Ball Remotality In Banach Spaces And Related Topics

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Dedicated to all those people who love Banach spaces . . .

Acknowledgments

To produce a faithful acknowledgment I have to go back about ten years. It was an article published in a daily newspaper on the ever discussed Fermat's last theorem. It was like a fairy tale to me : how a man reached his dream that was waiting for him for three hundred years. So let's give a sincere thanks to the writer of that article.

Though it is not possible to make a single list of all those people who have much or less contribution in my research, I mention some of them.

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Contents

Chapter 1

Introduction

Let us fix some notations and conventions first. Unless otherwise stated, most of our results hold for both real ($\mathbb{R}$) or complex ($\mathbb{C}$) scalars. We will denote the scalar field by $\mathbb{F}$.

The closed unit ball and the unit sphere of a Banach space X will be denoted by B_X and S_X respectively. By a subspace, we will mean a norm closed linear subspace. We denote by $NA(X)$ the set of all $x^* \in X^*$ which attain their norm on B_X. We will identify $x \in X$ with its canonical image in X^{**}.

For a closed and bounded set C in a Banach space X, the farthest distance map ϕ_C is defined as

$$\phi_C(x) = \sup\{\|z - x\| : z \in C\}, \quad x \in X.$$

ϕ_C is a Lipschitz continuous convex function. For $x \in X$, we denote the set of points in C farthest from x by $F_C(x)$, *i.e.*,

$$F_C(x) = \{z \in C : \|z - x\| = \phi_C(x)\}$$

Note that this set may be empty. Let $R(C, X) = \{x \in X : F_C(x) \neq \emptyset\}$. We will write $R(C)$ when there is no confusion about the ambient space. Call a closed and bounded set C remotal if $R(C, X) = X$ and densely remotal if $R(C, X)$ is norm dense in X.

Clearly, a compact set is remotal. The study of densely remotal sets was initiated by Edelstein [19] who proved that any closed and bounded set in a uniformly convex space is densely remotal. Asplund [3] extended this to show that any closed and bounded set in a reflexive locally uniformly convex Banach space is densely remotal. In [39], Zizler generalized Asplund's result by showing that if X^* is an Asplund space with a LUR dual norm, then any closed and bounded set in X is densely remotal. Then Lau [30] showed that

Theorem 1.0.1. [30, Theorem 2.3] *Any weakly compact set in any Banach space is densely remotal with respect to any equivalent norm.*

Deville and Zizler [17] proved a partial converse of this result :

Theorem 1.0.2. [17, Proposition 4] *Let X be a Banach space and C is a closed bounded convex subset of X. If C is densely remotal for every equivalent renorming on X, then C is weakly compact.*

They also proved that

Theorem 1.0.3. [17, Proposition 3] *If X has the Radon-Nikodým Property (RNP), every w*-compact set in X^* is densely remotal with respect to any equivalent dual norm.*

The survey article [14] contains many results on the existence of nearest and farthest points of sets and its relation with some geometric properties of Banach spaces.

Note that B_X is always a remotal set in any Banach space X. So it is natural to ask what happens in case of B_Y for a subspace Y? This issue was addressed in a recent paper [10].

Definition 1.0.4. Let us call a subspace Y of a Banach space X

(a) ball remotal (BR), if B_Y is remotal in X;

(b) densely ball remotal (DBR), if B_Y is densely remotal in X.

The main object of this thesis is to study these properties more extensively. As noted in [10], it follows from the results noted above that :

(a) Any finite-dimensional subspace is BR.

(b) Any reflexive subspace is DBR.

(c) If X^* is an Asplund space with a LUR dual norm, then any subspace of X^* is DBR.

(d) If X has the RNP, then for any w*-closed subspace of X^* is DBR.

The point of departure in [10] is the observation that the space c_0 is DBR in ℓ_∞. This was proved by first showing that $\phi_{B_{c_0}}(x) = \|x\| + 1$ for all $x \in \ell_\infty$ and hence $R(B_{c_0}, \ell_\infty) = NA(\ell_1)$ and then appealing to the Bishop-Phelps Theorem.

They observed that the nice expression of $\phi_{B_{c_0}}$ is shared by a class of subspaces.

Let Y be a subspace of X. Clearly, $\phi_{B_Y}(x) \leq \phi_{B_X}(x) = \|x\| + 1$ for all $x \in X$.

Definition 1.0.5. Let us call a subspace Y of a Banach space X a $(*)$-subspace of X if

$$\phi_{B_Y}(x) = \|x\| + 1 \quad \text{for all } x \in X.$$

We will encounter these subspaces rather frequently in this thesis. We have the complete description of $R(B_Y, X)$ in this case [10] (see Proposition 2.3.13 below).

In [10], the authors used a different definition and proved that their definition implies ours. They, however, used only the above expression for ϕ_{B_Y} in most applications. We show in Chapter 2 that the two definitions are actually equivalent.

A natural example of $(*)$-subspace is X as a subspace of X^{**}. Since $c_0^{**} = \ell_\infty$, it follows that c_0 is a $(*)$-subspace of ℓ_∞. To prove that $R(B_{c_0}, \ell_\infty) = NA(\ell_1)$, the authors in [10] also used the fact that c_0 is an M-embedded space, *i.e.*, it is an M-ideal in its bidual.

Definition 1.0.6. [26] A subspace Y of a Banach space X is an M-ideal in X if there is a projection P on X^* with $\ker(P) = Y^\perp$ and for all $x^* \in X^*$, $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$.

They showed that for any two Banach space X and Y, the space $\mathcal{K}(X, Y)$ of all compact operators from X to Y, is $(*)$-subspace of the space $\mathcal{L}(X, Y)$ of all bounded operators from X to Y. And they showed that :

(a) For a large class of Banach spaces that include all reflexive spaces, $C(K)$ spaces and $L^1(\mu)$ spaces, $\mathcal{K}(X)$ is DBR in $\mathcal{L}(X)$.

(b) If X is a rotund Banach space with the RNP, and μ is the Lebesgue measure on $[0, 1]$, then $\mathcal{K}(L^1(\mu), X)$ is DBR in $\mathcal{L}(L^1(\mu), X)$.

They also noted that for a compact Hausdorff space K, the space $C(K, X)$ of all X-valued continuous functions on K is a $(*)$-subspace of the space $WC(K, X)$ of all continuous functions from K to X when X is endowed with the weak topology. And if X is reflexive, $C(K, X)$ is DBR in $WC(K, X)$.

Coming to stability results, they observed that many natural summands are BR. For example,

(a) Any subspace of a Hilbert space is BR.

(b) For any measure space (Ω, Σ, μ), $L^1(\mu)$ is BR in $L^1(\mu)^{**}$. In particular, ℓ_1 is BR in ℓ_1^{**}.

Now, c_0 is an M-ideal as well as a DBR subspace of ℓ_∞. The paper [10] has more examples of this phenomenon. Hence, it is natural to ask if any M-ideal is DBR, or, more specifically, is any M-embedded space DBR in its bidual. Both the questions have been answered in the negative in [10] (see Remark 7.2.13).

Observe that if $z \in B_Y$ is farthest from $x \in X$, then it is nearest from any point on the line $[x, z]$ extended beyond z. So a natural question is : Is there any relation between ball proximinality as introduced in [9] and ball remotality? Since an M-ideal is ball proximinal [29, Corollary 5.1.1, p 86], the above example shows that a ball proximinal subspace need not be DBR. We will show later a DBR subspace too need not be ball proximinal, or even proximinal (Remark 4.2.34).

We now provide a chapter-wise summary of the principal results of this thesis.

In Chapter 2, we obtain several characterizations of $(*)$-subspaces, including the equivalence of our definition with that of [10]. In the process, we obtain a farthest distance formula, which is also of independent interest. We completely characterize $(*)$- and DBR/BR subspaces of a Banach space. In this chapter, we also characterize 1-dimensional $(*)$-subspaces. It turns out that this depends on the existence of a strong unitary (Definition 2.4.2), a notion related to that of geometric unitaries studied in [8].

In Chapter 3, we study ball remotality in the classical Banach spaces c_0, c and ℓ_∞. The farthest distance formula of Chapter 2 takes a simpler form in these spaces. Using this, we characterize $(*)$- and DBR subspaces of c_0, c and ℓ_∞. We observe that one can prove that c_0 is DBR in ℓ_∞ without using the Bishop-Phelps Theorem. In the process, we prove that

(a) For a subspace $Y \subseteq c_0$, the following are equivalent :

- (i) Y is $(*)$- and DBR in c_0
- (ii) Y is $(*)$- and BR in c_0
- (iii) Y is $(*)$- and DBR in c
- (iv) Y is $(*)$- and DBR in ℓ_∞.

(b) In particular, c_0 is a $(*)$- and DBR subspace of both c and ℓ_∞.

(c) If a subspace of c or ℓ_∞ contains the constant sequence **1**, then it is $(*)$- and BR.

(d) c and $\widehat{c}$, the canonical image of c, are $(*)$- and BR in ℓ_∞.

We also show that c_0 has no finite dimensional $(*)$-subspace. We characterize all hyperplanes in c_0 which are $(*)$- and DBR in terms of the defining linear functionals.

Then we come to the space ℓ_1 and observe that ℓ_1 with its usual norm provides a simple non-reflexive example of a Banach space in which every subspace is DBR (Corollary 3.3.2). The existence of such spaces was observed in [10] with a more involved argument. We show that a hyperplane in ℓ_1 is BR if and only if it contains a strong unitary.

We note that most of the above results extend without difficulty to subspaces of $c_0(\Gamma)$, $\ell_\infty(\Gamma)$ and $\ell_1(\Gamma)$ for some arbitrary index set Γ. However, if Γ is uncountable, a hyperplane in $\ell_1(\Gamma)$ is always $(*)$- and BR.

From sequence spaces, we shift our attention to function spaces. In Chapter 4, we study ball remotality of subspaces of the space $C(K)$ of all scalar-valued continuous functions on K, where K is a compact Hausdorff space. We characterize $(*)$- and $(*)$- & DBR/BR subspaces of $C(K)$ in terms of the density of certain subsets of K. In the process, we prove that any Banach space embeds isometrically as a $(*)$- and DBR subspace of some $C(K)$ space.

We also study boundaries of a general subspace Y of $C(K)$. In particular, we relate the Choquet boundary of Y with other boundaries, in the process recapturing some classical results. We show that if Y is a subspace of co-dimension n in $C(K)$, then any closed boundary for Y can miss at most n points of K. In particular, if K has no isolated points, then any finite co-dimensional subspace cannot have any proper closed boundary.

Applying these results to the question of DBR subspaces, we show that an infinite compact Hausdorff space K has no isolated point if and only if any finite co-dimensional subspace, in particular, any hyperplane in $C(K)$ is DBR. We characterize $(*)$- and DBR hyperplanes in $C(K)$ in terms of the defining measures. We show that a Banach space X is reflexive if and only if X is a DBR subspace of any superspace. We also prove that any M-ideal or any closed self-adjoint subalgebra of $C(K)$ is DBR.

In Chapter 5, we study ball remotality of M-ideals in some function spaces and function algebras. Isolating a common feature of M-ideals in subspaces of $C(K)$, we define an *Urysohn pair* (A, D).

Definition 1.0.7. Let K be a compact Hausdorff space, $A \subseteq C(K)$ a subspace and $D \subseteq K$ a closed set. We say that (A, D) is an *Urysohn pair* if

For any $t_0 \in K \setminus D$, there exists $f \in A$ such that $\|f\|_\infty = 1$, $f|_D \equiv 0$ and $f(t_0) = 1$.

In Theorem 5.2.4, we show that for an Urysohn pair (A, D), the subspace $Y = \{f \in A : f|_D \equiv 0\}$ forms a DBR subspace of A. As corollaries, we show that :

(a) Any M-ideal in $C(K)$ is DBR, recapturing our earlier result with a new proof.

(b) For a locally compact Hausdorff space L, any M-ideal in the space $C_0(L)$ of all scalar-valued continuous functions on L "vanishing at infinity", is a DBR subspace.

(c) Any M-ideal in the disc algebra $\mathbb{A}$ is DBR in $\mathbb{A}$.

In Chapter 5, we also consider the Banach space $A_{\mathbb{F}}(Q)$ of scalar-valued affine continuous functions, where Q is a compact convex set in some locally convex topological vector space E. We denote by $\partial_e Q$ the set of all extreme points of Q. Our main result in this chapter is that if Q is a Choquet simplex and $\overline{\partial_e Q} \setminus \partial_e Q$ is at most finite, then any M-ideal is a DBR subspace of $A_{\mathbb{F}}(Q)$. Some variants of this result are also considered.

In Chapter 6, we explore the stability of the properties $(*)$, BR and DBR. These properties are better behaved with respect to superspaces than subspaces. A p-summand is a $(*)$-subspace if and only if $p = 1$.

Coming to sequence spaces, we show that the c_0- or the ℓ_p-sum $(1 < p \leq \infty)$ of Y_α's is a $(*)$-/$(*)$- and DBR/$(*)$- and BR subspace in the corresponding sum of X_α's if and only if each Y_α is such a subspace in X_α. In the process, we answer [10, Question 2.17] in the affirmative. On the other hand, if at least one Y_α is a $(*)$-/$(*)$- and DBR/$(*)$- and BR subspace of X_α, then the ℓ_1-sum of Y_α's is such a subspace of the corresponding sum of X_α's.

Coming to function spaces, we show that Y is a $(*)$-/$(*)$- and DBR/$(*)$- and BR subspace of X if and only if $C(K, Y)$ is such a subspace of $C(K, X)$. For BR, the $(*)$- assumption may also be removed. If Y is a $(*)$-/$(*)$- and DBR subspace of X and (Ω, Σ, μ) is a probability space, then the space $L_1(\mu, Y)$ of Y-valued Bochner integrable functions is such a subspace of $L_1(\mu, X)$.

In Chapter 7, we study ball remotality of a Banach space X in its bidual. In particular, we consider the following properties :

Definition 1.0.8. We will say that a Banach space X

(a) is BR in its bidual (BRB) if $R(B_X, X^{**}) = X^{**}$.

(b) is DBR in its bidual (DBRB) if $\overline{R(B_X, X^{**})} = X^{**}$.

(c) has remotally spanned bidual (RSB) if $\overline{\text{span}}(R(B_X, X^{**})) = X^{**}$.

(d) is anti-remotal in its bidual (ARB) if $R(B_X, X^{**}) = X$.

It is clear that reflexivity $\Rightarrow BRB \Rightarrow DBRB \Rightarrow RSB$. We show that none of the converse holds. We show that a Banach space having a strong unitary is BRB, producing a large class of non-reflexive examples. We show that X is wALUR [7] if and only if X is rotund and ARB. We also obtain characterizations of reflexivity in terms of these phenomena. For example, we show that a separable Banach space is reflexive if and only if it is BRB/DBRB/RSB in every equivalent renorming. Some stability results are also obtained.

CHAPTER 2

$(*)$-subspaces and the farthest distance formula

2.1 Summary of results

As a generalization of $(*)$-subspaces, we introduce $(*)$-subsets and obtain several characterizations. In the process, we obtain a farthest distance formula for a closed bounded balanced subset of a Banach space, which is also of independent interest. We completely characterize $(*)$- and DBR/BR subspaces of a Banach space. In this chapter, we also characterize 1-dimensional $(*)$-subspaces. It turns out that this depends on the existence of a strong unitary (Definition 2.4.2), a notion related to that of geometric unitaries studied in [8].

2.2 The farthest distance formula

Notation 1. Let $\mathbb{T} = \{z \in \mathbb{F} : |z| = 1\}$. Define $sgn : \mathbb{F} \to \mathbb{T}$ by

$$sgn(z) = \begin{cases} 1 & \text{if} \quad z = 0 \\ |z|/z & \text{if} \quad z \neq 0 \end{cases}$$

That is, for any $z \in \mathbb{F}$, $|sgn(z)| = 1$ and $sgn(z) \cdot z = |z|$.

Definition 2.2.1. For $x \in X$, let $D(x) = \{x^* \in S_{X^*} : x^*(x) = \|x\|\}$.

We say that $A \subseteq B_{X^*}$ is a norming set for X if $\|x\| = \sup\{|x^*(x)| : x^* \in A\}$ for all $x \in X$.

We say that $B \subseteq S_{X^*}$ is a boundary for X if for every $x \in X$, there exists $x^* \in B$ such that $\|x\| = |x^*(x)|$.

Theorem 2.2.2. *Let C be a closed, bounded and balanced subset of a Banach space X. For $x^* \in X^*$, let $\|x^*\|_C = \sup_{z \in C} |x^*(z)|$.*

(a) Let $A \subseteq B_{X^*}$ be a norming set for X. Then for any $x \in X$,

$$\phi_C(x) = \sup\{|x^*(x)| + \|x^*\|_C : x^* \in A\}. \tag{2.1}$$

(b) If $A \subseteq S_{X^*}$ is a boundary for X, then $x \in R(C)$ if and only if there exists $x^* \in A$ and $z \in C$ such that the supremum in (2.1) is attained at x^* and $\|x^*\|_C = |x^*(z)|$.

Proof. (a). Let $x \in X$.

$$\begin{aligned} \phi_C(x) &= \sup_{z\in C}\|x-z\| = \sup_{z\in C}\sup_{x^*\in A}|x^*(x-z)| = \sup_{x^*\in A}\sup_{z\in C}|x^*(x-z)| \\ &= \sup_{x^*\in A}\sup_{z\in C}[|x^*(x)| + |x^*(z)|] \quad \text{(since } C \text{ is balanced)} \\ &= \sup_{x^*\in A}[|x^*(x)| + \|x^*\|_C]. \end{aligned}$$

(b). Suppose $x^* \in A$ and $z \in C$ are such that the sup in (2.1) is attained at x^* and $\|x^*\|_C = |x^*(z)|$. Then for some $\alpha \in \mathbb{T}$,

$$\phi_C(x) = |x^*(x)| + \|x^*\|_C = |x^*(x)| + |x^*(z)| = |x^*(x - \alpha z)| \leq \|x - \alpha z\| \leq \phi_C(x).$$

Since C is balanced, $\alpha z \in C$ and hence, $\alpha z \in F_C(x)$ and $x \in R(C)$.

This argument does not need A to be a boundary.

Conversely, suppose $x_0 \in R(C)$. Let $z_0 \in C$ be such that $\|x_0 - z_0\| = \phi_C(x_0)$. Since A is a boundary, there exists $x^* \in A$ such that $\|x_0 - z_0\| = |x^*(x_0 - z_0)|$, then

$$|x^*(x_0)|+\|x^*\|_C \geq |x^*(x_0)|+|x^*(z_0)| \geq |x^*(x_0-z_0)| = \|x_0-z_0\| = \phi_C(x_0) \geq |x^*(x_0)|+\|x^*\|_C$$

Hence, equality must hold everywhere. This completes the proof. □

Corollary 2.2.3. *Let X be a Banach space and $Y \subseteq X$ a subspace.*

(a) Let $A \subseteq B_{X^*}$ be a norming set for X. Then for any $x \in X$,

$$\phi_{B_Y}(x) = \sup\{|x^*(x)| + \|x^*|_Y\| : x^* \in A\}. \tag{2.2}$$

(b) If $A \subseteq S_{X^*}$ is a boundary for X, then $x \in R(B_Y)$ if and only if there exists $x^* \in A$ and $z \in B_Y$ such that the sup *in (2.2) is attained at x^* and $\|x^*|_Y\| = |x^*(z)|$.*

2.3 Characterization of $(*)$-subsets

Definition 2.3.1. Let $C \subseteq B_X$ be closed and $\sup_{z\in C}\|z\| = 1$. We call C a $(*)$-subset of X if for all $x \in X$, $\phi_C(x) = \|x\| + 1$.

Definition 2.3.2. For a closed and balanced subset $C \subseteq X$ with $\sup_{x\in C}\|x\| = 1$, define

$$A_C = \{x^* \in S_{X^*} : \|x^*\|_C = 1\}$$

If Y is a subspace of X, we will simply write A_Y for A_{B_Y}.

Definition 2.3.3. Let K be a subset of a vector space E. A nonempty set $M \subseteq K$ is said to be an extremal subset of K if $x_1, x_2 \in K$, $0 < \lambda < 1$, and $\lambda x_1 + (1-\lambda)x_2 \in M$ implies $x_1, x_2 \in M$. A convex extremal set is called a face. A singleton face is an extreme point.

Proposition 2.3.4. *For a closed and bounded set $C \subseteq X$, if $F_C(x) \neq \emptyset$, $F_C(x)$ is a norm closed extremal subset of C, but need not be a face.*

Proposition 2.3.5. *If $C \subseteq X$ is closed and balanced with $\sup_{x\in C}\|x\| = 1$, then A_C is a norm closed extremal subset of B_{X^*}, but is not a face.*

Proof. Clearly $\big|\|x^*\|_C - \|y^*\|_C\big| \leq \|x^* - y^*\|$ and hence, the function $\|\cdot\|_C$ is norm continuous. It follows that A_C is a norm closed set.

Let $x_1^*, x_2^* \in B_{X^*}$ and $0 < \lambda < 1$ be such that $\lambda x_1^* + (1-\lambda)x_2^* \in A_C$. Then there exists $(y_n) \subseteq C \subseteq B_X$ such that $\lim_n[(\lambda x_1^* + (1-\lambda)x_2^*)(y_n)] = 1$. It follows that

$$\lim_n x_1^*(x_n) = 1 \quad \text{and} \quad \lim_n x_2^*(y_n) = 1.$$

And hence, $x_1^*, x_2^* \in A_C$.

Since A_C is $\mathbb{T}$-invariant, A_C cannot be convex. □

We will also need the following lemma repeatedly.

Lemma 2.3.6. *If $A \subseteq S_{X^*}$ is such that $\{x \in X : D(x) \cap A \neq \emptyset\}$ is norm dense in X, then A is a norming set for X.*

Proof. Let $x \in X$ and $\varepsilon > 0$. Find $z \in \{x \in X : D(x) \cap A \neq \emptyset\}$ such that $\|x - z\| < \varepsilon/2$. Let $z^* \in D(z) \cap A$. Then

$$|z^*(x)| = |z^*(z) - z^*(z-x)| \geq \|z\| - \|z - x\| > \|x\| - \varepsilon/2 - \varepsilon/2 = \|x\| - \varepsilon.$$

Since ε is arbitrary, A norms x. □

Definition 2.3.7. For $f : X \to \mathbb{R}$, we define the subdifferential of f at an $x \in X$ as

$$\partial f(x) = \{x^* \in X^* : Re\, x^*(z-x) \leq f(z) - f(x), \text{ for all } z \in X\}$$

As a simple consequence of Hahn-Banach theorem, we have for each continuous convex function f on X, $\partial f(x)$ is a nonempty w*-compact, convex set in X^*.

We will also need the following result of Lau [30] (see also [16, Proposition II.2.7]). This needs the scalars to be real. For any bounded set K, any $x \in X$, and any $x^* \in \partial\phi_K(x)$, we have $\|x^*\| \leq 1$, and hence, $\sup_{z\in K} x^*(x-z) \leq \phi_K(x)$. Moreover, the set

$$G(K) = \{x \in X : \sup_{z\in K} x^*(x-z) = \phi_K(x) \text{ for all } x^* \in \partial\phi_K(x)\}$$

is a dense G_δ in X.

Now we return to arbitrary scalars and to our main result of this chapter.

Theorem 2.3.8. *Let $C \subseteq B_X$ be closed and balanced with $\sup_{x\in C}\|x\| = 1$. Then the following are equivalent :*

(a) *A_C is a norming set for X.*

(b) *C is a $(*)$-subset of X.*

(c) *There is a dense set $G \subseteq X$ such that $\phi_C(x) = \|x\| + 1$ for all $x \in G$.*

(d) *$\{x \in X : D(x) \subseteq A_C\}$ is a dense G_δ set in X.*

(e) *$\{x \in X : D(x) \cap A_C \neq \emptyset\}$ contains a dense G_δ set in X.*

(f) *$\{x \in X : D(x) \cap A_C \neq \emptyset\}$ is dense in X.*

(g) *For every boundary B for X, $B \cap A_C$ is a norming set for X.*

(h) *For some boundary B for X, $B \cap A_C$ is a norming set for X.*

(i) *Intersection of all balls containing C equals B_X.*

Proof. Clearly, $(d) \Rightarrow (e) \Rightarrow (f)$, $(g) \Rightarrow (h) \Rightarrow (a)$ and $(b) \Leftrightarrow (i)$. Since ϕ_C and $\|\cdot\|$ are both norm continuous, $(b) \Leftrightarrow (c)$.

$(f) \Rightarrow (a)$ follows from Lemma 2.3.6 with $A = A_C$.

$(a) \Rightarrow (b)$. By Theorem 2.2.2(a) with $A = A_C$, we get

$$\phi_C(x) = \sup\{|x^*(x)| + \|x^*\|_C : x^* \in A_C\} = \sup\{|x^*(x)| + 1 : x^* \in A_C\} = \|x\| + 1.$$

$(b) \Rightarrow (d)$. First let us assume that the scalars are real.

It is easy to see that if $f(x) = \|x\|$, then $\partial f(x) = D(x)$. Thus, $(b) \Rightarrow \partial\phi_C(x) = D(x)$.

By the result of [30] quoted above, $G(C) = \{x \in X : \text{for all } x^* \in D(x), \sup_{z\in C} x^*(x-z) = \|x\| + 1\}$ is a dense G_δ subset of X.

Claim : $G(C) = \{x \in X : D(x) \subseteq A_C\}$.

Let $x \in G(C)$ and $x^* \in D(x)$. Then $\sup_{z\in C} x^*(x - z) = \|x\| + 1$. Since C is balanced, it follows that $\|x^*\|_C = 1$. Hence $x^* \in A_C$.

Conversely, if $x \in X$ is such that $D(x) \subseteq A_C$ and $x^* \in D(x)$, then $\sup_{z\in C} x^*(x - z) = x^*(x) + \|x^*\|_C = \|x\| + 1$. Thus, $x \in G(C)$. This proves the claim.

If the scalars are complex, consider the real restriction $X_\mathbb{R}$ of X. Recall that $x^* \to Re\ x^*$ establishes a real linear isometry between $(X^*)_\mathbb{R}$ and $(X_\mathbb{R})^*$.

If (b) holds, then $\partial\phi_C(x) = D'(x) = \{Re\ x^* \in S_{X^*_\mathbb{R}} : Re\ x^*(x) = \|x\|\}$. Let $A'_C = \{Re\ x^* \in S_{X^*_\mathbb{R}} : \|Re\ x^*\|_C = 1\}$. Again, by the above arguments, $G' = \{x \in X_\mathbb{R} : D'(x) \subseteq A'_C\}$

is a dense G_δ set in $X_{\mathbb{R}}$. As $x \to x$ is an isometry from $X_{\mathbb{R}}$ to X, it follows that the set $G = \{x \in X : D(x) \subseteq A_C\}$ is a dense G_δ subset of X.

$(d) \Rightarrow (g)$. Since B is a boundary, for every $x \in X$, $\mathbb{T}B \cap D(x) \neq \emptyset$. Since $\{x \in X : D(x) \subseteq A_C\} \subseteq \{x \in X : D(x) \cap [\mathbb{T}B \cap A_C] \neq \emptyset\}$, by (d), the right hand set is dense in X. By Lemma 2.3.6, $\mathbb{T}B \cap A_C$ is norming for X. Since A_C is $\mathbb{T}$-invariant, $B \cap A_C$ is norming for X. □

Putting $C = B_Y$ in Theorem 2.3.8 we obtain the following characterization theorem:

Theorem 2.3.9. *For a subspace Y of a Banach space X, the following are equivalent :*

- (a) *A_Y is a norming set for X.*
- (b) *Y is a $(*)$-subspace of X.*
- (c) *There is a dense set $G \subseteq X$ such that $\phi_{B_Y}(x) = \|x\| + 1$ for all $x \in G$.*
- (d) *$\{x \in X : D(x) \subseteq A_Y\}$ is a dense G_δ set in X.*
- (e) *$\{x \in X : D(x) \cap A_Y \neq \emptyset\}$ contains a dense G_δ set in X.*
- (f) *$\{x \in X : D(x) \cap A_Y \neq \emptyset\}$ is dense in X.*
- (g) *For every boundary B for X, $B \cap A_Y$ is a norming set for X.*
- (h) *For some boundary B for X, $B \cap A_Y$ is a norming set for X.*
- (i) *Intersection of all balls containing B_Y equals B_X.*

Remark 2.3.10. In [10], the authors used (a) as the definition of a $(*)$-subspace and proved $(a) \Rightarrow (b)$.

Here are some natural examples of $(*)$-subspaces, as observed in [10].

Example 2.3.11. (a) X is a $(*)$-subspace of X^{**}, since $S_{X^*} \subseteq A_X$ (See Chapter 7).

- (b) If $Y \subseteq Z \subseteq X$ and Y is a $(*)$-subspace of X, then Z is a $(*)$-subspace of X and Y is a $(*)$-subspace of Z.
- (c) For any two Banach spaces X and Y, $\mathcal{K}(X, Y)$ is a $(*)$-subspace of $\mathcal{L}(X, Y)$. To see this, note that $A_{\mathcal{K}(X,Y)} \supseteq \{x \otimes y^* : x \in S_X,\ y^* \in S_{Y^*}\}$, which already norms $\mathcal{L}(X, Y)$.
- (d) For a compact Hausdorff space K, $C(K, X)$ is a $(*)$-subspace of $WC(K, X)$.

Remark 2.3.12. Recall that a Banach space X has the Mazur Intersection Property (MIP) if every closed bounded convex set in X is the intersection of closed balls containing it. From (i) above, it follows that a space with the MIP cannot have a proper $(*)$-subspace. On the other hand, it has been noted in [10, Proposition 2.8] that a wLUR Banach space also has no proper $(*)$-subspace.

It now follows from Corollary 2.2.3 that

Proposition 2.3.13. [10, Proposition 2.10] *If Y is a $(*)$-subspace of X, then $x_0 \in R(B_Y)$ if and only if there exists $x^* \in S_{X^*}$ and $y \in S_Y$ such that*

$$x^*(x_0) = \|x_0\| \quad \textit{and} \quad x^*(y) = 1.$$

As noted in [10, Remark 2.11], it follows from the result above that for $(*)$-subspaces, $R(B_Y)$ is closed under scalar multiplications. We will show later in Example 5.2.11 that this need not be true if Y is not a $(*)$-subspace.

Since the sets involved are $\mathbb{T}$-invariant, we actually get

Corollary 2.3.14. *If Y is a $(*)$-subspace of X, then $x_0 \in R(B_Y)$ if and only if there exists $x^* \in S_{X^*}$ and $y \in S_Y$ such that*

$$|x^*(x_0)| = \|x_0\| \quad \textit{and} \quad |x^*(y)| = 1.$$

Proposition 2.3.15. *Let $N_Y = \{x^* \in S_{X^*} : x^*(y) = 1$ for some $y \in S_Y\}$. Then*

(a) Y is $(*)$ and DBR if and only if $\{x \in X : N_Y \cap D(x) \neq \emptyset\}$ is dense in X. In particular, if Y is $(*)$- and DBR, then N_Y is norming for X.

(b) Y is a $(*)$ and BR subspace of X if and only if N_Y is a boundary for X.

Proof. Clearly, $N_Y \subseteq A_Y$. If $\{x \in X : N_Y \cap D(x) \neq \emptyset\}$ is dense in X; in particular, if N_Y is a boundary for X, then by Theorem 2.3.9, Y is a $(*)$-subspace.

On the other hand, if Y is a $(*)$-subspace of X then, by Proposition 2.3.13, $R(B_Y) = \{x \in X : N_Y \cap D(x) \neq \emptyset\}$. Hence the result.

The last part of (a) follows from the first part and Lemma 2.3.6. □

Corollary 2.3.16. *If Y is a $(*)$-subspace and $A_Y \cap NA(X) = N_Y$, then Y is DBR in X.*

Proof. Clearly, $N_Y \subseteq A_Y \cap NA(X)$. Note that $NA(X)$ is a $\mathbb{T}$-invariant boundary for X. So, if Y is a $(*)$-subspace then as in the proof of Theorem 2.3.8 $(d) \Rightarrow (g)$, $\{x \in X : A_Y \cap NA(X) \cap D(x) \neq \emptyset\}$ is dense in X. Since $A_Y \cap NA(X) = N_Y$ and $R(B_Y) = \{x \in X : N_Y \cap D(x) \neq \emptyset\}$, Y is DBR. □

Corollary 2.3.17. *(a) If $Y \subseteq Z \subseteq X$ and Y is a $(*)$- and BR subspace of X, then Z is a $(*)$- and BR subspace of X.*

(b) If $Y \subseteq Z \subseteq X$ and Y is a $(*)$- and DBR subspace of X, then Z is a $(*)$- and DBR subspace of X.

Remark 2.3.18. However, if $Y \subseteq Z \subseteq X$ and Y is a $(*)$- and DBR subspace of X, then we do not know if Y must be a $(*)$- and DBR subspace of Z. Also if $Y \subseteq Z \subseteq X$ and Y is a DBR subspace of X, then Z need not be a DBR subspace of X (Example 4.2.50).

Proposition 2.3.19. *(a) If Y is a $(*)$-subspace of a strictly convex Banach space X, then $R(B_Y) = Y$.*

(b) A strictly convex space cannot have a proper $(*)$ and DBR subspace.

(c) A reflexive strictly convex space has no proper $(*)$ subspace.

(d) Any Hilbert space, the spaces $L_p([0,1])$ and ℓ_p, $1 < p < \infty$ has no proper $(*)$ subspace.

Proof. (a). Let X be a strictly convex space and Y be a $(*)$-subspace of X.

Let $x \in R(B_Y)$. We may assume $\|x\| = 1$. Then there exists $y \in B_Y$ such that $\|x + y\| = \phi_{B_Y}(x) = 2$. Since X is strictly convex, $x = y$.

Now, $(a) \Rightarrow (b) \Rightarrow (c)$ by Theorem 1.0.1 and $(c) \Rightarrow (d)$. □

Remark 2.3.20. (d) also follows from [10, Proposition 2.8] since these spaces are LUR.

2.4 Strong unitaries and 1-dimensional $(*)$-subspaces

It may seem that a $(*)$-subspace must be somewhat large. This, however, is not the case. A Banach space may even have 1-dimensional $(*)$-subspaces.

Theorem 2.4.1. *Let X be a Banach space and $x_0 \in S_X$. The following are equivalent :*

(a) $D(x_0)$ is a norming set for X.

(b) $D(x_0)$ is a boundary for X.

(c) $\mathbb{F}x_0$ is a $(*)$-subspace of X.

Proof. $(a) \Leftrightarrow (c)$. Observe that if $Y = \mathbb{F}x_0$, then for any $x^* \in X^*$, $\|x^*|_Y\| = |x^*(x_0)|$. It follows that $A_Y = \{x^* \in S_{X^*} : |x^*(x_0)| = 1\} = \mathbb{T}D(x_0)$.

Thus, Y is a $(*)$-subspace of $X \Leftrightarrow D(x_0)$ is a norming set for X.

Since $D(x_0)$ is w*-compact, $(a) \Leftrightarrow (b)$. □

Definition 2.4.2. Let X be a Banach space. Let us call $x_0 \in S_X$ a strong unitary if the equivalent conditions in Theorem 2.4.1 are satisfied.

For the origin of this terminology and related results, see [8] or the survey article [33].

Corollary 2.4.3. *If $x_0 \in S_X$ is a strong unitary in X and Y is a subspace with $x_0 \in Y$, then*

(a) x_0 is a strong unitary in Y.

(b) *Y is $(*)$ and BR in X.*

Example 2.4.4. Here are some natural examples of strong unitaries in Banach spaces.

(a) The constant sequence **1** in c or ℓ_∞ is a strong unitary.

(b) The canonical unit vectors in ℓ_1 are strong unitaries.

(c) Any unimodular function in $C(K)$ is a strong unitary in $C(K)$.

(d) A commutative C^* algebra $\mathcal{A}$ with identity contains strong unitaries. To see this, note that the Gelfand transform induces an isometric $(*)$-isomorphism from $\mathcal{A}$ onto $C(\Sigma)$, where Σ is the maximal ideal space of $\mathcal{A}$ [15, Theorem VIII.2.1].

On the other hand, if $\mathcal{A}$ is a commutative C^* algebra *without* identity then it does not contain any strong unitaries. Indeed, we show later (Corollary 4.3.8) that such a space has no finite-dimensional $(*)$-subspaces.

There are also 2-dimensional $(*)$-subspaces that do not contain a strong unitary.

Example 2.4.5. Consider the subspace $Y \subseteq c$ spanned by $x = (\sin \frac{1}{n})$ and $y = (\cos \frac{1}{n})$. Taking vectors of the form $\sin \frac{1}{k} \cdot x + \cos \frac{1}{k} \cdot y$, one can see that A_Y contains all the coordinate functionals. Hence, Y is a $(*)$-subspace.

This example appears in a related context in [4, Example 2.34].

CHAPTER 3

Ball remotality in some classical Banach spaces

3.1 Summary of results

In this chapter, we study ball remotality in the classical Banach spaces $c_0(\Gamma)$, $c(\Gamma)$, $\ell_\infty(\Gamma)$ for some arbitrary index set Γ. The farthest distance formula—Theorem 2.2.2—takes a simpler form in these spaces. Using this, we first characterize $(*)$- and densely remotal subsets (Theorem 3.2.3) and thereby, $(*)$- and DBR subspaces of $c_0(\Gamma)$, $c(\Gamma)$ and $\ell_\infty(\Gamma)$ (Corollary 3.2.4). In the process, we prove that (Corollaries 3.2.5 and 3.2.6) :

(a) For $Y \subseteq c_0(\Gamma)$, the following are equivalent :

(1) Y is $(*)$- and DBR in $c_0(\Gamma)$

(2) Y is $(*)$- and BR in $c_0(\Gamma)$

(3) Y is $(*)$- and DBR in $c(\Gamma)$

(4) Y is $(*)$- and DBR in $\ell_\infty(\Gamma)$.

(b) In particular, $c_0(\Gamma)$ is a $(*)$- and DBR subspace of both $c(\Gamma)$ and $\ell_\infty(\Gamma)$.

(c) If a subspace of $c(\Gamma)$ or $\ell_\infty(\Gamma)$ contains the constant vector $\mathbf{1}$, then it is $(*)$- and BR.

(d) $c(\Gamma)$ and $\widehat{c}(\Gamma)$, the canonical image of $c(\Gamma)$, are $(*)$- and BR in $\ell_\infty(\Gamma)$.

We also show that $c_0(\Gamma)$ has no finite dimensional $(*)$-subspace (Theorem 3.2.8). We characterize all hyperplanes in $c_0(\Gamma)$ which are $(*)$- and DBR in terms of the defining linear functionals (Theorem 3.2.16).

Then we come to the space $\ell_1(\Gamma)$ and observe that $\ell_1(\Gamma)$ with its usual norm provides a simple non-reflexive example of a Banach space in which every subspace is DBR (Corollary 3.3.2). We show that a hyperplane in ℓ_1 is BR if and only if it contains a strong unitary (Theorem 3.3.6). However, if Γ is uncountable, a hyperplane in $\ell_1(\Gamma)$ is always $(*)$- and BR (Theorem 3.3.8).

3.2 Ball remotality in $c_0(\Gamma), c(\Gamma)$ and $\ell_\infty(\Gamma)$

Notation 2. Let Γ be an arbitrary index set. Define

(a) $c_0(\Gamma) = \{x = (x_\gamma)_{\gamma\in\Gamma}$: given $\varepsilon > 0$, there exists a finite subset $\Gamma_1 \subseteq \Gamma$ such that $|x_\gamma| < \varepsilon$ for all $\gamma \notin \Gamma_1\}$.

(b) $\ell_\infty(\Gamma) = \{x = (x_\gamma)_{\gamma\in\Gamma} : \|x\|_\infty = \sup_{\gamma\in\Gamma} |x_\gamma| < \infty\}$.

(c) $\ell_1(\Gamma) = \{x = (x_\gamma)_{\gamma\in\Gamma}$: there exists $M > 0$ such that $\sum_{\gamma\in A} |x_\gamma| \le M$, for all finite subsets A of $\Gamma\}$.

(d) $c(\Gamma) = \{x = (x_\gamma)_{\gamma\in\Gamma}$: there exists $\lambda \in \mathbb{C}$ such that for all $\varepsilon > 0$ there exists a finite subset $\Gamma_1 \subseteq \Gamma$ such that $|x_\gamma - \lambda| < \varepsilon$ for all $\gamma \notin \Gamma_1\}$.

If $\Gamma = \mathbb{N}$, we get back the classical sequence spaces c_0, ℓ_∞, ℓ_1 and c.

Let $\{e_\gamma\}$ denote the canonical unit vectors in $X = c_0(\Gamma)$, $c(\Gamma)$ or $\ell_\infty(\Gamma)$ and $\{e_\gamma^*\}$ is the coordinate functionals in $\ell_1(\Gamma) \subseteq X^*$. Note that $\{e_\gamma^* : \gamma \in \Gamma\} \subseteq \ell_1(\Gamma)$ is a boundary for $X = c_0(\Gamma)$ and is a norming set for $X = c(\Gamma)$ or $\ell_\infty(\Gamma)$.

Remark 3.2.1. (a) It is clear that $c(\Gamma) = C(\Gamma_\infty)$, where $\Gamma_\infty = \Gamma \cup \{\infty\}$ is the one-point compactification of Γ endowed with discrete topology. Clearly, $c_0(\Gamma)$ is identified with the subspace of $C(\Gamma_\infty)$ that "vanish at" ∞.

(b) Clearly, $c(\Gamma)$ is a subspace of $\ell_\infty(\Gamma)$. Since $c(\Gamma)^{**} = \ell_\infty(\Gamma)$, there is also a canonical embedding $\widehat{c}(\Gamma)$ of $c(\Gamma)$ in $\ell_\infty(\Gamma)$. However, due to the nature of the action of $\ell_1(\Gamma)$ on $c(\Gamma)$, $\widehat{c}(\Gamma) \neq c(\Gamma)$. For a fixed $\gamma_0 \in \Gamma$, it can be shown that $\widehat{c}(\Gamma) = \{x \in c(\Gamma) : x_{\gamma_0} = \lambda\}$ where $\lambda \in \mathbb{C}$ corresponds to x as in the definition of $c(\Gamma)$.

Proposition 3.2.2. *Let X be one of $c_0(\Gamma)$, $c(\Gamma)$ or $\ell_\infty(\Gamma)$. Let $C \subseteq X$ be closed, bounded and balanced. For $\gamma \in \Gamma$, let*

$$M_\gamma = \sup_{z\in C} |z_\gamma| \tag{3.1}$$

Then

(a) *for any $x_0 = (x_\gamma) \in X$,*

$$\phi_C(x_0) = \sup\{|x_\gamma| + M_\gamma : \gamma \in \Gamma\}. \tag{3.2}$$

(b) *If there exist $\gamma \in \Gamma$ and $z \in C$ such that $\phi_C(x_0) = |x_\gamma| + M_\gamma = |x_\gamma| + |z_\gamma|$, then $x_0 \in R(C)$. If $X = c_0(\Gamma)$, the converse is also true.*

If $X = c(\Gamma)$ and $C \subseteq c(\Gamma)$ is closed, bounded and balanced. Then

(c) *for any $x_0 \in c(\Gamma)$,*

$$\phi_C(x_0) = \sup\{|x_\gamma| + M_\gamma : \gamma \in \Gamma_\infty\}, \tag{3.3}$$

where $M_\infty = \sup_{z\in C} |z_\infty|$.

(*d*) $x_0 \in R(C, c(\Gamma))$ *if and only if there exist* $\gamma \in \Gamma_\infty$ *and* $z \in C$ *such that* $\phi_C(x_0) = |x_\gamma| + M_\gamma = |x_\gamma| + |z_\gamma|$.

Proof. For (a) and (b), apply Theorem 2.2.2 with $A = \{e^*_\gamma : \gamma \in \Gamma\} \subseteq \ell_1(\Gamma) \subseteq X^*$.

For (c) and (d), take $A = \{e^*_\gamma : \gamma \in \Gamma_\infty\} \subseteq c(\Gamma)^*$. □

Theorem 3.2.3. *Let X be one of the spaces $c_0(\Gamma)$, $c(\Gamma)$ or $\ell_\infty(\Gamma)$ and $C \subseteq X$ be closed, balanced and* $\sup_{x\in C}\|x\| = 1$.

(*a*) *C is a $(*)$-subset of X if and only if for all $\gamma \in \Gamma$, $M_\gamma = \sup_{z\in C}|z_\gamma| = 1$.*

(*b*) *C is a $(*)$-subset and densely remotal if and only if for all $\gamma \in \Gamma$, there exists $y \in C$ such that $|y_\gamma| = 1$.*

(*c*) *A $(*)$-subset $C \subseteq c_0(\Gamma)$ is densely remotal if and only if it is remotal.*

(*d*) *If $X = c(\Gamma)$ or $\ell_\infty(\Gamma)$ and C contains the constant vector* **1**, *then C is remotal.*

Proof. (a). Necessity follows from Proposition 3.2.2.

If for some $\alpha \in \Gamma$, $M_\alpha < 1$, let $0 < \delta < 1 - M_\alpha$. Then for all $y \in C$, $|y_\alpha| \leq M_\alpha < 1 - \delta$. Therefore, for all $y \in C$, $\|e_\alpha - y\|_\infty = \max\{|1 - y_\alpha|, \sup_{\beta\neq\alpha}|y_\beta|\} \leq 1 + |y_\alpha| \leq 2 - \delta$. So $\phi_C(e_\alpha) < 2 = \|e_\alpha\|_\infty + 1$.

(b). If C is a $(*)$-subset and densely remotal, then by (a), $M_\gamma = 1$ for all $\gamma \in \Gamma$. Suppose there exists $\alpha \in \Gamma$ such that for all $y \in C$, $|y_\alpha| < 1$.

Claim : If $z \in X$ such that $|z_\alpha| = \|z\|_\infty > \sup_{\gamma\neq\alpha}|z_\gamma|$, then $z \notin R(C)$.

Otherwise, there exists $y \in C$, $\|z - y\|_\infty = \|z\|_\infty + 1 = |z_\alpha| + 1$. For any $\gamma \neq \alpha$, $|z_\gamma - y_\gamma| \leq |z_\gamma| + 1$. It follows that $\sup_{\gamma\neq\alpha}|z_\gamma - y_\gamma| \leq \sup_{\gamma\neq\alpha}|z_\gamma| + 1 < |z_\alpha| + 1$. Therefore, we must have $|y_\alpha| = 1$. This proves the claim.

Now, if $\|z - e_\alpha\|_\infty < 1/3$, then $\sup_{\gamma\neq\alpha}|z_\gamma| < 1/3$ and $|z_\alpha| > 2/3$ and hence, by the claim, $z \notin R(C)$. Hence C cannot be densely remotal.

Conversely, if for all $\gamma \in \Gamma$, there exists $y \in C$ such that $|y_\gamma| = 1$, then clearly $M_\gamma = 1$ for all $\gamma \in \Gamma$ and this value is attained. Thus, by (a), C is a $(*)$-subset.

Let $R = \{x \in X : \|x\|_\infty = |x_\gamma| \text{ for some } \gamma \in \Gamma\}$. By Proposition 3.2.2, $R \subseteq R(B_Y)$.

If $X = c_0(\Gamma)$, $R = c_0(\Gamma)$ and (c) follows. If $X = \ell_\infty(\Gamma)$ or $c(\Gamma)$, and $x \notin R$, let $\varepsilon > 0$.

Let $\|x\|_\infty = m$. There exists $\alpha \in \Gamma$ such that $m - \varepsilon < |x_\alpha| \leq m$. Define $z = (z_\gamma)$ by the following

$$z_\gamma = \begin{cases} x_\gamma & \text{if } \gamma \neq \alpha \\ sgn(x_\alpha)^{-1} m & \text{if } \gamma = \alpha \end{cases}$$

then $z \in R$ and $\|z - x\|_\infty = |x_\alpha - z_\alpha| = m - |x_\alpha| < \varepsilon$. Hence, R is dense in X.

(d). Suppose $\mathbf{1} \in C$. If $x \notin R$, then there exists a sequence $\{\gamma_n\} \subseteq \Gamma$ such that $|x_{\gamma_n}| \to \|x\|_\infty$. Passing to a subsequence, if necessary, we may assume that $\{x_{\gamma_n}\}$ is convergent, to

$x_0 \in \mathbb{F}$, say. Then $\|x\|_\infty = |x_0|$. Let $\alpha = sgn(x_0)^{-1}$. Now

$$\|x + \alpha\mathbf{1}\|_\infty \geq \lim_n |x_{\gamma_n} + \alpha| = |x_0 + \alpha| = |x_0| + 1 = \|x\|_\infty + 1.$$

Hence $x \in R(C)$. □

Corollary 3.2.4. *Let Y be a subspace of $X = c_0(\Gamma)$, $c(\Gamma)$ or $\ell_\infty(\Gamma)$. Let A_Y and N_Y be as in Definition 2.3.2 and Proposition 2.3.15 respectively. Then*

(a) Y is a $(*)$-subspace of X if and only if $e^*_\gamma \in A_Y$ for all $\gamma \in \Gamma$.

(b) Y is $(*)$- and DBR in X if and only if $e^*_\gamma \in N_Y$ for all $\gamma \in \Gamma$, that is, for all $\gamma \in \Gamma$, there exists $y \in B_Y$ such that $|y_\gamma| = 1$.

Corollary 3.2.5. *For $Y \subseteq c_0(\Gamma)$, the following are equivalent :*

(a) Y is $(*)$- and DBR in $c_0(\Gamma)$

(b) Y is $(*)$- and BR in $c_0(\Gamma)$

(c) Y is $(*)$- and DBR in $c(\Gamma)$

(d) Y is $(*)$- and DBR in $\ell_\infty(\Gamma)$.

Corollary 3.2.6. *(a) $c_0(\Gamma)$ is a $(*)$- and DBR subspace of both $c(\Gamma)$ and $\ell_\infty(\Gamma)$.*

(b) If a subspace of $c(\Gamma)$ or $\ell_\infty(\Gamma)$ contains the constant vector $\mathbf{1}$, then it is $(*)$- and BR.

(c) $c(\Gamma)$ and $\widehat{c}(\Gamma)$ are $(*)$- and BR in $\ell_\infty(\Gamma)$.

(d) $c_0(\Gamma)$ is not BR in $c(\Gamma)$ or $\ell_\infty(\Gamma)$.

Proof. (c). Since the constant vector $\mathbf{1} \in c(\Gamma)$, $c(\Gamma)$ is $(*)$- and BR in $\ell_\infty(\Gamma)$.

By Remark 3.2.1, $\widehat{c}(\Gamma)$ is a subspace of $\ell_\infty(\Gamma)$ that contains $\mathbf{1}$. Hence, $\widehat{c}(\Gamma)$ is also $(*)$- and BR in $\ell_\infty(\Gamma)$.

(d). Let $X = c(\Gamma)$ or $\ell_\infty(\Gamma)$.

CLAIM : If $R = \{x \in X : \|x\|_\infty = |x_\gamma| \text{ for some } \gamma \in \Gamma\}$, then $R = R(B_{c_0(\Gamma)})$.

As noted above, $R \subseteq R(B_{c_0(\Gamma)})$. If $x \in R(B_{c_0(\Gamma)})$, there exists $y \in B_{c_0(\Gamma)}$ such that $\|x - y\|_\infty = \|x\|_\infty + 1$. Since $y \in B_{c_0(\Gamma)}$, there is a finite set Γ_1 such that $\|y_\alpha\| < 1/2$ for all $\alpha \notin \Gamma_1$. It follows that for all $\alpha \notin \Gamma_1$,

$$|x_\alpha - y_\alpha| \leq |x_\alpha| + |y_\alpha| < \|x\|_\infty + 1/2 < \|x\|_\infty + 1 = \|x - y\|_\infty.$$

Therefore, $\|x - y\|_\infty = \sup_{\alpha\in\Gamma_1} \|x_\alpha - y_\alpha\|$ and the supremum is attained at some $\alpha \in \Gamma_1$. It follows that $|x_\alpha| = \|x\|_\infty$, proving the claim.

Clearly, $R \neq X$. Hence the result. □

Remark 3.2.7. Note that if $X = c(\Gamma)$ and $Y = c_0(\Gamma)$, then $\mathbf{1} \in R(B_Y)$ and $X = \operatorname{span}(Y \cup \{\mathbf{1}\})$, but Y is not BR. Thus, even for a $(*)$-subspace, $R(B_Y)$ need not be a subspace of X.

A strong unitary is necessarily an extreme point. Since $c_0(\Gamma)$ has no extreme points, it has no strong unitaries, and hence, it has no 1-dimensional $(*)$-subspaces. Indeed, stronger result holds.

Theorem 3.2.8. *$c_0(\Gamma)$ has no finite-dimensional $(*)$-subspaces.*

Proof. If $Y \subseteq c_0(\Gamma)$ is finite-dimensional, then S_Y is compact. Hence there exists $\alpha_0 \in \Gamma$ such that $|y_{\alpha_0}| < 1/2$ for all $y \in S_Y$. Hence, the result follows from Corollary 3.2.4. □

We have essentially proved that

Corollary 3.2.9. *A compact subset of $c_0(\Gamma)$ cannot be a $(*)$-subset.*

We will need the following result from [10]. We include the proof for completeness.

Lemma 3.2.10. [10, Lemma 3.1] *Suppose $X = Y \oplus Z$ and there exists a monotone map $\varrho : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that if $x = y + z$, then $\|x\| = \varrho(\|y\|, \|z\|)$.*

(a) *Let $E \subseteq Y$ and $F \subseteq Z$ be remotal sets, then $E + F$ is remotal in X. In particular, B_Y and B_Z are remotal in X.*

(b) *Let $E \subseteq Y$ and $F \subseteq Z$ be densely remotal sets, then $E + F$ is densely remotal in X if the convergence in X is equivalent to the component-wise convergence.*

Proof. (a). Let $x_0 = y_0 + z_0$. If $e_0 \in F_E(y_0)$ and $f_0 \in F_F(z_0)$, then for any $e \in E$ and $f \in F$,

$$\|y_0 + z_0 - e - f\| = \varrho(\|y_0 - e\|, \|z_0 - f\|) \leq \varrho(\|y_0 - e_0\|, \|z_0 - f_0\|) = \|y_0 + z_0 - e_0 - f_0\|$$

by the monotonicity of ϱ.

(b). Let $x_0 = y_0 + z_0$ and $\varepsilon > 0$. By assumption, there exists $\delta > 0$ such that $B(x_0, \varepsilon) \supseteq B(y_0, \delta) \oplus B(z_0, \delta)$. Let $y_1 \in R(E) \cap B(y_0, \delta)$ and $z_1 \in R(F) \cap B(z_0, \delta)$. If $e_1 \in F_E(y_1)$ and $f_1 \in F_F(z_1)$, then by (a), $e_1 + f_1 \in F_{E+F}(y_1 + z_1)$. That is, $x_1 = y_1 + z_1 \in R(E + F)$ and $\|x_0 - x_1\| < \varepsilon$. □

As an immediate corollary, we get

Theorem 3.2.11. *Any M-ideal in c_0 is BR.*

Proof. It is well known [26] that any M-ideal in c_0 is of the form $\{x \in c_0 : x_n = 0 \text{ for all } n \in J\}$ for some $J \subseteq \mathbb{N}$ and therefore, is an M-summand. By Lemma 3.2.10, it is BR. □

Question 3.2.12. *Is there any subspace of c_0 that is DBR but not BR?*

Theorem 3.2.13. *Let $\{X_i : i \in \Lambda\}$ be a family of reflexive Banach spaces and $X = \bigoplus_{c_0} X_i$. If Y is any finite co-dimensional subspace of X, where the linear functionals are finitely supported, then Y is DBR in X.*

Proof. Let $Y = \bigcap_{i=1}^n \ker x_i^*$ and each $x_i^* = (x_{ij}^*)_{j\in\Lambda} \in X^* = \bigoplus_{\ell_1} X_i^*$ is nonzero only at finitely many $\alpha \in \Lambda$. Thus, there exist a finite set $J \subseteq \Lambda$ such that $x_{ij}^* = 0$ for $j \notin J$, $i = 1, 2, \ldots, n$. Then

$$Y = \{(x_j) \in X : \sum_{j\in J} x_{ij}^*(x_j) = 0, \quad i = 1, 2, \ldots, n\}$$

Let $X_J = (\bigoplus_{j\in J})_{\ell_\infty} X_j$ and $Z_J = (\bigoplus_{j\notin J})_{c_0} X_j$. Then $X = X_J \oplus_\infty Z_J$. Let $Y_J = \{(x_j) \in X_J : \sum_{j\in J} x_{ij}^*(x_j) = 0, i = 1, 2, \ldots, n\}$. Then $Y = Y_J \oplus_\infty Z_J$.

Since X_J is reflexive, Y_J is DBR in X_J. And since $B_Y = B_{Y_J} \oplus_\infty B_{Z_J}$, by Lemma 3.2.10($b$), Y is DBR in X. □

Corollary 3.2.14. *If Y is a finite co-dimensional subspace of $c_0(\Gamma)$ where the linear functionals are finitely supported then Y is BR in $c_0(\Gamma)$.*

Proof. If each $X_i = \mathbb{F}$, X_J and hence Y_J are finite dimensional, making Y_J BR in X_J. Thus, the result follows again from Lemma 3.2.10. □

Remark 3.2.15. From [10, Theorem 3.6], the authors conclude that for any proximinal finite co-dimensional subspace $Y \subseteq c_0$ is DBR. As noted there, the proximinality of Y implies that linear functionals are finitely supported. Thus, by Corollary 3.2.14, Y must, in fact, be BR.

Coming to hyperplanes in $c_0(\Gamma)$ not covered above, we have

Theorem 3.2.16. *Let $a = (a_\gamma) \in S_{\ell_1(\Gamma)}$ be such that $a_\gamma \neq 0$ for infinitely many $\gamma \in \Gamma$. Then $Y = \ker a$ is $(*)$- and BR in $c_0(\Gamma)$ if and only if $|a_\gamma| < 1/2$ for all $\gamma \in \Gamma$.*

Proof. Let $\gamma_0 \in \Gamma$ be such that $|a_{\gamma_0}| = \sup_\gamma |a_\gamma|$. Observe that, for any $\gamma \neq \gamma_0$, if we define

$$y_\alpha = \begin{cases} -sgn(a_\gamma) & \text{if } \alpha = \gamma \\ |a_\gamma|/a_{\gamma_0} & \text{if } \alpha = \gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

then $y \in B_Y$ and $|y_\gamma| = 1$.

It follows from Corollary 3.2.4 that Y is $(*)$- and BR in $c_0(\Gamma)$ if and only if $e_{\gamma_0}^* \in N_Y$, *i.e.*, there exists $y \in B_Y$ such that $|y_{\gamma_0}| = 1$.

Now, $y \in B_Y$ implies $\sum_\gamma a_\gamma y_\gamma = 0$, and hence,

$$|a_{\gamma_0}| = |a_{\gamma_0} y_{\gamma_0}| = \left|\sum_{\gamma\neq\gamma_0} a_\gamma y_\gamma\right| \leq \sum_{\gamma\neq\gamma_0} |a_\gamma y_\gamma| \leq \sum_{\gamma\neq\gamma_0} |a_\gamma| = 1 - |a_{\gamma_0}| \tag{3.4}$$

It follows that $|a_{\gamma_0}| \leq 1/2$. If $|a_{\gamma_0}| = 1/2$, equality must hold everywhere in (3.4). Since $y \in c_0(\Gamma)$ and (a_γ) is infinitely supported, this is impossible. Therefore, $|a_{\gamma_0}| < 1/2$.

Conversely, suppose $|a_{\gamma_0}| < 1/2$. Define $b \in \ell_1(\Gamma)$ by

$$b_\gamma = \begin{cases} a_\gamma & \text{if } \gamma \neq \gamma_0 \\ 0 & \text{if } \gamma = \gamma_0 \end{cases}$$

Then $\|b\|_1 = \sum_{\gamma \neq \gamma_0} |a_\gamma| = 1 - |a_{\gamma_0}| > 1/2 > |a_{\gamma_0}|$. So there exists $z \in B_{c_0(\Gamma)}$ such that $\sum b_\gamma z_\gamma = |a_{\gamma_0}|$. Define $y \in c_0(\Gamma)$ by

$$y_\gamma = \begin{cases} z_\gamma & \text{if } \gamma \neq \gamma_0 \\ -sgn(a_{\gamma_0}) & \text{if } \gamma = \gamma_0 \end{cases}$$

Then $y \in S_Y$ and $|y_{\gamma_0}| = 1$. $\square$

Remark 3.2.17. If $|a_{\gamma_0}| = 1/2$, since infinitely supported elements of $\ell_1(\Gamma)$ are non-norm attaining on c_0, arguing as in Theorem 4.2.38 below, we can show that Y is a $(*)$-subspace, but not DBR.

For subspaces of $\ell_\infty(\Gamma)$, we have

Proposition 3.2.18. (a) *Any w*-closed subspace of $\ell_\infty(\Gamma)$ is DBR.*

(b) *If $c_0(\Gamma) \subseteq Y \subseteq \ell_\infty(\Gamma)$, then Y is $(*)$- and DBR in $\ell_\infty(\Gamma)$.*

(c) *If $\Lambda \in \partial_e(B_{\ell_\infty(\Gamma)^*})$, then $Y = \ker \Lambda$ is a DBR subspace of $\ell_\infty(\Gamma)$.*

Proof. (a). If $Y \subseteq \ell_\infty(\Gamma)$ is w*-closed, then B_Y is w*-compact and therefore, Y is DBR by Theorem 1.0.3, since $\ell_1(\Gamma)$ has RNP.

(b). Since $c_0(\Gamma)$ is $(*)$- and DBR in $\ell_\infty(\Gamma)$, so is Y, by Corollary 2.3.17.

(c). Recall that $\ell_\infty(\Gamma)^* = \ell_1(\Gamma) \oplus_1 c_0(\Gamma)^\perp$. Since $\Lambda \in \partial_e(B_{\ell_\infty(\Gamma)^*})$, either $\Lambda \in \ell_1(\Gamma)$ or $\Lambda \in c_0(\Gamma)^\perp$ [26, Lemma I.1.5]. And the results follows from (a) or (b). $\square$

Remark 3.2.19. A complete characterization of $(*)$- and DBR hyperplanes in $\ell_\infty(\Gamma)$ can be obtained from Theorem 4.2.41 below by identifying $\ell_\infty(\Gamma)$ with $C(\beta\mathbb{N})$. However, since we do not know any simple description of norm-attaining functionals on $\ell_\infty(\Gamma)$, we only derive a sufficient condition for $(*)$- and DBR hyperplanes in $\ell_\infty(\Gamma)$.

Theorem 3.2.20. *If $\Lambda \in \ell_\infty(\Gamma)^*$, and $|\Lambda(e_\gamma)| < \frac{1}{2}\|\Lambda|_{c_0(\Gamma)}\|$ for all $\gamma \in \Gamma$, then $\ker(\Lambda)$ is $(*)$- and DBR in $\ell_\infty(\Gamma)$.*

Proof. Since $\ell_\infty(\Gamma)^* = \ell_1(\Gamma) \oplus_1 c_0(\Gamma)^\perp$, $\Lambda = \Lambda_1 + \Lambda_2$ for some $\Lambda_1 \in c_0(\Gamma)^\perp$ and $\Lambda_2 \in \ell_1(\Gamma)$ and $\|\Lambda\| = \|\Lambda_1\| + \|\Lambda_2\|$. Let $a = \Lambda_2/\|\Lambda_2\| \in S_{\ell_1(\Gamma)}$. Since $\Lambda(e_\gamma) = \Lambda_2(e_\gamma) < \frac{1}{2}\|\Lambda_2\|$ we have $|a_\gamma| < 1/2$ for all $\gamma \in \Gamma$. Then by Theorem 3.2.16, $Y = \ker a = \ker \Lambda_2$ in $c_0(\Gamma)$ is $(*)$- and BR in $c_0(\Gamma)$. Also $\ker(\Lambda) \supseteq \ker(\Lambda_1) \cap \ker(\Lambda_2) \supseteq c_0(\Gamma) \cap \ker(\Lambda_2) = Y$. Since Y is a $(*)$- and DBR subspace in $c_0(\Gamma)$, it is also $(*)$- and DBR in $\ell_\infty(\Gamma)$ (Corollary 3.2.5). By Corollary 2.3.17, $\ker(\Lambda)$ is a $(*)$- and DBR subspace in $\ell_\infty(\Gamma)$. ☐

Similar to Theorem 3.2.16, we have

Theorem 3.2.21. *Let $a = (a_\gamma) \in S_{\ell_1(\Gamma)}$ and $Y = \ker a \subseteq \ell_\infty(\Gamma)$. Then Y is a $(*)$-subspace in $\ell_\infty(\Gamma)$ if and only if $|a_\gamma| \leq 1/2$ for all $\gamma \in \Gamma$.*

Proof. By Proposition 3.2.18(a), Y is DBR. Hence, by Corollary 3.2.4(b), Y is a $(*)$-subspace in $\ell_\infty(\Gamma)$ if and only if for all $\gamma \in \Gamma$, there exists $y \in B_Y$ such that $|y_\gamma| = 1$.

The proof now is essentially that of Theorem 3.2.16, modulo the following observation. If $|a_\gamma| = 1/2$ for some $\gamma \in \Gamma$, define $y \in \ell_\infty(\Gamma)$ by

$$y_\alpha = \begin{cases} sgn(a_\gamma) & \text{if } \alpha = \gamma \\ -sgn(a_\alpha) & \text{if } \alpha \neq \gamma \end{cases}$$

Then $y \in B_Y$ and $|y_\gamma| = 1$ for all $\gamma \in \Gamma$. ☐

3.3 Ball remotality in $\ell_1(\Gamma)$

Now let $\{e_\gamma\}$ denote the canonical basis of $\ell_1(\Gamma)$. Note that each e_γ is a strong unitary.

Theorem 3.3.1. *Any infinite dimensional subspace Y of $\ell_1(\Gamma)$ is a $(*)$- and DBR subspace of $\ell_1(\Gamma)$.*

Proof. For any finite set $\Lambda \subseteq \Gamma$, let $X_\Lambda = \text{span}\{e_\gamma : \gamma \in \Lambda\}$ and $Z_\Lambda = \overline{\text{span}}\{e_\gamma : \gamma \notin \Lambda\}$. Clearly, $\ell_1(\Gamma) = X_\Lambda \oplus_1 Z_\Lambda$.

Let $A = \{x = (x_\gamma) \in \ell_1(\Gamma) : x_\gamma = 0 \text{ for all but finitely many } \gamma \in \Gamma\}$.

If $x \in A$, $x \in X_\Lambda$ for some finite set $\Lambda \subseteq \Gamma$. Since Y is infinite dimensional and Z_Λ is of finite co-dimension, there exists $y \in Y \cap Z_\Lambda$ such that $\|y\|_1 = 1$. Then $\|x + y\|_1 = \|x\|_1 + \|y\|_1 = \|x\|_1 + 1$. It follows that $\phi_{B_Y}(x) = \|x\|_1 + 1$ and $x \in R(B_Y)$.

Since A is dense in $\ell_1(\Gamma)$, Y is a $(*)$- and DBR subspace of $\ell_1(\Gamma)$. ☐

Corollary 3.3.2. *Every subspace of $\ell_1(\Gamma)$ is a DBR subspace.*

Remark 3.3.3. It was noted in [10] that there are non-reflexive Banach spaces in which every subspace is DBR. Indeed, if X^{**} is separable, then X^* is an Asplund space and has an equivalent LUR dual norm [16, Theorem II.2.6]. Hence, by a result of [39] quoted in the Introduction, any subspace Y of X^* is DBR. However, such a space need not be reflexive (see *e.g.*, [18, p 214]). Clearly, the space ℓ_1 with its natural norm produces a much simpler example.

Remark 3.3.4. Analogous result for closed bounded balanced sets appears to be difficult. For example, [17, Proposition 1] produces a $(*)$-subset of ℓ_1 that has no farthest points.

Lemma 3.3.5. *Let Y be a $(*)$-subspace of $\ell_1(\Gamma)$. Then $x \in R(B_Y)$ if and only if there exists $y \in S_Y$ such that for all $\gamma \in \Gamma$, either $x_\gamma = 0$ or $y_\gamma = 0$ or $sgn(y_\gamma) = sgn(x_\gamma)$.*

Proof. For $\alpha, \beta \in \mathbb{F}$, it is easy to see that $|\alpha+\beta| = |\alpha| + |\beta|$ if and only if either $\alpha = 0$ or $\beta = 0$ or $sgn(\alpha) = sgn(\beta)$.

Since Y is a $(*)$-subspace, $-y \in F_{B_Y}(x)$

$$\begin{aligned} &\iff \|x+y\|_1 = \sum_\gamma |x_\gamma + y_\gamma| = \sum_\gamma (|x_\gamma| + |y_\gamma|) = \|x\|_1 + 1 \\ &\iff |x_\gamma + y_\gamma| = |x_\gamma| + |y_\gamma| \text{ for all } \gamma \in \Gamma \\ &\iff x_\gamma = 0 \text{ or } y_\gamma = 0 \text{ or } sgn(y_\gamma) = sgn(x_\gamma) \text{ for all } \gamma \in \Gamma \end{aligned}$$

□

We now characterize BR hyperplanes in ℓ_1.

Theorem 3.3.6. *Let $a = (a_n) \in \ell_\infty$ and $Y = \ker a \subseteq \ell_1$. Then the following are equivalent :*

(a) *Y is BR in ℓ_1*

(b) *$a_n = 0$ for some $n \geq 1$.*

(c) *Y contains a strong unitary.*

Proof. $(b) \Rightarrow (c)$. If $a_n = 0$ for some $n \geq 1$, then $e_n \in Y$, a strong unitary.

$(c) \Rightarrow (a)$ follows from Corollary 2.4.3(b).

$(a) \Rightarrow (b)$. By Theorem 3.3.1, Y is a $(*)$-subspace.

If $a_n \neq 0$ for all $n \geq 1$, let $z = (sgn(a_n)/2^n)$, then $z \in S_{\ell_1}$ and $z \notin Y$.

If $z \in R(B_Y)$, then by Lemma 3.3.5, there exists $y \in S_Y$ such that for all $n \geq 1$, either $y_n = 0$ or $sgn(y_n) = sgn(z_n) = (sgn(a_n))^{-1}$. Thus $\sum_n a_n y_n = \sum_n |a_n||y_n| \neq 0$. A contradiction since $y \in Y$.

Hence, $z \notin R(B_Y)$ and Y is not BR. □

Remark 3.3.7. On the contrary, c_0 has $(*)$- and BR hyperplanes by Theorem 3.2.16, but no strong unitaries.

If Γ is uncountable, the situation is quite different.

Theorem 3.3.8. *If Γ is an uncountable set, then any finite co-dimensional subspace of $\ell_1(\Gamma)$ is BR.*

Proof. The argument is similar to Theorem 3.3.1.

For any countable set $\Lambda \subseteq \Gamma$, let $X_\Lambda = \overline{\text{span}}\{e_\gamma : \gamma \in \Lambda\}$ and $Z_\Lambda = \overline{\text{span}}\{e_\gamma : \gamma \notin \Lambda\}$. Clearly, $\ell_1(\Gamma) = X_\Lambda \oplus_1 Z_\Lambda$.

If $x \in \ell_1(\Gamma)$, $x \in X_\Lambda$ for some countable set $\Lambda \subseteq \Gamma$. Since Y is finite co-dimensional, there exists $y \in Y \cap Z_\Lambda$ such that $\|y\|_1 = 1$. Then $\|x + y\|_1 = \|x\|_1 + \|y\|_1 = \|x\|_1 + 1$. It follows that $x \in R(B_Y)$. □

CHAPTER 4

Ball remotality in $C(K)$

4.1 Summary of results

In this chapter, we study ball remotality of subspaces of $C(K)$. We characterize $(*)$-, $(*)$- and DBR/BR subspaces of $C(K)$ in terms of the density of certain subsets of K. As before, we first prove the results for closed bounded balanced subsets. In the process, we prove that any Banach space embeds isometrically as a $(*)$- and DBR subspace of some $C(K)$ space.

In subsection 4.2.1, we study boundaries of a subspace of $C(K)$. In particular, we relate the Choquet boundary with other boundaries, in the process recapturing some classical results. We also show that if Y is a subspace of co-dimension n in $C(K)$, then any closed boundary for Y can miss at most n points of K. In particular, if K has no isolated points, then any finite co-dimensional subspace cannot have any proper closed boundary.

Applying these results to the question of DBR subspaces, in subsection 4.2.2, we show that an infinite compact Hausdorff space K has no isolated point if and only if any finite co-dimensional subspace, in particular, any hyperplane is DBR in $C(K)$ (Theorem 4.2.38). We characterize $(*)$- and DBR hyperplanes in $C(K)$ in terms of the defining measures (Theorem 4.2.41). We also show that a Banach space X is reflexive if and only if X is a DBR subspace of any superspace in which it embeds isometrically as a hyperplane (Corollary 4.2.44).

In subsection 4.2.3, we obtain some partial results in the remaining cases. As applications, we prove that any M-ideal or any closed $*$-subalgebra of $C(K)$ is a DBR subspace.

In Section 4.3, we extend some of these results to the space $C_0(L)$, where L is a locally compact Hausdorff space.

4.2 Ball remotality of subspaces in $C(K)$

Notation 3. Let K be a compact Hausdorff space. We denote by $C(K)$ the space of all $\mathbb{F}$-valued continuous functions on K with the sup norm. Recall that $C(K)^* = M(K)$, the space of all regular Borel measures on K with the total variation norm. For $\mu \in M(K)$, we will write

$$\mu(f) = \int_K f d\mu, \quad f \in C(K).$$

For $t \in K$, let δ_t be Dirac measure at t, and for a subspace Y of $C(K)$, let $e_t = \delta_t|_Y$.

Since $\{\delta_t : t \in K\}$ is a boundary for $C(K)$, it follows from Theorem 2.2.2 that

Theorem 4.2.1. *Let $C \subseteq C(K)$ be closed, bounded and balanced set. For $t \in K$, let $M_t = \sup_{g\in C} |g(t)|$. Then*

(a) *for any $f \in C(K)$,*

$$\phi_C(f) = \sup\{|f(t)| + M_t : t \in K\}. \tag{4.1}$$

(b) *$f \in R(C)$ if and only if there exists $t_0 \in K$ and $g \in C$ such that $M_{t_0} = |g(t_0)|$ and $\phi_C(f) = |f(t_0)| + M_{t_0} = |f(t_0)| + |g(t_0)|$.*

Lemma 4.2.2. *$T \subseteq K$ is dense in K if and only if $\{\delta_t : t \in T\}$ is a norming set for $C(K)$.*

Proof. Clearly, if T is dense in K, then $\{\delta_t : t \in T\}$ is a norming set for $C(K)$.

Conversely, if T is not dense in L, there exists $t_0 \in K \setminus \overline{T}$. Get $g \in C(K)$ such that $0 \le g \le 1$, $g(t_0) = 1$ and $g|_{\overline{T}} \equiv 0$. Clearly, $\{\delta_t : t \in T\}$ cannot norm this g. □

Remark 4.2.3. For a locally compact Hausdorff space L, essentially the same argument shows that $T \subseteq L$ is dense in L if and only if $\{\delta_t : t \in T\}$ is a norming set for $C_0(L)$.

Definition 4.2.4. Let $C \subseteq C(K)$ be a closed balanced subset with $\sup_{g\in C} \|g\|_\infty = 1$. Let

$$\begin{aligned} T' &= \{t \in K : \delta_t \in A_C\} = \{t \in K : M_t = 1\}. \\ T_0 &= \{t \in K : |g(t)| = 1 \text{ for some } g \in C\}. \end{aligned}$$

where A_C is as in Definition 2.3.2.

In addition to Theorem 2.3.8, we have

Theorem 4.2.5. *For a closed balanced subset $C \subseteq C(K)$ with $\sup_{g\in C} \|g\|_\infty = 1$, the following are equivalent :*

(a) *C is a $(*)$-subset of $C(K)$.*

(b) *T' is dense in K.*

(c) *T' is residual, i.e., contains a dense G_δ set in K.*

Proof. $(c) \Rightarrow (b) \Rightarrow (a)$ is clear.

$(a) \Rightarrow (c)$. Let $F(t) = M_t$. Then F is clearly lower semi-continuous (lsc).

CLAIM : $T' = \{t \in K : F \text{ is continuous at } t\}$.

Let $t_0 \in T' = \{t \in K : F(t) = 1\}$ and $t_\alpha \to t_0$. Since F is lsc and $F \le 1$,

$$1 = F(t_0) \le \liminf_\alpha F(t_\alpha) \le \limsup_\alpha F(t_\alpha) \le 1.$$

Hence F is continuous at t_0.

Conversely, suppose F is continuous at some $t_0 \notin T'$. Then $F(t_0) < 1$. Let $0 < \varepsilon < (1 - F(t_0))/2$. By continuity, there is an open neighbourhood U of t_0 such that $|F(t) - F(t_0)| < \varepsilon$ for all $t \in U$. Let $f \in C(K)$ be such that $f(K) \subseteq [0,1]$, $f(t_0) = 1$ and $f|_{K \setminus U} \equiv 0$.

By (a), $\phi_C(f) = \|f\|_\infty + 1 = 2$. Therefore, there exists $g \in C$ such that $\|f - g\|_\infty > 2 - \varepsilon$. It follows that $f - g$ must attain its norm at some $t_1 \in U$. But

$$|g(t_1)| \leq M_{t_1} = F(t_1) < F(t_0) + \varepsilon.$$

Thus,

$$\|f - g\|_\infty = |f(t_1) - g(t_1)| \leq |f(t_1)| + |g(t_1)| < 1 + F(t_0) + \varepsilon < 2 - \varepsilon,$$

a contradiction that proves the claim.

Now by [23], points of continuity of an lsc function on a compact Hausdorff space forms a residual set. Hence T' is residual. □

Remark 4.2.6. (a) If the scalars are real, then $(a) \Rightarrow (c)$ can also be proved using Theorem 2.3.8(i) and the characterization of sets in $C(K)$ that are intersection of closed balls [32, Proposition 4.1].

(b) A much simpler and direct proof of $(a) \Rightarrow (b)$ can also be obtained from Theorem 2.3.8 and Lemma 4.2.2, as $\{\delta_t : t \in K\}$ is a boundary for $C(K)$.

Proposition 4.2.7. *Let $C \subseteq C(K)$ be a closed balanced subset with $\sup_{g \in C} \|g\|_\infty = 1$. Let*

$$A = \{f \in C(K) : f(t) = \|f\|_\infty \text{ for some } t \in T_0\}.$$

Then $A \subseteq R(C)$. If C is a $(*)$-subset, then $A = R(C)$.

Proof. Let $f \in A$ and $t \in T_0$ be such that $|f(t)| = \|f\|_\infty$. By definition of T_0, there exists $g \in C$ such that $|g(t)| = 1$. By Theorem 4.2.1(b), $f \in R(C)$.

Conversely, if $f \in R(C)$, then by Theorem 4.2.1(b) there exists $t_0 \in K$ and $g \in C$ such that $\phi_C(f) = |f(t_0)| + M_{t_0}$ and $|g(t_0)| = M_{t_0}$. Since C is a $(*)$-subset, $\phi_C(f) = \|f\|_\infty + 1$. Hence $|f(t_0)| = \|f\|_\infty$ and $M_{t_0} = |g(t_0)| = 1$. It follows that $t_0 \in K_0$ and hence, $f \in A$. □

Lemma 4.2.8. *Let $L \subseteq K$ be such that $\overline{L} = K$. Then for any Banach space X, $f \in C(K, X)$ and $\varepsilon > 0$, there exists $g \in C(K, X)$ such that g attains its norm on L and $\|f - g\|_\infty < \varepsilon$.*

Proof. Let $\|f\|_\infty = M$. There exists $x_0 \in f(K)$ such that $\|x_0\| = M$.

Since L is dense in K, there exists $u \in L$ such that $\|f(u) - x_0\| < \varepsilon$.

Let $x_1 = f(u)$. Choose r_0 such that $\|x_1 - x_0\| < r_0 < \varepsilon$ and split X into three disjoint regions :

$$\begin{aligned} X_1 &= \{x \in X : \|x - x_0\| > \varepsilon\}, \\ X_2 &= \{x \in X : \|x - x_0\| \leq r_0\} \text{ and} \\ X_3 &= \{x \in X : r_0 < \|x - x_0\| \leq \varepsilon\}. \end{aligned}$$

Define $\phi : X_1 \cup X_2 \to X$ as follows :

$$\phi(x) = x \quad \text{if } x \in X_1, \qquad \phi(x) = x_0 \quad \text{if } x \in X_2.$$

To define ϕ on X_3, notice that any point in X_3 is of the form $x_0 + ry$ for some $r \in (r_0, \varepsilon]$ and $y \in S_X$. Define $h : [r_0, \varepsilon] \to [0, \varepsilon]$ by $h(r) = \frac{r - r_0}{\varepsilon - r_0}\varepsilon$ and define $\phi : X_3 \to X$ by $\phi(x_0 + ry) = x_0 + h(r)y$.

CLAIM: $\phi : X \to X$ is continuous.

It clearly suffices to check the continuity of ϕ on $\overline{X}_3$.

Let $(z_n), z_0 \in \overline{X}_3$ such that $z_n \to z_0$. Then $z_n = x_0 + r_n y_n$ and $z_0 = x_0 + ry$, for some $r_n, r \in [r_0, \varepsilon]$ and $y_n, y \in S_X$.

Clearly, $r_n \to r$, and since $r \geq r_0 > 0$, $\frac{r_n}{r} y_n \to y$. Therefore, $\|y_n - y\| \leq \left\|y_n - \frac{r_n}{r} y_n\right\| + \left\|\frac{r_n}{r} y_n - y\right\| \to 0$ as $n \to \infty$. Now

$$\begin{aligned} \|\phi(z_n) - \phi(z_0)\| &= \|h(r_n)y_n - h(r)y\| \leq |h(r_n) - h(r)| + |h(r)|\|y_n - y\| \\ &\leq \varepsilon \left(\left| \frac{r_n - r}{\varepsilon - r_0} \right| + \|y_n - y\| \right) \to 0 \end{aligned}$$

as $n \to \infty$. This proves the claim.

Define $g : K \to X$ by $g = \phi \circ f$.

Note that $g(K) \subseteq \phi(MB_X) \subseteq MB_X$ as the last set is convex and ϕ maps a point z of X_3 to a point on the straight line $[z, x_0]$. It follows that $\|g\|_\infty \leq M = \|x_0\| = \|\phi(x_1)\| = \|g(u)\|$. Thus, g attains its norm on L. Moreover,

$$\begin{aligned} \|f - g\|_\infty &\leq \sup\{\|x - \phi(x)\| : x \in X\} = \sup\{\|x - \phi(x)\| : x \in X_2 \cup X_3\} \\ &= \max\{\sup\{\|x - \phi(x)\| : x \in X_2\}, \sup\{\|x - \phi(x)\| : x \in X_3\}\} \\ &= \max\{r_0, \sup\{|r - h(r)| : r \in (r_0, \varepsilon]\}\} \\ &= \max\left\{r_0, \sup\left\{\frac{r_0(\varepsilon - r)}{\varepsilon - r_0} : r \in (r_0, \varepsilon]\right\}\right\} \leq r_0 < \varepsilon. \end{aligned}$$

Thus completes the proof. □

Remark 4.2.9. For a locally compact Hausdorff space L, the above proof still works if $C(K, X)$ is replaced by $C_0(L, X)$.

Theorem 4.2.10. *Let $C \subseteq C(K)$ be a closed balanced subset with $\sup_{g\in C} \|g\|_\infty = 1$. Then C is a $(*)$- and densely remotal subset of $C(K)$ if and only if T_0 is dense in K.*

Proof. Since $T_0 \subseteq T'$, if T_0 is dense in K, then by Theorem 4.2.5, C is a $(*)$-subset of $C(K)$.

Putting $L = T_0$ in Lemma 4.2.8, we get that the set A in Proposition 4.2.7 is dense in $C(K)$. And hence, C is densely remotal.

Conversely, if C is a $(*)$-subset, by Proposition 4.2.7, $A = R(C)$. It follows that for any $f \in R(C)$, $\|f\|_\infty = \|f|_{T_0}\|_\infty$. If C is densely remotal, then it follows that $\|f\|_\infty = \|f|_{T_0}\|_\infty$ for all $f \in C(K)$. Hence T_0 is dense in K. □

Coming to subspaces of $C(K)$, we will use the following notations :

Definition 4.2.11. Let Y be a subspace of $C(K)$. Let

$$\begin{aligned} K' &= \{t \in K : \delta_t \in A_Y\} = \{t \in K : \|e_t\| = 1\}. \\ K_0 &= \{t \in K : |g(t)| = 1 \text{ for some } g \in S_Y\} = \{t \in K : \delta_t \in N_Y\} \end{aligned}$$

where A_Y and N_Y are as in Definition 2.3.2 and Proposition 2.3.15 respectively. Clearly, $K_0 \subseteq K'$.

Putting $C = B_Y$ in the above results, we obtain

Theorem 4.2.12. *Let Y be a subspace of $C(K)$.*

(a) *Then for any $f \in C(K)$,*

$$\phi_{B_Y}(f) = \sup\{|f(t)| + \|e_t\| : t \in K\}. \tag{4.2}$$

(b) *$f \in R(B_Y)$ if and only if there exists $t_0 \in K$ such that the supremum in (4.2) is attained at t_0 and $e_{t_0} \in NA(Y)$.*

Theorem 4.2.13. *Let Y be a subspace of $C(K)$. The following are equivalent :*

(a) *Y is a $(*)$-subspace of $C(K)$.*

(b) *K' is dense in K.*

(c) *K' is residual in K.*

Proposition 4.2.14. *Let Y be a subspace of $C(K)$. Let*

$$A = \{f \in C(K) : f(t) = \|f\|_\infty \text{ for some } t \in K_0\}.$$

Then $A \subseteq R(B_Y)$. If Y is a $(*)$-subspace, then $A = R(B_Y)$.

And our main characterization theorem becomes

Theorem 4.2.15. *Let Y be a subspace of $C(K)$. Then Y is a $(*)$- and DBR subspace of $C(K)$ if and only if K_0 is dense in K.*

Now we are ready for few further consequences.

Corollary 4.2.16. *Any Banach space X is a $(*)$- and DBR subspace of $C(K)$, where $K = (B_{X^*}, w^*)$ if X is infinite dimensional, and $K = (S_{X^*}, norm)$, if X is finite dimensional.*

Proof. If X is finite dimensional, it is BR in any superspace.

If X is infinite dimensional, let $K = (B_{X^*}, w^*)$. Then X embeds isometrically as a subspace of $C(K)$. Now notice that $K_0 = NA(X) \cap S_{X^*}$, which is norm dense in S_{X^*} by Bishop-Phelps Theorem, and hence w*-dense in B_{X^*}. □

The following result should be known. The proof is, nevertheless, included.

Proposition 4.2.17. *For a compact Hausdorff space K, the following are equivalent :*

(a) *K is first countable.*

(b) *Each singleton in K is a G_δ set in K.*

(c) *For any $t_0 \in K$, there exists $f \in C(K)$ such that $f(t_0) = 1$ and $0 \leq f(t) < 1$ for all $t \neq t_0$.*

Proof. $(a) \Rightarrow (b)$ is clear.

$(b) \Rightarrow (a)$. Let $t_0 \in K$. By (b), there exists open sets $\{U_n\}$ such that $\{t_0\} = \cap_n U_n$. Now for each $n \geq 1$, get an open set V_n such that $t_0 \in V_n \subseteq \overline{V}_n \subseteq U_n$. Let $W_n = \cap_{i=1}^n V_i$. We will show that $\{W_n\}$ is a local base at t_0.

Claim: Let U be an open neighbourhood of t_0. Then $\overline{W}_n \subseteq U$ for some $n \geq 1$.

If not, then $R_n = \overline{W}_n \setminus U \neq \emptyset$ for all n. Now, $\{R_n\}$ is a decreasing sequence of closed sets, hence $\cap_n R_n \neq \emptyset$. But $\overline{W}_n \subseteq \cap_{i=1}^n \overline{V}_i \subseteq \cap_{i=1}^n U_i$. Hence $\cap_n R_n \subseteq \cap_n \overline{W}_n \subseteq \cap_n U_n = \{t_0\}$. This implies $t_0 \notin U$, a contradiction.

$(b) \Rightarrow (c)$. Let $t_0 \in K$. By (b), there exists open sets $\{U_n\}$ such that $\{t_0\} = \cap_n U_n$. Get $\{f_n\} \subseteq C(K)$ such that $f_n : K \to [0,1]$, $f_n(t_0) = 1$ and $f_n(U_n^c) = 0$. Define $f(t) = \sum_n 2^{-n} f_n(t)$. Clearly, $f \in C(K)$ and $f(t_0) = 1$. If $t \in K$ and $t \neq t_0$, there exists U_m such that $t \notin U_m$. So $f_m(t) = 0$, and hence, $f(t) < 1$.

$(c) \Rightarrow (b)$. If such an $f \in C(K)$ exists, then $\{t_0\} = f^{-1}(\{1\})$ is a G_δ set in K. □

Remark 4.2.18. Note that if K is metrizable, then the above conditions hold. But a first countable compact Hausdorff space need not be metrizable (see *e.g.*, [21, Exercise 3.2.E]).

Proposition 4.2.19. *(a) If K_0 is finite, then Y is BR.*

(b) If $K_0 = K$, then Y is $(*)$- and BR in $C(K)$. If K is first countable, then the converse is also true.

Proof. (a). If K_0 is finite, it is closed and clearly, Y embeds isometrically into $C(K_0)$. It follows that Y is finite dimensional, and hence, BR.

(b). If $K_0 = K$, it follows from Theorem 4.2.13 that Y is a $(*)$-subspace. Moreover, in Proposition 4.2.14, $A = C(K)$ and therefore, Y is BR.

If Y is $(*)$- and BR, let $t_0 \in K$. By Proposition 4.2.17, there exists $f \in C(K)$ such that $f(t_0) = 1$ and $0 \leq f(t) < 1$ for all $t \neq t_0$. Since $f \in R(B_Y)$, by Proposition 4.2.14, $t_0 \in K_0$, that is, $K_0 = K$. □

Corollary 4.2.20. *(a) If $\mathbf{1} \in Y \subseteq C(K)$ then Y is $(*)$- and BR in $C(K)$.*

(b) $Y = \{g \in C[0,1] : \int_0^1 g(t)dt = 0\}$ is $(*)$- and BR in $C[0,1]$.

Remark 4.2.21. (a) also follows from the fact that Y contains the constant function $\mathbf{1}$, which is a strong unitary.

If the scalars are real, (b) produces an example of a subspace of $C(K)$ which is $(*)$- and BR but does not contain a strong unitary.

4.2.1 On boundaries of subspaces of $C(K)$

Some of the results in this section may be folklore, but we have not found them recorded anywhere, hence we include proofs.

Definition 4.2.22. Let Y be a subspace of $C(K)$. A set $B \subseteq K$ is said to be a boundary for Y if for every $g \in Y$, there exists $t \in B$ such that $|g(t)| = \|g\|_\infty$.

Clearly, K_0 defined above is a boundary for Y and $K_1 = \overline{K_0}$ is a closed boundary.

We first recall some standard, and some not-so-standard, results on the Choquet boundary of a subspace Y of $C(K)$. We follow the terminology from [35].

Let Y be a subspace of $C(K)$ which separates points of K and contains the constants. The state space of Y is defined as

$$\mathcal{S}_Y = \{\Lambda \in S_{Y^*} : \Lambda(\mathbf{1}) = 1\}.$$

This is a weak*-compact convex subset of B_{Y^*}. The evaluation map $e : K \to (B_{Y^*}, w^*)$ defined by $e(t) = e_t$ is a homeomorphism of K into $\mathcal{S}_Y$. The Choquet boundary of Y is defined to be the set :

$$\partial Y = \{t \in K : e_t \in \partial_e \mathcal{S}_Y\}.$$

Definition 4.2.23. Let Y be a subspace of $C(K)$. Any $\mu \in M(K)$ satisfying

$$f(t) = \mu(f) \quad \text{for all } f \in Y,$$

will be called a representing measure for t.

An useful characterization of ∂Y in this situation is given by

Proposition 4.2.24. [34, Proposition 6.2, page 29] *Let Y be a subspace of $C(K)$ that separates points of K and contains constants. Then $t \in \partial Y$ if and only if $\mu = \delta_t$ is the only probability measure on K representing t.*

It follows that for every $t \in \partial Y$, δ_t is the unique Hahn-Banach, *i.e.*, norm-preserving extension of e_t to $C(K)$.

Now if Y separates points of K but does not contain the constants, we cannot define the state space. To define the Choquet boundary in this case, we observe that if $\Lambda \in \partial_e B_{Y^*}$, then Λ has a Hahn-Banach extension that is in $\partial_e B_{C(K)^*}$. Therefore, there exists $t \in K$ and $\alpha \in \mathbb{T}$ (not necessarily unique) such that $\Lambda = \alpha e_t$. Then $e_t \in \partial_e B_{Y^*}$, and

Definition 4.2.25. The Choquet boundary of Y is defined to be the set :

$$\partial Y = \{t \in K : e_t \in \partial_e B_{Y^*}\}.$$

This definition coincides with the earlier definition when Y contains the constants. We note that the same definition works even for general subspaces of $C(K)$ and that is what we work with. It is clear that ∂Y is a boundary for Y.

It is well-known that when Y separates points of K and contains the constants, ∂Y is contained in any closed boundary [34, Proposition 6.4, p 30]. We now relate the Choquet boundary with other closed boundaries for a general subspace Y of $C(K)$.

Theorem 4.2.26. *Let Y be a subspace of $C(K)$ and $B \subseteq K$ a closed boundary for Y. Then*

(a) *$e(\partial Y) \subseteq \mathbb{T}e(B)$.*

(b) *If Y contains the constants and separates points of K, then $\partial Y \subseteq B$.*

(c) *If $\{|f| : f \in Y\}$ separates points of K then also $\partial Y \subseteq B$.*

(d) *If K_0, as in Definition 4.2.11, is closed, then $\partial Y \subseteq K_0$.*

Proof. (a). The map $e : K \to (B_{Y^*}, w^*)$ is clearly continuous and hence, $\mathbb{T}e(B)$ is a w*-compact subset of B_{Y^*}. Since $e(B)$ is a norming set for Y, by separation arguments,

$$B_{Y^*} = \overline{co}^{w^*}(\mathbb{T}e(B)).$$

By Milman's theorem [34, Proposition 1.5, page 6], we have

$$e(\partial Y) \subseteq \partial_e B_{Y^*} \subseteq \mathbb{T}e(B).$$

(b). If $t \in \partial Y$, then by (a), there are $\gamma \in \mathbb{T}$ and $b \in B$ such that $g(t) = \gamma g(b)$ for all $g \in Y$. Taking $g \equiv 1 \in Y$, we get $\gamma = 1$. Now, since Y separates points, $t = b$ and (b) follows.

(c). By (a), if $t \in \partial Y$, then there is $b \in B$ such that

$$|g(t)| = |g(b)| \quad \text{for all } g \in Y. \tag{4.3}$$

Now the hypothesis implies $t = b$, *i.e.*, $\partial Y \subseteq B$.

(d) follows from (4.3) and the definition of K_0. □

Remark 4.2.27. (a) If $Y = \{f \in C(K) : f|_D \equiv 0\}$, where $D \subseteq K$ is a closed set, then $K' = K \setminus D$ and points of K' are separated by non-negative functions in Y. Therefore, (c) holds.

(b) Though $\partial Y \subseteq K_0$ when K_0 is closed, the two sets need not be equal. For example, if $Y = \{f \in C[0,1] : f(0) = \int_0^1 f(t)\,dt\}$, then $K_0 = [0,1]$ is closed but $0 \notin \partial Y$ as it has a representing measure other than δ_0, namely, the Lebesgue measure on $[0,1]$.

Lemma 4.2.28. *Let $B \subseteq K$ be a closed boundary for Y. For any $t \in K$, there exists $\mu_t \in M(B)$ such that $\|\mu_t\| = \|e_t\|$ and μ_t represents t on B.*

If Y separates points of K, the map $t \to \mu_t$ is one-one.

Proof. Since B is a closed boundary for Y, the map $g \to g|_B$ is an isometry between Y and $Y|_B \subseteq C(B)$. Therefore, e_t induces a functional $\Lambda \in (Y|_B)^*$ with $\|\Lambda\| = \|e_t\|$. Any norm preserving extension of Λ on $C(B)$ corresponds to a $\mu_t \in C(B)^*$ such that $\|\Lambda\| = \|\mu_t\|$.

The last statement in the lemma is obvious. □

Theorem 4.2.29. *If Y is a subspace of co-dimension n in $C(K)$ and $B \subseteq K$ is a closed boundary for Y, then $K \setminus B$ contains at most n distinct points.*

In particular, if K has no isolated points, then $B = K$.

Proof. Suppose there are $(n+1)$ distinct points $t_1, t_2, \ldots, t_{n+1}$ in $K \setminus B$. Let μ_i be a representing measure for t_i on B. If $e_{t_i} = 0$ for some i, then $\mu_i = 0$ and $\delta_{t_i} \in Y^\perp$. Since each μ_i has no point mass outside of B, it is clear that the measures $\mu_i - \delta_{t_i}$ are linearly independent. Since each $\mu_i - \delta_{t_i} \in Y^\perp$, this contradicts the fact that Y has co-dimension n.

Now, if $K \setminus B$ is nonempty, it contains at most n points and necessarily these points are isolated. Thus, if K has no isolated points, then $B = K$. □

Since ∂Y is a boundary, we obtain

Corollary 4.2.30. *If Y is a subspace of co-dimension n in $C(K)$, then $K \setminus \overline{\partial Y}$ contains at most n distinct points. And if K has no isolated points, then $\overline{\partial Y} = K$.*

Remark 4.2.31. The stronger result that the set $K \setminus \partial Y$ itself contains at most n points, was proved, under the additional assumption that Y separates points of K, in [24, Lemma 5.6, Theorem 7.3] and in full generality in [5, Proposition 3.1]. Our argument is significantly simpler.

4.2.2 Finite co-dimensional subspaces of $C(K)$

Coming back to DBR subspaces, let Y be a subspace of finite co-dimension in $C(K)$ and $K_1 = \overline{K}_0$. Now, Theorem 4.2.29 yields

Corollary 4.2.32. *If Y has co-dimension n, then there can be at most n distinct points in $K \setminus K_1$. And if K has no isolated points, then $K_1 = K$.*

And therefore,

Theorem 4.2.33. *If K has no isolated points, then any finite co-dimensional subspace of $C(K)$ is a $(*)$- and DBR subspace.*

Remark 4.2.34. Observe that if K is infinite and has no isolated points, *e.g.*, $K = [0, 1]$, then $C(K)$ clearly has hyperplanes that are not proximinal. Thus, DBR subspaces need not be proximinal. And BR hyperplanes in ℓ_1 (see Theorem 3.3.6) produce examples that even BR subspaces need not be proximinal.

It also follows that

Corollary 4.2.35. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-atomic measures, then $Y = \cap_{i=1}^{n} \ker(\lambda_i)$ is a $(*)$- and DBR subspace of $C(K)$.*

Proof. If $K \setminus K_1$ is nonempty, let $t_1, t_2, \ldots, t_m \in K \setminus K_1$ for some $m \leq n$.

Let μ_i be a representing measure for t_i on K_1. Then $\mu_i - \delta_{t_i} \in Y^\perp$. It follows that at least some elements of $Y^\perp$ must put nonzero mass on the points $t_1, t_2, \ldots, t_m$. Hence the result follows. □

Theorem 4.2.36. *If Y is of co-dimension n and $K \setminus K_1$ contains exactly n points, then K_0 is closed. Moreover, $\partial Y = \overline{\partial Y} = K_0 = K_1$.*

Proof. For simplicity, we give the proof for $n = 2$ as no new ideas are required for other values of n.

Let $t_1, t_2 \in K \setminus K_1$ with representing measures μ_1, μ_2 respectively. Let $Y_i = \ker(\mu_i - \delta_{t_i})$, $i = 1, 2$. Then $Y = Y_1 \cap Y_2$.

Find $f_1, f_2 \in C(K)$ such that $f_i(t_j) = \delta_{ij}$ and $f_i|_{K_1} = 0$ for $i, j = 1, 2$. Then $(\mu_i - \delta_{t_i})(f_j) = -\delta_{ij}$. It follows that for $f \in C(K)$, if we put $g = f + (\mu_1 - \delta_{t_1})(f) \cdot f_1 + (\mu_2 - \delta_{t_2})(f) \cdot f_2$, then $g \in Y$.

Now define $f \in C(K)$ by $f|_{K_1} = 1$, $f(t_1) = f(t_2) = 0$ and consider $g \in Y$ as above. Then $g|_{K_1} = 1$ and $g(t_i) = \mu_i(K_1)$, $i = 1, 2$. Thus, $\|g\|_\infty = \max\{1, |\mu_1(K_1)|, |\mu_2(K_1)|\}$.

Since g attains its norm only on $K_0 \subseteq K_1$, we must have $|\mu_1(K_1)| < 1$, $|\mu_2(K_1)| < 1$ and $K_1 \subseteq K_0$ as $g \in S_Y$ and $g|_{K_1} \equiv 1$, and so, K_0 is closed.

Now by Theorem 4.2.26 (d), $\partial Y \subseteq \overline{\partial Y} \subseteq K_0$. If $K_0 \setminus \partial Y$ were non-empty, there would be more than n points outside ∂Y contradicting [5, Proposition 3.1]. □

Remark 4.2.37. What happens if Y is of co-dimension n but $K \setminus K_1$ has fewer than n points? We don't know the answer but a look at some examples seems to suggest that if $t \notin K_1$, then $e_t \notin \partial_e B_{Y^*}$, and so $\overline{\partial Y} \subseteq K_1$.

Theorem 4.2.38. *For an infinite compact Hausdorff space K, the following are equivalent :*

(a) *K has no isolated point.*

(b) *Any finite co-dimensional subspace of $C(K)$ is DBR.*

(c) *Any hyperplane in $C(K)$ is DBR.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ is clear from Theorem 4.2.33.

$(c) \Rightarrow (a)$. Suppose $t_0 \in K$ is an isolated point. Then $K = T \cup \{t_0\}$, where T is closed. Since $C(T)$ is non-reflexive, there exists $\mu \in S_{C(T)^*}$ such that μ is not norm attaining on $C(T)$. Now let $Y = \ker(\delta_{t_0} - \mu)$, *i.e.*,

$$Y = \{f \in C(K) : f(t_0) = \mu(f|_T)\}.$$

It follows that given any $h \in C(T)$, if we define $f : K \to \mathbb{F}$ as

$$f(t) = \begin{cases} h(t) & \text{if} \quad t \in T \\ \mu(h) & \text{if} \quad t = t_0 \end{cases}$$

then $f \in Y$ and $\|f\|_\infty = \|h\|_\infty$. Thus, $T \subseteq K_0$.

CLAIM : $\|e_{t_0}\| = 1$, but $t_0 \notin K_0$.

Since $\|\mu\| = 1$, there exists $(h_n) \subseteq S_{C(T)}$ such that $\mu(h_n) \to 1$. If we define the corresponding $f_n \in S_Y$ as above, then $f_n(t_0) \to 1$. Thus, $\|e_{t_0}\| = 1$.

On the other hand, since μ is not norm attaining on $C(T)$, $t_0 \notin K_0$.

It follows that $K_0 = T$ and $K' = K$. Therefore, Y is a $(*)$-subspace, and hence, cannot be a DBR subspace of $C(K)$. □

Remark 4.2.39. (a) If K is finite, $C(K)$ is finite dimensional and hence, any subspace is BR, but any point of K is also isolated.

(b) Since $K = T \cup \{t_0\}$, $C(K) = C(T) \oplus_\infty \mathbb{F}$. Therefore by Lemma 3.2.10, $C(T)$ is BR in $C(K)$. Clearly, Y is isometric to $C(T)$, but Y is not even DBR in $C(K)$. This emphasizes the fact that this property not only depends on the norm, but also on how Y 'sits' in X.

A simple example of the above phenomenon is given by

Example 4.2.40. Let $K = [0,1] \cup \{2\}$ and μ be the measure on $[0,1]$ defined by $\mu = \lambda|_{[0,1/2]} - \lambda|_{[1/2,1]}$, where λ denotes the Lebesgue measure on $[0,1]$. Now define

$$Y = \{F \in C(K) : F(2) = \int_0^{\frac{1}{2}} F(x)dx - \int_{\frac{1}{2}}^1 F(x)dx\}.$$

It is easy to see that μ is not norm attaining on $C[0,1]$. It follows that $K_0 = [0,1]$ and $K' = K$. Therefore, Y is a $(*)$-subspace of $C(K)$. But Y cannot be DBR in $C(K)$.

Similar examples can be constructed for any finite co-dimension. For example for $n = 2$, let $K = [0,1] \cup \{-1\} \cup \{2\}$, and let $Y = \ker(\mu_1) \cap \ker(\mu_2)$ where

$$\begin{aligned} \mu_1 &= \frac{1}{2} \cdot \delta_2 + \lambda|_{[\frac{1}{2},\frac{3}{4}]} - \lambda|_{[\frac{3}{4},1]} \\ \mu_2 &= \frac{1}{2} \cdot \delta_{-1} + \lambda|_{[\frac{1}{4},\frac{1}{2}]} - \lambda|_{[0,\frac{1}{4}]} \end{aligned}$$

One can check as before that Y is a $(*)$-subspace, but not DBR in $C(K)$.

Now we can characterize $(*)$- and DBR hyperplanes in $C(K)$.

Theorem 4.2.41. *If $\mu \in S_{C(K)^*}$, then $Y = \ker(\mu)$ is not $(*)$- and DBR in $C(K)$ if and only if the following conditions hold :*

(a) There is an isolated point $t_0 \in K$ such that $|\mu(\{t_0\})| \geq 1/2$.

(b) If we write $\mu = \alpha\delta_{t_0} + \nu$ and $|\alpha| = 1/2$, then ν is not norm attaining on $C(K \setminus \{t_0\})$.

Proof. First assume Y is not $(*)$- and DBR in $C(K)$.

Then $K_1 \neq K$ and hence, by Theorem 4.2.36, K_0 is closed and there exists exactly one isolated point $t_0 \in K$ such that $t_0 \notin K_0$ and $K = K_0 \cup \{t_0\}$.

Now we can write $\mu = \alpha\delta_{t_0} + \nu$, where ν is supported on K_0. Moreover, $1 = \|\mu\| = |\alpha| + \|\nu\|$.

If $|\alpha| < 1/2$, then $\|\nu\| = 1 - |\alpha| > 1/2 > |\alpha|$. So there exists $g \in B_{C(K_0)}$ such that $\nu(g) = |\alpha|$. Define $f \in B_{C(K)}$ by

$$f(t) = \begin{cases} g(t) & \text{if } t \in K_0 \\ -sgn(\alpha) & \text{if } t = t_0 \end{cases}$$

Clearly $f \in S_Y$ and $|f(t_0)| = 1$, which implies $t_0 \in K_0$. This contradiction ensures that $|\alpha| \geq 1/2$.

Now suppose $|\alpha| = 1/2$, then $\|\nu\| = 1/2$. If ν is norm attaining on $C(K_0)$, we can get $g \in B_{C(K_0)}$ such that $\nu(g) = \|\nu\| = 1/2 = |\alpha|$ and hence, $f \in B_{C(K)}$ as defined above satisfies $f \in S_Y$ and $|f(t_0)| = 1$. This again implies $t_0 \in K_0$. A contradiction!

Conversely assume that (a) and (b) hold. It is enough to prove that $t_0 \notin K_0$.

If $t_0 \in K_0$, then there exists $f \in B_Y$ with $|f(t_0)| = 1$. It follows that

$$|\alpha| = |\alpha f(t_0)| = |\nu(f)| \leq \|\nu\| = 1 - |\alpha|$$

which implies $|\alpha| \leq 1/2$. This together with (a) implies $|\alpha| = 1/2$, and hence, $\|\nu\| = 1/2$. It follows that $|\nu(f)| = |\alpha| = 1/2$. Thus ν is norm attaining, contradicting (b). Hence $t_0 \notin K_0$. □

Remark 4.2.42. In the above, if $|\alpha| = \|\nu\| = 1/2$ and ν is norm attaining on $C(K \setminus \{t_0\})$, then $Y = \ker(\mu)$ is actually $(*)$- and BR. Indeed, from the above proof, it follows that $t_0 \in K_0$. Now define g on $B_{C(K)}$ by

$$g(t) = \begin{cases} 1 & \text{if } t \neq t_0 \\ -\nu(1)/\alpha & \text{if } t = t_0 \end{cases}$$

Since $|\nu(1)| \leq \|\nu\| = |\alpha| = 1/2$, $g \in S_Y$ and therefore, $K = K_0$.

We now obtain a characterization of reflexivity. For a subspace Y of a Banach space X, let A_Y and N_Y be as in Definition 2.3.2 and Proposition 2.3.15 respectively.

Theorem 4.2.43. *Let X be a non-reflexive Banach space. Then there exists a Banach space Z and a hyperplane Y in Z such that X is isometric to Y and Y is not a DBR subspace of Z.*

Proof. Define $Z = X \oplus_\infty \mathbb{F}$. Since X is non-reflexive, there exists $x_0^* \in S_{X^*} \setminus NA(X)$.

Let $Y = \{(x, x_0^*(x)) : x \in X\}$. Clearly, Y is a hyperplane in Z. Since $\|x_0^*\| = 1$, Y is isometric to X.

CLAIM 1 : Y is a $(*)$-subspace of Z.

Clearly, $\{(x^*, 0) : x^* \in S_{X^*}\} \subseteq A_Y$. And since $\|x_0^*\| = 1$, it also follows that $(0, 1) \in A_Y$. Thus A_Y is norming for Z.

CLAIM 2 : $N_Y = \{(x^*, 0) : x^* \in NA(X) \cap S_{X^*}\}$.

Let $z^* = (x^*, \alpha) \in N_Y$. Then for some $x \in S_X$,

$$1 = |(x^*, \alpha)(x, x_0^*(x))| = |x^*(x) + \alpha x_0^*(x)| \leq |x^*(x)| + |\alpha| \cdot |x_0^*(x)| \leq \|x^*\| + |\alpha| = 1.$$

Since $x_0^* \notin NA(X)$, $\alpha = 0$ and $|x^*(x)| = \|x^*\|$. Hence the claim.

But clearly, N_Y cannot be norming for Z and hence, by Proposition 2.3.16, Y cannot be a DBR subspace of Z. □

Corollary 4.2.44. *For a Banach space X, the following are equivalent :*

(a) *X is reflexive.*

(b) *X is a DBR subspace of any superspace.*

(c) *X is a DBR subspace of any superspace in which it embeds isometrically as a hyperplane.*

4.2.3 Other DBR subspaces of $C(K)$

Let Y be a subspace of $C(K)$. We may assume $K_1 \neq K$. Note that $g \mapsto g|_{K_1}$ is an isometric embedding of Y into $C(K_1)$, and $B_Y|_{K_1} = B_{(Y|_{K_1})}$ is densely remotal in $C(K_1)$.

Theorem 4.2.45. *Let $K_2 = \overline{K \setminus K_1}$. Suppose $B_Y|_{K_2}$ is remotal in $C(K_2)$, then Y is a DBR subspace of $C(K)$.*

Proof. Let $h \in C(K)$ and $\varepsilon > 0$. Let $h_1 = h|_{K_1}$. Since $B_Y|_{K_1}$ is densely remotal in $C(K_1)$, there is some $f_1 \in C(K_1)$ such that $\|f_1 - h_1\|_\infty < \varepsilon$ and $f_1 \in R(B_Y|_{K_1})$. By Tietze's extension theorem, there is $f \in C(K)$ such that $\|f - h\|_\infty < \varepsilon$ and $f|_{K_1} = f_1$. Let $g_1 \in B_Y$ be such that for all $g \in B_Y$, $\|(f+g)|_{K_1}\|_\infty \leq \|(f+g_1)|_{K_1}\|_\infty$. Let $f_2 = f|_{K_2}$. Since $B_Y|_{K_2}$ is remotal in $C(K_2)$, there exist $g_2 \in B_Y$ such that $\|(f+g)|_{K_2}\|_\infty \leq \|(f+g_2)|_{K_2}\|_\infty$ for all $g \in B_Y$. Then for all $g \in B_Y$, $\|f+g\|_\infty = \max\{\|(f+g)|_{K_1}\|_\infty, \|(f+g)|_{K_2}\|_\infty\} \leq \max\{\|(f+g_1)|_{K_1}\|_\infty, \|(f+g_2)|_{K_2}\|_\infty\}$. Now, depending on which of the two terms on the RHS is bigger, either $-g_1$ or $-g_2$ is farthest from f in B_Y. Hence Y is DBR in $C(K)$. □

Remark 4.2.46. If Y is finite co-dimensional, K_2 is finite and hence, as soon as $B_Y|_{K_2}$ is closed in $C(K_2)$, it is remotal and Theorem 4.2.45 applies. However, as Theorem 4.2.38 shows, $B_Y|_{K_2}$ need not be closed in $C(K_2)$.

Interchanging the roles of K_1 and K_2 in the above argument, we also obtain

Theorem 4.2.47. *Suppose K_0 is closed and $B_Y|_{K_2}$ is densely remotal in $C(K_2)$, then Y is a DBR subspace of $C(K)$.*

Corollary 4.2.48. *Suppose for all $g \in Y$, $g|_{K_2} \equiv 0$, then Y is a DBR subspace of $C(K)$.*

Theorem 4.2.49. *If Y is an M-ideal in $C(K)$, then Y is a DBR subspace of $C(K)$.*

Proof. Recall that any M-ideal in $C(K)$ is of the form $Y = J_D = \{f \in C(K) : f|_D \equiv 0\}$ for some closed set $D \subseteq K$ (see [26, Example 1.4 (a)]).

It is easy to see that in this case, $K_0 = K \setminus D$ and therefore, $K_2 = D$. Thus, the result follows from Corollary 4.2.48. □

Example 4.2.50. We now give an example to show that if $Y \subseteq Z \subseteq X$ and Y is a DBR subspace of X, but Z need not be DBR in X.

Let $K = [0,1] \cup \{2\}$ and

$$Z = \{F \in C(K) : F(2) = 2\int_0^{\frac{1}{4}} F(x)dx - 2\int_{\frac{1}{4}}^{\frac{1}{2}} F(x)dx\}.$$

As before, Z is a $(*)$-subspace but not DBR in $C(K)$. Let $D = [0, 1/2] \cup \{2\}$, then J_D is an M-ideal in $C(K)$, and hence, DBR. Also, $J_D \subseteq Z \subseteq C(K)$, but Z is not DBR in $C(K)$.

Theorem 4.2.51. *Let $\{\mu_n\}$ be countable family of regular Borel measures on K. Let $S(\mu_n)$ denote the support of μ_n. Suppose*

(a) for each $n \geq 1$, $K \setminus S(\mu_n)$ is dense in K, and

(b) $\cup_n S(\mu_n)$ is a closed subset of K.

Then $Y = \cap_n \ker(\mu_n)$ is a DBR subspace of $C(K)$.

Proof. Let $D = \cup_n S(\mu_n)$. Let $Z = \{f \in C(K) : f|_D \equiv 0\}$. By Baire Category Theorem, $K \setminus D$ is dense in K. Therefore, by Theorem 4.2.15 and Theorem 4.2.49, Z is a DBR $(*)$-subspace of $C(K)$. Since $Z \subseteq Y \subseteq C(K)$, Y is also a DBR $(*)$-subspace of $C(K)$. □

Proposition 4.2.52. *Let K and S be compact Hausdorff spaces, $\sigma : K \to S$ a continuous onto map, and $s_0 \in S$. Then*

$$Y = \{h \circ \sigma : h \in C(S) \text{ and } h(s_0) = 0\}$$

is a DBR subspace of $C(K)$.

Proof. Since σ is onto, $\|h \circ \sigma\|_K = \|h\|_S$ for $h \in C(S)$.

Let $D = \sigma^{-1}(\{s_0\})$. If $t \notin D$, then there is $h \in C(S)$ such that $h(S) \subseteq [0,1]$, $h(s_0) = 0$ and $h(\sigma(t)) = 1$. It follows that $g = h \circ \sigma \in Y$ and $\|g\|_K = 1$.

Thus $K_0 = K \setminus D$ and therefore, $K_2 = D$. Clearly, $Y|_{K_2} \equiv 0$ and the result again follows from Corollary 4.2.48. □

Theorem 4.2.53. *Any closed self-adjoint subalgebra $\mathcal{A}$ of $C(K)$ is a DBR subspace of $C(K)$.*

Proof. Let $\mathcal{A}$ be a closed self-adjoint subalgebra of $C(K)$.

If $\mathcal{A}$ contains the unit, i.e., the constant function $\mathbf{1}$, then by Corollary 4.2.20, $\mathcal{A}$ is a BR subspace of $C(K)$.

If $\mathcal{A}$ does not contain the unit, then it follows from [36, 38] that there is a compact Hausdorff space S, $s_0 \in S$ and a continuous onto map $\sigma : K \to S$ such that $\mathcal{A} = \{h \circ \sigma : h \in C(S) \text{ and } h(s_0) = 0\}$. Now by Proposition 4.2.52, $\mathcal{A}$ is a DBR subspace of $C(K)$. □

4.3 Ball remotality of subspaces in $C_0(L)$

Let L be a locally compact Hausdorff space. Let $C \subseteq C_0(L)$ be closed, bounded and balanced set. Since $\{\delta_t : t \in L\}$ is a boundary for $C_0(L)$, it follows from Theorem 2.2.2 that

Theorem 4.3.1. *Let $C \subseteq C_0(L)$ be closed, bounded and balanced. For $t \in L$, let $M_t = \sup_{g\in C} |g(t)|$. Then*

(a) for any $f \in C_0(L)$,

$$\phi_C(f) = \sup\{|f(t)| + M_t : t \in L\}. \tag{4.4}$$

(b) $f \in R(C)$ if and only if there exists $t_0 \in L$ and $g \in C$ such that the supremum in (4.4) is attained at t_0 and $M_{t_0} = |g(t_0)|$.

Definition 4.3.2. Let $C \subseteq C_0(L)$ be a closed balanced subset with $\sup_{g\in C} \|g\|_\infty = 1$. Let

$$\begin{aligned} L' &= \{t \in L : \delta_t \in A_C\} = \{t \in L : M_t = 1\}. \\ L_0 &= \{t \in L : |g(t)| = 1 \text{ for some } g \in C\}. \end{aligned}$$

Theorem 4.3.3. *Let $C \subseteq C_0(L)$ be a closed balanced subset with $\sup_{g\in C} \|g\|_\infty = 1$. Then C is a $(*)$-subset of $C_0(L)$ if and only if L' is dense in L.*

Proof. Since $B = \{\delta_t : t \in L\}$ is a boundary for $C_0(L)$, by Theorem 2.3.8, C is a $(*)$-subset of $C_0(L) \Leftrightarrow B \cap A_C = \{\delta_t : t \in L'\}$ is a norming set for $C_0(L) \Leftrightarrow L'$ is dense in L, by Remark 4.2.3. □

Similar to Proposition 4.2.7, we also have

Proposition 4.3.4. *Let $C \subseteq C_0(L)$ be a closed balanced subset with $\sup_{g\in C} \|g\|_\infty = 1$. Let*

$$A = \{f \in C_0(L) : f(t) = \|f\|_\infty \text{ for some } t \in L_0\}.$$

Then $A \subseteq R(C)$. If C is a $(*)$-subset, then $A = R(C)$.

Similar to Theorem 4.2.10, we obtain

Theorem 4.3.5. *Let $C \subseteq C_0(L)$ be a closed balanced subset with $\sup_{g\in C} \|g\|_\infty = 1$. C is $(*)$-subset and densely remotal if and only if $\overline{L}_0 = L$.*

Coming to subspaces, we get

Theorem 4.3.6. *(a) $Y \subseteq C_0(L)$ is a $(*)$-subspace if and only if $\{t \in L : \delta_t \in A_Y\}$ is dense in L.*

(b) $Y \subseteq C_0(L)$ is a $(*)$- and DBR subspace if and only if $\{t \in L : \delta_t \in N_Y\}$ is dense in L.

Similar to c_0, we have

Theorem 4.3.7. *$C_0(L)$ has no finite dimensional $(*)$-subspace.*

Proof. If $Y \subseteq C_0(L)$ is finite-dimensional, then S_Y is compact.

Let $\{g_1, g_2, \ldots, g_n\}$ be a finite $1/4$-net in S_Y. Then $K = \bigcup_{k=1}^{n}\{t \in L : |g_k(t)| \geq 1/4\}$ is compact in L. Therefore, $L \setminus K$ is a nonempty open set.

If $g \in S_Y$, then $\|g - g_k\|_\infty < 1/4$ for some k. Now if $t \in L \setminus K$, then $|g(t)| < |g_k(t)| + 1/4 < 1/2$. It follows that $\|\delta_t|_Y\| \leq 1/2$ and hence, $\{t \in L : \delta_t \in A_Y\}$ cannot be dense in L. □

Corollary 4.3.8. *If $\mathcal{A}$ is a commutative C^*-algebra without identity then $\mathcal{A}$ has no finite dimensional $(*)$-subspace.*

Proof. If $\mathcal{A}$ is a commutative C^*-algebra without identity then there exists a locally compact Hausdorff space Σ (viz. the state space of $\mathcal{A}$) and the Gelfand transform induces an isometric $(*)$-isomorphism from $\mathcal{A}$ onto $C_0(\Sigma)$ [15, Page 237, Corollary 2.2]. The result now follows from the above theorem. □

CHAPTER 5

Ball remotality of M-ideals in some function spaces

5.1 Summary of results

In this chapter, we study ball remotality of M-ideals in some function spaces and function algebras. Isolating a common feature of M-ideals in function spaces, we define an *Urysohn pair* (A, D) (Definition 5.2.2), where A is a subspace of $C(K)$ and $D \subseteq K$ a closed set.

In Theorem 5.2.4, we show that for an Urysohn pair (A, D), the functions in A that vanishes on D forms a DBR subspace of A. As corollaries, we show that :

(a) Any M-ideal in $C(K)$ is DBR, recapturing Theorem 4.2.49 with a new proof.

(b) If L is a locally compact Hausdorff space, then any M-ideal in $C_0(L)$ is a DBR subspace.

(c) Any M-ideal in the disc algebra $\mathbb{A}$ is DBR.

Some generalizations of the last result are also obtained.

In this chapter, we also consider M-ideals in $A_{\mathbb{F}}(Q)$, where Q is a compact convex set in some locally convex topological vector space E. Our main result in this chapter is that if Q is a Choquet simplex and $\overline{\partial_e Q} \setminus \partial_e Q$ is at most finite, then any M-ideal is a DBR subspace of $A_{\mathbb{F}}(Q)$. Some variants of this result are also considered.

5.2 Urysohn pair

Let Y be an M-ideal of X, then as in [26, Remark I.1.13], we can identify Y^* as a subspace of X^*. Moreover, $X^* = Y^\perp \oplus_1 Y^*$. As a consequence of Corollary 2.2.3, we get the following :

Theorem 5.2.1. *Let Y be an M-ideal in a Banach space X. Then*

$$\phi_{B_Y}(x) = \max[d(x, Y), \|x|_{Y^*}\| + 1].$$

Proof. Since $X^* = Y^\perp \oplus_1 Y^*$, $S_{Y^\perp} \cup S_{Y^*}$ is a boundary for X. So we have from Corollary 2.2.3,

$$\begin{aligned}\phi_{B_Y}(x) &= \sup\{|x^*(x)| + \|x^*|_Y\| : x^* \in S_{Y^\perp} \cup S_{Y^*}\}\\ &= \max[\sup\{|x^*(x)| + \|x^*|_Y\| : x^* \in S_{Y^\perp}\}, \sup\{|x^*(x)| + \|x^*|_Y\| : x^* \in S_{Y^*}\}]\\ &= \max[\sup\{|x^*(x)| : x^* \in S_{Y^\perp}\}, \sup\{|x^*(x)| + 1 : x^* \in S_{Y^*}\}]\\ &= \max[d(x, Y), \|x|_{Y^*}\| + 1].\end{aligned}$$

□

Notation 4. If K is compact Hausdorff space and X is a Banach space, $C(K, X)$ will denote the space of all X-valued continuous functions on K with the norm $\|f\|_\infty = \sup_{t\in K} \|f(t)\|$. It is well-known that $C(K, X) = C(K) \otimes_\varepsilon X$, the injective tensor product of $C(K)$ and X.

(a) If $x \in X$, define $1 \otimes x \in C(K, X)$ by $1 \otimes x(t) = x$, for all $t \in K$.

(b) If $f \in C(K)$ and $x \in X$, define $f \otimes x \in C(K, X)$ by $f \otimes x(t) = f(t)x$.

(c) If $t \in K$ and $x^* \in X^*$, define $\delta_t \otimes x^* \in C(K, X)^*$ by $\delta_t \otimes x^*(f) = x^*(f(t))$.

Definition 5.2.2. Let K be a compact Hausdorff space, X a Banach space, $A \subseteq C(K, X)$ a subspace, and $D \subseteq K$ a closed set. We say that (A, D) is an *Urysohn pair* if :

For any $t_0 \in K \setminus D$ and $x \in S_X$, there exists $f \in S_A$ such that $f|_D \equiv 0$ and $f(t_0) = x$.

Proposition 5.2.3. *Let (A, D) be an Urysohn pair and $Y = \{f \in A : f|_D \equiv 0\}$. Then*

(a) *for any $f \in A$, $\phi_{B_Y}(f) = \max\{\|f|_D\|_\infty, \|f|_{K\setminus D}\|_\infty + 1\}$.*

(b) *$f \in R(B_Y)$ if and only if $\phi_{B_Y}(f) = \|f|_D\|_\infty$ or $\|f|_{K\setminus D}\|_\infty = \|f(t_0)\|$ for some $t_0 \in K \setminus D$.*

Proof. (a). Clearly, $B = \{\delta_t \otimes x^* : t \in K, x^* \in NA(X)\}$ is a boundary for $C(K, X)$. Note that if $t \in D$, $\|\delta_t \otimes x^*|_Y\| = 0$ for any $x^* \in NA(X)$. Now, if $t \notin D$ and $x^* \in NA(X)$, let $x_0 \in S_X$ such that $x^*(x_0) = 1$. Since (A, D) is an Urysohn pair, there exists $g \in S_Y$ such that $g(t) = x_0$. Thus, $\|\delta_t \otimes x^*|_Y\| = 1$. It follows from Corollary 2.2.3 that for any $f \in C(K, X)$,

$$\begin{aligned}\phi_{B_Y}(f) &= \sup\{|\delta_t \otimes x^*(f)| + \|\delta_t \otimes x^*|_Y\| : t \in K, x^* \in NA(X)\}\\ &= \max\left[\sup\{|x^*(f(t))| + \|\delta_t \otimes x^*|_Y\| : t \in D, x^* \in NA(X)\},\right.\\ &\qquad \left.\sup\{|x^*(f(t))| + \|\delta_t \otimes x^*|_Y\| : t \in K \setminus D, x^* \in NA(X)\}\right]\\ &= \max\left[\sup\{|x^*(f(t))| : t \in D, x^* \in NA(X)\},\right.\\ &\qquad \left.\sup\{|x^*(f(t))| + 1 : t \in K \setminus D, x^* \in NA(X)\}\right]\\ &= \max\left[\sup\{\|f(t)\| : t \in D\}, \sup\{\|f(t)\| + 1 : t \in K \setminus D\}\right]\\ &= \max\left[\|f|_D\|_\infty, \|f|_{K\setminus D}\|_\infty + 1\right]\end{aligned}$$

(b). If $\phi_{B_Y}(f) = \|f|_D\|_\infty$, then $\phi_{B_Y}(f) = d(f,Y)$ and therefore, any $g \in B_Y$ is farthest from f. Thus, $f \in R(B_Y)$.

If $\phi_{B_Y}(f) = \|f|_{K\setminus D}\|_\infty + 1$ and there is $t_0 \in K \setminus D$ such that $\|f(t_0)\| = \|f|_{K\setminus D}\|_\infty$. Then by Corollary 2.2.3(b), $f \in R(B_Y)$.

Conversely, let $f \in R(B_Y)$. If $\|f|_D\|_\infty \geq 1 + \|f|_{K\setminus D}\|_\infty$, then $\phi_{B_Y}(f) = \|f|_D\|_\infty$ and we are done.

If $\|f|_D\|_\infty < 1 + \|f|_{K\setminus D}\|_\infty$, then $\phi_{B_Y}(f) = \|f|_{K\setminus D}\|_\infty + 1$. By Corollary 2.2.3(b), there exist $t_0 \in K$ and $x^* \in NA(X)$ such that $\|f|_{K\setminus D}\|_\infty + 1 = |x^*(f(t_0))| + \|\delta_{t_0} \otimes x^*|_Y\|$. It follows that $t_0 \notin D$ and $\|f|_{K\setminus D}\|_\infty = |x^*(f(t_0))| = \|f(t_0)\|$. □

Theorem 5.2.4. *If A is a subspace of $C(K,X)$ and $D \subseteq K$ is a closed set such that (A,D) is an Uryshon pair, then $Y = \{f \in A : f|_D \equiv 0\}$ is a DBR subspace of A.*

Proof. If $\phi_{B_Y}(f) = \|f|_D\|_\infty$ or $\|f|_{K\setminus D}\|_\infty = \|f(t_0)\|$ for some $t_0 \in K \setminus D$, then by Proposition 5.2.3(b), $f \in R(B_Y)$.

Let $K_1 = \overline{K \setminus D}$. Suppose $\phi_{B_Y}(f) = \|f|_{K_1}\|_\infty + 1 > \|f|_D\|_\infty$ and $\|f(t_0)\| = \|f|_{K_1}\|_\infty$ only for some $t_0 \in \partial D$. In this case, we will show that $f \in \overline{R(B_Y)}$.

Let $\varepsilon > 0$. We may assume that $\|h|_{K_1}\|_\infty + 1 > \|h|_D\|_\infty$ whenever $\|h - f\| \leq \varepsilon$. Consider $U = \{t \in K : \|f(t) - f(t_0)\| < \varepsilon/2\}$. Since $t_0 \in \partial D$, there exists $t_1 \in K \setminus D$ such that $t_1 \in U$. Let $x_0 = f(t_1)/\|f(t_1)\|$ if $f(t_1) \neq 0$. Otherwise choose $x_0 \in S_X$ arbitrarily.

Since (A,D) is an Urysohn pair, there exists $g \in S_Y$ such that $g(t_1) = x_0$. Set $h = f + \varepsilon g$. Then $\|h - f\|_\infty \leq \varepsilon$. Therefore, $\|h|_{K_1}\|_\infty + 1 > \|h|_D\|_\infty$. Thus, $\phi_{B_Y}(h) = \|h|_{K_1}\|_\infty + 1$. Moreover, $\|h(t_1)\| = \|f(t_1) + \varepsilon \cdot x_0\| = \|f(t_1)\| + \varepsilon \geq \|f(t_0)\| + \varepsilon/2 > \|f|_{K_1}\|_\infty$.

Let $s_0 \in K_1$ be such that $\|h(s_0)\| = \|h|_{K_1}\|_\infty$. If $s_0 \in \partial D$, then $\|h|_{K_1}\|_\infty = \|h(s_0)\| = \|f(s_0)\| \leq \|f|_{K_1}\|_\infty < \|h(t_1)\| \leq \|h|_{K_1}\|_\infty$, a contradiction. Hence, $s_0 \in K \setminus D$. By Proposition 5.2.3(b), $h \in R(B_Y)$. Since $\varepsilon > 0$ is arbitrary, $f \in \overline{R(B_Y)}$. □

Proposition 5.2.5. *If A is a subspace of $C(K)$ and $D \subseteq K$ is a closed set such that (A,D) is an Uryshon pair, then for any Banach space X, $(A \otimes_\varepsilon X, D)$ is an Uryshon pair as a subspace of $C(K,X)$. In particular, $Y = \{f \in A \otimes_\varepsilon X : f|_D \equiv 0\}$ is a DBR subspace of $A \otimes_\varepsilon X$.*

Proof. Let $t_0 \in K \setminus D$ and $x \in S_X$. Since (A,D) is an Uryshon pair, there exists $f \in S_A$ such that $f|_D \equiv 0$ and $f(t_0) = 1$. Then $g = f \otimes x \in A \otimes_\varepsilon X$ has the required properties. □

If $D \subseteq K$ is closed, $(C(K), D)$ is clearly an Uryshon pair. And by Proposition 5.2.5, so is $(C(K,X), D)$. Therefore, we obtain

Theorem 5.2.6. *Let $Y = \{f \in C(K,X) : f|_D \equiv 0\}$ for some closed set $D \subseteq K$. Then Y is a DBR subspace of $C(K,X)$.*

Corollary 5.2.7. *If X has no non-trivial M-ideals, any M-ideal in $C(K,X)$ is DBR.*

Proof. If X has no non-trivial M-ideals, any M-ideal in $C(K,X)$ is of the form $\{f \in C(K,X) : f|_D \equiv 0\}$ for some closed set $D \subseteq K$ [26, Corollary VI.3.4]. Hence the result. □

Specializing to $X = \mathbb{F}$, we recapture Theorem 4.2.49.

Corollary 5.2.8. *An M-ideal in $C(K)$ is a DBR subspace.*

Corollary 5.2.9. *Let $t_0 \in K$ and $A = \{f \in C(K) : f(t_0) = 0\}$. Any M-ideal in A is DBR.*

Proof. Let $Y \subseteq A$ be an M-ideal. Since A is an M-ideal in $C(K)$, it follows from [26, Proposition I.1.17 (b)] that there is a closed set $D \subseteq K$ such that $Y = \{f \in C(K) : f|_D \equiv 0\}$. Clearly, $t_0 \in D$ and $Y = \{f \in A : f|_D \equiv 0\}$. Thus, (A, D) is an Urysohn pair. □

Corollary 5.2.10. *Let L be a locally compact Hausdorff space. Any M-ideal in $C_0(L)$ is DBR.*

Proof. Let K be the one-point compactification of L and let t_0 be the "point at infinity". Then this is a special case of Corollary 5.2.9. □

Example 5.2.11. We now show that if Y is not a $(*)$-subspace then $R(B_Y)$ need not be closed under scalar multiplication.

Let $X = C[0,1]$ and $Y = \{f \in C[0,1] : f|_{[0,\frac{1}{2}]} \equiv 0\}$. Let $f \in C[0,1]$ be defined by

$$f(x) = \begin{cases} 2 - 3x & \text{if} \quad x \in [0, \frac{1}{2}] \\ 1 - x & \text{if} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

Then $2 = \|f|_{[0,\frac{1}{2}]}\|_\infty > \|f|_{(\frac{1}{2},1]}\|_\infty + 1 = \frac{3}{2}$. By Proposition 5.2.3(b), $f \in R(B_Y)$.

On the other hand, $\|\frac{1}{2}f|_{[0,\frac{1}{2}]}\|_\infty = 1 < \|\frac{1}{2}f|_{(\frac{1}{2},1]}\|_\infty + 1 = \frac{5}{4}$. Since $\|\frac{1}{2}f|_{(\frac{1}{2},1]}\|_\infty = \frac{1}{4}$ is attained only at $\frac{1}{2} \notin (\frac{1}{2}, 1]$, again from Proposition 5.2.3(b), it follows that $\frac{1}{2}f \notin R(B_Y)$.

Theorem 5.2.4 can also be used to obtain sufficient conditions for an M-ideal in a Banach space to be a DBR subspace.

Proposition 5.2.12. *Let X be a Banach space and $Y \subseteq X$ an M-ideal. Suppose there exists a w*-closed subset $T \subseteq B_{X^*}$ such that*

(a) T is a norming set for X.

(b) Let $T_1 = \{x^* \in T : x^* \in Y^\perp\}$ and $T_2 = T \setminus T_1$. Suppose $T_2 \subseteq N_Y$.

Then Y is a DBR subspace of X.

Proof. Since T is w*-compact and norming, $X \subseteq C(T)$. Arguing as in the proof of [26, Theorem I.1.18], $Y = \{x \in X : x|_{T_1} \equiv 0\}$. And, (b) implies (X, T_1) is an Urysohn pair. □

Corollary 5.2.13. *Let Y be an M-ideal in a Banach space X. Suppose $T = NA(X) \cap \partial_e B_{X^*}$ is w^*-closed and $T \cap A_Y \subseteq N_Y$. Then Y is a DBR subspace of X.*

Question 5.2.14. *Is there any example of this phenomenon other than those described here?*

Remark 5.2.15. Definition 5.2.2 is apparently stronger than necessary. Instead of the separation *for all* $t_0 \in K \setminus D$, it often suffices to have the separation for sufficiently many $t_0 \in K \setminus D$. The results of Section 5.3 illustrate this phenomenon.

5.2.1 Application to the disc algebra and its generalizations

In this section, we work with complex scalars. Let $\mathbb{A}$ be the disc algebra, *i.e.*, the Banach space of continuous functions on the closed unit disc $\mathbb{D}$ that are analytic on the open unit disc. By the maximum modulus principle $\mathbb{A}$ can be realized as a subspace of $C(\mathbb{T})$.

Let λ be the Lebesgue measure on $\mathbb{T}$. Recall that [26, Example 1.4 (b)] any nontrivial M-ideal in $\mathbb{A}$ is of the form $Y = \{a \in \mathbb{A} : a|_D \equiv 0\}$, where $D \subseteq \mathbb{T}$ is a closed set with $\lambda(D) = 0$. We will need the following result [27, p 81].

Theorem 5.2.16. (Rudin) *Let $D \subseteq \mathbb{T}$ be a closed set with $\lambda(D) = 0$ and $F \in C(D)$. Then there exists $f \in \mathbb{A}$ such that $f|_D \equiv F$ and $|f(z)| \leq \|F\|_\infty$ for all $z \in \mathbb{D}$.*

Theorem 5.2.17. *The M-ideals in the disc algebra are DBR subspaces.*

Proof. If $Y \subseteq \mathbb{A}$ is an M-ideal, then there is a closed set $D \subseteq \mathbb{T}$ of Lebesgue measure zero such that $Y = \{a \in \mathbb{A} : a|_D \equiv 0\}$.

CLAIM : $(\mathbb{A}, D)$ is an Urysohn pair.

Let $t_0 \in \mathbb{T} \setminus D$. Define $g \in C(D \cup \{t_0\})$ by $g|_D \equiv 0$ and $g(t_0) = 1$. Now by Theorem 5.2.16, there exists $f \in \mathbb{A}$ with $f|_{D \cup \{t_0\}} = g$ and $\|f\|_\infty = 1$.

This proves the claim and hence, the theorem. □

We now isolate the crucial property of the disc algebra that makes the above proof work.

Proposition 5.2.18. *If $\mu \in \mathbb{A}^\perp$, then $\mu \ll \lambda$. In particular, every $\mu \in \mathbb{A}^\perp$ is non-atomic.*

Proof. If $\mu \in \mathbb{A}^\perp$, all the negative Fourier coefficients of μ vanish. Therefore, by F. and M. Riesz Theorem [37, Theorem 17.13], $\mu \ll \lambda$. Since λ is non-atomic, so is μ. □

Now we are ready for a generalization of Theorem 5.2.17. We will need the following strengthening of Rudin's theorem obtained by combining [25, Theorem II.12.5] with the remark immediately following its proof.

Theorem 5.2.19. *Let A be a subspace of $C(K)$. Let D be a closed subset of K such that $\mu|_D = 0$ for all $\mu \in A^\perp$. Let $f \in C(D)$ and $p : K \to (0, \infty)$ be continuous such that $|f(t)| \leq p(t)$ for all $t \in D$. Then there is $g \in A$ such that $g|_D \equiv f$ and $|g(t)| \leq p(t)$ for all $t \in K$.*

We will also need some results from subsection 5.3.1 below.

Theorem 5.2.20. *(a) Let K be a compact Hausdorff space, $A \subseteq C(K)$ a subspace such that every $\mu \in A^\perp$ is non-atomic. Then the Choquet boundary $\partial A = K$.*

(b) If $D \subseteq K$ is a closed set such that $|\mu|(D) = 0$ for all $\mu \in A^\perp$, then

$$Y = \{a \in A : a|_D \equiv 0\}$$

is an M-ideal in A.

(c) Y is a DBR subspace of A.

Proof. (a). First note that the hypothesis implies A separates points of K.

If possible, let $t \in K \setminus \partial A$. By Hustad's theorem (see Theorem 5.3.14 below), there exists a boundary measure μ_t such that $\|\delta_t|_A\| = \|\mu_t\|$ and

$$f(t) = \mu_t(f), \qquad f \in A.$$

So $\mu_t - \delta_t \in A^\perp$, but $(\mu_t - \delta_t)(\{t\}) = -1$, since a boundary measure has no point mass outside ∂A (see Theorem 5.3.15 below), contradicting our hypothesis about $A^\perp$.

(b). To show that Y is an M-ideal, we follow the argument in [26, p 4]. If $p \in A^*$, find $\sigma \in M(K)$ such that $\|\sigma\| = \|p\|$ and $\sigma|_A = p$. Define $q = (\chi_D \sigma|_A) \in A^*$, where χ_D denotes the characteristic function of D. Then $P : A^* \to A^*$ defined by $P(p) = q$ is unambiguous, for if $p = \widetilde{\sigma}|_A$ is another representation for p, then $(\widetilde{\sigma} - \sigma) \in A^\perp$ and it follows that $|(\widetilde{\sigma} - \sigma)|(D) = 0$ whence $\chi_D \sigma = \chi_D \widetilde{\sigma}$. We have

$$\|p\| = \|\sigma\| = \|\chi_D \sigma\| + \|\chi_{D^c} \sigma\| \geq \|q\| + \|p - q\| \geq \|p\|$$

and it follows that P is an L-projection with $\ker(P) = Y^\perp$.

(c). By Theorem 5.2.4, it suffices to show that (A, D) is an Urysohn pair. So let $t_0 \in K \setminus D$. Define $f \in C(D \cup \{t_0\})$ by $f(t_0) = 1$ and $f|_D = 0$. All $\mu \in A^\perp$ being non-atomic, $|\mu|(D \cup \{t_0\}) = 0$. Hence by Theorem 5.2.19, there exists $g \in A$ with $g|_{D \cup \{t_0\}} \equiv f$ and $\|g\|_\infty = \|f\|_\infty = 1$. ☐

Remark 5.2.21. (a) The result in (a) above is suggested by [24, Corollary 5.11], where it is assumed that $\mathbf{1} \in A$, a simpler situation.

(b) It seems highly unlikely, given hypothesis (a) alone, that all M-ideals in A are of the form given in (b), unlike the disc algebra case.

Corollary 5.2.22. *Let K be a compact Hausdorff space, $\mu \in S_{M(K)}$ non-atomic. Let $D \subseteq K$ be a closed set such that $|\mu|(D) = 0$. Let $A = \{f \in C(K) : \mu(f) = 0\}$. Then $Y = \{g \in A : g|_D \equiv 0\}$ is an M-ideal as well as a DBR subspace of A.*

5.3 M-ideals in $A_{\mathbb{F}}(Q)$

5.3.1 Preliminaries

Here we briefly recall some notions and results that will be needed in our discussion. [1] and [34] are standard references for the background and unexplained terminology of this section. In this section, Q will always denote a compact convex set in some locally convex topological vector space E and K a compact Hausdorff space. $A_{\mathbb{F}}(Q)$ is the Banach space of scalar-valued affine continuous functions. $A_{\mathbb{R}}(Q)$ will be denoted simply by $A(Q)$. We denote by

(1) $\mathcal{S}(Q)$, the set of all continuous convex functions on Q.

(2) $M^+(K)$, the set of all non-negative regular Borel measures on K.

(3) $M_1^+(K)$, the set of all regular Borel probability measures on K.

Definition 5.3.1. [34]

(a) Let μ be a regular probability measure on Q. A point $x \in E$ (if it exists!) is said to be *represented* by μ if $f(x) = \mu(f)$ for all $f \in E^*$. Also, one says that x is the *resultant* or *barycenter* of μ and writes $x = r(\mu)$.

(b) Let $\mu \in M^+(Q)$ and $S \subseteq Q$ a Borel set. We say that μ is supported by S if $\mu(Q \setminus S) = 0$.

With this terminology, the Krein-Milman Theorem can be reformulated as :
For each $x \in Q$, there exists $\mu \in M_1^+(Q)$ such that $x = r(\mu)$ and μ is supported by $\overline{\partial_e Q}$.

Theorem 5.3.2. (Bauer's Maximum Principle) *Suppose $f : Q \to \mathbb{R}$ is convex and upper semi-continuous (usc). Then f attains its maximum on Q at some point of $\partial_e Q$.*

Definition 5.3.3. If $f : Q \to \mathbb{R}$ is a bounded function, the upper envelope of f is defined as

$$\widehat{f}(x) = \inf\{h(x) : h \in A(Q), f \leq h\}, \quad x \in Q.$$

Here are some very well-known properties of $\widehat{f}$ [34, Chapter 3]:

(a) $\widehat{f}$ is concave, bounded and usc (hence Borel measurable).

(b) $f \leq \widehat{f}$ and if f is concave and usc then $f = \widehat{f}$.

(c) If f and g are bounded, then (i) $\widehat{f+g} \leq \widehat{f} + \widehat{g}$; (ii) $|\widehat{f} - \widehat{g}| \leq \|f - g\|$; (iii) $\widehat{f+g} = \widehat{f} + g$ if $g \in A(Q)$; (iv) if $r > 0$, then $\widehat{rf} = r\widehat{f}$.

(d) If $\{f_\alpha\} \subseteq C_{\mathbb{R}}(Q)$ and $f_\alpha \downarrow f$, then $\widehat{f_\alpha} \downarrow \widehat{f}$ [1, p 23].

Definition 5.3.4. For $\mu, \nu \in M_1^+(Q)$, say that $\mu \sim \nu$ if $\mu(f) = \nu(f)$ for all $f \in A(Q)$.

Proposition 5.3.5. [34, Proposition 3.1] *If $f : Q \to \mathbb{R}$ is continuous, then for each $x \in Q$, $\widehat{f}(x) = \sup\{\mu(f) : \mu \sim \delta_x\}$.*

Proposition 5.3.6. [1, Proposition I.4.1] *For $x \in Q$, the following are equivalent :*

(a) $x \in \partial_e Q$.

(b) *$f(x) = \hat{f}(x)$ for all $f : Q \to \mathbb{R}$ continuous.*

(c) *$f(x) = \hat{f}(x)$ for all $f : Q \to \mathbb{R} \cup \{-\infty\}$ usc.*

Definition 5.3.7. The Choquet ordering on $M^+(Q)$ is defined as : For $\lambda, \mu \in M^+(Q)$, $\lambda \prec \mu$ if $\lambda(f) \leq \mu(f)$ for all $f \in \mathcal{S}(Q)$. By a judicious use of Zorn's lemma it can be shown that :

If $\lambda \in M^+(Q)$, then there exists a maximal measure $\mu \in M^+(Q)$ such that $\lambda \prec \mu$.

Proposition 5.3.8. (Mokobodzki) [1, Proposition I.4.5] *For $\mu \in M^+(Q)$, the following are equivalent :*

(a) *μ is a maximal element in $M^+(Q)$.*

(b) *$\mu(\hat{f}) = \mu(f)$ for all $f \in C_{\mathbb{R}}(Q)$.*

(c) *$\mu(\hat{f}) = \mu(f)$ for all $f \in \mathcal{S}(Q)$.*

(d) *$\mu(\hat{f}) = \mu(f)$ for all usc f.*

Theorem 5.3.9. *If μ is a maximal probability measure on Q, then $\mu(\{x\}) = 0$ for all $x \in Q \setminus \partial_e Q$.*

Proof. If $x \in Q \setminus \partial_e Q$, then from Proposition 5.3.6, it follows that there exists $f \in C_{\mathbb{R}}(Q)$ such that $f(x) < \widehat{f}(x)$. Now $g = \widehat{f} - f \geq 0$ and from Proposition 5.3.8, it follows that $\int_Q g d\mu = 0$. Hence $\mu(\{x\}) = 0$. □

In a finite dimensional vector space, a closed bounded set is a simplex if it is a closed convex hull of affinely independent points. There are many ways of defining a Choquet simplex (or, simply a simplex) in infinite dimensions. We will consider the following characterization theorem as our definition of a Choquet simplex.

Theorem 5.3.10. (Choquet) [34] *A compact convex set Q is a Choquet simplex if and only if every $x \in Q$ is the barycenter of a unique maximal probability measure on Q.*

The following characterization of a simplex follows from [1, Theorem II.3.7 & 3.8].

Theorem 5.3.11. *The following are equivalent :*

(i) *Q is a simplex.*

(ii) *$\mu(f) = \widehat{f}(x)$ for all $x \in Q$, $\mu \in M_1^+(Q)$ such that $r(\mu) = x$ and $f \in \mathcal{S}(Q)$.*

(iii) *$\widehat{f} + \widehat{g} = \widehat{f+g}$ for all $f, g \in \mathcal{S}(Q)$.*

(iv) *For all $f \in \mathcal{S}(Q)$, $\widehat{f}$ is an affine function.*

(v) *For all usc convex $f : Q \to \mathbb{R} \cup \{-\infty\}$, $\widehat{f}$ is an affine function.*

We have the following separation theorem in a simplex.

Theorem 5.3.12. (Edwards) [1, Theorem II.3.10] *If f and $-g$ are usc convex functions on a simplex K with $f \leq g$, then there exists an $h \in A(K)$ such that $f \leq h \leq g$.*

Definition 5.3.13. [24, Definition 7.1] Let A be a subspace of $C(K)$ which separates points of K. Given a nonzero $\nu \in M(K)$, define $\mu = |\nu| \circ e^{-1}$ on B_{A^*} as follows : If $S \subseteq B_{A^*}$ is a Borel set, then $\mu(S) = |\nu|(\{t \in K : e_t \in S\})$.

A nonzero $\nu \in M(K)$ is called a boundary measure if $|\nu| \circ e^{-1}$ is a maximal measure on B_{A^*}. $\nu = 0$ is also considered a boundary measure.

The following is a generalization of Hustad's theorem [34, Theorem 1.2].

Theorem 5.3.14. [24, Theorem 7.3] *Let A be a subspace of $C(K)$ which separates points. To each $\Lambda \in A^*$ there corresponds $\nu \in M(K)$ such that*

(a) *$\|\Lambda\| = \|\nu\|$.*

(b) *$\Lambda(h) = \nu(h)$, for all $h \in A$.*

(c) *ν is a boundary measure.*

Theorem 5.3.15. [24, Lemma 5.5] *Let K be a compact Hausdorff space and A be a subspace of $C(K)$ which separates points. If μ is a boundary measure then $\mu(\{t\}) = 0$ for all $t \in K \setminus \partial A$.*

Proof. If μ is a boundary measure, then $|\mu| \circ e^{-1}$ is a maximal measure on B_{A^*}. If $t \notin \partial A$ then $e_t \in B_{A^*} \setminus \partial_e B_{A^*}$, hence the result follows from Theorem 5.3.9. □

Let $A \subseteq X$, then $co(A)$ will denote the convex hull of A.

Definition 5.3.16. If $G \subseteq Q$, its *complementary set* G' is the union of all faces of Q that are disjoint from G.

In general, G' need not be convex, but if it is convex, it is a face.

Proposition 5.3.17. [1, Proposition II.6.5] *If F is a closed face of Q and F' is its complementary set, then every $x \in Q$ can be written as a convex combination*

$$x = \lambda y + (1-\lambda)z \quad \text{with } y \in F, z \in F', \quad \text{and } \lambda = \widehat{\chi}_F(x).$$

It follows that $\widehat{\chi}_F^{-1}(1) = F$, $\widehat{\chi}_F^{-1}(0) = F'$ and $Q = co(F \cup F')$. In particular, F' is a G_δ set.

Definition 5.3.18. A closed face F of Q is said to be a split face if it's complementary set F' is a face of Q. Equivalently, a closed face F of Q is a split face if there is another face F' of Q (called the complementary face) such that every $x \in Q \setminus (F \cup F')$ can be written uniquely as a convex combination

$$x = \lambda y + (1-\lambda)z \quad \text{with } y \in F, z \in F'.$$

It is well-known that a closed face in a simplex is a split face [1, Theorem II.6.22].

Proposition 5.3.19. [1, Corollary II.6.8] *If F and G are non-empty split-faces of Q, then $F \cap G$ and $co(F \cup G)$ are also split faces of Q.*

Definition 5.3.20. A Borel subset $B \subseteq Q$ is said to be measure convex if $\mu \in M_1^+(Q)$ and $\mu(B) = 1$ imply that $r(\mu) \in B$.

If F is a split face, then both F and F' are measure convex [1, Corollary II.6.11].

We will need the following extension result.

Theorem 5.3.21. [2, Theorem 2.10.5(ii)] *Let G be a split face of Q and $-f, g \in \mathcal{S}(Q)$. If $g \leq 0 < f$ and $a_1 \in A(G)$ with $g|_G \leq a_1 \leq f|_G$, then there exists $a \in A(Q)$ such that $a|_G = a_1$ and $g \leq a \leq f$.*

We will also need the following result

Theorem 5.3.22. (Mokobodzki) [1, Proposition I.5.1] *Let $f : Q \to \mathbb{R} \cup \{-\infty\}$ be an usc function. For every $x \in Q$ we have*

$$f(x) = \inf\{g(x) : g \in \mathcal{S}(Q),\, g > f\}.$$

5.3.2 Main Results

Let Q be a compact convex set in a locally convex topological vector space E. Recall that any M-ideal in $A_{\mathbb{C}}(Q)$ is of the form $Y = \{f \in A_{\mathbb{F}}(Q) : f|_F \equiv 0\}$, where F is a closed split face of Q [26, Example I.1.4 (c)].

Lemma 5.3.23. *Let f be an usc convex function on a Choquet simplex Q. If $t_\infty \in Q$ and λ is the unique maximal probability measure representing t_∞, then*

$$\widehat{f}(t_\infty) = \lambda(f).$$

Proof. By Theorem 5.3.22, there exists $\{f_\alpha\} \subseteq \mathcal{S}(Q)$ such that $f_\alpha \downarrow f$ and hence $\widehat{f_\alpha} \downarrow \widehat{f}$. Since Q is a simplex, $\widehat{f_\alpha}$ and $\widehat{f}$ are affine functions (Theorem 5.3.11). Hence combining Theorem 5.3.11 and [1, (2.3), p 10], we get

$$\widehat{f}(t_\infty) = \lim_\alpha \widehat{f_\alpha}(t_\infty) = \lim_\alpha \lambda(f_\alpha) = \lambda(f).$$

□

We will need the following basic lemma repeatedly.

Lemma 5.3.24. (a) *Let Q be a compact convex set such that each point of $\partial_e Q$ is split. If F is a closed split face of Q, F' its complementary face and $t_1, t_2, \ldots, t_n \in \partial_e Q \setminus \partial_e F$, then there exists $b \in A(Q)$ such that $0 \le b \le 1$, $b|_F \equiv 0$, $b(t_j) = 1$, $1 \le j \le n$.*

(b) *If Q is a Choquet simplex and $t_\infty^1, t_\infty^2, \ldots, t_\infty^m \in F'$ with $\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}}(t_\infty^i) < 1/2$, $1 \le i \le m$, then the above b can be chosen to have the additional property that $b(t_\infty^i) < 1/2$, $1 \le i \le m$.*

Proof. (a). By Proposition 5.3.19, $G = \mathrm{co}(F \cup \{t_1, t_2, \ldots, t_n\})$ is a closed split face of Q and clearly there exists $a_1 \in A(G)$ such that $0 \le a_1 \le 1$, $a_1|_F \equiv 0$, $a_1(t_i) = 1$, $1 \le i \le n$. Now apply Theorem 5.3.21 with $g \equiv 0$, $f \equiv 1$ to extend a_1 to $a \in A(Q)$ such that $a|_G = a_1$ and $0 \le a \le 1$.

(b). Since $\chi_{\{t_1,t_2,\ldots,t_n\}}$ is usc convex and Q is a simplex, by Theorem 5.3.11, $\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}}$ is an usc affine function. From the definition, it is easily checked that

$$\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}} = \widehat{\chi}_{co\{t_1,t_2,\ldots,t_n\}}$$

and $\mathrm{co}\{t_1, t_2, \ldots, t_n\}$ being a closed split face of Q, it follows by Proposition 5.3.17 that

$$\begin{aligned}
\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}}(t_i) &= 1, \qquad 1 \le i \le n \\
\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}}(x) &= 0 \quad \text{if } x \in \partial_e Q \setminus \{t_1, t_2, \ldots, t_n\} \text{ and} \\
\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}}(y) &< 1 \quad \text{if } y \in Q \setminus co\{t_1, t_2, \ldots, t_n\}.
\end{aligned}$$

Since $\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}}(t^j_\infty) < 1/2$ for $1 \leq j \leq m$, there exists $h_j \in A(Q)$ such that $h_j > \chi_{\{t_1,t_2,\ldots,t_n\}}$ and $h_j(t^j_\infty) < 1/2$. Now $h_j - \chi_{\{t_1,t_2,\ldots,t_n\}}$ is a strictly positive lsc function on Q, hence there exists $\alpha > 0$ such that $h_j - \chi_{\{t_1,t_2,\ldots,t_n\}} \geq \alpha$ on Q. Thus, $h_j \geq \left(\alpha + \chi_{\{t_1,t_2,\ldots,t_n\}}\right)\widehat{} = \alpha + \widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}} > \widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}}$ on Q. Note that the $a \in A(Q)$ constructed in (a) satisfies $\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}} \leq a$. By Theorem 5.3.12, we can find $b \in A(Q)$ such that

$$\widehat{\chi}_{\{t_1,t_2,\ldots,t_n\}} \leq b \leq \min\{h_1, h_2, \ldots, h_m, a\}$$

and this b will have all the required properties. □

Similar to Proposition 5.2.3, we have

Lemma 5.3.25. *Let Q be a compact convex set such that each point of $\partial_e Q$ is split and F is a closed split face of Q. Let $Y = \{f \in A_{\mathbb{F}}(Q) : f|_F \equiv 0\}$. Then*

(a) $\phi_{B_Y}(f) = \max\{\|f|_{\partial_e F}\|_\infty, \|f|_{\partial_e Q \setminus \partial_e F}\|_\infty + 1\}$.

(b) *$f \in R(B_Y)$ if and only if $\phi_{B_Y}(f) = \|f|_{\partial_e F}\|_\infty$ or $\|f|_{\partial_e Q \setminus \partial_e F}\|_\infty = |f(t_0)|$ for some $t_0 \in \partial_e Q \setminus \partial_e F$.*

Proof. (a). By Theorem 5.3.2, $\partial_e Q$ is a boundary for $A_{\mathbb{F}}(Q)$. And $\partial_e F = \partial_e Q \cap F$.

Note that if $t_0 \in \partial_e F$, $\|\delta_{t_0}|_Y\| = 0$. And if $t_0 \in \partial_e Q \setminus \partial_e F$, by Lemma 5.3.24, there exists $g \in S_Y$ such that $g(t_0) = 1$. Thus, $\|\delta_{t_0}|_Y\| = 1$. Now the rest of proof is similar to Proposition 5.2.3. □

Theorem 5.3.26. *Suppose Q be a compact convex set such that all points of $\partial_e Q$ are split and $\partial_e Q$ is closed. Then any M-ideal is a DBR subspace of $A_{\mathbb{F}}(Q)$.*

Proof. If $Y \subseteq A_{\mathbb{F}}(Q)$ is an M-ideal, then there is a closed split face F of Q such that $Y = \{f \in A_{\mathbb{F}}(Q) : f|_F \equiv 0\}$.

Since $\partial_e Q$ is closed, $A_{\mathbb{F}}(Q) \subseteq C(\partial_e Q)$. It follows from Lemma 5.3.24 that $(A_{\mathbb{F}}(Q), \partial_e F)$ is an Urysohn pair. Hence the result. □

Remark 5.3.27. This again recaptures Theorem 4.2.49, since $C(K) \cong A_{\mathbb{F}}(M_1^+(K))$, and $M_1^+(K)$ is a Bauer simplex, that is, a Choquet simplex with $\partial_e Q$ closed.

Our main result in this section is the following :

Theorem 5.3.28. *Let Q be a Choquet simplex such that $\overline{\partial_e Q} \setminus \partial_e Q$ is at most finite. Then any M-ideal is a DBR subspace of $A_{\mathbb{F}}(Q)$.*

Proof. Let $Y = \{f \in A_{\mathbb{F}}(Q) : f|_F \equiv 0\}$, where F is a closed (necessarily split) face of Q. If $\partial_e Q$ is closed, the result follows from Theorem 5.3.26.

For notational simplicity, let us put $\partial_e Q \setminus \partial_e F = \partial_e Q \setminus \partial_e F$.

Case 1 : Suppose $\overline{\partial_e Q} \setminus \partial_e Q$ is a singleton, $\{t_\infty\}$, say.

Let $f \in A_{\mathbb{F}}(Q)$. If $f \notin R(B_Y)$, then by Lemma 5.3.25, $\|f|_{\partial_e Q \setminus \partial_e F}\|_\infty + 1 > \|f|_{\partial_e F}\|_\infty$. And if $t_0 \in \overline{\partial_e Q \setminus \partial_e F}$ is such that $|f(t_0)| = \|f|_{\partial_e Q \setminus \partial_e F}\|_\infty$, then $t_0 \notin \partial_e Q \setminus \partial_e F$.

We will show that $f \in \overline{R(B_Y)}$. Let $\varepsilon > 0$. Here and anywhere else in the sequel, we will always choose $\varepsilon > 0$ such that $\|h|_{\partial_e Q \setminus \partial_e F}\|_\infty + 1 > \|h|_{\partial_e F}\|_\infty$, whenever $\|h - f\|_\infty \leq \varepsilon$.

Let $U = \{t \in \partial_e Q : |f(t) - f(t_0)| < \varepsilon/2\}$. Then there exists $t_1 \in (\partial_e Q \setminus \partial_e F) \cap U$.

Get $a \in A_{\mathbb{F}}(Q)$ such that $a : Q \to [0,1]$, $a|_F \equiv 0$ and $a(t_1) = 1$. Let $\alpha = sgn(f(t_1))$ and $f_1 = f + \varepsilon\alpha^{-1}a$. Then $\|f_1 - f\|_\infty \leq \varepsilon$. And hence, $\|f_1|_{\partial_e Q \setminus \partial_e F}\|_\infty + 1 > \|f_1|_{\partial_e F}\|_\infty$. Moreover, $|f_1(t_1)| = |f(t_1)| + \varepsilon > |f(t_0)| + \varepsilon/2 > \|f|_{\partial_e Q \setminus \partial_e F}\|_\infty$.

Let $s_1 \in \overline{\partial_e Q \setminus \partial_e F}$ be such that $|f_1(s_1)| = \|f_1|_{\partial_e Q \setminus \partial_e F}\|_\infty$. If $s_1 \in \partial_e Q \setminus \partial_e F$, then $f_1 \in R(B_Y)$ and we are done. If $s_1 \in F$, then $\|f_1|_{\partial_e Q \setminus \partial_e F}\|_\infty = |f_1(s_1)| = |f(s_1)| \leq \|f|_{\partial_e Q \setminus \partial_e F}\|_\infty < |f_1(t_1)| \leq \|f_1|_{\partial_e Q \setminus \partial_e F}\|_\infty$, a contradiction.

Thus we may assume $s_1 = t_\infty$. Since $s_1 \notin F$, we may assume $t_\infty \notin F$. We will show that in this case, $f_1 \in \overline{R(B_Y)}$.

Let λ be the (unique) maximal probability measure representing t_∞. Since F is measure convex, and $t_\infty \notin F$, it follows that $0 \leq \lambda(F) < 1$. There are two cases to consider :

CASE 1.1 : $0 < \lambda(F) < 1$.

Let $U = \{t \in \partial_e Q : |f_1(t_\infty) - f_1(t)| < \varepsilon\lambda(F)\}$. Then there exists $t_1 \in (\partial_e Q \setminus \partial_e F) \cap U$. By Lemma 5.3.24, find $a \in A_{\mathbb{F}}(Q)$, $a : Q \to [0,1]$, $a(t_1) = 1$, $a|_F \equiv 0$ and set $f_2 = f_1 + \varepsilon\alpha^{-1}a$, where $\alpha = sgn(f_1(t_1))$.

Since χ_F is an usc function on Q and λ is maximal, it follows from Proposition 5.3.8 that $\lambda(\{x \in Q : \hat{\chi}_F(x) > \chi_F(x)\}) = 0$. Now it follows from Proposition 5.3.17 that $\{x \in Q : \hat{\chi}_F(x) > \chi_F(x)\} = Q \setminus (F \cup F')$. Hence λ is supported by $F \cup F'$. Therefore, $a(t_\infty) = \int_{F \cup F'} a d\lambda = \int_{F'} a d\lambda \leq \lambda(F')$. And therefore

$$\begin{aligned} |f_2(t_\infty)| &= |f_1(t_\infty) + \varepsilon\alpha^{-1}a(t_\infty)| \leq |f_1(t_\infty)| + \varepsilon\lambda(F') \\ &< |f_1(t_1)| + \varepsilon\lambda(F) + \varepsilon\lambda(F') = |f_1(t_1)| + \varepsilon = |f_2(t_1)| \end{aligned}$$

Let $s_2 \in \overline{\partial_e Q \setminus \partial_e F}$ be such that $|f_2(s_2)| = \|f_2|_{\partial_e Q \setminus \partial_e F}\|_\infty$. It follows that $s_2 \neq t_\infty$, and hence, $s_2 \in \partial_e Q$. Now again, if $s_2 \in F$, then $|f_2(s_2)| = |f_1(s_2)| \leq |f_1(t_\infty)| < |f_1(t_1)| + \varepsilon = |f_2(t_1)|$, a contradiction. Thus, $s_2 \in \partial_e Q \setminus \partial_e F$ and $f_2 \in R(B_Y)$.

CASE 1.2 : $\lambda(F) = 0$.

So $\lambda(F') = 1$ and since F' is measure convex, $t_\infty \in F'$.

Let $\lambda = \sum_{n=1}^{\infty} \alpha_n \delta_{x_n} + \lambda'$ be the decomposition of λ into atomic and non-atomic parts. By Theorem 5.3.9, $x_n \in \partial_e Q$.

Let $U = \{t \in \partial_e Q : |f_1(t_\infty) - f_1(t)| < \varepsilon/2\}$. Since U is necessarily infinite, if at most finitely many $x_n \in U$, there is $t_1 \in U$ such that $\lambda(\{t_1\}) = 0 < 1/2$. On the other hand, if infinitely many $x_n \in U$, then since $\sum \alpha_n \leq 1$, $\lambda(\{x_n\}) = \alpha_n \to 0$ as $n \to \infty$. And hence, putting $t_1 = x_n$ for some sufficiently large n, we get $t_1 \in U$ with $\lambda(\{t_1\}) < 1/2$. Thus, in both the cases, we have by Lemma 5.3.23,

$$\widehat{\chi}_{t_1}(t_\infty) = \lambda(\chi_{t_1}) = \lambda(\{t_1\}) < 1/2.$$

By Lemma 5.3.24, find $a \in A(Q)$, $a : Q \to [0,1]$, $a(t_1) = 1$, $a|_F \equiv 0$, $a(t_\infty) < 1/2$. Define $f_2 = f_1 + \varepsilon\alpha^{-1}a$ as before, where $\alpha = sgn(f_1(t_1))$. Then

$$|f_2(t_\infty)| \leq |f_1(t_\infty)| + \varepsilon a(t_\infty) < |f_1(t_1)| + \varepsilon/2 + \varepsilon/2 = |f_2(t_1)|.$$

Hence again as in Case 1.1, if $s_2 \in \overline{\partial_e Q \setminus \partial_e F}$ is such that $|f_2(s_2)| = \|f_2|_{\partial_e Q \setminus \partial_e F}\|_\infty$, then $s_2 \in \partial_e Q \setminus \partial_e F$ and $f_2 \in R(B_Y)$.

Case 2 : Now, suppose that $\overline{\partial_e Q} \setminus \partial_e Q = \{t_\infty^1, t_\infty^2\}$.

If both f_1, f_2 constructed in Case 1 (for $t_\infty = t_\infty^1$) are not in $R(B_Y)$, then $s_1, s_2 \in \overline{\partial_e Q} \setminus \partial_e Q$, and $s_1 \neq s_2$. Moreover, $s_1, s_2 \notin F$. Thus, we may assume that $t_\infty^1, t_\infty^2 \in \overline{\partial_e Q \setminus \partial_e F} \setminus F$. We will show that in this case, at least one of $f_1, f_2 \in \overline{R(B_Y)}$. For convenience, let f be one of f_1 or f_2. We may assume that $|f(t_\infty^1)| = \|f|_{\partial_e Q \setminus \partial_e F}\|_\infty \geq |f(t_\infty^2)|$ and $1 + \|f|_{\partial_e Q \setminus \partial_e F}\|_\infty > \|f|_{\partial_e F}\|_\infty$.

For $i = 1, 2$, let λ_i be the unique maximal probability measure representing t_∞^i. As before, $0 \leq \lambda_i(F) < 1$. There are three cases to consider :

CASE 2.1 : $0 < \lambda_1(F), \lambda_2(F) < 1$.

If $|f(t_\infty^1)| = |f(t_\infty^2)|$, let $\varepsilon > 0$ be arbitrary. And if $|f(t_\infty^1)| > |f(t_\infty^2)|$, let $0 < \varepsilon' < |f(t_\infty^1)| - |f(t_\infty^2)|$ and $0 < \varepsilon < \varepsilon'/(\lambda_1(F) + \lambda_2(F))$.

As in Case 1.1, we can find $t_1, t_2 \in \partial_e Q \setminus \partial_e F$ such that

$$|f(t_\infty^i) - f(t_i)| < \varepsilon\lambda_i(F), \quad i = 1, 2.$$

If $|f(t_\infty^1)| = |f(t_\infty^2)|$, interchanging the role of t_∞^1 and t_∞^2 if necessary, we may assume $|f(t_1)| \geq |f(t_2)|$. And if $|f(t_\infty^1)| > |f(t_\infty^2)|$, note that

$$\begin{aligned} |f(t_2)| &< |f(t_\infty^2)| + \varepsilon\lambda_2(F) < |f(t_\infty^1)| - \varepsilon' + \varepsilon\lambda_2(F) \\ &< |f(t_1)| + \varepsilon\lambda_1(F) - \varepsilon' + \varepsilon\lambda_2(F) < |f(t_1)| \end{aligned}$$

By Lemma 5.3.24, find $b \in A(Q)$ such that $0 \leq b \leq 1$, $b(t_1) = b(t_2) = 1$ and $b|_F \equiv 0$. As before, $b(t_\infty^i) \leq \lambda_i(F')$.

Define $h = f + \alpha^{-1}\varepsilon b$, where $\alpha = sgn(f(t_1))$. Then

$$|h(t^i_\infty)| \leq |f(t^i_\infty)| + \varepsilon b(t^i_\infty) < |f(t_i)| + \varepsilon\lambda_i(F) + \varepsilon\lambda_i(F') = |f(t_i)| + \varepsilon$$

Thus, $|h(t^1_\infty)| < |f(t_1)| + \varepsilon = |h(t_1)|$ and $|h(t^2_\infty)| < |f(t_2)| + \varepsilon \leq |f(t_1)| + \varepsilon = |h(t_1)|$.

Again, let $s \in \overline{\partial_e Q \setminus \partial_e F}$ be such that $|h(s)| = \|h|_{\partial_e Q\setminus\partial_e F}\|_\infty$. It follows that $s \neq t^i_\infty$, and hence, $s \in \partial_e Q$. Now again, if $s \in F$, then $|h(s)| = |f(s)| \leq |f(t^1_\infty)| < |f(t_1)| + \varepsilon = |h(t_1)|$, a contradiction. Thus, $s \in \partial_e Q \setminus \partial_e F$ and $h \in R(B_Y)$ as before.

CASE 2.2 : $0 < \lambda_1(F) < 1$ and $\lambda_2(F) = 0$.

This implies $\lambda_2(F') = 1$ and hence, $t^2_\infty \in F'$ as in Case 1.2.

If $|f(t^1_\infty)| = |f(t^2_\infty)|$, let $\varepsilon > 0$ be arbitrary. And if $|f(t^1_\infty)| > |f(t^2_\infty)|$, let $0 < \varepsilon' < |f(t^1_\infty)| - |f(t^2_\infty)|$ and $0 < \varepsilon < \varepsilon'/(\lambda_1(F) + 1/2)$. Let

$$\begin{aligned} U_1 &= \left\{t \in \partial_e Q : |f(t^1_\infty) - f(t)| < \varepsilon\lambda_1(F)\right\} \text{ and} \\ U_2 &= \left\{t \in \partial_e Q : |f(t^2_\infty) - f(t)| < \varepsilon/2\right\} \end{aligned}$$

As before, we can find $t_i \in (\partial_e Q \setminus \partial_e F) \cap U_i$ such that $\lambda_2(\{t_1, t_2\}) < 1/2$.

The function $\chi_{\{t_1,t_2\}} = \max\{\chi_{t_1}, \chi_{t_2}\}$ is an usc convex function and hence by Theorem 5.3.11, $\widehat{\chi}_{\{t_1,t_2\}}$ is an usc affine function on Q and therefore, $\widehat{\chi}_{\{t_1,t_2\}}(t^2_\infty) = \lambda_2(\chi_{\{t_1,t_2\}}) = \lambda_2(\{t_1, t_2\}) < 1/2$.

Note also that $\widehat{\chi}_{\{t_1,t_2\}}(t_1) = \widehat{\chi}_{\{t_1,t_2\}}(t_2) = 1$. By Lemma 5.3.24, we can find $b \in A(Q)$, $0 \leq b \leq 1$, $b|_F \equiv 0$, $b(t_1) = b(t_2) = 1$, $b(t^2_\infty) < 1/2$. As before, $b(t^1_\infty) \leq \lambda_1(F')$.

As before, we may assume (or prove) that $|f(t_1)| \geq |f(t_2)|$.

Define $h(t) = f(t) + \alpha^{-1}\varepsilon b(t)$, (where $\alpha = sgn(f(t_1))$). Then

$$\begin{aligned} |h(t^1_\infty)| &= |f(t^1_\infty) + \alpha^{-1}\varepsilon b(t^1_\infty)| < |f(t_1)| + \varepsilon\lambda_1(F) + \varepsilon\lambda_1(F') \\ &= |f(t_1)| + \varepsilon = |h(t_1)| \text{ and} \\ |h(t^2_\infty)| &= |f(t^2_\infty) + \alpha^{-1}\varepsilon b(t^2_\infty)| \leq |f(t^2_\infty)| + \varepsilon/2 \\ &< |f(t_2)| + \varepsilon \leq |f(t_1)| + \varepsilon = |h(t_1)| \end{aligned}$$

Again, let $s \in \overline{\partial_e Q \setminus \partial_e F}$ be such that $|h(s)| = \|h|_{\partial_e Q\setminus\partial_e F}\|_\infty$. It follows that $s \neq t^i_\infty$, and hence, $s \in \partial_e Q$. Now again, if $s \in F$, then $|h(s)| = |f(s)| \leq |f(t^1_\infty)| < |f(t_1)| + \varepsilon = |h(t_1)|$, a contradiction. Thus, $s \in \partial_e Q \setminus \partial_e F$ and $h \in R(B_Y)$ as before.

CASE 2.3 : Similar arguments work in the case $\lambda_1(F) = 0$ and $0 < \lambda_2(F) < 1$,

CASE 2.4 : $\lambda_1(F') = \lambda_2(F') = 1$ (if and only if $t^1_\infty, t^2_\infty \in F'$).

For $i = 1, 2$, as before, find $t_i \in \{t \in \partial_e Q \setminus \partial_e F : |f(t^i_\infty) - f(t)| < \varepsilon/2\}$ such that $\widehat{\chi}_{\{t_1,t_2\}}(t^i_\infty) = \lambda_i(\{t_1, t_2\}) < 1/2$ and find $b \in A(Q)$ by Lemma 5.3.24 such that $0 \leq b \leq$

1, $b|_F \equiv 0$, $b(t_i) = 1$, $b(t^i_\infty) < 1/2$. Now define h as before and draw the appropriate conclusions. Hence we are done.

Finally, it is clear from the above analysis that at each finite step we can similarly use Lemma 5.3.24 to construct an approximating function that is either in $R(B_Y)$ or attains its maximum modulus over $\partial_e Q \setminus \partial_e F$ at a *new* point of $\overline{\partial_e Q} \setminus \partial_e Q$. If $\overline{\partial_e Q} \setminus \partial_e Q$ is finite, at some stage there will be no more *new* points available and therefore, the approximating function must be in $R(B_Y)$, completing the proof. □

Remark 5.3.29. We don't know whether Theorem 5.3.28 is true when $\overline{\partial_e Q} \setminus \partial_e Q$ is infinite. But there is a possibility that our proof may fail. The proof depends on finding points $t_i \in \partial_e Q \setminus \partial_e F$ 'near' the points $t^i_\infty \in \overline{\partial_e Q} \setminus \partial_e Q$ from which the function b was constructed with the help of Lemma 5.3.24. In the present situation, however, there are an infinite number of such t_i's and it may no longer be true that $\overline{co}\{F \cup \{t_1, t_2, \ldots\}\}$ is a split face as Størmer's axiom no longer holds [1, Theorem II.7.19] and therefore, Lemma 5.3.24 cannot be used.

Here is an easy example which satisfies the assumptions of Theorem 5.3.28.

Example 5.3.30. Let $A = \{f \in C[0,1] : \int_0^1 f d\lambda = f(0)\}$ where λ is Lebesgue measure in $[0,1]$. It is quite easy to see that the Choquet boundary $\partial A = [0,1] \setminus \{0\} = (0,1]$. Hence the state space S_A is a Choquet simplex with $\overline{\partial_e S_A} \setminus \partial_e S_A$ singleton.

An interesting variant of Theorem 5.3.28 is the following :

Theorem 5.3.31. *Suppose Q is a metrizable compact convex set such that all points of $\partial_e Q$ are split and $\overline{\partial_e Q} \setminus \partial_e Q$ is finite. Then any M-ideal is a DBR subspace of $A_{\mathbb{F}}(Q)$.*

Proof. The proof is more-or-less the same as that of Theorem 5.3.28 with some simplifications, sketched below, due to metrizability of Q, thus avoiding the pathologies associated with boundary measures [1, Proposition II.3.17].

If $\partial_e Q$ is closed, this is Theorem 5.3.26. Metrizability is not needed.

Suppose Q is metrizable and $\overline{\partial_e Q} \setminus \partial_e Q$ is finite. Let $Y = \{f \in A_{\mathbb{F}}(Q) : f|_F \equiv 0\}$, where F is a split face of Q. Harking back to the proof of Theorem 5.3.28, we need only consider the function f_1 constructed in Case 1, $|f_1(s_1)| = \|f_1|_{\partial_e Q \setminus \partial_e F}\|_\infty$, $s_1 \in \overline{\partial_e Q} \setminus \partial_e Q$, $s_1 \notin F$.

As before, we have two cases to consider : (1) If λ is a (not necessarily unique) maximal probability measure representing s_1 with $0 < \lambda(F) < 1$, i.e., $s_1 \notin F'$. This is handled as in Case 1.1 (needing only Lemma 5.3.24 (a)) and the subsequent inductive analysis. (2) $\lambda(F') = 1$, i.e., $s_1 \in F'$ for all maximal representing measures λ. We have

$$|f_1(s_1)| = \left| \int_{F'} f_1(t) d\lambda \right| \leq \int_{F'} |f_1(t)|\, d\lambda = \int_{\partial_e Q \setminus \partial_e F} |f_1(t)|\, d\lambda$$

by metrizability. This implies $|f_1(t)| = |f_1(s_1)|$ a.e. $[\lambda]$ on $\partial_e Q \setminus \partial_e F$, hence there is $s \in \partial_e Q \setminus \partial_e F$ with $|f_1(s)| = \|f_1|_{\partial_e Q\setminus\partial_e F}\|_\infty$ and $f_1 \in R(B_Y)$ by Lemma 5.3.25. □

Remark 5.3.32. Examples are furnished by the state spaces of uniform algebras as their extreme points are split. This follows from [20, Theorem 1] and the remarks immediately preceding it.

Although we are unable to prove Theorem 5.3.28 in full generality, it is interesting and relevant to observe that the following is true :

Theorem 5.3.33. *If Q is a metrizable simplex and $Y \subseteq A_{\mathbb{F}}(Q)$ is an M-ideal, then $\overline{span}(R(B_Y)) = A_{\mathbb{F}}(Q)$.*

Proof. Let $Y = \{f \in A_{\mathbb{F}}(Q) : f|_F \equiv 0\}$, where F is a closed (necessarily split) face of Q. Let $\Sigma := \{f \in A(Q) : \|f|_{\partial_e Q\setminus\partial_e F}\|_\infty + 1 > \|f|_{\partial_e F}\|_\infty, f(t_0) = \|f|_{\partial_e Q\setminus\partial_e F}\|_\infty$ for some $t_0 \in \partial_e Q \setminus \partial_e F$ and $0 \leq f \leq 1\}$. By Lemma 5.3.25, it suffices to show that $\overline{span}(\Sigma) = A_{\mathbb{F}}(Q)$.

The proof is simpler for $\mathbb{F} = \mathbb{R}$, so we argue with $\mathbb{F} = \mathbb{C}$.

Let $\Phi \in \Sigma^\perp \subseteq A_{\mathbb{C}}(Q)^*$. By Theorem 5.3.14, there is a complex boundary measure μ such that $\|\Phi\| = \|\mu\|$ and $\Phi = \mu|_{A_{\mathbb{C}}(Q)}$. Write $\mu = \mu_1 + i\mu_2$ and observe that both μ_1 and μ_2 are real boundary measures. If $f \in \Sigma$ then $\mu(f) = \mu_1(f) + i\mu_2(f) = 0$, so that $\mu_1(f) = \mu_2(f) = 0$ for all $f \in \Sigma$. We will prove that $\mu_1 = \mu_2 = 0$ and hence, $\mu = 0$. Let $\mu_1 = \mu_1^+ - \mu_1^-$ be the Hahn decomposition of μ_1. μ_1^+ and μ_1^- are easily verified to be maximal measures, hence μ_1^+, μ_1^- live on $F \cup F'$ and have disjoint supports. Let $A = S_{\mu_1^+} \cap F'$ and $B = S_{\mu_1^-} \cap F'$, where S_ν denotes the support of ν. We claim that

$$\mu_1^+(A) = \mu_1^-(B) = 0. \tag{5.1}$$

First note that by metrizability, both $A, B \subseteq \partial_e Q \cap F' = \partial_e Q \setminus \partial_e F$. Let $\varepsilon > 0$, take an arbitrary compact set $C_1 \subseteq A$ and a compact set $C_2 \subseteq B$ such that $\mu_1^-[B \setminus C_2] < \varepsilon$. Observe that $co(F \cup (\overline{co}(C_2)))$ and $\overline{co}(C_1)$ are disjoint closed split faces of Q. Indeed, by [2, Lemma 3.1.6], both $\overline{co}(C_1)$ and $\overline{co}(C_2)$ are closed faces of Q, hence split faces as Q is a simplex. Consequently, $co(F \cup (\overline{co}(C_2)))$ is also a closed split face (by Corollary 5.3.19). If $x \in \overline{co}(C_1) \cap [co(F \cup (\overline{co}(C_2)))]$, x can be represented by a (maximal) measure on C_1 as also by a (maximal) measure on $\partial_e F \cup C_2$. But the last two sets are disjoint, hence x would have two distinct maximal representing measures contradicting the fact that Q is a simplex.

Now, arguments similar to Lemma 5.3.24 enable us to find $f \in A(Q)$, $0 \leq f \leq 1$, $f = 1$ on $\overline{co}(C_1)$, $f = 0$ on $co(F \cup (\overline{co}(C_2)))$. Clearly $f \in \Sigma$, hence, $\mu_1(f) = 0$. Thus, $\int_A f d\mu_1^+ = \int_B f d\mu_1^+$, i.e.,

$$\mu_1^+(C_1) + \int_{A\setminus C_1} f d\mu_1^+ = \int_{B\setminus C_2} f d\mu_1^-$$

Thus, $\mu_1^+(C_1) < \varepsilon$. This implies $\mu_1^+(A) = 0$ as ε is arbitrary. Similarly, $\mu_1^-(B) = 0$. This proves the claim.

Now, let $A' = S_{\mu_1^+} \cap F$ and $B' = S_{\mu_1^-} \cap F$. Take $\varepsilon > 0$, an arbitrary compact set $C \subseteq A' \cap \partial_e Q$ and a compact set $D \subseteq B' \cap \partial_e Q$ such that $\mu_1^-[(B' \setminus D)] < \varepsilon$. Now, $F_C := \overline{co}(C)$ and $F_D := \overline{co}(D)$ are disjoint split faces. Take any $p \in \partial_e Q \setminus F$ and observe that $co(F_C \cup \{p\})$ is a split face disjoint from F_D. Again, by arguments similar to Lemma 5.3.24, we can find $f \in A(Q)$ such that $0 \leq f \leq 1$, $f = 1$ on $co(F_C \cup \{p\})$, $f = 0$ on F_D. Again, $f \in \Sigma$ and we see that

$$\int_{A'} f d\mu_1^+ = \int_{B'} f d\mu_1^-$$

Thus,

$$\mu_1^+(C) + \int_{A' \setminus C} f d\mu_1^+ = \int_{B' \setminus D} d\mu_1^- < \varepsilon.$$

Conclude from this and (5.1) that $\mu_1^+ = \mu_1^- = 0$. Similarly for μ_2^+ and μ_2^-, hence $\mu_1 = \mu_2 = 0$. □

Question 5.3.34. *Is the above result true for non-metrizable simplexes?*

Remark 5.3.35. A variant of the disc algebra $\mathbb{A}$ considered in [24] is the algebra $\mathbb{A}_1 = \{f \in \mathbb{A} : f(1) = \lambda(f)\}$, where λ is the Lebesgue measure on $\mathbb{T}$. As proved in [24], $\partial \mathbb{A}_1 = \mathbb{T} \setminus \{1\}$, $\mathbb{A}_1^\perp = \{h\lambda : h \in H_0^1(\lambda)\} \oplus \{\alpha(\lambda - \delta_1) : \alpha \in \mathbb{C}\}$ and state space of $\mathbb{A}_1$ is a simplex. Observe that a subspace $Y \subseteq \mathbb{A}_1$ is an M-ideal if and only if it is of the form $Y = \{f \in \mathbb{A}_1 : f|_D = 0\}$ for some closed subset $D \subseteq \mathbb{T}$ with $\lambda(D) = 0$ and $D \cap \{1\} = \emptyset$. Indeed, if Y is of the above form, then arguments similar to Theorem 5.2.20(b) (using the above description of $\mathbb{A}_1^\perp$) shows that Y is an M-ideal in $\mathbb{A}_1$. Conversely, if $Y \subseteq \mathbb{A}_1$ is an M-ideal, arguments of the proof of [26, Example I.1.14 (b)] for the disc algebra produces a closed set $D \subseteq \mathbb{T}$ with $\lambda(D) = 0$ such that $Y = \{f \in \mathbb{A}_1 : f|_D = 0\}$. Note that $1 \notin D$ as $\delta_1 \notin \partial_e B_{\mathbb{A}_1^*}$. Now again, by Theorem 5.2.19, given $t_1 \notin D \cup \{1\}$, there exists $f \in \mathbb{A}_1$ such that $f(t_1) = 1$, $f|_D = 0$ and $\|f\|_\infty = 1$.

By this observation, the fact that 1 is the only point in $\overline{\partial \mathbb{A}_1} \setminus \partial \mathbb{A}_1$ and λ is the only (maximal) representing measure of $\{1\}$, the same method of proof—in fact, simpler—as in Theorem 5.3.28, Case 1.2 shows that all M-ideals in $\mathbb{A}_1$ are DBR subspaces.

The close similarity between the proof of Theorem 5.3.28 and the proof just sketched seem to suggest that one should be able to deduce these results (for algebras) directly from Theorem 5.3.28 since the associated state spaces in both cases are simplexes. However, we don't quite see how to do this, the principal difficulty being that a general uniform algebra A cannot be realized as $A_{\mathbb{C}}(Q)$ on its state space Q and therefore ball remotality (which depends heavily on the norm) doesn't seem to follow by this approach.

CHAPTER 6

Stability results

6.1 Summary of results

In this chapter, we explore the stability of the properties $(*)$, BR and DBR. These properties are better behaved with respect to superspaces than subspaces. We have seen before that any p-summand is BR, but a p-summand is a $(*)$-subspace if and only if $p = 1$ (Theorem 6.2.2).

Coming to sequence spaces, we show that the c_0- or the ℓ_p-sum $(1 < p \leq \infty)$ of Y_α's is a $(*)$-/$(*)$- and DBR/$(*)$- and BR subspace in the corresponding sum of X_α's if and only if each Y_α is such a subspace in X_α. In the process, we answer [10, Question 2.17] in the affirmative.

On the other hand, if at least one Y_α is a $(*)$-/$(*)$- and DBR/$(*)$- and BR subspace of X_α, then the ℓ_1-sum of Y_α's is such a subspace of the corresponding sum of X_α's.

Coming to function spaces, we obtain a formula for $\phi_{B(CK,Y)}$ in $C(K,X)$, characterize $R(B(CK,Y), C(K,X))$ and show that

(a) Y is a $(*)$-subspace of X if and only if $C(K,Y)$ is a $(*)$-subspace of $C(K,X)$.

(b) Y is a BR subspace of X if and only if $C(K,Y)$ is a BR subspace of $C(K,X)$.

(c) Y is a $(*)$- & DBR subspace of X if and only if $C(K,Y)$ is a $(*)$- & DBR subspace of $C(K,X)$.

If Y is a $(*)$-/$(*)$- and DBR subspace of X and (Ω, Σ, μ) is a probability space, then the space $L_1(\mu, Y)$ of Y-valued Bochner integrable functions is such a subspace of $L_1(\mu, X)$.

6.2 Subspaces, etc.

We have noted in Example 2.3.11 and Corollary 2.3.17 that

Proposition 6.2.1. (a) *If $Y \subseteq Z \subseteq X$ and Y is a $(*)$-subspace of X, then Z is a $(*)$-subspace of X and Y is a $(*)$-subspace of Z.*

(b) *If $Y \subseteq Z \subseteq X$ and Y is a $(*)$- and BR subspace of X, then Z is a $(*)$- and BR subspace of X and Y is a $(*)$- and BR subspace of Z.*

(c) *If $Y \subseteq Z \subseteq X$ and Y is a $(*)$- and DBR subspace of X, then Z is a $(*)$- and DBR subspace of X. However, it is not clear if Y must be a $(*)$- and DBR subspace of Z.*

Theorem 6.2.2. *For $1 \leq p \leq \infty$, let Y be a p-summand in X, that is, there is a subspace $Z \subseteq X$ such that $X = Y \oplus_p Z$. Then Y is a $(*)$-subspace of X if and only if $p = 1$.*

Proof. Let $x_0 = y_0 + z_0$ for some $y_0 \in Y$ and $z_0 \in Z$. We have for $1 \leq p < \infty$,

$$\phi^p_{B_Y}(x_0) = \sup_{y \in B_Y} \|x_0 - y\|^p = \sup_{y \in B_Y} [\|y_0 - y\|^p + \|z_0\|^p] = (\|y_0\| + 1)^p + \|z_0\|^p$$

Similarly, for $p = \infty$,

$$\phi_{B_Y}(x_0) = \max\{\|y_0\| + 1, \|z_0\|\}.$$

Now if $p = 1$ then $\phi_{B_Y}(x_0) = \|y_0\| + \|z_0\| + 1 = \|x_0\| + 1$ and hence Y is a $(*)$-subspace of X.

On the other hand, if $p \neq 1$, $y_0 = 0$ and $z_0 \neq 0$, then

$$\phi_{B_Y}(x_0) = \begin{cases} [1 + \|z_0\|^p]^{1/p} & \text{if} \quad 1 < p < \infty \\ \max\{1, \|z_0\|\} & \text{if} \quad p = \infty \end{cases} < \|z_0\| + 1 = \|x_0\| + 1.$$

Hence Y cannot be a $(*)$-subspace of X. □

Remark 6.2.3. (a) Any p-summand in a Banach space is however BR (Lemma 3.2.10).

(b) It follows that a 1-summand in a Banach space is always a $(*)$- and BR subspace.

(c) It also follows that any Banach space embed isometrically as a $(*)$- and BR hyperplane in some superspace Z. Just take $Z = X \oplus_1 \mathbb{F}$. Compare this with Corollary 4.2.16 and Corollary 4.2.44.

(d) If a subspace Y is of finite dimension or co-dimension, more generally if Y is complemented in X, then X can be so renormed that the norm on Y remains unchanged and it becomes a $(*)$- and BR subspace of X. We do not know if this is true for any subspace.

(e) However, if $Y \subseteq X$ and $y_0 \in S_Y$, X can clearly be renormed to make y_0 a strong unitary and hence Y a $(*)$- BR subspace.

Example 6.2.4. It is clear from Proposition 6.2.1 that a $(*)$-subspace is intersection of $(*)$-hyperplanes. The converse is not true. By Theorem 3.3.1, any hyperplane in ℓ_1 is $(*)$- and DBR. However, the intersection of all of them is clearly $\{0\}$.

Similarly, intersection of two $(*)$- and DBR subspaces need not be either a $(*)$- or a DBR subspace. In ℓ_1, for $m \geq 1$, let $Y = \text{span}\{e_i : 1 \leq i \leq m\}$ and $Z = \overline{\text{span}}\{e_i : i > m\}$, then clearly Y and Z are two $(*)$- and BR subspaces in ℓ_1, but $Y \cap Z = \{0\}$.

6.3 Sequence spaces

Notation 5. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces. For $1 \le p \le \infty$, let $\widetilde{X}_p = \bigoplus_{\ell_p} X_\alpha$, and $\widetilde{X}_0 = \bigoplus_{c_0} X_\alpha$. If $\Lambda = \mathbb{N}$ and $X_\alpha = X$ for all α, then

(a) $c_0(X) = \bigoplus_{c_0} X = \{(x_n) : (x_n) \subseteq X, \lim_n \|x_n\| = 0\}$.

(b) $\ell_p(X) = \bigoplus_{\ell_p} X = \{(x_n) : (x_n) \subseteq X, (\|x_n\|) \in \ell_p\}$, $1 \le p \le \infty$.

(c) $c(X) = \{(x_n) : (x_n) \subseteq X, \lim_n x_n$ exists$\}$.

Proposition 6.3.1. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be subspaces. If at least one Y_α is a $(*)$-subspace of X_α, then $\widetilde{Y}_1$ is a $(*)$-subspace in $\widetilde{X}_1$.*

Proof. If Y_α is a $(*)$-subspace in X_α, then for any $x \in \widetilde{X}_1$, $\|x\|_1 + 1 \ge \phi_{B_{\widetilde{Y}_1}}(x) \ge \phi_{B_{Y_\alpha}}(x_\alpha) + \sum_{\beta \ne \alpha} \|x_\beta\| = \|x_\alpha\| + 1 + \sum_{\beta \ne \alpha} \|x_\beta\| = \|x\|_1 + 1$. □

Proposition 6.3.2. *Let X_α, Y_α be as above. If at least one Y_α is a $(*)$- and BR (DBR) subspace of X_α, then $\widetilde{Y}_1$ is a $(*)$- and BR (DBR) subspace of $\widetilde{X}_1$.*

Proof. By Proposition 6.3.1, $\widetilde{Y}_1$ is a $(*)$-subspace in $\widetilde{X}_1$. And it is easy to see that if $x \in \widetilde{X}_1$ such that $x_\alpha \in R(B_{Y_\alpha})$ for some $\alpha \in \Lambda$, then $x \in R(B_{\widetilde{Y}_1})$. The BR part follows.

Now let $x \in \widetilde{X}_1$ and $\varepsilon > 0$. Let $y_\alpha \in R(B_{Y_\alpha})$ such that $\|y_\alpha - x_\alpha\| < \varepsilon$. Defining $z \in \widetilde{X}_1$ by

$$z_\beta = \begin{cases} y_\alpha & \text{if } \beta = \alpha \\ x_\beta & \text{otherwise} \end{cases},$$

it follows $\|x - z\|_\infty = \|x_\alpha - y_\alpha\| < \varepsilon$ and $z \in R(B_{\widetilde{Y}_1})$. Hence the DBR part follows. □

Coming to infinite sums, we can show that

Theorem 6.3.3. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be subspaces such that $Y_\alpha \ne \{0\}$ for infinitely many $\alpha \in \Lambda$. Then $\widetilde{Y}_1$ is a $(*)$- and DBR subspace of $\widetilde{X}_1$.*

Proof. The proof is similar to that of Theorem 3.3.1.

Let $A = \{x = (x_\gamma) \in \widetilde{X}_1 : x_\gamma = 0$ for all but finitely many $\gamma \in \Gamma\}$. If $x \in A$, there exists a finite set $\Lambda_1 \subseteq \Lambda$ such that $x_\alpha = 0$ for $\alpha \notin \Lambda_1$. Since infinitely many $Y_\alpha \ne \{0\}$, we can find $\beta \notin \Lambda_1$ and $y_\beta \in S_{Y_\beta}$. Define $y \in B_{\widetilde{Y}_1}$ by putting $y_\alpha = 0$ if $\alpha \ne \beta$. It follows that $\phi_{B_{\widetilde{Y}_1}}(x) \ge \|x + y\|_1 = \|x\|_1 + 1$ and hence $\phi_{B_{\widetilde{Y}_1}}(x) = \|x\|_1 + 1$ and $x \in R(B_Y)$. Since A is dense in $\widetilde{X}_1$, $\widetilde{Y}_1$ is a $(*)$- and DBR subspace of $\widetilde{X}_1$. □

Theorem 6.3.4. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be subspaces. Then the following are equivalent :*

(a) *Each Y_α is a $(*)$-subspace in X_α.*

(b) $\widetilde{Y}_0$ *is a* $(*)$*-subspace in* $\widetilde{X}_0$.

(c) $\widetilde{Y}_\infty$ *is a* $(*)$*-subspace in* $\widetilde{X}_\infty$.

(d) $\widetilde{Y}_0$ *is a* $(*)$*-subspace in* $\widetilde{X}_\infty$.

Proof. $(a) \Rightarrow (d)$. Let $x \in \widetilde{X}_\infty$ and $\varepsilon > 0$. There exists $\alpha \in \Lambda$ such that $\|x_\alpha\| > \|x\|_\infty - \varepsilon/2$. Get $y_\alpha \in B_{Y_\alpha}$ such that $\|x_\alpha - y_\alpha\| > \|x_\alpha\| + 1 - \varepsilon/2$. Define $y \in B_{\widetilde{Y}_0}$ by

$$y_\beta = \begin{cases} y_\alpha & \text{if } \beta = \alpha \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

Then

$$\|x - y\|_\infty \geq \|x_\alpha - y_\alpha\| > \|x_\alpha\| + 1 - \varepsilon/2 > \|x\|_\infty + 1 - \varepsilon.$$

Hence, $\widetilde{Y}_0$ is a $(*)$-subspace of $\widetilde{X}_\infty$.

$(d) \Rightarrow (b)$ and (c). Since $\widetilde{Y}_0 \subseteq \widetilde{X}_0 \subseteq \widetilde{X}_\infty$ and $\widetilde{Y}_0 \subseteq \widetilde{Y}_\infty \subseteq \widetilde{X}_\infty$, the result follows from Example 2.3.11(b).

(b) or $(c) \Rightarrow (a)$. Let $\widetilde{X}$ and $\widetilde{Y}$ stand for either $\widetilde{X}_0$ and $\widetilde{Y}_0$ or $\widetilde{X}_\infty$ and $\widetilde{Y}_\infty$ as the case may be. Suppose $\widetilde{Y}$ is a $(*)$-subspace of $\widetilde{X}$.

Fix $\alpha \in \Lambda$. Let $x_\alpha \in X_\alpha$. Define $x \in \widetilde{X}$ by

$$x_\beta = \begin{cases} x_\alpha & \text{if } \beta = \alpha \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

Then

$$\|x_\alpha\| + 1 = \phi_{B_Y}(x) \leq \max\{\phi_{B_{Y_\alpha}}(x_\alpha), 1\} \leq \|x_\alpha\| + 1.$$

It follows that $\phi_{B_{Y_\alpha}}(x_\alpha) = \|x_\alpha\| + 1$. Hence Y_α is a $(*)$-subspace of X_α. □

Lemma 6.3.5. *Let* $\{X_\alpha : \alpha \in \Lambda\}$ *be a family of Banach spaces and* $Y_\alpha \subseteq X_\alpha$ *be* $(*)$*-subspaces. Then* $x \in R(B_{\widetilde{Y}_0}, \widetilde{X}_\infty)$ *if and only if there exists* $\alpha \in \Lambda$ *such that* $\|x_\alpha\| = \|x\|_\infty$ *and* $x_\alpha \in R(B_{Y_\alpha})$.

Proof. If $\alpha \in \Lambda$ is such that $\|x_\alpha\| = \|x\|_\infty$ and $x_\alpha \in R(B_{Y_\alpha})$, get $y_\alpha \in B_{Y_\alpha}$ such that $\|x_\alpha - y_\alpha\| = \|x_\alpha\| + 1$. Define $y \in B_{\widetilde{Y}_0}$ as in (6.1). Then

$$\|x\|_\infty + 1 \geq \|x - y\|_\infty \geq \|x_\alpha - y_\alpha\| = \|x_\alpha\| + 1 = \|x\|_\infty + 1.$$

Conversely, if $x \in R(B_{\widetilde{Y}_0})$, find $y \in B_{\widetilde{Y}_0}$ such that $\|x - y\|_\infty = \|x\|_\infty + 1$. Since $y \in B_{\widetilde{Y}_0}$, there is a finite set Λ_1 such that $\|y_\alpha\| < 1/2$ for all $\alpha \notin \Lambda_1$. It follows that for all $\alpha \notin \Lambda_1$,

$$\|x_\alpha - y_\alpha\| \leq \|x_\alpha\| + \|y_\alpha\| \leq \|x\|_\infty + \frac{1}{2} < \|x\|_\infty + 1 = \|x - y\|_\infty.$$

Therefore, $\|x - y\|_\infty = \sup_{\alpha \in \Lambda_1} \|x_\alpha - y_\alpha\|$ and the supremum is attained at some $\alpha \in \Lambda_1$. It clearly follows that $\|x_\alpha\| = \|x\|_\infty$ and $x_\alpha \in R(B_{Y_\alpha})$. □

Theorem 6.3.6. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be subspaces. Then the following are equivalent :*

(a) *Each Y_α is $(*)$- and DBR in X_α.*

(b) *$\widetilde{Y}_0$ is $(*)$- and DBR in $\widetilde{X}_0$.*

(c) *$\widetilde{Y}_\infty$ is $(*)$- and DBR in $\widetilde{X}_\infty$.*

(d) *$\widetilde{Y}_0$ is $(*)$- and DBR in $\widetilde{X}_\infty$.*

Proof. $(a) \Rightarrow (d)$. $\widetilde{Y}_0$ is a $(*)$-subspace of $\widetilde{X}_0$ follows from Theorem 6.3.4.

To prove $\widetilde{Y}_0$ is DBR in $\widetilde{X}_\infty$, let $x \in \widetilde{X}_\infty$ and $\varepsilon > 0$.

Let $\alpha \in \Lambda$ be such that $\|x_\alpha\| > \|x\|_\infty - \varepsilon/3$. Since Y_α is DBR in X_α, there exists $u_\alpha \in R(B_{Y_\alpha})$ such that $\|x_\alpha - u_\alpha\| < \varepsilon/3$. Define $z \in \widetilde{X}_\infty$ by

$$z_\beta = \begin{cases} \frac{\|x\|_\infty}{\|u_\alpha\|} u_\alpha & \text{if } \beta = \alpha \\ x_\beta & \text{if otherwise} \end{cases}$$

Clearly, $\|z_\alpha\| = \|x\|_\infty = \|z\|_\infty$ and by Proposition 2.3.13, $z_\alpha \in R(B_{Y_\alpha})$. Hence, by Lemma 6.3.5, $z \in R(B_{\widetilde{Y}_0})$. Moreover,

$$\begin{aligned} \|x - z\|_\infty &= \|x_\alpha - z_\alpha\| = \left\| x_\alpha - \frac{\|x\|_\infty}{\|u_\alpha\|} u_\alpha \right\| \\ &\leq \left\| x_\alpha - \frac{\|x_\alpha\|}{\|u_\alpha\|} u_\alpha \right\| + (\|x\|_\infty - \|x_\alpha\|) \\ &\leq \|x_\alpha - u_\alpha\| + \big| \|x_\alpha\| - \|u_\alpha\| \big| + (\|x\|_\infty - \|x_\alpha\|) < \varepsilon \end{aligned}$$

$(d) \Rightarrow (b)$ and (c). Since $\widetilde{Y}_0 \subseteq \widetilde{X}_0 \subseteq \widetilde{X}_\infty$ and $\widetilde{Y}_0 \subseteq \widetilde{Y}_\infty \subseteq \widetilde{X}_\infty$, the result follows from Corollary 2.3.17.

(b) or $(c) \Rightarrow (a)$. Let $\widetilde{X}$ and $\widetilde{Y}$ stand for either $\widetilde{X}_0$ and $\widetilde{Y}_0$ or $\widetilde{X}_\infty$ and $\widetilde{Y}_\infty$ as the case may be. Suppose $\widetilde{Y}$ is $(*)$- and DBR in $\widetilde{X}$.

Fix $\alpha \in \Lambda$. That Y_α is a $(*)$-subspace of X_α follows from Theorem 6.3.4.

If for some $\alpha \in \Lambda$, Y_α is not DBR in X_α, then there exist $x_\alpha \in X_\alpha$ and $\varepsilon > 0$ such that $B(x_\alpha, \varepsilon) \cap R(B_{Y_\alpha}) = \emptyset$.

Define $x \in \widetilde{X}$ as in (6.2). Let $\delta < \min\{\varepsilon, \|x_\alpha\|/3\}$.

Claim : $B(x, \delta) \cap R(B_Y) = \emptyset$.

If $z \in B(x, \delta) \cap R(B_{\widetilde{Y}})$, then $\|z_\alpha\| > 2\|x_\alpha\|/3$ and $\|z_\beta\| < \|x_\alpha\|/3$ if $\beta \neq \alpha$. Hence, $\|z\|_\infty = \|z_\alpha\| > 2 \sup_{\beta \neq \alpha} \|z_\beta\|$.

Let $y \in B_{\widetilde{Y}}$ be such that $\|z + y\|_\infty = \|z\|_\infty + 1 = \|z_\alpha\| + 1$.

If $\|z + y\|_\infty = \|z_\alpha + y_\alpha\|$, then $z_\alpha \in B(x_\alpha, \varepsilon) \cap R(B_{Y_\alpha})$, a contradiction. So, $\|z + y\|_\infty > \|z_\alpha + y_\alpha\|$.

Choose $\eta > 0$ such that $\eta < \|z\|_\infty/2$ and $\|z+y\|_\infty > \|z_\alpha + y_\alpha\| + \eta$. Find $\beta \in \Lambda$ such that $\|z_\beta + y_\beta\| > \|z+y\|_\infty - \eta$. It follows that $\beta \neq \alpha$ and $\|z_\beta\| + \|y_\beta\| \geq \|z_\beta + y_\beta\| > \|z\|_\infty + 1 - \eta > \|z_\alpha\|/2 + 1$. Since $\|y_\beta\| \leq 1$, $\|z_\beta\| \geq \|z_\alpha\|/2$, again a contradiction. □

Remark 6.3.7. It follows that $\widetilde{X}_0$ is DBR in $\widetilde{X}_\infty$, answering [10, Question 2.17].

If the subspaces are not assumed to be $(*)$-, we have

Theorem 6.3.8. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be DBR subspaces. Then*

(a) *$\widetilde{Y}_0$ is DBR in $\widetilde{X}_0$.*

(b) *$\widetilde{Y}_\infty$ is DBR in $\widetilde{X}_\infty$.*

Proof. (a). Let $x \in \widetilde{X}_0$. Then $\|x\|_\infty = \|x_\alpha\|$ for some $\alpha \in \Lambda$. Let $0 < \varepsilon < \|x_\alpha\|$. There exists a finite set $\Lambda_1 \subseteq \Lambda$, such that if $\beta \notin \Lambda_1$ then $\|x_\beta\| < \varepsilon$. For $\beta \in \Lambda_1$, since Y_β is DBR in X_β, find $z_\beta \in R(B_{Y_\beta})$ such that $\|x_\beta - z_\beta\| < \varepsilon$ and get $y_\beta \in B_{Y_\beta}$ such that $\|z_\beta + y_\beta\| = \phi_{B_{Y_\beta}}(z_\beta)$.

Define $z \in \widetilde{X}_0$ by putting z_β as above for $\beta \in \Lambda_1$ and $z_\beta = 0$ if $\beta \notin \Lambda_1$. It is easy to see that

$$\phi_{B_{\widetilde{Y}_0}}(z) \leq \max\{\max\{\phi_{B_{Y_\beta}}(z_\beta) : \beta \in \Lambda_1\}, 1\}$$

Fix $\alpha \notin \Lambda_1$ and $y_\alpha \in S_{Y_\alpha}$. Define $y \in B_{\widetilde{Y}_0}$ by

$$y_\beta = \begin{cases} y_\beta & \text{if } \beta \in \Lambda_1, \quad y_\beta \text{ as above} \\ y_\alpha & \text{if } \beta = \alpha \\ 0 & \text{if otherwise} \end{cases}$$

Then $\|z+y\|_\infty = \max\{\max\{\phi_{B_{Y_\beta}}(z_\beta) : \beta \in \Lambda_1\}, 1\} = \phi_{B_{\widetilde{Y}_0}}(z)$. And hence, $z \in R(B_{\widetilde{Y}_0})$ and $\|x - z\| < \varepsilon$.

(b). Let $x \in \widetilde{X}_\infty$. Observe that $\phi_{B_{\widetilde{Y}_\infty}}(x) = \sup_\beta \phi_{B_{Y_\beta}}(x_\beta)$. For $\beta \in \Lambda$, since Y_β is DBR in X_β, find $z_\beta \in R(B_{Y_\beta})$ such that $\|x_\beta - z_\beta\| < \varepsilon$ and get $y_\beta \in B_{Y_\beta}$ such that $\|z_\beta + y_\beta\| = \phi_{B_{Y_\beta}}(z_\beta)$.

Now if $z \in \widetilde{X}_\infty$ and $y \in \widetilde{Y}_\infty$ are defined with these values, it follows that $\|z+y\|_\infty = \phi_{B_{\widetilde{Y}_\infty}}(z) = \sup_\beta \phi_{B_{Y_\beta}}(z_\beta)$ and $\|x - z\| < \varepsilon$. □

Theorem 6.3.9. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be $(*)$- and BR subspaces. Then*

(a) *$\widetilde{Y}_0$ is $(*)$- and BR in $\widetilde{X}_0$.*

(b) *$\widetilde{Y}_\infty$ is $(*)$- and BR in $\widetilde{X}_\infty$.*

Proof. (a) follows from Lemma 6.3.5. (b) follows from the proof of Theorem 6.3.8(b), with $\varepsilon = 0$. □

Remark 6.3.10. Since c_0 in ℓ_∞ is DBR but not BR, $\widetilde{Y}_0$ need not be BR in $\widetilde{X}_\infty$.

Theorem 6.3.11. *If Y is $(*)$-subspace of X, then $c(Y)$ is $(*)$-subspace of $c(X)$.*

Proof. Since $c_0(Y) \subseteq c(Y) \subseteq c(X) \subseteq \ell_\infty(X)$, the result follows from Proposition 6.2.1. □

Corollary 6.3.12. *If X is any Banach space then $c(X)$ is a $(*)$- and DBR subspace of $\ell_\infty(X)$.*

Theorem 6.3.13. *If Y is $(*)$- and DBR (BR) in X, then $c(Y)$ is $(*)$- and DBR (BR) in $c(X)$.*

Proof. Since $c_0(Y) \subseteq c(Y) \subseteq c(X) \subseteq \ell_\infty(X)$, the DBR part follows from Theorem 6.3.6.

To prove the BR part, let $x \in c(X)$. Let $x_0 = \lim_n x_n$. If $\|x\|_\infty$ is attained at some $n \in \mathbb{N}$, then the c_0 argument works. Otherwise, $\|x\|_\infty = \lim_n \|x_n\| = \|x_0\|$. Let $y_0 \in F_{B_Y}(x_0)$. Define $y \in B_{c(Y)}$ as the constant sequence $y_n = y_0$. Then

$$\|x - y\|_\infty \geq \lim_n \|x_n - y_n\| = \|x_0 - y_0\| = \|x_0\| + 1 = \|x\|_\infty + 1$$

That is, $y \in F_{B_{c(Y)}}(x)$. □

Remark 6.3.14. It follows that for any Banach space X, $c_0(X)$ is $(*)$- and DBR in both $c(X)$ and $\ell_\infty(X)$. And $c(X)$ is $(*)$- and DBR in $\ell_\infty(X)$.

We now come to $1 < p < \infty$.

Theorem 6.3.15. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be $(*)$-subspaces. Let $1 < p < \infty$. Then $\widetilde{Y}_p$ is a $(*)$-subspace of $\widetilde{X}_p$.*

Proof. Let $x = (x_\alpha) \in S_{\widetilde{X}_p}$. We may write $x_\alpha = u_\alpha z_\alpha$, where $u_\alpha = \|x_\alpha\| \geq 0$, $z_\alpha \in S_{X_\alpha}$ and $(u_\alpha) \in S_{\ell_p}$.

Put $v_\alpha = u_\alpha^{p-1}$. Then $(v_\alpha) \in S_{\ell_q}$ and $\sum_\alpha u_\alpha v_\alpha = 1$.

Let $\varepsilon > 0$. Find $x_\alpha^* \in A_{Y_\alpha}$ such that $x_\alpha^*(z_\alpha) > 1 - \varepsilon$.

Define $x^* = (v_\alpha x_\alpha^*)$, then

$$x^*(x) = \sum_\alpha u_\alpha v_\alpha x_\alpha^*(z_\alpha) > (1 - \varepsilon) \sum_\alpha u_\alpha v_\alpha = 1 - \varepsilon,$$

as $u_\alpha, v_\alpha \geq 0$.

CLAIM : $x^* \in A_{\widetilde{Y}_p}$.

Let $\eta > 0$. Let $y_\alpha \in S_{Y_\alpha}$ such that $x_\alpha^*(y_\alpha) > 1 - \eta$. Define $y = (u_\alpha y_\alpha)$. Then $y \in S_{\widetilde{Y}_p}$. Now

$$x^*(y) > (1 - \eta) \sum_\alpha u_\alpha v_\alpha = 1 - \eta,$$

as before. □

Lemma 6.3.16. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be $(*)$-subspaces. Let $1 < p < \infty$. Then $x = (x_\alpha) \in R(B_{\widetilde{Y}_p})$ if and only if for every $\alpha \in \Lambda$, $x_\alpha \in R(B_{Y_\alpha})$.*

Proof. Let $x \in R(B_{\widetilde{Y}_p})$ such that $\|x\|_p = 1$. By Proposition 2.3.13, there exists $x^* \in S_{\widetilde{X}_p^*}$ and $y \in S_{\widetilde{Y}_p}$ such that

$$x^*(x) = 1 = x^*(y).$$

Let z stands for either x or y. Now,

$$1 = |\sum_\alpha x^*_\alpha(z_\alpha)| \leq \sum_\alpha |x^*_\alpha(z_\alpha)| \leq \sum_\alpha \|x^*_\alpha\|\|z_\alpha\| \leq (\sum_\alpha \|x^*_\alpha\|^q)^{1/q}(\sum_\alpha \|z_\alpha\|^p)^{1/p} = 1$$

It follows that $|x^*_\alpha(z_\alpha)| = \|x^*_\alpha\|\|z_\alpha\|$, and hence,

$$|\frac{x^*_\alpha}{\|x^*_\alpha\|}(x_\alpha)| = \|x_\alpha\| \quad \text{and} \quad |\frac{x^*_\alpha}{\|x^*_\alpha\|}(\frac{y_\alpha}{\|y_\alpha\|})| = 1.$$

It follows that $\|x_\alpha - \frac{y_\alpha}{\|y_\alpha\|}\| = \|x_\alpha\| + 1$. So that $x_\alpha \in R(B_{Y_\alpha})$.

Conversely, let $x \in \widetilde{X}_p$ such that for every $\alpha \in \Lambda$, $x_\alpha \in R(B_{Y_\alpha})$. We may assume $\|x\|_p = 1$. Write $x_\alpha = u_\alpha z_\alpha$, where $u_\alpha = \|x_\alpha\| \geq 0$, $z_\alpha \in R(B_{Y_\alpha}) \cap S_{X_\alpha}$ and $(u_\alpha) \in S_{\ell_p}$.

By Proposition 2.3.13, there exists $x^*_\alpha \in S_{X^*_\alpha}$ and $y_\alpha \in S_{Y_\alpha}$ such that

$$x^*_\alpha(z_\alpha) = 1 = x^*_\alpha(y_\alpha).$$

Put $v_\alpha = u_\alpha^{p-1}$. Define $x^* = (v_\alpha x^*_\alpha)$ and $y = (u_\alpha y_\alpha)$. Then $x^* \in S_{\widetilde{X}_p^*}$, $y \in S_{\widetilde{Y}_p}$ and

$$x^*(x) = 1 = x^*(y).$$

By Proposition 2.3.13 again, $x \in R(B_{\widetilde{Y}_p})$. □

Theorem 6.3.17. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $Y_\alpha \subseteq X_\alpha$ be subspaces. Let $1 < p < \infty$. Then $\widetilde{Y}_p$ is a $(*)$- and DBR/BR subspace of $\widetilde{X}_p$ if and only if each Y_α is a $(*)$- and DBR/BR subspaces of X_α.*

Proof. If each Y_α are $(*)$- and DBR subspaces of X_α, then by Theorem 6.3.15, $\widetilde{Y}_p$ is a $(*)$-subspace of $\widetilde{X}_p$.

Let $x = (x_\alpha) \in \widetilde{X}_p$. We may write $x_\alpha = u_\alpha z_\alpha$, where $u_\alpha = \|x_\alpha\|$, $z_\alpha \in S_{X_\alpha}$ and $(u_\alpha) \in \ell_p$.

Let $\varepsilon > 0$. Since each $R(B_{Y_\alpha})$ is dense in X_α, there exists $w_\alpha \in R(B_{Y_\alpha})$ such that $\|w_\alpha\| = 1$ and $\|z_\alpha - w_\alpha\| < \varepsilon$. Define $w = (u_\alpha w_\alpha)$. By Lemma 6.3.16, $w \in R(B_{Y_p})$ and $\|x - w\|_p < \varepsilon$.

Conversely, if $\widetilde{Y}_p$ is a $(*)$- and DBR subspace of $\widetilde{X}_p$, then for each $\alpha \in \Lambda$, the projection of $R(B_{\widetilde{Y}_p})$ onto the α-component is dense in X_α.

Now, as argued in Lemma 6.3.16, if $x \in R(B_{\widetilde{Y}_p})$, then $\phi_{B_{Y_\alpha}}(x_\alpha) = \|x_\alpha\| + 1$ and $x_\alpha \in R(B_{Y_\alpha})$. By density of such x_α's, Y_α is a $(*)$- and DBR subspace of X_α.

The proof of $(*)$- and BR is simpler. □

Theorem 6.3.18. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Asplund spaces. Then any w*-compact subset of $(\widetilde{X}_0)^{**} = (X_\alpha^{**})_\infty$ is densely remotal and hence any w*-closed subspace of $(X_\alpha^{**})_\infty$ is DBR.*

Proof. Follows from Theorem 1.0.3, since $(\widetilde{X}_0)^* = (X_\alpha^*)_1$ has the RNP. □

6.4 Spaces of functions

Theorem 6.4.1. *Let K be a compact Hausdorff space, X a Banach space and Y a subspace of X.*

(a) For $f \in C(K,X)$, $\phi_{B_{C(K,Y)}}(f) = \sup_{t\in K} \phi_{B_Y}(f(t))$.

(b) $f \in R(B_{C(K,Y)})$ if and only if there exists $t_0 \in K$ such that $\phi_{B_{C(K,Y)}}(f) = \phi_{B_Y}(f(t_0))$ and $f(t_0) \in R(B_Y)$.

Proof. (a). If $g \in B_{C(K,Y)}$, then for any $t \in K$, $g(t) \in B_Y$. On the other hand, if $z \in B_Y$, then $1 \otimes z \in B_{C(K,Y)}$. It follows that

$$\begin{aligned}\phi_{B_{C(K,Y)}}(f) &= \sup_{g\in B_{C(K,Y)}} \|f-g\|_\infty = \sup_{g\in B_{C(K,Y)}} \sup_{t\in K} \|f(t)-g(t)\| \\ &= \sup_{t\in K} \sup_{g\in B_{C(K,Y)}} \|f(t)-g(t)\| = \sup_{t\in K} \phi_{B_Y}(f(t))\end{aligned}$$

(b). Let $f \in R(B_{C(K,Y)})$ then there exists $g \in B_{C(K,Y)}$ such that $\|f-g\|_\infty = \phi_{B_{C(K,Y)}}(f)$. Let $t_0 \in K$ be such that $\|f(t_0)-g(t_0)\| = \|f-g\|_\infty$. Then

$$\|f(t_0)-g(t_0)\| = \|f-g\|_\infty = \sup_{t\in K} \phi_{B_Y}(f(t)) \geq \phi_{B_Y}(f(t_0)) \geq \|f(t_0)-g(t_0)\|.$$

Hence $f(t_0) \in R(B_Y)$.

Conversely, suppose there exists $t_0 \in K$ such that $\phi_{B_{C(K,Y)}}(f) = \phi_{B_Y}(f(t_0))$ and $f(t_0) \in R(B_Y)$. Then there exists $y_0 \in B_Y$ such that $\|f(t_0)-y_0\| = \phi_{B_Y}(f(t_0))$. Now $1 \otimes y_0 \in B_{C(K,Y)}$ and clearly $\|f - 1\otimes y_0\|_\infty \geq \|f(t_0)-y_0\| = \phi_{B_Y}(f(t_0)) = \phi_{B_{C(K,Y)}}(f)$. Hence, $f \in R(B_{C(K,Y)})$. □

Corollary 6.4.2. *Let K be a compact Hausdorff space, X a Banach space and Y a subspace of X. Then Y is a $(*)$-subspace of X if and only if $C(K,Y)$ is a $(*)$-subspace of $C(K,X)$.*

Theorem 6.4.3. *Let K be a compact Hausdorff space, X a Banach space and Y a subspace of X. Then Y is a BR subspace of X if and only if $C(K,Y)$ is a BR subspace of $C(K,X)$.*

Proof. Since ϕ_{B_Y} is continuous, if $f \in C(K,X)$, then there exists $t_0 \in K$ such that $\sup_{t\in K} \phi_{B_Y}(f(t)) = \phi_{B_Y}(f(t_0))$. If Y is a BR subspace of X, $f(t_0) \in R(B_Y)$. Therefore, by Theorem 6.4.1 (b), $f \in R(B_{C(K,Y)})$.

Conversely, let $x_0 \in X$. If $C(K,Y)$ is a BR subspace of $C(K,X)$, $1 \otimes x_0 \in R(B_{C(K,Y)})$. Therefore, by Theorem 6.4.1 (b), $x_0 \in R(B_Y)$. □

Corollary 6.4.4. *Let K be a compact Hausdorff space, X a Banach space and Y a subspace of X. Then the following are equivalent :*

(a) *Y is a $(*)$- and BR subspace of X.*

(b) *$C(K,Y)$ is a $(*)$- and BR subspace of $C(K,X)$.*

For DBR, we will need the following modification of Lemma 4.2.8.

Lemma 6.4.5. *Let $D \subseteq X$ be a dense set. For $f \in C(K,X)$ and $\varepsilon > 0$, there exists $g \in C(K,X)$, $z \in D$ and $t \in K$ such that $g(t) = z$, $\|z\| = \|g\|_\infty$ and $\|f - g\|_\infty < \varepsilon$.*

Proof. Let $\|f\|_\infty = M$. Let $t_0 \in K$ be such that $\|f(t_0)\| = M$ and let $f(t_0) = x_0$.

Since D is dense in X, there exists $x_1 \in D$, $\|x_1\| > \|x_0\|$ such that $\|x_1 - x_0\| < \varepsilon/3$.

As in Lemma 4.2.8, split X into three disjoint regions :

$$\begin{aligned} X_1 &= \{x \in X : \|x - x_0\| > \varepsilon\}, \\ X_2 &= \{x \in X : \|x - x_0\| \leq \varepsilon/3\} \text{ and} \\ X_3 &= \{x \in X : \varepsilon/3 < \|x - x_0\| \leq \varepsilon\}, \end{aligned}$$

and define $\phi : X_1 \cup X_2 \to X$ as follows :

$$\phi(x) = x \quad \text{if } x \in X_1, \qquad \phi(x) = x_1 \quad \text{if } x \in X_2.$$

As before, any point in X_3 is of the form $x_0 + ry$ for some $r \in (\varepsilon/3, \varepsilon]$ and $y \in S_X$. Define $h : [\varepsilon/3, \varepsilon] \to [0, \varepsilon]$ by $h(r) = \frac{r - \varepsilon/3}{\varepsilon - \varepsilon/3}\varepsilon$ and define $\phi : X_3 \to X$ by $\phi(x_0 + ry) = x_1 + h(r)y$.

An argument similar to Lemma 4.2.8 proves that $\phi : X \to X$ is continuous. Define $g : K \to X$ by $g = \phi \circ f$.

As before, $g(K) \subseteq \phi(MB_X) \subseteq \|x_1\| B_X$. It follows that $x_1 = g(t_0) \in D$ and $\|g\|_\infty = \|x_1\| = \|g(t_0)\|$. Moreover,

$$\begin{aligned} \|f - g\|_\infty &\leq \sup\{\|x - \phi(x)\| : x \in X\} = \sup\{\|x - \phi(x)\| : x \in X_2 \cup X_3\} \\ &= \max\{\sup\{\|x - \phi(x)\| : x \in X_2\}, \sup\{\|x - \phi(x)\| : x \in X_3\}\} \\ &= \max\{\sup\{\|x - x_1\| : \|x - x_0\| \leq \varepsilon/3\}, \\ &\qquad \sup\{\|(x_0 - x_1) + (r - h(r))y\| : r \in (\varepsilon/3, \varepsilon], y \in S_X\}\} \\ &\leq \max\{2\varepsilon/3, \varepsilon/3 + \sup\{|r - h(r)| : r \in (\varepsilon/3, \varepsilon]\}\} \leq 2\varepsilon/3 < \varepsilon. \end{aligned}$$

This completes the proof. □

Theorem 6.4.6. *Let K be a compact Hausdorff space, X a Banach space and Y a subspace of X. Then the following are equivalent :*

(a) *Y is a $(*)$- and DBR subspace of X.*

(b) *$C(K,Y)$ is a $(*)$- and DBR subspace of $C(K,X)$.*

Proof. The $(*)$- part follows from Corollary 6.4.2.

$(a) \Rightarrow (b)$. Let $f \in C(K,X)$. Let $t_0 \in K$ be such that $\|f(t_0)\| = \|f\|_\infty$ and put $x_0 = f(t_0)$.

Let $\varepsilon > 0$. By Lemma 6.4.5 with $D = R(B_Y)$, there exists $g \in C(K,X)$, $x_1 \in R(B_Y)$ and $t \in K$ such that $g(t) = x_1$, $\|x_1\| = \|g\|_\infty$ and $\|f-g\|_\infty < \varepsilon$. Hence by Theorem 6.4.1 (b), $g \in R(B_{C(K,Y)})$. Hence the result follows.

$(b) \Rightarrow (a)$. Let $x_0 \in X$ and $\varepsilon > 0$. Consider $1 \otimes x_0 \in C(K,X)$. Let $f \in R(B_{C(K,Y)})$ be such that $\|1 \otimes x_0 - f\|_\infty < \varepsilon$. By Theorem 6.4.1 (b), there exists $t \in K$ such that $f(t) \in R(B_Y)$. Then $\|x_0 - f(t)\| \leq \|1 \otimes x_0 - f\|_\infty < \varepsilon$. Since ε is arbitrary, the result follows. □

Remark 6.4.7. Arguments similar to $(b) \Rightarrow (a)$ above shows that if $C(K,Y)$ is a DBR subspace of $C(K,X)$, then Y is a DBR subspace of X.

Question 6.4.8. *Can we prove the converse without assuming the $(*)$- property?*

Coming to spaces of Bochner integrable functions, let (Ω, Σ, μ) be a probability space and G be a closed bounded subset of a Banach space X. Our first result is a simpler proof of [28, Theorem 1.1]. The assumption that G is remotal is also not needed.

Theorem 6.4.9. *If $f \in L_1(\mu, X)$, then $\phi_{L_1(\mu,G)}(f) = \int_\Omega \phi_G(f(t))d\mu$.*

Proof. If $f \in L_1(\mu, X)$ and $g \in L_1(\mu, G)$, then

$$\|f-g\|_1 = \int_\Omega \|f(t)-g(t)\|d\mu \leq \int_\Omega \phi_G(f(t))d\mu$$

Hence,

$$\phi_{L_1(\mu,G)}(f) \leq \int_\Omega \phi_G(f(t))d\mu$$

Now, let $f = \sum_{i=1}^n x_i \chi_{A_i} \in L_1(\mu, X)$ be a simple function. Without loss of generality, we may assume $\sum_{i=1}^n \mu(A_i) = 1$. Given $\varepsilon > 0$, there exists $y_i \in G$ such that $\|x_i - y_i\| > \phi_G(x_i) - \varepsilon$. Let $g = \sum_{i=1}^n y_i \chi_{A_i}$. Then $g \in L_1(\mu, G)$ and

$$\begin{aligned} \|f-g\|_1 &= \sum_{i=1}^n \|x_i - y_i\|\mu(A_i) > \sum_{i=1}^n (\phi_G(x_i) - \varepsilon)\mu(A_i) \\ &= \sum_{i=1}^n \phi_G(x_i)\mu(A_i) - \varepsilon = \int_\Omega \phi_G(f(t))d\mu - \varepsilon. \end{aligned}$$

Since ε is arbitrary,

$$\phi_{L_1(\mu,G)}(f) = \int_\Omega \phi_G(f(t))d\mu.$$

Since ϕ_G is Lipschitz, so is the map $f \mapsto \int_\Omega \phi_G(f(t))d\mu$ from $L_1(\mu, X)$ to $\mathbb{R}^+$. In particular, it is continuous. Since simple functions are dense in $L_1(\mu, X)$, the result follows. □

Theorem 6.4.10. *[28, Corollary 1.2] If $f \in L_1(\mu, X)$, then $g \in L_1(\mu, G)$ is farthest from f if and only if $g(t) \in G$ is farthest from $f(t)$ a.e. $[\mu]$.*

Proof. Let $g \in L_1(\mu, G)$ be farthest from f. By above theorem, this means

$$\int_\Omega \|f(t) - g(t)\| d\mu = \int_\Omega \phi_G(f(t))d\mu.$$

Since $\|f(t) - g(t)\| \leq \phi_G(f(t))$ a.e. $[\mu]$, this implies $\|f(t) - g(t)\| = \phi_G(f(t))$ a.e. $[\mu]$.

Conversely, if $g \in L_1(\mu, G)$ is such that $\|f(t) - g(t)\| = \phi_G(f(t))$ a.e. $[\mu]$, then $\|f - g\|_1 = \int_\Omega \|f(t) - g(t)\| d\mu = \int_\Omega \phi_G(f(t))d\mu = \phi_{L_1(\mu,G)}(f)$. That is, g is farthest from f. □

Corollary 6.4.11. *Let $G \subseteq B_X$ be closed and $\sup_{z \in G} \|z\| = 1$. The following are equivalent :*

- *(a) G is a $(*)$-subset of X.*
- *(b) $L_1(\mu, G)$ is a $(*)$-subset of $L_1(\mu, X)$.*

Theorem 6.4.12. *Let $G \subseteq X$ be a closed bounded set. The following are equivalent :*

- *(a) G is densely remotal in X.*
- *(b) $L_1(\mu, G)$ is densely remotal in $L_1(\mu, X)$.*

Proof. $(a) \Rightarrow (b)$. This is proved in [10, Lemma 3.7]. We include the details for completeness.

Let $f = \sum_1^n x_i \chi_{A_i}$ be a simple function in $L^1(\mu, X)$. Since $R(G)$ is dense, given $\varepsilon > 0$, we can get a simple function $g = \sum_1^n y_i \chi_{A_i}$ with $y_i \in R(G)$ and $\|f - g\|_1 < \varepsilon$. For each i, let $z_i \in G$ be farthest from y_i, then $h = \sum_1^n z_i \chi_{A_i}$ is point-wise farthest from g and hence, by Theorem 6.4.10, it is farthest from g in $L^1(\mu, G)$ as well.

$(b) \Rightarrow (a)$. Let $x \in X$. Consider $x\chi_\Omega \in L_1(\mu, X)$.

Given $\varepsilon > 0$ there exists $f \in L_1(\mu, X)$, and $g \in L_1(\mu, G)$ such that $\|x\chi_\Omega - f\|_1 < \varepsilon$ and g is farthest from f. By Theorem 6.4.10, it follows that $\|f(t) - g(t)\| = \phi_G(f(t))$ a.e. $[\mu]$.

Now there exists $A \in \Sigma$ with $\mu(A) > 0$ such that $\|x - f(t)\| < \varepsilon$ and $\|f(t) - g(t)\| = \phi_G(f(t))$ for all $t \in A$. Put $z = f(t)$ for some $t \in A$. Then $z \in R(G)$ and $\|x - z\| < \varepsilon$. □

Corollary 6.4.13. *If $L_1(\mu, G)$ is a remotal subset of $L_1(\mu, X)$, then G is a remotal subset of X.*

Since $L_1(\mu, B_Y) \subseteq B_{L_1(\mu,Y)}$, we have

Corollary 6.4.14.

- *(a) If Y is a $(*)$-subspace of X, then $L_1(\mu, Y)$ is a $(*)$-subspace of $L_1(\mu, X)$.*
- *(b) If Y is $(*)$- and DBR in X, then $L_1(\mu, Y)$ is $(*)$- and DBR in $L_1(\mu, X)$.*

CHAPTER 7

Ball remotality of X in X^{**}

7.1 Summary of results

In this chapter, we study ball remotality of a Banach space X in its bidual. In particular, we consider the following properties mentioned in the Introduction :

Definition 7.1.1. We will say that a Banach space X

(a) is BR in its bidual (BRB) if $R(B_X) = X^{**}$.

(b) is DBR in its bidual (DBRB) if $\overline{R(B_X)} = X^{**}$.

(c) has remotally spanned bidual (RSB) if $\overline{span}(R(B_X)) = X^{**}$.

(d) is anti-remotal in its bidual (ARB) if $R(B_X) = X$.

It is clear that reflexivity $\Rightarrow BRB \Rightarrow DBRB \Rightarrow RSB$. We show that none of the converse holds. We show that a Banach space having a strong unitary is BRB, producing a large class of non-reflexive examples. We show that X is wALUR ([7], see Definition 7.2.11) if and only if X is rotund and ARB. We also obtain characterizations of reflexivity in terms of these phenomena. For example, we show that a separable Banach space is reflexive if and only if it is BRB/DBRB/RSB in every equivalent renorming.

In stability results, We show that the ℓ_1-sum of a finite family of Banach spaces is BRB/DBRB if at least one coordinate space is such. c_0-sum of a family of Banach spaces is DBRB if and only if each coordinate space is DBRB. And ℓ_p-sum $(1 < p < \infty)$ of a family of Banach spaces is DBRB/BRB/ARB/RSB if and only if each coordinate space is such.

7.2 Main results

We will use the following notations in this chapter.

Notation 6. (a) We will consider the duality map only on S_X, that is, $D : S_X \to S_{X^*}$ is a set-valued map defined as $D(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}$, $x \in S_X$.

(b) The inverse duality map $D^{-1} : S_{X^*} \to S_X$ is a set-valued map defined as $D^{-1}(x^*) = \{x \in S_X : x^*(x) = 1\}$. Note that this set is empty unless $x^* \in D(S_X) = NA(X) \cap S_{X^*}$.

(c) D_n will denote the duality map from $S_{X^{(n)}}$ to $S_{X^{(n+1)}}$. Similarly, we can define D_n^{-1}.

(d) For a subspace Y of a Banach space X and $y^* \in Y^*$, let $\mathrm{HB}(y^*)$ be the set of all Hahn-Banach (i.e., norm preserving) extensions of y^* to X. Note that, by Hahn-Banach Theorem, $\mathrm{HB}(y^*)$ is always nonempty.

When we consider X as a subspace of X^{**}, then

$$A_X = \{x^{***} \in S_{X^{***}} : \|x^{***}|_X\| = 1\} = \mathrm{HB}(S_{X^*}) \supseteq S_{X^*},$$

which is clearly norming for X^{**}. Therefore, X is always a $(*)$-subspace of X^{**}.

Let $R = R(B_X, X^{**})$ and $R_1 = R \cap S_{X^{**}}$. By Proposition 2.3.13, R is closed under scalar multiplications. Therefore, $R = \mathbb{F}R_1$. By Proposition 2.3.13 again, we have

Proposition 7.2.1. *$R_1 = D_2^{-1}(D_2(S_X)) = D_2^{-1}(HB[D(S_X)])$.*

Proposition 7.2.2. *If X is a Banach space X, then*

(a) *X is BRB if and only if $R_1 = S_{X^{**}}$ if and only if $D_2(S_X)$ is a boundary for X^{**}.*

(b) *X is DBRB if and only if $\overline{R}_1 = S_{X^{**}}$.*

(c) *X has RSB if and only if $\overline{span}(R_1) = X^{**}$.*

(d) *X is ARB if and only if $R_1 = S_X$.*

The following set has been introduced in [10]. Let

$$NA_2(X) = \{x^{**} \in X^{**} : x^{**}(x^*) = \|x^{**}\| \text{ for some } x^* \in NA(X) \cap S_{X^*}\}.$$

It is easy to see that $NA_2(X) \subseteq R$ [10, Proposition 2.13]. The following is a mild improvement.

Definition 7.2.3. We call a Banach space X is weakly Hahn-Banach smooth if every $x^* \in NA(X)$ has a unique norm-preserving extension to all of X^{**}.

Proposition 7.2.4. *If X is weakly Hahn-Banach smooth then $R = NA_2(X)$.*

Proof. Clearly, $NA_2(X) \cap S_{X^{**}} = D_2^{-1}(D(S_X)) \subseteq D_2^{-1}(D_2(S_X)) = R_1$. And if X is weakly Hahn-Banach smooth, then $R_1 = D_2^{-1}(\mathrm{HB}[D(S_X)]) = D_2^{-1}(D(S_X)) = NA_2(X) \cap S_{X^{**}}$. □

Remark 7.2.5. (a) In [10], this result is proved under the additional assumption that B_{X^*} is w*-sequentially compact.

(b) Clearly, a reflexive space is BRB. On the other hand, if X is weakly Hahn-Banach smooth and BRB, then it follows that X is reflexive. However, BRB does not characterize reflexivity.

Let $A \subseteq X$, then $co(A)$ will denote the convex hull of A and $aco(A)$ will denote the absolute convex hull of A, that is, $aco(A) = co(\mathbb{T}A)$. We will need the following Lemma.

Lemma 7.2.6. *Let A be a w*-compact convex subset of B_{X^*}, then $\overline{aco}^{w^*}(A) = \overline{aco}^{\|\cdot\|}(A)$.*

Proof. If $\mathbb{F} = \mathbb{R}$, since A is w*-compact and convex, $aco(A) = co(A \cup -A)$ is w*-compact, and hence norm closed. Thus,

$$\overline{aco}^{w^*}(A) = aco(A) = \overline{aco}^{\|\cdot\|}(A).$$

If $\mathbb{F} = \mathbb{C}$, this follows from [31, Lemma 2.1]. We include the details for completeness.

Let $\mathbb{D}$ be the closed unit disc in $\mathbb{C}$. Note that $aco(A) = co(\mathbb{T}A) = co(\mathbb{D}A)$ as well. Given $\delta > 0$, we can find $z_1, z_2, \ldots, z_n \in \mathbb{C}$ such that $\mathbb{D} \subseteq \mathbb{D}_\delta = co\{z_1, z_2, \ldots, z_n\} \subseteq (1+\delta)\mathbb{D}$. Hence

$$aco(A) \subseteq co(\mathbb{D}_\delta A) \subseteq (1+\delta)aco(A).$$

Now since A is w*-compact and convex and $co(\mathbb{D}_\delta A) = co(\bigcup_{j=1}^{n} z_j A)$, $co(\mathbb{D}_\delta A)$ is w*-compact, hence w*-closed. It follows that

$$\overline{aco}^{w^*}(A) \subseteq co(\mathbb{D}_\delta A) \subseteq (1+\delta)(\overline{aco}^{\|\cdot\|}(A))$$

and hence,

$$\overline{aco}^{w^*}(A) \subseteq (1+\delta)(\overline{aco}^{\|\cdot\|}(A)).$$

Since $\delta > 0$ is arbitrary,

$$\overline{aco}^{w^*}(A) \subseteq \overline{aco}^{\|\cdot\|}(A)$$

The other inclusion is obvious. □

Corollary 7.2.7. *If $A \subseteq B_{X^*}$ is a w*-compact convex set that is norming for X, then it is norming for X^{**} as well.*

Proof. If $A \subseteq B_{X^*}$ is w*-compact convex and norming for X, then $\overline{aco}^{w^*}(A) = B_{X^*}$. From the above lemma it follows that $\overline{aco}^{\|\cdot\|}(A) = B_{X^*}$. Hence A is norming for X^{**} as well. □

Theorem 7.2.8. *If x_0 is a strong unitary in X, then it is also a strong unitary in X^{**}.*

Proof. Let x_0 be a strong unitary in X. Then $D(x_0) \subseteq S_{X^*}$ is a w*-compact convex norming set for X. From the above corollary, it follows that $D(x_0)$, and hence $D_2(x_0)$, is a norming set for X^{**}. □

Corollary 7.2.9. *Any Banach space containing a strong unitary is BRB. And hence, each of the following spaces is BRB :*

(a) *$C(K)$ and any subspace of $C(K)$ containing an unimodular function, in particular, any function space;*

(b) *ℓ_∞ and any subspace of ℓ_∞ containing the constant sequence 1, in particular, c;*

(c) *ℓ_1 and any subspace of ℓ_1 containing any of the canonical unit vectors.*

Theorem 7.2.10. *Reflexivity $\Rightarrow BRB \Rightarrow DBRB \Rightarrow RSB$ and none of the converse holds.*

Proof. It is clear that reflexivity $\Rightarrow BRB \Rightarrow DBRB \Rightarrow RSB$.

By Corollary 7.2.9, BRB $\not\Rightarrow$ reflexivity. Since c_0 is DBRB, but not BRB, DBRB $\not\Rightarrow$ BRB.

Let X_1 be ARB and X_2 be BRB and let $X = X_1 \oplus_\infty X_2$. We show that X is RSB, but not DBRB. It follows from Lemma 6.3.5 that

$$R_1 = [S_{X_1} \times B_{X_2^{**}}] \cup [B_{X_1^{**}} \times S_{X_2^{**}}].$$

Thus R_1 is a closed subset of $S_{X^{**}}$ and if $x_1^{**} \in S_{X_1^{**}} \setminus S_{X_1}$ and $x_2^{**} \in B_{X_2^{**}} \setminus S_{X_2^{**}}$, then $(x_1^{**}, x_2^{**}) \notin R_1$. Thus, X is not DBRB.

Nevertheless, $\mathrm{span}(R_1) = X^{**}$. That is, X is RSB.

To see this, note that if $(x_1^{**}, x_2^{**}) \in S_{X^{**}} \setminus R_1$, then $1 = \|x_1^{**}\| > \|x_2^{**}\|$ and $x_1^{**} \notin X_1$. If $x_2^{**} \neq 0$, put $u^{**} = x_2^{**}/\|x_2^{**}\|$, and if $x_2^{**} = 0$, take any $u^{**} \in S_{X_2^{**}}$. Then $(x_1^{**}, \pm u^{**}) \in R_1$ and $(x_1^{**}, x_2^{**}) \in co(\{(x_1^{**}, \pm u^{**})\})$. □

We recall the following definitions from [7].

Definition 7.2.11. We say that $x \in S_X$ is :

(a) A rotund point of B_X if $\|y\| = \left\|\frac{x+y}{2}\right\| = 1$ implies $x = y$.

(b) An LUR point of B_X if for any $\{x_n\} \subseteq B_X$ the condition

$$\lim_n \left\|\frac{x_n + x}{2}\right\| = 1$$

implies $x_n \to x$ in norm.

(c) An ALUR (resp. wALUR) point of B_X if for any $\{x_n\} \subseteq B_X$ and $\{x_m^*\} \subseteq B_{X^*}$, the condition

$$\lim_m \lim_n x_m^* \left(\frac{x_n + x}{2}\right) = 1$$

implies $x_n \to x$ in norm (resp. in the weak topology).

We say that a Banach space X has one of the above properties if every point of S_X has the same property.

Notice that any $x^{**} \in X^{**} \setminus X$ is in R if and only if it is *not rotund* in some direction in S_X. Indeed, we have

Theorem 7.2.12. *X is wALUR if and only if X is rotund and ARB.*

Proof. We recall [7, Corollary 8] that $x \in S_X$ is a wALUR point of B_X if and only if x is a rotund point of $B_{X^{**}}$.

Let X be wALUR. Then X is clearly rotund. Now, let $x^{**} \in X^{**} \setminus X$ with $\|x^{**}\| = 1$. Then $\phi_{B_X}(x^{**}) = 2$. If there exists $x \in B_X$ such that $\|x^{**} + x\| = 2$, then x is not a rotund point of $B_{X^{**}}$. It follows that $x^{**} \notin R$. Thus, $R = X$.

To prove the converse, let $x \in S_X$. If $x^{**} \in B_{X^{**}}$ is such that $\left\|\frac{x^{**}+x}{2}\right\| = 1$, then $-x \in F_{B_X}(x^{**})$. Since $R = X$, $x^{**} \in X$. Now since X is rotund, it follows that $x^{**} = x$. Therefore, x is a rotund point of $B_{X^{**}}$. □

Remark 7.2.13. Consider the quotient space $X = C(\mathbb{T})/\mathbb{A}$, where $\mathbb{A}$ is the disc algebra. Then it is known that X is an M-embedded space and $X^* = H_0^1$ is a smooth space [26]. It follows that X is wALUR [6] and hence, X is ARB. This shows that an M-ideal need not be DBR and an M-embedded space need not be DBRB [10].

Theorem 7.2.14. *The following are equivalent :*

(a) *X is reflexive.*

(b) *X is WCG and BRB for every equivalent renorming on X.*

(c) *X is WCG and DBRB for every equivalent renorming on X.*

(d) *X is WCG and RSB for every equivalent renorming on X.*

Proof. Clearly, $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. Since $RSB + ARB \Rightarrow$ reflexivity and every WCG Banach space has an LUR renorming [22, Theorem 11.20], $(d) \Rightarrow (a)$ follows from Theorem 7.2.12. □

Question 7.2.15. *Can the assumption that X is WCG be dropped or at least weakened?*

Corollary 7.2.16. *For a separable Banach space X, the following are equivalent :*

(a) *X is reflexive.*

(b) *X is BRB for every equivalent renorming on X.*

(c) *X is DBRB for every equivalent renorming on X.*

(d) *X is RSB for every equivalent renorming on X.*

Remark 7.2.17. Since reflexivity is separably determined, the answer to Question 7.2.15 would be clearly positive if these properties were hereditary. It however follows from the fact that any $C(K)$ space is BRB (Corollary 7.2.9) that they are not. On the other hand, it is easy to see that ARB is hereditary. In particular, if X is ARB, it does not contain isometric copies of c_0 or ℓ_1.

Coming to stability results, it follows from results of Chapter 6 (Proposition 6.3.1(b), Theorem 6.3.6, Lemma 6.3.16 and Theorem 6.3.17, in particular) that

Corollary 7.2.18. *If $\{X_1, X_2, \ldots, X_n\}$ is a finite family of Banach spaces such that at least one X_i is BRB/DBRB, then so is $\bigoplus_{\ell_1} X_i$.*

Corollary 7.2.19. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces. Then $\bigoplus_{c_0} X_\alpha$ is DBRB if and only if each X_α is DBRB.*

Corollary 7.2.20. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of Banach spaces and $1 < p < \infty$. Then $\bigoplus_{\ell_p} X_\alpha$ is DBRB/BRB/ARB/RSB if and only if each X_α is such.*

Remark 7.2.21. (a) It follows again from Corollary 7.2.18 that BRB or DBRB are not hereditary properties.

(b) Since c_0 is DBRB, but not BRB, there is no analogue of Corollary 7.2.19 for BRB.

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