Some aspects of toric topology

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To my family

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Contents

Chapter 0

Introduction

The main goal of this thesis is to study the topology of torus actions on manifolds and orbifolds. In algebraic geometry actions of the torus $(\mathbb{C}^*)^n$ on algebraic varieties with nice properties produce bridges between geometry and combinatorics see [Dan78], [Oda88] and [Ful93]. We see a similar bridge called *moment map* for Hamiltonian action of compact torus on symplectic manifolds see [Aud91] and [Gui94]. In particular whenever the manifold is compact the image of moment map is a simple polytope, the orbit space of the action. A topological counterpart called quasitoric manifolds, a class of topological manifolds with compact torus action having combinatorial orbit space, were introduced by Davis and Januskiewicz in [DJ91]. A class of examples of quasitoric manifolds are nonsingular projective toric varieties, introduced by M. Demazure [Dem70]. There are many properties of quasitoric manifolds akin to that of nonsingular complete toric varieties. The combinatorial formula for the cohomology ring of a nonsingular complete toric variety is analogous to the formula for quasitoric manifolds. Their K-theories described by P. Sankaran and V. Uma in [SU03] and [SU07] are also similar. The survey [BP02] is a good reference for many interesting developments and applications of quasitoric manifolds.

Inspired by the work of [DJ91] we generalize these quasitoric manifolds to quasitoric orbifolds with compact torus action. We have studied structures and topological invariants of quasitoric orbifolds. In addition, we have introduced a class of n-dimensional orbifolds with $\mathbb{Z}_2^{n-1}$-action with nice combinatorial description. We have also given two applications of quasitoric manifolds to cobordism theory. This section briefly introduces the main ingredients of this thesis. We will meet all in much greater detail in the following chapters.

We recall the definitions and topological invariants namely homology groups, cohomology rings and Chern classes of quasitoric manifolds in Chapter 1. A quasitoric manifold M^{2n} is an even dimensional smooth manifold with a locally standard action of the compact torus $\mathbb{T}^n = U(1)^n$ such that the orbit space has the structure of an

n-dimensional simple polytope. Cohomology rings of these manifolds can be computed using equivariant cohomology [DJ91]. As a special case, one obtain the cohomology rings of nonsingular projective toric varieties without recourse to algebraic geometry. Buchstaber and Ray [BR01] showed the existence of smooth and stable almost complex structure on quasitoric manifolds. We present a different proof of this following [Poddf].

In Chapter 2 we recall the definitions of small covers and orbifolds. The category of small covers was introduced by Davis and Januszkiewicz [DJ91]. Following them we discuss some basic theory about small covers. The remaining sections of this chapter describe the definition, tangent bundle and orbifold fundamental group of orbifolds following [ALR07]. Orbifolds were introduced by Satake [Sat57], who called them V-manifolds. Precisely, orbifolds are singular spaces that locally look like the quotient of an open subset of Euclidean space by an action of a finite group.

In Chapter 3 we study topological invariants and stable almost complex structure on quasitoric orbifolds. We discuss equivalent definitions of quasitoric orbifolds, one is an axiomatic definition of a quasitoric orbifold via locally standard action and the other is constructive. Our constructive definition uses the combinatorial model $(Q, N, \{\lambda_i\})$, where Q is a simple polytope, N is a free $\mathbb{Z}$-module of finite rank, λ_i is an assignment of a vector in N to each facet F_i of Q satisfying certain conditions. Let $\widehat{N}$ be the submodule of N generated by the vectors λ_i. We construct orbifold universal cover and compute the orbifold fundamental group of quasitoric orbifold.

Theorem 0.0.1 (Theorem 3.2, [PS10])**.** *The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{X})$ of the quasitoric orbifold $\mathcal{X}$ is isomorphic to $N/\widehat{N}$.*

We compute the homology of quasitoric orbifolds with coefficients in $\mathbb{Q}$. For this we need to generalize the notion of CW-complex a little. We introduce the notion of **q**-CW complex where an open cell is the quotient of an open disk by action of a finite group. We compute the rational cohomology ring of a quasitoric orbifold and show that it is isomorphic to a quotient of the Stanley-Reisner face ring of the base polytope Q, Theorem 3.8.2.

We show the existence of a stable almost complex structure on a quasitoric orbifold corresponding to any given omniorientation. The universal orbifold cover of the quasitoric orbifold is used here. As in the manifold case, we show that the cohomology ring is generated by the first Chern classes of some complex rank one orbifold vector bundles, canonically associated to facets of Q, see Section 3.10. We compute the top Chern number of an omnioriented quasitoric orbifold. We give a necessary condition for existence of torus invariant almost complex structure. Whether this condition is also sufficient remains open. Finally we compute the Chen-Ruan cohomology groups of an almost complex quasitoric orbifold.

In Chapter 4 we introduce some n-dimensional smooth orbifolds on which there is $\mathbb{Z}_2^{n-1}$-action having a simple polytope as the orbit space. We call these orbifolds small orbifolds. The classification problem of small orbifolds remains open. We show that the orbifold universal cover of n-dimensional ($n > 2$) small orbifold is $\mathbb{R}^n$. We also show the space $\mathcal{Z}$, constructed in Lemma 4.4 of [DJ91] associated to a simple n-polytope Q ($n > 2$), is diffeomorphic to $\mathbb{R}^n$ if there is an s-characteristic function on Q (Theorem 3.3, [Sar10b]). The converse is an interesting open question. One application of this result is the following. The space $\mathcal{Z}$ corresponding to the n-simplex ($n > 2$) is homeomorphic to the n-sphere. So there does not exist any s-characteristic function of n-simplex. Consequently there does not exist any small orbifold with the n-dimensional simplex as orbit space when $n > 2$. We calculate the orbifold fundamental group of n-dimensional small orbifolds. We compute the homology groups of small orbifolds in terms of *h-vector* (see [DJ91]) of the polytope. When Q is an even dimensional simple polytope then small orbifolds over Q are orientable. We compute the cohomology rings of even dimensional small orbifolds.

In Chapter 5 we compute the $\mathbb{T}^2$-cobordism group in the following category: the objects are all 4-dimensional quasitoric manifolds and morphisms are $\mathbb{T}^2$ equivariant maps between quasitoric 4-manifolds. In this chapter we introduce a particular type of polytope, which we call *edge-simple* polytope.

Definition 0.0.2. *An n-dimensional convex polytope P is called an n-dimensional edge-simple polytope if each edge of P is the intersection of exactly $(n-1)$ codimension one faces of P.*

The study of topology and combinatorics of these polytopes would be interesting. However we have not dealt with these questions in this thesis.

We introduce the notion of *isotropy* function on the set of facets $\mathcal{F}(P)$ of an n-dimensional edge-simple polytope P.

Definition 0.0.3. *A function $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^{n-1}$ is called an isotropy function of the edge-simple polytope P if the set of vectors $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_{n-1}})\}$ form a basis of $\mathbb{Z}^{n-1}$ whenever the intersection of the facets $\{F_{i_1}, \ldots, F_{i_{n-1}}\}$ is an edge of P.*

Deleting a suitable neighborhood of each vertex of P we get a simple n-polytope P_Q. We construct an $(2n-1)$-manifold with quasitoric boundary from simple n-polytope P_Q and an isotropy function λ of P. There is a natural $\mathbb{T}^{n-1}$-action on these manifolds with quasitoric boundary having P_Q as the orbit space. We show that these manifolds with quasitoric boundary are orientable and compute their Euler characteristic.

Now consider $n = 2$. Let $\pi : M \to Q$ be a quasitoric 4-manifold with the characteristic pair (Q, η). Here Q is a simple 2-polytope and η is the assignment, called characteristic function of M, of isotropy group to each facet of Q. Suppose the number of codimension one faces of Q is m. We construct an edge-simple 3-polytope $P_{\mathcal{E}}$ such that $P_{\mathcal{E}}$ has exactly one vertex which is the intersection of m codimension one faces and other vertices of $P_{\mathcal{E}}$ are intersection of 3 codimension one faces. Extending the characteristic function η we define an isotropy function λ of $P_{\mathcal{E}}$. The pair $(P_{\mathcal{E}}, \lambda)$ helps to construct an oriented $\mathbb{T}^2$ manifold W with boundary

$$\partial W = M + k_1 \mathbb{CP}^2 + k_2 \overline{\mathbb{CP}}^2$$

where k_1, k_2 are some integers. Thus we obtain the following lemma and theorem.

Lemma 0.0.4 (Lemma 6.1, [Sar10c])**.** *The $\mathbb{T}^2$-cobordism class of a Hirzebruch surface is trivial. In particular, oriented cobordism class of a Hirzebruch surface is also trivial.*

Lemma 0.0.5. *Any 4-dimensional quasitoric manifold is equivariantly cobordant to the disjoint union $\sqcup_1^l \mathbb{CP}^2$ for some l, where the $\mathbb{T}^2$-action on different copies of $\mathbb{CP}^2$ may be distinct.*

Theorem 0.0.6. *The set $\{[\mathbb{CP}^2_{\xi'}] \, : \, [\xi]_{eq} \in SL(2, \mathbb{Z})/ \sim_{eq}\}$ is a set of generators of the oriented torus cobordism group CG_2.*

In Chapter 6, we give a new proof of the fact that the oriented cobordism class of $\mathbb{CP}^{2k+1}$ is trivial for each $k \geq 0$ (Theorem 5.1 in [Sar10a]). The strategy of our proof is to first construct an odd dimensional compact orientable manifold with boundary, where the boundary is a disjoint union of three quasitoric manifolds. This involves an adaptation of the usual combinatorial method for constructing quasitoric manifolds. Moreover the combinatorial data is carefully chosen so that exactly one of the boundary components is $\mathbb{CP}^{2k+1}$ while the other two components are identifiable by an orientation reversing homeomorphism.

Chapter 1

Quasitoric manifolds

1.1 Introduction

In these chapter we recall the definitions and topological invariants of quasitoric manifolds. We follow [DJ91] for basic definitions, examples and a topological classification. We compute the homology group of quasitoric manifolds following [DJ91]. The next two sections deal with the study of orientability of quasitoric manifolds and the concept of connected sum of quasitoric manifolds respectively. The computation of the cohomology ring of quasitoric manifolds are explained following the work [DJ91]. The work of Buchstaber and Ray [BR01] on the existence of smooth and stable almost complex structure are the object of Section 1.9. Following this up one can show the existence of Chern classes as an application of the stable almost complex structure and describe the formulae for top Chern number.

1.2 Definition and examples

Quasitoric manifolds are essentially an even dimensional manifolds M^{2n}. The torus $\mathbb{T}^n = U(1)^n$ action on M^{2n} must be effective and locally resemble the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$ up to an automorphism of $\mathbb{T}^n$.

Definition 1.2.1. *An n-dimensional simple polytope in $\mathbb{R}^n$ is a convex polytope where exactly n bounding hyperplanes meet at each vertex. The codimension one faces of convex polytope are called facets.*

The ready examples of simple polytopes are simplices and cubes. Through out the thesis Q stands for simple polytope.

Definition 1.2.2. *An action of $\mathbb{T}^n$ on a $2n$-dimensional manifold M^{2n} is said to be locally standard if every point $y \in M$ has a $\mathbb{T}^n$-stable open neighborhood U_y and a home-*

omorphism $\psi : U_y \to V$, where V is a $\mathbb{T}^n$-stable open subset of $\mathbb{C}^n$ and an isomorphism $\delta_y : \mathbb{T}^n \to \mathbb{T}^n$ such that $\psi(t \cdot x) = \delta_y(t) \cdot \psi(x)$ for all $(t, x) \in \mathbb{T}^n \times U_y$.

Definition 1.2.3. *A $\mathbb{T}^n$-manifold M^{2n} is called quasitoric manifold over a simple polytope Q if the following conditions are satisfied:*

1. *the $\mathbb{T}^n$ action is locally standard,*
2. *there is a projection map $\mathfrak{p} : M^{2n} \to Q$ constant on $\mathbb{T}^n$ orbits which maps every l-dimensional orbit to a point in the relative interior of a l-dimensional face of Q.*

Definition 1.2.4. *A quasitoric manifold having a smooth structure such that the action of torus is smoothly locally standard is called smooth quasitoric manifold.*

Example 1.2.5. *Consider the complex projective space $\mathbb{CP}^2 \cong (\mathbb{C}^3 - \{0\})/\mathbb{C}^* \cong S^5/S^1$. Denote the coordinates on $\mathbb{C}^3$ by (z_1, z_2, z_3). So we may represent S^5 by the set*

$$\{|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$$

and S^1 by the subgroup $\{\alpha \cdot I_3 : |\alpha| = 1\}$ of $U(1)^3$ where I_3 is a rank 3 identity matrix.

The points of $\mathbb{T}^2 \cong U(1)^3/S^1$ and $\mathbb{CP}^2$ can be identified to the class $[t_1 : t_2 : t_3]$ and $[z_1 : z_2 : z_3]$ respectively. The natural action of $\mathbb{T}^2$ on $\mathbb{CP}^2$ and $\mathbb{C}^2$ are the following maps $\mathbb{T}^2 \times \mathbb{CP}^2 \to \mathbb{CP}^2$ and $\mathbb{T}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$, defined by

$$\begin{aligned} &([t_1 : t_2 : t_3], [z_1 : z_2 : z_3]) \to [t_1 z_1 : t_2 z_2 : t_3 z_3] \\ &\text{and } ([t_1 : t_2 : t_3], (z_1', z_2')) \to (t_1 {t_3}^{-1} z_1', t_2 {t_3}^{-1} z_2') \end{aligned} \tag{1.2.1}$$

respectively. Clearly $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ are the only fixed points of $\mathbb{T}^2$ action on $\mathbb{CP}^2$. Let $U_j = \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_j \neq 0\}$ for $j = 1, 2, 3$ and

$$\psi_1 : U_1 \to \mathbb{C}^2,\ \psi_2 : U_2 \to \mathbb{C}^2,\ \psi_3 : U_3 \to \mathbb{C}^2 \tag{1.2.2}$$

be the maps defined by

$$\begin{aligned} &\psi_1([z_1 : z_2 : z_3]) = (z_2/z_1, z_3/z_1),\ \psi_2([z_1 : z_2 : z_3]) = (z_1/z_2, z_3/z_2) \\ &\text{and } \psi_3([z_1 : z_2 : z_3]) = (z_1/z_3, z_2/z_3) \end{aligned} \tag{1.2.3}$$

respectively. The subsets U_1, U_2 and U_3 are $\mathbb{T}^2$-stable covering open subsets of $\mathbb{CP}^2$. The maps ψ_1, ψ_2 ψ_3 are homeomorphisms. The map ψ_1 satisfy the following relation.

$$\psi_1([t_1 : t_2 : t_3] \cdot [z_1 : z_2 : z_3]) = ((t_2 t_1^{-1} z_2)/z_1, (t_3 t_1^{-1} z_3)/z_1) = [t_2, t_3, t_1] \cdot (z_2/z_1, z_3/z_1)$$

Let δ_1, δ_2 and δ_3 are automorphism of $\mathbb{T}^2$ defined by

$$\delta_1([t_1 : t_2 : t_3]) = [t_2, t_3, t_1],\ \delta_2([t_1 : t_2 : t_3]) = [t_1, t_3, t_2] \text{ and } \delta_3([t_1 : t_2 : t_3]) = [t_1, t_2, t_3]$$

respectively. Hence we get the following relation for all $[t_1 : t_2 : t_3] \in \mathbb{T}^2$,

$$\psi_1([t_1 : t_2 : t_3] \cdot [z_1 : z_2 : z_3]) = \delta_1([t_1 : t_2 : t_3]) \cdot \psi_1([z_1 : z_2 : z_3]), \ \forall \ [z_1 : z_2 : z_3] \in U_1,$$

So ψ_1 *is a* δ_1*-equivariant map. Similarly we can show that* ψ_i *is a* δ_i*-equivariant map for* $i = 2, 3$*. Hence the natural* $\mathbb{T}^2$ *action on* $\mathbb{CP}^2$ *is a locally standard action. The orbit space can be identified to the triangle* $\triangle^2 = \{(|z_1|^2, |z_2|^2) \in \mathbb{R}^2 : |z_1|^2 + |z_2|^2 \leq 1\}$*. In fact the projective space* $\mathbb{CP}^n$ *is a quasitoric manifold over the* n*-dimensional simplex* $\triangle^n$ *for each* n*.*

Example 1.2.6. *In example 1.2.5 we show that the projective space* $\mathbb{CP}^1$ *is a quasitoric manifold over the interval* $I^1 = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$*. Then the space* $(\mathbb{CP}^1)^n$ *is a quasitoric manifold over the standard cube* I^n*. Here the torus* $\mathbb{T}^n$ *action on* $(\mathbb{CP}^1)^n$ *is the diagonal action.*

By the definition 1.2.3 a zero dimensional orbit maps to a vertex of Q. Since the $\mathbb{T}^n$ action is locally standard, the fixed point sets are parameterized by the vertex set of Q. Let F be a codimension k face of Q. We denote its relative interior by $int(F)$. The space $\mathfrak{p}^{-1}(int(F))$ is a trivial $\mathbb{T}^{n-k}$-bundle over $\mathfrak{p}^{-1}(int(F))$. The isotropy group

$$\mathbb{T}^n_x := \{t \in \mathbb{T}^n : tx = x\} \tag{1.2.4}$$

at each point $x \in \mathfrak{p}^{-1}(int(F))$ is locally constant on $\mathfrak{p}^{-1}(int(F))$. Since $\mathfrak{p}^{-1}(int(F))$ is product of two connected sets and so,

$$\mathbb{T}^n_x = \mathbb{T}^n_y \text{ for all points } x, y \in \mathfrak{p}^{-1}(int(F)). \tag{1.2.5}$$

These common isotropy group is denoted by $\mathbb{T}^n_F$. The group $\mathbb{T}^n_F$ is isomorphic to $\mathbb{T}^k$. Hence the orbit of each point over the relative interior of codimension-k face is the $(n-k)$-dimensional subtorus of $\mathbb{T}^n$.

Let $F_1, \ldots, F_m$ be the facets of Q, denoted by $\mathcal{F}(Q)$. So the isotropy subgroup of the preimage $\mathfrak{p}^{-1}(int(F_j))$ is the 1-dimensional subgroup $\mathbb{T}^n_{F_j}$ of $\mathbb{T}^n$. Each F_j is also a simple polytope. Let $M_j^{2(n-1)}$ be the $\mathbb{T}^n_{F_j}$ fixed subset of M^{2n}. Locally standardness of the manifold M^{2n} imply that the space $\mathfrak{p}^{-1}(F_j) = M_j^{2(n-1)}$ is a $\mathbb{T}^{n-1} \cong \mathbb{T}^n/\mathbb{T}^n_{F_j}$ manifold. The $\mathbb{T}^{n-1}$ action on $M_j^{2(n-1)}$ is locally standard and restriction of $\mathfrak{p}$ on $M_j^{2(n-1)}$ satisfy the condition (2) of 1.2.3. Hence $M_j^{2(n-1)}$ is a $2(n-1)$-dimensional quasitoric manifold over F_j, called the characteristic submanifold of M^{2n} corresponding to F_j. The isotropy subgroup $\mathbb{T}^n_{F_j}$ may be identified to the elements of $\mathbb{T}^n$ as

$$\mathbb{T}^n_{F_j} = \{(e^{2\pi i \lambda_{1j} r}, \ldots, e^{2\pi i \lambda_{nj} r}) \in \mathbb{T}^n \ ; \ \forall \ r \in \mathbb{R}\} \tag{1.2.6}$$

for some primitive vector $\lambda_j = (\lambda_{1j}, \ldots, \lambda_{nj}) \in \mathbb{Z}^n$. The correspondence $\mathbb{T}^n_{F_j}$ to the vector λ_j is one-to-one upto a sign. Hence we can define a function

$$\lambda : \mathcal{F}(Q) \to \mathbb{Z}^n \text{ by } \lambda(F_j) = \lambda_j, \tag{1.2.7}$$

called the characteristic function of M^{2n}. The vectors λ_j's are called the characteristic vector corresponding to F_j.

Since Q is a simple polytope, the codimension-k face F is the intersection of unique collection of k facets $F_{j_1}, \ldots, F_{j_k}$. Then the isotropy group $\mathbb{T}^n_F$ is $\mathbb{T}^n_{F_{j_1}} \times \ldots \times \mathbb{T}^n_{F_{j_k}}$. The group $\mathbb{T}^n_F$ is a direct summand of $\mathbb{T}^n$, since comparing the action of $\mathbb{T}^n$ on M^{2n} to the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$ we can conclude that the span of $\lambda_{j_1}, \ldots, \lambda_{j_k}$ in $\mathbb{Z}^n$ is a k-dimensional direct summand of $\mathbb{Z}^n$. In particular, when unique collection of n facets $F_{j_1}, \ldots, F_{j_n}$ meet at a vertex of Q the corresponding characteristic vectors $\lambda_{j_1}, \ldots, \lambda_{j_n}$ form a basis of $\mathbb{Z}^n$.

Example 1.2.7. *Suppose in example 1.2.5 the projection* $\mathfrak{p} : \mathbb{CP}^2 \to \triangle^2$ *maps*

$$[1, 0, 0] \to (0, 0) = O,\ [0, 1, 0] \to (1, 0) = A \text{ and } [0, 0, 1] \to (0, 1) = B.$$

Hence the characteristic submanifolds are $\mathfrak{p}^{-1}(OA) = \{[z_1 : z_2 : 0] \in \mathbb{CP}^2\}$, $\mathfrak{p}^{-1}(OB) = \{[z_1 : 0 : z_3] \in \mathbb{CP}^2\}$ *and* $\mathfrak{p}^{-1}(AB) = \{[0 : z_2 : z_3] \in \mathbb{CP}^2\}$. *Suppose*

$$a = [1/\sqrt{2} : 1/\sqrt{2} : 0],\ b = [1/\sqrt{2} : 0 : 1/\sqrt{2}] \text{ and } c = [0 : 1/\sqrt{2} : 1/\sqrt{2}].$$

Then $\mathbb{T}^2_a = \mathbb{T}^2_{OA}$, $\mathbb{T}^2_b = \mathbb{T}^2_{OB}$ *and* $\mathbb{T}^2_c = \mathbb{T}^2_{AB}$. *Suppose the isomorphism* $\mathbb{T}^2 \cong U(1)^3/S^1$ *is given by* $(t_1, t_2) \to [t_1 : t_2 : 1]$. *We can show that*

$$\mathbb{T}^2_a = \{(e^{2\pi i r}, e^{2\pi i r}) \in \mathbb{T}^2 : r \in \mathbb{R}\},\ \mathbb{T}^2_b = \{(e^{2\pi i 0}, e^{2\pi i r}) \in \mathbb{T}^2 : r \in \mathbb{R}\}$$

$$\text{and } \mathbb{T}^2_c = \{(e^{2\pi i r}, e^{2\pi i 0}) \in \mathbb{T}^2 : r \in \mathbb{R}\}.$$

Hence the characteristic function of $\mathbb{CP}^2$ *associated to the* $\mathbb{T}^2$ *action of example 1.2.5 is the function* $\lambda : \mathcal{F}(\triangle^2) \to \mathbb{Z}^2$ *such that* $\lambda(OA) = (1, 1)$, $\lambda(OB) = (0, 1)$, $\lambda(AB) = (1, 0)$.

We give the definition of characteristic model combinatorially followed by the construction of quasitoric manifold from this characteristic model. Let Q be an n-dimensional simple polytope and $\mathcal{F}(Q)$ be the set of facets of Q.

Definition 1.2.8. *A function* $\xi : \mathcal{F}(Q) \to \mathbb{Z}^n$ *is called characteristic function if the span of* $\xi(F_{j_1}), \ldots, \xi(F_{j_k})$ *is a* k*-dimensional direct summand of* $\mathbb{Z}^n$ *whenever the intersection* $F_{j_1} \cap \ldots \cap F_{j_k}$ *is nonempty.*

The vectors $\xi_j := \xi(F_j)$ *are called characteristic vectors and the pair* (Q, ξ) *is called a characteristic model.*

The orbit space of the standard $\mathbb{T}^n$ action on $\mathbb{C}^n$ is the positive octant $\mathbb{R}^n_{\geq 0}$. Consider the canonical projection $\mathfrak{p_c} : \mathbb{T}^n \times \mathbb{R}^n_{\geq 0} \to \mathbb{C}^n$ given by

$$(t_1, \ldots, t_n) = t \times (x_1, \ldots, x_n) = x \longmapsto (t_1x_1, \ldots t_nx_n) = tx. \tag{1.2.8}$$

Let $\mathfrak{p_c}(t, x) = \widetilde{x}$. The fiber over $\widetilde{x} \in \mathbb{R}^n$ of the projection $\mathfrak{p_c}$ is the isotropy group of $\widetilde{x}$ of the $\mathbb{T}^n$ action on $\mathbb{C}^n$. Denote this isotropy group by $\mathbb{T}^n_x$. So we can identify $\mathbb{C}^n$ with the quotient space $\mathbb{T}^n \times \mathbb{R}^n_{\geq}/\sim_0$, where the equivalence relation $\sim_0$ is defined by

$$(t, x) \sim_0 (s, y) \text{ if and only if } x = y \text{ and } ts^{-1} \in \mathbb{T}^n_x. \tag{1.2.9}$$

Let for each vertex $v \in Q$, $C_v = \{p \in F : F \text{ is a face of } Q \text{ not containing } v\}$. Let U_v be the open subset of Q complement to the set C_v. Hence U_v is diffeomorphic to $\mathbb{R}^n_{\geq 0}$ as manifold with corner. We consider an identifications on faces of the product $\mathbb{T}^n \times U_v$ similarly to the standard equivalence relation $\sim_0$ for each vertex v. Gluing them naturally one can reconstruct a quasitoric manifold from any characteristic model.

Theorem 1.2.9 (Subsection 1.5, [DJ91])**.** *A quasitoric manifold can be constructed from a characteristic model.*

Proof. Let Q be a simple n-polytope and (Q, ξ) be a characteristic model. A codimension-k face F of the polytope Q is the intersection $F_{j_1} \cap \ldots \cap F_{j_k}$ of unique collection of k facets $F_{j_1}, \ldots, F_{j_k}$ of Q. Let $\mathbb{Z}(F)$ be the submodule of $\mathbb{Z}^n$ generated by the characteristic vectors $\xi(F_{j_1}), \ldots, \xi(F_{j_k})$. The module $\mathbb{Z}(F)$ is a direct summand of $\mathbb{Z}^n$. Therefore the torus $\mathbb{T}_F := (\mathbb{Z}(F) \otimes_{\mathbb{Z}} \mathbb{R})/\mathbb{Z}(F)$ is a direct summand of $\mathbb{T}^n$. Define $\mathbb{Z}(Q) = (0)$ and $\mathbb{T}_Q$ to be the trivial subgroup of $\mathbb{T}^n$. If $p \in Q$, p belongs to relative interior of a unique face F of Q.

Define an equivalence relation $\sim$ on the product $\mathbb{T}^n \times Q$ by

$$(t, p) \sim (s, q) \text{ if and only if } p = q \text{ and } s^{-1}t \in \mathbb{T}_F \tag{1.2.10}$$

where F is the unique face containing p in its relative interior. Let

$$M(Q, \xi) := (\mathbb{T}^n \times Q)/\sim$$

be the quotient space. The natural left action of $\mathbb{T}^n$ on $\mathbb{T}^n \times Q$ is given by

$$t \times (s \times p) \mapsto ts \times p \text{ for all } t, s \in \mathbb{T}^n \text{ and } p \in Q. \tag{1.2.11}$$

This action induces a natural action of $\mathbb{T}^n$ on $M(Q, \xi)$. Then $M(Q, \xi)$ is a $\mathbb{T}^n$-space.

The projection onto the second factor of $\mathbb{T}^n \times Q$ descends to the quotient map

$$\mathfrak{p} : M(Q, \xi) \to Q \text{ defined by } \mathfrak{p}([t, p]) = p. \tag{1.2.12}$$

So the orbit space of this $\mathbb{T}^n$ action on $M(Q, \xi)$ is the polytope Q. We show that the space $M(Q, \xi)$ has the structure of a quasitoric manifold. The explanations are discussed in the following paragraph.

Consider an open neighborhood U_v of the vertices v of Q where U_v is defined in the paragraph before the statement of the theorem 1.2.9. Let

$$M_v(Q, \xi) := \mathfrak{p}^{-1}(U_v) = (\mathbb{T}^n \times U_v)/\sim . \tag{1.2.13}$$

Let the facets $F_{j_1}, \ldots, F_{j_n}$ meet at the vertex v. So the facets of U_v are $F_{j_1}, \ldots, F_{j_n}$. Suppose $\phi : U_v \to \mathbb{R}^n_{\geq 0}$ is a diffeomorphism such that ϕ sends the facets $F_{j_1}, \ldots, F_{j_n}$ to the facets

$$\{x_1 = 0\} \cap \mathbb{R}^n_{\geq 0}, \ \ldots, \ \{x_n = 0\} \cap \mathbb{R}^n_{\geq 0} \text{ of } \mathbb{R}^n_{\geq 0}$$

respectively. Where $x_j = 0$ is the j-th coordinate hyperplane in $\mathbb{R}^n$ for $j = 1, \ldots, n$.

Let δ_v be the automorphism of $\mathbb{T}^n$ corresponding to the automorphism of $\mathbb{Z}^n$ obtained by sending the basis vectors $\xi_{j_1}, \ldots, \xi_{j_n}$ to the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{Z}^n$ respectively. Since the quotient maps $\mathfrak{p}$ and $\mathfrak{p}_{\mathfrak{c}}$ of equation 1.2.8 and 1.2.12 respectively are continuous surjections and ϕ is a diffeomorphism, the following commutative diagram ensure us that the lower horizontal map ϕ_v is a homeomorphism.

$$\begin{array}{ccc} (\mathbb{T}^n \times U_v) & \xrightarrow{\delta_v \times \phi} & (\mathbb{T}^n \times \mathbb{R}^n_{\geqslant 0}) \\ \mathfrak{p}\big\downarrow & & \mathfrak{p}_{\mathfrak{c}}\big\downarrow \\ M_v(Q, \xi) & \xrightarrow{\phi_v} & \mathbb{C}^n \end{array} \tag{1.2.14}$$

Again the commutativity of the following diagram show that the space $M_v(Q, \xi)$ is $\mathbb{T}^n$-stable and $\mathbb{T}^n$ action on $M_v(Q, \xi)$ satisfy the relation $\phi_v(t \cdot z) = \delta_v(t) \cdot \phi_v(z)$.

$$\begin{array}{ccc} \mathbb{T}^n \times (\mathbb{T}^n \times U_v) & \xrightarrow{Id \times \delta_v \times \phi} & \mathbb{T}^n \times (\mathbb{T}^n \times \mathbb{R}^n_{\geqslant 0}) \\ Id \times \mathfrak{p}\big\downarrow & & Id \times \mathfrak{p}_{\mathfrak{c}}\big\downarrow \\ \mathbb{T}^n \times M_v(Q, \xi) & \xrightarrow{Id \times \phi_v} & \mathbb{T}^n \times \mathbb{C}^n \\ \mathfrak{a}_{\mathfrak{v}}\big\downarrow & & \mathfrak{a}_{\mathfrak{s}}\big\downarrow \\ M_v(Q, \xi) & \xrightarrow{\phi_v} & \mathbb{C}^n \end{array} \tag{1.2.15}$$

Where $\mathfrak{a}_{\mathfrak{v}}$ is the restriction of the $\mathbb{T}^n$ action on $M(Q, \xi)$ to the subset $M_v(Q, \xi)$ and $\mathfrak{a}_{\mathfrak{s}}$ is the standard $\mathbb{T}^n$ action on $\mathbb{C}^n$. Since $\{U_v\}_{v \in V(Q)}$ is an open covering of Q, the $\mathbb{T}^n$

stable subsets $\{M_v(Q,\xi)\}$ is an open covering of $M(Q,\xi)$. From the diagram 1.2.15 we get that every l-dimensional orbit in $M_v(Q,\xi)$ maps to a point in the relative interior of a l-dimensional face of U_v. Hence (Q,ξ) is a $2n$-dimensional quasitoric manifold. □

Definition 1.2.10. *The manifold $M(Q,\xi)$ is called the quasitoric manifold derived from the characteristic model (Q,ξ).*

Remark 1.2.11. *The signs of the characteristic vectors $\{\xi_j\}$ in the characteristic model do not affect the groups $\mathbb{T}_F$. Hence they do not change the homeomorphism class of $M(Q,\xi)$. However the signs of the characteristic vectors ξ_j affect the action of $\mathbb{T}^n$. If the polytope of a characteristic model is replaced by a diffeomorphic polytope, we derive the same quasitoric manifold modulo an equivariant homeomorphism.*

In the following examples we use two notions, namely orientation and connected sum of quasitoric manifolds which we discuss in the sections 1.6 and 1.7.

Example 1.2.12. *Let Q be a triangle $\triangle^2$ in $\mathbb{R}^2$. The possible characteristic maps are indicated by the following Figure 1.1. The quasitoric manifold corresponding to the*

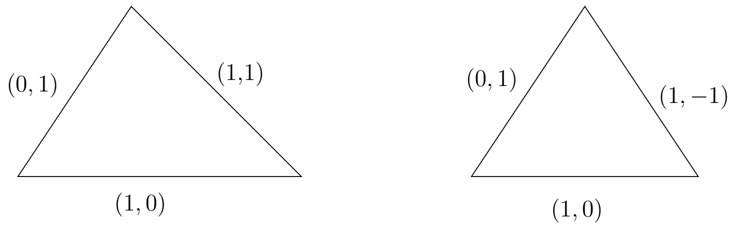


Figure 1.1: The characteristic models corresponding to a triangle.

first characteristic model is $\mathbb{CP}^2$ with the usual $\mathbb{T}^2$ action. The orientation on $\mathbb{CP}^2$ is the standard orientation. The second correspond to the same $\mathbb{T}^2$ action with the reverse orientation on $\mathbb{CP}^2$, we denote it by $\overline{\mathbb{CP}}^2$. The similar considerations can apply whenever Q is an n-dimensional simplex $\triangle^n$. So the quasitoric manifold over $\triangle^n$ is either $\mathbb{CP}^n$ or $\overline{\mathbb{CP}}^n$.

Example 1.2.13. *Suppose Q is combinatorially a square in $\mathbb{R}^2$. In this case there are many possible characteristic maps. Some examples are given by the Figure 1.2.*

The first characteristic pairs may construct an infinite family of 4-dimensional quasitoric manifolds, denote them by M^4_k for each $k \in \mathbb{Z}$. The manifolds $\{M^4_k : k \in \mathbb{Z}\}$ are equivariantly distinct. Let $L(k)$ be the complex line bundle over $\mathbb{CP}^1$ with the first Chern class k. The complex manifold $\mathbb{CP}(L(k) \oplus \mathbb{C})$ is the Hirzebruch surface for the integer k, where $\mathbb{CP}(\cdot)$ denotes the projectivisation of a complex bundle. So each Hirzebruch surface is the total space of the bundle $\mathbb{CP}(L(k) \oplus \mathbb{C}) \to \mathbb{CP}^1$ with fiber $\mathbb{CP}^1$. In [Oda88] the author shows that with the natural action of $\mathbb{T}^2$ on $\mathbb{P}(L(k) \oplus \mathbb{C})$ it is equivariantly

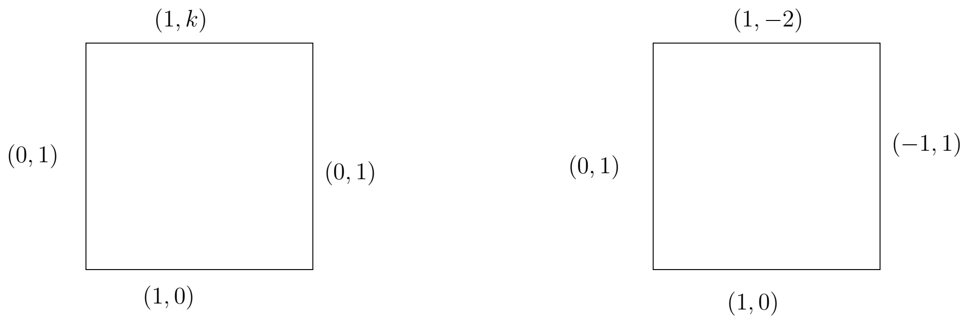


Figure 1.2: Some characteristic models corresponding to a square.

homeomorphic to M_k^4 *for each* k. *That is, with respect to the* $\mathbb{T}^2$*-action, Hirzebruch surfaces are quasitoric manifolds where the orbit space is a combinatorial square and the corresponding characteristic map is described in Figure 1.2.*

Note that the second combinatorial model gives the quasitoric manifold $\mathbb{CP}^2 \# \mathbb{CP}^2$, *the equivariant connected sum of* $\mathbb{CP}^2$.

Remark 1.2.14. *Orlik and Raymond ([OR70], p. 553) showed that any* 4*-dimensional quasitoric manifold* M^4 *over* 2*-dimensional simple polytope is an equivariant connected sum of some copies of* $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ *and* M_k^4 *for some* $k \in \mathbb{Z}$.

This classification result is used in Chapter 5. In Chapter 2 we show that there do not exist any combinatorial model corresponding to some simple polytope.

Lemma 1.2.15 ([Dav78]). *Suppose* $\mathfrak{p} : M^{2n} \to Q$ *is a smooth quasitoric manifold. Then there exists a continuous section* $s : Q \to M^{2n}$.

Proof. We give an outline of the proof. Suppose E is a smooth vector bundle of rank k over a manifold M'. Let E_0 denote the complement of the zero section. The positive real numbers $\mathbb{R}_{>0}$ act on E and E_0 by fiberwise scalar multiplication. Consider the $\mathbb{R}_{>0}$ action on the product $E_0 \times [0, \infty)$ defined by $r(x, u) = (xr^{-1}, ru)$. So the quotient space

$$C_+E := E_0 \times_{\mathbb{R}_{>0}} [0, \infty)$$

is a smooth bundle over M' with fibers homeomorphic to $S^{k-1} \times [0, \infty)$. Denote the image of (x, u) in C_+E by $[x, u]$. The boundary of C_+E is the sphere bundle

$$E_0 \times_{\mathbb{R}_{>0}} \{0\} \cong E_0/\mathbb{R}_{>0}.$$

There is a canonical projection map $\mathfrak{pr} : C_+E \to E$ defined by $\mathfrak{pr}[x, u] = ux$ which is a diffeomorphism away from the zero section. Replacing E by C_+E is called a *real blow-up*. The projection $\mathfrak{pr}$ may be called a *real blow-down*.

We can stratify $M^{2n} = \bigsqcup M(F^\circ, \xi)$ where F varies over all faces of Q. Then a neighborhood of $M(F^\circ, \xi)$ in M^{2n} is diffeomorphic to a $\mathbb{R}^{2k}$-bundle over $M(F^\circ, \xi)$, where

k is the codimension of F. We start with the minimal strata (vertices) progressively blow-up M^{2n} along the strata of increasing dimension to finally get a smooth manifold with boundary $\widehat{M}$. The precise description may be found in Chapter 4 of [Dav78]. One can show that $\widehat{M}$ is homeomorphic to $\mathbb{T}^n \times Q$. The canonical blow-downs $\mathfrak{pr}$ are combined to give a continuous map $\mu : \widehat{M} \to M^{2n}$. Now choose a continuous section $\widehat{s} : Q \to \widehat{M}$ and compose with μ to get a continuous section $s : Q \to M^{2n}$. □

Remark 1.2.16. *We will show in Section 1.9 that every quasitoric manifold has a smooth structure. However the above lemma remains valid even if we drop the smoothness assumption.*

Corollary 1.2.17 (Lemma 1.8, [DJ91])**.** *Suppose M^{2n} is a quasitoric manifold with characteristic model (Q, ξ). Suppose $M(Q, \xi)$ is a quasitoric manifold derived from (Q, ξ). Then $M(Q, \xi)$ is equivariantly homeomorphic to M^{2n}.*

Proof. Let $s : Q \to M^{2n}$ be a continuous section. Consider the composition map

$$g : \mathbb{T}^n \times Q \xrightarrow{Id \times s} \mathbb{T}^n \times M^{2n} \xrightarrow{\cdot} M^{2n}, \tag{1.2.16}$$

where $\cdot$ is the $\mathbb{T}^n$ action on M^{2n}. From the locally standardness of $\mathbb{T}^n$ action it is clear that the fiber $g^{-1}(x)$ of each $x \in M^{2n}$ is the isotropy group of x. Hence the map g factors through the maps $\mathfrak{p}$ and f in following commutative diagram.

$$\begin{array}{ccc} \mathbb{T}^n \times Q & \xrightarrow{\mathfrak{p}} & (\mathbb{T}^n \times Q)/\sim \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ M^{2n} & = & M^{2n} \end{array} \tag{1.2.17}$$

Clearly the map f is a bijection. The diagram gives that f is an equivariant map. Again the locally standard property of $\mathbb{T}^n$ action ensure that f is a homeomorphism. □

Definition 1.2.18. *Let M_1^{2n} and M_2^{2n} be quasitoric manifolds whose associated base polytope is Q. Let δ be an automorphism of $\mathbb{T}^n$. A map $f : M_1^{2n} \to M_2^{2n}$ is called a δ-equivariant homeomorphism if f is a homeomorphism and satisfies $f(t \cdot x) = \delta(t) \cdot f(x)$ for all $(x, t) \in M_1^{2n} \times \mathbb{T}^n$.*

Two δ-equivariant homeomorphisms $f : M_1^{2n} \to M_2^{2n}$ and $g : M_1^{2n} \to M_2^{2n}$ are said to be *equivalent* if there exist equivariant homeomorphisms $h_j : M_j^{2n} \to M_j^{2n}$, for $j = 1, 2$, such that the following diagram is commutative.

$$\begin{array}{ccc} M_1^{2n} & \xrightarrow{f} & M_2^{2n} \\ {\scriptstyle h_1}\big\downarrow & & {\scriptstyle h_2}\big\downarrow \\ M_1^{2n} & \xrightarrow{g} & M_2^{2n} \end{array} \tag{1.2.18}$$

For the automorphism δ we also define the δ-*translation* of a characteristic model (Q, ξ) to be the pair $(Q, \delta' \circ \xi)$, where δ' is an automorphism of $\mathbb{Z}^n$ induced by the automorphism δ. We can determine quasitoric manifolds over a fixed polytope up to δ-equivariant homeomorphism class using the following lemma.

Lemma 1.2.19 (Proposition 2.6 of [BR01]). *For any automorphism δ of $\mathbb{T}^n$, the assignment of characteristic models defines a bijection between equivalence classes of δ-equivariant homeomorphisms of quasitoric manifolds over Q and δ-translations of characteristic model (Q, ξ).*

Proof. First we show that the inverse assignment is given by consisting the quasitoric manifolds derived from characteristic models (Q, ξ) and $(Q, \delta' \circ \xi)$. To each δ-translation $(Q, \xi) \to (Q, \delta' \circ \xi)$ we associate the δ-equivariant diffeomorphism

$$(\delta \times Id)^{\sim} : (\mathbb{T}^n \times Q)/ \sim \to (\mathbb{T}^n \times Q)/ \sim_{\delta},$$

where

$$(t, q) \sim_{\delta} (u, q) \text{ if and only if } tu^{-1} \in \delta'(\xi)(\mathbb{T}_F). \tag{1.2.19}$$

Here F is the unique face containing p in its relative interior. It is clear from the definitions 1.2.10 and 1.2.19 that $(\delta \times Id)^{\sim}$ descends to the original δ-translation $(Q, \xi) \to (Q, \delta'(\xi))$ of characteristic models.

Conversely, let $f : M_1^{2n} \to M_2^{2n}$ be a δ-equivariant homeomorphism of quasitoric manifolds over Q. This diffeomorphism descends to a δ-translation of characteristic models. Let $(\delta \times Id)^{\sim}$ be the δ-equivariant homeomorphism derived from the corresponding δ-translation of characteristic models. The preferred section $s_1 : Q \to M_1^{2n}$ automatically extends to an equivariant homeomorphism $S_1 : (\mathbb{T}^n \times Q)/ \sim \to M_1^{2n}$. Let $s_2 = f \circ s_1$. Since f is a δ-equivariant homeomorphism, $s_2 : Q \to M_2^{2n}$ is a section. The section s_2 extends to an eqivariant homeomorphism $S_2 : (\mathbb{T}^n \times Q)/ \sim_{\delta} \to M_2^{2n}$. Thus $f \circ S_1 = S_2 \circ (\delta \times Id)^{\sim}$. Hence f and $(\delta \times Id)^{\sim}$ are equivalent, as required. □

1.3 Invariant closed submanifolds

Corresponding to the faces of the polytope Q there are certain $\mathbb{T}^n$-invariant submanifolds of M^{2n}. If F is a face of Q of codimension-k, then define $M(F, \xi) := \mathfrak{p}^{-1}(F)$. Define $\mathbb{Z}^{\perp}(F) = \mathbb{Z}^n/\mathbb{Z}(F)$. Let $\varrho_F : \mathbb{Z}^n \to \mathbb{Z}^{\perp}(F)$ be the projection homomorphism. Let $J(F) \subset \mathcal{F}(Q)$ be the index set of facets of Q, other than F in case $k = 1$, that intersect F. Observe that $J(F)$ indexes the set of facets of the $(n-k)$-dimensional polytope F. Let $\{H_j : j \in J(F)\}$ be the set of all facets of F. So $H_j = F \cap F_j$ for some facets F_j of Q, $\quad j \in J(F)$. If ξ is the characteristic function of M^{2n} over Q, the assignment

$$\xi_F(H_j) := \varrho_F \circ \xi(F_j),\ j \in J(F), \tag{1.3.1}$$

defines the characteristic function ξ_F of the quasitoric manifold $M(F, \xi_F)$. With subspace topology on $M(F, \xi)$, $M(F, \xi)$ is equivariantly homeomorphic to $M(F, \xi_F)$. Hence $M(F, \xi)$ is a quasitoric manifold of dimension $2n - 2k$. One can show that if M' is an invariant closed submanifold of M^{2n} then $M' = M(F, \xi)$ for some face F of Q.

Definition 1.3.1. *When F is a facet of Q, the space $M(F, \xi_F)$ is called a characteristic submanifold of M^{2n} corresponding to F.*

1.4 Face vectors and face ring of polytopes

The face vector or *f-vector* is an important concept in the combinatorics of polytopes. Let $\mathfrak{L}$ be a simplicial n-polytope and f_j be the number of j-dimensional faces of $\mathfrak{L}$. The integer vector $f(\mathfrak{L}) = (f_0, \ldots, f_{n-1})$ is called the *f-vector* of the simplicial polytope $\mathfrak{L}$. Let h_i be the coefficients of t^{n-i} in the polynomial

$$(t-1)^n + \Sigma_0^{n-1} f_i (t-1)^{n-1-i}. \tag{1.4.1}$$

The vector $h(\mathfrak{L}) = (h_0, \ldots, h_n)$ is called *h-vector* of $\mathfrak{L}$. Obviously $h_0 = 1$, and $\Sigma_0^n h_i = f_{n-1}$. The *f-vector* and *h-vector* of a simple n-polytope Q is the *f-vector* and *h-vector* of its dual simplicial polytope Q^* respectively, that is

$$f(Q) = f(Q^*) \text{ and } h(Q) = h(Q^*).$$

Hence for a simple n-polytope Q,

$$f(Q) = (f_0, \ldots, f_{n-1}), \tag{1.4.2}$$

where f_j is the number of codimension $(j+1)$ faces of Q. Then $h_n = 1$ and $\Sigma_1^n h_i$ is the number of vertices of Q. The face vectors are a combinatorial invariant of polytopes, that is it depends only on the face poset of the polytope.

Let $w_1, \ldots, w_m$ be the vertices of a simplicial complex $\mathfrak{L}$. Let $\mathcal{R}$ be a commutative ring with unity. Consider the polynomial ring $\mathcal{R}[w_1, \ldots, w_m]$ where the w_i's are indeterminates. Let $I_{\mathfrak{L}}$ be the homogeneous ideal generated by all monomials of the form $w_{i_1} \ldots w_{i_l}$ such that the set of vertices $\{w_{i_1}, \ldots, w_{i_l}\}$ does not span a simplex in $\mathfrak{L}$.

Definition 1.4.1. *The face ring or Stanley-Reisner ring of $\mathfrak{L}$ with coefficients in $\mathcal{R}$ is the quotient ring $\mathcal{R}[w_1, \ldots, w_m]/I_{\mathfrak{L}}$, denoted by $SR(\mathfrak{L}, \mathcal{R})$.*

We define the face ring of a polytope Q. Let $\mathfrak{L}$ be the simplicial complex dual to Q.

Definition 1.4.2. *The face ring or Stanley-Reisner ring $SR(Q, \mathcal{R})$ of Q over $\mathcal{R}$ is the ring $SR(\mathfrak{L}, \mathcal{R})$. The face ring is graded by declaring the degree of each v_i.*

1.5 Homology groups of quasitoric manifolds

Following [DJ91] we compute the homology groups of quasitoric manifolds with coefficients in $\mathbb{Z}$. The combinatorial type of the orbit space (essentially a simple convex polytope) of quasitoric manifold makes the computation of the singular homology groups easier. First we decompose the polytope into a disjoint union of relative open subsets such that this collections correspond to the set of vertices of polytope bijectively. Corresponding to each of these relative open subsets we get an even dimensional cell. These cells give a perfect CW-complex structure on the manifold. We decompose the polytope using the notion of index of a vertex that will describe the degree of each cell. Though there is no canonical choice of index, the homology group of any degree can be computed up to isomorphism.

Let $M^{2n} \xrightarrow{\mathfrak{p}} Q$ be a quasitoric manifold of dimension $2n$. Suppose Q is a simple polytope in $\mathbb{R}^n$. Choose a vector $z \in \mathbb{R}^n$ which is not perpendicular to any line joining two vertices of Q. Let $\zeta : \mathbb{R}^n \to \mathbb{R}$ be the linear functional defined by

$$\zeta(x) := \langle x, z \rangle \text{ for all } x \in \mathbb{R}^n. \tag{1.5.1}$$

Where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. Choice of z distinguishes the vertices linearly according to ascending value of ζ. Since ζ is linear, we make the 1-skeleton of Q into a directed graph by orienting each edge such that ζ increases along it. For each vertex v of Q the number of incident edges that point towards v is called its *index*, denoted by $\mathfrak{f}(v)$.

Let F_v denote the smallest face of Q which contains the inward pointing edges incident to v. Since Q is simple polytope and ζ is a linear functional distinguishing the vertices of Q, such a face F_v exist uniquely corresponding to each vertex v. Then $\dim F_v = \mathfrak{f}(v)$ and if F' is a face of Q with top vertex v then F' is a face of F_v. By top vertex we mean that $\mathfrak{f}(v) > \mathfrak{f}(u)$ for all vertices u other than v. Let $\widehat{F}_v$ be the relative open subset F_v obtain by deleting all faces of F_v not containing v. So $Q = \sqcup_v \widehat{F}_v$, where v run over the vertices of Q. The space $\widehat{F}_v$ is diffeomorphic to the positive octant $\mathbb{R}^{\mathfrak{f}(v)}_{\geq 0}$.

Some combinatorial arguments show that the number of vertices v with $\mathfrak{f}(v) = j$ is h_j for $j = 0, \ldots, n$. For each vertex v put $e_v = \mathfrak{p}^{-1}(\widehat{F}_v)$. By the locally standard property at the fixed point corresponding to the vertex v we can show

$$e_v \cong (\mathbb{T}_{F_v} \times \mathbb{T}^{\mathfrak{f}(v)} \times \widehat{F}_v)/\sim \ \cong (\mathbb{T}^{\mathfrak{f}(v)} \times \mathbb{R}^{\mathfrak{f}(v)}_{\geq 0})/\sim_0 \ \cong D^{2\mathfrak{f}(v)}, \tag{1.5.2}$$

where $D^{2\mathfrak{f}(v)}$ is an open disk in $\mathbb{R}^{2\mathfrak{f}(v)}$. Hence e_v is a $2\mathfrak{f}(v)$-dimensional cell in M^{2n}.

So M^{2n} can be given the structure of a CW-complex structure as follows. Define the $2k$-skeleton

$$M_{2k} := \bigcup_{\mathfrak{f}(v)=k} M(F_v, \xi) \text{ for } 0 \leq k \leq n. \tag{1.5.3}$$

Define $M_{2k+1} = M_{2k}$ for $0 \leq k \leq n-1$, and $M_{2n} = M^{2n}$. The $2k$-skeleton M_{2k} can be obtained from M_{2k-1} by attaching those cells e_v for which $\mathfrak{f}(v) = k$. The attaching maps are to be described.

Define $\mathbb{Z}^{\perp}(F) := \mathbb{Z}^n/\mathbb{Z}(F)$. Then

$$\mathbb{T}(F)^{\perp} := \mathbb{Z}^{\perp}(F) \otimes \mathbb{R}/\mathbb{Z}^{\perp}(F) \cong \mathbb{T}^n/\mathbb{T}(F). \tag{1.5.4}$$

Let $\sim$ be the equivalence relation such that $M(F_v, \xi) = F_v \times \mathbb{T}(F_v)^{\perp}/\sim$. The disk $\overline{D}^{2\mathfrak{f}(v)})$ can be identified with $\mathbb{T}(F_v)^{\perp} \times F_v/\approx$ where

$$(t, p) \approx (s, q) \text{ if } p = q \text{ and } s^{-1}t \in \mathbb{T}(F')/\mathbb{T}(F_v) \tag{1.5.5}$$

where F' is the minimal face of F_v containing p whose top vertex is v. The attaching map $S^{2\mathfrak{f}(v)-1} \to M_{2\mathfrak{f}(v)-1}$ is the natural quotient map from

$$(F_v - \widehat{F}_v) \times \mathbb{T}(F_v)^{\perp}/\approx \,\to (F_v - \widehat{F}_v) \times \mathbb{T}(F_v)^{\perp}/\sim. \tag{1.5.6}$$

Hence M^{2n} is a CW-complex with no odd dimensional cells and with $\mathfrak{f}^{-1}(k) = h_k$ number of $2k$ dimensional cells. Hence by cellular homology theory

$$H_p(M^{2n}; \mathbb{Z}) = \begin{cases} \bigoplus_{h_k} \mathbb{Z} & \text{if } 0 \leq p \leq n \text{ and } p \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.5.7}$$

1.6 Orientation of quasitoric manifolds

Let $M^{2n} \xrightarrow{\mathfrak{p}} Q$ be a $2n$-dimensional quasitoric manifold. From the previous section 1.5 we get that the top homology group of M^{2n} is $\mathbb{Z}$. So the manifold M^{2n} is orientable. In fact a choice of orientation on $\mathbb{T}^n$ and Q gives an orientation on M^{2n}. We fix the standard orientation on $\mathbb{T}^n$. Hence an orientation on $Q \subset \mathbb{R}^n$ determines an orientation on M^{2n}.

Suppose the manifold M^{2n} has a smooth structure. Clearly the isotropy group of a characteristic submanifold is a circle subgroup of $\mathbb{T}^n$. So there is a natural S^1 action on the normal bundle of that characteristic submanifold. Thus the normal bundle has a complex structure and consequently an orientation. Whenever the sign of the characteristic vector of a facet is reverse, we get the opposite orientation on the normal bundle. An orientation on the normal bundle together with an orientation on M^{2n} induces an orientation on the characteristic submanifold. A structure called *omniorientation* provides a combinatorial description for a stable complex structure on quasitoric manifold.

Definition 1.6.1. *An omniorientation on a quasitoric manifold M^{2n} is a choice of orientation for M^{2n} as well as for each characteristic submanifold of M^{2n}.*

If the polytope Q has m facets, there are 2^{m+1} possible omniorientations for M^{2n}. From the above discussion we get the following remark.

Remark 1.6.2. *A choice of omniorientation is equivalent to a choice of orientation for Q and a choice of sign for each characteristic vector.*

We will apply the same terminology in the case of quasitoric orbifolds in Chapter 3.

Example 1.6.3. *Consider a triangle whose edges in counterclockwise order have characteristic vectors $(1,0),(0,1)$ and $(1,1)$. The corresponding manifold is $\mathbb{CP}^2$.*

Now consider the orientation reversing map on $\mathbb{R}^2$ that maps $(1,0) \mapsto (0,1)$ and $(0,1) \mapsto (1,-1)$. Then $(1,1) \mapsto (1,0)$. Rotating the triangle observe that $\overline{\mathbb{CP}}^2$ has characteristic vectors $(1,0),(0,1),(1,-1)$. There are other choices of characteristic vectors for $\overline{\mathbb{CP}}^2$ such as those given by applying an orientation reversing automorphism of $\mathbb{R}^2$ to the standard one.

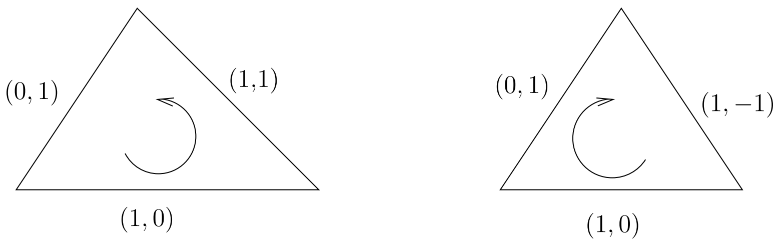


Figure 1.3: Omniorientation of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$.

1.7 Equivariant connected sums

Following [BR01] we define connected sum of polytopes to perform the equivariant connected sums of quasitoric manifolds. Let Q_1, Q_2 be two n-dimensional polytopes in $\mathbb{R}^n$. Consider the polyhedral template

$$\Gamma = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_2, \ldots, x_n \text{ and } x_2 + \ldots + x_n \leq 1\}.$$

Let

$$G_j = \{(x_1, x_2, \ldots, x_n) \in \Gamma : x_j = 0\} \text{ for } 2 \leq j \leq n,$$

and

$$G_1 = \{(x_1, x_2, \ldots, x_n) \in \Gamma : x_2 + \ldots + x_n = 1\}.$$

So $G_1, G_2, \ldots, G_n$ are the facets of Γ. The sets $\Gamma, G_1, \ldots, G_n$ are divided into two halves, namely positive and negative halves, determined by the sign of the coordinate x_1. Let v and w are two distinct vertices of Q_1 and Q_2 respectively. Considering the local orientation at $v, w \in \mathbb{R}^n$, we order the facets of Q_1 meeting at v as F'_j and the facets of Q_2 meeting at w as F''_j for $1 \leq j \leq n$. Let C_v and C_w are union of facets not containing the vertices v and w of Q_1 and Q_2 respectively. Suppose ϕ_{Q_1} is a projective transformation which maps v to $x_1 = +\infty$ and embeds Q_1 into Γ such that following two conditions are satisfied;

1. The hyperplane defining F'_j is identified with the hyperplane defining G_j, for each $1 \leq j \leq n$.
2. The images of the hyperplanes defining C_v under the map ϕ_{Q_1} belong to the negative half of Γ.

Suppose $v_1, v_2, \ldots, v_n$ are vertices of C_v such that there is an edge joining v and v_j for for each $1 \leq j \leq n$. We may define the map ϕ_{Q_1} in the following way. Let

$$\triangle^{n-1} = \{(-1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} : 0 \leq x_2, \ldots, x_n \text{ and } x_2 + \ldots + x_n \leq 1\}.$$

Let ϕ'_{Q_1} be an affine equivalence mapping which sends v and $v_1, v_2, \ldots, v_n$ to $(1, 0, \ldots, 0)$ and vertices of $\triangle^{n-1}$ respectively. Consider the map A defined by $A(x) = x/(1 - x_1)$. Then the composition $\phi_{Q_1} = A \circ \phi'_{Q_1}$ of maps ϕ'_{Q_1} and A is the required projective transformation. Similarly we can choose ϕ_{Q_2} such that it sends w to $x_1 = -\infty$ and identifies the hyperplanes defining F''_j and G_j in such a way that the images of the hyperplanes defining C_w belong to the positive half of Γ. We define the connected sum $Q_1 \#_{v;w} Q_2$ of Q_1 at v and Q_2 at w to be the n-dimensional simple polytope determined by all the hyperplanes in $\phi_{Q_1}(C_v)$ and $\phi_{Q_2}(C_w)$ together with G_j for $1 \leq j \leq n$. The connected sum is defined only up to combinatorial equivalence. Different choices for the vertices v and w or the orderings for F'_j and F''_j may affect the combinatorial type of resulting polytope. When the choices are clear, we use the abbreviation $Q_1 \# Q_2$.

Let $M(Q_1, \xi)$ and $M(Q_2, \mu)$ be two quasitoric manifolds over Q_1 and Q_2 with fixed points x and y corresponding to the vertices $v \in Q_1$ and $w \in Q_2$ respectively. We state the following Lemma and Theorem of Buchstaber and Ray without the proof.

Lemma 1.7.1 (Lemma 6.7, [BR01])**.** *Up to δ-translation, we may assume that ξ identifies $\mathbb{T}_{F_j}$ with the j-th coordinate subtorus $\mathbb{T}_j$, for each $1 \leq j \leq n$.*

Applying the previous lemma to both ξ and μ we can define a characteristic function ξ_μ of $Q_1 \# Q_2$

$$\xi_\mu = \begin{cases} \xi(F) & \text{if } F \text{ is a facet in } C_v \\ \mathbb{T}_j & \text{for } F = G_j \text{ and } 1 \leq j \leq n \\ \mu(F) & \text{if } F \text{ is a facet in } C_w. \end{cases} \tag{1.7.1}$$

Theorem 1.7.2 (Theorem 6.9, [BR01]). *The quasitoric manifold $M(Q_1 \# Q_2, \xi_\mu)$ is equivariantly homeomorphic to the connected sum of $M(Q_1, \xi)$ at x and $M(Q_2, \mu)$ at y.*

Example 1.7.3. *Let the triangles $Q_1 = ABC$ and $Q_2 = DEF$ be the orbit space of quasitoric manifolds $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ respectively. Suppose the characteristic vectors along AB, BC, CA, DE, EF and FD are $(1,0), (0,1), (1,1), (0,1), (1,-1)$ and $(1,0)$ respectively. To perform an equivariant connected sum we fix the vertices B and D of Q_1 and Q_2 respectively. Cut off the corners at these vertices by open halves H_1' and H_2' (see the Figure 1.4). Straighten out the remaining portions of the lines AB', CB'' to make them perpendicular to AC. We do the same for the lines ED', FD''. Now identify $B'B''$ with $D'D''$ such that $A, B' = D', E$ lie on a line (say AE) and $C, B'' = D'', F$ lie on the line (say CF). So we get a quadrilateral $Q = AEFC$. Let us retain $(1,0)$, $(1,1)$ and $(0,1)$ as characteristic vectors for AE, AC and CF respectively. That means the characteristic vector $(0,1)$ of DE is mapped to $(1,0)$ and the characteristic vector $(1,0)$ of FD is mapped to $(0,1)$. These determine an orientation reversing isomorphism of $\mathbb{R}^2$. Using this we get that the characteristic vector $(1,-1)$ of EF should transform to the characteristic vector $(-1,1)$ for EF in Q. Thus the quadrilateral $AEFC$ with characteristic vectors $(1,0), (1,-1), (0,1), (1,1)$ as in the Figure 1.4 represents $\mathbb{CP}^2 \# \mathbb{CP}^2$.*

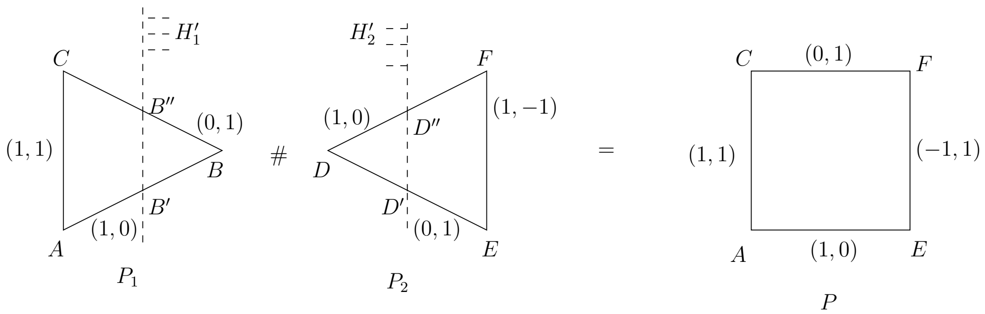


Figure 1.4: Equivariant connected sum for $\mathbb{CP}^2$ and $\mathbb{CP}^2$.

It is well known that $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not have any almost complex structure. This example then shows that not every quasitoric manifold is a toric variety. Note that the quasitoric manifold M_1^4, defined in example 1.2.13, is the equivariant connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

Considering the omniorientation we can construct the omnioriented connected sum of quasitoric manifolds similarly. The Figure 1.5 describe how to perform an omniori-

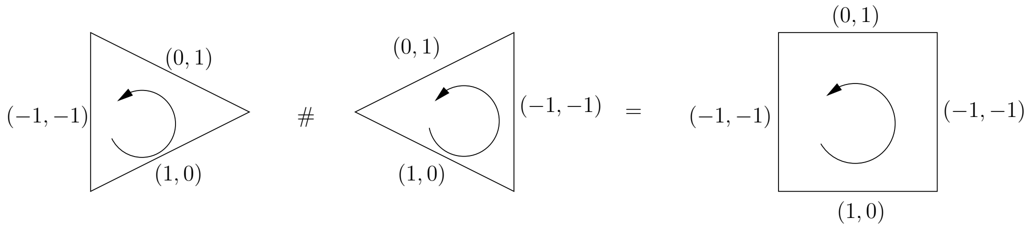


Figure 1.5: Omnioriented connected sum for $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$.

ented connected sum of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$.

1.8 Cohomology ring

We compute the cohomology ring of quasitoric manifold M^{2n} over simple n-polytope Q following [DJ91]. The main idea is to make use of the equivariant cohomology of M^{2n} with respect to the $\mathbb{T}^n$ action. Consider the Borel space

$$BQ := E\mathbb{T}^n \times_{\mathbb{T}^n} M^{2n}$$

of the $\mathbb{T}^n$ action on M^{2n}. Where $E\mathbb{T}^n$ is a contractible space and $\mathbb{T}^n$ action on $E\mathbb{T}^n$ is free. Let 2 be the degree of each $w_i \in SR(Q, \mathbb{Z})$. All homology and cohomology modules in this section will have coefficients in $\mathbb{Z}$. We show that the cohomology ring $H^*(BQ, \mathbb{Z})$ is isomorphic to the face ring $SR(Q, \mathbb{Z})$.

We study the Leray spectral sequence of the fiber bundle $M^{2n} \hookrightarrow BQ \longrightarrow B\mathbb{T}^n$. The spectral sequence degenerates at E_2. Knowledge of the cohomology ring of BQ and $B\mathbb{T}^n$, together with the cohomology group of M^{2n} is sufficient to deduce the cohomology ring of M^{2n}. In our approach we use the localization principle of Atiyah-Bott [AB84].

We show that M^{2n} is the union of $2n$-dimensional disks centered around fixed points of $\mathbb{T}^n$ action in the following. Let $\mathfrak{L}$ be the simplicial complex associated to the boundary of the dual polytope of Q. Then there is a bijective correspondence between $(n-1)$-dimensional faces of $\mathfrak{L}$ and the vertices of Q. Also Q is the cone on the barycentric subdivision of $\mathfrak{L}$. So Q can be written as the union of cubes Q_v where v varies over the vertices of Q. Recall the projection map $\mathfrak{p} : M^{2n} \to Q$. Define

$$M_v := \mathfrak{p}^{-1}(Q_v).$$

Then $M^{2n} = \bigcup_v M_v$. The space BQ has a corresponding decomposition as follows. We regard the k-cube $[0, 1]^k$ as the orbit space of standard k-dimensional torus action on the $2k$-disk

$$\overline{D}^{2k} = \{(z_1, \cdots, z_k) \in \mathbb{C}^k : |z_i| \leq 1\}. \tag{1.8.1}$$

Define

$$BQ_v = E\mathbb{T}^n \times_{\mathbb{T}^n} M_v \simeq E\mathbb{T}^n \times_{\mathbb{T}^n} \overline{D}^{2n}. \tag{1.8.2}$$

Then $BQ = \bigcup_v BQ_v$. Let $\mathfrak{b}_\mathfrak{p} : BQ \to B\mathbb{T}^n$ be the Borel map which is a fibration with fiber M^{2n}. So we have a homomorphism ${\mathfrak{b}_\mathfrak{p}}^* : H^*(B\mathbb{T}^n, \mathbb{Z}) \to H^*(BQ, \mathbb{Z})$ induced by $\mathfrak{b}_\mathfrak{p}$.

Theorem 1.8.1 (Theorem 4.8, [DJ91])**.** *The homomorphism ${\mathfrak{b}_\mathfrak{p}}^*$ is a surjection and induces an isomorphism of graded rings $H^*(BQ, \mathbb{Z}) \cong SR(Q, \mathbb{Z})$.*

Proof. Let $F_{i_1}, \ldots, F_{i_n}$ be the facets meeting at a vertex v of the polytope Q. Then $BQ_v = E\mathbb{T}^n \times_{\mathbb{T}^n} (\overline{D}^{2n})$ is a $\overline{D}^{2n}$ fiber bundle over $B\mathbb{T}^n$. The associated complex vector bundle is $\gamma_v : E\mathbb{T}^n \times_{\mathbb{T}^n} \mathbb{C}^n \to B\mathbb{T}^n$. Regard $B\mathbb{T}^n$ as the product of n copies of $BU(1)$. Let p_j be the projection from $\prod_{j=1}^n BU(1)$ to the j-th coordinate. Denote the universal complex line bundle over $BU(1)$ by $\gamma_{1,\infty}$. Since the action of $\mathbb{T}^n$ on $\mathbb{C}^n$ is diagonal, $\gamma_v = \bigoplus_j p_j^*(\gamma_{1,\infty})$. That is $p_j^*(\gamma_{1,\infty})$ corresponds to j-th coordinate line in $\mathbb{C}^n$. So without confusion, we may set

$$c_1(p_j^*(\gamma_{1,\infty})) = w_{i_j} \in H^2(B\mathbb{T}^n; \mathbb{Z}).$$

Note that $H^*(B\mathbb{T}^n, \mathbb{Z}) = \mathbb{Z}[w_{i_1}, \ldots, w_{i_n}]$. Since $\overline{D}^{2n}$ is contractible, $H^*(BQ_v; \mathbb{Z}) = H^*(B\mathbb{T}^n; \mathbb{Z}) = \mathbb{Z}[w_{i_1}, \cdots, w_{i_n}]$. We compute $H^*(BQ, \mathbb{Z})$ by gluing the spaces BQ_v with the Mayer-Vietoris argument. We need the cohomology of

$$BQ_{S_v} = E\mathbb{T}^n \times_{\mathbb{T}^n} \partial(\overline{D}^{2n}) = E\mathbb{T}^n \times_{\mathbb{T}^n} S^{2n-1}.$$

The fiber bundle $\mathfrak{b}_s : BQ_{S_v} \to B\mathbb{T}^n$ can be identified with the sphere bundle of the vector bundle γ_v. By the Whitney product formula we get that $c_n(\gamma_v) = w_{i_1} \cdots w_{i_n}$. Hence $e := w_{i_1} \cdots w_{i_n}$ is the Euler class of the sphere bundle $\mathfrak{b}_s$.

Now consider the Gysin exact sequence for sphere bundles (see [MS74])

$$\cdots \to H^*(BQ_{S_v}) \to H^*(B\mathbb{T}^n) \xrightarrow{\cup e} H^{*+2n}(B\mathbb{T}^n) \xrightarrow{\mathfrak{b}_s^*} H^{*+2n}(BQ_{S_v}) \to \cdots \tag{1.8.3}$$

Since the map $\cup e$ is an injection in equation 1.8.3, by exactness of this sequence the map $\mathfrak{b}_s^*$ is a surjection and one get the following commutative diagram

$$\begin{array}{ccccc} 0 \to H^*(B\mathbb{T}^n) & \xrightarrow{\cup e} & H^{*+2n}(B\mathbb{T}^n) & \xrightarrow{\mathfrak{p}_s^*} & H^{*+2n}(BQ_{S_v}) \to 0 \\ id \big\downarrow & & id \big\downarrow & & \\ \mathbb{Z}[w_{i_1}, \cdots, w_{i_n}] & \xrightarrow{w_{i_1} \ldots w_{i_n}} & \mathbb{Z}[w_{i_1}, \ldots, w_{i_n}] & & \end{array} \tag{1.8.4}$$

Hence from diagram (1.8.4) $H^*(BQ_{S_v}) = \mathbb{Z}[w_{i_1}, \cdots, w_{i_n}]/(w_{i_1} \ldots w_{i_n})$. Now applying the Mayer-Vietoris argument for cohomology the theorem can be obtained. □

Let $\mathfrak{h}_N : \mathbb{N} \to \mathbb{N}$ be the Hilbert function of the face ring $SR(Q, \mathbb{Z})$. Stanley (Proposition 3.2, [Sta75]) shows that

$$\mathfrak{h}_N(k) = \begin{cases} \Sigma_{j=0}^{l-1} f_i \; {}^{l-1}C_j & \text{if } k = 2l \\ 1 & \text{if } k = 1 \\ 0 & \text{if } k \text{ is odd} \end{cases} \tag{1.8.5}$$

Here f_j's and h_j's are *f-vectors* and *h-vectors* of Q respectively defined in Section 1.4. So the Poincaré series of $SR(Q, \mathbb{Z})$ is $\Sigma_{k=0}^{\infty} \mathfrak{h}_N(k)$. Using the relations between *f-vectors* and *h-vectors* of Q see [BP02], one can show that there is an identity of formal power series

$$(1 - t^2)^n (\Sigma_{k=0}^{\infty} \mathfrak{h}_N(k)) = \Sigma_0^n h_j t^{2j}. \tag{1.8.6}$$

Theorem 1.8.2 (Corollary 4.13, [DJ91])**.** *Let $\iota : M^{2n} \to BQ$ be an inclusion of the fiber. The induced map $\iota^* : H^*(BQ, \mathbb{Z}) \to H^*(M^{2n}, \mathbb{Z})$ is a surjection.*

Proof. Consider the Leray-Serre spectral sequence of the fibration $\mathfrak{p} : BQ \to B\mathbb{T}^n$ with fiber M^{2n}. The E_2-terms of this sequence is $E_2^{p,q} = H^p(B\mathbb{T}^n; H^q(M^{2n}))$. Since $B\mathbb{T}^n$ is simply connected we have $E_2^{p,q} = H^p(B\mathbb{T}^n) \otimes H^q(M^{2n})$. The Poincaré series of the E_r-terms is by definition

$$\sum_k (\sum_{p+q=k} \text{dim} E_r^{p,q}) t^k. \tag{1.8.7}$$

The Poincaré series of $H^*(B\mathbb{T}^n)$ is $1/(1-t^2)^n$. The Poinearé series of $H^*(M^{2n})$ is

$$h(t) := h_0 + h_1 t^2 + \ldots + h_n t^{2n}.$$

Hence, the Poinearé series of E_2 is $h(t)/(1-t^2)^n$. It turns out from Theorem 1.8.1 and equation 1.8.6 that the Poincaré series of E_2 equals to the Poincaré series of $H^*(BQ)$ or E_∞. However since E_∞ is an iterated subquotient of E_2 and they have the same Poincaré series, we get the following equality; $E_2^{p,q} = E_\infty^{p,q}$. Hence it follows that $H^*(BQ, \mathbb{Z}) \cong H^*(B\mathbb{T}^n, \mathbb{Z}) \otimes H^*(M^{2n}, \mathbb{Z})$ as $\mathbb{Z}$-modules. Thus the map $\iota^* : H^*(BQ) \to H^*(M^{2n})$ is a surjection. $\square$

Recall that $\mathcal{F}(Q)$ is the set of facets of the polytope Q. Let m be the cardinality of $\mathcal{F}(Q)$. Consider the standard local model $(\mathbb{R}^m_{\geq 0}, \epsilon)$ for $\mathbb{C}^m$, where ϵ corresponds to the assignment of standard basis elements of $\mathbb{Z}^m$ to the facets of $\mathbb{R}^m_{\geq 0}$. Let $\mathfrak{p}_s : \mathbb{C}^m \to \mathbb{R}^m_{\geq 0}$ be the projection map. Embed the polytope Q in $\mathbb{R}^m_{\geq 0}$ by the map $d_{\mathcal{F}} : Q \to \mathbb{R}^m$ where the i-th coordinate of $d_{\mathcal{F}}(q)$ is the Euclidean distance $d(q, F_i)$ of q from the hyperplane of the i-th facet $F_i \in \mathcal{F}(Q)$ in $\mathbb{R}^n$. Define the *moment angle complex* $Z(Q)$ as follows.

$$Z(Q) := \mathfrak{p}_s^{-1}(d_{\mathcal{F}}(Q)). \tag{1.8.8}$$

Let $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^n$ be the map of $\mathbb{Z}$-modules which maps the standard generator e_i of $\mathbb{Z}^m$ to the characteristic vector ξ_i. Let K denote the kernel of this map. The sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^m \stackrel{\Lambda}{\longrightarrow} \mathbb{Z}^n \longrightarrow 0 \tag{1.8.9}$$

splits and we can write $\mathbb{Z}^m = K \oplus \mathbb{Z}^n$. The torus $\mathbb{T}_K := (K \otimes \mathbb{R})/K$ is a subtorus of $\mathbb{T}^m$ and we have a split exact sequence

$$1 \longrightarrow \mathbb{T}_K \longrightarrow \mathbb{T}^m \stackrel{\Lambda_*}{\longrightarrow} \mathbb{T}^n \longrightarrow 1 \tag{1.8.10}$$

Denote the $\mathbb{Z}$-module $\mathbb{Z}^m$ by N. For any face F of Q let $N(F)$ be the submodule of N generated by the basis vectors e_i such that $d_{\mathcal{F}}(F)$ intersects the i-th facet of $\mathbb{R}^m_{\geq 0}$, that is the coordinate hyperplane $\{x_i = 0\}$. Note that image of $N(F)$ under Λ is precisely $\mathbb{Z}(F)$, so that the preimage $\Lambda^{-1}(\mathbb{Z}(F)) = K \cdot N(F)$. Consider the exact sequence

$$0 \longrightarrow \frac{K \cdot N(F)}{N(F)} \longrightarrow \frac{N}{N(F)} \stackrel{\Lambda}{\longrightarrow} \frac{\mathbb{Z}^n}{\mathbb{Z}(F)} \longrightarrow 0 \tag{1.8.11}$$

Since the characteristic vectors corresponding to the facets whose intersection is F are linearly independent, it follows from the definition of K and Λ that $K \cap N(F) = \{0\}$. Hence by the second isomorphism theorem we have a canonical isomorphism

$$\frac{K \cdot N(F)}{N(F)} \cong K \tag{1.8.12}$$

So the equation 1.8.11 yields

$$0 \longrightarrow K \longrightarrow \frac{N}{N(F)} \stackrel{\Lambda}{\longrightarrow} \frac{\mathbb{Z}^n}{\mathbb{Z}(F)} \longrightarrow 0 \tag{1.8.13}$$

We obtain the following split exact sequence of tori

$$0 \longrightarrow \mathbb{T}_K \longrightarrow \mathbb{T}(F; N)^{\perp} \longrightarrow \mathbb{T}(F; \mathbb{Z}^n)^{\perp} \longrightarrow 0 \tag{1.8.14}$$

where $\mathbb{T}(F; \mathbb{Z}^n)^{\perp}$ is the fiber of $\mathfrak{p} : M^{2n} \to Q$ and $\mathbb{T}(F; N)^{\perp}$ is the fiber of $\mathfrak{p}_s : Z(Q) \to Q$ over any point in the relative interior of the arbitrary face F. It follows that M^{2n} is a quotient of $Z(Q)$ by the above action of $\mathbb{T}_K$. This action of $\mathbb{T}_K$ is same as the restriction of its action on $\mathbb{C}^m$ as a subtorus of $\mathbb{T}^m$.

Denote the standard basis of N by $\{e_j : 1 \leq j \leq m\}$. The dual $N^* := Hom_{\mathbb{Z}}(N, \mathbb{Z})$ of N is the character group of $\mathbb{T}^m$. That is any character is uniquely represented by $\sum a_j e_j^*$ where $\{e_j^* : 1 \leq j \leq m\}$ denote the basis of L^* dual to $\{e_j\}$ and $a_j \in \mathbb{Z}$. Denote the irreducible representation of $\mathbb{T}^m$ corresponding to e_j^* by $\mathbb{C}(e_j^*)$. So the irreducible representation of $\mathbb{T}^m$ corresponding to the character $\sum a_j e_j^*$ is $\mathbb{C}(\sum a_j e_j^*) := \bigotimes_j \mathbb{C}(e_j^*)^{a_j}$.

Define a line bundle $\nu(F_i) := Z(Q) \times_{\mathbb{T}_K} \mathbb{C}(e_i^*)$ corresponding to each facet F_i of Q. The following Lemma's are proved in [DJ91]. We prove these Lemma's in a different way following [Poddf].

Lemma 1.8.3. *$c_1(\nu(F_i)) = \iota^*(w_i)$.*

Proof. Note that

$$E\mathbb{T}^m \times_{\mathbb{T}^m} (Z(Q) \times \mathbb{C}(e_i^*)) = E\mathbb{T}^m \times_{\mathbb{T}^n} (Z(Q) \times_{\mathbb{T}_K} \mathbb{C}(e_i^*)) \simeq E\mathbb{T}^n \times_{\mathbb{T}^n} \nu(F_i).$$

Let $x \in M(F_i, \xi)$ be a fixed point of $\mathbb{T}^n$ action on M^{2n}. Then as $\mathbb{T}^n$-representations, $\nu(F_i)|_x = \mathbb{C}(\xi_i^*)$. The line bundle $E\mathbb{T}^n \times_{\mathbb{T}^n} \mathbb{C}(\xi_i^*)$ over $B\mathbb{T}^n$ is equal to $pr_i^*(\gamma_{1,\infty})$. Let $\bar{\iota} : x \hookrightarrow M^{2n}$ be the inclusion map. So there is an associated umkehrungs homomorphism in equivariant cohomology $\bar{\iota}_* : H^*(B\mathbb{T}^n) \to H^*(BQ)$. Let $\mathfrak{b}_\mathfrak{c} : BQ \to B\mathbb{T}^n$ is the equivariant version to the collapsing map $M^{2n} \mapsto \{x\}$. The map $\bar{\iota}_*$ can be identified with the map ${\mathfrak{b}_\mathfrak{c}}^* : H^*(B\mathbb{T}^n) \to H^*(BQ)$. In Theorem 1.8.1 we have identified $\bar{\iota}_* c_1(pr_i^* \gamma_{1,\infty})$ with $w_i \in H^*(BQ)$. We consider the inclusion $\iota : M^{2n} \hookrightarrow BQ$ as a fiber of $\mathfrak{b}_\mathfrak{c}$. Then we have the following commutative diagram

$$\begin{array}{ccccc} \nu(F_i) & \longrightarrow & E\mathbb{T}^n \times_{\mathbb{T}^n} \nu(F_i) & \xrightarrow{\kappa} & E\mathbb{T}^n \times_{\mathbb{T}^n} \mathbb{C}(\xi_i^*) \\ \downarrow & & \downarrow & & \downarrow \\ N & \xrightarrow{\iota} & BQ & \xrightarrow{\mathfrak{b}_\mathfrak{c}} & B\mathbb{T}^n. \end{array} \tag{1.8.15}$$

Thus $c_1(\nu(F_i) = \iota^* {\mathfrak{b}_\mathfrak{c}}^* c_1(pr_i^*(\gamma_{1,\infty})) = \iota^*(w_i)$. □

Lemma 1.8.4. *The line bundle $\bigotimes_i \nu(F_i)^{a_i}$ over M^{2n} is trivial if and only if the vector $(a_1, \ldots, a_m)$ belongs to the row space of the matrix of Λ.*

Proof. Since $\nu(F_i) = Z(Q) \times_{\mathbb{T}_K} \mathbb{C}(e_i^*)$, $\bigotimes_i \nu(F_i)^{a_i} = Z(Q) \times_{\mathbb{T}_K} \mathbb{C}(\sum a_i e_i^*)$. This line bundle is trivial if and only if the character $\sum a_i e_i^*$ restricts to the trivial character on $\mathbb{T}_K$. This holds if and only if $\sum a_i e_i^*(u) = 0$ for all $u \in K = Ker(\Lambda)$. This is equivalent to saying that $(a_1, \ldots, a_m)$ is a linear combination of rows of Λ. □

Corollary 1.8.5. *$\sum_{i=1}^m a_i \iota^*(w_i) = 0$ whenever $(a_1, \ldots, a_m)$ is in the row space of Λ.*

The above calculations in terms of characters imply that ${\mathfrak{b}_\mathfrak{c}}^* : H^2(B\mathbb{T}^n) \to H^2(BQ)$ can be identified with the map $\Lambda^* : (\mathbb{Z}^n)^* \to (\mathbb{Z}^m)^*$, where Λ^* denotes the dual of the characteristic map Λ. So we have the following short exact sequence,

$$\begin{array}{ccccc} 0 \to H^2(B\mathbb{T}^n) & \xrightarrow{{\mathfrak{b}_\mathfrak{c}}^*} & H^2(BQ) & \xrightarrow{\iota^*} & H^2(N) \to 0 \\ \| & & \| & & \\ (\mathbb{Z}^n)^* & \xrightarrow{\Lambda^*} & (\mathbb{Z}^m)^*. & & \end{array}$$

Let $\mathcal{J}$ be the homogeneous ideal in $\mathbb{Z}[w_1, \ldots, w_m]$ generated by the polynomials $\{\xi^j := \sum_{i=1}^m a_{j_i} w_i | 1 \leq j \leq n\}$, where $(a_{j_1}, \ldots, a_{j_m})$ denotes the j-th row of Λ. The element ξ^j can be identified with the image of the j-th generator of $H^2(B\mathbb{T}^n)$ under Λ^*. Let $\overline{\mathcal{J}}$ be the image of $\mathcal{J}$ in $SR(Q, \mathbb{Z})$. Since $\iota^* : SR(Q, \mathbb{Z}) \to H^*(M^{2n}, \mathbb{Z})$ is onto and $\overline{\mathcal{J}}$ is belongs to its kernel, ι^* induces a surjection $SR(Q, \mathbb{Z})/\overline{\mathcal{J}} \to H^*(M^{2n}, \mathbb{Z})$.

Theorem 1.8.6 (Theorem 4.14, [DJ91]). *Let M^{2n} be the quasitoric manifold associated to the characteristic model (Q, ξ). Then $H^*(M^{2n}; \mathbb{Z})$ is the quotient of the face ring $SR(Q, \mathbb{Z})$ by $\overline{\mathcal{J}}$. That is $H^*(M^{2n}; \mathbb{Z}) = \mathbb{Z}[w_1, \ldots, w_m]/(\mathcal{I} + \mathcal{J})$.*

Proof. We know that the cohomology ring $H^*(B\mathbb{T}^n, \mathbb{Z})$ is a polynomial ring on n generators. The face ring $H^*(BQ, \mathbb{Z}) \cong H^*(B\mathbb{T}^n) \otimes H^*(M^{2n})$ as $\mathbb{Z}$-modules. Also the map $\mathfrak{b}_\mathfrak{c}{}^* : H^*(B\mathbb{T}^n, \mathbb{Z}) \to H^*(BQ, \mathbb{Z})$ is an injection and $\overline{\mathcal{J}}$ is identified with the image of $\mathfrak{b}_\mathfrak{c}{}^*$. Thus $H^*(M^{2n}) = H^*(BQ, \mathbb{Z})/\overline{\mathcal{J}} = \mathbb{Z}[w_1, \ldots, w_m]/(\mathcal{I} + \mathcal{J})$, where $\mathcal{I}$ is an ideal defined in definition 1.4.2. □

1.9 Smooth and stable complex structure

In Section 6 of [BR01] the authors describe the existence of smooth equivariant connected sum operation for quasitoric manifolds. In this section we follow the paper [BR01] and the lecture note [Poddf]. We will realize the quasitoric manifold M^{2n} as the quotient of an open subset of $\mathbb{C}^m$. We follow the notation of Section 1.8. Identify $\mathbb{R}^m$ with the space of functions $\mathbb{R}^{\mathcal{F}(Q)}$. Consider the thickening $\mathcal{W} \subset \mathbb{R}^m_{\geq 0}$ of the image $d_\mathcal{F}(Q)$, defined by

$$\mathcal{W} = \{f : \mathcal{F}(Q) \to \mathbb{R}_{\geq 0} | f^{-1}(0) \in \mathfrak{L}_F(Q)\} \tag{1.9.1}$$

where $\mathfrak{L}_F(Q)$ denotes the face lattice of Q. Let V_Q be the n-dimensional linear subspace of $\mathbb{R}^m$ parallel to $d_\mathcal{F}(Q)$ and $V_Q^\perp$ be its orthogonal complement in $\mathbb{R}^m$. The group $G := \exp(V_Q^\perp)$ acts naturally on $\mathbb{R}^m$ and $\mathcal{W}$ by coordinatewise multiplication. We want to produce a thickening $\mathcal{Q} \subset \mathcal{W}$ of $d_\mathcal{F}(Q)$ which will be close to a G principal bundle over $d_\mathcal{F}(Q)$.

Lemma 1.9.1 (Lemma 5.1, [Poddf]). *The tangent spaces to the orbits of G-action on $\mathbb{R}^m$ form an integrable distribution.*

Proof. Let $(y_1, \ldots, y_m)$ be the standard coordinates on $\mathbb{R}^m$. Let $*$ be the binary operation of coordinatewise multiplication of two vectors in $\mathbb{R}^m$. Let the vectors $c_j = (c_{j_1}, \ldots, c_{j_m})$, $1 \leq j \leq m - n$, form a orthogonal basis for the subspace $V_Q^\perp$ of $\mathbb{R}^m$. Suppose q is any fixed point in $\mathbb{R}^m$ with coordinate vector $y(q) = (y_1(q), \ldots, y_m(q))$. Then the vectors $y(q) * c_j$, $1 \leq j \leq m - n$, span the tangent space to the orbit through q of G-action. Clearly the Lie bracket of any two vector fields $y(q) * c_j = \sum y_i(q) c_{j_i} \frac{\partial}{\partial y_i}$ and $y(q) * c_k = \sum y_i c_{k_i} \frac{\partial}{\partial y_i}$ is zero. Thus the distribution is integrable. □

Definition 1.9.2. *We will denote the distribution consisting of tangent vectors to orbits of G-action on $\mathbb{R}^m$ by χ.*

Consider the following decomposition of the space $\mathcal{W}$ as in [BR01]. Observe that each point $q \in Q$ determines a function $d_{\mathcal{F}}(q) : \mathcal{F}(Q) \to \mathbb{R}$, where $d(q, F_i)$ is the Euclidean distance between p and the hyperplane containing the facet $F_i \in \mathcal{F}(Q)$ for $i = 1, \ldots, m$. These maps produce an embedding $d_{\mathcal{F}}(Q) \subset \mathbb{R}^m$ of Q. For any subset $\mathcal{G}(Q) \subset \mathcal{F}(Q)$, we realize $\mathbb{R}^{\mathcal{G}(Q)}$ as a subspace of $\mathbb{R}^{\mathcal{F}(Q)}$ by choosing those coordinates y_i to be zero for which $F_i \in \mathcal{F}(Q) - \mathcal{G}(Q)$. Let F be a face (may be empty) of Q. Denote the set of facets of Q that contain F by $\mathcal{F}_F$. Let $\mathcal{G}_F = \mathcal{F}(Q) - \mathcal{F}_F$. Then $\mathcal{W}$ is the union of open cones $\bigcup \mathbb{R}^{\mathcal{G}_F}_{>0}$.

Lemma 1.9.3 (Lemma 5.2, [Poddf])**.** *For any face F of Q the orbits of G-action define a foliation on $\mathbb{R}^{\mathcal{G}_F}_{>0}$.*

Proof. Clearly the open cone $\mathbb{R}^{\mathcal{G}_F}_{>0}$ is an invariant subset under the G-action. Let $q \in \mathbb{R}^{\mathcal{G}_F}_{>0}$ be any point. Let B be a matrix whose row vectors c_j form a basis for $V_Q^{\perp}$. Consider the matrix $B(q)$ whose rows are $y(q) * c_j$. Then the row vectors of $B(q)$ span the vector space $\chi(q)$. Denote the columns of the matrix B by β_i, $1 \leq i \leq m$. Then the i-th column of $B(q)$ is $y_i(q)\beta_i$. This Lemma is clear if F is an empty face.

Now suppose F is a vertex v of Q. Let $d_{\mathcal{F}}(v) = q$. Then exactly n coordinates of $y(q)$ are zero. Without loss of generality we may assume that the facets $F_1, \ldots, F_n$ of Q meet at v. So we get that $y_1(q) = \ldots = y_n(q) = 0$. Note that the vectors $u_1, \ldots, u_n$ tangent to the edges of $d_{\mathcal{F}}(Q)$ meeting at q form a basis of V_Q. By our assumption each vector u_i has 0 in the first n positions except for the i-th position. A vector z belongs to $V_Q^{\perp}$ if and only if it satisfies the following system of linear equations;

$$z \cdot u_j = 0, \quad \text{for each } 1 \leq j \leq n. \tag{1.9.2}$$

Let $e_1, \ldots, e_m$ be the standard basis of $\mathbb{R}^m$. Solving the above system we get a basis $c_1, \ldots, c_{m-n}$ of $V_Q^{\perp}$ where c_k has the form $c_{k_1}e_1 + \ldots + c_{k_n}e_n + e_{k+n}$ for $1 \leq k \leq m-n$.

Hence we can assume that $\beta_{n+1}, \ldots, \beta_m$ are linearly independent. Since the coordinates $\{y_i(q) : n+1 \leq i \leq m\}$ are each positive we get $y_{n+1}(q)\beta_{n+1}, \ldots, y_m(q)\beta_m$ are linearly independent. Hence $B(q)$ has the rank $m-n$. Therefore the integrable distribution χ has constant rank and corresponds to a foliation.

The argument for faces of higher dimension is similar and follows from the zero dimensional case. □

Lemma 1.9.4 (Lemma 5.3, [Poddf])**.** *The integrable distribution χ forms a foliation on a neighborhood $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ in $\mathbb{R}^m$.*

Proof. It is enough to show that the distribution χ has constant rank $m-n$ in a neighborhood of each point q in $\mathbb{R}^{\mathcal{G}_F}_{>0}$ where $F \subsetneq Q$ is a face of Q. Without loss of generality we may assume that $y_i(q) > 0$ for all $i \geq n+1$ and that $\beta_{n+1}, \ldots, \beta_m$ are linearly independent. Let U be a small open ball in $\mathbb{R}^m$ around q. Let s be any point in U. Then we may assume that $y_i(s) > 0$ for all $i \geq n+1$. Hence $y_{n+1}(s)\beta_{n+1}, \ldots, y_m(s)\beta_m$ are linearly independent. Hence the vector space $\chi(s)$ has the rank $m-n$. □

Lemma 1.9.5 (Lemma 5.4, [Poddf]). *The orbits of G-action are transverse to $d_{\mathcal{F}}(Q)$.*

Proof. Let q be a point in $d_{\mathcal{F}}(Q)$ with coordinate vector $y(q)$. Then a tangent vector to the orbit of G-action through q has the form $c * y(q)$ where $c = (c_1, \ldots, c_m) \in V_Q^{\perp}$. The inner product $\langle c, c * y(q) \rangle = \sum c_i^2 y_i(q)$. This is a strictly positive quantity. Thus the set $\chi(q) \bigcap V_Q$ is singleton. Since $\chi(q)$ and V_Q have complementary dimensions, they are transversal. □

Definition 1.9.6. *Define $\mathcal{Q}$ to be the union of all G orbits that pass through $d_{\mathcal{F}}(Q)$.*

Lemma 1.9.7 (Lemma 5.5, [Poddf]). *The space $\mathcal{Q}$ is an open subset of $\mathbb{R}^m_{\geq 0}$.*

Proof. Let $y \in \mathcal{Q}$ and X_y denote the G orbit through y. By definition of $\mathcal{Q}$, X_y meets $d_{\mathcal{F}}(Q)$ at some point y_1. Let α be any path in X_y from y_1 to y. Let T be a transversal to the foliation χ on $\widetilde{\mathcal{W}}$ at y. Let $\widetilde{Q} := \widetilde{\mathcal{W}} \bigcap (V_Q + y(q))$ where q is any point on $d_{\mathcal{F}}(Q)$. Since $\widetilde{Q}$ is transversal to χ at y_1, there exists small open set $U_1 \subset \widetilde{Q}$ around y_1 which maps diffeomorphically onto a small open set $U \subset T$ around y via the holonomy of the foliation χ along α. Note that each leaf of χ that intersects $\mathcal{W}$ lies completely in $\mathcal{W}$. Hence the holonomy of χ along α maps $U_1 \cap d_{\mathcal{F}}(Q)$ onto $U \cap \mathcal{W}$. Thus there exists a small foliation chart $X \subset \widetilde{\mathcal{W}}$ around y such that every plaque in $X \cap \mathcal{W}$ lies in a leaf that hits $d_{\mathcal{F}}(Q)$. Therefore the neighborhood $X \cap \mathcal{W}$ of y in $\mathcal{W}$ is contained in $\mathcal{Q}$. Thus $\mathcal{Q}$ is open in $\mathcal{W}$. Since $\mathcal{W}$ is open in $\mathbb{R}^m_{\geq 0}$, so is the space $\mathcal{Q}$. □

Let O_m be the origin of $\mathbb{R}^m$. Let $\mathcal{Q}'$ be a small tubular neighborhood of $d_{\mathcal{F}}(Q)$ in $\mathcal{Q}$ such that:

1. $\mathcal{Q}'$ is diffeomorphic to the product of $Q \times S$ where S is an open neighborhood of the identity in G,
2. $\mathcal{Q}'$ is bounded and the Euclidean distance from $\mathcal{Q}'$ to O_m is positive.

Being a foliation local triviality is provided and global triviality follows from contractibility of $d_{\mathcal{F}}(Q)$. Denote the restriction of the foliation χ on $\mathcal{Q}'$ by the same.

Consider the group $\Upsilon = G \times \mathbb{T}_K$. So Υ is a subgroup of $(\mathbb{C}^*)^m$. Let $\mathcal{Z}' := \mathfrak{p}_s^{-1}(\mathcal{Q}')$. The map $\mathfrak{p}_s : \mathcal{Z}' \to \mathcal{Q}'$ is smooth and transversal to χ. Also $\mathbb{T}_K$ acts on $\mathcal{Z}'$ freely. These two fact induce a foliation, say χ', on $\mathcal{Z}'$. Let $\mathcal{M}$ be the leaf space of this foliation. In fact this is a fiber bundle over $\mathcal{M}$. From the choice of $\mathcal{Q}'$ we get that each fiber of $\mathcal{Z}'$ over $\mathcal{M}$ is diffeomorphic to $S \times \mathbb{T}_K$ where S is an open neighborhood of the identity in

G. We use Lie groupoid theory to show a smooth structure on $\mathcal{M}$. Now we give the definition of Lie groupoid following [ALR07].

Definition 1.9.8. *A Lie groupoid $\mathcal{G}$ consists of two smooth Hausdorff manifolds, one is a set of objects $\mathcal{G}_0$ and another is a set of invertible arrows $\mathcal{G}_1$, together with the following smooth maps where the maps s, t are submersions.*

1. *The source map $s : \mathcal{G}_1 \to \mathcal{G}_0$ which assigns each arrow g to its source $s(g)$.*
2. *The target map $t : \mathcal{G}_1 \to \mathcal{G}_0$ which assigns each arrow g to its target $t(g)$. For an arrow $g \in \mathcal{G}_1$, we write $g : y \to x$ to indicate that $s(g) = y$ and $t(g) = x$.*
3. *The composition map $\mathfrak{c} : \mathcal{G}_1 \times_{(s,t)} \mathcal{G}_1 \to \mathcal{G}_1$, where*

$$\mathcal{G}_1 \times_{(s,t)} \mathcal{G}_1 = \{(g, h) \in \mathcal{G}_1 \times \mathcal{G}_1 : s(h) = t(g)\},$$

is defined in the following. If $g : x \to y$ and $h : y \to z$ are two arrows then we can define their composition arrow $hg : x \to z$. That is the composition map $\mathfrak{c}$ is defined by $\mathfrak{c}(g, h) = hg$ and is required to be associative.

4. *The identity map $\mathfrak{i} : \mathcal{G}_0 \to \mathcal{G}_1$ such that $s\mathfrak{i}(x) = x = t\mathfrak{i}(x)$ and $g\mathfrak{i}(x) = g = \mathfrak{i}(y)g$ for all $x, y \in \mathcal{G}_0$ and $g \in \mathcal{G}_1$ with $g : x \to y$.*
5. *An inverse map $\mathfrak{u} : \mathcal{G}_1 \to \mathcal{G}_1$, written by $\mathfrak{u}(g) = g^{-1}$, is a two-sided inverse for the composition. That is, if $g : x \to y$ then $g^{-1} : y \to x$ such that $g^{-1}g = \mathfrak{i}(x)$ and $gg^{-1} = \mathfrak{i}(y)$.*

Example 1.9.9. *Let M be a connected manifold. Then the fundamental groupoid $\Pi(M)$ of M is the groupoid with the space of objects $\Pi(M)_0 = M$ and each homotopy class g of paths from x to y is an arrow $g : x \to y$.*

Example 1.9.10. *Let $\mathfrak{K}$ be a Lie group which acts smoothly on a manifold M from the left. Define a Lie groupoid $\mathfrak{K} \ltimes M$ by setting $(\mathfrak{K} \ltimes M)_0 = M$ and $(\mathfrak{K} \ltimes M)_1 = \mathfrak{K} \times M$, where the source map $s : \mathfrak{K} \times M \to M$ is the projection onto the second factor and the target map $t : \mathfrak{K} \times M \to M$ is the group action. Let $g : x_1 \to k_1x_1$ and $h : x_2 \to k_2x_2$ be two arrows in $(\mathfrak{K} \ltimes M)_1$ such that $k_1x_1 = x_2$. The composition map is defined by $hg : x_1 \to (k_2k_1)x_1$. The identity map $\mathfrak{i} : M \to \mathfrak{K} \times M$ is defined by $\mathfrak{i}(x) = (1_K, x)$. This groupoid is called the translation groupoid associated to the group action. The unit groupoid is the translation groupoid for the action of the trivial group. Also by taking M to be a point we can view any Lie group $\mathfrak{K}$ as a Lie groupoid with a single object.*

Definition 1.9.11. *Let $\mathcal{G}$ be a Lie groupoid with the set of objects $\mathcal{G}_0$ and the set of arrows $\mathcal{G}_1$. For a point $x \in \mathcal{G}_0$, the set of all arrows $g : x \to x$ is a Lie group. Denote it by $\mathcal{G}_x$, called the isotropy group at x. The set $ts^{-1}(x)$ of targets of arrows out of x*

is called the orbit of x. The orbit space $|\mathcal{G}|$ of $\mathcal{G}$ is the quotient space of $\mathcal{G}_0$ under the equivalence relation $x \sim y$ if and only if x and y are in the same orbit. Conversely, $\mathcal{G}$ is called a groupoid presentation of $|\mathcal{G}|$.

Definition 1.9.12. *A Lie groupoid $\mathcal{G}$ is called proper if the map $(s,t) : \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is a proper map.*

Note that in a proper Lie groupoid $\mathcal{G}$, every isotropy group is compact (see Proposition 1.37 of [ALR07]).

Theorem 1.9.13 (Theorem 5.6, [Poddf])**.** *The leaf space $\mathcal{M}$ of the foliation χ' can be identified with the quasitoric manifold M^{2n}. Thus M^{2n} has a smooth structure.*

Proof. Let $\mathcal{G}$ be the Lie groupoid with the set of objects $\mathcal{G}_0 = \mathcal{Z}'$ and the set of arrows $\mathcal{G}_1 = \{(z,g) : z \in \mathcal{Z}',\, g \in \Upsilon,\, z \cdot g \in \mathcal{Z}'\}$. Since the group G acts on $\mathbb{R}^m$ coordinatewise and $\mathbb{T}_K$ is compact, this is a proper Lie groupoid. Clearly isotropy group of each $x \in \mathcal{G}_0$ is trivial. So the set $ts^{-1}(x)$, the orbit of x in $\mathcal{G}$, is diffeomorphic to $S \times \mathbb{T}_K$ for each $x \in \mathcal{G}_0$. That is $ts^{-1}(x)$ the leaf of the foliation χ' on $\mathcal{G}_0$. Thus $\mathcal{M}$ is the orbit space of $\mathcal{G}$. So there exists an embedded submanifold U_x transversal to $ts^{-1}(x)$ at x and a neighborhood $S_u \subset S$ of identity in G such that the map $U_x \times S_u \to \mathcal{G}_0$ given by the action is diffeomorphic onto its image. So the space $\mathcal{M}$ has a natural smooth structure (see Chapter 5, [MM03]). Topologically $\mathcal{M}$ can be identified with the quotient $Z(P)/\mathbb{T}_K$ that is M^{2n}. As the $\mathbb{T}^m \subset (\mathbb{C}^*)^m$ action on $\mathcal{Z}'$ is smooth, the induced $\mathbb{T}^n$ action on $\mathcal{M}$ is smooth. Hence $\mathcal{M}$ is a smooth quasitoric manifold with same characteristic pair. Therefore by the classification result 1.2.17 it is equivariantly homeomorphic to M^{2n}. □

The following Theorem was first proved by Buschtabar and Ray in [BR01]. We give a different proof following the lecture note [Poddf].

Theorem 1.9.14. *A quasitoric manifold M^{2n} has a stable complex structure.*

Proof. Let $\mathcal{T}\mathcal{Z}'$ be the tangent bundle of $\mathcal{Z}'$. Υ acts naturally on $\mathcal{T}\mathcal{Z}'$. Define the space $\mathcal{D} := \mathcal{T}\mathcal{Z}'/\Upsilon$, so that a point of $\mathcal{D}$ is a field of tangent vectors to $\mathcal{Z}'$, defined along one of its fibers, and invariant under Υ. Let $\mathcal{U}$ denote the subbundle of $\mathcal{T}\mathcal{Z}'$ formed by vectors tangential to the fibers of $\mathcal{Z}'$ over $\mathcal{M}$. Then Υ acts on $\mathcal{U}$ and define $\mathcal{R} = \mathcal{U}/\Upsilon$. The arguments of Atiyah [Ati57] apply, with complex analytic replaced by smooth. Therefore $\mathcal{D}$ has a natural vector bundle structure over M^{2n} and $\mathcal{R}$ is a subbundle of $\mathcal{D}$. Moreover the following sequence of vector bundles, where $\mathcal{T}M^{2n}$ denotes the tangent bundle of M^{2n}, is exact.

$$0 \to \mathcal{R} \to \mathcal{D} \to \mathcal{T}M^{2n} \to 0 \tag{1.9.3}$$

The standard complex structure on $\mathbb{C}^m$ restricts to a complex structure J on $\mathcal{T}\mathcal{Z}'$. The action of Υ commutes with J. Thus $\mathcal{D}$ inherits a complex structure, which we again denote by J.

Let $\overline{\Upsilon}$ denote the Lie algebra of Υ. Since every fiber of $\mathcal{Z}'$ can be identified as a Lie groupoid to a neighborhood of the identity in Υ, and Υ is commutative, following [Ati57] we obtain that $\mathcal{R}$ is isomorphic to $\overline{\Upsilon} \times M^{2n}$. Thus we get a complex structure on $\mathcal{T}M^{2n} \oplus (\mathbb{R}^{2m-2n} \times M^{2n})$, i.e. a stable complex structure on M^{2n}. □

Remark 1.9.15. *Note that each omniorientation determines a stable complex structure by the above procedure. That is there is a canonical choice of stable complex structure only if omniorientation is fixed.*

The space $\mathcal{T}\mathcal{Z}'$ splits naturally into a direct sum of m complex line bundles corresponding to the complex coordinate directions. These directions correspond to the facets of Q. We get a corresponding splitting $\mathcal{D} = \bigoplus \nu(F_i)$.

1.10 Chern classes

The existence of stable almost complex structure on M^{2n} implies that Chern classes can be defined. These classes depend on the choice of omniorientation on M^{2n}.

Definition 1.10.1. *The total Chern class of an omnioriented quasitoric manifold M^{2n} is defined to be the total Chern class of its stable tangent bundle $\mathcal{D}$, $c(\mathcal{T}M^{2n}) := c(\mathcal{D})$.*

Since $\mathcal{D} = \bigoplus \nu(F_i)$, by the Whitney product formula we obtain

$$c(\mathcal{T}M^{2n}) = \prod_{i=1}^{m}(1 + c_1(\nu(F_i))) = \prod_{i=1}^{m}(1 + w_i). \tag{1.10.1}$$

Note that in our notation, w_i depends on the characteristic vectors λ_i and if we want to compare different omniorientations the signs of w_is have to be adjusted with respect to some fixed choice.

From the combinatorial information of the combinatorial model we can calculate the Chern numbers of M^{2n} by localization. In a different approach, Panov [Pan01] uses results from index theory to give a beautiful formula for the χ_y-genus of M^{2n}. To present the formula we need to first introduce some notation.

Suppose v is a vertex of simple polytope Q which is the intersection of n facets $F_{i_1}, \ldots, F_{i_n}$. To each facet F_{i_k} assign the unique edge E_k such that $E_k \cap F_{i_k} = v$. Let e_k be a vector along E_k with origin v. Order the facets at v such that $e_1, \ldots, e_n$ form a positively oriented basis of $\mathbb{R}^n$. The characteristic vectors associated to these facets are also ordered accordingly. Adopting this convention on ordering for each vertex, we make the following definition.

Definition 1.10.2. *The sign* $\operatorname{sign}(v)$ *of a vertex* $v = F_{i_1} \cap \ldots \cap F_{i_n}$ *is defined to be the determinant of the* $n \times n$ *matrix* $\Lambda_{(v)} := [\lambda_{i_1} \cdots \lambda_{i_n}]$.

Let E be an edge of Q. Recall the module $\mathbb{Z}(E)$ from Theorem 1.2.9. Let α be a generator of the module $\mathbb{Z}(E)^{\perp} := \{v \in \mathbb{Z}^n \mid \langle v, w\rangle = 0 \, \forall \, w \in \mathbb{Z}(E)\}$. We refer to α as the edge vector corresponding to E. It is determined up to choice of sign. For a given omniorientation, the sign of α may be locally fixed at a vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of $E = F_{i_2} \cap \ldots \cap F_{i_n}$ by requiring that $\langle \alpha, \lambda_{i_1}\rangle > 0$. Then each vertex has n well defined edge vectors $\alpha_1, \ldots, \alpha_n$ (ordered according to the convention discussed above).

Definition 1.10.3. *Let* $\eta \in \mathbb{Z}^n$ *be a primitive vector such that* $\langle \alpha, \eta\rangle \neq 0$ *for any edge vector* α *of* Q. *Then define the index of a vertex* $v = F_{i_1} \cap \ldots \cap F_{i_n}$ *of* Q *with respect to* η *to be*

$$\operatorname{ind}_{\eta}(v) := \#\{1 \leq k \leq n \mid \langle \alpha_k, \eta\rangle < 0\}.$$

Theorem 1.10.4 (Theorem 3.1, [Pan01])**.** *Let* M^{2n} *be an omnioriented quasitoric manifold. For any primitive vector* $\eta \in \mathbb{Z}^n$ *such that* $\langle \alpha_k, \eta\rangle \neq 0$ *for any edge vector* α_k *of* Q*, the* χ_y*-genus of* M^{2n} *may be calculated as*

$$\chi_y(X) = \sum_{v} (-y)^{\operatorname{ind}_{\eta}(v)} \operatorname{sign}(v).$$

For the values 1, 0, 1 of y the χ_y-genus specializes to the top Chern number, the Todd genus and signature or L-genus of M^{2n} respectively. Thus we readily obtain formulae for these important invariants from the above theorem. For instance, formula for the top Chern number of quasitoric manifold M^{2n} over Q is,

$$c_n(M^{2n}) = \sum_{v} \operatorname{sign}(v). \tag{1.10.2}$$

Chapter 2

Small covers and orbifolds

2.1 Introduction

The category of small covers were introduced by Davis and Januszkiewicz [DJ91]. Following the paper [DJ91] we discuss some basic theory about small covers.

Orbifolds were introduced by Satake [Sat57], who called them V-manifolds. Orbifolds are singular spaces that are locally look like as a quotient of an open subset of Euclidean space by an action of a finite group. Following [ALR07], we provide a definition of effective orbifolds. We recall the tangent bundle and Orbifold fundamental group of an orbifold.

2.2 Small covers

Small covers are real analog of quasitoric manifolds. Let N_s be an n-dimensional manifold and $\rho : \mathbb{Z}_2^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the standard action.

Definition 2.2.1. *An action $\eta : \mathbb{Z}_2^n \times N_s \to N_s$ is said to be locally standard if the followings hold.*

1. *Every point $y \in N_s$ has a $\mathbb{Z}_2^n$-stable open neighborhood U_y, that is $\eta(\mathbb{Z}_2^n \times U_y) = U_y$.*
2. *There exists a homeomorphism $\psi : U_y \to V$, where V is a $\mathbb{Z}_2^n$-stable (that is $\rho(\mathbb{Z}_2^n \times V) = V$) open subset of $\mathbb{R}^n$.*
3. *There exists an isomorphism $\delta_y : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ such that $\psi(\eta(t, x)) = \rho(\delta_y(t), \psi(x))$ for all $(t, x) \in \mathbb{Z}_2^n \times U_y$.*

Definition 2.2.2. *A closed n-dimensional manifold N_s is said to be a small cover if there is an effective $\mathbb{Z}_2^n$-action on N_s such that*

1. *the action is a locally standard action,*

2. the orbit space of the action is a simple polytope.

Example 2.2.3. *Suppose* $\mathbb{T}^1 = \{(x, y) \in \mathbb{R}^2 : |x|^2 + |y|^2 = 1\}$ *is the unit circle in* $\mathbb{R}^2$*. Let* l *be a line in* $\mathbb{R}^2$ *passing through the origin. The group* $\mathbb{Z}_2$ *acts on* $\mathbb{T}^1$ *by a reflection along the line* l*. Denote this action by*

$$\rho_l : \mathbb{Z}_2 \times \mathbb{T}^1 \to \mathbb{T}^1. \tag{2.2.1}$$

Clearly the orbit space is an interval and the action is locally standard.

Example 2.2.4. *Consider the* n*-fold product* $\mathbb{T}^n = (\mathbb{T}^1)^n \subset (\mathbb{R}^2)^n$*. An action of* $\mathbb{Z}_2^n$ *defined on* $\mathbb{T}^n$ *by*

$$\rho_n((g_1, g_2, \ldots, g_n), (t_1, t_2, \ldots, t_n)) = (\rho_{l_1}(g_1, t_1), \ldots, \rho_{l_n}(g_n, t_n)), \tag{2.2.2}$$

where l_i *is a line belongs to the* i*-th component of* $(\mathbb{R}^2)^n$*, for* $i = 1, 2, \ldots, n$*. The action* ρ_n *is locally standard and the orbit space is the standard* n*-cube in* $\mathbb{R}^n$*. So* $\mathbb{T}^n$ *is a small cover over the* n*-cube.*

Example 2.2.5. *The natural action of* $\mathbb{Z}_2^n$ *defined on the real projective space* $\mathbb{RP}^n$ *by*

$$(g_1, \ldots, g_n) \cdot [x_0, x_1, \ldots, x_n] \to [x_0, g_1 x_1, \ldots, g_n x_n] \tag{2.2.3}$$

is locally standard and the orbit space is diffeomorphic as manifold with corners to the standard n*-simplex. Hence* $\mathbb{RP}^n$ *is a small cover over the* n*-simplex* $\triangle^n$*.*

Remark 2.2.6. *We can define an equivariant connected sum of small covers following the Section 1.7 of Chapter 1. The equivariant connected sum of* n*-dimensional finitely many small covers is also a small cover. For example, the connected sum* $\mathbb{RP}^n \# \mathbb{RP}^n \# \ldots \# \mathbb{RP}^n$ *of* k *copies of* $\mathbb{RP}^n$ *is a small cover over the connected sum* $\triangle^n \# \triangle^n \# \ldots \# \triangle^n$ *(*k *times) of* $\triangle^n$*.*

Let $\mathcal{F}(Q) = \{F_1, \ldots, F_m\}$ be the set of facets of a simple n-polytope Q. We denote the underlying additive group of the vector space $\mathbb{F}_2^n$ by $\mathbb{Z}_2^n$.

Definition 2.2.7. *The function* $\beta : \mathcal{F}(Q) \to \mathbb{F}_2^n$ *is called* $\mathbb{Z}_2$*-characteristic function on* Q *if the span of* $\{\beta(F_{j_1}), \ldots, \beta(F_{j_l})\}$ *is an* l*-dimensional subspace of* $\mathbb{F}_2^n$ *whenever the intersection of the facets* $F_{j_1}, \ldots, F_{j_l}$ *is nonempty.*

The vectors $\beta(F_j) = \beta_j$ *are called* $\mathbb{Z}_2$*-characteristic vectors and the pair* (Q, β) *is called* $\mathbb{Z}_2$*-characteristic pair.*

We show that associated to a small cover there exists a $\mathbb{Z}_2$-characteristic pair. Then we construct a small cover from a $\mathbb{Z}_2$-characteristic pair.

Let $\varsigma : N_s \to Q$ be a small cover over n-polytope Q. By the locally standard property of the $\mathbb{Z}_2^n$-action on N_s we can show that each $N_{F_j} = \varsigma^{-1}(F_j)$ is a connected $(n-1)$-dimensional submanifold of N_s for $j = 1, \ldots, m$. The submanifold N_{F_j} is fixed pointwise by the subgroup G_j ($\cong \mathbb{Z}_2$) of $(\mathbb{Z}_2)^n$. So we can correspond each facet F_j to the subgroup G_j. Let $\beta_j \in (\mathbb{Z}_2)^n$ be the nonidentity element of G_j. Hence we can define a function

$$\beta : \mathcal{F}(Q) \to \mathbb{F}_2^n \text{ by } \beta(F_j) = \beta_j. \tag{2.2.4}$$

If the intersection of the facets $F_{j_1}, \ldots, F_{j_k}$ is nonempty then $F = F_{j_1} \cap \ldots \cap F_{j_k}$ is a codimension-k face of Q. Then the isotropy group G_F of the submanifold $\varsigma^{-1}(F) \subset N_s$ is the subgroup of $\mathbb{Z}^n$ generated by $\beta_{j_1}, \ldots, \beta_{j_k}$. Let v be a vertex of F and $y = \varsigma^{-1}(v)$. Comparing the action of $\mathbb{Z}_2^n$ on a $\mathbb{Z}_2^n$-stable neighborhood of y in N_s to the standard action we get that G_F is a k-dimensional subspace of $\mathbb{F}_2^n$. So we can assign a unique subgroup G_F to each face F of Q. Hence (Q, β) is a $\mathbb{Z}_2$-characteristic pair.

Let (Q, β) be a $\mathbb{Z}_2$-characteristic function. Let G_F be the subgroup of $\mathbb{Z}_2^n$ generated by $\{\beta_{j_1}, \ldots, \beta_{j_l}\}$ whenever $F = F_{j_1} \cap \ldots \cap F_{j_l}$. Define an equivalence relation $\sim_z$ on $\mathbb{Z}_2^n \times Q$ by

$$(t, p) \sim_z (s, q) \text{ if } p = q \text{ and } s - t \in G_F \tag{2.2.5}$$

where $F \subset Q$ is the unique face whose relative interior contains p. Let

$$N(Q, \beta) = (\mathbb{Z}_2^n \times Q)/\sim_z$$

be the quotient space. Following the proof of the theorem 1.2.9 we can show that the quotient space $N(Q, \beta)$ is a manifold. The action of $\mathbb{Z}_2^n$ by the left translations descends to a locally standard $\mathbb{Z}_2^n$-action on $N(Q, \beta)$. The projection onto the second factor of $\mathbb{Z}_2^n \times Q$ descends to a projection $\varsigma_\beta : N(Q, \beta) \to Q$. Hence $N(Q, \beta)$ is an n-dimensional small cover over Q.

Theorem 2.2.8 (Proposition 1.8, [DJ91])**.** *Let $\varsigma : N_s \to Q$ be a small cover over Q and the function $\beta : \mathcal{F}(Q) \to \mathbb{F}_2^n$ defined in 2.2.4 be its $\mathbb{Z}_2$-characteristic function. Let $\varsigma_\beta : N(Q, \beta) \to Q$ be the constructed small cover from the pair (Q, β). Then there exists an equivariant homeomorphism from N_s to $N(Q, \beta)$ covering the identity over Q. Hence small cover is determined up to equivalence over Q by its $\mathbb{Z}_2$-characteristic function.*

Remark 2.2.9. *The constructive definition of small cover give an idea to introduce the notion of small orbifolds, see Chapter 4.*

Example 2.2.10. *Let Q^2 be the standard 2 simplex in $\mathbb{R}^2$. The only possible $\mathbb{Z}_2$-characteristic vectors are described by the Figure 2.1. The product $\mathbb{Z}_2^2 \times Q^2$ is 4 copies of Q^2. Identifying the faces of $\mathbb{Z}_2^2 \times Q^2$ according to the equivalence relation $\sim_z$ we can show that the small cover $N(Q^2, \beta)$ is the real projective space $\mathbb{RP}^2$. By theorem 2.2.8,*

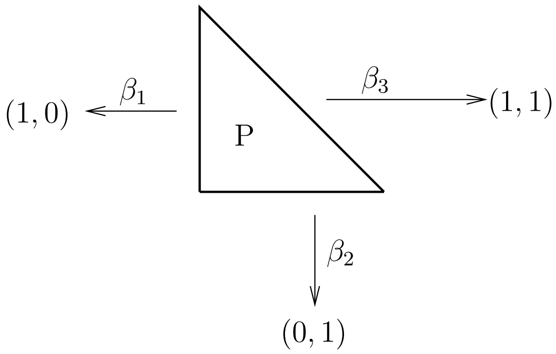


Figure 2.1: The $\mathbb{Z}_2$-characteristic function corresponding to a triangle.

we can show that the small cover $N(\triangle^n, \beta)$ corresponding to the $\mathbb{Z}_2$-characteristic pair $(\triangle^n, \beta)$ is equivariantly homeomorphic to $\mathbb{RP}^n$.

Remark 2.2.11. *Suppose λ is the characteristic function of a $2n$-dimensional quasitoric manifold $N = N(Q, \lambda)$. Consider the involution $\bar{\tau}$ on $\mathbb{T}^n \times Q$ defined by $(t, p) \to (t^{-1}, p)$. The fixed point set of $\bar{\tau}$ is $\mathbb{Z}_2^n \times Q$. The involution $\bar{\tau}$ descends to the involution τ on $N(Q, \lambda)$ with the fixed point set homeomorphic to $N(Q, \beta)$, where $\beta : \mathcal{F}(Q) \to \mathbb{F}_2^n$ is the mod 2 reduction of $\lambda : \mathcal{F}(Q) \to \mathbb{Z}^n$. For example, the fixed point set of the complex conjugation on the complex projective space $\mathbb{CP}^n$ is the real projective space $\mathbb{RP}^n$.*

Following [DJ91], we give some examples of simple polytopes on which there exist no $\mathbb{Z}_2$-characteristic function.

Example 2.2.12. *For each integer n and $k \geq (n+1)$, cyclic polytope is defined as the convex hull of k distinct points on the moment curve $\wp : \mathbb{R} \to \mathbb{R}^n$ defined by $\wp(t) = (t, t^2, \ldots, t^n) \in \mathbb{R}^n$. We denote this cyclic polytope by C_k^n. The Vandermonde determinant identity gives that no $(n+1)$ vertices of C_k^n lie on a common affine hyperplane. Hence C_k^n is a simplicial n-polytope with k vertices.*

Let $n \geq 4$ and $k \geq 2^n$. Let Q_k^n be the dual polytope of C_k^n. Since $n \geq 4$, the 1-skeleton of C_k^n is a complete graph. So for any two facets $F_i, F_j \in \mathcal{F}(Q_k^n)$, the intersection $F_i \cap F_j$ is a nonempty face of Q_k^n. Suppose there exist a $\mathbb{Z}_2$-characteristic function

$$\beta : \mathcal{F}(Q_k^n) \to \mathbb{F}_2^n.$$

So $\beta(F_i)$ and $\beta(F_j)$ are distinct nonzero vectors in $\mathbb{F}_k^n$. This contradicts to the definition of $\mathbb{Z}_2$-characteristic function. Hence, there can be no such function β when $k > 2^n$. Therefore there does not exists a $\mathbb{Z}_2$-characteristic function on the set of facets of Q_k^n.

Hence by remark 2.2.11, when $n \geq 4$ and $k \geq 2^n$ the polytope Q_k^n cannot be the orbit space of a quasitoric manifold.

2.3 Classical effective orbifolds

Following [ALR07] we give the definition of classical effective orbifolds. Let Y be a Hausdorff topological space.

Definition 2.3.1. *An n-dimensional orbifold chart on an open subset $V \subseteq Y$ is given by a triple $(\widetilde{V}, H, \zeta)$ where*

1. *$\widetilde{V}$ is a connected open subset of $\mathbb{R}^n$,*
2. *H is a finite subgroup of smooth automorphisms of $\widetilde{V}$,*
3. *ζ is a map from $\widetilde{V}$ to Y such that ζ is H-invariant map and induces a homeomorphism from $\widetilde{V}/H$ onto V.*

Definition 2.3.2. *An embedding $\xi : (\widetilde{V}, H, \zeta) \to (\widetilde{U}, G, \varphi)$ between two orbifold charts is a smooth embedding $\xi : \widetilde{V} \to \widetilde{U}$ of manifolds with $\zeta \circ \varphi = \xi$.*

Definition 2.3.3. *Two orbifold charts $(\widetilde{V}, H, \zeta)$ on $V = \zeta(\widetilde{V}) \subseteq Y$ and $(\widetilde{U}, G, \varphi)$ on $U = \varphi(\widetilde{U}) \subseteq Y$ with a point $x \in V \cap U$ are locally compatible if there exists an open neighborhood $W \subseteq V \cap U$ of x and an orbifold chart $(\widetilde{W}, K, \mu)$ on W such that there are smooth embeddings $(\widetilde{W}, K, \mu) \to (\widetilde{V}, H, \zeta)$ and $(\widetilde{W}, K, \mu) \to (\widetilde{U}, G, \varphi)$.*

Definition 2.3.4. *A smooth orbifold atlas on Y is a family $\mathcal{V} = \{(\widetilde{V}, H, \zeta)\}$ of locally compatible orbifold charts such that $\{\varphi(\widetilde{V}) : \widetilde{V} \in \mathcal{V}\}$ is an open cover of Y.*

Definition 2.3.5. *An atlas $\mathcal{V}$ is a refinement of an atlas $\mathcal{U}$ if for any chart $(\widetilde{V}, H, \zeta) \in \mathcal{V}$ there exists an embedding $\xi : (\widetilde{V}, H, \zeta) \to (\widetilde{U}, G, \varphi)$ into some chart $(\widetilde{U}, G, \varphi) \in \mathcal{U}$.*

Two orbifold atlases are said to be equivalent if they have a common refinement. Denote the equivalence class of an atlas $\mathcal{V}$ by $[\mathcal{V}]$.

Definition 2.3.6. *Let Y be a paracompact Hausdorff space equipped with an equivalence class $[\mathcal{V}]$ of n-dimensional smooth orbifold atlases. The pair $(Y, \mathcal{V})$, denoted by $\mathcal{Y}$, is called an effective smooth orbifold of dimension n.*

Throughout this section we assume that all orbifolds are effective. We enlist some observations about the definition.

Observation 2.3.7.

1. *For each orbifold chart $(\widetilde{V}, H, \zeta)$ the group H is acting smoothly and effectively on $\widetilde{V}$. In particular H acts freely on a dense open subset of $\widetilde{V}$. With this property $\mathcal{Y}$ is called a reduced orbifold.*
2. *A linear chart is the triple $(\mathbb{R}^n, H, \zeta)$, where H is a finite subgroup of $O(n)$. The group H acts on $\mathbb{R}^n$ via an orthogonal representation. Since smooth actions are locally smooth, any orbifold has an atlas consisting of linear charts.*

3. *Given two embeddings of orbifold charts* $\xi_1, \xi_2 : (\widetilde{V}, H, \zeta) \hookrightarrow (\widetilde{U}, G, \varphi)$*, there exists a unique* $g \in G$ *such that* $\xi_2 = g \circ \xi_1$*. The proof follows from Lemma* 2.11 *in [MM03].*

   As a consequence, an embedding of orbifold charts $\xi : (\widetilde{V}, H, \zeta) \hookrightarrow (\widetilde{U}, G, \varphi)$ *induces a monomorphism* $\xi_{HG} : H \to G$ *of groups.*

4. *Given an embedding* $\xi : (\widetilde{V}, H, \zeta) \hookrightarrow (\widetilde{U}, G, \varphi)$*, if there exists* $g \in G$ *such that* $\xi(\widetilde{V}) \cap g \circ \xi(\widetilde{V}) \neq \phi$ *then* $g \in Im(\xi_{HG})$ *and so* $\xi(\widetilde{V}) = g \circ \xi(\widetilde{V})$*.*

5. *If* $(\widetilde{V}, H, \zeta)$ *and* $(\widetilde{U}, G, \varphi)$ *are two charts for the same orbifold structure on* $V \subseteq Y$ *and if* $\widetilde{V}$ *is simply connected, then there exists an embedding* $\xi : (\widetilde{V}, H, \zeta) \hookrightarrow (\widetilde{U}, G, \varphi)$ *whenever* $\zeta(\widetilde{V}) \subset \varphi(\widetilde{U}) \subseteq V$*.*

6. *Every orbifold atlas for* Y *is contained in a unique maximal atlas.*

7. *If the finite group actions on all the charts are free, then* Y *is locally Euclidean, hence a manifold.*

Definition 2.3.8. *Let* $\mathcal{Y} = (Y, \mathcal{V})$ *be an orbifold and* $y \in Y$*. Let* $(\widetilde{V}, H, \zeta)$ *be an orbifold chart so that* $y = \varphi(x) \in \varphi(\widetilde{V}) \subset Y$*. The local group at* y *is defined by the group*

$$H_y = \{h \in H : h \cdot x = x\}.$$

The group H_y is uniquely determined up to conjugacy. We use the notion of local group to define the singular set of the orbifold $\mathcal{Y}$ in the following definition.

Definition 2.3.9. *A point* $y \in Y$ *is called a smooth point if the group* H_y *is trivial, otherwise* y *is called singular point. The set of singular points of an orbifold* $\mathcal{Y} = (Y, \mathcal{V})$ *is called its singular set, denoted by* $\Sigma\mathcal{Y}$*. That is,*

$$\Sigma\mathcal{Y} = \{y \in Y : H_y \neq 1\}.$$

Definition 2.3.10. *Let* $H \times M \to M$ *be an smooth and effective action of a finite group* H *on a smooth manifold* M*. The associated orbifold* $\mathcal{Y} = (M/H,\ \mathcal{V})$ *is called an effective global quotient, where* $\mathcal{V}$ *is constructed from a manifold atlas using the locally smooth structure.*

Example 2.3.11. *Let* H *be a finite subgroup of* $GL_n(\mathbb{C})$ *and let* $Y = \mathbb{C}^n/H$*. This is a singular complex manifold called a quotient singularity.* Y *has the structure of an algebraic variety, arising from the algebra of* H*-invariant polynomials on* $\mathbb{C}^n$*. If* $H \subset SL_n(\mathbb{C})$*, the quotient* $\mathbb{C}^n/H$ *is called Gorenstein.*

Example 2.3.12. *Consider*

$$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : \Sigma_i |z_i|^2 = 1\}, \tag{2.3.1}$$

the circle $S^1 \ni \mathfrak{a}$ *act on* S^{2n+1} *by*

$$\mathfrak{a}(z_0, \ldots, z_n) = (\mathfrak{a}^{a_0} z_0, \ldots, \mathfrak{a}^{a_n} z_n),$$

where the a_i*'s are coprime integers. The quotient space*

$$W\mathbb{P}(a_0, \ldots, a_n) = S^{2n+1}/S^1$$

has an orbifold structure, denoted by $\mathcal{W}\mathbb{P}(a_0, \ldots, a_n)$*. This orbifold is called a weighted projective space. The orbifold* $\mathcal{W}\mathbb{P}(1, a)$ *is the famous teardrop. It is well known that these orbifolds are non-global quotient orbifold. We will show in Chapter 3 that teardrops are quasitoric orbifolds.*

Example 2.3.13. *Orbifold Riemann surfaces are generalization from the teardrop. These are a fundamental class of examples in orbifold theory. We need to specify the isolated singular points and the order of the local group at each one. Let* y_i *is a singular point with order* n_i*. Then the local chart at* y_i *is* $(D^2, \mathbb{Z}_{n_i}, \zeta_{y_i})$ *where* D^2 *is an open ball centered at origin and* ζ_{y_i} *is the orbit map of the action* $\mathfrak{e} \cdot z = \mathfrak{e} z$ *for a generator* $\mathfrak{e}$ *of* $\mathbb{Z}_{n_i}$*.*

Let Σ *be an orbifold Riemann surface of genus* g *and* k *singular points. Thurston [Thu3m] has shown that* Σ *is a global quotient if and only if* $g + 2k \geq 3$ *or* $g = 0$ *and* $k = 2$ *with* $n_1 = n_2$*. An orbifold Riemann surface can be expressed as a orbit space* $M^3/\mathbb{T}^1$ *for some* 3*-dimensional Seifert fiber manifold* M^3 *with an effective action of* $\mathbb{T}^1$*.*

Next we define the notion of smooth maps between orbifolds.

Definition 2.3.14. *Let* $\mathcal{Y} = (Y, \mathcal{V})$ *and* $\mathcal{W} = (W, \mathcal{U})$ *be two orbifolds. A map* $f : Y \to W$ *is called smooth if for any point* $y \in Y$ *there are charts* $(\widetilde{V}, H, \zeta)$ *containing* y *and* $(\widetilde{U}, G, \varphi)$ *containing* $f(y)$*, such that* f *maps* $V = \zeta(\widetilde{V})$ *into* $U = \varphi(\widetilde{U})$ *and* f *can be lifted to a smooth map* $\widetilde{f} : \widetilde{V} \to \widetilde{U}$ *with* $\varphi \circ \widetilde{f} = f \circ \zeta$*.*

Using this we can define the notion of diffeomorphism of orbifolds.

Definition 2.3.15. *Two orbifolds* $\mathcal{Y}$ *and* $\mathcal{W}$ *are diffeomorphic if there are smooth maps of orbifolds* $f : Y \to W$ *and* $g : W \to Y$ *with* $f \circ g = 1_W$ *and* $g \circ f = 1_Y$*.*

2.4 Tangent bundle of orbifolds

In this section we define the tangent of an orbifold and related notions following [ALR07]. We show the identifications of orbifold charts to yield the original orbifold.

Let $\mathcal{Y} = (Y, \mathcal{V})$ be an orbifold. Let $(\widetilde{V}, H, \zeta)$ and $(\widetilde{U}, G, \varphi)$ be two orbifold charts with $y \in \zeta(\widetilde{V}) \cap \varphi(\widetilde{U})$. So by definition there is a chart $(\widetilde{W}, K, \mu)$ and embeddings

$$\xi_1 : (\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{V}, H, \zeta) \text{ and } \xi_2 : (\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{U}, G, \varphi).$$

These two embeddings give rise the following equivariant diffeomorphisms

$$\xi_1^{-1} : \xi_1(\widetilde{W}) \to \widetilde{W} \text{ and } \xi_2 : \widetilde{W} \to \xi_2(\widetilde{W}). \tag{2.4.1}$$

between K-spaces. Here K acts on $\xi_1(\widetilde{W})$ and $\xi_2(\widetilde{W})$ via the subgroups ${\xi_1}_{KG}(K) \subseteq G$ and ${\xi_2}_{KH}(K) \subseteq H$ respectively. The composition of these maps give an equivariant diffeomorphism

$$\xi_2 \circ \xi_1^{-1} : \xi_1(\widetilde{W}) \to \xi_2(\widetilde{W}). \tag{2.4.2}$$

between K-spaces. Hence we can glue the sets $\widetilde{V}/H$ and $\widetilde{U}/G$ according to the equivalence relation $\sim_t$ defined by

$$\zeta(\widetilde{v}) \sim_t \varphi(\widetilde{u}) \text{ if } \xi_2 \circ \xi_1^{-1}(\widetilde{v}) = \widetilde{u}. \tag{2.4.3}$$

Let

$$\widehat{Y} = \bigsqcup_{\widetilde{V} \in \mathcal{V}} (\widetilde{V}/H)/ \sim_t \tag{2.4.4}$$

be the space obtained by gluing the sets $\{\widetilde{V}/H : (\widetilde{V}, H, \zeta) \in \mathcal{V}\}$. So we get a homeomorphism $\Phi : \widehat{Y} \to Y$ induced by the maps $\zeta : \widetilde{V} \to Y$.

We may consider the function $\xi_2 \circ \xi_1^{-1}$ as a transition function. Suppose there exist another two embeddings

$$\xi_1' : (\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{V}, H, \zeta),\ \xi_2' : (\widetilde{W}, K, \mu) \hookrightarrow (\widetilde{U}, G, \varphi).$$

We have observed in 3 of 2.3.7 that there exist unique $h \in H$ and $g \in G$ such that

$$\xi_1' = h \circ \xi_1 \text{ and } \xi_2' = g \circ \xi_2.$$

Hence $g \circ (\xi_2 \circ \xi_1^{-1}) \circ h^{-1}$ is the resulting transition function.

Using this explicit computations, we construct the tangent bundle of the orbifold Y. Given an orbifold chart $(\widetilde{V}, H, \zeta)$, we consider the tangent bundle $\mathcal{T}\widetilde{V}$ of the manifold $\widetilde{V}$. By the observation 1 of 2.3.7 the group H acts smoothly on $\widetilde{V}$. Hence H also acts smoothly on $\mathcal{T}\widetilde{V}$. Suppose $(\widetilde{v}, \alpha)$ is an element of $\mathcal{T}\widetilde{V}$, then $h \in H$ acts by

$$h \cdot (\widetilde{v}, \alpha) = (h \cdot \widetilde{v}, Dh_{\widetilde{v}}(\alpha)). \tag{2.4.5}$$

Let $\mathcal{T}V = \mathcal{T}\widetilde{V}/H$ and $\zeta_{TV} : \mathcal{T}\widetilde{V} \to \mathcal{T}V$ be the orbit map. So the triple $(\mathcal{T}\widetilde{V}, H, \zeta_{TU})$ is an orbifold chart on $\mathcal{T}U$. The projection map $\mathcal{T}\widetilde{V} \to \widetilde{V}$ is an equivariant map. This map induces a natural projection $p_V : \mathcal{T}V \to V$. We describe the fibers of the map p_V. If $\widetilde{y} \in \widetilde{V}$ and $y = \zeta(\widetilde{y}) \in V$, then

$$p_V^{-1}(y) = \{H(z, v) : z = \widetilde{y}\} \subset \mathcal{T}V. \tag{2.4.6}$$

We claim that $p_V^{-1}(y)$ is homeomorphic to the orbit space $\mathcal{T}_{\widetilde{y}}\widetilde{U}/H_y$ of H_y action on $\mathcal{T}_{\widetilde{y}}\widetilde{U}$, where H_y denotes the isotropy group of the H-action on $\widetilde{V}$ at $\widetilde{y}$. Define the map

$$f_y : p_V^{-1}(y) \to \mathcal{T}_{\widetilde{y}}\widetilde{V}/H_y \text{ by } f_y(H(\widetilde{y}, \alpha)) = H_y\alpha. \tag{2.4.7}$$

Observe that

$$H(\widetilde{y}, \alpha_1) = H(\widetilde{y}, \alpha_2) \text{ if and only if there exists a } h \in H \text{ such that } h \cdot (\widetilde{y}, \alpha_1) = (\widetilde{y}, \alpha_2).$$

Again

$$h \cdot (\widetilde{y}, \alpha_1) = (\widetilde{y}, \alpha_2) \text{ if and only if } h \in H_y \text{ and } D_{\widetilde{y}}h(\alpha_1) = \alpha_2.$$

This is equivalent to the assertion that $H_y\alpha_1 = H_y\alpha_2$. So f_y is both well defined and injective. From the following commutative diagram it is clear that f_y is a surjection and continuous map.

$$\begin{array}{ccc} \mathcal{T}\widetilde{V} & \longrightarrow & \widetilde{V} \\ \zeta_{TV}\big\downarrow & & \zeta\big\downarrow \\ \mathcal{T}V & \xrightarrow{p_V} & V \end{array} \tag{2.4.8}$$

Hence we established our claim. The fiber $p_V^{-1}(y)$ is a quotient of the form $\mathbb{R}^n/H_0$, where $H_0 \subset GL_n(\mathbb{R})$ is a finite group. So the fiber $p_V^{-1}(y)$ may not be a vector space. Hence we have constructed a bundle-like object $\mathcal{T}V$ over V.

Definition 2.4.1. *The map $p_V : \mathcal{T}V \to V$ is called an orbifold tangent bundle associated to the orbifold chart $(\widetilde{V}, H, \zeta)$.*

It may be clear how to construct the tangent bundle on an orbifold $\mathcal{Y} = (Y, \widetilde{V})$. Let

$$\mathcal{T}\mathcal{V} = \{(\mathcal{T}\widetilde{V}, H, \zeta_{TV}) : (\widetilde{V}, H, \zeta) \in \mathcal{V}\}.$$

We need to identify the orbifold tangent bundles $\mathcal{T}V \to V$ associated to the charts $(\mathcal{V}, H, \zeta)$. We observe that the gluing maps $\xi_{12} = \xi_2 \circ \xi_1^{-1}$ for orbifold charts of $\mathcal{V}$ in equation 2.4.2 are smooth. We may use the equivariant differential

$$D\xi_{12} : \mathcal{T}\xi_1(\widetilde{W}) \to \mathcal{T}\xi_2(\widetilde{W}) \tag{2.4.9}$$

as the transition functions to identify the bundles $\mathcal{T}V \to V$ and $\mathcal{T}U \to U$. Let $\mathcal{T}Y$ be the resulting identification space

$$(\bigsqcup_{\widetilde{V}\in\mathcal{V}} \mathcal{T}V)/\sim_T . \tag{2.4.10}$$

We consider the minimal topology on $\mathcal{T}Y$ such that each inclusion $\mathcal{T}V \to \mathcal{T}Y$ is homeomorphic onto an open subset of $\mathcal{T}Y$. With this topology, $\mathcal{T}Y$ has an orbifold structure. $\mathcal{T}\mathcal{V}$ is a family of locally compatible orbifold charts such that the collection $\{\zeta_{TV}(\mathcal{T}\widetilde{V}) : (\widetilde{V}, H, \zeta) \in \mathcal{V}\}$ is an open cover of $\mathcal{T}Y$. So the family $\mathcal{T}\mathcal{U}$ is an orbifold atlas on $\mathcal{T}Y$. By the identification $\sim_T$, there exists a continuous surjection $p_Y : \mathcal{T}Y \to Y$ such that $p_Y|_{\mathcal{T}V} = p_V$. Let $\mathcal{T}\mathcal{Y} = (\mathcal{T}Y, \mathcal{T}\mathcal{V})$.

Definition 2.4.2. *The triple $(\mathcal{T}\mathcal{Y}, p_Y, \mathcal{Y})$ is called the orbifold tangent bundle of the orbifold $\mathcal{Y}$.*

We summarize the above computations in the next proposition.

Proposition 2.4.3 ([ALR07], Proposition 1.21)**.** *The tangent bundle of an n-dimensional orbifold $\mathcal{Y}$ has the structure of a $2n$-dimensional orbifold. Also the map p_Y is a smooth map of orbifolds with fibers $\mathcal{T}_{\widetilde{y}}\widetilde{V}/H_y$.*

Now we define the frame bundle of an orbifold $\mathcal{Y}$. Note that for each local chart $(\widetilde{V}, H, \zeta)$ we can define an H-invariant inner product on $\mathcal{T}\widetilde{V}$. Let $O(\mathcal{T}_{\widetilde{y}}\widetilde{V})$ be the orthogonal transformations of $\mathcal{T}_{\widetilde{y}}\widetilde{V}$. We can construct the frame manifold

$$Fr(\widetilde{V}) = \{(\widetilde{y}, A) : A \in O(\mathcal{T}_{\widetilde{y}}\widetilde{V})\} \tag{2.4.11}$$

and the induced left H-action on $Fr(\widetilde{V})$ is given by

$$h \cdot (\widetilde{y}, A) = (h\widetilde{y}, Dh_{\widetilde{y}}A). \tag{2.4.12}$$

Since the H-action on $\widetilde{V}$ is an effective action, the H-action on the frame manifold $Fr(\widetilde{V})$ is free. So the quotient space $Fr(\widetilde{V})/H$ is a smooth manifold. Denote the orbit of $(\widetilde{y}, A)$ by $[\widetilde{y}, A]$. There is a right $O(n)$ action on $Fr(\widetilde{V})/H$ induced from the natural translation action on $Fr(\widetilde{V})$, given by

$$[\widetilde{y}, A] \cdot B = [\widetilde{y}, AB] \text{ for } B \in O(n). \tag{2.4.13}$$

Observe that this action is transitive on the fibers. Indeed, $[\widetilde{y}, A] = [\widetilde{y}, I] \cdot A$. The isotropy subgroup for this orbit consists of those orthogonal matrices A such that

$$(\widetilde{y}, A) = (h\widetilde{y}, Dh_{\widetilde{y}}I)$$

for some $h \in H$. This is equivalent to say that $h \in H_y$ and $A = Dh_{\widetilde{y}}$. So the differential establishes an injection $H_y \to O(\mathcal{T}_{\widetilde{y}}\widetilde{V})$. Hence we conclude that H_y is the isotropy subgroup of the $O(n)$-action on $Fr(\widetilde{V})/H$. The fiber is the associated homogeneous space $O(n)/H_y$.

Consider the orbit space of the action 2.4.13 on $Fr(\widetilde{V})/H$. Clearly this orbit space is homeomorphic to V. So we obtain the natural projection $Fr(\widetilde{V})/H \to V$. Let $Fr(\mathcal{Y})$ be the space obtained by identifying the local charts $Fr(\widetilde{V})/H \to UV$ using the $O(n)$-transition functions obtained from the tangent bundle of $\mathcal{Y}$. Let $p_{Fr} : Fr(\mathcal{Y}) \to Y$ be the induced continuous surjection.

Definition 2.4.4. *The triple $(Fr(\mathcal{Y}), p_{Fr}, \mathcal{Y})$ is called the frame bundle of an orbifold $\mathcal{Y} = (Y, \mathcal{V})$.*

This frame bundle has some useful properties, which we summarize below.

Theorem 2.4.5 ([ALR07], Theorem 1.23)**.** *For a given orbifold $\mathcal{Y}$, its frame bundle $Fr(\mathcal{Y})$ is a smooth manifold with a smooth, effective and almost free $O(n)$-action. The orbifold $\mathcal{Y}$ is naturally isomorphic to the resulting quotient orbifold $Fr(\mathcal{Y})/O(n)$.*

The following is a very important consequence of the theorem 2.4.5.

Corollary 2.4.6 ([ALR07], Corollary 1.24)**.** *Every smooth effective n-dimensional orbifold $\mathcal{Y}$ is diffeomorphic to a quotient orbifold for a smooth, effective and almost free $O(n)$-action on a smooth manifold M.*

Definition 2.4.7. *Let $\mathcal{Y} = (Y, \mathcal{V})$ denote an orbifold with tangent bundle $(\mathcal{T}\mathcal{Y}, p_Y, \mathcal{Y})$.*

1. *A non-degenerate symmetric 2-tensor of $S^2(T\mathcal{Y})$ is called a Riemannian metric on $\mathcal{Y}$.*

2. *An almost complex structure on $\mathcal{Y}$ is an endomorphism $J : T\mathcal{Y} \to T\mathcal{Y}$ such that $J^2 = -Id$.*

3. *A stable almost complex structure on $\mathcal{Y}$ is an endomorphism*

$$J : T\mathcal{Y} \oplus (Y \times \mathbb{R}^k) \to T\mathcal{Y} \oplus (Y \times \mathbb{R}^k)$$

   such that $J^2 = -Id$ for some positive integer k.

4. *We call $\mathcal{Y}$ a complex orbifold if all the defining maps are holomorphic.*

Using the frame bundle of an orbifold, we see that techniques applicable to quotient spaces of almost free smooth action of a compact Lie group will yield results about orbifolds. For example, we have the following proposition. The proof of this proposition can be found in [AP93].

Proposition 2.4.8 ([ALR07], Proposition 1.28)**.** *If a compact, connected Lie group G acts smoothly and almost freely on an orientable, connected, compact manifold M, then $H^*(M/G;\mathbb{Q})$ is a Poincaré duality algebra. Hence, if $\mathcal{Y}$ is a compact, connected, orientable orbifold, then $H^*(Y;\mathbb{Q})$ will satisfy Poincaré duality.*

2.5 Orbifold fundamental group

The goal of this section is to provide an idea how one can compute the orbifold fundamental group of an effective smooth orbifold.

A covering orbifold or orbifold cover of an n-dimensional orbifold $\mathcal{Y}$ is a smooth map of orbifolds $\mathfrak{p}:\mathcal{X}\to\mathcal{Y}$ whose associated continuous map (also denoted by $\mathfrak{p}$) $\mathfrak{p}:X\to Y$ between underlying spaces satisfies the following condition:

Each point $y\in Y$ has a neighborhood $V\cong\widetilde{V}/H$ with $\widetilde{V}$ homeomorphic to a connected open set in $\mathbb{R}^n$, for which each component U_i of $\mathfrak{p}^{-1}(V)$ is homeomorphic to $\widetilde{V}/H_i$ for some subgroup $H_i\subset H$ such that the natural map $\mathfrak{p}_i:\widetilde{V}/H_i\to\widetilde{V}/H$ corresponds to the restriction of $\mathfrak{p}$ on U_i.

Definition 2.5.1. *Given an orbifold cover $\mathfrak{p}:\mathcal{X}\to\mathcal{Y}$, a diffeomorphism $\mathfrak{h}:\mathcal{X}\to\mathcal{X}$ is called a deck transformation if $\mathfrak{p}\circ\mathfrak{h}=\mathfrak{p}$.*

Definition 2.5.2. *An orbifold cover $\mathfrak{p}:\mathcal{X}\to\mathcal{Y}$ is called a universal orbifold cover of $\mathcal{Y}$ if given any orbifold cover $\mathfrak{p}_1:\mathcal{W}\to\mathcal{Y}$, there exists an orbifold cover $\mathfrak{p}_2:\mathcal{X}\to\mathcal{W}$ such that $\mathfrak{p}=\mathfrak{p}_1\circ\mathfrak{p}_2$.*

Every orbifold has a universal orbifold cover which is unique up to diffeomorphism, see [Thu3m]. The corresponding group of deck transformations is called the *orbifold fundamental group* of $\mathcal{Y}$ and denoted by $\pi_1^{\mathrm{orb}}(\mathcal{Y})$.

Example 2.5.3 (Hurwitz cover)**.** *Suppose that $\mathfrak{p}:\Sigma_1\to\Sigma_2$ is a holomorphic map between two compact orbifold Riemann surfaces Σ_1, Σ_2. Usually, $\mathfrak{p}$ is not a covering map. Instead, it ramifies in finitely many points $z_1,\ldots,z_k\in\Sigma_2$. Hence the restriction map, namely,*

$$\mathfrak{p}:\Sigma_1-\{\cup_i\mathfrak{p}^{-1}(z_i)\}\to\Sigma_2-\{z_1,\ldots,z_k\} \tag{2.5.1}$$

is a manifold covering map. Suppose that the preimage of z_i is $y_{i_1},\ldots,y_{i_{j_i}}$. Let n_{i_l} be the ramification order at y_{i_l}. That is, under suitable coordinate system near y_{i_l}, the map $\mathfrak{p}$ can be written as $z\to z^{n_{i_l}}$.

We assign an orbifold structure on Σ_1 and Σ_2 as follows. We first assign an orbifold structure at y_{i_p} with order n_{i_p}. Let n_i be the largest common factor of the n_{i_p}'s. Then we assign an orbifold structure at z_i with order n_i. One readily verifies that under these assignments, $\mathfrak{p}:\Sigma_1\to\Sigma_2$ becomes an orbifold cover. The map $\mathfrak{p}:\Sigma_1\to\Sigma_2$ is referred to as a Hurwitz cover.

Example 2.5.4. *If $\mathcal{Y} = M/H$ is a global quotient and $\widetilde{M} \to M$ is a universal cover, then $\widetilde{M} \to M \to \mathcal{Y}$ is the orbifold universal cover of $\mathcal{Y}$. This gives an extension of groups*

$$1 \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(\mathcal{Y}) \to H \to 1. \tag{2.5.2}$$

This implies that an orbifold $\mathcal{Y}$ can not be a global quotient if $\pi_1^{\mathrm{orb}}(\mathcal{Y})$ is trivial, unless $\mathcal{Y}$ is itself a manifold.

Definition 2.5.5. *An orbifold is a good orbifold if its orbifold universal cover is smooth manifold.*

Observation 2.5.6. *The following observation is very useful in computations of orbifold fundamental groups. Suppose that $\mathfrak{p} : \mathcal{X} \to \mathcal{Y}$ is an orbifold universal cover. Then the restriction*

$$\mathfrak{p} : \mathcal{X} - \mathfrak{p}^{-1}(\Sigma\mathcal{X}) \to \mathcal{Y} - \Sigma\mathcal{Y} \tag{2.5.3}$$

is a manifold cover. The covering group H of this cover is the orbifold fundamental group $\pi_1^{orb}(\mathcal{Y})$. The sets $\Sigma\mathcal{X}$ and $\Sigma\mathcal{Y}$ are the singular points of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Therefore, $\mathcal{Y} = \mathcal{X}/H$ and there is a surjective homomorphism

$$q_{\mathfrak{p}} : \pi_1(\mathcal{Y} - \Sigma\mathcal{Y}) \to H. \tag{2.5.4}$$

In general, there is no reason to expect that $q_{\mathfrak{p}}$ will be an isomorphism. However, to compute the group $\pi_1^{orb}(\mathcal{Y})$, we can start with the group $\pi_1(\mathcal{Y} - \Sigma\mathcal{Y})$ and then specify the additional relations if required.

Example 2.5.7. *Consider the orbifold Riemann surface Σ_g of genus g with k orbifold points $O_k = \{x_1, \ldots, x_k\}$ of orders $n_1, \ldots, n_k$. Then, according to [Sco83], (p. 424) a presentation for its orbifold fundamental group is given by*

$$\pi_1^{orb}(\Sigma_g) = \{\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k, \sigma_1, \ldots, \sigma_k : \sigma_1 \ldots \sigma_k \Pi_1^g[\alpha_i, \beta_i] = 1, \sigma_i^{n_i} = 1\}, \tag{2.5.5}$$

where α_i and β_i are the generators of $\pi_1(\Sigma_g)$ and σ_i are the generators of $\Sigma_g - O_k$ represented by a loop around each orbifold point. Note that $\pi_1^{orb}(\Sigma_g)$ is obtained from $\pi_1(\Sigma_g - O_k)$ by introducing the relations $\sigma^{m_i} = 1$.

Consider the special case when $\Sigma = \widetilde{\Sigma}/H$, where H is a finite group of automorphisms of $\widetilde{\Sigma}$. In this case, the orbifold fundamental group is isomorphic to $\pi_1(EH \times_H \widetilde{\Sigma})$, which in turn fits into a group extension

$$1 \to \pi_1(\widetilde{\Sigma}) \to \pi_1^{\mathrm{orb}}(\Sigma) \to H \to 1. \tag{2.5.6}$$

In other words, the orbifold fundamental group is a virtual surface group. This will be true for any good orbifold Riemann surface.

Chapter 3

Quasitoric orbifolds

In this chapter we study structures and invariants of quasitoric orbifolds. In particular, we discuss the constructive and axiomatic definitions of quasitoric orbifolds. We compute the orbifold fundamental group of these orbifolds. We determine whether any quasitoric orbifold can be the quotient of a smooth manifold by a finite group action or not. To calculate the rational homology groups of quasitoric orbifolds we need to generalize the usual CW-complex little bit. The cohomology ring of a quasitoric orbifold with coefficient in $\mathbb{Q}$ is computed in this chapter. We prove existence of stable almost complex structure and describe the Chen-Ruan cohomology groups of an almost complex quasitoric orbifold.

3.1 Definition by construction and orbifold structure

For any $\mathbb{Z}$-module N denote $N \otimes_{\mathbb{Z}} \mathbb{R}$ by N_R. Let N be a free $\mathbb{Z}$-module of rank n. The quotient $\mathbb{T}_N = N_R/N$ is a compact n-dimensional torus. Suppose N' is a free submodule of N of rank n'. Let $\mathbb{T}_{N'}$ denote the torus N'_R/N'. Let $j : N'_R \to N_R$ and $j_* : \mathbb{T}_{N'} \to N_R/N'$ be the natural inclusions. The inclusion $i : N' \to N$ induces a homomorphism

$$i_* : N_R/N' \to N_R/N = \mathbb{T}_N \text{ defined by } i_*(a + N') = a + N$$

on cosets. Denote the composition $i_* \circ j_* : \mathbb{T}_{N'} \to \mathbb{T}_N$ by $\xi_{N'}$ and also denote the image of $\xi_{N'}$ by $Im(\xi_{N'})$. $Ker(i_*) \simeq N/N'$. If $n' = n$, then j_* is identity and i_* is a surjection. In this case $\xi_{N'} : \mathbb{T}_{N'} \to \mathbb{T}_N$ is a surjection group homomorphism with kernel $G_{N'} = N/N'$, a finite abelian group.

A $2n$-dimensional quasitoric orbifold may be constructed from the following data: a simple polytope Q of dimension n with set of facets $\mathcal{F}(Q) = \{F_i : i \in \{1, \ldots, m\} = I\}$, a free $\mathbb{Z}$-module N of rank n and a dicharacteristic function, defined below.

Definition 3.1.1. *Let there exists an assignment of a vector λ_i in N to each facet F_i of Q such that whenever $F_{i_1} \cap \ldots \cap F_{i_k} \neq \emptyset$ the corresponding vectors $\lambda_{i_1}, \ldots, \lambda_{i_k}$ are linearly independent over $\mathbb{Z}$. The function $\lambda : \mathcal{F}(Q) \to N$ defined by $\lambda(F_i) = \lambda_i$ is called a dicharacteristic function of Q.*

These data will be referred to as a *combinatorial model* and abbreviated as $(Q, N, \{\lambda_i\})$. The vector λ_i is called the dicharacteristic vector corresponding to the i-th facet.

Example 3.1.2. *The quasitoric orbifolds associated to the first and second combinatorial model has* 1 *and* 3 *singular points respectively.*

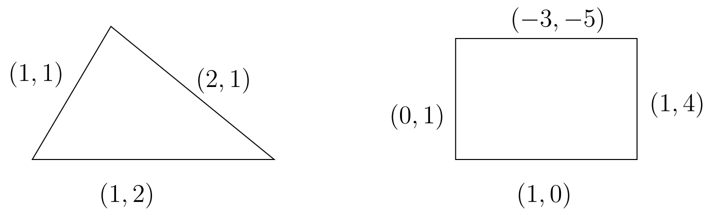


Figure 3.1: Some dicharacteristic function of triangle and receptacle.

We give the constructive definition of quasitoric orbifolds below. Each face F of Q of codimension $k \geq 1$ is the intersection of a unique set of k facets $F_{i_1}, \ldots, F_{i_k}$. Let $I(F) = \{i_1, \ldots, i_k\} \subset I$. Let $N(F)$ denote the submodule of N generated by the characteristic vectors $\{\lambda_j : j \in I(F)\}$. So $\mathbb{T}_{N(F)} = N(F)_R / N(F)$ is a torus of dimension k. We will adopt the convention that $\mathbb{T}_{N(Q)} = 1$.

Define an equivalence relation $\sim$ on the product $Q \times \mathbb{T}_N$ by

$$(p, t) \sim (q, s) \text{ if } p = q \text{ and } s^{-1}t \in Im(\xi_{N(F)}), \tag{3.1.1}$$

where F is the unique face whose relative interior contains p. Let $X = Q \times \mathbb{T}_N / \sim$ be the quotient space. Let $\mathfrak{q} : Q \times \mathbb{T}_N \to X$ denote the quotient map. Then X is a $\mathbb{T}_N$-space and let $\pi : X \to Q$ defined by $\pi([p, t]^{\sim}) = p$ be the associated map to the orbit space Q. The space X has the structure of an orbifold, which we explain next.

Pick open neighborhoods U_v of the vertices v of Q such that U_v is the complement in Q of all facets that do not contain v. Let

$$X_v = \pi^{-1}(U_v) = U_v \times \mathbb{T}_N / \sim .$$

For a face F of Q containing v the inclusion $\{\lambda_i : i \in I(F)\}$ in $\{\lambda_i : i \in I(v)\}$ induces an inclusion of $N(F)$ in $N(v)$ whose image will be denoted by $N(v, F)$. Since $\{\lambda_i : i \in I(F)\}$ extends to a basis $\{\lambda_i : i \in I(v)\}$ of $N(v)$, the natural map from the torus

$$\mathbb{T}_{N(v,F)} = N(v, F)_R / N(v, F) \text{ to } \mathbb{T}_{N(v)} = N(v)_R / N(v)$$

defined by $a + N(v,F) \mapsto a + N(v)$ is an injection. We will identify its image with $\mathbb{T}_{N(v,F)}$. Denote the canonical isomorphism $\mathbb{T}_{N(F)} \to \mathbb{T}_{N(v,F)}$ by $i(v,F)$.

Define an equivalence relation $\sim_v$ on $U_v \times \mathbb{T}_{N(v)}$ by

$$(p,t) \sim_v (q,s) \text{ if } p = q \text{ and } s^{-1}t \in \mathbb{T}_{N(v,F)}$$

where F is the face whose relative interior contains p. Then $W_v = U_v \times \mathbb{T}_{N(v)}/ \sim_v$ is θ-equivariantly diffeomorphic to an open ball in $\mathbb{C}^n$ where $\theta : \mathbb{T}_{N(v)} \to U(1)^n$ is an isomorphism, see [DJ91]. Note that the map $\xi_{N(F)}$ factors as $\xi_{N(F)} = \xi_{N(v)} \circ i(v,F)$. Since $i(v,F)$ is an isomorphism, $t \in \mathbb{T}_{N(v,F)}$ if and only if $\xi_{N(v)}(t) \in Im\xi_{N(F)}$. Hence the map $\xi_{N(v)} : \mathbb{T}_{N(v)} \to \mathbb{T}_N$ induces a map

$$\xi_v : W_v \to X_v \text{ defined by } \xi_v([(p,t)]^{\sim_v}) = [(p, \xi_{N(v)}(t))]^{\sim}$$

on equivalence classes.The group $G_v = N/N(v)$, the kernel of $\xi_{N(v)}$, is a finite subgroup of $\mathbb{T}_{N(v)}$ and therefore has a natural smooth, free action on $\mathbb{T}_{N(v)}$ induced by the group operation. This induces smooth action of G_v on W_v. This action is not free in general. Since $\mathbb{T}_N \cong \mathbb{T}_{N(v)}/G_v$, X_v is homeomorphic to the quotient space W_v/G_v. So (W_v, G_v, ξ_v) is an orbifold chart on X_v. To show the compatibility of these charts as v varies, we introduce some additional charts.

For any proper face E of dimension $k \geq 1$ define $U_E = \bigcap U_v$, where the intersection is over all vertices v that belong to E. Let $X_E = \pi^{-1}(U_E)$. For a face F containing E there is an injective homomorphism $\mathbb{T}_{N(F)} \to \mathbb{T}_{N(E)}$ whose image we denote by $\mathbb{T}_{N(E,F)}$.

Let

$$N^*(E) = (N(E) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap N \quad \text{and} \quad G_E = N^*(E)/N(E). \tag{3.1.2}$$

G_E is a finite group. Let $\xi_{*,E} : \mathbb{T}_{N(E)} \to \mathbb{T}_{N^*(E)}$ be the natural homomorphism. The map $\xi_{*,E}$ has kernel G_E. Denote the quotient $N/N^*(E)$ by $N^{\perp}(E)$. It is a free $\mathbb{Z}$-module and $N \cong N^*(E) \oplus N^{\perp}(E)$. Fixing a choice of this isomorphism (or fixing an inner product on N) we may regard $N^{\perp}(E)$ as a submodule of N. Consequently $\mathbb{T}_N = \mathbb{T}_{N^*(E)} \times \mathbb{T}_{N^{\perp}(E)}$.

Define an equivalence relation $\sim_E$ on $U_E \times \mathbb{T}_{N(E)} \times \mathbb{T}_{N^{\perp}(E)}$ by

$$(p_1, t_1, s_1) \sim_E (p_2, t_2, s_2) \text{ if } p_1 = p_2, \ s_1 = s_2 \text{ and } t_2^{-1}t_1 \in \mathbb{T}_{N(E,F)}$$

where F is the face whose relative interior contains p_1. Let

$$W_E = U_E \times \mathbb{T}_{N(E)} \times \mathbb{T}_{N^{\perp}(E)}/ \sim_E .$$

It is diffeomorphic to $\mathbb{C}^{n-k} \times (\mathbb{C}^*)^k$. There is a natural map $\xi_E : W_E \to X_E$ induced by $\xi_{*,E} : \mathbb{T}_{N(E)} \to \mathbb{T}_{N^*(E)}$ and the identity maps on U_E and $\mathbb{T}_{N^{\perp}(E)}$. The triple

(W_E, G_E, ξ_E) is an orbifold chart on X_E.

Given E, fix a vertex v of Q contained in E. $N(v) = N(E) \oplus N'$ where N' is the free submodule of $N(v)$ generated by the dicharacteristic vectors $\{\lambda_j : j \in I(v) - I(E)\}$. Consequently $\mathbb{T}_{N(v)} = \mathbb{T}_{N(E)} \times \mathbb{T}_{N'}$. We can, without loss of generality, assume that $N' \subset N^{\perp}(E)$. Thus we have a covering homomorphism $\mathbb{T}_{N'} \to \mathbb{T}_{N^{\perp}(E)}$. For a point $x = [p, t, s] \in X_E$, choose a small neighborhood B of s in $\mathbb{T}_{N^{\perp}(E)}$ such that B lifts to $\mathbb{T}_{N'}$. Choose any such lift and denote it by $l : B \to \mathbb{T}_{N'}$. Let

$$W_x = U_E \times \mathbb{T}_{N(E)} \times B / \sim_E .$$

So (W_x, G_E, ξ_E) is an orbifold chart on a neighborhood of x, and it is induced by (W_E, G_E, ξ_E). The natural map $W_x \hookrightarrow W_v$ induced by the map l and the identification $\mathbb{T}_{N(v)} = \mathbb{T}_{N(E)} \oplus \mathbb{T}_{N'}$, and the natural injective homomorphism $G_E \hookrightarrow G_v$ induce an embedding of orbifold charts

$$(W_x, G_E, \xi_E) \to (W_v, G_v, \xi_v).$$

The existence of these embeddings shows that the charts $\{(W_v, G_v, \xi_v) : v \text{ vertex of } Q\}$ are compatible and form part of a maximal $2n$-dimensional orbifold atlas $\mathcal{A}$ for X. We denote the pair $\{X, \mathcal{A}\}$ by $\mathcal{X}$. We say that $\mathcal{X}$ is the quasitoric orbifold associated to the combinatorial model $(Q, N, \{\lambda_i\})$.

Remark 3.1.3. *Note that the orbifold $\mathcal{X}$ is a reduced orbifold. Also note that changing the sign of a dicharacteristic vector gives rise to a diffeomorphic orbifold.*

In the case of a quasitoric orbifold $\mathcal{X}$, for any $x \in X$, $\pi(x)$ belongs to the relative interior of a uniquely determined face E^x of Q. The isotropy group $G_x = G_{E^x}$ (see (3.1.2)). We adopt the convention that $G_Q = 1$.

Definition 3.1.4. *A quasitoric orbifold is called primitive if all its dicharacteristic vectors are primitive.*

Note that in a primitive quasitoric orbifold the local group actions are devoid of complex reflections (that is, maps which have one as an eigenvalue with multiplicity $n-1$) and the classification theorem of [Pri67] for germs of complex orbifold singularities applies.

3.2 Axiomatic definition of quasitoric orbifolds

Analyzing the structure of the quasitoric orbifold associated to a combinatorial model, we make the following axiomatic definition. This is a generalization of the axiomatic definition of a quasitoric manifold using the notion of locally standard action, as mentioned in the introduction.

Definition 3.2.1. *A $2n$-dimensional quasitoric orbifold $\mathcal{Y}$ is an orbifold whose underlying topological space Y has a $\mathbb{T}_N$ action, where N is a fixed free $\mathbb{Z}$-module of rank n, such that the orbit space is (diffeomorphic to) a simple n-dimensional polytope Q. Denote the projection map from Y to Q by $\pi : Y \to Q$. Furthermore every point $x \in Y$ has*

A_1. *a $\mathbb{T}_N$-invariant neighborhood V,*

A_2. *an associated free $\mathbb{Z}$-module N' of rank n with an isomorphism $\theta : \mathbb{T}_{N'} \to U(1)^n$ and an injective module homomorphism $\iota : N' \to N$ which induces a surjective covering homomorphism $\xi_{N'} : \mathbb{T}_{N'} \to \mathbb{T}_N$,*

A_3. *an orbifold chart (W, G, ξ) over V where W is θ-equivariantly diffeomorphic to an open set in $\mathbb{C}^n$, $G = Ker\xi_{N'}$ and $\xi : W \to V$ is an equivariant map i.e. $\xi(t \cdot y) = \xi_{N'}(t) \cdot \xi(y)$ inducing a homeomorphism between W/G and V.*

It is obvious that a quasitoric orbifold defined constructively from a combinatorial model satisfies the axiomatic definition. We now demonstrate that a quasitoric orbifold defined axiomatically is associated to a combinatorial model.

Take any facet F_i of Q and let F_i^0 be its relative interior. By the characterization of local charts in A_3), the isotropy group of the $\mathbb{T}_N$ action at any point x in $\pi^{-1}(F_i^0)$ is a locally constant circle subgroup of $\mathbb{T}_N$. It is the image under $\xi_{N'}$ of a circle subgroup of $\mathbb{T}_{N'}$. Thus it determines a locally constant vector, up to choice of sign, λ_i in N. Since $\pi^{-1}(F_i^0)$ is connected, we get a characteristic vector λ_i, unique up to sign, for each facet of Q. That the characteristic vectors corresponding to all facets of Q which meet at a vertex are linearly independent follows from the fact that their preimages under the appropriate ι form a basis of N'. Thus we recover a combinatorial model $(Q, N, \{\lambda_i\})$ starting from $\mathcal{Y}$.

Definition 3.2.2. *Call the triple $(Q, N, \{\lambda_i\})$ a combinatorial model of $\mathcal{Y}$.*

Remark 3.2.3. *Similarly to the quasitoric manifolds case we can prove that a quasitoric orbifold has a smooth structure. In [GP09], the authors give an explicit smooth orbifold charts for 4-dimensional quasitoric orbifolds.*

Lemma 3.2.4. *Let $\mathcal{X}$ be the quasitoric orbifold obtained from the combinatorial model $(Q, N, \{\lambda_i\})$ of $\mathcal{Y}$ by the construction 3.1. Then $\mathcal{X}$ and $\mathcal{Y}$ are diffeomorphic orbifolds.*

Proof. The hard part is to show the existence of $\mathbb{T}_N$-equivariant a continuous map from $X \to Y$. This can be done following Lemma 1.2.15 and Corollary 1.2.17. The idea is to stratify $\mathcal{Y}$ according to *normal orbit type*, see Davis [Dav78]. Here we need to use the fact that the orbifold $\mathcal{Y}$ being reduced, is the quotient of a compact smooth manifold by the foliated action of a compact Lie group. Then one can *blow up* (see [Dav78]) the singular strata of Y to get a manifold $\widehat{Y}$ equivariantly diffeomorphic to $\mathbb{T}_N \times Q$.

One has to modify the arguments of Davis slightly in the orbifold case. The important thing is that by the differentiable slice theorem each singular stratum has a neighborhood diffeomorphic to its orbifold normal bundle, and is thus equipped with a fiberwise linear structure so that the constructions of Davis go through. Finally there is a collapsing map $\widehat{Y} \to Y$ and by composition with the above diffeomorphism a map $\mathbb{T}_N \times Q \to Y$. It is easily checked that this map induces a continuous equivariant map $X \to Y$. □

Definition 3.2.5. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be quasitoric orbifolds whose associated base polytope Q and free $\mathbb{Z}$-module N are identical. Let θ be an automorphism of $\mathbb{T}_N$. A map $\mathbf{f} : \mathcal{X}_1 \to \mathcal{X}_2$ of quasitoric orbifolds is called a θ-equivariant diffeomorphism if $\mathbf{f}$ is an diffeomorphism of orbifolds and the induced map on underlying spaces $f : X_1 \to X_2$ satisfies $f(t \cdot x) = \theta(t) \cdot f(x)$ for all $x \in X_1$ and $t \in \mathbb{T}_N$.*

Definition 3.2.6. *Two θ-equivariant diffeomorphisms $\mathbf{f}, \mathbf{g} : \mathcal{X}_\mathbf{1} \to \mathcal{X}_\mathbf{2}$ are said to be equivalent if there exists equivariant diffeomorphisms $\mathbf{h_i} : \mathcal{X}_i \to \mathcal{X}_i$, for $i = 1, 2$, such that $\mathbf{g} \circ \mathbf{h_1} = \mathbf{h_2} \circ \mathbf{f}$.*

We also define, for θ as above, the θ-translation of a combinatorial model $(Q, N, \{\lambda_i\})$ to be the combinatorial model $(Q, N, \{\theta'(\lambda_i)\})$, where θ' is an automorphism of N induced by θ.

The following lemma classifies quasitoric orbifolds over a fixed polytope up to θ-equivariant diffeomorphism.

Lemma 3.2.7. *For any automorphism θ of $\mathbb{T}_N$, the assignment of combinatorial model defines a bijection between equivalence classes of θ-equivariant diffeomorphisms of quasitoric orbifolds and θ-translations of combinatorial models.*

Proof. Proof is similar to the Lemma 1.2.19, which we discuss in details. Note that the existence of a preferred section $s : Q \to Y$ for an axiomatically defined quasitoric orbifold $\mathcal{Y}$ follows from the blow up construction in the proof of Lemma 3.2.4. □

3.3 Characteristic subspaces

Of special importance are certain $\mathbb{T}_N$-invariant subspaces of X corresponding to the faces of the polytope Q. If F is a face of Q of codimension k, then define $X(F) := \pi^{-1}(F)$. With subspace topology, $X(F)$ is a quasitoric orbifold of dimension $2n - 2k$. Recall that

$$N^*(F) = (N(F) \otimes_\mathbb{Z} \mathbb{Q}) \cap N \text{ and } N^\perp(F) = N/N^*(F).$$

Let $\varrho_F : N \to N^{\perp}(F)$ be the projection homomorphism. Let $J(F) \subset I$ be the index set of facets of Q, other than F in case $k = 1$, that intersect F. Note that $J(F)$ indexes the set of facets of the $n - k$ dimensional polytope F. The combinatorial model for $X(F)$ is given by $(F, N^{\perp}(F), \varrho_F \circ \lambda_{|J(F)})\})$.

Definition 3.3.1. *$X(F)$ is called a characteristic subspace of X, if F is a facet of Q.*

Example 3.3.2. *Let Q^1 be the 1-polytope with vertices v_1, v_2 and $N = \mathbb{Z}$. Let $\lambda_1 = \lambda(v_1) = 1$ and $\lambda_2 = \lambda(v_1) = a \in \mathbb{Z} - \{-1, 0, 1\}$. So $(Q^1, N, \{\lambda_i\})$ is a combinatorial model. The quasitoric orbifold associated to this model is the weighted projective space $\mathcal{W}\mathbb{P}(1, a)$.*

3.4 Orbifold fundamental group

We first give a canonical construction of a quasitoric orbifold cover $\mathcal{O}$ for any given quasitoric orbifold $\mathcal{X}$. We will prove later that $\mathcal{O}$ is the universal orbifold cover of $\mathcal{X}$. Let $\widehat{N}$ be the submodule of N generated by the characteristic vectors of $\mathcal{X}$.

Definition 3.4.1. *Let $\widehat{\lambda}_i$ denote the characteristic vector λ_i as an element of $\widehat{N}$. Let $\mathcal{O}$ be the quasitoric orbifold associated to the combinatorial model $(Q, \widehat{N}, \{\widehat{\lambda}_i)\}$. Denote the corresponding equivalence relation by $\widehat{\sim}$ so that the underlying topological space of $\mathcal{O}$ is $O = Q \times \mathbb{T}_{\widehat{N}}/\widehat{\sim}$. Denote the quotient map $Q \times \mathbb{T}_{\widehat{N}} \to O$ by $\widehat{\pi}$.*

Proposition 3.4.2. *The quasitoric orbifold $\mathcal{O}$ is an orbifold cover of the quasitoric orbifold $\mathcal{X}$ with deck group $N/\widehat{N}$.*

Proof. The inclusion $\iota : \widehat{N} \hookrightarrow N$ induces a surjective group homomorphism

$$\iota_* : \mathbb{T}_{\widehat{N}} = (\widehat{N} \otimes \mathbb{R})/\widehat{N} \to \mathbb{T}_N = (N \otimes \mathbb{R})/N$$

with kernel $N/\widehat{N}$. In fact for any face F of Q we have commuting diagram

$$\begin{array}{ccc} \mathbb{T}_{\widehat{N}(F)} & \xrightarrow{\xi_{\widehat{N}(F)}} & \mathbb{T}_{\widehat{N}} \\ \iota_0 \big\downarrow & & \iota_* \big\downarrow \\ \mathbb{T}_{N(F)} & \xrightarrow{\xi_{N(F)}} & \mathbb{T}_N \end{array} \tag{3.4.1}$$

where $\widehat{N}(F)$ is $N(F)$ viewed as a sublattice of $\widehat{N}$ and ι_0 is an isomorphism induced by ι. Thus there is an induced surjective map

$$\iota_1 : \mathbb{T}_{\widehat{N}}/Im(\xi_{\widehat{N}(F)}) \to \mathbb{T}_N/Im(\xi_{N(F)}). \tag{3.4.2}$$

We obtain a torus equivariant map $f : O \to X$ defined fiberwise by 3.4.2, that is, for any point $q \in Q$ belonging to the relative interior of the face F, the restriction of $f : \widehat{\pi}^{-1}(q) \to \pi^{-1}(q)$ matches ι_1.

The map f lifts to a smooth map of orbifolds $\mathbf{f} : \mathcal{O} \to \mathcal{X}$. Consider orbifold charts on $\mathcal{X}$ and $\mathcal{O}$ corresponding to vertex v. Identifying $\widehat{N}(v)$ and $\widehat{N}(v, F)$ with $N(v)$ and $N(v, F)$ respectively, we note that

$$\widehat{W}_v = U_v \times \mathbb{T}_{\widehat{N}(v)}/\widehat{\sim}_v \text{ may be identified with } W_v = U_v \times T_{N(v)}/\sim_v.$$

Hence $O_v = W_v/\widehat{G}_v$ and $f : O_v \to X_v$ is given by the projection $W_v/\widehat{G}_v \to W_v/G_v$ where $\widehat{G}_v = \widehat{N}/N(v)$ is a subgroup of $G_v = N/N(v)$. So $\mathbf{f} : \mathcal{O} \to \mathcal{X}$ is in fact an orbifold covering. The deck group for this covering is clearly $N/\widehat{N}$. □

Theorem 3.4.3. *The quasitoric orbifold $\mathcal{O}$ is the orbifold universal cover of the quasitoric orbifold $\mathcal{X}$. The orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{X})$ of $\mathcal{X}$ is isomorphic to $N/\widehat{N}$.*

Proof. Let Σ denote the singular loci of $\mathcal{X}$ (refer to definition 2.3.9). The set Σ has real codimension at least 2 in X. Note that $\pi(\Sigma)$ is a union of faces of Q. Let $Q_{\Sigma} = Q - \pi(\Sigma)$. Observe that

$$X - \Sigma = \pi^{-1}(Q_{\Sigma}) = (Q_{\Sigma} \times \mathbb{T}_N)/\sim .$$

Since Q_{Σ} is contractible, $\pi_1(Q_{\Sigma} \times \mathbb{T}_N) \cong \pi_1(\mathbb{T}_N) \cong N$. When we take quotient of $Q_{\Sigma} \times \mathbb{T}_N$ by the equivalence relation $\sim$, certain elements of this fundamental group are killed. Precisely, if Q_{Σ} contains a point p which belongs to the intersection of certain facets $F_1, \ldots, F_k$ of Q, then the elements $\lambda_1, \ldots, \lambda_k$ of N given by the corresponding characteristic vectors map to the identity element of $\pi_1(X - \Sigma)$.

Let $I(\Sigma)$ be the collection of facets of Q that have nonempty intersection with Q_{Σ}. Let $N(\Sigma)$ be the submodule generated by those λ_i for which $i \in I(\Sigma)$. Then the argument above suggests that

$$\pi_1(X - \Sigma) = N/N(\Sigma).$$

Indeed, this can be established easily by systematic use of the Seifert-Van Kampen theorem.

It is instructive to first do the proof in the case $\mathcal{X}$ is primitive (see Definition 3.1.4). Here $G_{F_i} = 1$ (see 3.1.2) for each facet F_i. Hence $I(\Sigma) = I$ and $N_{\Sigma} = \widehat{N}$. Therefore $\pi_1(X - \Sigma) = N/\widehat{N}$. Hence by Proposition 3.4.2,

$$f_0 : O - f^{-1}(\Sigma) \to X - \Sigma,$$

where f_0 is the restriction of f, is the universal covering.

Now if $\mathbf{p} : \mathcal{W} \to \mathcal{X}$ is any orbifold cover then the induced map

$$p_0 : W - p^{-1}(\Sigma) \to X - \Sigma$$

is a manifold cover. Since $p^{-1}(\Sigma)$ has real codimension at least two in W, $W - p^{-1}(\Sigma)$ is connected. Hence f_0 factors through p_0 and $\mathbf{f}$ factors through $\mathbf{p}$. Let

$$g_0 : O - f^{-1}(\Sigma) \to W - p^{-1}(\Sigma)$$

be the covering map such that $f_0 = p_0 \circ g_0$. We'll show that g_0 can be extend to an orbifold covering $g : O \to W$ such that $f = p \circ g$. Locally the orbifold cover $f : O \to X$ is given by

$$f : O_v = W_v/\widehat{G}_v \to W_v/G_v = X_v$$

for each vertex $v \in Q$. Also we have the commutative diagram

$$\begin{array}{ccc} O_v - f^{-1}\Sigma & \xrightarrow{g_v} & p^{-1}(X_v) - p^{-1}(\Sigma) \\ {\scriptstyle f_v}\big\downarrow & & {\scriptstyle p_v}\big\downarrow \\ X_v - \Sigma & = & X_v - \Sigma \end{array}$$

where $f_v,\ g_v,\ p_v$ are restrictions of $f,\ g,\ p$ respectively. The orbifold $\mathcal{O}$ is compact, Hausdorff, second countable topological space, so by Urysohn metrization theorem O is a metric space with metric d_o and the topology induced by d_o is the topology of O.

Let $G_1 = \pi_1(W - p^{-1}(\Sigma))$. Define

$$d_2 : W - p^{-1}(\Sigma) \times W - p^{-1}(\Sigma) \to [0, \infty) \text{ by } d_2(y_1, y_2) = d_1(G_1x_1, G_1x_2),$$

where $x_i \in O$ and $g_0(x_i) = y_i, i = 1, 2$. Then d_2 is a metric on $W - p^{-1}(\Sigma)$. The topology induced by d_2 on $W - p^{-1}(\Sigma)$ is quotient topology which is subspace topology on $W - p^{-1}(\Sigma)(\subseteq W)$. Suppose $(\widetilde{W}, \widetilde{d_2})$ is metric completion of $(W - p^{-1}(\Sigma), d_2)$. So $W - p^{-1}(\Sigma)$ is a dense open subset of $\widetilde{W}$. Since $W - p^{-1}(\Sigma)$ is a dense open subset of compact spaces W, W is compact subset of $\widetilde{W}$ and the topology induced by $\widetilde{d_2}$ on W is the topology of W.

Now suppose $x \in O_v \cap f^{-1}(\Sigma)$. From the construction of $\mathcal{O}$ there exist a Cauchy sequence $\{x_n\}$ in $O_v - f^{-1}(\Sigma) \subset O$ converging to x. Since g_0 is a covering map

$$d_2(g_0(x_l), g_0(x_m)) = d_2(G_1x_l, G_1x_m) \leq d_1(x_l, x_m)$$

Hence $\{g_0(x_n)\}$ is a Cauchy sequence in $W - p^{-1}(\Sigma)$ and converge to $y \in W$. Define $g(x) = y$. So g_0 can be extend to $g : O \to W$.

$p(y) = p \circ g(x)$

$= p \circ lit_{n\to\infty} g(x_n)$
$= lit_{n\to\infty} p \circ g(x_n)$; since p is continuous map
$= lit_{n\to\infty} f(x_n) = f(x)$.

Hence **f** factors through **p**. From the above commutative diagram it is clear that $g : O \to W$ is an orbifold cover.

For the general case we will use an argument which is similar to that of Scott [Sco83] for orbifold Riemann surfaces. The underlying idea also appeared in remarks after Proposition 13.2.4 of Thurston [Thu3m].

The group $N/\widehat{N}$ is naturally a quotient of $\pi_1(X - \Sigma) = N/N(\Sigma)$ and the corresponding projection homomorphism has kernel $K = \widehat{N}/N(\Sigma)$. Consider the manifold covering $f_0 : f^{-1}(X - \Sigma) \to X - \Sigma$ obtained by restricting the map $f : O \to X$. Note that

$$\pi_1(f^{-1}(X - \Sigma)) = K$$

and the deck group of f_0 is $N/\widehat{N}$. Let $\mathcal{W}$ be any orbifold covering of $\mathcal{X}$ with projection map **p**. Then $W_0 = W - p^{-1}(\Sigma)$ is a covering of $X - \Sigma$ in the usual sense. We claim that $\pi_1(W_0)$ contains K as a subgroup.

Let $\bar{\lambda}_i$ denote the image of λ_i in $N/N(\Sigma)$. Obviously $\{\bar{\lambda}_i, : i \in I - I(\Sigma)\}$ generate K. Physically such a $\bar{\lambda}_i$ can be represented by the conjugate of a small loop c_i in $X - \Sigma$ going around some point $x_i \in \pi^{-1}(F_i^\circ)$ once in a plane transversal to $\pi^{-1}(F_i)$, where F_i° denotes the relative interior of the facet F_i.

The point x_i has a neighborhood U in X homeomorphic to $\mathbb{C}^{n-1} \times (\mathbb{C}/G_{F_i})$. Therefore a connected component V of the preimage $p^{-1}(U) \subset W$ is homeomorphic to $\mathbb{C}^{n-1} \times (\mathbb{C}/G'_{F_i})$ where G'_{F_i} is a subgroup of G_{F_i}. We may assume, without loss of generality, that

$$c_i \text{ lies in the plane } \{0\} \times \mathbb{C}/G_{F_i}.$$

By the definition of G_{F_i}, $\bar{\lambda}_i$ is trivial in G_{F_i} and hence in G'_{F_i}. Identifying G_{F_i} with the deck group of the covering $\mathbb{C}^* \to \mathbb{C}^*/G_{F_i}$, we infer that c_i lifts to a loop in $\mathbb{C}^*$ and consequently in $\mathbb{C}^*/G'_{F_i}$. Hence c_i lifts to a loop in $V - p^{-1}(\Sigma)$. Thus each generator and therefore every element of K is represented by a loop in W_0. This induces a homomorphism

$$K \to \pi_1(W_0).$$

This homomorphism is injective since K is a subgroup of the fundamental group of the space $X - \Sigma$ which has W_0 as a cover.

For any orbifold covering $\mathcal{W}$ of $\mathcal{X}$, the associated covering W_0 of $X - \Sigma$ admits a covering by $f^{-1}(X - \Sigma) \subset O$ since $\pi_1(f^{-1}(X - \Sigma)) = K$ is a normal subgroup of $\pi_1(W_0)$. Thus $\mathcal{O}$ is an orbifold cover of $\mathcal{W}$. Hence $\mathcal{O}$ is the universal orbifold cover of $\mathcal{X}$ and H is the orbifold fundamental group of $\mathcal{X}$. □

Remark 3.4.4. *Note that the orbifold fundamental group of a quasitoric orbifold is always a finite group. It follows that a quasitoric orbifold is a global quotient if and only if its orbifold universal cover is a smooth manifold. Therefore Theorem 3.4.3 yields a rather easy method for determining if a quasitoric orbifold is a global quotient or not. For example, by Theorem 3.4.3* $\pi_1^{orb}(\mathcal{W}\mathbb{P}(1,a)) = 1$ *which imply that* $\mathcal{W}\mathbb{P}(1,a)$ *is not a global quotient.*

Example 3.4.5. *If* $\widehat{N} = N$*, then* $\mathcal{X}$ *is not a global quotient unless* $\mathcal{X}$ *is a manifold. For instance, let* Q *be a* 2*-dimensional simplex with characteristic vectors* $(1,1)$, $(1,-1)$, $(-1,0)$ *and let* $\mathcal{X}$ *be the quasitoric orbifold corresponding to this model. Then* $N = \widehat{N}$*, but* $\mathcal{X}$ *has an orbifold singularity at* $\pi^{-1}(v)$ *where* $v = F_1 \cap F_2$*. Therefore* $\mathcal{X}$ *is not a global quotient.*

3.5 q-Cellular homology groups

We introduce the notion of **q**-CW complex where an open cell is the quotient of an open disk by action of a finite group. Otherwise the construction mirrors the construction of usual CW complex given in Hatcher [Hat02]. We show that our **q**-cellular homology of a **q**-CW complex is isomorphic to its singular homology with coefficients in $\mathbb{Q}$.

Definition 3.5.1. *Let* G *be a finite group acting linearly, preserving orientation, on an* n*-dimensional disk* $\overline{D}^n$ *centered at the origin. Such an action preserves* $S^{n-1} = \partial\overline{D}^n$*. We call the quotient* $\overline{D}^n/G$ *an* n*-dimensional* **q***-disk. Call* S^{n-1}/G *a* **q***-sphere.*

An n*-dimensional* **q***-cell* $e^n_G = e^n(G)/G$ *is defined to be a copy of* D^n/G *where* $\overline{e}^n(G)$ *is* G*-equivariantly homeomorphic to* $\overline{D}^n$*. We will denote the boundary of* $\overline{e}^n(G)$ *by* S^{n-1} *without confusion.*

Start with a discrete set X_0 , where points are regarded as 0-dimensional **q**-cells. Inductively, form the n-dimensional **q**-skeleton X_n from X_{n-1} by attaching n-dimensional **q**-cells $e^n_{G_\alpha}$ via continuous maps

$$\phi_\alpha : S^{n-1}_\alpha/G_\alpha \to X_{n-1}.$$

This means that X_n is the quotient space of the disjoint union $X_{n-1} \sqcup_\alpha \overline{e}^n_{G_\alpha}$ of X_{n-1} with a finite collection of n-dimensional **q**-disks $\overline{e}^n_\alpha(G_\alpha)/G_\alpha$ under the identification

$$x \sim_n \phi_\alpha(x) \text{ for } x \in S^{n-1}_\alpha/G_\alpha.$$

Assume $X = X_n$ for some finite n. The topology of X is the quotient topology built inductively. We call a space X constructed in this way a finite **q**-CW complex.

By Proposition 2.22 and Corollary 2.25 of [Hat02],

$$H_p((X_n, X_{n-1});\mathbb{Q}) = \bigoplus_{\alpha} \widetilde{H}_p(\frac{\overline{D}^n_\alpha/G_\alpha}{S^{n-1}_\alpha/G_\alpha};\mathbb{Q}) \tag{3.5.1}$$

Note that

$$\widetilde{H}_p(\frac{\overline{D}^n_\alpha/G_\alpha}{S^{n-1}_\alpha/G_\alpha};\mathbb{Q}) = \begin{cases} H_{p-1}(S^{n-1}_\alpha/G_\alpha;\mathbb{Q}) & \text{if } p \geq 2 \\ 0 & \text{otherwise} \end{cases} \tag{3.5.2}$$

Lemma 3.5.2. *Let $\overline{D}^n/G$ be a* $\mathbf{q}$*-disk. Then S^{n-1}/G is a $\mathbb{Q}$-homology sphere.*

Proof. S^{n-1} admits a simplicial G-complex structure. Apply Theorem 2.4 of Bredon [Bre72] and Poincaré duality for orbifolds. □

Lemma 3.5.3. *If X is a* $\mathbf{q}$*-CW complex, then*

1. $H_p((X_n, X_{n-1});\mathbb{Q}) = \begin{cases} 0 & \text{for } p \neq n \\ \oplus_{i\in I_n}\mathbb{Q} & \text{for } p = n \end{cases}$
   where I_n is the set of n-dimensional $\mathbf{q}$*-cells in X.*
2. $H_p(X_n;\mathbb{Q}) = 0$ *for $p > n$. In particular,* $H_p(X;\mathbb{Q}) = 0$ *for* $p > \dim(X)$.
3. *The inclusion $i : X_n \hookrightarrow X$ induces an isomorphism $i_* : H_p(X_n;\mathbb{Q}) \to H_p(X;\mathbb{Q})$ if $p < n$.*

Proof. Proof is similar to the proof of Lemma 2.3.4 of [Hat02]. The key ingredient is Lemma 3.5.2. □

Using Lemma 3.5.3 we can define $\mathbf{q}$-cellular chain complex $(H_p(X_p, X_{p-1}), d_p)$ and $\mathbf{q}$-cellular groups $H_p^{\mathbf{q}\text{-}CW}(X;\mathbb{Q})$ of X in the same way as cellular chain complex is defined in [Hat02], page 139.

Theorem 3.5.4. $H_p^{\mathbf{q}\text{-}CW}(X;\mathbb{Q}) \cong H_p(X;\mathbb{Q})$ *for all p.*

Proof. Proof is similar to the proof of Theorem 2.35 of [Hat02]. □

3.6 Rational homology of quasitoric orbifolds

Following Goresky [Gor78] one may obtain a CW structure on a quasitoric orbifold. However it is too complicated for easy computation of homology. We now follow the main ideas of the computation for the manifold case as in Section 1.5 to compute the rational homology groups of quasitoric orbifold X over Q. First we construct a $\mathbf{q}$-CW structure on X. We adhere the notations of Section 1.5.

Let $v \in Q$ be a vertex of index $\mathfrak{f}(v) = k$. We put $\mathbf{e}_v = \pi^{-1}(\widehat{F}_v)$. Then $\mathbf{e}_v$ is a contractible subspace of $X(F_v)$ homeomorphic to the quotient of an open disk $D^{2\mathfrak{f}(v)}$ in $\mathbb{R}^{2\mathfrak{f}(v)}$ by a finite group $G(v)$ determined by the orbifold structure on $X(F_v)$ described in Section 3.3. $\widehat{F}_v$ is homeomorphic to the intersection of the unit disk in $\mathbb{R}^{\mathfrak{f}(v)}$ with $\mathbb{R}^{\mathfrak{f}(v)}_{\geq 0}$. Since the action of the group $G(v)$ is obtained from a combinatorial model, see Section 3.3, $\mathbf{e}_v$ is a $2\mathfrak{f}(v)$-dimensional **q**-cell.

X can be given the structure of a **q**-CW complex as follows. Define the k-skeleton $X_{2k} := \bigcup_{\mathfrak{f}(v)=k} X(F_v)$ for $0 \leq k \leq n$. $X_{2k+1} = X_{2k}$ for $0 \leq k \leq n-1$ and $X_{2n} = X$. X_{2k} can be obtained from X_{2k-1} by attaching those **q**-cells $\mathbf{e}_v$ for which $\mathfrak{f}(v) = k$. The attaching maps are to be described. Let $\sim$ be the equivalence relation such that

$$X(F_v) = F_v \times \mathbb{T}_{N^{\perp}(F_v)}/\sim.$$

The **q**-disk $\overline{D}^{2\mathfrak{f}(v)}/G(v)$ can be identified with $F_v \times \mathbb{T}_{N^{\perp}(F_v)}/\approx$ where

$$(p,t) \approx (q,s) \text{ if } p = q \in F' \text{ for some face } F' \text{ whose top vertex is } v \text{ and } (p,t) \sim (q,s).$$

The attaching map $\phi_v : S^{2\mathfrak{f}(v)-1}/G(v) \to X_{2\mathfrak{f}(v)-1}$ is the natural quotient map

$$(F_v - \widehat{F}_v) \times \mathbb{T}_{N^{\perp}(F_v)}/\approx \to (F_v - \widehat{F}_v) \times \mathbb{T}_{N^{\perp}(F_v)}/\sim.$$

So X is a **q**-CW complex with no odd dimensional cells and with $\mathfrak{f}^{-1}(k) = h_k$ number of $2k$-dimensional **q**-cells. Hence by **q**-cellular homology theory

$$H_p^{\mathbf{q}\text{-}CW}(X;\mathbb{Q}) = \begin{cases} \bigoplus_{\mathfrak{f}^{-1}(p/2)} \mathbb{Q} & \text{if } p \leq n \text{ and } p \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{3.6.1}$$

Hence by Theorem 3.5.4

$$H_p(X;\mathbb{Q}) = \begin{cases} \bigoplus_{h_{(p/2)}} \mathbb{Q} & \text{if } p \leq n \text{ and } p \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{3.6.2}$$

3.7 Gysin sequence for q-sphere bundle

Let $\rho : E \to B$ be a rank n vector bundle with paracompact base space B. Restricting ρ to the space E_0 of nonzero vectors in E, we obtain an associated projection map $\rho_0 : E_0 \to B$. Fix a finite group G and a representation of G on $\mathbb{R}^n$. Such a representation induces a fiberwise linear action of G on E and E_0. Consider the two fiber bundles

$$\rho^G : E/G \to B \text{ and } \rho_0^G : E_0/G \to B.$$

There exist natural fiber bundle maps $f_1 : E \to E/G$ and $f_2 : E_0 \to E_0/G$. These induce isomorphisms

$$f_1^* : H^p(E/G) \to H^p(E) \text{ and } f_2^* : H^p(E_0/G) \to H^p(E)$$

for each p. The second isomorphism is obtained by applying Theorem 2.4 of [Bre72] fiberwise and then using Kunneth formula, Mayer-Vietoris sequence and a direct limit argument as in the proof of Thom isomorphism in [MS74]. The commuting diagram

$$\begin{array}{ccccc} E_0 & \xrightarrow{i_1} & E & \xrightarrow{j_1} & (E, E_0) \\ f_2 \downarrow & & f_1 \downarrow & & \downarrow f_3 \\ E_0/G & \xrightarrow{i_2} & E/G & \xrightarrow{j_2} & (E/G, E_0/G) \end{array}$$

induces a commuting diagram of two exact rows

$$\begin{array}{ccccccc} \cdots \to H^{p-1}(E_0) & \xrightarrow{\delta_1^*} & H^p(E, E_0) & \xrightarrow{j_1^*} & H^p(E) & \xrightarrow{i_1^*} & H^p(E_0) \to \cdots \\ f_2^* \uparrow & & f_3^* \uparrow & & \uparrow f_1^* & & \uparrow f_2^* \\ \cdots \to H^{p-1}(E_0/G) & \xrightarrow{\delta_2^*} & H^p(E/G, E_0/G) & \xrightarrow{j_2^*} & H^p(E/G) & \xrightarrow{i_2^*} & H^p(E_0/G) \to \cdots \end{array}$$

By the five lemma f_3^* is an isomorphism. Using the Thom isomorphism $\cup u : H^{p-n}(E) \to H^p(E, E_0)$ we get the isomorphism

$$\cup u_G : H^{p-n}(E/G) \to H^p(E/G, E_0/G) \text{ where } \cup u_G = {f_3^*}^{-1} \circ \cup u \circ f_1^*.$$

Substituting the isomorphic module $H^{p-n}(E/G)$ in place of $H^p(E/G, E_0/G)$ in the second row of the above diagram, we obtain an exact sequence

$$\cdots \to H^{p-n}(E/G) \xrightarrow{g} H^p(E/G) \to H^p(E_0/G) \to H^{p-n+1}(E/G) \to \cdots$$

where $g = j_2^* \circ \cup u_G$. The pull back of cohomology class $u_G|(E/G)$ in $H^n(B)$ by the zero section of ρ^G will be called the Euler class $\mathfrak{e}$ of ρ^G. Now substitute the isomorphic cohomology ring $H^*(B)$ in place of $H^*(E/G)$ in the above sequence. This yields the Gysin exact sequence for the $\mathbf{q}$-sphere bundle $\rho_0^G : E_0/G \to B$

$$\cdots \to H^{p-n}(B; \mathbb{Q}) \xrightarrow{\cup e} H^p(B; \mathbb{Q}) \to H^p(E_0/G; \mathbb{Q}) \to H^{p-n+1}(B; \mathbb{Q}) \to \cdots \quad (3.7.1)$$

Remark 3.7.1. *Euler classes of $\rho : E \to B$ and $\rho^G : E/G \to B$ are the same since f_1^* is an isomorphism.*

3.8 Cohomology ring of quasitoric orbifolds

Again we will modify some technical details but retain the broad framework of the argument in [DJ91] to get the anticipated answer. All homology and cohomology modules in this section will have coefficients in $\mathbb{Q}$. Let $\mathfrak{L}$ be the simplicial complex associated to the boundary of the dual polytope of Q. Then Q is the cone on the barycentric subdivision of $\mathfrak{L}$. Q can be split into cubes Q_σ where σ varies over $(n-1)$-dimensional faces of $\mathfrak{L}$. These correspond bijectively to vertices of Q. We regard the k-cube as the orbit space of standard k-dimensional torus action on the $2k$-disk

$$\overline{D}^{2k} = \{(z_1, \cdots, z_k) \in \mathbb{C}^k : |z_i| \leq 1\} \tag{3.8.1}$$

Define

$$BQ_\sigma = E\mathbb{T}_N \times_{\mathbb{T}_N} ((Q_\sigma \times \mathbb{T}_N)/\sim) \simeq E\mathbb{T}_N \times_{\mathbb{T}_N} (\overline{D}^{2n}/G_\sigma),$$

where $G_\sigma = G_{v_\sigma}$, v_σ being the vertex in Q dual to σ. If σ_1 is another $(n-1)$ simplex in L such that $\sigma \cap \sigma_1$ is an $(n-2)$ simplex then BQ_σ and BQ_{σ_1} are glued along the common part of the boundaries of Q_σ and Q_{σ_1}. In this way BQ_σ fit together to yield

$$BQ = E\mathbb{T}_N \times_{\mathbb{T}_N} X.$$

Let $\mathfrak{p} : BQ \to B\mathbb{T}_N$ be the Borel map which is a fibration with fiber X. The fibration $\mathfrak{p} : BQ \to B\mathbb{T}_N$ induces a homomorphism

$$\mathfrak{p}^* : H^*(B\mathbb{T}_N; \mathbb{Q}) \to H^*(BQ; \mathbb{Q}).$$

The face ring $SR(Q, \mathbb{Q})$ is graded by declaring the degree of each w_i to be 2. The following result resembles Theorem 4.8 of [DJ91].

Theorem 3.8.1. *Let Q be an n-polytope and $SR(Q,\mathbb{Q})$ be the face ring of Q. The map $\mathfrak{p}^* : H^*(B\mathbb{T}_N;\mathbb{Q}) \to H^*(BQ;\mathbb{Q})$ is surjective and induces an isomorphism of graded rings $H^*(BQ;\mathbb{Q}) \cong SR(Q,\mathbb{Q})$.*

Proof. Suppose σ is an $(n-1)$-simplex in L with vertices $w_1, \ldots, w_n$. Note that there is a one-to-one correspondence between facets of Q meeting at v_σ and vertices of σ. Let Q_σ be the corresponding n-cube in Q. Then $BQ_\sigma = E\mathbb{T}_N \times_{\mathbb{T}_N} (\overline{D}^{2n}/G_\sigma)$ is a $\overline{D}^{2n}/G_\sigma$ fiber bundle over $B\mathbb{T}_N$. Hence

$$E\mathbb{T}_N \times_{\mathbb{T}_N} (S^{2n-1}/G_\sigma) \to B\mathbb{T}_N$$

give the associated **q**-sphere bundle $\mathfrak{p}_\sigma : BQ_{\partial\sigma} \to B\mathbb{T}_N$. Also consider the disk bundle

$$\mathfrak{r} : E\mathbb{T}_N \times_{\mathbb{T}_N} \overline{D}^{2n} \to B\mathbb{T}_N.$$

It is bundle homotopic to the complex vector bundle

$$\mathfrak{r}' : E\mathbb{T}_N \times_{\mathbb{T}_N} \mathbb{C}^n \to B\mathbb{T}_N.$$

Since $\mathbb{T}_N$ acts diagonally on $\mathbb{C}^n$, the last bundle is the sum of line bundles $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$

where $\mathcal{L}_j$ corresponds to j-th coordinate direction in $\mathbb{C}^n$ and hence to w_j. Without confusion, we set

$$c_1(\mathcal{L}_i) = w_i \in H^2(B\mathbb{T}_N; \mathbb{Q}).$$

By the Whitney product formula $c_n(\mathfrak{r}') = w_1 \cdots w_n$. Hence from Section 3.7 the Euler class of the $\mathbf{q}$-sphere bundle $\mathfrak{p}_\sigma$ is $\mathfrak{e} = w_1 \cdots w_n$.

Now consider the Gysin exact sequence for $\mathbf{q}$-sphere bundles

$$\begin{aligned} \cdots \to H^*(BQ_{\partial\sigma}) \to H^*(B\mathbb{T}_N) \xrightarrow{\cup \mathfrak{e}} H^{*+2n}(B\mathbb{T}_N) \xrightarrow{\mathfrak{p}_\sigma^*} H^{*+2n}(BQ_{\partial\sigma}) \\ \to H^{*+2n}(B\mathbb{T}_N) \to \cdots \end{aligned} \tag{3.8.2}$$

Since the map $\cup\mathfrak{e}$ is injective, by exactness $\mathfrak{p}_\sigma^*$ is surjective and we get the following diagram

$$\begin{array}{ccccc} 0 \to H^*(B\mathbb{T}_N) & \xrightarrow{\cup \mathfrak{e}} & H^{*+2n}(B\mathbb{T}_N) & \xrightarrow{\mathfrak{p}_\sigma^*} & H^{*+2n}(BQ_{\partial\sigma}) \to 0 \\ id \downarrow & & id \downarrow & & \\ \mathbb{Q}[w_1, \cdots, w_n] & \xrightarrow{w_1 \ldots w_n} & \mathbb{Q}[w_1, \ldots, w_n]. & & \end{array} \tag{3.8.3}$$

Hence from diagram (3.8.3) $H^*(BQ_{\partial\sigma}) = \mathbb{Q}[w_1, \cdots, w_n]/(w_1 \ldots w_n)$. Since $\overline{D}^{2n}/G_\sigma$ is contractible, $H^*(BQ_\sigma; \mathbb{Q}) = H^*(B\mathbb{T}_N; \mathbb{Q}) = \mathbb{Q}[w_1, \cdots, w_n]$. Using induction on the dimension of L and an application of the Mayer-Vietoris sequence we get the conclusion of the Theorem. □

Let $j : X \to BQ$ be inclusion of the fiber. Consider the Serre spectral sequence of the fibration $\mathfrak{p} : BQ \to B\mathbb{T}_N$ with fiber X. It has E_2-term

$$E_2^{p,q} = H^p(B\mathbb{T}_N; H^q(X)) = H^p(B\mathbb{T}_N) \otimes H^q(X).$$

Using the formula for Poincaré series of X it can be proved that this spectral sequence degenerates, $E_2^{p,q} = E_\infty^{p,q}$ (see Theorem 1.8.2). So it follows that $H^*(BQ, \mathbb{Q}) \cong H^*(B\mathbb{T}^n, \mathbb{Q}) \otimes H^*(M^{2n}, \mathbb{Q})$ as $\mathbb{Q}$-modules. Hence $j^* : H^*(BQ, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ is surjective.

We have natural identifications $H_2(BQ) = \mathbb{Q}^m$ and $H_2(B\mathbb{T}_N) = \mathbb{Q}^n$. Here $\mathbb{Q}^m$ is regarded as the $\mathbb{Q}$ vector space with basis corresponding to the set of codimension one faces of Q. The map

$$\mathfrak{p}_* : H_2(BQ) \to H_2(B\mathbb{T}_N)$$

is naturally identified with the characteristic map $\Lambda_\lambda : \mathbb{Q}^m \to \mathbb{Q}^n$ that sends w_i, the i-th standard basis vector of $\mathbb{Q}^m$, to λ_i. The map $\mathfrak{p}^* : H^2(B\mathbb{T}_N) \to H^2(BQ)$ is then identified with the dual map $\Lambda_\lambda^* : (\mathbb{Q}^n)^* \to (\mathbb{Q}^m)^*$. Regarding the map Λ_λ as an $n \times m$

matrix λ_{ij}, the matrix for Λ_λ^* is the transpose. Column vectors of Λ_λ^* can then be regarded as linear combinations of $w_1, \dots, w_m$. Define

$$\lambda^i = \lambda_{i1} w_1 + \cdots + \lambda_{im} w_m. \tag{3.8.4}$$

We have a short exact sequence

$$\begin{array}{ccccc} 0 \to H^2(B\mathbb{T}_N) & \xrightarrow{\mathfrak{p}^*} & H^2(BQ) & \xrightarrow{j^*} & H^2(X) \to 0 \\ \Vert & & \Vert & & \\ (\mathbb{Q}^n)^* & \xrightarrow{\Lambda_\lambda^*} & (\mathbb{Q}^m)^*. & & \end{array}$$

Let $\mathcal{J}$ be the homogeneous ideal in $\mathbb{Q}[w_1, \dots, w_m]$ generated by the λ^i and let $\overline{\mathcal{J}}$ be its image in $SR(Q, \mathbb{Q})$. Since $j^* : SR(Q, \mathbb{Q}) \to H^*(X)$ is onto and $\overline{\mathcal{J}}$ is in its kernel, j^* induces a surjection $SR(Q, \mathbb{Q})/\overline{\mathcal{J}} \to H^*(X)$.

Theorem 3.8.2. *Let $\mathcal{X}$ be the quasitoric orbifold associated to the combinatorial model (Q, N, λ). Then $H^*(X; \mathbb{Q})$ is the quotient of the face ring of Q by $\overline{\mathcal{J}}$; i.e., $H^*(X; \mathbb{Q}) = \mathbb{Q}[w_1, \dots, w_m]/(\mathcal{I} + \mathcal{J})$.*

Proof. We know that $H^*(B\mathbb{T}_N)$ is a polynomial ring on n generators, and $H^*(BQ)$ is the face ring. Since the spectral sequence degenerates, $H^*(BQ) \simeq H^*(B\mathbb{T}_N) \otimes H^*(X)$. Furthermore, $\mathfrak{p}^* : H^*(B\mathbb{T}_N) \to H^*(BQ)$ is injective and $\overline{\mathcal{J}}$ is identified with the image of $\mathfrak{p}^*$. Thus $H^*(X) = H^*(BQ)/\overline{\mathcal{J}} = \mathbb{Q}[w_1, \dots, w_m]/(\mathcal{I} + \mathcal{J})$. □

3.9 Stable almost complex structure

Buchstaber and Ray [BR01] have shown the existence of a stable almost complex structure on omnioriented quasitoric manifolds. We generalize their result to omnioriented quasitoric orbifolds (see Section 1.6 for definition). Let m be the cardinality of I, the set of facets of the polytope Q. We will realize the orbifold X as the quotient of the action of an appropriate subgroup of $(\mathbb{C}^*)^m$ on an open set of $\mathbb{C}^m$. Consider the natural combinatorial model $(\mathbb{R}^m_{\geq 0}, L \cong \mathbb{Z}^m, \{e_i\})$ for $\mathbb{C}^m$, where e_i is the i-th standard vector of $\mathbb{Z}^m$. Let

$$\pi_s : \mathbb{C}^m \to \mathbb{R}^m_{\geq 0}$$

be the projection map corresponding to taking modulus coordinatewise. Embed the polytope Q in $\mathbb{R}^m_{\geq 0}$ by the map

$$d_{\mathcal{F}} : Q \to \mathbb{R}^m$$

where the i-th coordinate of $d_{\mathcal{F}}(p)$ is the Euclidean distance $(d(p, F_i))$ of p from the hyperplane of the i-th facet F_i in $\mathbb{R}^n$. Consider the thickening $W^{\mathbb{R}}(Q) \subset \mathbb{R}^m_{\geq 0}$ of $d_{\mathcal{F}}(Q)$, defined by

$$W^{\mathbb{R}}(Q) = \{f : I \to \mathbb{R}_{\geq 0} | f^{-1}(0) \in \mathfrak{L}_F(Q)\} \tag{3.9.1}$$

where $\mathfrak{L}_F(Q)$ denotes the face lattice of Q.

Denote the n-dimensional linear subspace of $\mathbb{R}^m$ parallel to $d_{\mathcal{F}}(Q)$ by V_Q and its orthogonal complement by $V_Q^{\perp}$. As a manifold with corners, $W^{\mathbb{R}}(Q)$ is canonically diffeomorphic to the Cartesian product $d_{\mathcal{F}}(Q) \times \exp(V_Q^{\perp})$ (see [BR01], Proposition 3.4).

Define the spaces $W(Q)$ and $\mathcal{Z}(Q)$ as follows.

$$W(Q) := \pi_s^{-1}(W^{\mathbb{R}}(Q)), \qquad \mathcal{Z}(Q) := \pi_s^{-1}(d_{\mathcal{F}}(Q)). \tag{3.9.2}$$

$W(Q)$ is an open subset of $\mathbb{C}^m$ and there is a canonical diffeomorphism

$$W(Q) \cong \mathcal{Z}(Q) \times \exp(V_Q^{\perp}). \tag{3.9.3}$$

Let $\Lambda : L \to N$ be the map of $\mathbb{Z}$-modules which maps the standard generator e_i of L to the dicharacteristic vector λ_i. Let K denote the kernel of this map. Recall the submodule $\widehat{N}$ of N generated by the dicharacteristic vectors and the orbifold universal cover $\mathcal{O}$ from Section 3.4. Since the $\mathbb{Z}$-modules L and $\widehat{N}$ are free, the sequence

$$0 \longrightarrow K \longrightarrow L \xrightarrow{\Lambda} \widehat{N} \longrightarrow 0 \tag{3.9.4}$$

splits and we may write $L = K \oplus \widehat{N}$. Hence $K_R \cap L = K$ and applying the second isomorphism theorem for groups we can consider the torus $\mathbb{T}_K := K_R/K$ to be a subgroup of $\mathbb{T}_L$. In fact we get a split exact sequence

$$1 \longrightarrow \mathbb{T}_K \longrightarrow \mathbb{T}_L \xrightarrow{\Lambda_*} \mathbb{T}_{\widehat{N}} \longrightarrow 1 \tag{3.9.5}$$

For any face F of Q let $L(F)$ be the sublattice of L generated by the basis vectors e_i such that $d_{\mathcal{F}}(F)$ intersects the i-th facet of $\mathbb{R}^m_{\geq 0}$, that is the coordinate hyperplane $\{x_i = 0\}$. Note that image of $L(F)$ under Λ is precisely $\widehat{N}(F)$, so that the preimage $\Lambda^{-1}(\widehat{N}(F)) = K \cdot L(F)$. Consider the exact sequence

$$0 \longrightarrow \frac{K \cdot L(F)}{L(F)} \longrightarrow \frac{L}{L(F)} \xrightarrow{\Lambda} \frac{\widehat{N}}{\widehat{N}(F)} \longrightarrow 0. \tag{3.9.6}$$

Since the dicharacteristic vectors corresponding to the facets whose intersection is F are linearly independent, it follows from the definition of K and Λ that $K \cap L(F) = \{0\}$. Hence by the second isomorphism theorem we have a canonical isomorphism

$$\frac{K \cdot L(F)}{L(F)} \cong K. \tag{3.9.7}$$

So 3.9.6 yields

$$0 \longrightarrow K \longrightarrow \frac{L}{L(F)} \xrightarrow{\Lambda} \frac{\widehat{N}}{\widehat{N}(F)} \longrightarrow 0. \tag{3.9.8}$$

In general $\dfrac{\widehat{N}}{\widehat{N}(F)}$ is not a free $\mathbb{Z}$-module. Let $\widehat{N}'(F) = (\widehat{N}(F) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \widehat{N}$. Define

$$\Lambda' = \Lambda \circ \phi \tag{3.9.9}$$

where ϕ is the canonical projection

$$\phi : \frac{\widehat{N}}{\widehat{N}(F)} \longrightarrow \frac{\widehat{N}}{\widehat{N}'(F)}. \tag{3.9.10}$$

Since $\dfrac{\widehat{N}}{\widehat{N}'(F)}$ is free, the following exact sequence splits

$$0 \longrightarrow \frac{\Lambda^{-1}(\widehat{N}'(F))}{L(F)} \longrightarrow \frac{L}{L(F)} \xrightarrow{\Lambda'} \frac{\widehat{N}}{\widehat{N}'(F)} \longrightarrow 0. \tag{3.9.11}$$

Denoting the modules in 3.9.11 by $\overline{K}$, $\overline{L}$ and $\overline{N}$ respectively we obtain a split exact sequence of tori

$$0 \longrightarrow \mathbb{T}_{\overline{K}} \xrightarrow{\theta_1} \mathbb{T}_{\overline{L}} \xrightarrow{\Lambda'_*} \mathbb{T}_{\overline{N}} \longrightarrow 0. \tag{3.9.12}$$

Note that K is a submodule of same rank of the free module $\overline{K}$ and there is a natural exact sequence

$$0 \longrightarrow \frac{\widehat{N}'(F)}{\widehat{N}(F)} \longrightarrow \mathbb{T}_K \xrightarrow{\theta_2} \mathbb{T}_{\overline{K}} \longrightarrow 0. \tag{3.9.13}$$

The composition

$$\theta_1 \circ \theta_2 : \mathbb{T}_K \longrightarrow \mathbb{T}_{\bar{L}} \tag{3.9.14}$$

defines a natural action of $\mathbb{T}_K$ on $\mathbb{T}_{\overline{L}}$ with isotropy $\widehat{G}_F = \widehat{N}'(F)/\widehat{N}(F)$ and quotient $\mathbb{T}_{\overline{N}}$.

Since $\mathbb{T}_{\overline{N}}$ is the fiber of $\widehat{\pi} : O \to Q$ and $\mathbb{T}_{\overline{L}}$ is the fiber of $\pi_s : \mathcal{Z}(Q) \to Q$ over any point in the relative interior of the arbitrary face F, it follows O is quotient of $\mathcal{Z}(Q)$ by the above action of $\mathbb{T}_K$. This action of $\mathbb{T}_K$ is same as the restriction of its action on $\mathbb{C}^m$ as a subgroup of $\mathbb{T}_L$ and hence $(\mathbb{C}^*)^m$. By 3.9.3 it follows that O is the quotient of the open set $W(Q)$ in $\mathbb{C}^m$ by the action of the subgroup $\mathbb{T}_K \times \exp(V_Q^{\perp})$ of $(\mathbb{C}^*)^m$,

$$O = \frac{W(Q)}{\mathbb{T}_K \times \exp(V_Q^{\perp})}. \tag{3.9.15}$$

The induced action of $\widehat{H} := \mathbb{T}_K \times \exp(V_Q^{\perp})$ on the real tangent bundle $\mathcal{T}W(Q)$ of

$W(Q)$ commutes with the almost complex structure

$$J : \mathcal{T}W(Q) \to \mathcal{T}W(Q)$$

obtained by restriction of the standard almost complex structure on $\mathcal{T}\mathbb{C}^m$. Therefore the quotient $\mathfrak{W}$ of $\mathcal{T}W(Q)$ by $\widehat{H}$ has the structure of an almost complex orbibundle (or orbifold vector bundle) over $\mathcal{O}$. Moreover this quotient splits, by an Atiyah sequence ([Ati57]), as the direct sum of a trivial rank $2(n-m)$ real bundle $\widehat{\mathfrak{h}}$ over O corresponding to the Lie algebra of $\widehat{H}$ and the orbifold tangent bundle $\mathcal{T}\mathcal{O}$ of $\mathcal{O}$. The existence of a stable almost complex structure on $\mathcal{T}\mathcal{O}$ is thus established.

The tangent bundle $\mathcal{T}\mathbb{C}^m$ splits naturally into a direct sum of m complex line bundles corresponding to the complex coordinate directions which of course correspond to the facets of Q. We get a corresponding splitting

$$\mathcal{T}W(Q) = \oplus C_F.$$

The bundles C_F are invariant under J as well $\widehat{H}$. Therefore the quotient of C_F by $\widehat{H}$ is a complex orbibundle $\widehat{\nu}(F)$ of rank one on $\mathcal{O}$ and

$$\widehat{\mathfrak{W}} = \oplus\widehat{\nu}(F).$$

It is not hard to see that the natural action of $\mathbb{T}_{\widehat{N}}$ on $\widehat{\mathfrak{W}}$ commutes with the almost complex structure on it. The quotient

$$\mathfrak{W} := \widehat{\mathfrak{W}}/(N/\widehat{N})$$

is an orbibundle on X with an induced almost complex structure since $(N/\widehat{N})$ is a subgroup of $\mathbb{T}_{\widehat{N}}$. Furthermore $\mathcal{T}\mathcal{X}$ is the quotient of $\mathcal{T}\mathcal{O}$ by $N/\widehat{N}$. Therefore

$$\mathfrak{W} = \mathcal{T}\mathcal{X} \oplus \mathfrak{h}$$

where $\mathfrak{h}$ is the quotient of $\widehat{\mathfrak{h}}$ by $N/\widehat{N}$. Since the action of $\mathbb{T}_{\widehat{N}}$ and hence $N/\widehat{N}$ on $\widehat{\mathfrak{h}}$ is trivial, $\mathfrak{h}$ is a trivial vector bundle on X. Hence the almost complex structure on $\mathfrak{W}$ induces a stable almost complex structure on $\mathcal{T}\mathcal{X}$. We also have a decomposition

$$\mathfrak{W} = \oplus\nu(F)$$

where the orbifold line bundle $\nu(F) := \widehat{\nu}(F)/(N/\widehat{N})$.

3.10 Line bundles and cohomology

Recall the manifold $\mathcal{Z}(Q)$ of dimension $m+n$ defined in equation 3.9.2. Let $B_LQ = E\mathbb{T}_L \times_{\mathbb{T}_L} \mathcal{Z}(Q)$. Since $O = \mathcal{Z}(Q)/\mathbb{T}_K$,

$B_LQ = E\mathbb{T}_L \times_{\mathbb{T}_L} \mathcal{Z}(Q) = E\mathbb{T}_L \times_{\mathbb{T}_K} \mathcal{Z}(Q)/(\mathbb{T}_L/\mathbb{T}_K) = E\mathbb{T}_L \times (\mathcal{Z}(Q)/\mathbb{T}_K)/(\mathbb{T}_{\widehat{N}}) \simeq E\mathbb{T}_{\widehat{N}} \times_{\mathbb{T}_{\widehat{N}}} O = E\mathbb{T}_{\widehat{N}} \times_{\mathbb{T}_N} O/(N/\widehat{N}) \simeq E\mathbb{T}_N \times_{\mathbb{T}_N} X = BQ$.

Let $w_1, \ldots, w_m$ be the generators of $H^2(BQ)$ as in Section 3.8 and let F_i denote the facet of Q corresponding to w_i. Let

$$\alpha_i : \mathbb{T}_L \to \mathbb{T}^1$$

be the projection onto the i-th factor and $\mathbb{C}(\alpha_i)$ denote the corresponding 1-dimensional representation space of $\mathbb{T}_L$. Define

$$L_i = E\mathbb{T}_L \times_{\mathbb{T}_L} \widetilde{L_i},$$

where $\widetilde{L_i} = \mathbb{C}(\alpha_i) \times \mathcal{Z}(Q)$ is the trivial equivariant line bundle over $\mathcal{Z}(Q)$. Then L_i is an orbifold line bundle over BQ. Let $c_1(L_i)$ be the first Chern class of L_i in $H^2(BQ;\mathbb{Q})$. We will show that $c_1(L_i) = w_i$.

Since the i-th factor of $\mathbb{T}_L$ acts freely on $\mathcal{Z}(Q) - \pi_s^{-1}(F_i)$, the restriction of L_i to $BQ - BF_i$ is trivial. Consider the following commutative diagram

$$\begin{array}{ccc} \iota^*(L_i) & \longrightarrow & L_i \\ \downarrow & & \downarrow \\ (BQ - BF_i) & \xrightarrow{\iota} & BQ \end{array}$$

where ι is inclusion map. By naturality $c_1(\iota^*(L_i)) = \iota^*(c_1(L_i))$. Since the bundle $\iota^*(L_i)$ over $BQ - BF_i$ is trivial $\iota^*(c_1(L_i)) = c_1(\iota^*(L_i)) = 0$. It is easy to show that

$$B(Q - F_i) = E\mathbb{T}_L \times_{\mathbb{T}_L} (\pi_s^{-1}(Q - F_i)) \simeq BQ - BF_i.$$

From the proof of Theorem 3.8.1 it is evident that $H^*(BQ - BF_i;\mathbb{Q}) \cong SR(Q - F_i, \mathbb{Q})$. Hence $H^2(BQ - BF_i, \mathbb{Q}) = \bigoplus_{j \neq i} \mathbb{Q}w_j$. The map

$$\iota^* : H^2(BQ;\mathbb{Q}) \to H^2(BQ - BF_i;\mathbb{Q})$$

is a surjective homomorphism with kernel $\mathbb{Q}w_i$ implying $c_1(L_i) \in \mathbb{Q}w_i$. Naturality axiom ensures, as follows, that $c_1(L_i)$ is nonzero, so that we can identify $c_1(L_i)$ with w_i.

Let F be an edge in F_i. Then

$$BF := E\mathbb{T}_L \times_{\mathbb{T}_L} (\pi_s^{-1}(F)) \simeq E\mathbb{T}_N \times_{\mathbb{T}_N} (\pi^{-1}(F)) = (E\mathbb{T}_N \times_{\mathbb{T}_F} \pi^{-1}(F))/(\mathbb{T}_N/\mathbb{T}_F)$$

$$= (E\mathbb{T}_N \times (\pi^{-1}(F)/\mathbb{T}_F))/(\mathbb{T}_N/\mathbb{T}_F) \simeq E(\mathbb{T}_N/\mathbb{T}_F) \times_{\mathbb{T}_N/\mathbb{T}_F} \pi^{-1}(F) \simeq ES^1 \times_{S^1} S^2,$$

where $\mathbb{T}_F$ is the isotropy subgroup of F in $\mathbb{T}_N$ and action of S^1 on S^2 is corresponding action of $\mathbb{T}_N/\mathbb{T}_F$ on $\pi^{-1}(F)$. Let $L_i(F)$ is the pullback of orbibundle L_i. Using Thom isomorphism and cohomology exact sequence obtained from

$$BF \xrightarrow{s} L_i(F) \to (L_i(F), BF)$$

where s is zero section of L_i bundle, we can show $c_1(L_i(F))$ is nonzero. Since $c_1(L_i(F))$ is pullback of $c_1(L_i)$, $c_1(L_i)$ is nonzero. Hence $c_1(L_i) = w_i$. Note that if F_i is the facet of Q corresponding to L_i,

$$L_i = E\mathbb{T}_L \times_{\mathbb{T}_L} \widetilde{L}_i = E\mathbb{T}_{\widehat{N}} \times_{\mathbb{T}_{\widehat{N}}} (\widetilde{L}_i/\mathbb{T}_K) = E\mathbb{T}_{\widehat{N}} \times_{\mathbb{T}_{\widehat{N}}} \widehat{\nu}(F_i) = E\mathbb{T}_{\widehat{N}} \times_{\mathbb{T}_N} \widehat{\nu}(F_i)/(N/\widehat{N})$$

$\simeq E\mathbb{T}_N \times_{\mathbb{T}_N} \nu(F_i)$. Let

$$\mathfrak{j} : \nu(F_i) \hookrightarrow L_i$$

be the inclusion of fiber covering $j : X \hookrightarrow BQ$. Then $\mathfrak{j}^*(L_i) = \nu(F_i)$. Hence

$$c_1(\nu(F_i)) = j^* c_1(L_i) = j^* w_i.$$

Hence by Theorem 3.8.2 the first Chern classes of the bundles $\nu(F_i)$ generate the cohomology ring of X. We also obtain the formula for the total Chern class of $\mathcal{TX}$ with the stable almost complex structure determined by the given dicharacteristic.

$$c(\mathcal{T}) = \prod_{i=1}^{m} (1 + c_1(\nu(F_i))) \tag{3.10.1}$$

3.11 Chern numbers

Chern numbers of an omnioriented quasitoric orbifold, with the induced stable almost complex structure, can be computed using standard localization formulae, given for instance in Chapter 9 of [CK99]. The fixed points of the $\mathbb{T}_N$ action correspond to the vertices of Q. While computing the numerator contributions at a vertex, one needs to recall that $\mathbb{T}_N$ action on the bundle $\mathfrak{h}$ is trivial. We will give a formula for the top Chern number below. In the manifold case similar formula was obtained by Panov in [Pan01]. In principle any Hirzebruch genus associated to a series may be computed similarly.

Fix an orientation for $\mathcal{X}$ by choosing orientations for $Q \subset \mathbb{R}^n$ and $\mathbb{T}_N$. We order the facets or equivalently the dicharacteristic vectors at each vertex in a compatible manner as follows.

Suppose the vertex v of Q is the intersection of facets $F_{i_1}, \ldots, F_{i_n}$. To each of these facets F_{i_k} assign the unique edge E_k of Q such that

$$F_{i_k} \cap E_k = v.$$

Let $\hat{e}_k$ be a vector along E_k with origin at v. Then $\hat{e}_1, \ldots, \hat{e}_n$ is a basis of $\mathbb{R}^n$ which is oriented depending on the ordering of the facets. We will assume the ordering $F_{i_1}, \ldots, F_{i_n}$ to be such that $\hat{e}_1, \ldots, \hat{e}_n$ is positively oriented.

For each vertex v, let $\Lambda_{(v)}$ be the matrix

$$\Lambda_{(v)} = [\lambda_{i_1} \ldots \lambda_{i_n}]$$

whose columns are ordered as described above. Let $\sigma(v) := \det\Lambda_{(v)}$. Then we obtain the following formula for the top Chern number,

$$c_n(\mathcal{X}) = \Sigma_v \frac{1}{\sigma(v)}. \tag{3.11.1}$$

Remark 3.11.1. *If the stable almost complex structure of an omnioriented quasitoric orbifold admits a reduction to an almost complex structure, then $\sigma(v)$ is positive for each vertex v. This follows from comparing orientations, taking $\mathcal{X}$ to be oriented according to the almost complex structure. The converse is true in the case of quasitoric manifolds, see subsection 5.4.2 of [BP02]. The orbifold case remains unsolved at the moment.*

3.12 Chen-Ruan cohomology groups

We refer the reader to [CR04, ALR07] for definition and motivation of the Chen-Ruan cohomology groups of an almost complex orbifold. They may roughly be thought of as a receptacle for a suitable Chern character from orbifold or equivariant K-theory. Briefly, the Chen-Ruan cohomology with coefficients in $\mathbb{Q}$ is the direct sum of the cohomology of the underlying space and the cohomology of certain subspaces of it called *twisted sectors* which are counted with multiplicities and rational degree shifts depending on the orbifold structure. The verification of the statements below is straightforward.

For an almost complex quasitoric orbifold $\mathcal{X}$, each twisted sector is a $\mathbb{T}_N$-invariant subspace $X(F)$ as described in Section 3.3. The contribution of $X(F)$ is counted with multiplicity one less than the order of the group G_F, corresponding to the nontrivial elements of G_F. However the degree shift of these contributions depend on the particular element of G_F to which the twisted sector corresponds. If

$$g = (a + N(F)) \in G_F$$

where $a \in N^*(F)$, then the degree shift $2\iota(g)$ can be calculated as follows.

Suppose $\lambda_1, \ldots, \lambda_k$ is the defining basis of $N(F)$. Then a can be uniquely expressed as

$$a = \sum_{i=1}^{k} q_i \lambda_i$$

where each q_i is a rational number in $[0, 1)$, and $\iota(g) = \sum_{i=1}^{k} q_i$. Note that the rational homology and hence rational cohomology of $X(F)$ can be computed using its combinatorial model given in Section 3.3.

Chapter 4

Small orbifolds over simple polytopes

4.1 Introduction

In this chapter we introduce some n-dimensional orbifolds on which there is a natural $\mathbb{Z}_2^{n-1}$ action having a simple polytope as the orbit space. We call these orbifolds small orbifolds. Small orbifolds are closely related to the notion of small covers.

We give the precise definition of small orbifold and show that they are smooth. We calculate the orbifold fundamental group of small orbifolds. We show that the universal orbifold cover of an n-dimensional ($n > 2$) small orbifold is diffeomorphic to $\mathbb{R}^n$. Theorem 4.3.4 shows that the space $\mathcal{Z}$, constructed in Lemma 4.4 of [DJ91], is diffeomorphic to $\mathbb{R}^n$ if there is an s-characteristic function (definition 4.2.1) of simple n-polytope. We compute the singular homology groups of small orbifold with integer coefficients. We establish a relation between the modulo 2 Betti numbers of a small orbifold and *h-vector* of the polytope. In the last section we discuss intersection theory of small orbifold and rewrite the Poincaré duality theorem for even dimensional small orbifold. We compute the singular cohomology groups and cohomology ring of even dimensional small orbifold.

4.2 Definition and orbifold structure

Let Q be a simple polytope of dimension n. Let $\mathcal{F}(Q) = \{F_i,\ i = 1, 2, \ldots, m\}$ be the set of facets of Q. Let $V(Q)$ be the set of vertices of Q. We denote the underlying additive group of the vector space $\mathbb{F}_2^{n-1}$ by $\mathbb{Z}_2^{n-1}$.

Definition 4.2.1. *A function $\vartheta : \mathcal{F}(Q) \to \mathbb{Z}_2^{n-1}$ is called an s-characteristic function of the polytope Q if the facets $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ intersect at a vertex of Q then the set*

$\{\vartheta_{i_1}, \vartheta_{i_2}, \ldots, \vartheta_{i_{k-1}}, \widehat{\vartheta}_{i_k}, \vartheta_{i_{k+1}}, \ldots, \vartheta_{i_n}\}$, *where* $\vartheta_i := \vartheta(F_i)$, *is a basis of* $\mathbb{F}_2^{n-1}$ *over* $\mathbb{F}_2$ *for each* k, $1 \leq k \leq n$. *We call the pair* (Q, ϑ) *an s-characteristic pair.*

Here the symbol $\widehat{}$ represents the omission of corresponding entry. We give examples of s-characteristic function in examples 4.2.4 and 4.2.5.

Now we give the constructive definition of small orbifold using the s-characteristic pair (Q, ϑ). Let F be a face of the simple polytope Q of codimension $k \geq 1$. Then

$$F = F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_k},$$

for some facets $F_{i_j} \in \mathcal{F}(Q)$ containing F. Let G_F be the subspace of $\mathbb{F}_2^{n-1}$ spanned by $\{\vartheta_{i_1}, \vartheta_{i_2}, \ldots, \vartheta_{i_k}\}$. Without any confusion we denote the underlying additive group of the subspace G_F by G_F. By the definition of ϑ, $G_v = \mathbb{Z}_2^{n-1}$ for each $v \in V(Q)$. So the s-characteristic function ϑ determines a unique subgroup of $\mathbb{Z}_2^{n-1}$ associated to each face of the polytope Q. Note that if $k < n$ then $G_F \cong \mathbb{Z}_2^k$. The subgroup G_F of $\mathbb{Z}_2^{n-1}$ is a direct summand.

Each point p of Q belongs to relative interior of a unique face $F(p)$ of Q. Define an equivalence relation $\sim_s$ on $\mathbb{Z}_2^{n-1} \times Q$ by

$$(a, p) \sim_s (b, q) \text{ if } p = q \text{ and } b - a \in G_{F(p)}. \tag{4.2.1}$$

Let $X(Q, \vartheta) = (\mathbb{Z}_2^{n-1} \times Q)/\sim_s$ be the quotient space. Then $X(Q, \vartheta)$ is a $\mathbb{Z}_2^{n-1}$-space with the orbit map

$$\pi : X(Q, \vartheta) \to Q \text{ defined by } \pi([a, p]^{\sim_s}) = p. \tag{4.2.2}$$

Let $\hat{\pi} : (\mathbb{Z}_2^{n-1} \times Q) \to X(Q, \vartheta)$ be the quotient map. Let B^n be the open ball of radius 1 in $\mathbb{R}^n$.

We claim that the space $X(Q, \vartheta)$ has a smooth orbifold structure. To prove our claim we construct a smooth orbifold atlas. We show that for each vertex v of Q there exists an orbifold chart $(B^n, \mathbb{Z}_2, \phi_v)$ of $X(Q, \vartheta)$ where $\phi_v(B^n)$ is an open subset $X_v(Q, \vartheta)$ of $X(Q, \vartheta)$ and $\{X_v(Q, \vartheta) : v \in V(Q)\}$ cover $X(Q, \vartheta)$. To show the compatibility of these charts as v varies over $V(Q)$, we introduce some additional orbifold charts to make this collection an orbifold atlas.

Let $v \in V(Q)$ and U_v be the open subset of Q obtained by deleting all faces of Q not containing v. Let

$$X_v(Q, \vartheta) := \pi^{-1}(U_v) = (\mathbb{Z}_2^{n-1} \times U_v)/\sim_s .$$

The subset U_v is diffeomorphic as manifold with corners to

$$B_1^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} : \Sigma_1^n x_j < 1\}. \tag{4.2.3}$$

Let $f_v : B_1^n \to U_v$ be a diffeomorphism. Suppose the facets

$$\{x_1 = 0\} \cap B_1^n,\ \{x_2 = 0\} \cap B_1^n,\ \ldots,\ \{x_n = 0\} \cap B_1^n$$

of B_1^n map to the facets $F_{i_1},\ F_{i_2},\ \ldots,\ F_{i_n}$ of U_v respectively under the diffeomorphism f_v. Then $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_n} = v$. Define an equivalence relation $\sim_0$ on the product $\mathbb{Z}_2^{n-1} \times B_1^n$ by

$$(a, x) \sim_0 (b, y) \text{ if } x = y \text{ and } b - a \in G_{F(f_v(x))}. \tag{4.2.4}$$

Let $Y_0 = (\mathbb{Z}_2^{n-1} \times B_1^n)/ \sim_0$ be the quotient space with the orbit map $\pi_0 : Y_0 \to B_1^n$. Let $\hat{\pi}_0 : \mathbb{Z}_2^{n-1} \times B_1^n \to Y_0$ be the quotient map. The diffeomorphism

$$id \times f_v : \mathbb{Z}_2^{n-1} \times B_1^n \to \mathbb{Z}_2^{n-1} \times U_v$$

descends to the following commutative diagram.

$$\begin{array}{ccc} \mathbb{Z}_2^{n-1} \times B_1^n & \xrightarrow{id \times f_v} & \mathbb{Z}_2^{n-1} \times U_v \\ {\scriptstyle \hat{\pi}_0}\big\downarrow & & {\scriptstyle \hat{\pi}_v}\big\downarrow \\ Y_0 & \xrightarrow{\hat{f}_v} & X_v(Q, \vartheta). \end{array} \tag{4.2.5}$$

Here $\hat{\pi}_v$ is the map $\hat{\pi}$ restricted to $\mathbb{Z}_2^{n-1} \times U_v$. It is easy to observe that the map $\hat{f}_v$ is a bijection. Since the maps $\hat{\pi}_v$ and $\hat{\pi}_0$ are continuous and the map $id \times f_v$ is a diffeomorphism, the map $\hat{f}_v$ is a homeomorphism.

Let $u \in [0, 1)$ and H_u be the hyperplane $\{\Sigma_1^n x_j = u\}$ in $\mathbb{R}^n$. Then $Q_0 = H_0 \cap B_1^n$ is the origin of $\mathbb{R}^n$ and

$$Q_u = H_u \cap B_1^n$$

is an $(n-1)$-simplex for each $u \in (0, 1)$. When $u \in (0, 1)$, the facets of Q_u are

$$\{F_{u_j} := \{x_j = 0\} \cap Q_u;\ j = 1, 2, \ldots, n\}.$$

The map

$$\vartheta_u : \{F_{u_j} : j = 1, \ldots, n\} \to \mathbb{Z}_2^{n-1} \text{ defined by } \vartheta_u(F_{u_j}) = \vartheta_{i_j} \tag{4.2.6}$$

satisfies the following condition.

If F_u is the intersection of unique l $(0 \leq l \leq n-1)$ facets $F_{u_{j_1}}, \ldots, F_{u_{j_l}}$ of Q_u then the vectors $\vartheta_u(F_{u_{j_1}}), \ldots, \vartheta_u(F_{u_{j_l}})$ are linearly independent vectors of $\mathbb{F}_2^{n-1}$.

Hence ϑ_u is a $\mathbb{Z}_2$-characteristic function (see definition 2.2.7) of a small cover over the polytope Q_u. Since Q_u is an $(n-1)$-simplex, the small cover corresponding to the $\mathbb{Z}_2$-characteristic pair (Q_u, ϑ_u) is equivariantly homeomorphic to the real projective space $\mathbb{RP}^{n-1}$, see Chapter 2. Here consider $\mathbb{RP}^{n-1}$ as the identification space

$$\{\overline{B}^{n-1}/\{x = -x\} : x \in \partial \overline{B}^{n-1}\}.$$

So at each point $(u, 0, \ldots, 0) \in B_1^n - \{0\}$ we get an equivariant homeomorphism

$$(\mathbb{Z}_2^{n-1} \times Q_u)/\sim_0 \cong \mathbb{RP}^{n-1}, \tag{4.2.7}$$

which sends the fixed point $[a, u]^{\sim_0}$ to the origin of $\overline{B}^{n-1}$. It is clear from the definition of the equivalence relation $\sim_0$ that at $(0, \ldots, 0) \in B_1^n$, $(\mathbb{Z}_2^{n-1} \times Q_0)/\sim_0$ is a point. Hence Y_0 is equivariantly homeomorphic to the open cone

$$(\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\}$$

on real projective space $\mathbb{RP}^{n-1}$. Consider the following map

$$S^{n-1} \times [0,1) \to B^n \;\text{ define by } ((x_1, x_2, \ldots, x_n), r) \to (rx_1, rx_2, \ldots, rx_n).$$

This map induces a homeomorphism $f : B^n \to (S^{n-1} \times [0,1))/S^{n-1} \times \{0\}$. The covering map $S^{n-1} \to \mathbb{RP}^{n-1}$ induces a projection map

$$\phi_0 : (S^{n-1} \times [0,1))/S^{n-1} \times \{0\} \to (\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\}.$$

Observe that this projection map ϕ_0 is nothing but the orbit map q of the antipodal action of $\mathbb{Z}_2$ on B^n. In other words the following diagram is commutative.

$$\begin{array}{ccc} B^n & \xrightarrow{f} & (S^{n-1} \times [0,1))/S^{n-1} \times \{0\} \\ {\scriptstyle q}\downarrow & & {\scriptstyle \phi_0}\downarrow \\ B^n/\mathbb{Z}_2 & \xrightarrow{\hat{f}} & (\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\} \end{array} \tag{4.2.8}$$

Since the map ϕ_0 is induced from the antipodal action on S^{n-1} the commutativity of the diagram ensure that the map $\hat{f}$ is a homeomorphism. Let ϕ_v be the composition of the following maps.

$$B^n \xrightarrow{q} B^n/\mathbb{Z}_2 \xrightarrow{\hat{f}} (\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\} \cong Y_0 \xrightarrow{\hat{f}_v} X_v(Q, \vartheta).$$

Hence $(B^n, \mathbb{Z}_2, \phi_v)$ is an orbifold chart of $X_v(Q, \vartheta)$ corresponding to the vertex v of the polytope.

Now we introduce some additional orbifold charts corresponding to each face F of codimension-k ($0 < k < n$) and the interior of polytope Q. Let

$$U_F = \bigcap U_v,$$

where the intersection is over all vertices v of F. Let $X_F(Q, \vartheta) := \pi^{-1}(U_F)$. Fix a vertex v of F. Consider the diffeomorphism $f_v : B_1^n \to U_v$. Observe that U_F can be obtained from U_v by deleting unique $n-k$ facets of U_v. Let $F_{l_1}, \ldots, F_{l_{n-k}}$ be the facets of U_v such that

$$U_F = U_v - \{F_{l_1} \cup \ldots \cup F_{l_{n-k}}\},$$

where $\{l_1, \ldots, l_{n-k}\} \subset \{1, 2, \ldots, n\}$. Let $B_F^n = f_v^{-1}(U_F)$. Let $\{x_{l_1} = 0\}, \ldots, \{x_{l_{n-k}} = 0\}$ be the coordinate hyperplanes in $\mathbb{R}^n$ such that

$$f_v(\{x_{l_1} = 0\} \cap B_1^n) = F_{l_1}, \ \ldots, \ f_v(\{x_{l_{n-k}} = 0\} \cap B_1^n) = F_{l_{n-k}}.$$

So $B_F^n = B_1^n - \{\{x_{l_1} = 0\} \cup \ldots \cup \{x_{l_{n-k}} = 0\}\}$. Then $\hat{f}_v(\pi_0^{-1}(B_F)) = X_F(Q, \vartheta)$.

Let $u \in (0,1)$ and $Q'_u = Q_u - \{x_{l_1} = 0\}$. Since (Q_u, ϑ_u) is a $\mathbb{Z}_2$-characteristic pair, there exist an equivariant homeomorphism from $(\mathbb{Z}_2^{n-1} \times Q'_u)/\sim_0$ to $B^{n-1} \subset \mathbb{R}^{n-1}$ such that $(\mathbb{Z}_2^{n-1} \times F_{u_j})/\sim_0$ maps to a coordinate hyperplane $H_j := \{x_{i_j} = 0\} \cap B^{n-1}$, for $j \in \{\{1, 2, \ldots, n\} - l_1\}$. Clearly $H_i \neq H_j$ for $i \neq j$.

Let $Q''_u = Q'_u - \{\{x_{l_2} = 0\} \cup \ldots \cup \{x_{l_{n-k}} = 0\}\}$. Then

$$(\mathbb{Z}_2^{n-1} \times Q''_u)/\sim_0 \ \cong B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\} \text{ and } B_F^n \cong (0,1) \times Q''.$$

So $\pi_0^{-1}(B_F^n) = (\mathbb{Z}_2^{n-1} \times B_F^n)/\sim_0$ is homeomorphic to

$$(0,1) \times \{(\mathbb{Z}_2^{n-1} \times Q''_u)/\sim_0\} \ \cong \ (0,1) \times \{B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\}\}.$$

By our assumption

$$(0,1) \times \{B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\}\} \hookrightarrow (\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\}.$$

So there exist two open subsets D_F, D'_F of B^n such that $D'_F = -D_F$ and the following restrictions are homeomorphism.

1. $\phi_0 \circ f|_{D_F} : D_F \to (0,1) \times \{B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\}\}$.
2. $\phi_0 \circ f|_{D'_F} : D'_F \to (0,1) \times \{B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\}\}$.

Hence the restriction $\phi_v|_{D_F} : D_F \to X_F(Q, \vartheta)$ is homeomorphism. Clearly

$$D_F \cong \{\{B^n \cap \{x_n > 0\}\} - \cup_{j=1, x_{l_j} \neq x_n}^{(n-k-1)} \{x_{l_j} = 0\}\}. \tag{4.2.9}$$

The set D_F is homeomorphic to an open ball in $\mathbb{R}^n$ if $k = n-1$. When $k = n-1$, F is an edge of the polytope Q. Let $E(Q)$ be the set of edges of the polytope Q and $e \in E(Q)$. Let $\phi_{e_v} = \phi_v|_{D_e} : D_e \to X_e(Q, \vartheta)$, where $v \in V(e)$. Hence $(D_e, \{0\}, \phi_{e_v})$ is an orbifold chart on $X_e(Q, \vartheta)$ for each $e \in E(Q)$ and $v \in V(e)$.

The set D_F is disjoint union of open sets $\{A_{F(i)} : i = 1, \ldots, 2(n-k-1)\}$ in $\mathbb{R}^n$ whenever $0 < k < n-1$. Here all $A_{F(i)}$ are homeomorphic to an open ball in $\mathbb{R}^n$. Let

$$\phi_{F_v(i)} = \phi_v|_{A_{F(i)}} : A_{F(i)} \to X_F(Q, \vartheta) \tag{4.2.10}$$

be the restriction of the map ϕ_v to the domain $A_{F(i)}$, where $v \in V(F)$. So for each $(i, v) \in \{1, 2, \ldots, 2(n-k-1)\} \times V(F)$, the triple $(A_{F(i)}, \{0\}, \phi_{F_v(i)})$ is an orbifold chart on the image of $\phi_{F_v(i)}$ in $X_F(Q, \vartheta) \subseteq X(Q, \vartheta)$.

Let Q^0 be the interior of Q and $X_Q(Q, \vartheta) = \pi^{-1}(Q^0)$. Hence

$$D_Q := \{\{B^n \cap \{x_n > 0\}\} - \cup_{j=1}^{n-1}\{x_j = 0\}\}$$

is homeomorphic to $X_Q(Q, \vartheta)$ under the restriction of ϕ_v on D_Q. The set D_Q is disjoint union of connected open sets $\{B_j : j = 1, \ldots, 2(n-1)\}$ in $\mathbb{R}^n$ where each B_j is homeomorphic to the open ball B^n. Let

$$\phi_{Q_v(j)} = \phi_v|_{B_j} : B_j \to X_Q(Q, \vartheta) \tag{4.2.11}$$

be the restriction of the map ϕ_v to the domain B_j. Hence for $(j, v) \in \{1, \ldots, 2(n-1)\} \times V(Q)$, $(B_j, \{0\}, \phi_{Q_v(j)})$ is an orbifold chart on the image of $\phi_{Q_v(j)}$ in $X_Q(Q, \vartheta)$. Let

$$\mathfrak{U}' = \{(B^n, \mathbb{Z}_2, \phi_v)\} \cup \{(D_e, \{0\}, \phi_{e_v})\} \cup \{(A_{F(i)}, \{0\}, \phi_{F_v(i)})\} \cup \{(B_j, \{0\}, \phi_{Q_v(j)})\} \tag{4.2.12}$$

where $v \in V(Q)$, $e \in E(Q)$, F run over the faces of codimension k $(0 < k < n-1)$, $i = 1, \ldots, 2(n-k-1)$ and $j = 1, \ldots, 2(n-1)$.

From the description of orbifold charts corresponding to each faces and interior of polytope it is clear that the collection $\mathcal{U}'$ is an orbifold atlas on $X(Q, \vartheta)$. Clearly the inclusions $D_e \hookrightarrow B^n$, $A_{F(i)} \hookrightarrow B^n$ and $B_j \hookrightarrow B^n$ induce the following smooth embeddings respectively:

$$(D_e, \{0\}, \phi_{e_v}) \hookrightarrow (B^n, \mathbb{Z}_2, \phi_v), (A_{F(i)}, \{0\}, \phi_{F_v(i)}) \hookrightarrow (B^n, \mathbb{Z}_2, \phi_v)$$

$$\text{and } (B_j, \{0\}, \phi_{Q_v(j)}) \hookrightarrow (B^n, \mathbb{Z}_2, \phi_v).$$

So $\mathfrak{U}'$ is a part of a maximal atlas $\mathfrak{U}$ for $X(Q, \vartheta)$. Thus $\mathcal{X}(Q, \vartheta) = (X, \mathfrak{U})$ is a smooth n-dimensional orbifold.

Definition 4.2.2. *We call the smooth orbifold $\mathcal{X}(Q, \vartheta)$ small orbifold corresponding to* *the s-characteristic pair (Q, ϑ).*

Remark 4.2.3. *1. The small orbifold $\mathcal{X}(Q, \vartheta)$ is reduced, that is, the group in each chart has an effective action. Singular set of the orbifold $\mathcal{X}(Q, \vartheta)$ is*

$$\Sigma\mathcal{X}(Q, \vartheta) = \{[t, v]^{\sim_s} \in X(Q, \vartheta) : v \in V(Q)\}.$$

We call an element of $\Sigma\mathcal{X}(Q, \vartheta)$ an orbifold point of $X(Q, \vartheta)$.

2. *We can not define an s-characteristic function for an arbitrary polytope. Later we will see some examples.*

3. *The small orbifold $X(Q, \vartheta)$ is compact and connected.*

Example 4.2.4. *Let Q^2 be a simple 2-polytope in $\mathbb{R}^2$. Define*

$$\vartheta : \mathcal{F}(Q^2) \to \mathbb{Z}_2 \text{ by } \vartheta(F) = 1, \forall F \in \mathcal{F}(Q^2). \tag{4.2.13}$$

So ϑ is the s-characteristic function of Q^2. The resulting quotient space $X(Q^2, \vartheta)$ is homeomorphic to the sphere S^2. These are the only cases where the identification space is a manifold.

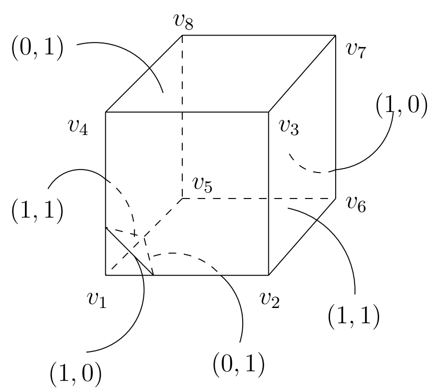


Figure 4.1: An s-characteristic function of I^3.

Example 4.2.5. *Let $I^3 = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1\}$ be the standard cube in $\mathbb{R}^3$. Let $v_1, \ldots, v_8$ be the vertices of I^3, see Figure 4.1. So the facets of I^3 are the following squares*

$$F_1 = v_1v_2v_3v_4, \; F_2 = v_1v_2v_6v_5, \; F_3 = v_1v_5v_8v_4, \; F_4 = v_2v_6v_3v_7,$$

$$F_5 = v_4v_3v_7v_8 \text{ and } F_6 = v_5v_6v_7v_8.$$

Define $\vartheta : \mathcal{F}(I^3) \to \mathbb{Z}_2^2$ by

$$\vartheta(F_1) = \vartheta(F_6) = (1, 0), \; \vartheta(F_2) = \vartheta(F_5) = (0, 1), \; \vartheta(F_3) = \vartheta(F_4) = (1, 1).$$

Hence ϑ is an s-characteristic function of I^3. Then

$$G_{F_1} = G_{F_6} = \{(0,0),(1,0)\},\ G_{F_2} = G_{F_5} = \{(0,0),(0,1)\},\ G_{F_3} = G_{F_4} = \{(0,0),(1,1)\}.$$

For other proper face F of I^3, $G_F = \mathbb{Z}_2^2$. Hence $\mathcal{X}(I^3, \vartheta)$ is a 3-dimensional small orbifold.

Observation 4.2.6. *Let F be a codimension-k $(0 < k < n-1)$ face of Q. Then F is a simple polytope of dimension $n-k$. Let $\mathcal{F}(F) = \{F'_{j_1}, \ldots, F'_{j_l}\}$ be the set of facets of F. So there exist unique facets $F_{j_1}, \ldots, F_{j_l}$ of Q such that*

$$F_{j_1} \cap F = F'_{j_1},\ \ \ldots,\ \ F_{j_l} \cap F = F'_{j_l}.$$

Fix an isomorphism $\mathfrak{b}$ from the quotient field $\mathbb{F}_2^{n-1}/G_F$ to $\mathbb{F}_2^{n-1-k}$. Define a function

$$\vartheta' : \mathcal{F}(F) \to \mathbb{Z}_2^{n-1-k} \text{ by } \vartheta'(F'_{j_m}) = \mathfrak{b}(\vartheta_{j_m} + G_F).$$

Observe that the function ϑ' is an s-characteristic function of F. Let $\sim'_s$ be the restriction of $\sim_s$ on $\mathbb{Z}_2^{n-1-k} \times F$. So $\mathcal{X}(F, \vartheta')$ is an $(n-k)$-dimensional smooth small orbifold associated to the s-characteristic pair (F, ϑ'). The orbifold $\mathcal{X}(F, \vartheta')$ is a suborbifold of $\mathcal{X}(Q, \vartheta)$. We have shown that for each edge e of Q, the set $X_e(Q, \vartheta)$ is homeomorphic to the open ball B^n. Let e' be an edge of F and $U'_{e'} = U_{e'} \cap F$. Hence

$$W_{e'} = (\mathbb{Z}_2^{n-1-k} \times U'_{e'})/ \sim'_s = (\mathbb{Z}_2^{n-1} \times U'_{e'})/ \sim_s$$

is homeomorphic to the open ball B^{n-k}.

4.3 Orbifold fundamental group

Let $\mathcal{X}(Q, \vartheta)$ be a small orbifold over simple n-polytope Q. The set of smooth points

$$M(Q, \vartheta) := X(Q, \vartheta) - \Sigma\mathcal{X}(Q, \vartheta)$$

of small orbifold $\mathcal{X}(Q, \vartheta)$ is an n-dimensional manifold. For each $v \in V(Q)$ we have

$$X_v(Q, \vartheta) - [0, v]^{\sim_s} \cong \mathbb{RP}^{n-1} \times I^0.$$

The sphere S^{n-1} is the double sheeted universal cover of $\mathbb{RP}^{n-1}$. So the universal cover of $X_v(Q, \vartheta) - [0, v]^{\sim_s}$ is $S^{n-1} \times I^0 \cong B^n - 0$. Actually the map $\phi_v : B^n \to X_v(Q, \vartheta)$ is the orbifold universal covering. Let e be an edge containing the vertex v of Q. Define $\bar{e} := e \cap U_v$.

Identifying the faces containing the edge $\bar{e}$ of U_v according to the equivalence relation

$\sim_s$ we get the quotient space $X_{\bar{e}}(U_v, \vartheta)$ homeomorphic to

$$B^n_e := \{(x_1, x_2, \ldots, x_n) \in B^n : x_n \geq 0\}.$$

The set $X_v(Q, \vartheta)$ is obtained from $X_{\bar{e}}(U_v, \vartheta)$ by identifying the antipodal points of the boundary of $X_{\bar{e}}(U_v, \vartheta)$ around the fixed point $[0, v]^{\sim_s}$. Identifying two copies of $X_{\bar{e}}(U_v, \vartheta)$ along their boundary via the antipodal map on the boundary we get a space homeomorphic to B^n.

Doing these identification associated to the orbifold points we obtain that the universal cover of $M(Q, \vartheta)$ is homeomorphic to $\mathbb{R}^n - N$ for some infinite subset N of $\mathbb{Z}^n$ where N depends on the polytope Q in $\mathbb{R}^n$. Let

$$\zeta : \mathbb{R}^n - N \to M(Q, \vartheta) \tag{4.3.1}$$

be the universal covering map. This map locally resemble the chart maps.

The chart maps $\phi_v : B^n \to X_v(Q, \vartheta)$ are uniformly continuous on a compact neighborhood of $0 \in B^n$ and Q is an n-polytope in $\mathbb{R}^n$. So for each $x \in N$ there exists a neighborhood $V_x \subset \mathbb{R}^n$ of x such that the restriction of the universal covering map ζ on $V_x - x$ is uniformly continuous. Hence the map ζ has a unique extension, say $\hat{\zeta}$, on their metric completion. The metric completion of $\mathbb{R}^n - N$ and $M(Q, \vartheta)$ are $\mathbb{R}^n$ and $X(Q, \vartheta)$ respectively. The map $\hat{\zeta}$ sends N onto $V(Q)$.

We show the map $\hat{\zeta}$ is an orbifold covering. Let $\varrho : \mathcal{Z} \to \mathcal{X}(Q, \vartheta)$ be an orbifold cover. Then the restriction $\varrho : Z - \Sigma\mathcal{Z} \to M(Q, \vartheta)$ is an honest cover. Hence there exist a covering map $\zeta_\varrho : \mathbb{R}^n - N \to Z - \Sigma\mathcal{Z}$ so that the following diagram is commutative.

$$\begin{array}{ccc} \mathbb{R}^n - N & \xrightarrow{\zeta_\varrho} & Z - \Sigma\mathcal{Z} \\ \zeta \big\downarrow & & \varrho \big\downarrow \\ M(Q, \vartheta) & \xrightarrow{id} & M(Q, \vartheta) \end{array} \tag{4.3.2}$$

Since the map ζ is locally uniformly continuous and the maps ζ_ϱ, ϱ are continuous, all the maps in the diagram 4.3.2 can be extended to their metric completion. That is we get a commutative diagram of orbifold coverings.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\hat{\zeta}_\varrho} & \mathcal{Z} \\ \hat{\zeta} \big\downarrow & & \hat{\varrho} \big\downarrow \\ \mathcal{X}(Q, \vartheta) & \xrightarrow{id} & \mathcal{X}(Q, \vartheta) \end{array} \tag{4.3.3}$$

Hence $\hat{\zeta} : \mathbb{R}^n \to \mathcal{X}(Q, \vartheta)$ is an orbifold universal cover of $\mathcal{X}(Q, \vartheta)$. Since the orbit map of antipodal action is smooth, the map $\hat{\zeta}$ is a smooth map. Thus we get the following.

Theorem 4.3.1. *The universal orbifold cover of an n-dimensional small orbifold is diffeomorphic to $\mathbb{R}^n$.*

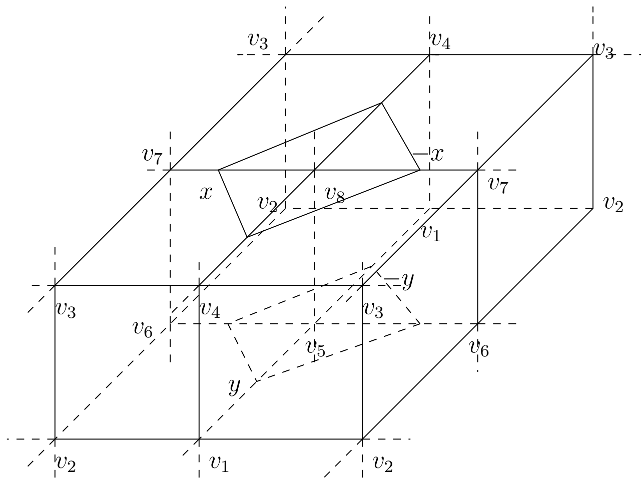


Figure 4.2: Identification of faces containing the edge v_5v_8 of I^3.

Example 4.3.2. *Recall the small orbifold $X(I^3, \vartheta)$ of example 4.2.5. The set of smooth points*

$$M(I^3, \vartheta) := X(I^3, \vartheta) - \Sigma\mathcal{X}(I^3, \vartheta)$$

is a 3-dimensional manifold. The universal cover of $M(I^3, \vartheta)$ is homeomorphic to $\mathbb{R}^3 - \mathbb{Z}^3$. To show this we need to observe how the faces of $\mathbb{Z}_2^2 \times I^3$ are identified by the equivalence relation $\sim_s$ (see equation 4.2.1) on $\mathbb{Z}_2^2 \times I^3$. For each $v \in V(I^3)$

$$X_v(I^3, \vartheta) - [a, v]^{\sim_s} \cong \mathbb{RP}^2 \times I^0.$$

The sphere S^2 is the double sheeted universal cover of $\mathbb{RP}^2$. So the universal cover of $X_v(I^3, \vartheta) - [a, v]^{\sim_s}$ is $S^2 \times I^0 \cong B^3 - 0$. Hence the identification of faces around each vertex of I^3 tells us that the universal cover of $M(I^3, \vartheta)$ is $\mathbb{R}^3 - \mathbb{Z}^3$. We illustrate the identification of faces by the Figure 4.2, where $x \sim_s -x$ on the upper face and $y \sim_s -y$ on the lower face in that figure.

We use the observation 2.5.6 to compute the orbifold fundamental group of $\mathcal{X}(Q, \vartheta)$. Let $\{\beta_1, \beta_2, \ldots, \beta_m\}$ be the standard basis of $\mathbb{Z}_2^m$. Define a map $\beta : \mathcal{F}(Q) \to \mathbb{Z}_2^m$ by $\beta(F_j) = \beta_j$. For each face $F = F_{j_1} \cap F_{j_2} \cap \ldots \cap F_{j_l}$, let H_F be the subgroup of $\mathbb{Z}_2^m$ generated by $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_l}$. Define an equivalence relation $\sim_\beta$ on $\mathbb{Z}_2^m \times Q$ by

$$(s, p) \sim_\beta (t, q) \text{ if and only if } p = q \text{ and } t - s \in H_F$$

where $F \subset Q$ is the unique face whose relative interior contains p. So the quotient space $N(Q, \beta) = (\mathbb{Z}_2^m \times Q)/\sim_\beta$ is an n-dimensional smooth manifold. $N(Q, \beta)$ is a $\mathbb{Z}_2^m$-space with the orbit map

$$\pi_u : N(Q, \beta) \to Q \text{ defined by } \pi_u([s,p]^{\sim_\beta}) = p.$$

We show Q has a smooth orbifold structure. Recall the open subset U_v of Q associated to each vertex $v \in V(Q)$. Note that open sets $\{U_v : v \in V(Q)\}$ cover Q. Let d be the Euclidean distance in $\mathbb{R}^n$. Let $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ be the facets of Q such that v is the intersection of $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$. For each $p \in U_v$, let

$$x_j(p) = d(p, F_{i_j}), \text{ for all } j = 1, 2, \ldots, n.$$

Let $B_v^n = \{(x_1(p), \ldots, x_n(p)) \in \mathbb{R}^n_{\geq 0} : p \in U_v\}$. So the map

$$f : U_v \to B_v^n \text{ defined by } p \to (x_1(p), \ldots, x_n(p))$$

gives a diffeomorphism from U_v to B_v^n. Consider the standard action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ with the orbit map

$$\xi : \mathbb{R}^n \to \mathbb{R}^n_{\geq 0}.$$

Then $\xi^{-1}(B_v^n)$ is diffeomorphic to B^n. Hence $(\xi^{-1}(B_v^n), f^{-1} \circ \xi, \mathbb{Z}_2^n)$ is a smooth orbifold chart on U_v. To show the compatibility of these charts as v varies over $V(Q)$, we can introduce some additional smooth orbifold charts to make this collection a smooth orbifold atlas. From the definition of $\sim_s$ it is clear that $\pi : X(Q, \vartheta) \to Q$ is a smooth orbifold covering.

Definition 4.3.3. *Let $\mathfrak{L}$ be the simplicial complex dual to Q. The right-angled Coxeter group Γ associated to Q is the group with one generator for each element of $V(\mathfrak{L})$ and relations between generators are the following; $a^2 = 1$ for all $a \in V(\mathfrak{L})$, $(ab)^2 = 1$ if $\{a, b\} \in E(\mathfrak{L})$.*

For each $p \in Q$, let $F(p) \subset Q$ be the unique face containing p in its relative interior. Let $F(p) = F_{j_1} \cap \ldots \cap F_{j_l}$. Let $a_{j_1}, \ldots, a_{j_l}$ be the vertices of $\mathfrak{L}$ dual to $F_{j_1}, \ldots, F_{j_l}$ respectively. Let $\Gamma_{F(p)}$ be the subgroup generated by $a_{j_1}, \ldots, a_{j_l}$ of Γ. Define an equivalence relation $\sim_c$ on $\Gamma \times Q$ by

$$(g, p) \sim_c (h, q) \text{ if } p = q \text{ and } h^{-1}g \in \Gamma_{F(p)}.$$

Let $Y = (\Gamma \times Q)/\sim_c$ be the quotient space. We follow this construction from [DJ91]. So Y is a Γ-space with the orbit map

$$\xi_\Gamma : Y \to Q \text{ defined by } \xi_\Gamma([g,p]^{\sim_c}) = p. \tag{4.3.4}$$

Then Y is an n-dimensional manifold and ξ_Γ is an orbifold covering. Since each facet is connected, whenever two generators of Γ commute the intersection of corresponding facets of Q is nonempty. From Theorems 10.1 and 13.5 of [Dav83], we get that Y is simply connected. Hence ξ_Γ is a universal orbifold covering and the orbifold fundamental group of Q is Γ.

Let H be the kernel of abelianization map $\Gamma \to \Gamma^{ab}$. The group H acts on Y freely and properly discontinuously. So the orbit space Y/H is a manifold. The space Y/H is called the universal abelian cover of Q. Note that $N(Q,\beta) = Y/H$. Let

$$\xi_\beta : Y \to N(Q,\beta) \tag{4.3.5}$$

be the corresponding orbit map.

Define a function $\bar{\vartheta} : \mathbb{Z}_2^m \to \mathbb{Z}_2^{n-1}$ by $\bar{\vartheta}(\beta_j) = \vartheta(F_j) = \vartheta_j$ on the basis of $\mathbb{Z}_2^m$. So $\bar{\vartheta}$ is a linear surjection. $\bar{\vartheta}$ induces a surjection

$$\widetilde{\vartheta} : N(Q,\beta) \to X(Q,\vartheta) \text{ defined by } \widetilde{\vartheta}([s,p]^{\sim_\beta}) = [s,p]^{\sim_c}. \tag{4.3.6}$$

That is the following diagram commutes.

$$\begin{array}{ccc} N(Q,\beta) & \xrightarrow{\widetilde{\vartheta}} & X(Q,\vartheta) \\ {\scriptstyle \hat{\pi}_u}\big\downarrow & & {\scriptstyle \hat{\pi}}\big\downarrow \\ Q & \xrightarrow{id} & Q \end{array} \tag{4.3.7}$$

From this commutative diagram we get $\widetilde{\vartheta}$ is a smooth orbifold covering of $X(Q,\vartheta)$. Hence the composition map

$$\widetilde{\vartheta} \circ \xi_\beta : Y \to X(Q,\vartheta)$$

is a smooth universal orbifold covering. From [Thu3m] and Theorem 4.3.1 we obtain the following necessary condition for existence of an s-characteristic function.

Theorem 4.3.4. *Let $\vartheta : \mathcal{F}(Q) \to \mathbb{Z}_2^{n-1}$ be an s-characteristic function of the n-polytope Q $(n > 2)$. Then the space Y is diffeomorphic to $\mathbb{R}^n$.*

Note that when Q is an n-simplex, Y is homeomorphic to the n-dimensional sphere S^n. So by this theorem there does not exist an s-characteristic function of n-simplex. Consequently there does not exist any small orbifold with the n-dimensional simplex as orbit space when $n > 2$.

Let ξ_ϑ be the following composition map

$$\Gamma \to \Gamma^{ab} \xrightarrow{\bar{\vartheta}} \mathbb{Z}_2^{n-1}. \tag{4.3.8}$$

Clearly $Ker(\xi_\vartheta)$, kernel of ξ_ϑ, acts on Y with the orbit map $\widetilde{\vartheta} \circ \xi_\beta$. Now using the observation 2.5.6, we get the following corollary.

Corollary 4.3.5. *The orbifold fundamental group of $X(Q, \vartheta)$ is $ker(\xi_\vartheta)$ which is a normal subgroup of the right-angled Coxeter group associated to the polytope Q.*

4.4 Homology and Euler characteristic

To calculate the singular homology groups of small orbifold $X(Q, \vartheta)$ we will construct a CW-structure on these orbifolds and describe how the cells are attached. We redefine index function precisely. Realize Q as a convex polytope in $\mathbb{R}^n$ and choose a linear functional

$$\phi : \mathbb{R}^n \to \mathbb{R} \tag{4.4.1}$$

which distinguishes the vertices of Q, as in the proof of Theorem 3.1 in [DJ91]. The vertices are linearly ordered according to ascending value of ϕ. We make the 1-skeleton of Q into a directed graph by orienting each edge such that ϕ increases along edges. For each vertex of Q define its index $ind_Q(v)$, as the number of incident edges that point towards v.

Definition 4.4.1. *A subset $\hat{Q} \subseteq Q$ of dimension k is called a proper subcomplex of Q if $\hat{Q}$ is connected and $\hat{Q}$ is the union of some k-dimensional faces of Q.*

In particular each face of Q is a proper subcomplex of Q. The 1-skeleton of a proper subcomplex $\hat{Q}$ is a subcomplex of the 1-skeleton of Q. The restriction of ϕ on the 1-skeleton of $\hat{Q}$ makes it a directed graph. We define index $ind_{\hat{Q}}(v)$ of each vertex v of $\hat{Q}$ as the number of incident edges in $\hat{Q}$ that point towards v. Let $V(\hat{Q})$ and $\mathfrak{F}(\hat{Q})$ denote the set of vertices and the set of faces of $\hat{Q}$ respectively. We construct a CW-structure on $X(Q, \vartheta)$ in the following. Let

$$I_Q = \{(u, e_u) \in V(Q) \times E(Q) : ind_Q(u) = n \text{ and } e_u \text{ is the edge joining the vertices}$$

$$u, x_u \text{ such that } \phi(u) > \phi(x_u) > \phi(x) \text{ for all } x \in V(Q) - \{u, x_u\}\}.$$

Let $U_{e_u} = U_u \cap U_{x_u}$ and $\hat{Q}^n = Q$. Then $W_{e_u} = (\mathbb{Z}_2^{n-1} \times U_{e_u})/ \sim_s$ is homeomorphic to the n-dimensional open ball $B^n \subset \mathbb{R}^n$. Let

$$\hat{Q}^{n-1} = Q - U_{e_u}. \tag{4.4.2}$$

Then $\hat{Q}^{n-1}$ is the union of facets not containing the edge e_u of Q. So $\hat{Q}^{n-1}$ is an $(n-1)$-dimensional proper subcomplex of Q and $V(Q) = V(\hat{Q}^{n-1})$. Let $v \in V(\hat{Q}^{n-1})$ with $ind_{\hat{Q}^{n-1}}(v) = n - 1$. Let $F_v^{n-1} \in \mathfrak{F}(\hat{Q}^{n-1})$ be the smallest face which contains the inward pointing edges incident to v in $\hat{Q}^{n-1}$. If v_1, v_2 are two vertices of $\hat{Q}^{n-1}$ with

$ind_{\hat{Q}^{n-1}}(v_1) = n-1 = ind_{\hat{Q}^{n-1}}(v_2)$ then $F^{n-1}_{v_1} \neq F^{n-1}_{v_2}$. Let

$$I_{\hat{Q}^{n-1}} = \{(v, e_v) \in V(Q) \times E(Q) : ind_{\hat{Q}^{n-1}}(v) = n-1 \text{ and } e_v \text{ is the edge joining the vertices } v, y_v \in V(F^{n-1}_v) \text{ such that } \phi(v) > \phi(y_v) > \phi(y) \text{ for all } y \in V(F^{n-1}_v) - \{v, y_v\}\}.$$

Let

$$U_{e_v} = U_v \cap U_{y_v} \cap F^{n-1}_v \text{ for each } (v, e_v) \in I_{\hat{Q}^{n-1}}.$$

From the observation 4.2.6, $W_{e_v} = (\mathbb{Z}^{n-1}_2 \times U_{e_v})/\sim_s$ is homeomorphic to the $(n-1)$-dimensional open ball $B^{n-1} \subset \mathbb{R}^{n-1}$. Let

$$\hat{Q}^{n-2} = Q - \{\{\bigcup_{(u,e_u) \in I_{\hat{Q}^n}} U_{e_u}\} \cup \{\bigcup_{(v,e_v) \in I_{\hat{Q}^{n-1}}} U_{e_v}\}\}. \tag{4.4.3}$$

So $\hat{Q}^{n-2}$ is an $(n-2)$-dimensional proper subcomplex of Q and $V(Q) = V(\hat{Q}^{n-2})$. Let $w \in V(\hat{Q}^{n-2})$ with $ind_{\hat{Q}^{n-2}}(w) = n-2$. Let $F^{n-2}_w \in \mathfrak{F}(\hat{Q}^{n-2})$ be the smallest face which contains the inward pointing edges incident to w in $\hat{Q}^{n-2}$. If w_1, w_2 are two vertices of $\hat{Q}^{n-2}$ with $ind_{\hat{Q}^{n-2}}(w_1) = n-1 = ind_{\hat{Q}^{n-2}}(w_2)$ then $F^{n-2}_{w_1} \neq F^{n-2}_{w_2}$. Let

$$I_{\hat{Q}^{n-2}} = \{(w, e_w) \in V(Q) \times E(Q) : ind_{\hat{Q}^{n-2}}(w) = n-2 \text{ and } e_w \text{ is the edge joining the vertices } w, z_w \in F^{n-2}_w \text{ such that } \phi(w) > \phi(z_w) > \phi(z) \text{ for all } z \in V(F^{n-2}_w) - \{w, z_w\}\}.$$

Let

$$U_{e_w} = U_w \cap U_{z_w} \cap F^{n-2}_w \text{ for each } (w, e_w) \in I_{\hat{Q}^{n-2}}.$$

From the observation 4.2.6, $W_{e_w} = (\mathbb{Z}^{n-1}_2 \times U_{e_w})/\sim_s$ is homeomorphic to the $(n-2)$-dimensional open ball $B^{n-2} \subset \mathbb{R}^{n-2}$.

Continuing this process we observe that $\hat{Q}^1 (\cong (\mathbb{Z}^{n-1}_2 \times \hat{Q}^1)/\sim_s)$ is a maximal tree of the 1-skeleton of Q and $\hat{Q}^0 = V(Q)$. Hence relative interior of each edge of $(\mathbb{Z}^{n-1}_2 \times \hat{Q}^1)/\sim_s$ is homeomorphic to the 1-dimensional ball in $\mathbb{R}$. So corresponding to each edge of polytope Q, we construct a cell of dimension ≥ 1 of $X(Q, \vartheta)$.

Recall the *h-vectors* h_i of the simple polytope Q. The integer h_{n-i} is the number of vertices $v \in V(Q)$ with $ind_Q(v) = i$. The Dehn-Sommerville relation is

$$h_i = h_{n-i} \ \forall \ i = 0, 1, \ldots, n,$$

see Theorem 1.20 of [BP02]. Hence the number of k-dimensional cells in $X(Q, \vartheta)$ is

$$|I_{\hat{Q}^k}| = \Sigma^n_k h_i. \tag{4.4.4}$$

We describe the attaching map for a k-dimensional cell. Here k-dimensional cells are

$$\{W_{e_v} : (v, e_v) \in I_{\hat{Q}^k}\}.$$

Let $(v, e_v) \in I_{\hat{Q}^k}$. Let $F_v^k \in \mathfrak{F}(\hat{Q}^k)$ be the smallest face containing the inward pointing edges to v in $\hat{Q}^k$. Define an equivalence relation $\sim_{e_v}$ on $\mathbb{Z}_2^{n-1} \times F_v^k$ by

$$(t, p) \sim_{e_v} (s, q) \text{ if } p = q \in F' \text{ and } s - t \in G_{F'} \tag{4.4.5}$$

where $F' \in \mathfrak{F}(F_v^k)$ is a face containing the edge e_v. The quotient space $(\mathbb{Z}_2^{n-1} \times F_v^k)/\sim_{e_v}$ is homeomorphic to the closure of open ball $B^k \subset \mathbb{R}^k$. The attaching map $\phi_{F_v^k}$ is the natural quotient map

$$\phi_{F_v^k} : S^{k-1} \cong (\mathbb{Z}_2^{n-1} \times (F_v^k - U_{e_v})/\sim_{e_v} \;\to (\mathbb{Z}_2^{n-1} \times (F_v^k - U_{e_v})/\sim_s . \tag{4.4.6}$$

Let $X_k = \bigcup_{i=1}^{k} \bigcup_{(v,e_v)\in I_{\hat{Q}^i}} \overline{W}_{e_v}$. Then X_k is the k-th skeleton of $X(Q, \vartheta)$ and

$$X(Q, \vartheta) = \bigcup_{k=1}^{n} X_k.$$

So we get a CW-complex structure on $X(Q, \vartheta)$ with $\Sigma_k^n h_i$ cells in dimension k, $0 \leq k \leq n$. Since singular homology and cellular homology are isomorphic, we compute cellular homology of $X(Q, \vartheta)$. To calculate cellular homology we compute the boundary map of the cellular chain complex for $X(Q, \vartheta)$. To compute the boundary map we need to compute the degree of the following composition map β_{we_v}

$$S^{k-1} \xrightarrow{\phi_{F_v^k}} (\mathbb{Z}_2^{n-1} \times (F_v^k - U_{e_v})/\sim_s \xrightarrow{q} \frac{X_{k-1}}{X_{k-2}} = \bigvee_{(w,e_w)\in I_{\hat{Q}^{k-1}}} S_w^{k-1} \xrightarrow{q_w} S_w^{k-1} \tag{4.4.7}$$

where F_v^k is a face of $\hat{Q}^k$ of dimension k $(k \geq 2)$, $S_w^{k-1} \cong S^{k-1}$ and q, q_w are the quotient maps. Clearly the above composition map β_{we_v} is either surjection or constant up to homotopy. When the map is constant the degree of the composition β_{we_v} is zero. We calculate the degree of the composition when it is surjection.

Let $(w, e_w) \in I_{\hat{Q}^{k-1}}$ be such that β_{we_v} is a surjection. Let z_w be the vertex of the edge e_w other than w. Let $F_w^{k-1} \in \mathfrak{F}(\hat{Q}^{k-1})$ be the smallest face which contains the inward pointing edges to w in $\hat{Q}^{k-1}$. Let

$$U_{e_w} = U_w \cap U_{z_w} \cap F_w^{k-1}.$$

So U_{e_w} is an open subset of F_w^{k-1} and U_{e_w} contains the relative interior of e_w. The face

$F_w^{k-1} \subset F_v^k - U_{e_v}$ is a facet of F_v^k. Note that

$$W_{e_w} = (\mathbb{Z}_2^{n-1} \times U_{e_w})/\sim_s \;= S_w^{k-1} - \{X_{k-2}/X_{k-2}\}.$$

The quotient group $G_{F_w^{k-1}}/G_{F_v^k}$ is isomorphic to $\mathbb{Z}_2$. Hence from equations 4.4.5 and 4.4.6 we get that $(\beta_{we_v})^{-1}(W_{e_w})$ has two components Y^1 and Y^2 in S^{k-1}. The restrictions $(\beta_{we_v})_{|Y^1}$ and $(\beta_{we_v})_{|Y^2}$, on Y^1 and Y^2 respectively, give homeomorphism to W_{e_w}. Let y_v be the vertex of the edge e_v other than v. Observe that

$$(\mathbb{Z}_2^{n-1} \times (F_v^k - \{U_{e_v} \cup \{v, y_v\}\})/\sim_{e_v} \;\cong I^0 \times S^{k-2}.$$

Hence from the definition of equivalence relation $\sim_s$, it is clear that Y^2 is the image of Y^1 under the map (possibly up to homotopy)

$$(id \times \mathfrak{a}) : I^0 \times S^{k-2} \to I^0 \times S^{k-2} \text{ defined by } (id \times \mathfrak{a})(r, x) = (r, -x), \tag{4.4.8}$$

where $I^0 = (0, 1) \subset \mathbb{R}$. The degree of $(id \times \mathfrak{a})$ is $(-1)^{k-1}$. Hence the degree of the composition map β_{we_v} is

$$d_{vw} = deg(\beta_{we_v}) = \begin{cases} 2 & \text{if } k \geq 2 \text{ is odd and } \beta_{a_v} \text{ is a surjection} \\ 0 & \text{if } k \geq 2 \text{ is even and } \beta_{a_v} \text{ is a surjection} \\ 0 & \text{if } \beta_{a_v} \text{ is constant.} \end{cases} \tag{4.4.9}$$

Hence the cellular chain complex of the constructed CW-complex of $X(Q, \vartheta)$ is

$$0 \to \mathbb{Z} \xrightarrow{d_n} \oplus_{|I_{\hat{Q}^{n-1}}|}\mathbb{Z} \to \cdots \xrightarrow{d_3} \oplus_{|I_{\hat{Q}^2}|}\mathbb{Z} \xrightarrow{d_2} \oplus_{|I_{\hat{Q}^1}|}\mathbb{Z} \xrightarrow{d_1} \oplus_{|I_{\hat{Q}^0}|}\mathbb{Z} \to 0 \tag{4.4.10}$$

where d_k is the boundary map of the cellular chain complex. If $k \geq 2$ the formula of d_k is

$$d_k(W_{e_v}) = \sum_{(w,e_w)\in I_{\hat{Q}^{k-1}}} d_{vw} W_{e_w}, \tag{4.4.11}$$

where $(v, e_v) \in I_{\hat{Q}^k}$ and d_{vw} is the degree of the composition map β_{we_v}. Hence the map d_k is represented by the following matrix with entries

$$\{d_{vw} : (v, e_v) \in I_{\hat{Q}^k}, (w, e_w) \in I_{\hat{Q}^{k-1}}\}. \tag{4.4.12}$$

So the map d_k is the zero matrix for all even k. When $k \geq 2$ is odd, the map d_k is an injection and the image of the map d_k is the submodule generated by

$$\{ \sum_{(w,e_w)\in I_{\hat{Q}^{k-1}}} d_{vw} W_{e_w} : (v, e_v) \in I_{\hat{Q}^k}\}. \tag{4.4.13}$$

Hence the quotient module $(\oplus_{|I_{\hat{Q}^{k-1}}|}\mathbb{Z})/Imd_k$ is isomorphic to

$$(\bigoplus_{h_k}\mathbb{Z}) \oplus (\bigoplus_{\Sigma_{k+1}^{n}h_i}\mathbb{Z}_2).$$

The 1-skeleton X_1 is a tree with $\Sigma_0^n h_i$ vertices and $\Sigma_1^n h_i$ edges. The boundary map d_1 is an injection and image of d_1 is $\Sigma_1^n h_i$ dimensional direct summand of $\oplus_{|I_{\hat{Q}^0}|}\mathbb{Z}$ over $\mathbb{Z}$. Hence $(\oplus_{|I_{\hat{Q}^0}|}\mathbb{Z})/d_1(\oplus_{|I_{\hat{Q}^1}|}\mathbb{Z})$ is isomorphic to $\mathbb{Z}$. From the previous calculation we have proved the following theorem.

Theorem 4.4.2. *The singular homology groups of the small orbifold $X(Q,\vartheta)$ with coefficients in $\mathbb{Z}$ is*

$$H_k(X(Q,\vartheta),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \text{ and if } k=n \text{ even} \\ (\oplus_{h_k}\mathbb{Z}) \oplus (\oplus_{\Sigma_{k+1}^{n}h_i}\mathbb{Z}_2) & \text{if } k \text{ is even, } 0<k<n \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.4.3. *If Q is an even dimensional simple polytope then the small orbifold over Q is orientable.*

Corollary 4.4.4. *The singular homology groups of the orbifold $X(Q,\vartheta)$ with coefficients in $\mathbb{Q}$ is*

$$H_k(X(Q,\vartheta),\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k=0 \text{ and if } k=n \text{ even} \\ \oplus_{h_k}\mathbb{Q} & \text{if } k \text{ is even, } 0<k<n \\ 0 & \text{otherwise.} \end{cases}$$

With coefficients in $\mathbb{Z}_2$ the cellular chain complex 4.4.10 is

$$0 \to \mathbb{Z}_2 \xrightarrow{0} \oplus_{|I_{\hat{Q}^{n-1}}|}\mathbb{Z}_2 \xrightarrow{0} \cdots \xrightarrow{0} \oplus_{|I_{Q^1}|}\mathbb{Z}_2 \xrightarrow{d_1} \oplus_{|I_{\hat{Q}^0}|}\mathbb{Z}_2 \xrightarrow{0} 0 \tag{4.4.14}$$

Where d_1 is an injection. Hence $(\oplus_{|I_{\hat{Q}^0}|}\mathbb{Z}_2)/d_1(\oplus_{|I_{\hat{Q}^1}|}\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$. So we get the following corollary.

Corollary 4.4.5. *The singular homology groups of the orbifold $X(Q,\vartheta)$ with coefficients in $\mathbb{Z}_2$ is*

$$H_k(X(Q,\vartheta),\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k=0 \text{ and if } k=n \\ \oplus_{\Sigma_k^n h_i}\mathbb{Z}_2 & \text{if } 1<k<n \\ 0 & \text{if } k=1. \end{cases}$$

Remark 4.4.6. *The k-th modulo 2 Betti number $b_k(X(Q,\vartheta))$ of small orbifold $X(Q,\vartheta)$ is zero when $k=1$ and $b_k(X(Q,\vartheta)) = \Sigma_k^n h_i$ if $1<k\leq n$ and $b_0(X(Q,\vartheta)) = h_0 = 1$. Hence modulo 2 Euler characteristic of $X(Q,\vartheta)$ is*

$$\mathfrak{X}(X(Q,\vartheta)) = h_0 + \Sigma_{k=2}^{n}(-1)^k\Sigma_k^n h_i = \Sigma_0^{[n/2]}h_{2i}. \tag{4.4.15}$$

Observe that $b_k(X(Q,\vartheta)) \neq b_{n-k}(X(Q,\vartheta))$ if $1 \leqslant k < n$. Hence the Poincaré Duality for small orbifolds is not true with coefficients in $\mathbb{Z}_2$.

4.5 Cohomology ring of small orbifolds

We have shown that the even dimensional small orbifolds are compact, connected, orientable. Let $\mathcal{X}(Q,\vartheta)$ be an even dimensional small orbifold over the polytope Q. Hence by the Proposition 2.4.8 we get that the cohomology ring of $\mathcal{X}(Q,\vartheta)$ satisfy the Poincaré duality with coefficients in rationals.

We rewrite Poincaré duality for small orbifolds using the intersection theory. The purpose is to show the cup product in cohomology ring is Poincaré dual to intersection, see equation 4.5.9. The proof is akin to the proof of Poincaré duality for oriented closed manifolds proved in [GH78]. To show these we construct a **q**-CW complex structure on $X(Q,\vartheta)$. The **q**-CW complex structure on a Hausdorff topological space is discussed in Section 3.5. Similarly to the **q**-cellular homology case, we can show that **q**-cellular cohomology of a **q**-CW complex is isomorphic to its singular cohomology with coefficients in rationals.

Let Q be an n-dimensional simple polytope where n is even and $\pi : X(Q,\vartheta) \to Q$ be a small orbifold over Q. Let Q' be the second barycentric subdivision of Q. Let

$$\{\eta^k_\alpha : \alpha \in \Lambda(k) \text{ and } k = 0,1,\ldots,n\} \tag{4.5.1}$$

be the simplices in Q'. Here k is the dimension of η^k_α and $\Lambda(k)$ is an index set. Let $(\eta^k_\alpha)^0$ be the relative interior of k-dimensional simplex η^k_α.

Definition 4.5.1. *A subset $Y \subseteq X(Q,\vartheta)$ is said to be relatively open subset of dimension k if for each point $y \in Y$ there exist an orbifold chart $(\widetilde{U}, G, \psi)$ such that $\psi(V) \ni y$ is an open subset of Y, for some k-dimensional submanifold V of $\widetilde{U}$.*

Then $(\pi^{-1})(\eta^k_\alpha)^0$ is disjoint union of the following relatively open subsets

$$\{(\sigma^k_{\alpha_i})^0 \subset X(Q,\vartheta) : i = 1,\ldots,\alpha(k)\}$$

for some natural number $\alpha(k)$. Here $\sigma^k_{\alpha_i}$ is the closure of $(\sigma^k_{\alpha_i})^0$ in $X(Q,\vartheta)$. The restriction of π on $\sigma^k_{\alpha_i}$ is a homeomorphism onto the simplex η^k_α for $i = 1,\ldots,\alpha(k)$. Then the collection

$$\{\sigma^k_{\alpha_i} : i = 1,\ldots,\alpha(k) \text{ and } \alpha \in \Lambda(k) \text{ and } k = 0,1,\ldots,n\} \tag{4.5.2}$$

gives a simplicial decomposition of the small orbifold $X(Q,\vartheta)$. So

$$\mathcal{K} = \{\sigma^k_{\alpha_i}, \partial\}_{\alpha_i,k} \tag{4.5.3}$$

is a simplicial complex of $X(Q, \vartheta)$.

Definition 4.5.2. *The transversality of two relatively open subsets U and V of $X(Q, \vartheta)$ at $p \in U \cap V$ is defined as follows:*

1. *If p is a smooth point of $X(Q, \vartheta)$, we say U intersect V transversely at p whenever $T_p(U) + T_p(V) = T_p(X(Q, \vartheta))$.*
2. *If p is an orbifold point of $X(Q, \vartheta)$ there exist an orbifold chart $(B^n, \mathbb{Z}_2, \phi_v)$ such that $\phi_v(0) = p$. We say U intersect V transversely at p whenever $T_0(\phi_v^{-1}(U)) + T_0(\phi_v^{-1}(V)) = T_0(B^n)$.*

Let $\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ be two simplices of dimension k_1 and k_2 respectively in the simplicial complex $\mathcal{K}$ of $X(Q, \vartheta)$.

Definition 4.5.3. *We say $\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ intersect transversely at $p \in \sigma_{\alpha_i}^{k_1} \cap \rho_{\beta_j}^{k_2}$ if there exist two relatively open subsets $U \subset X(Q, \vartheta)$ and $V \subset X(Q, \vartheta)$ containing $\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ respectively such that dim$(U) = k_1$, dim$(V) = k_2$ and U intersect V transversely at p.*

Let U and V be two complementary dimensional relatively open subset of $X(Q, \vartheta)$ that intersect transversely at $p \in U \cap V$.

Definition 4.5.4. *Define the intersection index of U and V at p to be 1 if there exist oriented bases $\{\xi_1, \ldots, \xi_{k_1}\}$ and $\{\eta_1, \ldots, \eta_{k_2}\}$ for $T_p(U)$ $(T_0(\phi_v^{-1}(U)))$ and $T_p(V)$ $(T_0(\phi_v^{-1}(V)))$ respectively such that $\{\xi_1, \ldots, \xi_{k_1}, \eta_1, \ldots, \eta_{k_2}\}$ is an oriented basis for $T_pX(Q, \vartheta)(T_0B^n)$ whenever p is smooth (respectively orbifold) point of $X(Q, \vartheta)$. Otherwise the intersection index of U and V at p is -1.*

Since antipodal action on B^n (as n is even) is orientation preserving there is no ambiguity in the above definition. Let

$$A = \Sigma n_{\alpha_i} \sigma_{\alpha_i}^{k_1} \text{ and } B = \Sigma m_{\beta_j} \rho_{\beta_j}^{k_2}$$

be two cycles of the simplicial complex $\mathcal{K}$ such that $n = k_1 + k_2$ and they intersect transversely.

Definition 4.5.5. *Define the intersection number of A and B is the sum of the intersection indixes (counted with multiplicity) at their intersection points.*

The number is finite since A and B are closed subsets of compact space $X(Q, \vartheta)$. We show that the intersection number depends only on the homology class of the cycle. Let $\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ be two simplices in $\mathcal{K}$ with $k_1 + k_2 = n$. From the construction of the simplicial complex $\mathcal{K}$ we make some observations.

Observation 4.5.6.

1. *$\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ can not contain different orbifold points whenever their intersection is nonempty.*

2. *Each* $\sigma^{k_1}_{\alpha_i}$ *and* $\rho^{k_2}_{\beta_j}$ *can contain at most one orbifold point.*

3. *If* $\sigma^{k_1}_{\alpha_i}$ *and* $\rho^{k_2}_{\beta_j}$ *contain an orbifold point or not, whenever their intersection is nonempty, we can find a* $\mathbb{Z}_2$*-invariant smooth homotopy*

$$\mathcal{G} : [0,1] \times X_v(Q,\vartheta) \to X_v(Q,\vartheta)$$

fixing the orbifold point of $X_v(Q,\vartheta)$ *such that* $\mathcal{G}(0 \times U^{k_1}_{\alpha_i})$ *and* $\mathcal{G}(1 \times V^{k_2}_{\beta_j})$ *intersect transversely where* $U^{k_1}_{\alpha_i}$ *and* $V^{k_2}_{\beta_j}$ *containing* $\sigma^{k_1}_{\alpha_i}$ *and* $\rho^{k_2}_{\beta_j}$ *respectively are suitable relatively open subsets of* $X_v(Q,\vartheta)$ *and* $dim U^{k_1}_{\alpha_i} = k_1$, $dim V^{k_2}_{\beta_j} = k_2$.

Let $\sigma^{k_1}_{\alpha_0} + \ldots + \sigma^{k_1}_{\alpha_{k_1}}$ be the boundary of (k_1+1)-simplex $\sigma^{k_1+1}_{\alpha}$. The observations 4.5.6 also hold for the simplices $\sigma^{k_1+1}_{\alpha}$ and $\rho^{k_2}_{\beta_j}$ although $k_1 + 1 + k_2 = n+1$. If $\mathcal{G}'$ is the smooth homotopy and $\mathcal{G}'(0 \times U^{k_1+1}_{\alpha}) \cap \mathcal{G}'(1 \times V^{k_2}_{\beta_j})$ is nonempty then the subset

$$\mathcal{G}'(0 \times U^{k_1+1}_{\alpha}) \cap \mathcal{G}'(1 \times V^{k_2}_{\beta_j})$$

of $X(Q,\vartheta)$ is a collection of piecewise smooth curves. After lifting a curve to an orbifold chart (if necessary), using the similar arguments as in [GH78] we can show that intersection number of $\sigma^{k_1}_{\alpha_0} + \ldots + \sigma^{k_1}_{\alpha_{k_1}}$ and $\rho^{k_2}_{\beta_j}$ is zero. Integrating these computation to the boundary $A = \Sigma n_{\alpha_i}\sigma^{k_1}_{\alpha_i}$ and the cycle $B = \Sigma m_{\beta_j}\rho^{k_2}_{\beta_j}$ we ensure that the intersection number of A and B is zero.

Let $\mathcal{K}' = \{\tau^k_{\alpha_i}, \partial\}$ be the first barycentric subdivision of the complex $\mathcal{K}$. Now we construct the dual **q**-cell decomposition of the complex $\mathcal{K}$. For each vertex $\sigma^0_{\alpha_i}$ in the complex $\mathcal{K}$, let

$$*\,\sigma^0_{\alpha_i} = \bigcup_{\sigma^0_{\alpha_i} \in \tau^n_{\beta_j}} \tau^n_{\beta_j} \tag{4.5.4}$$

be the n-dimensional **q**-cell which is the union of the n-simplices $\tau^n_{\beta_j} \in \mathcal{K}'$ containing $\sigma^0_{\alpha_i}$ as a vertex. Then for each k-simplex $\sigma^k_{\alpha_i}$ in the decomposition $\mathcal{K}$, let

$$*\,\sigma^k_{\alpha_i} = \bigcap_{\sigma^0_{\beta_j} \in \tau^n_{\alpha_i}} *\sigma^0_{\beta_j} \tag{4.5.5}$$

be the intersection of the n-dimensional **q**-cells associated to the $k+1$ vertices of $\sigma^k_{\alpha_i}$. The **q**-cells $\{\Delta^{n-k}_{\alpha_i} = *\sigma^k_{\alpha_i}\}$ give a **q**-cell decomposition of $X(Q,\vartheta)$, called the dual **q**-cell decomposition of $\mathcal{K}$. So the dual **q**-cell decomposition $\{\Delta^{n-k}_{\alpha}\}$ is a **q**-CW structure on $X(Q,\vartheta)$.

From the description of dual **q**-cells it is clear that $\Delta^{n-k}_{\alpha_i}$ intersects $\sigma^k_{\alpha_i}$ transversely when dimension of $\sigma^k_{\alpha_i}$ is greater than zero. $\Delta^n_{\alpha_i}$ is a quotient space of the antipodal action on a symmetric convex polyhedral centered at origin in $\mathbb{R}^n$. Since the antipodal action on $\mathbb{R}^n$ (n even) preserve orientation of $\mathbb{R}^n$, we can define the intersection number of $\sigma^0_{\alpha_i}$ and $\Delta^n_{\alpha_i}$ to be 1. We consider the orientation on the dual **q**-cell $\{\Delta^n_{\alpha_i}\}$ such that the intersection number of $\sigma^k_{\alpha_i}$ and $\Delta^{n-k}_{\alpha_i}$ is 1.

Using the same argument as Grifiths and Harris have made in the proof of Poincaré duality theorem in [GH78], we can prove the following relation between boundary operator ∂ on the cell complex $\{\sigma^k_{\alpha_i}\}$ and coboundary operator δ on the dual **q**-cell complex $\{\Delta^{n-k}_{\alpha_i}\}$ when dimension of $\sigma^k_{\alpha_i}$ is greater than one,

$$\delta(\{\Delta^{n-k}_{\alpha_i}\}) = (-1)^{n-k+1} * (\partial\sigma^k_{\alpha_i}). \tag{4.5.6}$$

Let $\{\sigma^k_{\alpha_i}\} = < x, y > \in \mathcal{K}$ be a one simplex with the vertices x, y. The orientation on $\{\sigma^k_{\alpha_i}\}$ comes from the orientation of $X(Q, \vartheta)$. Since we are considering **q**-cell structure on $X(Q, \vartheta)$, define $\delta(\{\Delta^{n-1}_{\alpha_i}\}) = *\sigma^0_y - *\sigma^0_x$. So we get a map $\sigma^k_{\alpha_i} \to \Delta^{n-k}_{\alpha_i}$ which induces an isomorphism

$$\xi'_k : H_k(X(Q, \vartheta), \mathbb{Q}) \to H^{n-k}_{\mathbf{q}\text{-}CW}(X(Q, \vartheta), \mathbb{Q}), \tag{4.5.7}$$

where $H^{n-k}_{\mathbf{q}\text{-}CW}(X(Q, \vartheta), \mathbb{Q})$ is $n-k$ th **q**-cellular cohomology group. Hence we have the following theorem for even dimensional small orbifold.

Theorem 4.5.7 (Poincaré duality)**.** *Let $X(Q, \vartheta)$ be an even dimensional small orbifold. The intersection pairing*

$$H_k(X(Q, \vartheta), \mathbb{Q}) \times H_{n-k}(X(Q, \vartheta), \mathbb{Q}) \to \mathbb{Q}$$

is nonsingular; that is, any linear functional $H_{n-k}(X(Q, \vartheta), \mathbb{Q}) \to \mathbb{Q}$ is expressible as the intersection with some class $\Theta \in H_k(X(Q, \vartheta), \mathbb{Q})$. There is an isomorphism ξ'_k from $H_k(X(Q, \vartheta), \mathbb{Q})$ to $H^{n-k}(X(Q, \vartheta), \mathbb{Q})$.

Using this Poincaré duality theorem for even dimensional small orbifold we can calculate the cohomology groups of small orbifold $X(Q, \vartheta)$.

Theorem 4.5.8. *The singular cohomology groups of the even dimensional small orbifold $X(Q, \vartheta)$ with coefficients in $\mathbb{Q}$ is*

$$H^k(X(Q, \vartheta), \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 0 \text{ and if } k = n \text{ even} \\ \oplus_{h_k} \mathbb{Q} & \text{if } k \text{ is even, } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

We can also define a product $\mu_{k_1 k_2}$ similarly as in [GH78] but some care is needed at orbifold points. The product

$$\mu_{k_1 k_2} : H_{n-k_1}(X(Q, \vartheta), \mathbb{Q}) \times H_{n-k_2}(X(Q, \vartheta), \mathbb{Q}) \to H_{n-k_1-k_2}(X(Q, \vartheta), \mathbb{Q}) \tag{4.5.8}$$

on the homology of $X(Q,\vartheta)$ in arbitrary dimensions satisfying the following commutative diagram.

$$\begin{array}{ccc} H_{n-k_1}(X(Q,\vartheta),\mathbb{Q}) \times H_{n-k_2}(X(Q,\vartheta),\mathbb{Q}) & \xrightarrow{\mu_{k_1k_2}} & H_{n-k_1-k_2}(X(Q,\vartheta),\mathbb{Q}) \\ \xi_{n-k_1}\times\xi_{n-k_2}\downarrow & & \xi_{n-k_1-k_2}\downarrow \\ H^{k_1}(X(Q,\vartheta),\mathbb{Q}) \times H^{k_2}(X(Q,\vartheta),\mathbb{Q}) & \xrightarrow{\mathfrak{u}} & H^{k_1+k_2}(X(Q,\vartheta),\mathbb{Q}) \end{array} \tag{4.5.9}$$

where the lower horizontal map $\mathfrak{u}$ is the cup product in cohomology ring.

We write some observations about the transversality of faces of an n-dimensional polytope Q (n even). Let F and F' be two faces of Q. F and F' intersect transversely if $\text{codim}(F \cap F') = \text{codim}F + \text{codim}F'$. Since Q is simple polytope, the following two properties are satisfied.

Property 1. *Let F be a $2k$-dimensional face of Q and u be a vertex of F. Then there is a unique $(n-2k)$-dimensional face F' of Q such that F and F' meet at u transversely.*

Property 2. *Let F be a face of codimension $2k$. Then there is k many distinct faces of codimension two such that they intersect transversely at each point of F.*

Lemma 4.5.9. *Let $\pi : X(Q,\vartheta) \to Q$ be an even dimensional small orbifold and $X(F,\vartheta') = \pi^{-1}(F)$ for each face F of Q. Then*

1. *For each $2k$-dimensional face F of Q, the homology class represented by $X(F,\vartheta')$, denoted by $[X(F,\vartheta')]$, is not zero in $H_*(X(Q,\vartheta),\mathbb{Q})$.*
2. *The cohomology ring $H^*(X(Q,\vartheta),\mathbb{Q})$ is generated by 2-dimensional classes.*

Proof. The space $X(F,\vartheta')$ is a $2k$-dimensional suborbifold of $X(Q,\vartheta)$, for each $2k$-dimensional ($0 \leq 2k \leq n$) face F of Q. By Corollary 4.4.4 we get that the homology in degree $2k$ of $X(Q,\vartheta)$ is generated by the classes of form $[X(F,\vartheta')]$, where F is a $2k$-dimensional face.

By equation 4.5.9, the dual of $X(F \cap F',\vartheta')$ is the cup product of the dual of $[X(F,\vartheta')]$ and the dual of $[X(F',\vartheta')]$, if F and F' intersect transversely and otherwise the dual of $X(F \cap F',\vartheta')$ is zero.

The property 1 tells that there is an $(n-2k)$-dimensional face F' which intersects F transversely at a vertex of Q. Since the homology classes $[X(F,\vartheta')]$ and $[X(F',\vartheta')]$ are dual in intersection pairing of Poincaré duality, they are both nonzero. This proves (1) of the above Lemma.

In theorem 4.5.8 we show the odd dimensional cohomology group is zero. The cohomology in degree $2k$ is generated by Poincaré duals of classes of the form $[X(F,\vartheta')]$, $\text{codim}F = 2k$. By property 2, F is the transverse intersection of distinct faces of codimension two. Hence, the Poincaré dual of $[X(F,\vartheta')]$ is the product of cohomology classes of dimension 2. This proves (2) of the above Lemma. □

Recall the index function ind_Q from Section 4.4. Let $\hat{F}_v \in \mathfrak{F}(Q)$ be the smallest face containing the inward pointing edges incident to the vertex v of Q. Let w be the Poincaré dual of class of the form $[X(\hat{F}_v, \vartheta')]$, also denoted by $[v]$. Let $\{v_1, v_2, \ldots, v_r\}$ be the set of vertices of Q such that $ind_Q(v_i) = n-2$. We show that $\{w_1, w_2, \ldots, w_r\}$ is a minimal generating set of $H^*(X(Q,\vartheta), \mathbb{Q})$.

Let $A_j = \{v \in V(Q) : ind_Q(v) = j$. Let $U_{\hat{F}_v}$ be the open subset of $\hat{F}_v$ obtained by deleting all faces of $\hat{F}_v$ not containing the vertex v. From Section 5.3 it is clear that $\pi^{-1}(U_{\hat{F}_v})$ is homeomorphic to the orbit space $B^j/\mathbb{Z}_2$, where $\mathbb{Z}_2$ action on B^j is antipodal. So $\pi^{-1}(U_{\hat{F}_v})$ is j-dimensional **q**-cell in $X(Q,\vartheta)$. Clearly

$$X(Q,\vartheta) = \bigcup_{v \in V(Q)} \pi^{-1}(U_{\hat{F}_v}).$$

This gives a **q**-CW structure on $X(Q,\vartheta)$. From Theorem 1.20 of [BP02], we get the number of j-dimensional cells is h_{n-j}, cardinality of A_j. So the corresponding **q**-cellular chain complex gives that $\{[v] : v \in A_j\}$ is a basis of $H_j(X(Q,\vartheta), \mathbb{Q})$ if j is even. Theorem 4.5.8 tells that $\{w = \xi_j([v]) : v \in A_j\}$ is a basis of $H^{n-j}(X(Q,\vartheta), \mathbb{Q})$ if j is even.

Let F be a codimension $2k$ face of Q with top vertex v of index $n-2k$. By property 2 F is unique intersection of k many distinct codimension 2 faces $\hat{F}_{v_{i_1}}, \ldots, \hat{F}_{v_{i_k}}$ with top vertices $v_{i_1}, \ldots, v_{i_k} \in \{v_1, v_2, \ldots, v_r\}$ respectively. Hence $w_{i_1} \ldots w_{i_k} = w$ in $H^*(X(Q,\vartheta), \mathbb{Q})$. Consider the polynomial ring $\mathbb{Q}[w_1, w_2, \ldots, w_r]$. Let the map

$$\mu_{n_{i_1} \ldots n_{i_l}} : H_{n_{i_1}}(X(Q,\vartheta), \mathbb{Q}) \times \cdots \times H_{n_{i_l}}(X(Q,\vartheta), \mathbb{Q}) \to H_{n - n_{i_1} - \cdots - n_{i_l}}(X(Q,\vartheta), \mathbb{Q}) \tag{4.5.10}$$

be defined by the repeated application of the product map $\mu_{n_{i_1} n_{i_2}}$. Let I be the ideal of $\mathbb{Q}[w_1, w_2, \ldots, w_r]$ generated by the following elements

$$S = \begin{cases} w_{i_1} w_{i_2} \ldots w_{i_l} & \text{if } n_{i_1}, \ldots, n_{i_l} \text{ are even and } \mu_{n_{i_1} \ldots n_{i_l}}(v_{i_1}, v_{i_2}, \ldots, v_{i_l}) = 0 \\ & \text{in } H_{n - \{n_{i_1} + \cdots + n_{i_l}\}}(X(Q,\vartheta), \mathbb{Q}) \\ \prod_1^{l_1} w_{i_k} - \prod_1^{l_2} w_{j_l} & \text{if } \mu_{n_{i_1} \ldots n_{i_{l_1}}}(v_{i_1}, v_{i_2}, \ldots, v_{i_{l_1}}) = \mu_{n_{j_1} \ldots n_{j_{l_2}}}(v_{j_1}, v_{j_2}, \ldots, v_{j_{l_2}}) \text{ in} \\ & H_{n - \{n_{i_1} + \cdots + n_{i_l}\}}(X(Q,\vartheta), \mathbb{Q}) \text{ for some } n_{i_1}, \ldots, n_{i_{l_1}}, n_{j_1}, \ldots, \\ & n_{j_{l_2}} \text{ such that } n_{i_1} + \ldots + n_{i_{l_1}} = n_{j_1} + \ldots + n_{j_{l_2}} \end{cases} \tag{4.5.11}$$

The Poincaré Duality theorem and intersection theory ensure that the relations among w_i's are exactly as described above. Hence we have the following theorem.

Theorem 4.5.10. *The cohomology ring of an even dimensional small orbifold $X(Q,\vartheta)$ over the simple polytope Q is isomorphic to the quotient ring $\mathbb{Q}[w_1, w_2, \ldots, w_r]/I$.*

Chapter 5

$\mathbb{T}^2$-cobordism of quasitoric 4-manifolds

5.1 Introduction

In this chapter we introduce the notion of edge-simple polytope. We give the brief definition of some manifolds with quasitoric and small cover boundary in a constructive way. There is a natural torus action on these manifolds with quasitoric boundary having a simple convex polytope as the orbit space. Interestingly, we show that such a manifold with quasitoric boundary could be viewed as the quotient space of a quasitoric manifold corresponding to a certain circle action on it. We show these manifolds with quasitoric boundary are orientable and compute their Euler characteristic.

We consider the following category: the objects are all quasitoric manifolds and morphisms are torus equivariant maps between quasitoric manifolds. We compute the $\mathbb{T}^2$-cobordism group of 4-dimensional manifolds in this category. We show that the $\mathbb{T}^2$-cobordism group of 4-dimensional quasitoric manifolds is generated by the $\mathbb{T}^2$-cobordism classes of the complex projective space $\mathbb{CP}^2$, see Theorem 5.7.7. We also show that $\mathbb{T}^2$-cobordism class of a Hirzebruch surface is trivial, see Lemma 5.7.3. The main tool is the theory of quasitoric manifolds.

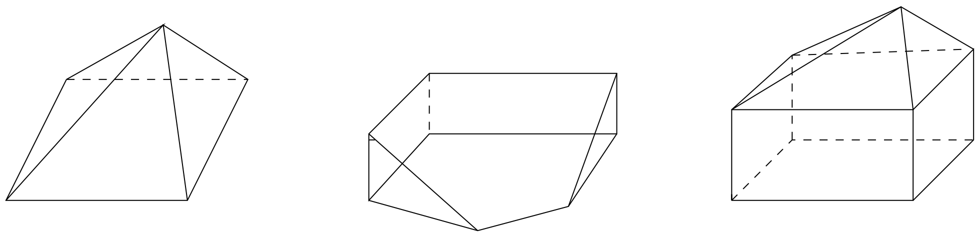
5.2 Edge-Simple Polytopes

In this section we introduce a particular type of polytopes, which we call an edge-simple polytopes.

Definition 5.2.1. *An n-dimensional convex polytope P is called an n-dimensional edge-simple polytope if each edge of P is the intersection of exactly $(n-1)$ facets of P.*

Example 5.2.2. *1. An n-dimensional simple polytope is an n-dimensional edge-simple polytope.*

2. The following convex polytopes are edge-simple polytopes of dimension 3.



3. The dual polytope of a 3*-dimensional simple polytope is a* 3*-dimensional edge-simple polytope. This result is not true for higher dimensional polytopes, that is if P is a simple polytope of dimension $n \geq 4$ the dual polytope of P may not be an edge-simple polytope. For example the dual of the* 4*-dimensional standard cube in $\mathbb{R}^4$ is not an edge-simple polytope.*

Proposition 5.2.3. ***(a)*** *If P is a* 2*-dimensional simple polytope then the suspension SP on P is an edge-simple polytope and SP is not a simple polytope.*

(b) *If P is an n-dimensional simple polytope then the cone CP on P is an $(n+1)$-dimensional edge-simple polytope.*

Proof. ***(a)*** Let P be a 2-dimensional simple polytope with m vertices $\{v_i \colon i \in I = \{1, 2, \ldots, m\}\}$ and m edges $\{e_i \colon i \in I\}$. Let a and b be the other two vertices of SP. Then facets of SP are the cone $(Ce_i)_x$ on e_i at $x = a, b$. Edges of SP are $\{xv_i \colon x = a, b$ and $i \in I\} \cup \{e_i : i \in I\}$. The edge xv_i is the intersection of $(Ce_{i_1})_x$ and $(Ce_{i_2})_x$ if $v_i = e_{i_1} \cap e_{i_2}$ for $x = a, b$ and $e_i = (Ce_i)_a \cap (Ce_i)_b$. Hence SP is an edge-simple polytope. Each vertex v_i of P is the intersection of 4 facets of SP. So SP is not a simple polytope.

(b) Let P be an n-dimensional simple polytope in $\mathbb{R}^n \times 0 \subseteq \mathbb{R}^{n+1}$ with m facets $\{F_i \colon i \in I = \{1, 2, \ldots, m\}\}$ and k vertices $\{v_1, v_2, \ldots, v_k\}$. Assume that the cone are taken at a fixed point a in $\mathbb{R}^{n+1} - \mathbb{R}^n$ lying above the centroid of P. Then facets of CP are $\{(CF_i) \colon i = 1, 2, \ldots, m\} \cup \{P\}$. Edges of CP are $\{av_i = C(\{v_i\}) \colon i = 1, 2, \ldots, k\} \cup \{e_l : e_l$ is an edge of $P\}$. Since P is a simple polytope, each vertex v_i of P is the intersection of exactly n facets of P, namely $\{v_i\} = \cap_{j=1}^{n} F_{i_j}$ and each edge e_l is the intersection of unique collection of $(n-1)$ facets $\{F_{l_1}, \ldots, F_{l_{n-1}}\}$. Then $C\{v_i\} = \cap_{j=1}^{n} CF_{i_j}$ and $e_l = P \cap CF_{l_1} \cap CF_{l_2} \cap \ldots \cap CF_{l_{n-1}}$. That is $C\{v_i\}$ and $\{e_l\}$ are the intersection of exactly n facets of CP. Hence CP is an $(n+1)$-dimensional edge-simple polytope. □

Cut off a neighborhood of each vertex $v_i, i = 1, 2, \ldots, k$ of an n-dimensional edge-simple polytope $P \subset \mathbb{R}^n$ by an affine hyperplane $H_i, i = 1, 2, \ldots, k$ in $\mathbb{R}^n$ such that $H_i \cap H_j \cap P$ are empty sets for $i \neq j$. Then the remaining subset of the polytope P is a simple polytope of dimension n, denote it by Q_P. Suppose $P_{H_i} = H_i \cap P = H_i \cap Q_P$ for $i = 1, 2, \dots k$. Then P_{H_i} is a facet of Q_P called the facet corresponding to the vertex v_i for each $i = 1, \dots, k$. Since each vertex of P_{H_i} is an interior point of an edge of P and P is an edge-simple polytope, P_{H_i} is an $(n-1)$-dimensional simple polytope for each $i = 1, 2, \dots, k$.

Lemma 5.2.4. *Let F be a codimension $l < n$ face of P. Then F is the intersection of unique set of l facets of P.*

Proof. The intersection $F \cap Q_P$ is a codimension l face of Q_P not contained in $\cup_{i=0}^{k}\{P_{H_i}\}$. Since Q_P is a simple polytope, $F \cap Q_P = \cap_{j=1}^{l} F'_{i_j}$ for some facets $\{F'_{i_1}, \dots, F'_{i_1}\}$ of Q_P. Let F_{i_j} be the unique facet of P such that $F'_{i_j} \subseteq F_{i_j}$. Then $F = \cap_1^l F_{i_j}$. Hence each face of P of codimension $l < n$ is the intersection of unique set of l facets of P. □

Remark 5.2.5. *If v_i is the intersection of facets $\{F_{i_1}, \dots, F_{i_l}\}$ of P for some positive integer l, the facets of P_{H_i} are $\{P_{H_i} \cap F_{i_1}, \dots, P_{H_i} \cap F_{i_l}\}$.*

5.3 Manifolds with quasitoric boundary

Definition 5.3.1. *A manifold with quasitoric boundary is a manifold with boundary where the boundary is a disjoint union of some quasitoric manifolds.*

Let P be an edge-simple polytope of dimension n with m facets $F_1, \dots, F_m$ and k vertices $v_1, \dots, v_k$. Let e be an edge of P. Then e is the intersection of unique collection of $(n-1)$ facets $\{F_{i_j} : j = 1, \dots, (n-1)\}$. Suppose $\mathcal{F}(P) = \{F_1, \dots, F_m\}$.

Definition 5.3.2. *The function $\lambda \colon \mathcal{F}(P) \to \mathbb{Z}^{n-1}$ is called an isotropy function of the edge-simple polytope P if the set of vectors $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_{n-1}})\}$ form a basis of $\mathbb{Z}^{n-1}$ whenever the intersection of the facets $\{F_{i_1}, \dots, F_{i_{n-1}}\}$ is an edge of P.*

The vectors $\lambda_i := \lambda(F_i)$ are called isotropy vectors and the pair (P, λ) is called an isotropy pair.

We define some isotropy functions of the edge-simple polytopes I^3 and P_0 in examples 5.3.5 and 5.3.6 respectively.

Remark 5.3.3. *It may not possible to define an isotropy function on the set of facets of arbitrary edge-simple polytopes. For example there does not exist an isotropy function of the standard n-simplex $\triangle^n$ for each $n \geq 3$.*

We construct a manifold with quasitoric boundary from the isotropy pair (P, λ). Let F be a face of P of codimension $l < n$. Then F is the intersection of a unique collection of l facets $F_{i_1}, F_{i_2}, \dots, F_{i_l}$ of P. Let $\mathbb{T}_F$ be the torus subgroup of $\mathbb{T}^{n-1}$ corresponding to the submodule generated by $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_l}$ in $\mathbb{Z}^{n-1}$. Assume $\mathbb{T}_v = \mathbb{T}^{n-1}$ for each vertex

v of P and $\mathbb{T}_P = 1$. We define an equivalence relation $\sim_b$ on the product $\mathbb{T}^{n-1} \times P$ as follows.

$$(t, p) \sim_b (u, q) \text{ if and only if } p = q \text{ and } tu^{-1} \in \mathbb{T}_F \tag{5.3.1}$$

where $F \subseteq P$ is the unique face containing p in its relative interior. We denote the quotient space $(\mathbb{T}^{n-1} \times P)/\sim_b$ by $X(P, \lambda)$. The space $X(P, \lambda)$ is not a manifold except when P is a 2-dimensional polytope. If P is 2-dimensional polytope the space $X(P, \lambda)$ is homeomorphic to the 3-dimensional sphere.

But whenever $n > 2$ we can construct a manifold with boundary from the space $X(P, \lambda)$. We restrict the equivalence relation $\sim_b$ on the product $(\mathbb{T}^{n-1} \times Q_P)$ where $Q_P \subset P$ is a simple polytope as constructed in Section 5.2 corresponding to the edge-simple polytope P. Let

$$W(Q_P, \lambda) = (\mathbb{T}^{n-1} \times Q_P)/\sim_b \quad \subset X(P, \lambda) \tag{5.3.2}$$

be the quotient space. The natural action of $\mathbb{T}^{n-1}$ on $W(Q_P, \lambda)$ is induced by the group operation in $\mathbb{T}^{n-1}$.

Theorem 5.3.4. *The space $W(Q_P, \lambda)$ is a manifold with boundary. The boundary is a disjoint union of quasitoric manifolds.*

For each edge e of P, $e' = e \cap Q_P$ is an edge of the simple polytope Q_P. Let $U_{e'}$ be the open subset of Q_P obtained by deleting all facets of Q_P that does not contain e' as an edge. Then the set $U_{e'}$ is diffeomorphic to $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ where I^0 is the open interval $(0, 1)$ in $\mathbb{R}$. The facets of $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ are $I^0 \times \{x_1 = 0\}, \ldots, I^0 \times \{x_{n-1} = 0\}$ where $\{x_j = 0,\ j = 1, 2, \ldots, n-1\}$ are the coordinate hyperplanes in $\mathbb{R}^{n-1}$. Let $F'_{i_1}, \ldots, F'_{i_{n-1}}$ be the facets of Q_P such that $\cap_{j=1}^{n-1} F'_{i_j} = e'$. Suppose the diffeomorphism

$$\mathfrak{g} \colon U_{e'} \to I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$$

sends $F'_{i_j} \cap U_{e'}$ to $I^0 \times \{x_j = 0\}$ for all $j = 1, 2, \ldots, n-1$. Define an isotropy function λ_e on the set of all facets of $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ by $\lambda_e(I^0 \times \{x_j = 0\}) = \lambda_{i_j}$ for all $j = 1, 2, \ldots, n-1$. We define an equivalence relation $\sim_e$ on $(\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})$ as follows.

$$(t, b, x) \sim_e (u, c, y) \text{ if and only if } (b, x) = (c, y) \text{ and } tu^{-1} \in \mathbb{T}_{\mathfrak{g}(F)}. \tag{5.3.3}$$

where $\mathfrak{g}(F) \subseteq I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ is the unique face containing (b, x) in its relative interior, for a unique face F of $U_{e'}$ and $\mathbb{T}_{\mathfrak{g}(F)} = \mathbb{T}_F$. So for each $a \in I^0$ the restriction of λ_e on $\{(\{a\} \times \{x_j = 0\}) : j = 1, 2, \ldots, n-1\}$ define a characteristic function (see definition 1.2.8) on the set of facets of $\{a\} \times \mathbb{R}^{n-1}_{\geqslant 0}$. From the constructive definition of quasitoric manifold given in [DJ91] it is clear that the quotient space $\{a\} \times (\mathbb{T}^{n-1} \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e$ is diffeomorphic to $\{a\} \times \mathbb{R}^{2(n-1)}$. Hence the quotient space

$$(\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e \;=\; I^0 \times (\mathbb{T}^{n-1} \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e \;\cong\; I^0 \times \mathbb{R}^{2(n-1)}.$$

Since the quotient maps

$$\pi : (\mathbb{T}^{n-1} \times U_{e'}) \to (\mathbb{T}^{n-1} \times U_{e'})/\sim_b$$

and

$$\pi_e : (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}) \to (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e$$

are continuous surjection and $\mathfrak{g}$ is a diffeomorphism, the following commutative diagram ensure that the lower horizontal map $\mathfrak{g}_e$ is a homeomorphism.

$$\begin{array}{ccccc} (\mathbb{T}^{n-1} \times U_{e'}) & \xrightarrow{id \times \mathfrak{g}} & (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}) & & \\ \pi\big\downarrow & & \pi_e\big\downarrow & & \\ (\mathbb{T}^{n-1} \times U_{e'})/\sim_b & \xrightarrow{\mathfrak{g}_e} & (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e & \xrightarrow{\cong} & I^0 \times \mathbb{R}^{2(n-1)} \end{array} \tag{5.3.4}$$

Let v_1' and v_2' be the vertices of the edge e' of Q_P. Suppose $H_1 \cap e' = \{v_1'\}$ and $H_2 \cap e' = \{v_2'\}$, where H_1 and H_2 are affine hyperplanes as considered in Section 5.2 corresponding to the vertices v_1 and v_2 of e respectively. Let $U_{v_1'}$ and $U_{v_2'}$ be the open subset of Q_P obtained by deleting all facets of Q_P not containing v_1' and v_2' respectively. Hence there exist diffeomorphism $\mathfrak{g}^1 : U_{v_1'} \to [0,1) \times \mathbb{R}^{n-1}_{\geqslant 0}$ and $\mathfrak{g}^2 : U_{v_2'} \to [0,1) \times \mathbb{R}^{n-1}_{\geqslant 0}$ satisfying the same property as the map $\mathfrak{g}$. We get the following commutative diagram and homeomorphisms $\mathfrak{g}_e^j$ for $j = 1, 2$.

$$\begin{array}{ccccc} (\mathbb{T}^{n-1} \times U_{v_j'}) & \xrightarrow{id \times \mathfrak{g}^j} & (\mathbb{T}^{n-1} \times [0,1) \times \mathbb{R}^{n-1}_{\geqslant 0}) & & \\ \pi\big\downarrow & & \pi_e\big\downarrow & & \\ (\mathbb{T}^{n-1} \times U_{v_j'})/\sim_b & \xrightarrow{\mathfrak{g}_e^j} & (\mathbb{T}^{n-1} \times [0,1) \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e & \xrightarrow{\cong} & [0,1) \times \mathbb{R}^{2(n-1)} \end{array} \tag{5.3.5}$$

Hence each point of $(\mathbb{T}^{n-1} \times Q_P)/\sim_b$ has a neighborhood homeomorphic to an open subset of $[0,1) \times \mathbb{R}^{2(n-1)}$. So $W(Q_P, \lambda)$ is a manifold with boundary. From the above discussion the interior of $W(Q_P, \lambda)$ is

$$\cup_{e'} (\mathbb{T}^{n-1} \times U_{e'})/\sim_b \;=\; W(Q_P, \lambda) \smallsetminus \{(\mathbb{T}^{n-1} \times \sqcup_{i=1}^{k} P_{H_i})/\sim_b\}$$

and the boundary is $\sqcup_{i=1}^{k}\{(\mathbb{T}^{n-1} \times P_{H_i})/\sim_b\}$. Let $F(H)_{i_j}$ be a facet of P_{H_i}. So there exists a unique facet F_j of P such that $F(H)_{i_j} = F_j \cap Q_P \cap H_i$. The restriction of the function λ on the set of facets of P_{H_i} (namely $\lambda(F(H)_{i_j}) = \lambda_j$) give a characteristic function of a quasitoric manifold over P_{H_i}. Hence restricting the equivalence relation $\sim_b$ on $(\mathbb{T}^{n-1} \times P_{H_i})$ we get that the quotient space $W_i = (\mathbb{T}^{n-1} \times P_{H_i})/\sim_b$ is a quasitoric manifold over P_{H_i}. The boundary $\partial W(Q_P, \lambda)$ is the disjoint union $\sqcup_{i=1}^{k} W_i$, where W_i is a quasitoric manifold. So $W(Q_P, \lambda)$ is a manifold with quasitoric boundary. In Section 5.6 we have shown that these manifolds with quasitoric boundary are orientable.

Example 5.3.5. *An isotropy function of the standard cube I^3 is described in the following Figure 5.1. Here simple convex polytopes $P_{H_1}, \ldots, P_{H_8}$ are triangles. The restriction of the isotropy function on P_{H_i} gives that the space $(\mathbb{T}^2 \times P_{H_i})/\sim_b$ is the complex projective space either $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$. Since antipodal map in $\mathbb{R}^3$ is an orientation reversing map we can show that the disjoint union $\sqcup_{i=1}^{4} \mathbb{CP}^2 \sqcup_{i=1}^{4} \overline{\mathbb{CP}}^2$ is the boundary of $(\mathbb{T}^2 \times Q_{I^3})/\sim_b$.*

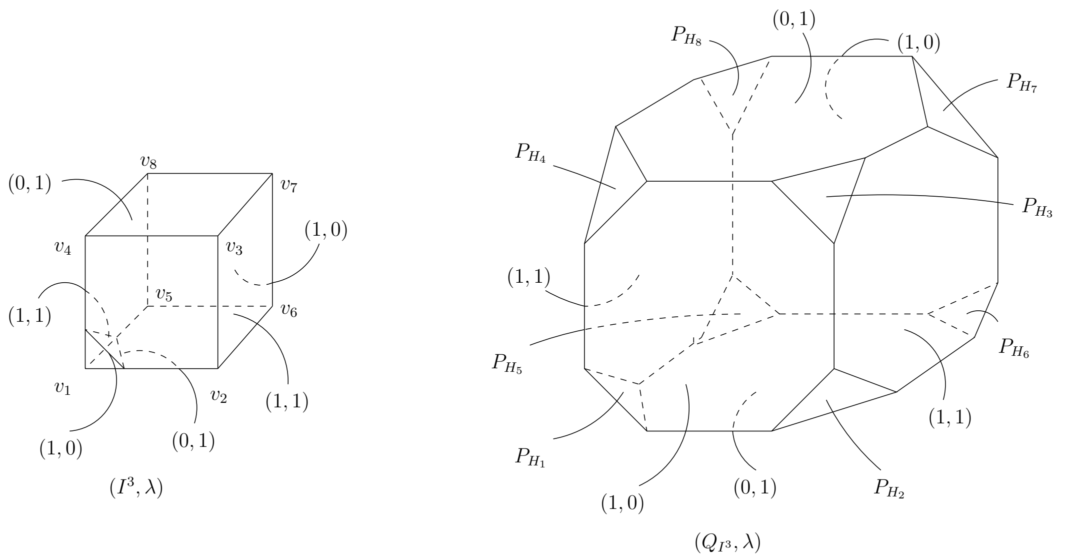


Figure 5.1: An isotropy function λ of the edge-simple polytope I^3

Example 5.3.6. *In the following Figure 5.2 we define an isotropy function of the edge-simple polytope P_0. Here simple polytopes $P_{H_1}, P_{H_2}, P_{H_3}, P_{H_4}$ are triangles and the simple polytope P_{H_5} is a rectangle. The restriction of the isotropy function on P_{H_i} gives that the space $(\mathbb{T}^2 \times P_{H_i})/\sim_b$ is either $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$ for each $i \in \{1, 2, 3, 4\}$ and $(\mathbb{T}^2 \times P_{H_5})/\sim_b$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$. Hence the space $\sqcup_{i=1}^{2} \sqcup \mathbb{CP}^2 \sqcup \overline{\mathbb{CP}}^2 \sqcup (\mathbb{CP}^1 \times \mathbb{CP}^1)$ is the boundary of $(\mathbb{T}^2 \times Q_{P_0})/\sim_b$, see Section 5.7.*

5.4 Manifolds with small cover boundary

Definition 5.4.1. *A manifold with small cover boundary is a manifold with boundary where the boundary is a disjoint union of some small covers.*

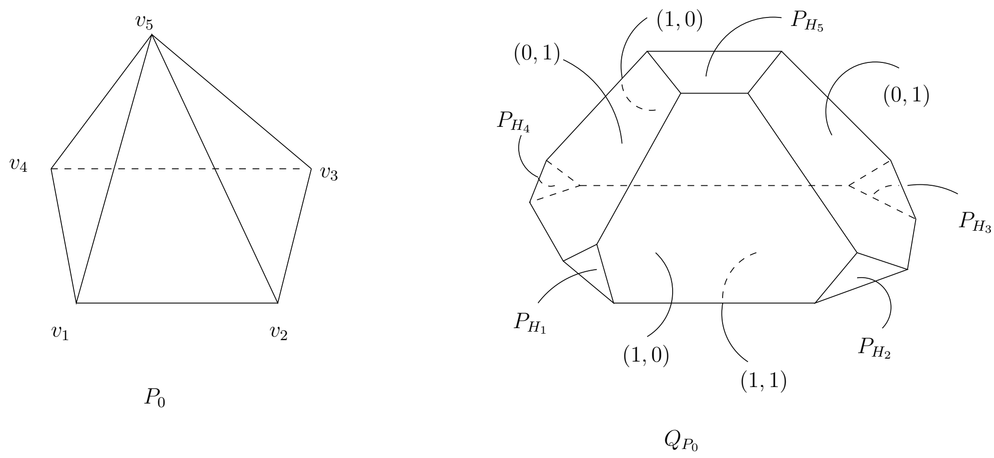


Figure 5.2: An isotropy function λ of the edge-simple polytope P_0

Definition 5.4.2. *The function $\lambda^s \colon \mathcal{F}(P) \to \mathbb{F}_2^{n-1}$ is called a $\mathbb{F}_2$-isotropy function of the edge-simple polytope P if the set of vectors $\{\lambda^s(F_{i_1}), \ldots, \lambda^s(F_{i_{n-1}})\}$ form a basis of $\mathbb{F}_2^{n-1}$ whenever the intersection of the facets $\{F_{i_1}, \ldots, F_{i_{n-1}}\}$ is an edge of P.*

The vectors $\lambda_i^s := \lambda^s(F_i)$ are called $\mathbb{F}_2$-isotropy vectors and the pair (P, λ^s) is called $\mathbb{F}_2$-isotropy pair.

We can construct a manifold with small cover boundary from the pair (P, λ^s). Assign each face F to the subgroup G_F of $\mathbb{Z}_2^{n-1}$ determined by the vectors $\lambda_{i_1}^s, \ldots, \lambda_{i_l}^s$ where F is the intersection of the facets $F_{i_1}, \ldots, F_{i_l}$. Let $\sim_s$ be an equivalence relation on $(\mathbb{Z}_2^{n-1} \times P)$ defined by the following.

$$(t, p) \sim_s (u, q) \text{ if and only if } p = q \text{ and } t - u \in G_F \tag{5.4.1}$$

where $F \subseteq P$ is the unique face containing p in its relative interior. Consider the restriction of $\sim_s$ on $(\mathbb{Z}_2^{n-1} \times Q_P)$. The quotient space $(\mathbb{Z}_2^{n-1} \times Q_P)/\sim_s \;\subset (\mathbb{Z}_2^{n-1} \times P)/\sim_s$, denoted by $S(Q_P, \lambda^s)$, is a manifold with boundary. This can be shown by the same arguments given in the Section 5.3. The boundary of this manifold is $\{(\mathbb{Z}_2^{n-1} \times \sqcup_{i=1}^k P_{H_i})/\sim_s\} = \sqcup_{i=1}^k \{(\mathbb{Z}_2^{n-1} \times P_{H_i})/\sim_s\}$. Clearly the restriction of the $\mathbb{Z}_2$-isotropy function λ^s on the set of facets of P_{H_i} gives the $\mathbb{Z}_2$-characteristic function of a small cover over P_{H_i}. So $(\mathbb{Z}_2^{n-1} \times P_{H_i})/\sim_s$ is a small cover for each $i = 0, \ldots, k$. Hence $S(Q_P, \lambda^s)$ is a manifold with small cover boundary.

5.5 Some observations

The set of facets of the simple convex polytope Q_P are

$$\mathcal{F}(Q_P) = \{P_{H_j} \colon j = 1, 2, \ldots, k\} \cup \{F_i' \colon i = 1, 2, \ldots, m\},$$

where $F'_i = F_i \cap Q_P$ for a unique facets F_i of P. Define the function $\xi \colon \mathcal{F}(Q_P) \to \mathbb{Z}^n$ as follows.

$$\xi(F) = \begin{cases} (0, \ldots, 0, 1) \in \mathbb{Z}^n & \text{if } F = P_{H_j} \text{ and } j \in \{1, \ldots, k\} \\ \lambda_i \in \mathbb{Z}^{n-1} \times \{0\} \subset \mathbb{Z}^n & \text{if } F = F_i \text{ and } i \in \{1, 2, \ldots, m\}. \end{cases} \tag{5.5.1}$$

So the function ξ satisfies the condition for the characteristic function (see definition 1.2.8) of a quasitoric manifold over the n-dimensional simple convex polytope Q_P. Hence from the characteristic model (Q_P, ξ) we can construct the quasitoric manifold $M(Q_P, \xi)$ over Q_P, see [DJ91]. There is a natural $\mathbb{T}^n$ action on $M(Q_P, \xi)$. Let $\mathbb{T}_H$ be the circle subgroup of $\mathbb{T}^n$ determined by the submodule $\{0\} \times \{0\} \times \ldots \times \{0\} \times \mathbb{Z}$ of $\mathbb{Z}^n$. Hence $W(Q_P, \lambda)$ is the orbit space of the circle $\mathbb{T}_H$ action on $M(Q_P, \xi)$. The orbit map $\mathfrak{g}_H \colon M(Q_P, \xi) \to W(Q_P, \lambda)$ is not a fiber bundle map.

Remark 5.5.1. *The manifold $S(Q_p, \lambda_s)$ with small cover boundary constructed in Section 5.4 is the orbit space of $\mathbb{Z}_2$ action on a small cover.*

5.6 Orientability of $W(Q_P, \lambda)$

Suppose $W = W(Q_P, \lambda)$. The boundary ∂W has a collar neighborhood in W. Hence by the Proposition 2.22 of [Hat02] we get $H_i(W, \partial W) = \widetilde{H}_i(W/\partial W)$ for all i. We show the space $W/\partial W$ has a CW-structure. Actually we show that corresponding to each edge of P there exist an odd-dimensional cell of $W/\partial W$. Realize Q_P as a simple polytope in $\mathbb{R}^n$ and choose a linear functional $\phi : \mathbb{R}^n \to \mathbb{R}$ which distinguishes the vertices of Q_P, as in the proof of Theorem 3.1 in [DJ91]. The vertices are linearly ordered according to ascending value of ϕ. We make the 1-skeleton of Q_P into a directed graph by orienting each edge such that ϕ increases along edges. For each vertex v of Q_P define its index, $ind(v)$, as the number of incident edges that point towards v. Suppose $\mathcal{V}(Q_P)$ is the set of vertices and $\mathcal{E}(Q_P)$ is the set of edges of Q_P. For each $j \in \{1, 2, \ldots, n\}$, let

$$I_j = \{(v, e_v) \in \mathcal{V}(Q_P) \times \mathcal{E}(Q_P) : ind(v) = j \text{ and } e_v \text{ is the incident edge that points}$$
$$\text{towards } v \text{ such that } e_v = e \cap Q_P \text{ for an edge } e \text{ of } P\}.$$

Suppose $(v, e_v) \in I_j$. Let F_{e_v} denote the smallest face of Q_P which contains the inward pointing edges incident to v. Then F_{e_v} is a unique face not contained in any P_{H_i}. Let U_{e_v} be the open subset of F_{e_v} obtain by deleting all faces of F_{e_v} not containing the edge e_v. The restriction of the equivalence relation $\sim_b$ on $(\mathbb{T}^{n-1} \times U_{e_v})$ gives that the quotient space $(\mathbb{T}^{n-1} \times U_{e_v})/\sim_b$ is homeomorphic to the open disk B^{2j-1}. Hence the quotient space $(W/\partial W)$ has a CW-complex structure with odd dimensional cells and one zero dimensional cell only. The number of $(2j-1)$-dimensional cell is $|I_j|$, the cardinality of I_j for $j = 1, 2, \ldots, n$. So we get the following theorem.

Theorem 5.6.1. $H_i(W, \partial W) = \begin{cases} \bigoplus_{|I_j|} \mathbb{Z} & \textit{if } i = 2j-1 \textit{ and } j \in \{1, \ldots, n\} \\ \mathbb{Z} & \textit{if } i = 0 \\ 0 & \textit{otherwise} \end{cases}$

When $j = n$ the cardinality of I_j is one. So $H_{2n-1}(W, \partial W) = \mathbb{Z}$. Hence W is an oriented manifold with boundary.

Example 5.6.2. *We adhere the notations of example 5.3.6. Here* $I_3 = \{(v_{14}, e_{v_{14}})\}$*,* $I_2 = \{(v_8, e_{v_8}), (v_{13}, e_{v_{13}}), (v_{15}, e_{v_{15}})\}$ *and* $I_1 = \{(v_3, e_{v_3}), (v_6, e_{v_6}), (v_9, e_{v_9})\}$*, see Figure 5.3. The face* $F_{e_{v_{13}}}$ *corresponding to the point* $(v_{13}, e_{v_{13}})$ *is* $v_0v_3v_5v_{13}v_{12}v_1$*. Thus we can give a CW-structure of* $W(Q_{P_0}, \lambda)/\partial W(Q_{P_0}, \lambda)$ *with one* 0*-cell, two* 1*-cells, three* 3*-cells and one* 5*-cell.*

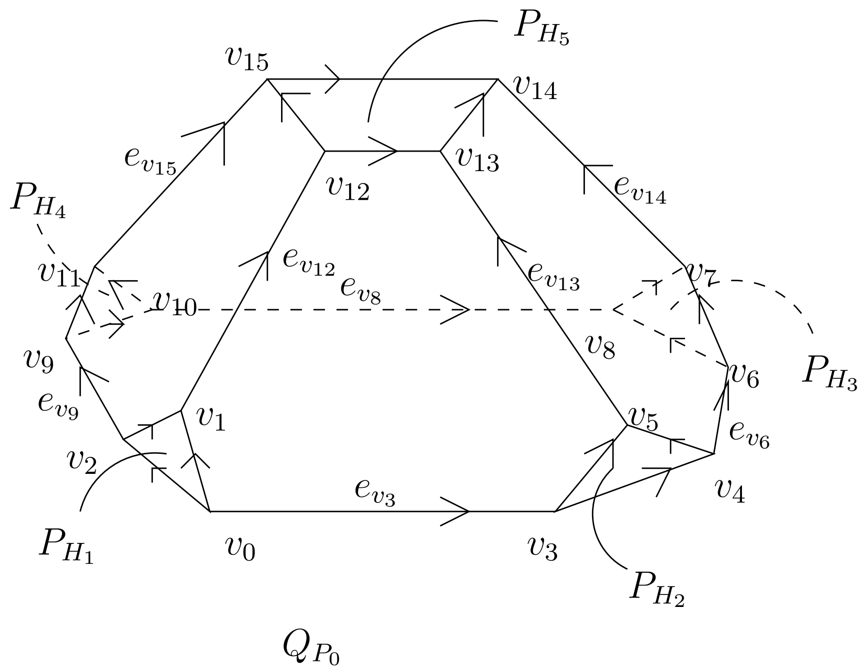


Figure 5.3: The index function of Q_{P_0}.

In [DJ91] the authors showed that the odd dimensional homology of quasitoric manifolds are zero. So $H_{2i-1}(\partial W) = 0$ for all i. Hence we get the following exact sequences for the collared pair $(W, \partial W)$.

$$\begin{array}{ccccccc}
0 \to H_{2n-1}(W) & \xrightarrow{j_*} & H_{2n-1}(W, \partial W) & \xrightarrow{\partial} & H_{2n-2}(\partial W) & \xrightarrow{i_*} & H_{2n-2}(W) \to 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 \to H_3(W) & \xrightarrow{j_*} & H_3(W, \partial W) & \xrightarrow{\partial} & H_2(\partial W) & \xrightarrow{i_*} & H_2(W) \to 0 \\
0 \to H_1(W) & \xrightarrow{j_*} & H_1(W, \partial W) & \xrightarrow{\partial} & H_0(\partial W) & \xrightarrow{i_*} & H_0(W) \twoheadrightarrow \mathbb{Z}
\end{array} \tag{5.6.1}$$

Where $\mathbb{Z} \cong H_0(W, \partial W)$. Let $(h_{i_0}, \ldots, h_{i_{n-1}})$ be the *h-vector* of P_{H_i}, for $i = 1, 2, \ldots, k$. The definition of *h-vector* of simple polytope is given in Section 1.4 of Chapter 1. Hence the Euler characteristic of the manifold W with quasitoric boundary is $\Sigma_{i=1}^{k} \Sigma_{j=0}^{n-1} h_{i_j} - \Sigma_{j=1}^{n-1} |I_j|$.

Fix the standard orientation on $\mathbb{T}^{n-1}$. Let $I_n = \{(v', e_{v'})\}$. Then the $(2n-1)$-dimensional cell $(\mathbb{T}^{n-1} \times U_{e_{v'}})/\sim \subset W$ represents a fundamental class of $W/\partial W$ with coefficient in $\mathbb{Z}$. Thus an orientation of $U_{e_{v'}}$ (hence of Q_P) determines an orientation of W. Note that an orientation of Q_P is induced by orienting the ambient space $\mathbb{R}^n$.

So the boundary orientation on P_{H_i} induced from the orientation of Q_P gives the orientation on the quasitoric manifold $W_i \subset \partial W$. In the next section we consider the orientation of Q's and Q_P's induced from the standard orientation of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively.

5.7 Torus cobordism of quasitoric manifolds

Let $\mathfrak{C}$ be the following category: the objects are all quasitoric manifolds and morphisms are torus equivariant maps between quasitoric manifolds. We are considering torus cobordism in this category only. Quasitoric manifolds are orientable manifolds, see the Section 1.5 of Chapter 1.

Definition 5.7.1. *Two $2n$-dimensional quasitoric manifolds M_1 and M_2 are said to be $\mathbb{T}^n$-cobordant if there exist an oriented $\mathbb{T}^n$ manifold W with boundary ∂W such that ∂W is $\mathbb{T}^n$ equivariantly homeomorphic to $M_1 \sqcup (-M_2)$ under an orientation preserving homeomorphism. Here $-M_2$ represents the reverse orientation of M_2.*

We denote the $\mathbb{T}^n$-cobordism class of quasitoric $2n$-manifold M by $[M]$.

Definition 5.7.2. *The n-th torus cobordism group is the group of all cobordism classes of $2n$-dimensional quasitoric manifolds with the operation of disjoint union. We denote this group by CG_n.*

Let $M \to Q$ be a 4-dimensional quasitoric manifold over the square Q with the characteristic function $\xi : \mathcal{F}(Q) \to \mathbb{Z}^2$. Consider that the orientation on M comes from the standard orientation on $\mathbb{T}^2$ and $Q \subset \mathbb{R}^2$. We construct an oriented $\mathbb{T}^2$ manifold W

with boundary ∂W, where ∂W is equivariantly homeomorphic to $-M \sqcup \sqcup_{k_1} \mathbb{CP}^2 \sqcup \sqcup_{k_2} \overline{\mathbb{CP}}^2$ for some integer k_1, k_2. To show this we construct a 3-dimensional edge-simple polytope $P_{\mathcal{E}}$ such that $P_{\mathcal{E}}$ has exactly one vertex O which is the intersection of 4 facets with $P_{\mathcal{E}} \cap H_O = Q$ and other vertices of $P_{\mathcal{E}}$ are intersection of 3 facets. We define an isotropy function λ, extending the characteristic function ξ of M, from the set of facets of $P_{\mathcal{E}}$ to $\mathbb{Z}^2$. Then $W(Q_{P_{\mathcal{E}}}, \lambda)$ is the required oriented $\mathbb{T}^2$ manifold with quasitoric boundary. We have done an explicit calculation in the following.

Let $Q = ABCD$ be a rectangle (see Figure 5.4) belongs to $\{(x, y, z) \in \mathbb{R}^3_{\geq 0} : x + y + z = 1\}$. Let $\xi : \{AB, BC, CD, DA\} \to \mathbb{Z}^2$ be the characteristic function for a quasitoric manifold M over $ABCD$ such that the characteristic vectors are

$$\xi(AB) = \xi_1,\ \xi(BC) = \xi_2,\ \xi(CD) = \xi_3 \text{ and } \xi(DA) = \xi_4.$$

We may assume that $\xi_1 = (0, 1)$ and $\xi_2 = (1, 0)$. From the classification results given in Section 1.2, it is enough to consider the following cases only.

$$\xi_3 = (0, 1) \text{ and } \xi_4 = (1, 0) \tag{5.7.1}$$

$$\xi_3 = (0, 1) \text{ and } \xi_4 = (1, k),\ k = 1 \text{ or } -1 \tag{5.7.2}$$

$$\xi_3 = (0, 1) \text{ and } \xi_4 = (1, k),\ k \in \mathbb{Z} - \{-1, 0, 1\} \tag{5.7.3}$$

$$\xi_3 = (-1, 1) \text{ and } \xi_4 = (1, -2) \tag{5.7.4}$$

For the case 5.7.1: In this case the edge-simple polytope $\widetilde{P}_1$, given in Figure 5.4, is the required edge-simple polytope. The isotropy vectors of $\widetilde{P}_1$ are given by

$$\lambda(OGH) = \xi_1,\ \lambda(OHI) = \xi_2,\ \lambda(OIJ) = \xi_3,\ \lambda(OGJ) = \xi_4 \text{ and } \lambda(GHIJ) = \xi_1 + \xi_2.$$

So we get an oriented $\mathbb{T}^2$ manifold $W(Q_{\widetilde{P}_1}, \lambda)$ with quasitoric boundary where the boundary is the quasitoric manifold $-M \sqcup \sqcup_{k_1} \mathbb{CP}^2 \sqcup \sqcup_{k_2} \overline{\mathbb{CP}}^2$ for some integers k_1, k_2. Note that orientation on $\widetilde{P}_1 \subset \mathbb{R}^3_{\geq 0}$ comes from the standard orientation of $\mathbb{R}^3$. Let A' and B' be the midpoints of GJ and HI respectively. Let $\mathcal{H}$ be the plane passing through O, A' and B' in $\mathbb{R}^3$. Since a reflection in $\mathbb{R}^3$ is an orientation reversing homeomorphism, it is easy to observe that the reflection on $\mathcal{H}$ induces the following orientation reversing equivariant homeomorphisms.

$$(\mathbb{T}^2 \times \widetilde{P}_{1_I})/\sim \to (\mathbb{T}^2 \times \widetilde{P}_{1_H})/\sim \text{ and } (\mathbb{T}^2 \times \widetilde{P}_{1_J})/\sim \to (\mathbb{T}^2 \times \widetilde{P}_{1_G})/\sim .$$

So $k_1 = k_2$. Since $[\overline{\mathbb{CP}}^2] = -[\mathbb{CP}^2]$, $[M] = 0[\mathbb{CP}^2]$.

For the case 5.7.2: In this case $|det(\xi_2, \xi_4)| = 1$. Let O be the origin of $\mathbb{R}^3$. Let C_Q be the open cone on rectangle $ABCD$ at the origin O. Let G, H, I, J be points on extended OA, OB, OC, OD respectively. Let E and F be two points in the interior of the open cones on AB and CD at O respectively such that $|OG| < |OE|$, $|OH| < |OE|$ and $|OI| < |OF|$, $|OJ| < |OF|$. Then the convex polytope $P_1 \subset C_Q$ on the set of vertices $\{O, G, E, H, I, F, J\}$ is an edge-simple polytope (see Figure 5.4) of dimension 3. Define a function, denote by λ, on the set of facets of P_1 by

$$\begin{aligned}&\lambda(OGEH) = \xi_1,\ \lambda(OHI) = \xi_2,\ \lambda(OJFI) = \xi_3,\ \lambda(OJG) = \xi_4,\\ &\lambda(HIFE) = \xi_4 \text{ and } \lambda(GJFE) = \xi_2.\end{aligned} \tag{5.7.5}$$

Hence λ is an isotropy function on the edge-simple polytope P_1. The boundary of the oriented $\mathbb{T}^2$ manifold $W(Q_{P_1}, \lambda)$ is the quasitoric manifold $M \sqcup \sqcup_{k_1} \mathbb{CP}^2 \sqcup \sqcup_{k_2} \overline{\mathbb{CP}}^2$ for some integers k_1, k_2. Similarly to the previous case we can show that a suitable reflection induces the following orientation reversing equivariant homeomorphisms.

$$(\mathbb{T}^2 \times P_{1_H})/\sim \to (\mathbb{T}^2 \times P_{1_I})/\sim,\ (\mathbb{T}^2 \times P_{1_E})/\sim \to (\mathbb{T}^2 \times P_{1_J})/\sim$$

and

$$(\mathbb{T}^2 \times P_{1_G})/\sim \to (\mathbb{T}^2 \times P_{1_J})/\sim .$$

So $k_1 = k_2$. Hence $[M] = 0[\mathbb{CP}^2]$.

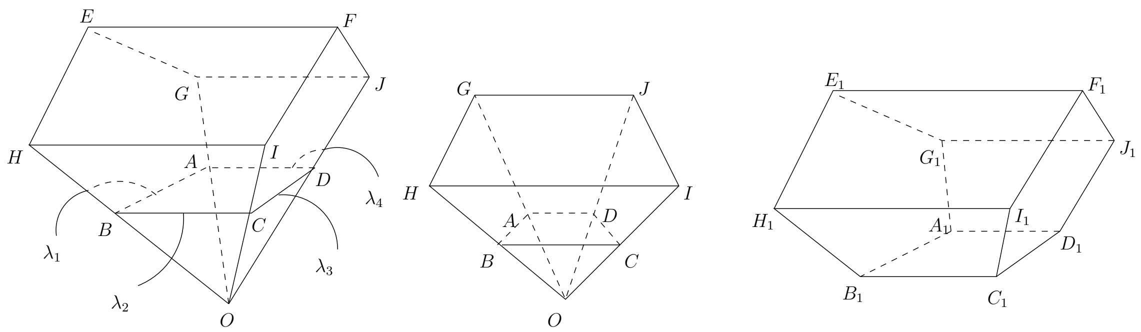
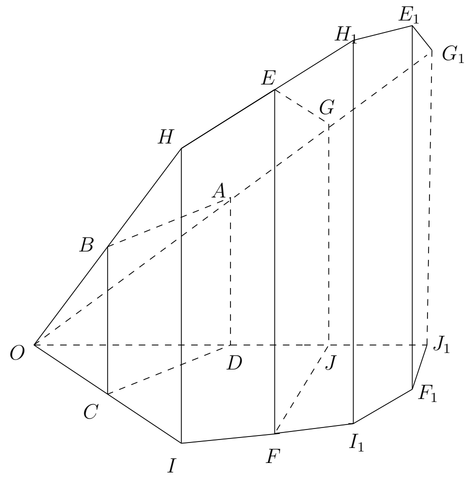


Figure 5.4: The edge-simple polytope $P_1, \widetilde{P}_1$ and the convex polytope P_1' respectively.

For the case 5.7.3: Suppose $det(\xi_2, \xi_4) = k > 1$. Define a function $\lambda^{(1)}$ on the set of facets of P_1 except $GEFJ$ by

$$\begin{aligned}&\lambda^{(1)}(OGEH) = \xi_1,\ \lambda^{(1)}(OHI) = \xi_2,\ \lambda^{(1)}(OIFJ) = \xi_3,\ \lambda^{(1)}(OGJ) = \xi_4,\\ &\text{and } \lambda^{(1)}(EHIF) = \xi_2 + \xi_1.\end{aligned} \tag{5.7.6}$$



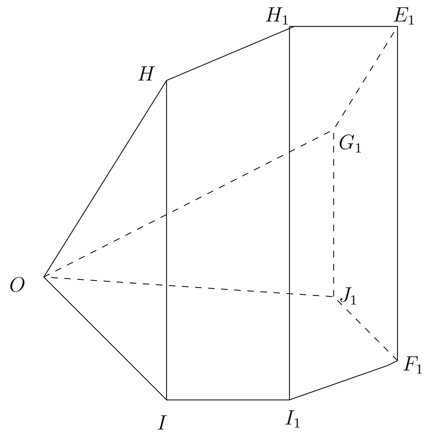


Figure 5.5: The edge-simple polytope P_2.

So the function $\lambda^{(1)}$ satisfies the condition of an isotropy function of the edge-simple polytope P_1 along each edge except the edges of the rectangle $GEFJ$. The restriction of the function $\lambda^{(1)}$ on the edges GE, EF, FJ, GJ of the rectangle $GEFJ$ gives the following equations,

$$\begin{aligned}&|det[\lambda^{(1)}(GE), \lambda^{(1)}(EF)]| = 1, \ |det[\lambda^{(1)}(EF), \lambda^{(1)}(FJ)]| = 1,\\ &|det[\lambda^{(1)}(FJ), \lambda^{(1)}(GJ)]| = 1, \ |det[\lambda^{(1)}(GJ), \lambda^{(1)}(GE)]| = 1\\ &\text{and } det[\lambda^{(1)}(EF), \lambda^{(1)}(GJ)] = k-1 < k.\end{aligned} \tag{5.7.7}$$

Let P_1' be a 3-dimensional convex polytope as in the Figure 5.4. Identifying the facet $GEFJ$ of P_1 and $A_1B_1C_1D_1$ of P_1' through a suitable diffeomorphism of manifold with corners such that the vertices G, E, F, J maps to the vertices A_1, B_1, C_1, D_1 respectively, we can form a new convex polytope P_2, see Figure 5.5. After the identification following holds.

1. The facet of P_1 containing GE and the facet of P_1' containing A_1B_1 make the facet $OHH_1E_1G_1$ of P_2.

2. The facet of P_1 containing EF and the facet of P_1' containing B_1C_1 make the facet HH_1I_1I of P_2.

3. The facet of P_1 containing FJ and the facet of P_1' containing C_1D_1 make the facet $OII_1F_1J_1$ of P_2.

4. The facet of P_1 containing JG and the facet of P_1' containing D_1A_1 make the facet OJ_1G_1 of P_2.

The polytope P_2 is an edge-simple polytope. We define a function $\lambda^{(2)}$ on the set of facets of P_2 except $G_1E_1F_1J_1$ by

$$\begin{aligned}
&\lambda^{(2)}(OHH_1E_1G_1) = \xi_1,\ \lambda^{(2)}(OIH) = \xi_2,\ \lambda^{(2)}(OII_1F_1J_1) = \xi_3, \\
&\lambda^{(2)}(OJ_1G_1) = \xi_4,\ \lambda^{(2)}(HH_1I_1I) = \xi_2 + \xi_1 \\
&\text{and } \lambda^{(2)}(H_1I_1F_1E_1) = \xi_2 + 2\xi_1.
\end{aligned} \tag{5.7.8}$$

So the function $\lambda^{(2)}$ satisfies the condition of an isotropy function of the edge-simple polytope P_2 along each edge except the edges of the rectangle $G_1E_1F_1J_1$. The restriction of the function $\lambda^{(2)}$ on the edges namely $G_1E_1, E_1F_1, F_1J_1, G_1J_1$ of the rectangle $G_1E_1F_1J_1$ gives the following equations,

$$\begin{aligned}
&|det[\lambda^2(G_1E_1), \lambda^2(E_1F_1)]| = 1,\ |det[\lambda^2(E_1F_1), \lambda^2(F_1J_1)]| = 1, \\
&|det[\lambda^2(F_1J_1), \lambda^2(G_1J_1)]| = 1,\ |det[\lambda^2(G_1J_1), \lambda^2(G_1E_1)]| = 1 \\
&\text{and } det[\lambda^2(E_1F_1), \lambda^2(G_1J_1)] = k-2 < k-1.
\end{aligned} \tag{5.7.9}$$

Proceeding in this way, at k-th step we construct an edge-simple polytope P_k with the function $\lambda^{(k)}$, extending the function $\lambda^{(k-1)}$, on the set of facets of P_k such that

$$\begin{aligned}
&\lambda^{(k)}(H_{k-2}H_{k-1}I_{k-1}I_{k-2}) = \xi_2 + (k-1)\xi_1 = \lambda^{(k-1)}(H_{k-2}I_{k-2}F_{k-2}E_{k-2}), \\
&\lambda^{(k)}(OG_{k-1}J_{k-1}) = \xi_4 = \lambda^{(k-1)}(OG_{k-2}J_{k-2}), \\
&\lambda^{(k)}(H_{k-1}I_{k-1}F_{k-1}E_{k-1}) = \xi_4 \text{ and } \lambda^{(k)}(G_{k-1}E_{k-1}F_{k-1}J_{k-1}) = \xi_2 + (k-1)\xi_1.
\end{aligned} \tag{5.7.10}$$

Observe that the function $\lambda := \lambda^{(k)}$ is an isotropy function of the edge-simple polytope P_k. So we get an oriented $\mathbb{T}^2$-manifold with boundary $W(Q_{P_k}, \lambda)$ where the boundary is the quasitoric manifold $M \sqcup \sqcup_{k_1}\mathbb{CP}^2 \sqcup \sqcup_{k_2}\overline{\mathbb{CP}}^2$ for some integers k_1, k_2.Similarly to the previous cases we can construct the following orientation reversing equivariant homeomorphisms.

$$(\mathbb{T}^2 \times P_{k_H})/\sim \to (\mathbb{T}^2 \times P_{k_I})/\sim,\ (\mathbb{T}^2 \times P_{k_{G_{k-1}}})/\sim \to (\mathbb{T}^2 \times P_{k_{J_{k-1}}})/\sim,$$

$$(\mathbb{T}^2 \times P_{k_{E_{k-1}}})/\sim \to (\mathbb{T}^2 \times P_{k_{F_{k-1}}})/\sim \text{ and } (\mathbb{T}^2 \times P_{k_{H_i}})/\sim \to (\mathbb{T}^2 \times P_{k_{I_i}})/\sim$$

for $i = 1, \ldots, k-1$. So $k_1 = k_2$. Hence $[M] = 0[\mathbb{CP}^2]$. If $k < -1$, similarly we can show $[M] = 0[\mathbb{CP}^2]$.

From the calculations for the cases 5.7.1, 5.7.2 and 5.7.3 we get the following lemma.

Lemma 5.7.3. *The $\mathbb{T}^2$-cobordism class of a Hirzebruch surface is trivial. In particular, oriented cobordism class of a Hirzebruch surface is also trivial.*

For the case 5.7.4: In this case $|det[\xi_1, \xi_3]| = 1$. Following case 5.7.2, we can construct an edge simple polytope P'' and an isotropy function λ over this edge-simple polytope, see Figure 5.6. Hence we can construct an oriented $\mathbb{T}^2$ manifold with quasitoric boundary $W(Q_{P''}, \lambda)$ where the boundary is $-M \sqcup \sqcup_{k_1}\mathbb{CP}^2 \sqcup \sqcup_{k_2}\overline{\mathbb{CP}}^2$ for some integers

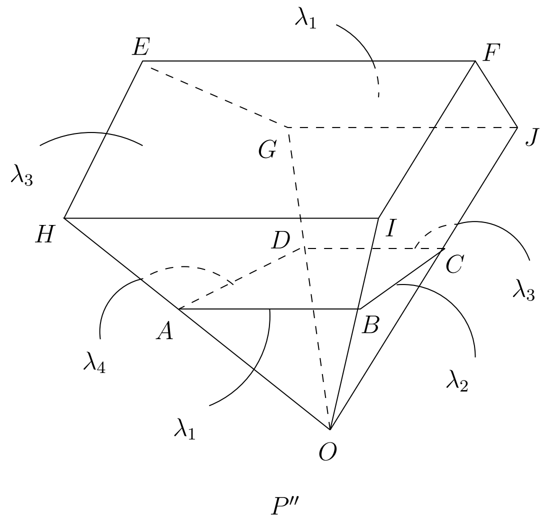


Figure 5.6: The edge-simple polytope P'' and an isotropy function λ associated to the case 5.7.4.

k_1, k_2. We may assume that 'the angles between the planes OHI and $HIFE$' and 'the angles between the planes $EFJG$ and $HIFE$' are equal. Clearly a suitable reflection induces the following orientation reversing equivariant homeomorphisms.

$$(\mathbb{T}^2 \times P''_H)/\sim \to (\mathbb{T}^2 \times P''_E)/\sim \text{ and } (\mathbb{T}^2 \times P''_I)/\sim \to (\mathbb{T}^2 \times P''_F)/\sim . \tag{5.7.11}$$

Let $\mathbb{CP}^2_J = (\mathbb{T}^2 \times P''_J)/\sim$ and $\mathbb{CP}^2_G = (\mathbb{T}^2 \times P''_G)/\sim$. Observe that the characteristic functions of the triangles P''_J and P''_G are differ by a non-trivial automorphism of $\mathbb{T}^2$ (or $\mathbb{Z}^2$). So $\mathbb{CP}^2_J$ and $\mathbb{CP}^2_G$ are complex projective space $\mathbb{CP}^2$ with two non-equivariant $\mathbb{T}^2$-actions. Hence $[M] = [\mathbb{CP}^2_J] + [\mathbb{CP}^2_G]$.

To compute the group CG_2 we use the induction on the number of facets of 2-dimensional simple convex polytope in $\mathbb{R}^2$. We rewrite the proof of well-known following lemma briefly.

Lemma 5.7.4. *The equivariant connected sum of two quasitoric manifolds is equivariant cobordant to the disjoint union of these two quasitoric manifolds.*

Proof. Let M_1 and M_2 be two quasitoric manifolds of dimension $2n$. Then $W_1 := [0,1] \times M_1$ and $W_2 := [0,1] \times M_2$ are oriented $\mathbb{T}^n$-manifolds with boundary such that

$$\partial W_1 = 0 \times (-M_1) \sqcup 1 \times M_1 \text{ and } \partial W_2 = 0 \times (-M_2) \sqcup 1 \times M_2.$$

Let $x_1 \in M_1$ and $x_2 \in M_2$ be two fixed points. Let $U_1 \subset W_1$ and $U_2 \subset W_2$ be two $\mathbb{T}^n$ invariant open neighborhoods of $1 \times x_1$ and $1 \times x_2$ respectively. Identifying $\partial U_1 \subset (W_1 - U_1)$ and $\partial U_2 \subset (W_2 - U_2)$ via an orientation reversing equivariant map we get the lemma. □

Now consider the case of a quasitoric manifold M over a convex 2-polytope P with m facets, where $m > 4$. By the classification result of 4-dimensional quasitoric manifold which is discussed in Section 1.2, M is one of the following equivariant connected sum.

$$M = N_1 \# \mathbb{CP}^2 \tag{5.7.12}$$

$$M = N_2 \# \overline{\mathbb{CP}}^2 \tag{5.7.13}$$

$$M = N_3 \# M^4_k \tag{5.7.14}$$

The quasitoric manifolds N_1, N_2 and N_3 are associated to the 2-polytopes Q_1, Q_2 and Q_3 respectively. The number of facets of Q_1, Q_2 and Q_3 are $m-1$, $m-1$ and $m-2$ respectively. The quasitoric manifold M^4_k is defined in Section 1.2 of Chapter 1. In previous calculations we have shown that $[M^4_k] = 0[\mathbb{CP}^2]$. So by the Lemma 5.7.4 we get either $[M] = [N_1] + [\mathbb{CP}^2]$ or $[M] = [N_2] - [\mathbb{CP}^2]$ or $[M] = [N_3]$. Thus using the induction on m, the number of facets of Q, we can prove the following.

Lemma 5.7.5. *Any 4-dimensional quasitoric manifold is equivariantly cobordant to the disjoint union $\sqcup_1^l \mathbb{CP}^2$ for some l, where the $\mathbb{T}^2$-action on different copies of $\mathbb{CP}^2$ may be distinct.*

We classify the equivariant cobordism classes of all $\mathbb{T}^2$-actions on $\mathbb{CP}^2$. Let Q be a triangle and $\{F_1, F_2, F_3\}$ be the edges (facets) of Q. Let $\xi : \{F_1, F_2, F_3\} \to \mathbb{Z}^2$ be a characteristic function such that $\xi(F_1) = (a_1, b_1)$ and $\xi(F_2) = (a_2, b_2)$. Because of the Corollary 1.2.17, we may assume that

$$det(\xi(F_1), \xi(F_2)) = |(a_1, b_1; a_2, b_2)| = 1$$

where $(a_1, b_1; a_2, b_2)$ is the 2×2 matrix in $SL(2, \mathbb{Z})$ with row vectors $\xi(F_1)$ and $\xi(F_2)$. We denote this matrix by ξ also. Then either $\xi(F_3) = (a_1 + a_2, b_1 + b_2)$, $\xi(F_3) = -(a_1 + a_2, b_1 + b_2)$, $\xi(F_3) = -(a_1 - a_2, b_1 - b_2)$ or $\xi(F_3) = (a_1 - a_2, b_1 - b_2)$. Let ξ' and ξ'' be two characteristic function defined respectively by,

$$\xi'(F_1) = (a_1, b_1), \xi'(F_2) = (a_2, b_2), \xi'(F_3) = (a_1 + a_2, b_1 + b_2)$$

and

$$\xi''(F_1) = (a_1, b_1), \xi''(F_2) = (a_2, b_2), \xi''(F_3) = (a_1 - a_2, b_1 - b_2).$$

Denote the quasitoric manifolds associated to the pairs (Q, ξ') and (Q, ξ'') by $\mathbb{CP}^2_{\xi'}$ and $\mathbb{CP}^2_{\xi''}$ respectively. The quasitoric manifolds associated to other posible characteristic function are equivariantly homeomorphic to either $\mathbb{CP}^2_{\xi'}$ or $\mathbb{CP}^2_{\xi''}$. Define an equivalence relation $\sim_{eq}$ on $SL(2,\mathbb{Z})$ by

$$(a_1, b_1; a_2, b_2) \sim_{eq} (-a_1, -b_1; -a_2, -b_2).$$

Denote the equivalence class of $\xi \in SL(2,\mathbb{Z})$ by $[\xi]_{eq}$. Using Corollary 1.2.17 we get the following classification.

Lemma 5.7.6. *A $\mathbb{T}^2$-actions on $\mathbb{CP}^2$ is equivariantly homeomorphic to either $\mathbb{CP}^2_{\xi'}$ or $\mathbb{CP}^2_{\xi''}$ for a unique $[\xi]_{eq} \in SL(2,\mathbb{Z})/\sim_{eq}$.*

Note that the natural $\mathbb{T}^2$-actions on $\mathbb{CP}^2_{\xi'}$ and $\mathbb{CP}^2_{\xi''}$ are same. Consider the linear map $L_\xi : \mathbb{Z}^2 \to \mathbb{Z}^2$, defined by $L_\xi(1,0) = (a_1, b_1), L_\xi(0,1) = (a_2, b_2)$. The map L_ξ induces orientation preserving homeomorphisms $\mathbb{CP}^2_s \to \mathbb{CP}^2_{\xi'}$ and $\overline{\mathbb{CP}}^2_s \to \mathbb{CP}^2_{\xi''}$. Thus $[\mathbb{CP}^2_{\xi'}] = -[\mathbb{CP}^2_{\xi''}]$. Observe that if $[\xi_1]_{eq} \neq [\xi_2]_{eq}$ then the corresponding characteristic functions are differ by δ_*, for some non trivial auto morphism $\delta : \mathbb{T}^2 \to \mathbb{T}^2$. So $[\mathbb{CP}^2_{\xi_1'}] \neq [\mathbb{CP}^2_{\xi_2'}]$. Hence we get the following.

Theorem 5.7.7. *The set $\{[\mathbb{CP}^2_{\xi'}] \ : \ [\xi]_{eq} \in SL(2,\mathbb{Z})/\sim_{eq}\}$ is a set of generators of the oriented torus cobordism group CG_2.*

Chapter 6

Oriented cobordism of $\mathbb{CP}^{2k+1}$

6.1 Introduction

In this chapter we have given a new construction of oriented manifold with the boundary $\mathbb{CP}^{2k+1}$ for each $k \geq 0$. The main tool is the theory of quasitoric manifolds. The strategy of our proof is to first construct some compact orientable manifolds with quasitoric boundary. Then identifying suitable boundary components using certain homeomorphisms we obtain oriented manifold with the boundary $\mathbb{CP}^{2k+1}$ for each $k \geq 0$, see Theorem 6.4.1.

6.2 Some manifolds with quasitoric boundary

Set $n = 2(k+1)$. Corresponding to each even $k \geq 0$ we construct a manifold with quasitoric boundary. Let $\{A_j : j = 0, \ldots, n\}$ be the standard basis of $\mathbb{R}^{n+1}$. Let

$$\triangle^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \; x_j \geq 0 \text{ and } \Sigma_0^n x_j = 1\}. \tag{6.2.1}$$

Then $\triangle^n$ is an n-dimensional simplex with vertices $\{A_j : j = 0, \ldots, n\}$ in $\mathbb{R}^{n+1}$. Define

$$\triangle_j^{n-1} = \{(x_0, x_1, \ldots, x_n) \in \triangle^n : x_j = 0\}. \tag{6.2.2}$$

So $\triangle_j^{n-1}$ is the facet of $\triangle^n$ not containing the vertex A_j. Let F be the largest face of $\triangle^n$ not containing the vertices $A_{j_1}, \ldots, A_{j_l}$. Then

$$F = \cap_{i=1}^{l} \triangle_{j_i}^{n-1} = \{(x_0, x_1, \ldots, x_n) \in \triangle^n : x_{j_i} = 0, \; i = 1, \ldots, l\}. \tag{6.2.3}$$

Define a function $\eta : \{\triangle_j^{n-1} : j = 0, \ldots, n\} \longrightarrow \mathbb{Z}^{n-1}$ as follows.

$$\eta(\triangle^{n-1}_{n-j}) = \begin{cases} (0,\dots,0,1,0,\dots,0) & \text{if } 0 \le j < \frac{n}{2}-1, \text{ here 1 is in the } (j+1)\text{-th place} \\ (1,\dots,1,1,0,\dots,0) & \text{if } j = \frac{n}{2}-1, \text{ here 1 occurs up to } \frac{n}{2}\text{-th place} \\ (0,\dots,0,1,0,\dots,0) & \text{if } \frac{n}{2} \le j < n, \text{ here 1 is in the } j\text{-th place} \\ (0,\dots,0,1,1,\dots,1) & \text{if } j = n, \text{ here 0 occurs up to } (\frac{n}{2}-1)\text{-th place} \end{cases} \tag{6.2.4}$$

Define

$$\eta_j := \eta(\triangle^{n-1}_{n-j}), \text{ for all } j = 0,1,\dots,n. \tag{6.2.5}$$

Example 6.2.1. *For $n = 4$, let $\triangle^4$ be the 4-simplex in $\mathbb{R}^5$ with vertices A_0, A_1, A_2, A_3, A_4 (see Figure 6.1). Define a function η from the set of facets of $\triangle^4$ to $\mathbb{Z}^3$ by,*

$$\eta(\triangle^3_{4-j}) = \begin{cases} (1,0,0) & \text{if } j = 0 \\ (1,1,0) & \text{if } j = 1 \\ (0,1,0) & \text{if } j = 2 \\ (0,0,1) & \text{if } j = 3 \\ (0,1,1) & \text{if } j = 4 \end{cases} \tag{6.2.6}$$

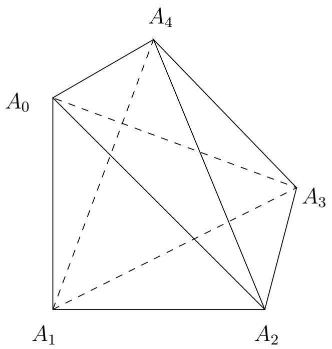


Figure 6.1: The 4-simplex $\triangle^4$

Suppose the faces F' and F'' of $\triangle^n$ are the intersection of facets $\{\triangle^{n-1}_n, \triangle^{n-1}_{n-1}, \dots, \triangle^{n-1}_{\frac{n}{2}}\}$ and $\{\triangle^{n-1}_{\frac{n}{2}}, \triangle^{n-1}_{\frac{n}{2}-1}, \dots, \triangle^{n-1}_0\}$ respectively. Then

$$F' = \{(x_0, x_1, \dots, x_n) \in \triangle^n : x_{\frac{n}{2}} = 0, \dots, x_n = 0\}, \tag{6.2.7}$$

$$F'' = \{(x_0, x_1, \dots, x_n) \in \triangle^n : x_0 = 0, \dots, x_{\frac{n}{2}} = 0\}. \tag{6.2.8}$$

Hence $\dim(F') = \dim(F'') = \frac{n}{2} - 1 \ge 1$. The set of vectors $\{\eta_0, \dots, \eta_{\frac{n}{2}}\}$ and $\{\eta_{\frac{n}{2}}, \dots, \eta_n\}$ are linearly dependent sets in $\mathbb{Z}^{n-1}$. But the submodules generated by the vectors $\{\eta_0, \dots, \widehat{\eta_j}, \dots, \eta_{\frac{n}{2}}\}$ and $\{\eta_{\frac{n}{2}}, \eta_{\frac{n}{2}+1}, \dots, \widehat{\eta_l}, \dots, \eta_n\}$ are $\frac{n}{2}$-dimensional direct summands of $\mathbb{Z}^{n-1}$ for each $j = 0, \dots, \frac{n}{2}$ and $l = \frac{n}{2}, \dots, n$ respectively. Here the symbol $\widehat{\ }$ represents the omission of the corresponding entry.

Suppose e is an edge of $\triangle^n$ not contained in $F' \cup F''$. Then $e = \cap_{j=1}^{n-1} \triangle^{n-1}_{n-l_j}$ for some $\{l_j : j = 1, \ldots, n-1\} \subset \{0, 1, \ldots, n\}$. Observe that $\{\eta_0, \ldots, \eta_{\frac{n}{2}}\} \nsubseteq \{\eta_{l_1}, \ldots, \eta_{l_{n-1}}\}$ and $\{\eta_{\frac{n}{2}}, \eta_{(\frac{n+2}{2})}, \ldots, \eta_n\} \nsubseteq \{\eta_{l_1}, \ldots, \eta_{l_{n-1}}\}$. Hence the set of vectors $\{\eta_{l_1}, \ldots, \eta_{l_{n-1}}\}$ form a basis of $\mathbb{Z}^{n-1}$.

Let r_1, r_2 be two positive real numbers such that $r_1 < r_2$ and $r_1 + 2r_2 < 1$. Consider the following affine hyperplanes in $\mathbb{R}^{n+1}$.

$$H_1 = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \ x_{\frac{n}{2}} + \cdots + x_n = r_2\}. \tag{6.2.9}$$

$$H_2 = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \ x_0 + \cdots + x_{\frac{n}{2}} = r_2\}. \tag{6.2.10}$$

$$H_3 = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \ x_{\frac{n}{2}} = 1 - r_1\}. \tag{6.2.11}$$

$$H = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \ x_0 + \cdots + x_n = 1\}. \tag{6.2.12}$$

Then $\triangle^n \subset H$ and the intersections $\triangle^n \cap H_1 \cap H_2$, $\triangle^n \cap H_1 \cap H_3$, $\triangle^n \cap H_3 \cap H_2$ are empty.

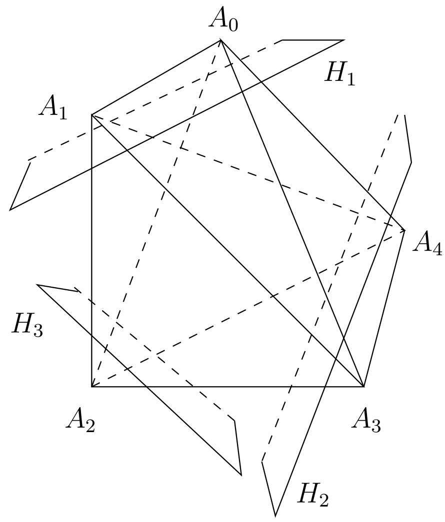


Figure 6.2: The 4-simplex $\triangle^4$ and the affine hyperplanes H_1, H_2 and H_3.

We cut off an open neighborhood of faces F', F'' and $\{A_{\frac{n}{2}}\}$ by affine hyperplanes $H_1 \cap H$, $H_2 \cap H$ and $H_3 \cap H$ respectively in H. Let H'_j be the closed half space associated to the affine hyperplane H_j such that the interior of half spaces H'_1, H'_2, H'_3 do not contain the faces $F', F'', \{A_{\frac{n}{2}}\}$ respectively. We illustrate such hyperplanes for the case $n = 4$ in Figure 6.2. Define

$$\triangle^n_Q = \triangle^n \cap H'_1 \cap H'_2 \cap H'_3, \ P_1 = \triangle^n \cap H_1, \ P_2 = \triangle^n \cap H_2 \text{ and } P_3 = \triangle^n \cap H_3. \tag{6.2.13}$$

The convex polytope $\triangle^n_Q$ is a simple convex polytope of dimension n and the polytopes P_1, P_2 and P_3 are also facets of $\triangle^n_Q$. The polytopes P_1, P_2 and P_3 are given by the following equations.

$$P_1 = \{(x_0, x_1, \dots, x_n) \in \triangle^n : x_0 + \dots + x_{\frac{n}{2}-1} = 1 - r_2 \text{ and } x_{\frac{n}{2}} + \dots + x_n = r_2\}. \quad (6.2.14)$$

$$P_2 = \{(x_0, x_1, \dots, x_n) \in \triangle^n : x_0 + \dots + x_{\frac{n}{2}} = r_2 \text{ and } x_{\frac{n}{2}+1} + \dots + x_n = 1 - r_2\}. \quad (6.2.15)$$

$$P_3 = \{(x_0, x_1, \dots, x_n) \in \triangle^n : x_{\frac{n}{2}} = 1 - r_1 \text{ and } x_0 + \dots + \widehat{x}_{\frac{n}{2}} + \dots + x_n = r_1\}. \quad (6.2.16)$$

By equations 6.2.14 and 6.2.15, the convex polytopes P_1 and P_2 are diffeomorphic to the product $\triangle^{\frac{n}{2}-1} \times \triangle^{\frac{n}{2}}$. From equation 6.2.16 P_3 is diffeomorphic to the simplex $\triangle^{n-1}$. The facets of P_1, P_2 and P_3 are given by the following equations respectively.

$$\triangle^{n-1}_j \cap P_1 = \{(x_0, x_1, \dots, x_n) \in P_1 : x_j = 0\} \text{ for all } j \in \{0, \dots, n\}. \quad (6.2.17)$$

$$\triangle^{n-1}_j \cap P_2 = \{(x_0, x_1, \dots, x_n) \in P_2 : x_j = 0\} \text{ for all } j \in \{0, \dots, n\}. \quad (6.2.18)$$

$$\triangle^{n-1}_j \cap P_3 = \{(x_0, x_1, \dots, x_n) \in P_3 : x_j = 0\} \text{ for all } j \in \{0, \dots, \widehat{\frac{n}{2}}, \dots, n\}. \quad (6.2.19)$$

Now we want to construct $(2n-1)$-dimensional manifold with quasitoric boundary. Let F be a face of $\triangle^n$ of codimension l. Then

$$F = \triangle^{n-1}_{n-j_1} \cap \ldots \cap \triangle^{n-1}_{n-j_l}$$

for a unique $\{j_1, \dots, j_l\} \subset \{0, 1, \dots, n\}$. Suppose $\mathbb{T}_F$ be the torus subgroup of $\mathbb{T}^{n-1}$ determined by the submodule generated by $\{\eta_{j_1}, \dots, \eta_{j_l}\}$ in $\mathbb{Z}^{n-1}$. Assume $\mathbb{T}_{\triangle^n} = \{1\}$. We define an equivalence relation $\sim_\eta$ on the product $\mathbb{T}^{n-1} \times \triangle^n$ as follows,

$$(s, p) \sim_\eta (t, q) \text{ if and only if } p = q \text{ and } ts^{-1} \in \mathbb{T}_F \quad (6.2.20)$$

where $F \subset \triangle^n$ is the unique face containing the point p in its relative interior. Restrict the equivalence relation $\sim_\eta$ on $\mathbb{T}^{n-1} \times \triangle^n_Q$. Define

$$W(\triangle^n_Q, \eta) := (\mathbb{T}^{n-1} \times \triangle^n_Q)/ \sim_\eta$$

to be the quotient space. So $W(\triangle^n_Q, \eta)$ is a $\mathbb{T}^{n-1}$-space. Let

$$\mathfrak{p} : W(\triangle^n_Q, \eta) \to \triangle^n_Q,$$

defined by $\mathfrak{p}([s, p]) = p$, be the corresponding orbit map.

Let η^1, η^2 and η^3 be the restriction of the function η on the set of facets of P_1, P_2

and P_3 respectively. Define

$$\eta^i_j := \eta^i(\triangle^{n-1}_j \cap P_i) = \begin{cases} \eta_j & \text{if } i = 1,2 \text{ and } j \in \{0,1,\ldots,n\} \\ \eta_j & \text{if } i = 3 \text{ and } j \in \{0,1,\ldots,\widehat{\frac{n}{2}},\ldots,n\}. \end{cases} \tag{6.2.21}$$

Let v be a vertex of P_i. So v belongs to the relative interior of a unique edge e_v of $\triangle^n$ not contained in $F' \cup F''$. If

$$e_v = \cap_{j=1}^{n-1} \triangle^{n-1}_{n-l_j}$$

for some $\{l_j : j = 1,\ldots,n-1\} \subset \{0,1,\ldots,n\}$, the vectors $\{\eta_{l_1},\ldots,\eta_{l_{n-1}}\}$ form a basis of $\mathbb{Z}^{n-1}$. So

$$v = \cap_{j=1}^{n-1} (\triangle^{n-1}_{n-l_j} \cap P_i)$$

and the vectors $\{\eta^i_{l_1},\ldots,\eta^i_{l_{n-1}}\}$ form a basis of $\mathbb{Z}^{n-1}$. So η^i defines the characteristic function of a quasitoric manifold $M(P_i,\eta^i)$ over P_i. Hence from the definition of equivalence relation $\sim_\eta$ we get that

$$M(P_i,\eta^i) = (\mathbb{T}^{n-1} \times P_i)/\sim_\eta \ \text{ for } i = 1,2,3. \tag{6.2.22}$$

Let U_i be the open subset of $\triangle^n_Q$ obtained by deleting all faces F of $\triangle^n_Q$ such that the intersection $F \cap P_i$ is empty. Then $\triangle^n_Q = U_1 \cup U_2 \cup U_3$. The space U_i is diffeomorphic as manifold with corners to $[0,1) \times P_i$. Let

$$\mathfrak{f}_i : U_i \to [0,1) \times P_i$$

be a diffeomorphism. From the definition of η and $\sim_\eta$ we get the following homeomorphisms

$$(\mathbb{T}^{n-1} \times \mathfrak{f}_i^{-1}(\{a\} \times P_i))/\sim \ \cong \{a\} \times M(P_i,\eta^i) \text{ for all } a \in [0,1). \tag{6.2.23}$$

Hence the space $\mathfrak{p}^{-1}(U_i)$ is homeomorphic to

$$(\mathbb{T}^{n-1} \times \mathfrak{f}_i^{-1}([0,1) \times P_i))/\sim_\eta \ \cong \ [0,1) \times M(P_i,\eta^i).$$

Since $W(\triangle^n_Q,\eta) = \mathfrak{p}^{-1}(U_1) \cup \mathfrak{p}^{-1}(U_2) \cup \mathfrak{p}^{-1}(U_3)$, the space $W(\triangle^n_Q,\eta)$ is a manifold with quasitoric boundary. The intersections $P_1 \cap P_2$, $P_2 \cap P_3$ and $P_1 \cap P_3$ are empty. Hence the boundary

$$\partial W(\triangle^n_Q,\eta) = M(P_1,\eta^1) \sqcup M(P_2,\eta^2) \sqcup M(P_3,\eta^3).$$

6.3 Orientability of $W(\triangle_Q^n, \eta)$

Fix the standard orientation on $\mathbb{T}^{n-1}$. Then the boundary orientations on P_1, P_2 and P_3 induced from the orientation of $\triangle_Q^n$ give the orientations of $M(P_1,\eta^1), M(P_2,\eta^2)$ and $M(P_3,\eta^3)$ respectively.

Let $W := W(\triangle_Q^n, \eta)$. The boundary ∂W has a collar neighborhood in W. Hence by the proposition 2.22 of [Hat02],

$$H_i(W, \partial W) = \widetilde{H}_i(W/\partial W) \text{ for all } i.$$

We show the space $W/\partial W$ has a CW-structure. Assuming $\triangle_Q^n \subset \mathbb{R}^n$, we choose a linear functional

$$\zeta : \mathbb{R}^n \to \mathbb{R}$$

which distinguishes the vertices of $\triangle_Q^n$, as in the Section 1.5 of Chapter 1. The vertices are linearly ordered according to ascending value of ζ. We make the 1-skeleton of $\triangle_Q^n$ into a directed graph by orienting each edge such that ζ increases along edges. For each vertex v of $\triangle_Q^n$ define its index $ind(v)$ as the number of incident edges that point towards v. Suppose $V(\triangle_Q^n)$ is the set of vertices and $E(\triangle_Q^n)$ is the set of edges of $\triangle_Q^n$. For each $j \in \{1, 2, \ldots, n\}$, let

$$I_j = \{(v, e_v) \in V(\triangle_Q^n) \times E(\triangle_Q^n) : ind(v) = j \textit{ and } e_v \textit{ is the incident edge that}$$

$$\textit{points towards } v \textit{ such that } e_v = e \cap \triangle_Q^n \textit{ for an edge } e \textit{ of } \triangle^n\}.$$

Suppose $(v, e_v) \in I_j$ and $F_v \subset \triangle_Q^n$ is the smallest face containing the inward pointing edges incident to v in $\triangle_Q^n$. Then $ind(v) = \dim(F_v)$. Let U_{e_v} be the open subset of F_v obtained by deleting all faces of F_v not containing the edge e_v. The restriction of the equivalence relation $\sim_\eta$ on $(\mathbb{T}^{n-1} \times U_{e_v})$ gives that the quotient space $(\mathbb{T}^{n-1} \times U_{e_v})/\sim_\eta$ is homeomorphic to the open disk $B^{2j-1} \subset \mathbb{R}^{2j-1}$.

Hence the quotient space $(W/\partial W)$ has a CW-complex structure with odd dimensional cells and one zero dimensional cell only. The number of $(2j-1)$-dimensional cell is $|I_j|$, the cardinality of I_j for $j = 1, 2, \ldots, n$. So we get the following theorem.

Theorem 6.3.1. $$H_i(W, \partial W) = \begin{cases} \bigoplus_{|I_j|} \mathbb{Z} & \textit{if } i = 2j-1 \textit{ and } j \in \{1, \ldots, n\} \\ \mathbb{Z} & \textit{if } i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

When $j = n$ the cardinality of I_j is one. So $H_{2n-1}(W, \partial W) = \mathbb{Z}$. Hence W is an oriented manifold with quasitoric boundary. From the definition 6.2.20 we get that the boundary orientation on $M(P_i, \eta^i)$ is same as the orientation on $M(P_i, \eta^i)$ as the quasitoric manifold, for all $i = 1, 2, 3$.

6.4 Oriented cobordism of $\mathbb{CP}^{2k+1}$

We show the quasitoric manifolds $M(P_1, \eta^1)$ and $M(P_2, \eta^2)$ are equivariantly homeomorphic up to an automorphism of $\mathbb{T}^{n-1}$. Consider the permutation

$$\rho : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$$

defined by

$$\rho(j) = \begin{cases} n-1-j & \text{if } 0 \leq j < \frac{n}{2} - 1 \text{ and } \frac{n}{2} < j < n \\ n & \text{if } j = \frac{n}{2} - 1 \\ \frac{n}{2} & \text{if } j = \frac{n}{2} \\ \frac{n}{2} - 1 & \text{if } j = n. \end{cases} \tag{6.4.1}$$

So ρ is an even or odd permutation if $n = 4l$ or $n = 4l + 2$ respectively. Define a linear automorphism Φ on $\mathbb{R}^{n+1}$ by

$$\Phi(x_0, \ldots, x_j, \ldots, x_n) = (x_{\rho(0)}, \ldots, x_{\rho(j)}, \ldots, x_{\rho(n)}). \tag{6.4.2}$$

Hence Φ is an orientation preserving or reversing diffeomorphism if $n = 4l$ or $n = 4l + 2$ respectively. From equations 6.2.14 and 6.2.15 it is clear that Φ maps P_1 diffeomorphically onto P_2. We denote the restriction of Φ on the faces of P_1 by Φ. Also from the equations 6.2.17 and 6.2.18 we get that Φ maps the facet $\triangle_j^{n-1} \cap P_1$ of P_1 diffeomorphically onto the facet $\triangle_{\rho(j)}^{n-1} \cap P_2$ of P_2. So

$$\Phi(\triangle_j^{n-1} \cap P_1) = \triangle_{\rho(j)}^{n-1} \cap P_2, \quad \text{for all } j = 0, \ldots, n. \tag{6.4.3}$$

Let $\alpha_1, \ldots, \alpha_{n-1}$ be the standard basis of $\mathbb{Z}^{n-1}$ over $\mathbb{Z}$. Let δ' be the linear automorphism of $\mathbb{Z}^{n-1}$ defined by

$$\delta'(\alpha_i) = \alpha_{n-i} \ \forall \ i = 1, \ldots, (n-1). \tag{6.4.4}$$

Hence

$$\delta'(\eta_i) = \eta_{\rho(i)} \text{ and } \delta'(\eta_{\rho(i)}) = \eta_i \text{ for } i = 0, 1, \ldots, n. \tag{6.4.5}$$

Let δ be the automorphism of $\mathbb{T}^{n-1}$ induced by δ'. Hence the automorphism δ is orientation reversing if $n = 4l$ and it is orientation preserving if $4l + 2$. From the equations 6.2.21, 6.4.3 and 6.4.5 we get that the following commutative diagram.

$$\begin{array}{ccc} \mathcal{F}(P_1) & \xrightarrow{\Phi} & \mathcal{F}(P_2) \\ \eta^1 \downarrow & & \eta^2 \downarrow \\ \mathbb{Z}^{n-1} & \xrightarrow{\delta} & \mathbb{Z}^{n-1}. \end{array}$$

So the diffeomorphism

$$\delta \times \Phi : \mathbb{T}^{n-1} \times P_1 \to \mathbb{T}^{n-1} \times P_2$$

induces a δ-equivariant orientation reversing homeomorphism

$$g_n : M(P_1, \eta^1) \to M(P_2, \eta^2).$$

From the definition 6.2.21 of the characteristic function η^3 we get that the quasitoric manifold $M(P_3, \eta^3)$ is equivariantly homeomorphic to $\mathbb{CP}^{n-1}$ if $n = 4l+2$ and $\overline{\mathbb{CP}}^{n-1}$ if $n = 4l$.

Define an equivalence relation $\sim_n$ on $W(\triangle^n_Q, \eta)$ by

$$x \sim_n y \text{ if and only if } x \in M(P_1, \eta^1) \text{ and } y = g_n(x). \tag{6.4.6}$$

So the quotient space $W(\triangle^n_Q, \eta)/\sim_n$ is an oriented manifold with boundary. The boundary of these manifold is $\mathbb{CP}^{n-1}$ if $n = 4l+2$ and the boundary is $\overline{\mathbb{CP}}^{n-1}$ if $n = 4l$.

So the quotient space $W(\triangle^n_Q, \eta)/\sim_n$ is an oriented manifold with boundary and the boundary is $\mathbb{CP}^{n-1}$. Hence we have proved the following theorem.

Theorem 6.4.1. *The complex projective space* $\mathbb{CP}^{2k+1}$ *is boundary of an oriented manifold, for all* $k \geq 0$.

Example 6.4.2. *We adhere to definition and notations given in the example 6.2.1. The faces* A_0A_1 *and* A_3A_4 *are the intersection of facets* $\{\triangle^3_4, \triangle^3_3, \triangle^3_2\}$ *and* $\{\triangle^3_2, \triangle^3_1, \triangle^3_0\}$ *respectively of* $\triangle^4$.

Here the polytopes P_1, P_2 *are prism and* P_3 *is* 3*-simplex, see the Figure 6.3. The restriction of* η *(namely* η^1, η^2 *and* η^3*) on the facets of* P_1, P_2 *and* P_3 *are given in following Figure 6.4.*

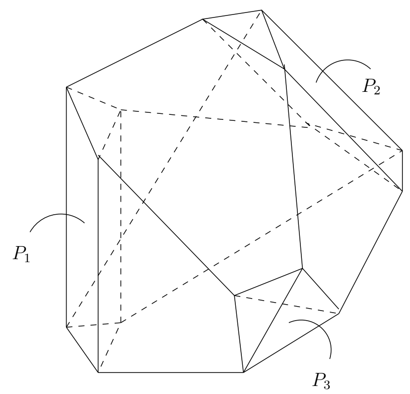


Figure 6.3: The simple convex polytope $\triangle^4_Q$ with the facets P_1, P_2 and P_3.

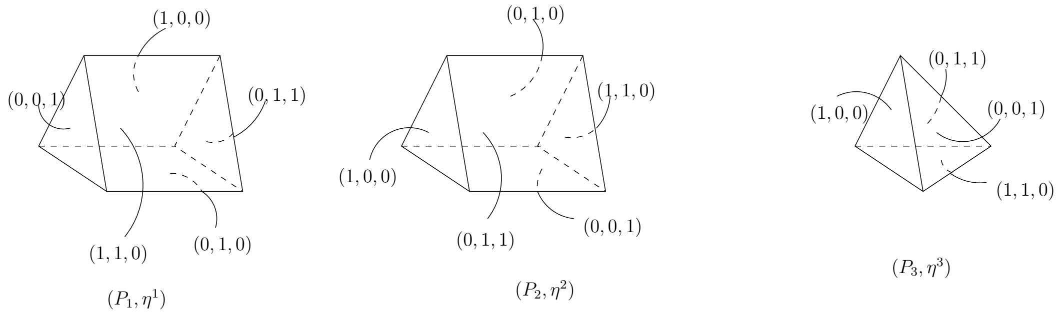


Figure 6.4: The characteristic functions η^1, η^2 and η^3 of P_1, P_2 and P_3 respectively.

Let δ' be the automorphism of $\mathbb{Z}^3$ defined by

$$\delta'(1,0,0) = (0,0,1),\ \delta'(0,1,0) = (0,1,0) \text{ and } \delta'(0,0,1) = (1,0,0).$$

Clearly the combinatorial pairs (P_1, η^1) and (P_2, η^2) give two δ-equivariantly homeomorphic quasitoric manifolds, namely $M(P_1, \eta^1)$ and $M(P_2, \eta^2)$ respectively. The combinatorial pair (P_3, η^3) gives the quasitoric manifold $\overline{\mathbb{CP}}^3$ over P_3. So the boundary of $W(\triangle^4_Q, \eta)$ is $M(P_1, \eta^1) \sqcup M(P_2, \eta^2) \sqcup \overline{\mathbb{CP}}^3$.

Hence after identifying $M(P_1, \eta^1)$ and $M(P_2, \eta^2)$ via an orientation reversing homeomorphism, we get an oriented manifold with boundary and the boundary is $\overline{\mathbb{CP}}^3$.

Now we briefly give the previous proof of Theorem 6.4.1 following a notes by Andrew J. Baker. Consider the unit sphere $S^{4k+3} \subset \mathbb{H}^{k+1}$ where $\mathbb{H}$ denotes the quaternions. This is acted on freely by the unit quaternions $S^3 \subset \mathbb{H}$ and its subgroup of unit complex numbers $S^1 \subset \mathbb{C} \subset \mathbb{H}$. Note that the conjugation action of S^3 on $\mathbb{H}$ restricted to the pure quaternions gives a realization of S^3 as $Spin(3)$ acting on $\mathbb{R}^3$ via the natural map to $SO(3)$. Furthermore, $S^1 \subset S^3$ is identifies with $Spin(2)$ and we have

$$Spin(3)/Spin(2) \cong SO(3)/SO(2) \cong S^2. \tag{6.4.7}$$

Also

$$\mathbb{HP}^k = S^{4k+3}/S^3; \mathbb{CP}^{2k+1} = S^{4k+3}/S^1; \tag{6.4.8}$$

and the natural map $\mathbb{CP}^{2k+1} \to \mathbb{HP}^k$ can be identified with the sphere bundle of

$$S^{4k+3} \times_{Spin(3)} \mathbb{R}^3 \to \mathbb{HP}^k.$$

Thus we have a commutative diagram

$$\begin{array}{ccc} \mathbb{CP}^{2k+1} & \stackrel{\hookrightarrow}{\longrightarrow} & S^{4k+3} \times_{Spin(3)} D^3 \\ \downarrow & & \downarrow \\ \mathbb{HP}^k & = & \mathbb{HP}^k \end{array} \tag{6.4.9}$$

in which $\mathbb{CP}^2k+1$ identifies with the boundary of $S^{4k+3} \times_{Spin(3)} D^3$. It is easy to show that cohomology groups of $W(\triangle^n_Q, \eta)/\sim_n$ $(n = 2(k+1))$ and $S^{4k+3} \times_{Spin(3)} D^3$ are different for all $k > 0$, . So we get two different construction of manifold with the boundary $\mathbb{CP}^{2k+1}$.

Remark 6.4.3. *We can give a nice CW-structure on $W(\triangle^n_Q, \eta)/\sim_n$ with one cell in dimension greater than zero and two cells in dimension zero from the combinatorial information. So the Theorem 6.4.1 may be helpful in some computations.*

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