On some properties of P-matrix sets

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Abstract

A nonempty set $\mathscr{C}$ in $\mathbb{R}^{n\times n}$ is said to have the row-**P**-property if every row representative of $\mathscr{C}$ is a **P**-matrix. We show that this property is equivalent to saying that for every nonzero x in $\mathbb{R}^n$ there is an index i with $x_i(Mx)_i > 0$ for all $M \in \mathscr{C}$. We relate this concept to the unique solvability of certain nonlinear complementarity problems. We also show that when $\mathscr{C}$ is compact and has the row-**P**-property, there exists a vector $u > 0$ such that $Mu > 0$ for all $M \in \mathscr{C}$.

1. Introduction

In 1962, Fiedler and Pták [1] introduced the concepts of **P** and $\mathbf{P}_0$-matrices. A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathbf{P}(\mathbf{P}_0)$-matrix if every principal minor of M is positive (respectively, nonnegative). Since then, numerous equivalent formulations of these concepts have been given, and moreover, the importance of these and related matrices (**M**-matrices, **S**-matrices, etc.,) in optimization, differential equations, statistics, and various areas have been well documented in the literature.

To motivate our results, we consider the following well-known equivalent conditions for a matrix $M \in \mathbb{R}^{n\times n}$ [2,1]:

(a) M is a **P**-matrix.

(b) For each $x \neq 0$ in $\mathbb{R}^n$, there is an index i such that $x_i(Mx)_i > 0$.

(c) For every $q \in \mathbb{R}^n$, the linear complementarity problem LCP(M, q) has a unique solution.

In this paper, we aim to extend this equivalence to a set of matrices. Concerning the equivalence (a) $\iff$ (b), we ask the following: Given a set $\mathscr{C}$ of $n \times n$ matrices, when does the condition (b) hold uniformly for all $M \in \mathscr{C}$? That is, for any nonzero $x \in \mathbb{R}^n$, when is there an index i such that

$$x_i(Mx)_i > 0 \quad \forall M \in \mathscr{C}? \tag{1}$$

In a recent paper, Rohn and Rex [3] have shown that when $\mathscr{C}$ is an interval in $\mathbb{R}^{n\times n}$, i.e.,

$$\mathscr{C} = \{M \in \mathbb{R}^{n\times n}: \underline{M} \leqslant M \leqslant \overline{M}\} \tag{2}$$

(where $\underline{M}$ and $\overline{M}$ are given $n \times n$ matrices and the inequality defined above is componentwise), for any nonzero x, the uniform inequality (1) holds for some index i if and only if every matrix in this interval is a **P**-matrix. In this paper, we introduce the concept of *row-**P**-property* for a set of matrices, see Section 2 for the definition, and show that this property is equivalent to: for any $x \neq 0$, there exists an index i satisfying (1). In [3], Rohn and Rex also show that the interval described in (2) consists of P-matrices if and only if certain finite set of 'extreme' matrices are P-matrices. We extend this result by showing that a compact convex set $\mathscr{C}$ in $\mathbb{R}^{n\times n}$ has the row-**P**-property if and only if the set of extreme points (matrices) of $\mathscr{C}$ has the row-**P**-property.

The row-**P**-property plays a role in the complementarity problems as well. In [4], for a finite set of matrices, this concept was called the **P**-property and was shown to be equivalent to the unique solvability in vertical complementarity problems. In this paper, we extend this equivalence to certain complementarity problems arising from a compact set $\mathscr{C}$ and a compact set of vectors.

An important property of a **P**-matrix M is the existence of a vector $u > 0$ such that $Mu > 0$. (This is the so-called **S**-property of M, introduced, once again, by Fiedler and Pták.) Given a set of matrices $\mathscr{C}$, we ask whether there is a vector $u > 0$ such that $Mu > 0$ for all $M \in \mathscr{C}$. It turns out, see Corollary 10 in Section 4, that the row-**P**-property plays a role here also: If $\mathscr{C}$ has the row-**P**-property and is compact, then such a vector u exists. We show in Theorem 11 that this 'uniform' S-property of $\mathscr{C}$ is equivalent to the row-**P**-property of $\mathscr{C}$ when $\mathscr{C}$ is compact and consists of **Z**-matrices.

2. Preliminaries

Consider a *nonempty* set $\mathscr{C}$ of matrices in $\mathbb{R}^{n\times n}$. A matrix $A \in \mathbb{R}^{n\times n}$ is called a *row representative* of $\mathscr{C}$ if for every $i = 1, 2, \ldots, n$, the ith row of A, denoted by

A_i, is the ith row of some matrix in $\mathscr{C}$. Let $\hat{\mathscr{C}}$ denote the set of all row representatives of $\mathscr{C}$, i.e.,

$$\hat{\mathscr{C}} := \{A \in \mathbb{R}^{n\times n}: \text{for each } i \in \{1,2,\ldots,n\}, A_i = M_i \text{ for some } M \in \mathscr{C}\}.$$

We call $\hat{\mathscr{C}}$ the *row-completion* of $\mathscr{C}$. It is clear that $\mathscr{C} \subseteq \hat{\mathscr{C}}$.

We say that $\mathscr{C}$ has the *row-***P***-property* (*row-*$\mathbf{P_0}$*-property*) if every matrix in $\hat{\mathscr{C}}$ is a $\mathbf{P}(\mathbf{P_0})$-matrix, i.e., every row representative A of $\mathscr{C}$ is a $\mathbf{P}(\mathbf{P_0})$-matrix. We note that if $\mathscr{C}$ has the row-**P**-property (row-$\mathbf{P_0}$-property), then every $M \in \mathscr{C}$ is a $\mathbf{P}(\mathbf{P_0})$-matrix since a matrix $M \in \mathscr{C}$ could be a row representative of itself. So in this case, we can think of $\mathscr{C}$ as a $\mathbf{P}(\mathbf{P_0})$-matrix set.

To see an example, first we recall that a matrix $M = [m_{ij}]$ is said to be (*row*) *strictly diagonally dominant* if for each $i = 1,2,\ldots,n$,

$$|m_{ii}| > \sum_{j\neq i} |m_{ij}|.$$

It is well known that if such a matrix has a positive diagonal, then the matrix is a **P**-matrix, see Section 3.3 in [2]. Now consider any set $\mathscr{C} \subseteq \mathbb{R}^{n\times n}$ in which every matrix is strictly diagonally dominant and has a positive diagonal. It is clear that any row representative of $\mathscr{C}$ is a **P**-matrix, i.e., $\mathscr{C}$ has the row-**P**-property. The following lemma is needed later.

Lemma 1. *If $\mathscr{C}$ is compact in $\mathbb{R}^{n\times n}$, then so is $\hat{\mathscr{C}}$.*

Proof. Let $\{A^k\}$ be a sequence in $\hat{\mathscr{C}}$. Let $i = 1$. For each $k = 1,2,\ldots$, there exists an $M^k \in \mathscr{C}$ such that $(A^k)_1 = (M^k)_1$. By compactness of $\mathscr{C}$, we may assume that a subsequence $\{M^{k_j}\}$ converges to a matrix $M^{(1)} \in \mathscr{C}$. This means that the subsequence $\{A^{k_j}\}$ has the property that $(A^{k_j})_1$ converges to $(M^{(1)})_1$. By renaming, we may assume that $(A^k)_1$ converges to $(M^{(1)})_1$. Now starting with the (new) sequence $\{A^k\}$, we argue as above and produce a subsequence of $\{A^k\}$, still called $\{A^k\}$, such that $(A^k)_2$ converges to $(M^{(2)})_2$ where $M^{(2)} \in \mathscr{C}$. We continue this argument for $i = 3,4,\ldots,n$ and conclude that an appropriate subsequence of $\{A^k\}$, which we call $\{A^k\}$ for simplicity, converges to a matrix $A \in \mathbb{R}^{n\times n}$ with the property that for each i, $A_i = (M^{(i)})_i$. Clearly $A \in \hat{\mathscr{C}}$ and we have proved that the given sequence $\{A^k\}$ has a convergent subsequence in $\hat{\mathscr{C}}$. This proves the compactness of $\hat{\mathscr{C}}$. □

3. The row-P-property

Theorem 2. *A set $\mathscr{C} \subseteq \mathbb{R}^{n\times n}$ has the row-***P***-property if and only if for any $x \neq 0$ in $\mathbb{R}^n$, there exists an index $j \in \{1,2,\ldots,n\}$ such that $x_j(Mx)_j > 0$ for every matrix $M \in \mathscr{C}$.*

Proof. Assume $\mathscr{C}$ has the row-**P**-property and suppose to the contrary that there exists an $x^* \neq 0$ in $\mathbb{R}^n$ such that for any index i, there exists an $M^{(i)} \in \mathscr{C}$ with $x_i^*(M^{(i)}x^*)_i \leqslant 0$. We construct a row representative A with its ith row given by $A_i = (M^{(i)})_i$. Then clearly $x_i^*(Ax^*)_i \leqslant 0$ for all indices i, contradicting our assumption that $\mathscr{C}$ has the row-**P**-property.

For the converse, take any row representative matrix A of $\mathscr{C}$ and any non-zero vector $x \in \mathbb{R}^n$. Then for every index i, there exists $M^{(i)} \in \mathscr{C}$ such that $A_i = (M^{(i)})_i$. Since there exists an index j with $x_j(Mx)_j > 0$ for every matrix $M \in \mathscr{C}$ by our assumption, it is clear that for that index j, $x_j(Ax)_j = x_j(M^{(j)}x)_j$ is also positive. Hence A is a **P**-matrix and so $\mathscr{C}$ has the row-**P**-property. □

Theorem 3. *Suppose that $\mathscr{C}$ is compact and convex in $\mathbb{R}^{n\times n}$. Let $\mathscr{E}$ denote the set of all extreme points of $\mathscr{C}$. Then $\mathscr{C}$ has the row-**P**-property if and only if $\mathscr{E}$ has the row-**P**-property.*

Proof. We first observe that $\mathscr{E}$ is nonempty and every matrix in $\mathscr{C}$ is a finite convex combination of matrices in $\mathscr{E}$, by Corollary 18.5.1 in [5]. Suppose that $\mathscr{E}$ has the row-**P**-property. Let $x \neq 0$ in $\mathbb{R}^n$. By Theorem 1, there exists an index i such that $x_i(Mx)_i > 0$ for all $M \in \mathscr{E}$. Now take any $A \in \mathscr{C}$. By writing A as a convex combination of a finite number of matrices in $\mathscr{E}$, we see that $x_i(Ax)_i > 0$. Since A is arbitrary, by Theorem 2, we conclude that $\mathscr{C}$ has the row-**P**-property. The other implication is obvious. □

We now specialize the above theorems to an interval $\mathscr{C}$ in $\mathbb{R}^{n\times n}$ described by (2). It is clear that $\mathscr{C}$ is compact and convex with the extreme point set $\mathscr{E} = \{M : M_{ij} = \underline{M}_{ij} \text{ or } \overline{M}_{ij}\}$ where M_{ij} denotes the (i,j)-entry in M. It is obvious that in this case, $\hat{\mathscr{C}} = \mathscr{C}$ and $\hat{\mathscr{E}} = \mathscr{E}$; hence $\mathscr{C}$ ($\mathscr{E}$) has the row-**P**-property if and only if each matrix in $\mathscr{C}$ (respectively, $\mathscr{E}$) is a **P**-matrix. Theorems 2 and 3 now give the following known results [3].

Corollary 4. *Suppose $\mathscr{C}$ is an interval in $\mathbb{R}^{n\times n}$ as described in* (2).

(i) *Every matrix in $\mathscr{C}$ is a **P**-matrix if and only if for any $x \neq 0$ in $\mathbb{R}^n$, there exists an index $j \in \{1, 2, \ldots, n\}$ such that $x_j(Mx)_j > 0$ for all $M \in \mathscr{C}$.*

(ii) *Every matrix in $\mathscr{C}$ is a **P**-matrix if and only if every matrix in the set $\{M : M_{ij} = \underline{M}_{ij} \text{ or } \overline{M}_{ij}\}$ is a **P**-matrix.*

Regarding the item (ii) above, it should be remarked that Rohn and Rex [3], in their characterization, use only 2^n matrices whereas our extreme point set has 2^{n^2} objects.

Here is an analog of Theorem 2 for $\mathbf{P}_0$-matrices.

Theorem 5. *A set $\mathscr{C} \subseteq \mathbb{R}^{n\times n}$ has the row-$\mathbf{P}_0$-property if and only if for any $x \neq 0$ in $\mathbb{R}^n$, there exists an index $j \in \{1, 2, \ldots, n\}$ with $x_j \neq 0$ and $x_j(Mx)_j \geqslant 0$ for every matrix $M \in \mathscr{C}$.*

Proof. Suppose $\mathscr{C}$ has the row-$\mathbf{P}_0$-property. Then for every $\varepsilon > 0$, $\mathscr{C} + \varepsilon I := \{M + \varepsilon I : M \in \mathscr{C}\}$ has the row-**P**-property. Fix any $x \neq 0$ and let $J = \{j : x_j \neq 0\}$. Then for any $\varepsilon > 0$, there exists an index $j_\varepsilon \in J$ such that $x_{j_\varepsilon}((M + \varepsilon I)x)_{j_\varepsilon} > 0$ for every matrix $M + \varepsilon I \in \mathscr{C} + \varepsilon I$. By considering an appropriate sequence $\varepsilon_k \downarrow 0$, we can obtain a single index $j \in J$ such that $x_j((M + \varepsilon_k I)x)_j > 0$ for every matrix $M \in \mathscr{C}$ and for every k. Letting $k \to \infty$, we get $x_j(Mx)_j \geqslant 0$ for all $M \in \mathscr{C}$ with $x_j \neq 0$.

The converse follows by arguing exactly the same way as in the second part of the proof of Theorem 2 with '$\geqslant$' in place of '$>$' and 'nonnegative' in place of 'positive'. □

4. Complementarity problems

In this section, we describe the row-**P**-property via unique solvability of certain complementarity problems. We assume that the set $\mathscr{C}$ is described by

$$\mathscr{C} := \{M^\alpha \in \mathbb{R}^{n \times n} : \alpha \in \Gamma\},$$

where Γ denotes a nonempty index set. Corresponding to this description, we consider a set of vectors in $\mathbb{R}^n$ given by

$$\mathrm{q} := \{q^\alpha \in \mathbb{R}^n : \alpha \in \Gamma\}.$$

As we have done for $\mathscr{C}$, we may also define the *row-completion of* q by

$$\hat{\mathrm{q}} := \{p \in \mathbb{R}^n : \text{for each } i \in \{1, 2, \ldots n\}, p_i = (q^\alpha)_i \text{ for some } q^\alpha \in \mathrm{q}\}$$

As in Lemma 1, it is easy to see that if q is compact, then so is $\hat{\mathrm{q}}$.

Assuming $\mathscr{C}$ and q *are bounded,* we define the function $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x) := \inf_\alpha (M^\alpha x + q^\alpha) \quad \text{(componentwise)}.$$

Then the complementarity problem, denoted by $\mathrm{CP}(\mathscr{C}, \mathrm{q})$, is to find a vector $x \in \mathbb{R}^n$ such that

$$x \wedge f(x) = 0,$$

where '$\wedge$' denotes the componentwise minimum. Note that when the index set Γ has finite number of elements, $\mathrm{CP}(\mathscr{C}, \mathrm{q})$ reduces to a vertical linear complementarity problem [4].

Lemma 6. *Consider $\mathscr{C}$ and* q *bounded, and let*

$$f(x) = \inf_\alpha (M^\alpha x + q^\alpha).$$

Then for every x and y in $\mathbb{R}^n$,

$$\inf_\alpha M^\alpha(x-y) \leqslant f(x) - f(y) \leqslant \sup_\alpha M^\alpha(x-y).$$

Moreover, f is continuous.

Proof. For any x and y in $\mathbb{R}^n$ and for any index $i \in \{1, 2, \ldots, n\}$, we have

$$\inf_\alpha (M^\alpha y + q^\alpha)_i + \inf_\alpha (M^\alpha (x-y))_i \leqslant \inf_\alpha [(M^\alpha y + q^\alpha) + (M^\alpha (x-y))]_i$$

and hence

$$\inf_\alpha (M^\alpha (x-y))_i \leqslant \inf_\alpha (M^\alpha x + q^\alpha)_i - \inf_\alpha (M^\alpha y + q^\alpha)_i. \tag{3}$$

By multiplying the above inequality by -1, and interchanging x and y, we get

$$\inf_\alpha (M^\alpha x + q^\alpha)_i - \inf_\alpha (M^\alpha y + q^\alpha)_i \leqslant \sup_\alpha (M^\alpha (x-y))_i. \tag{4}$$

Combining (3) and (4), we have the stated inequalities. The continuity of f follows easily from these inequalities since $\sup_\alpha \|M_\alpha\| < \infty$. $\square$

Theorem 7. *Suppose* $\mathscr{C}$ *is compact and* q *is bounded. If* $\mathscr{C}$ *has the row-***P***-property, then f is a* **P***-function, i.e., for all* $x \neq y$ *in* $\mathbb{R}^n$ *there exists an index j such that* $(x_j - y_j)(f_j(x) - f_j(y)) > 0$.

Proof. Let $x \neq y$ in $\mathbb{R}^n$ so that $z = x - y \neq 0$. Since $\mathscr{C}$ has the row-**P**-property, by Theorem 2, there exists an index j such that $z_j(M^\alpha z)_j > 0$ for every $M^\alpha \in \mathscr{C}$. If $z_j = x_j - y_j > 0$, then $(M^\alpha z)_j > 0$ and by compactness of $\mathscr{C}$, $\inf_\alpha (M^\alpha z)_j > 0$. From Lemma 6, we have $f_j(x) - f_j(y) \geqslant \inf_\alpha (M^\alpha z)_j > 0$. Likewise, if $z_j = x_j - y_j < 0$, then $\sup_\alpha (M^\alpha z)_j < 0$ and again by Lemma 6, $f_j(x) - f_j(y) \leqslant \sup_\alpha (M^\alpha z)_j < 0$. So in either case, we have an index j such that $(x_j - y_j)(f_j(x) - f_j(y)) > 0$. Thus, f is a **P**-function. $\square$

By slightly modifying the above proof we arrive at the following.

Corollary 8. *Suppose that* $\mathscr{C}$ *and* q *are bounded and* $\mathscr{C}$ *has the row-*$\mathbf{P}_0$*-property. Then f is a* $\mathbf{P}_0$*-function.*

Now we are ready for the main theorem of this section.

Theorem 9. *Suppose* $\mathscr{C}$ *is compact in* $\mathbb{R}^{n\times n}$. *Then the complementarity problem* CP($\mathscr{C}$, q) *has a unique solution for every compact* q *in* $\mathbb{R}^n$ *if and only if* $\mathscr{C}$ *has the row-***P***-property.*

Proof. Assume that CP($\mathscr{C}$, q) has a unique solution for every compact q. Suppose, if possible, that there exists $z \neq 0$ in $\mathbb{R}^n$ such that for every index i, there exists $M^{(i)} \in \mathscr{C}$ with $z_i(M^{(i)}z)_i \leqslant 0$. Defining, for any vector x, $x^+ := \max\{0, x\}$ and $x^- := x^+ - x$, we observe that $z^+ \neq z^-$ and $z_i^+ \wedge (M^{(i)}z)_i^+ = 0 = z_i^- \wedge (M^{(i)}z)_i^-$. We define q^α by $q^\alpha := (M^\alpha z)^+ - M^\alpha z^+ = (M^\alpha z)^- - M^\alpha z^-$. Then $M^\alpha z^+ + q^\alpha = (M^\alpha z)^+$ and $M^\alpha z^- + q^\alpha = (M^\alpha z)^-$; hence $f(z^+) = \inf_\alpha (M^\alpha z)^+ \geqslant 0$ and $f(z^-) = \inf_\alpha (M^\alpha z)^- \geqslant 0$. We see that for every index i,

$$0 \leqslant [z^+ \wedge f(z^+)]_i \leqslant z_i^+ \wedge (M^{(i)}z)_i^+ = 0$$

which shows that z^+ is a solution of CP($\mathscr{C}$, q). A similar argument shows that z^- is also a solution of the same problem. Since $z^+ \neq z^-$, we reach a contradiction to the uniqueness of solution to CP($\mathscr{C}$, q). Thus $\mathscr{C}$ has the row-**P**-property.

Now for the converse. Assume that $\mathscr{C}$ has the row-**P**-property and let q be compact. We show the existence of a solution to CP($\mathscr{C}$, q) using topological degree theory.

We first define the function $H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ by

$$H(x, t) := x \wedge [tf(x) + (1 - t)x].$$

By continuity of f (cf. Lemma 6), it is clear that the function $H(x, t)$ is continuous. We claim that the set

$$\mathscr{X} := \{x \in \mathbb{R}^n : H(x, t) = 0 \text{ for some } t \in [0, 1]\}$$

is bounded. To see the claim, suppose that there exist sequences $\{x^k\}$ with $\|x^k\| \to \infty$ and $\{t_k\} \subseteq [0, 1]$ such that $H(x^k, t_k) = 0$. Upon division by $\|x^k\|$, we obtain

$$\frac{x^k}{\|x^k\|} \wedge \left[t_k \frac{\inf_\alpha (M^\alpha x^k + q^\alpha)}{\|x^k\|} + (1 - t_k) \frac{x^k}{\|x^k\|} \right] = 0. \tag{5}$$

Without loss of generality, we may assume $x^k / \|x^k\| \to d \neq 0$ and $t_k \to t^*$. Also note that for each k, $\inf_\alpha (M^\alpha x^k + q^\alpha) = A^k x^k + p^k$ for some $A^k \in \hat{\mathscr{C}}$ and $p^k \in \hat{\mathrm{q}}$ by compactness of $\mathscr{C}$ and q. In view of Lemma 1, we may assume that $A^k \to A \in \hat{\mathscr{C}}$ and $p^k \to p \in \hat{\mathrm{q}}$. Upon letting $k \to \infty$, we get from Eq. (5),

$$d \wedge [t^* A + (1 - t^*)I]d = 0.$$

This says that d is a solution of the homogeneous linear complementarity problem LCP($t^*A + (1 - t^*)I$, 0). Since $\mathscr{C}$ has the row-**P**-property, A is a **P**-matrix, and so is $t^*A + (1 - t^*)I$. Then by Theorem 3.3.7 in [2], zero is the only solution of the above LCP, contradicting our assumption that $d \neq 0$. Hence the set $\mathscr{X}$ is bounded.

Now let Ω be a bounded open set in $\mathbb{R}^n \times [0,1]$ which contains the set $\mathscr{Z}$. Then by the homotopy invariance of the degree (Theorem 2.1.2 in [6]),

$$1 = \deg(I, \Omega, 0) = \deg(H(\cdot, 0), \Omega, 0) = \deg(H(\cdot, 1), \Omega, 0).$$

By Theorem 2.1.1 in [6], the function $H(x, 1) = x \wedge f(x)$ has a zero in Ω, i.e., CP($\mathscr{C}$, q) has a solution. Now for the uniqueness. We know from Theorem 7 that the function f is a **P**-function. For such a function f, it is well known (and easy to see) that there is at most one zero of the equation $x \wedge f(x) = 0$. Thus we have proved that CP($\mathscr{C}$, q) has a unique solution. This completes the proof. □

As a consequence of Theorem 9 given above, we have the following 'uniform' **S**-property.

Corollary 10. *Suppose $\mathscr{C}$ is compact in $\mathbb{R}^{n\times n}$ with the row-**P**-property. Then there exists a vector $u > 0$ such that $Mu > 0$ for all $M \in \mathscr{C}$.*

Proof. In Theorem 9, put $q^\alpha = -e$ for all α where e is the vector of ones in $\mathbb{R}^n$. Then the solution x to the complementarity problem will satisfy the conditions $x \geqslant 0$ and $M^\alpha x - e \geqslant 0$ for all α. By compactness of $\mathscr{C}$, we can perturb x to get $u > 0$ with $M^\alpha u > 0$ for all α. □

The following example shows that the conclusion of the above corollary may fail without the row-**P**-property.

Example. Let $\mathscr{C} = \{A, B\}$ where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Note that both A and B are **P**-matrices. Suppose that there is a vector $u > 0$ such that $Au > 0$ and $Bu > 0$. Then $(Au)_1 > 0$ and $(Bu)_2 > 0$ mean $u_1 - 2u_2 > 0$ and $-2u_1 + u_2 > 0$, hence $u_1 > 2u_2 > 4u_1$, contradicting $u_1 > 0$. Hence there is no such u.

In the classical LCP theory, the **P**-property is the same as the **S**-property for **Z**-matrices (Theorem. 3.11.10 in [2]). (Recall that a matrix is a **Z**-matrix if all of its off-diagonal entries are nonpositive.) This equivalence carries over for sets of **Z**-matrices in the following way.

Theorem 11. *Suppose $\mathscr{C} \subseteq \mathbb{R}^{n\times n}$ is a compact set of **Z**-matrices. Then the following are equivalent.*

(a) *$\mathscr{C}$ has the row-**P**-property.*

(b) *There exists a vector $u > 0$ such that $Mu > 0$ for all $M \in \mathscr{C}$.*

Proof. The implication (a) ⇒ (b) is given in the previous corollary. Assume that (b) holds. Let A be any row representative of $\mathscr{C}$. Clearly A is a **Z**-matrix. It follows from (b) that $Au > 0$. Thus by Theorem 3.11.10 in [2], A is a **P**-matrix. We conclude that $\mathscr{C}$ has the row-**P**-property. □

Throughout this paper, we have considered the row-**P**-property. By considering the column representatives instead of row representatives, one can define the concept of *column*-**P**-*property* for a set $\mathscr{C}$ in $\mathbb{R}^{n\times n}$. When $\mathscr{C}$ is finite, this property is related to the uniqueness of solutions in certain 'horizontal complementarity problems', see [7]. When $\mathscr{C}$ is an interval, these two properties coincide. In the general case, except for obvious analogs (obtained by considering transposes) of results presented in the paper, nothing much is known. We end this paper by noting that even when the set $\mathscr{C}$ consists of **Z**-matrices, the row-**P** and the column-**P**-properties may be different.

Example. Let $\mathscr{C} = \{A, B\}$ where

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that every row representative of $\mathscr{C}$ is a **P**-matrix as well as a **Z**-matrix so $\mathscr{C}$ has the row-**P**-property, yet the column representative

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$$

is not a **P**-matrix. i.e., $\mathscr{C}$ does not have the column-**P**-property.

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