$

In the practical examples $N = 1$ was used. The difference was that in the one-dimensional case the authors of [6] considered unbroken supersymmetry (and therefore $V_-(x)$ had one more state than $V_+(x)$), while in the three-dimensional case they chose to discuss broken supersymmetry, so the spectra of the partner potentials were identical.

3. An alternative SUSYQM construction of CES potentials

Here we use various SUSYQM transformations systematically to recapitulate the formalism of Section 2 and to put it into a more general context.

It is known from the theory of isospectral potentials that a potential $V_2(r)$ isospectral with a known potential $V_1(r)$ can be constructed by [12]

$$V_2(r) = V_1(r) - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\phi(r), \tag{8}$$

where $\phi(r)$ is a solution of the Schrödinger equation

$$-\frac{1}{2}\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}+V_1(r)\phi=\epsilon\phi. \tag{9}$$

Here ϵ is usually called factorization energy. Depending on the value of ϵ and the boundary conditions of the solution $\phi(r)$, $V_2(r)$ in (8) will have various properties. In the case of a radial problem (in three-dimensions) four types of transformations are possible. These are related four different types of *nodeless* solutions $\phi(r)$ of Eq. (9) and have been described in terms of SUSYQM [12,13,15]. The nodelessness of $\phi(r)$ guarantees that the resulting potential $V_2(r)$ does not have singularities for finite values of r (besides the origin), and this can be achieved whenever the factorization energy ϵ is below the ground-state energy of $V_1(r)$ [12]. We briefly summarize the basic characteristics of the four SUSYQM transformation types in Table 1.

Let us consider the radial harmonic oscillator as an example and solve (9) for $\phi(r)$ with

$$\begin{aligned}V_1(r)=V_+^{(0)}(r)&=\tfrac{1}{2}\left[W_0^2(r)+W_0'(r)\right]\\&=\frac{1}{2}r^2+\frac{\gamma(\gamma+1)}{2r^2}+\gamma+\frac{3}{2}.\end{aligned} \tag{10}$$

Here the superpotential is $W_0(r)=r+(\gamma+1)r^{-1}$, and the bound states of $V_1(r)$ are found at $E_n=2n+2\gamma+3$. The solution $\phi(r)$ can be searched for in the form

$$\phi(r)\simeq r^A\exp\left(\frac{B}{2}r^2\right)F(a,b;Cr^2), \tag{11}$$

where $F(a,b;z)$ is the confluent hypergeometric function [16]. Straightforward calculation shows that (9) transforms into the confluent hypergeometric equation if the following conditions hold:

$$A(A-1)=\gamma(\gamma+1),\quad B^2=1,\quad B=-C; \tag{12}$$

$$b=A+\tfrac{1}{2} \tag{13}$$

$$a=-\frac{\epsilon}{2C}+\frac{\gamma}{2C}+\frac{3}{4C}+\frac{A}{2}+\frac{1}{4}. \tag{14}$$

Recalling that besides $F(a,b;z)$, $z^{1-b}F(a-b+1,2-b;z)$ is a linearly independent solution of the same confluent hypergeometric function [16], the general solution $\phi(r)$ has the form

$$\begin{aligned}\phi(r)\simeq\exp\left(\frac{B}{2}r^2\right)\Bigg[&\alpha_1 r^{\gamma+1}F\left(\frac{\epsilon}{2C}+\frac{\gamma}{2C}\right.\\&\left.+\frac{3}{4C}+\frac{\gamma}{2}+\frac{3}{4},\gamma+\frac{3}{2};Cr^2\right)\\&+\alpha_2 r^{-\gamma}F\left(\frac{\epsilon}{2C}+\frac{\gamma}{2C}+\frac{3}{4C}\right.\\&\left.-\frac{\gamma}{2}+\frac{1}{4},-\gamma+\frac{1}{2};Cr^2\right)\Bigg].\end{aligned} \tag{15}$$

Note that the two terms in Eq. (15) are connected by the $\gamma\leftrightarrow 1-\gamma$ transformation, therefore it is enough to consider one of the solutions ($A=\gamma+1$ or $A=-\gamma$) of $A(A-1)=\gamma(\gamma+1)$ in Eq. (12). The solutions corresponding to the transformations T_1, T_2, T_3 and T_4 in Table 1 can then be identified by imposing the appropriate boundary conditions on $\phi(r)$.

Substituting the $\phi(r)$ function in Eq. (8) one obtains an expression for $V_2(r)$ in terms of $\phi'(r)$ and $\phi''(r)$. With the use of (9), $V_2(r)$ can be ex-

Table 1
SUSYQM transformations belonging to different types of solutions $\phi(r)$. The notation of Refs. [12,13,15] has been adapted to the potentials discussed here.

Transformation	T_1	T_2	T_3	T_4
ϵ	$\epsilon=E_0$	$\epsilon<E_0$	$\epsilon<E_0$	$\epsilon<E_0$
$\lim_{r\to 0}\phi$	$r^{\gamma+1}$	$r^{-\gamma}$	$r^{\gamma+1}$	$r^{-\gamma}$
$\lim_{r\to\infty}\phi$	convergent	divergent	divergent	convergent
Spectrum modification	deletes ground state	adds new ground state ($0<\gamma$ only)	none	none ($0<\gamma$ only)
Singularity modification	$(\gamma+1)r^{-2}$	$-\gamma r^{-2}$	$(\gamma+1)r^{-2}$	$-\gamma r^{-2}$

pressed in terms of $V_1(r)$, ϵ and ϕ'/ϕ. In this last expression the first-order derivatives of two confluent hypergeometric functions occur, each of which can be expressed in terms of another confluent hypergeometric function [16]. This means that $V_2(r)$ can be expressed in a somewhat complicated, but closed analytic form. A special situation occurs when $a=-N$ or $a-b+1=-M$ holds. In this case one of the confluent hypergeometric functions occuring in (15) reduces to an Nth or Mth order (generalized Laguerre [16]) polynomial of the argument. According to (14), this case corresponds to specific choices of the factorization energy ϵ. We note that in principle both confluent hypergeometric functions can reduce to a polynomial if both ϵ and γ has a specific values: $\epsilon=-C(M+N+1)-\gamma-\frac{3}{2}$, $\gamma=M-N+\frac{1}{2}$. Let us now consider the four transformations T_1, T_2, T_3 and T_4 one by one.

3.1. The T_1 case

The boundary condition at $r=0$ allows solutions only with $\alpha_2=0$ in (15) (if $\gamma>0$ holds), furthermore $B=-1$ is also required to fulfil the asymptotic boundary condition. The $a=-N$ condition leads to $\epsilon=2N+2\gamma+3$. This factorization energy corresponds to the bound-states energies of $V_1(r)=V_+^{(0)}(r)$ and $\phi(r)$ simply reproduces the physical wavefunctions. It is known that for $N=n\neq 0$ the transformed potential $V_2(r)$ has singularities at those locations, where the wavefunctions have nodes. The ground-state wavefunction with $n=0$, however, is nodeless, and the T_1 transformation then simply retrieves the classic SUSYQM transformation which eliminates the ground state of $V_1(r)$ and increases the value of γ with one unit.

3.2. The T_3 case

Similarly to the T_1 case only the term regular at the origin is allowed by the boundary condition at $r=0$ (i.e. $\alpha_2=0$), however, the asymptotic boundary condition requires $B=1$ in this case. The $a=-N$ polynomial condition then leads to the specific factorization energies $\epsilon=-2N$, which are always below the ground-state energy of $V_1(r)$, so the nodelessness of $\phi(r)$ is always secured. The $N=0$ choice recovers $V_2(r)$ as another oscillator with the same spectrum as $V_1(r)$: only the value of γ is increased with one unit and the energy is shifted downwards with one unit. The $N=1$ case results in the CES potential described in [6] (denoted by $V_-(r)$ there) up to an energy shift:

$$V_2(r)=\frac{1}{2}r^2+\frac{(\gamma+1)(\gamma+2)}{2r^2}+\gamma+\frac{1}{2}+\frac{4g_1^2r^2}{\left(1+g_1r^2\right)^2}-\frac{2g_1}{1+g_1r^2};$$

$$g_1=\frac{2}{2\gamma+3}. \tag{16}$$

The energy shift is two units here, and it appears both in the numerical constant in (16) (it is $1/2$ here and $7/2$ in [6]) and the factorization energy (-2 here and 0 in [6]). Similar, but more complicated isospectral potentials would arise from choosing $N>1$.

3.3. The T_4 case

In contrast with the previous two cases, the boundary condition at the origin now allows both the regular and the singular solution in (15). The ratio of the two coefficients, α_1 and α_2 appears as a new parameter in $V_2(r)$: the resulting potential family will have the same spectrum, but different shape. Similarly to the T_3 case, this one is usually also interpreted as a situation with broken supersymmetry, because the spectra of the partner potentials (and, of course, of the whole family) is identical.

In order to get a situation similar to the T_3 and T_1 cases, we can restrict this potential family to a single potential, i.e. to that with $\alpha_1=0$ in (15). The structure of $\phi(r)$ then becomes the same as before: it will contain only a single confluent hypergeometric function. With the loss of the generality, however, the nodelessness of $\phi(r)$ cannot be guaranteed in general, rather it has to be checked in each case separately.

Taking also into account the asymptotic boundary condition which now requires $B=-1$, we find that the $a-b+1=-N$ polynomial condition now leads to factorization energies $\epsilon=2N+2$. The $N=0$ choice again results in another harmonic oscillator potential, with γ decreased with one unit and with

an energy shift of one unit upwards. For $N=1$ a potential similar to that in [6] arises, whenever $\gamma > 1/2$ holds. (As we have mentioned already, this latter condition secures that the polynomial $F(-1,-\gamma+\frac{1}{2};r^2) = 1+2r^2/(2\gamma-1)$ remains nodeless, and there will be no singularities in the $V_2(r)$. In fact, the $2\gamma+3=E_0>\epsilon=2N+2$ condition also leads to $\gamma>1/2$ for $N=1$.) The potential is then

$$V_2(r) = \frac{1}{2}r^2 + \frac{\gamma(\gamma-1)}{2r^2} + \gamma + \frac{5}{2} + \frac{4g_1^2 r^2}{(1+g_1 r^2)^2} - \frac{2g_1}{1+g_1 r^2};$$

$$g_1 = \frac{2}{2\gamma-1}. \tag{17}$$

The functional form of $V_2(r)$ is essentially the same as that of (16), only the value of γ, the numerical constant and g_1 is different. Similarly to the T_3 case, further potentials isospectral with a harmonic oscillator can be constructed by choosing $N>1$, but the nodelessness of $\phi(r)$ has to be checked in each case.

3.4. The T_2 case

The situation here is the same as in the T_4 case: both the regular and the singular solutions are allowed by the boundary condition at the origin. This means, that we again have a whole family of potentials $V_2(r)$, which have the same spectrum and differ only in their shape. As before, we again restrict our attention to a particular member of this family, i.e. to the potential obtained with $\alpha_1=0$. Furthermore, we consider the polynomial condition $a-b+1=-N$, which leads to $\epsilon=-2N+2\gamma+1$.

For $N=0$,

$$V_2(r) = \frac{1}{2}r^2 + \frac{\gamma(\gamma-1)}{2r^2} + \frac{1}{2},$$

which corresponds to another harmonic oscillator potential with the γ value decreased by one unit and also shifted lower by one energy unit. Clearly, this corresponds to the usual SUSYQM transformation which inserts a new state (at $E=2\gamma+1$) below the ground state of $V_1(r)$. For $N=1$ we find that $F(-1,-\gamma+\frac{1}{2};-r^2 = 1-2r^2/(2\gamma-1)$, which has a node at a positive value of r, unless $\gamma<1/2$ holds. As in the T_4 case, here we have to check the nodelessness of $\phi(r)$ in each case, because it cannot be automatically guaranteed after we restricted the general solution by selecting $\alpha_1=0$ in (15). The functional form of $V_2(r)$ is the same as (17), but with $g_1=2/(1-2\gamma)$.

3.5. The one-dimensional case

We note that similarly to the radial equation, the one-dimensional case can also be handled in the present framework. The difference arises from the different boundary conditions. The various transformation types corresponding to solutions of a Schrödinger equation with different boundary conditions has been described in [14]. Here we only mention the example discussed in [6] for one dimension. In order to construct $V_2(r)$ with one more bound state than $V_1(r)$, the solution diverging in both directions has to be considered [14]. In general, such a solution should be constructed from the two linearly independent solutions of the Schrödinger equation, similarly to the situation seen in Sections 3.3 and 3.4. However, taking only one of these, $\phi(x) = \exp(\frac{x^2}{2})F(-1,\frac{1}{2};-x^2) = \exp(\frac{x^2}{2})(1+2x^2)$, we obtain the new state introduced for $V_2(x)$ as $\psi_0^{(2)}(x) \simeq 1/\phi(x)$. Note that $V_2(r)$ is symmetric with respect to the $x \leftrightarrow -x$ transformation. In the general case $V_2(r)$ would be asymmetric, similarly to the SUSYQM partner potentials of the one-dimensional harmonic oscillator constructed in Ref. [17].

4. Discussion

The relation of the two procedures outlined in Sections 2 and 3 can be interpreted in a simple way by noting that the partner potentials are linked by $V_+(r) - V_-(r) = W'(r)$ and $V_1(r) - V_2(r) = (\ln\phi(r))'$. From this

$$W(r) = (\ln\phi(r))' + c \tag{18}$$

follows. Direct integration of (2) and (7) with $W_0(r) = r + (\gamma+1)r^{-1}$, as in [6] and $c=0$, indeed, recovers the general solution $\phi(r)$ specific to the T_3 case:

$$\phi(r) \simeq r^{\gamma+1}\exp\left(\frac{r^2}{2}\right)\Pi_{i=0}^{N}\left(1+g_i r^2\right). \qquad (19)$$

In addition to the notation of [6], $g_0=0$ was also introduced for convenience. This function is also an Nth order polynomial, as expected from (11) for $a=-N$. In addition to the T_3 case, the situation should be the same for the other cases mentioned here, including also the one-dimensional case. We note that in the T_2, T_4 and in the one-dimensional cases discussed here, (19) is not the most general form of the solution, rather it is a specific member of a family of solutions obtained as the linear combination of two independent solutions.

In fact, all the $V_2(r)$ potentials derived from polynomial-type $\phi(r)$ solutions can be expressed by a common formula. Substituting $a=-N$ in (11) and combining it with (8) and (10) one gets

$$V_2(r) = \frac{1}{2}r^2 + \frac{\gamma(\gamma+1)+2A}{2r^2} + \gamma + \frac{3}{2} - B - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln F\left(-N, A+\frac{1}{2}; Cr^2\right). \qquad (20)$$

The solutions relevant to the T_1, T_2, T_3 and T_4 cases can then be obtained by substituting $[A,B,C] = [\gamma+1,-1,1]$, $[-\gamma,1,-1]$, $[\gamma+1,1,-1]$ and $[-\gamma,-1,1]$, respectively. In the $N=0$ case the last term in (20) cancels and $V_2(r)$ contains only terms characteristic of the three-dimensional harmonic oscillator potential. For $N=1$, $F(-1,A+\frac{1}{2};Cr^2) = 1+g_1 r^2$, with $g_1 = -2C/(2A+1)$, which gives rise to two new terms, formally identical ith the last two terms of (16).

Another interesting formula can also be derived if recalling (5) and (10), i.e. $V_1(r) = V_+(r) - \Delta$, which also indicates $V_2(r) = V_-(r) - \Delta$. From this $V_1(r) + V_2(r) = W^2(r) - 2\Delta$ directly follows. On the other hand, (8) and (9) lead to $V_2(r) + V_1(r) = 2\epsilon + (\phi'/\phi)^2$. Combinig these two formulas, we get

$$\left[W(r) - (\ln\phi(r))'\right]\left[W(r) + (\ln\phi(r))'\right] = 2(\epsilon+\Delta). \qquad (21)$$

This means, that $W(r) = \pm(\ln\phi(r))'$ always requires $\epsilon = -\Delta$ to hold. In this way Δ, the constant appearing in the Riccati Eq. (6) in Section 2 is related to the factorization energy used in the formulas in Section 3. Note that the $W = -(\ln\phi)'$ choice is also acceptable here, and it simply corresponds to the $V_+ \leftrightarrow V_+$ and $V_1 \leftrightarrow V_2$ replacements.

We stress that although the examples we presented here are related to the harmonic oscillator in three and one dimensions, our treatment can be applied to other types of potentials as well. Besides the relatively simple shape-invariant potential [18] (see also [19,8–11] for a list of them), any member of the more general Natanzon potential class [5] can also serve as the $V_+^{(0)}(r)$ reference potential in (5). However, for the first such applications some more thoroughly studied special Natanzon potentials could be the best candidates [20–28].

Acknowledgements

This work has been supported by by the OTKA grant No. F20689 (Hungary).

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