SOME EXTREMAL PROBLEMS AND CHARACTERIZATIONS IN THE THEORY OF GRAPHS

by

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CONTENTS

INTRODUCTION

Graph theory has become such a well known and widely applied subject that it is not necessary to give a general introduction to it. Instead we give below a summary of the results of the thesis chapterwise.

The study of extremal problems in Graph theory was started by Turan who determined the minimum independence number of a graph on n vertices with m edges. More recently Harary determined the maximum connectivity of a graph on n vertices with m edges.

In a paper concerning the degrees of the vertices of a graph, Hakimi posed the following two problems : determine the maximum number of cut vertices (cut edges) in a graph on n vertices with given degrees $d_1, d_2, \ldots, d_n$. We solve some related problems in Chapter 1.

A cut vertex (cut edge) of a graph is a vertex (edge) whose removal increases the number of components of the graph. In Chapter 1, we consider the determination of the range of the number of cut vertices (cut edges) in a graph on n vertices with m edges under each of the following two conditions.

(1) The degree of no vertex is less than d.

(2) The degree of no vertex is greater than d. The problem is solved completely for $d \leq 3$ and when $d = 4$ and $m = 2n$. The extremal graphs are characterized when $d \leq 2$. The range of the number of bridges in a matroid with m cells, rank n, and containing no circuit of length less than k is also determined.

The power of a vertex x of a connected graph G may be defined as the number of components of $G-x$. Necessary and sufficient conditions for positive integers $p_1, p_2, \ldots, p_n$ to be the powers of the vertices of some graph on n vertices (also with m edges) are given in Chapter 2. The maximum power of a vertex of a connected graph with n vertices and m edges is also determined. Necessary and sufficient conditions for the existence of an (undirected) graph with degrees $d_1, d_2, \ldots, d_n$ are deduced from the corresponding results of Gale, Ryser and Fulkerson for directed graphs. The same problem is also solved under each of the conditions (i) the graph is connected (ii) the graph is biconnected. These results have been obtained earlier, using different methods by Erdos and Gallai, Hakimi and Tutte.

The line graph of a graph G may be defined as the graph whose vertices are the edges of G, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent. Some simple properties and characterizations of line graphs are given in Chapter 3. In particular, a characterization of the line graph of the complete λ-partite graph $K_{n,n,\ldots,n}$ is given. This generalizes the results of Connor and Shrikhande on the line graphs of the triangular and L_2-association schemes.

In Chapter 4, we obtain a characterization of the graph $G(\lambda,n,m)$ defined as follows: take λ disjoint sets $S_1,\ldots,S_\lambda$, each with n elements and let $S = \bigcup_{i=1}^{\lambda} S_i$. The vertices of $G(\lambda, n, m)$ are all the unordered m-tuples of elements of S where in any such m-tuple at most one element of each S_i can be present. Two vertices of $G(\lambda, n, m)$ are joined by an edge if and only if the corresponding m-tuples of S have a common $(m-1)$-tuple. This graph $G(\lambda, n, m)$ includes the T_m graph characterized by Dowling and the cubic lattice graph characterized by Laskar as special cases. The problem is solved when $m \leq 5$ and $(\lambda-m+1)n$ exceeds two quadratic functions in m. Finally a similar characterization for the line graph of a BIB design with $\lambda=1$ is obtained.

CHAPTER I

THE NUMBER OF CUT VERTICES AND CUT EDGES IN A GRAPH

In this thesis we consider only finite undirected graphs with neither multiple edges nor loops. For notation and terminology, C. Berge [1] is generally followed.

A vertex x of a graph G is called a cut vertex if the graph $G-x$, obtained by deleting the vertex x and its incident edges, has more components than G. Thus if G is connected, then x is a cut vertex of G if and only if $G-x$ is disconnected. A vertex which is not a cut vertex is called a neutral vertex. If G represents a communication network, the centres corresponding to cut vertices are particularly important since their loss can destroy the unity and cohesion of the network.

If x is a cut vertex of a connected graph G and C is a component of $G-x$, the subgraph generated by $C \cup \{x\}$ is called a piece of G with respect to x.

A cut edge of a graph is defined similarly, as an edge whose removal increases the number of components of the graph. An edge which is not a cut edge is called a neutral edge. It is easily seen that an edge is neutral if and only if it belongs to some cycle. Also if (x, y) is a cut edge of a connected graph G, its removal leaves exactly two components, one containing x and the other containing y.

A connected graph is called biconnected if it has no cut vertex. A block of a connected graph is a maximal biconnected subgraph.

In this chapter we determine the ranges of the number of cut vertices and the number of cut edges in a graph on n vertices with m edges. This problem is solved also when some conditions are imposed on the degrees of the vertices. The range of the number of bridges in a matroid with m cells, rank n and without a circuit of length less than k is also determined.

1.1 Cut vertices

In this section we determine the range of the number of cut vertices in a graph on n vertices with m edges. We start with a simple result :

Lemma 1.1 A connected graph on n vertices has at most $n-2$ cut vertices. Further the only such graph with $n-2$ cut vertices is an elementary chain on n vertices.

This lemma can be proved easily using the concept of a spanning tree.

Lemma 1.2 In a connected graph on n vertices with r cut vertices, the maximum number of edges is $\binom{n-r}{2} + r$.

Proof: Let G be a connected graph on n vertices with r cut vertices and with the maximum number of edges. Then obviously every block of G is complete and the number t of blocks is not less than $r+1$. Let n_i be the number of vertices in the i-th block, $i = 1, 2, \ldots, t$. Then $n_i \geq 2$ and it can be proved by induction on t that

$$\sum_{i=1}^{t} n_i = n+t-1.$$

Thus the number of edges in G is not more than

$$\max \left\{ \sum_{i=1}^{t} \binom{n_i}{2} : \sum_{i=1}^{t} n_i = n+t-1,\ n_i \geq 2,\ t \geq r+1 \right\}$$

$$= \max_{t \geq r+1} \left\{ \binom{n+t-1-2t+2}{2} + t-1 \right\}$$

$$= \binom{n-r}{2} + r.$$

But a complete graph on $n-r$ vertices with an attached elementary chain of length r has n vertices, $\binom{n-r}{2} + r$ edges and r cut vertices. This completes the proof of the lemma.

Theorem 1.3 The maximum number of cut vertices in a connected graph on n vertices with m edges is $r = r(n,m)$ where

$$r(n,m) = \max \{ q : q \leq n-2,\ m \leq \binom{n-q}{2} + q \}$$

$$= \left[\frac{2n-3-\sqrt{8m-8n+9}}{2} \right] \qquad (1.1)$$

and $[x]$ denotes the largest integer less than or equal to x.

Proof : It is easy to prove that the two expressions for $r(n,m)$ given in (1.1) are equal by writing $\binom{n-q}{2} + q$ as a quadratic form in q.

Now, the present theorem follows from lemma 1.1 if $m = n-1$. So let $n \leq m \leq \binom{n}{2}$. By lemma 1.2, if a connected graph G on n vertices has $r+1$ or more cut vertices, then the number of edges in G is not more than

$$\binom{n-r-1}{2} + r + 1 < m.$$

Hence the number of cut vertices in any connected graph on n vertices with m edges is not more than r. To construct such a graph with exactly r cut vertices, take any biconnected graph on $n-r$ vertices with $m-r$ edges (this is possible since $n-r \leq m-r \leq \binom{n-r}{2}$) and attach to one of its vertices an elementary chain of length r. This proves the theorem.

The following result can be deduced easily from theorem 1.3 : a connected graph on n vertices with m edges and with (exactly) k cut vertices exists if and only if $0 \leq k \leq r(n,m)$ and $k \geq 1$ if $m = n-1$.

By an extremal graph in the following we mean a connected graph with $r(n,m)$ cut vertices where n and m are the number of vertices and the number of edges, respectively, of the graph. Now we determine all the extremal graphs on n vertices with m edges.

Lemma 1.4 In an extremal graph there cannot be more than two pieces with respect to any cut vertex.

Proof : If there are at least three pieces with respect to a cut vertex x of an extremal graph G, remove one of the pieces (without the vertex x itself) and attach it at a neutral vertex of the remaining graph. This increases the number of cut vertices in the graph, a contradiction which proves the lemma.

Lemma 1.5 The graph G consisting of two complete graphs, each on more than three vertices, attached by a common vertex, is not extremal.

<u>Proof</u> : Let r, s be the numbers of vertices in the two complete subgraphs of G, so that the number n of vertices in G is $r+s-1$. Then the number of edges in G is not more than

$$\max_{\substack{r,s \geq 4 \\ r+s=n+1}} \left\{\binom{r}{2} + \binom{s}{2}\right\} = \binom{4}{2} + \binom{n-3}{2}$$
$$=\binom{n-2}{2} + 2 + 7 - n.$$

Hence if G is extremal, by theorem 1.3, G has at least two cut vertices, a contradiction.

<u>Lemma 1.6</u> An extremal graph G without any cut edge has at most one cut vertex.

<u>Proof</u> : Adding new edges if necessary we make each block of G complete. The resulting graph H is also extremal since it has the same number, say r, of cut vertices as G but more edges. By lemma 1.4, the number of blocks in H is $r+1$. If n_i is the number of vertices in the i-th block, then $n_i \geq 3$ since H has no cut edge. Thus the number of edges in H is not more than

$$\max \left\{ \sum_{i=1}^{r+1} \binom{n_i}{2} : \sum_{i=1}^{r+1} n_i = n+r, \; n_i \geq 3 \right\}$$
$$=\binom{n-2r}{2} + 3r. \qquad (1.2)$$

The right side expression of (1.2) is not greater than

$$\binom{n-r-1}{2} + r + 1$$

whenever $r \geq 2$. But H has only r cut vertices and this gives a contradiction to theorem 1.3 if $r \geq 2$. This proves the lemma.

<u>Lemma 1.7</u> If an extremal graph G with n vertices and m edges has no cut edge, then either $m \geq \binom{n-1}{2} + 2$ or G consists of a complete graph and a triangle attached at some vertex.

<u>Proof</u> : Suppose $m < \binom{n-1}{2} + 2$. Then by theorem 1.3 and lemma 1.6, G has exactly one cut vertex. Now making each block of G complete, we get a graph, which by lemma 1.4 consists of two complete graphs attached by a common vertex. By lemma 1.5, at least one of these complete subgraphs is a triangle. If the other block is not complete in G, one edge of the triangle can be transfered to it so that the number of cut vertices increases. The impossibility of this proves the lemma.

<u>Theorem 1.8</u> Let $r = r(n,m)$ be given by (1.1). Then the extremal graphs on n vertices with m edges are the following :

(1) a graph consisting of a biconnected graph on $n-r$ vertices with $m-r$ edges to which elementary chains of total length r are attached at distinct vertices;

(2) a graph consisting of an elementary chain μ (which may be a single vertex) separating a complete graph at one end and a triangle at the other end, with elementary chains attached at distinct vertices not belong to μ, where the sum of the lengths of μ and all the terminal chains is $r-1$.

Proof : It is evident that the graphs (1) and (2) are extremal. Conversely let G be an extremal graph on n vertices with m edges. By successively removing a pendant vertex and its incident edge, we finally arrive at a subgraph H without pendant vertices such that G is obtained from H by attaching trees at some of the vertices of H. Each of those trees is an elementary chain, for otherwise, we can increase the number of cut vertices by replacing such a tree by an elementary chain with the same number of vertices. Obviously now H is also extremal.

If H has no cut vertex, then G is a graph of the type (1).

If H has at least one cut vertex, then there is a unique elementary chain μ (which may be a single vertex) separating blocks on more than two vertices. For otherwise, by suppressing every such chain and identifying its end vertices we get an extremal graph with at least two cut vertices and without any cut edge, a contradiction to lemma 1.6. By the same argument it follows from lemma 1.7 that H consists of an elementary chain μ separating a complete graph at one end and a triangle at the other end. Obviously now G is a graph of the type(2). This completes the proof of the theorem.

If we consider graphs on n vertices with m edges, which are not necessarily connected, we have

<u>Theorem 1.9</u> The minimum number of cut vertices in a graph on n vertices with m edges is zero except when $n = 3$ and $m = 2$. The maximum number of cut vertices in a graph on n vertices with m edges is 0, $m-1$ or $r(n,m)$ according as $m=0$, $1 \leq m \leq n-1$ or $m \geq n$. Further every value between the minimum and maximum is attained by some graph on n vertices with m edges.

The ''extremal graphs'' for the problem considered in theorem 1.9 can be described easily. It is enough to note that when $m \geq n-1$, a graph on n vertices, with m edges and with the maximum number of cut vertices is connected, for otherwise, we can remove some neutral edge and join two neutral vertices of two connected components, thereby increasing the number of cut vertices in the graph.

1.2. Cut edges

In this section we determine the range of the number of cut edges in a graph on n vertices with m edges.

<u>Lemma 1.10</u> The maximum number of cut edges in a connected graph on n vertices is $n-1$. This maximum is attained by any tree on n vertices and by no other graph.

This lemma can be proved easily using the concept of a spanning tree.

<u>Lemma 1.11</u> If $r \leq n-2$, the maximum number of edges in a connected graph G on n vertices with r cut edges is $\binom{n-r}{2} + r$.

<u>Proof</u>: Since $r \neq n-1$, G has at least one cycle and hence there are at least $r+1$ blocks, namely the r cut edges and another block on at least three vertices. Now the proof of the lemma is similar to that of lemma 1.2.

<u>Corollary.</u> A connected graph on n vertices cannot have exactly $n-2$ cut edges.

<u>Theorem 1.12</u> The maximum number of cut edges in a connected graph on n vertices with m edges is $n-1$ or $r(n,m)$ according as $m = n-1$ or $m \geq n$, where $r(n,m)$ is given by (1.1).

The proof of this theorem utilizes lemma 1.11 and is similar to that of theorem 1.3.

Now the following résult is easy to prove : a connected graph with n vertices, m edges and with (exactly) k cut edges exists if and only if $0 \leq k \leq r(n,m)$ if $m \geq n$ and $k = n-1$ if $m = n-1$.

<u>Theorem 1.13</u> Let $m \geq n$ and $r = r(n,m)$ be given by (1.1). Then a connected graph on n vertices with m edges and with r cut edges consists of a biconnected graph on $n-r$ vertices with $m-r$ edges to which trees with a total of r edges are attached at some of the vertices.

<u>Proof</u> : Let G be a connected graph on n vertices with m edges and with $r(n,m)$ cut edges. Then as shown in the proof of theorem 1.8, there exists a subgraph H without pendant vertices such that G is obtained from H by attaching trees at some of the vertices of H. If H has a cut edge, then by successively suppressing such edges and identifying their end vertices we finally get a connected graph H_0 without any cut edge. If n_0 and m_0 are the number of vertices and the number of edges respectively in H_0, then by theorem 1.12, $r(n_0,m_0) = 0$, but H_0 has a cut vertex. This contradiction shows that H has no cut edge. Thus H has $n-r$ vertices, $m-r$ edges and is biconnected. This completes the proof of the theorem.

When we consider graphs not necessarily connected, we have:

<u>Theorem 1.14</u> The minimum number of cut edges in a graph on n vertices with m edges is 0 or m according as $m \geq 3$ or $m < 3$. The maximum number of cut edges in a graph on n vertices with m edges is m or $r(n,m)$ according as $m \leq n-1$ or $m \geq n$ where $r(n,m)$ is given by (1.1). Further every value between the minimum and the maximum, excepting $m-1$ and $m-2$ when $m \leq n-1$, is attained by some graph on n vertices with m edges.

Here again the ''extremal graphs'' are easily described. It is enough to note that whenever $m \geq n-1$, a graph with n vertices, m edges and with the maximum number of cut edges is connected.

The results of sections 1.1 and 1.2 have appeared in [16].

1.3 Bridges in a matroid

In this section we determine the range of the number of bridges in a matroid with m cells, rank n and containing no circuit of length less than k.

A *matroid* is a pair $(M, \mathcal{G})$ where M is a finite set and $\mathcal{G}$ is a collection of subsets of M satisfying the following three conditions

(1) $\emptyset \in \mathcal{G}$

(2) any subset of a member of $\mathcal{G}$ is itself a member of $\mathcal{G}$

(3) for every subset A of M, all maximal subsets of A belonging to $\mathcal{G}$ have the same cardinality.

If $(M, \mathcal{G})$ is a matroid, elements of M are called *cells* and sets belonging to $\mathcal{G}$ are called *independent sets*. Subsets of M which do not belong to $\mathcal{G}$ are called *dependent* sets and a minimal dependent set is called a *circuit*. The length of a circuit is the number of cells in it. If A is a subset of M, a maximal independent subset of A is called a *basis* of A. The cardinality of any basis of M is called the *rank* of the matroid. A *bridge* is a cell which belongs to every basis of M.

If G is a graph, let M be the set of edges of G. Define a subset I of M to be independent if it does not contain any cycle. Then $(M, \mathcal{G})$ is a matroid and an edge u of G is a cut edge if and only if u is a bridge of $(M, \mathcal{G})$.

Thus the concepts matroid and bridge are extensions of the concepts graph and cut edge.

Let $n \leq m$ and $1 \leq k \leq n+1$. Then we have

<u>Theorem 1.15</u> The minimum number of bridges in a matroid with m cells, rank n and containing no circuit of length less than k is 0 or n according as $n < m$ or $n = m$. The maximum number of bridges in a matroid with m cells, rank n and containing no circuit of length less than k is $n-k+1$ or n according as $n < m$ or $n = m$. Further every value between the minimum and the maximum is attained by some such matroid.

<u>Proof</u> : To prove the first part of the theorem, we need only consider the matroid obtained by taking a set M of m elements and defining a subset I of M to be independent if and only if $|I| \leq n$. To prove the latter part, let $(M, \mathcal{G})$ be a matroid with m cells, rank n and containing no circuit of length less thank k. If $n = m$, the theorem is trivial, so let $n < m$. If possible let there be $n - k + 2$ bridges $x_1, x_2, \ldots, x_{n-k+2}$. Let $\{x_1, \ldots, x_{n-k+2}, y_1, \ldots, y_{k-2}\}$ be a basis of M and let z be a cell not in this basis. If $\{y_1, \ldots, y_{k-2}, z\}$ is independent, the rank of the matroid is greater than n ; otherwise there exists a circuit of length

less than k. This contradiction proves that $(M, \mathcal{G})$ can have at most $n-k+1$ bridges. To construct such a matroid with $n-r$ bridges, $r \geq k-1$, take a set M_0 of m elements and let $x_1, \ldots, x_{n-r}$ be any $n-r$ fixed elements of M_0. Then define a subset I of M_0 to be independent if I contains at most r elements not in $\{x_1, \ldots, x_{n-r}\}$. This completes the proof of the theorem.

1.4 Cut vertices with bounds on degrees

In this section we consider the problem of determining the range of the number of cut vertices in a graph on n vertices with m edges when certain restrictions are imposed on the degrees of the graph. More specifically we consider the following two cases :

(1) the minimum edgree $\geq d$,

(2) the maximum degree $\leq d$.

These problems are solved when $d \leq 3$ or $d = 4$ and $m = 2n$.

Let $A(n,m,d)$ and $B(n,m,d)$ denote the maximum number of cut vertices in the cases (1) and (2) respectively, under the additional constraint that the graph be connected.

<u>Theorem 1.16</u> Let $r = r(n,m)$ be given by (1.1), $\ell = m - \binom{n-r-1}{2} - r$ and $k = m-n+1$. Then

$$A(n,m,2) = \begin{cases} r & \text{if } r \geq 1,\ k \geq 2,\ \ell = 2 \\ r-1 & \text{if } r \geq 1,\ k \geq 2,\ \ell > 2 \\ 0 & \text{if } r = 0 \text{ or } k = 1. \end{cases}$$

<u>Proof</u>: The theorem is trivial if $k=1$ and follows from theorem 1.3 if $r=0$. So let $r \geq 1$, $k \geq 2$. If $\ell = 2$, the connected graph consisting of an elementary chain of length $r - 1$ separating a complete graph on $n-r-1$ vertices at one end and a triangle at the other end has r cut vertices and by theorem 1.3 it follows that $A(n,m,2) = r$. Now suppose that $\ell > 2$ and G is a connected graph on n vertices with m edges and with r cut vertices. Then by theorem 1.8, G consists of a biconnected subgraph H to which elementary chains of total length r are attached at some vertices. Thus G has a pendant vertex. So if $\ell > 2$, then $A(n,m,2) \leq r-1$ and equality follows from a simple construction.

The following result can be deduced easily from theorem 1.16: a connected graph on n vertices, with m edges, with minimum degree ≥ 2 and with (exactly) p cut vertices exists if and only if $0 \leq p \leq A(n,m,2)$.

For the determination of $A(n,m,3)$ we start with any connected graph G on n vertices with m edges and with minimum degree ≥ 3. Then by a (finite) sequence of modifications we reduce the graph G to a graph (to be described later) with $A(n,m,3)$ cut vertices in such a way that at every stage the number of cut vertices in the graph does not decrease, the graph remains connected, with n vertices, m edges and with minimum degree ≥ 3.

The following are the important steps in this reduction: first the graph is reduced until there are exactly two pieces with respect to any cut vertex. Then every block containing at least three cut vertices of the graph is reduced to a triangle. Next these triangle-blocks are brought together so that any two triangle-blocks are separated only by edges and other triangle-blocks. Now all the ''terminal chains'' (defined later) except possibly one are reduced until each of them consists of an edge and a block on four or five vertices at its end. Finally the number of triangle-blocks is increased as far as possible using the exceptional terminal chain and the exceptional terminal chain itself is replaced by one with the maximum number of cut vertices possible. The number of cut vertices in the final graph thus obtained is $A(n,m,3)$ whose value is given in theorem 1.17. We give below the details of the above reduction procedure.

If x is a cut vertex of a graph H, by removing a piece with respect to x we mean removing the edges and vertices of the piece excepting the vertex x itself. Suppose now that G has at least three pieces with respect to a cut vertex x. If after the removal of one of them, the degree of x is not less than three, we remove such a piece and attach it at a neutral vertex of the remaining graph. If there is no such piece, remove any one piece with respect to x, say L, suppress x by homeomorphism (i.e., suppress the vertex x and amalgamate its two incident edges), create a vertex x' by homeomorphism in a terminal block and attach L at x'. Repeating this as long as necessary we get a graph in which there are exactly two pieces with respect to any cut vertex.

Suppose now that there is a block H containing three or more cut vertices of the graph. After a suitable redistribution of the edges of H within H itself, we may assume that H has a Hamiltonian cycle. If a cut vertex x of the graph belongs to H and has degree in $H \geq 3$, we transfer the other piece at x to a neutral vertex of a terminal block. Thus we may take that a vertex belonging to H has degree two in H if and only if it is a cut vertex of the graph. Suppose now that H still contains three cut vertices of the graph.

If H is not a triangle, we may assume that H contains a vertex x such that there are exactly two vertices y and z of H adjacent to x and y,z are not adjacent. Then x can be dropped from H in an obvious way. Thus the graph is reduced until every block containing at least three cut vertices of the graph is a triangle.

If now there is a block C (which is not an edge) containing two cut vertices of the graph and separating two triangle-blocks, we remove C to one end as shown in figure 1.1.

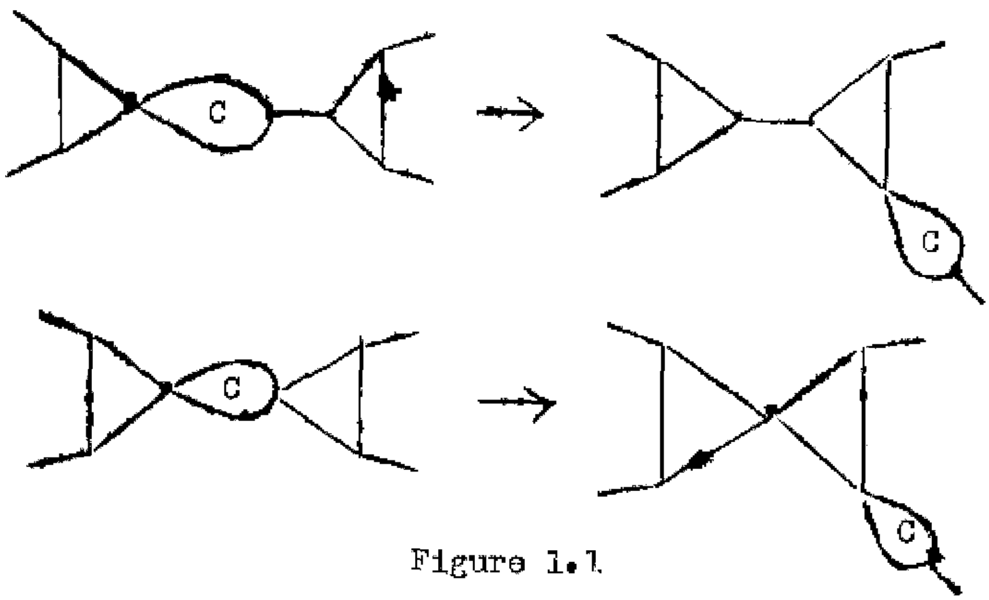

Figure 1.1

Suppose next that there are two triangle-blocks T_1 and T_2 with a common vertex x as in figure 1.2(a). If there is a cut edge (y,z) of the graph, not separating two triangle-blocks, then suppress the edge (y,z), amalgamate its end vertices and separate T_1 and T_2 by an edge. Otherwise, let the graph be as shown in figure 1.2(a). If the degree of a_i in A_i is

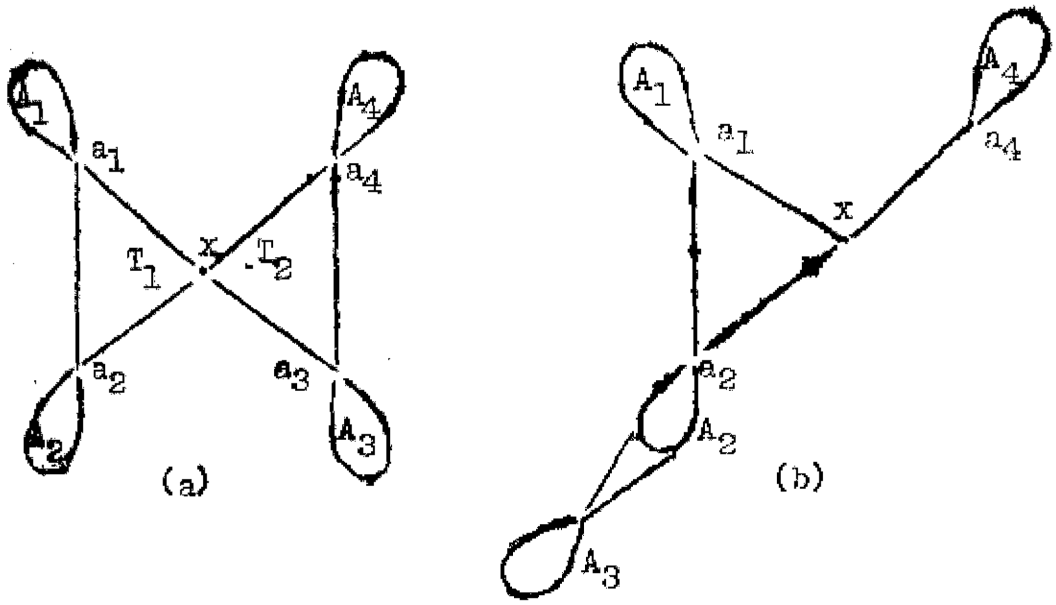

Figure 1.2

one for $i = 1,2,3,4$, then replace A_4 at a_4 by a terminal block of A_4 and put the rest of the graph A_4 between T_2 and A_3. Thus we take that the degree of a_4 in A_4 is at least two. Now suppress the edges (a_3,x), (a_3,a_4) and join a_3 to two vertices of a terminal block of A_2. Thus we reduce the graph until any two triangle-blocks are separated only by edges and other triangle-blocks.

In what follows, by a terminal chain of the graph, we shall mean a maximal subgraph not including two vertices of any triangle-block. Now the terminal chains of the graph can be collected at one place so that all of them except possibly one consist of either a single block or a cut edge and a block at its end. Suppose now that there are two terminal blocks C_1 and C_2 each having at least six vertices.

<u>Case (i)</u>. The number of vertices in C_1 is $2p (p \geq 3)$ and the number of vertices in C_2 is at least $2p$. If q is the number of edges in C_1, then obviously $3p \leq q \leq \binom{2p}{2}$. Now replace C_1 by a block on $2p-1$ vertices with edges between $3p-2$ and $\binom{2p-1}{2}$ in number and enlarge C_2 with the remaining vertex and the remaining edges (between 2 and $2p-1$ in number).

<u>Case (ii)</u>. The number of vertices in C_1 is $2p + 1$ $(p \geq 3)$ and the number of vertices in C_2 is at least $2p + 1$. In this case $3p + 1 \leq q \leq \binom{2p+1}{2}$. Now replace C_1 by a block on $2p-1$ vertices with edges between $3p - 2$ and $\binom{2p-1}{2}$ in number and enlarge C_2 with the remaining two vertices and the remaining edges (between 3 and $4p-1$ in number). When we have to enlarge C_2 with two vertices and three edges, we create two vertices by homeomorphism and join them.

Using the above procedure we reduce the graph until all terminal blocks except possibly one have four or five vertices. Suppose now that there are two terminal chains each consisting of a single block. Then we may assume that two such blocks

A_1 and A_2 occur at two vertices a_1, a_2 of a triangle-block $\{a_1,a_2,a_3\}$. Then suppress the edges (a_3,a_1), (a_3,a_2) and join a_3 to two vertices of A_2.

The graph we obtain at this stage consists of ℓ (say) triangle-blocks separated by $\ell - 1$ edges, with $\ell + 1$ of the $\ell + 2$ terminal chains consisting of a cut edge and a block on four or five vertices at its end. Let now $m = \frac{3n+k}{2}$ and $p = \min(k, \ell + 1)$. Then we reduce the terminal blocks of p of the non-exceptional terminal chains to complete graphs on four vertices. When in this process a block on 5 vertices is replaced by a complete graph on four vertices, we get one free vertex and between one and four free edges. If the number of free edges is at least two they can be used to enlarge the terminal block of the exceptional chain. If there is only one free edge, two cases arise. If there is a block in the exceptional chain containing a vertex with degree within the block ≥ 3, then this can easily be enlarged. Otherwise, two blocks (which are not edges) of the terminal chain will be attached by a common vertex and these two blocks can be separated using the free vertex and the free edge.

If now the exceptional chain has five or more blocks, it can be replaced by a graph of the type shown in figure 1.3. Here x is the vertex by which the exceptional chain

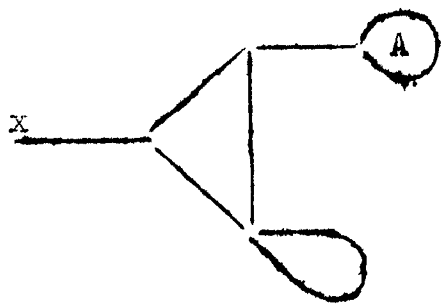

Figure 1.3

is attached to some triangle-block and **A** is a block on four or five vertices. Thus in our final graph we may take that the exceptional chain has at most four blocks and cannot be replaced by a graph with a greater number of cut vertices. If now n_1, m_1 are the number of vertices and the number of edges respectively in the exceptional chain, then obviously n_1, m_1 are given by (1.3), (1.4) and satisfy the conditions (1.5), (1.6). Here ℓ (≥ 0) denotes the number of triangle-blocks in the final graph and $p = \min(k, \ell + 1)$ where $m = \frac{3n+k}{2}$.

$$n_1 = n - \{3\ell + 4p + 5(\ell + 1 - p)\} + 1 = n - 8\ell + p - 4 \qquad (1.3)$$

$$m_1 = m - 3\ell - (\ell - 1) - (\ell + 1) - 6p - 7(\ell + 1 - p) = m - 12\ell + p - 7 \qquad (1.4)$$

$$\frac{1 + 3(n_1 - 1)}{2} \leq m_1 \leq \binom{n_1}{2} \; ; \; m_1 = 6 \text{ if } n_1 = 4 \qquad (1.5)$$

$$n_1 \geq 4 \qquad (1.6)$$

Further since the exceptional chain is ''extremal'', n_1 and m_1 satisfy one of the following four conditions.

$$n_1 < 11 \qquad (1.7)$$

$$n_1 = 11 \text{ and } m_1 = 16 \qquad (1.8)$$

$$n_1 = 12 \text{ and } m_1 = 17 \qquad (1.9)$$

$$n_1 \geq 11 \text{ and } m_1 > \binom{n_1-7}{2} + 11 \qquad (1.10)$$

Also the number of cut vertices in the exceptional chain (not counting the vertex by which the chain is attached to the rest of the graph) is given by :

$$\delta_{n_1,m_1} = \begin{cases} 0 & \text{if } m_1 \geq \binom{n_1-1}{2} + 2 \\ 1 & \text{if either } n_1 \leq 7 \text{ and } m_1 \leq \binom{n_1-1}{2}+1 \\ & \quad \text{or } n_1 = 8 \text{ and } m_1 = 11 \\ & \quad \text{or } n_1 \geq 8 \text{ and } \binom{n_1-4}{2}+8 \leq m_1 \leq \binom{n_1-1}{2}+1 \\ 2 & \text{if } n_1 \geq 8 \text{ and } \binom{n_1-5}{2}+9 \leq m_1 \leq \binom{n_1-4}{2} + 7 \\ 3 & \text{if } n_1 \geq 9 \text{ and } m_1 \leq \binom{n_1-5}{2}+8 \end{cases} \qquad (1.11)$$

If $\ell \geq 0$ exists such that n_1, m_1 defined by (1.3) and (1.4) satisfy conditions (1.5), (1.6) and one of the conditions (1.7) to (1.10), then ℓ is unique and the final graph we obtain has $4\ell + 1 + \delta_{n_1,m_1}$ cut vertices where δ_{n_1,m_1} is given by (1.11). If such a ℓ does not exist, then our final graph has no cut vertex. Thus we have proved the following

Theorem 1.17 If a non-negative integer λ exists such that n_1, m_1 defined by (1.3) and (1.4) satisfy the conditions (1.5), (1.6) and one of the conditions (1.7) to (1.10), then $A(n,m,3) = 4\lambda + 1 + \delta_{n_1,m_1}$ where δ_{n_1,m_1} is given by (1.11). Otherwise $A(n,m,3) = 0$.

Now the following result is not difficult to prove : a connected graph on n vertices, with m edges, with minimum degree ≥ 3 and with (exactly) q cut vertices exists if and only if $0 \leq q \leq A(n,m,3)$ and q is even if $m = \frac{3n}{2}$.

We determine the value of $A(n,2n,4)$ by a procedure similar to that used to prove theorem 1.17. Thus we start with an arbitrary connected graph G on n vertices which is regular of degree four. Obviously then there are exactly two pieces with respect to any cut vertex x and there are two edges incident to x in each of these pieces. Suppose now that there is a block C containing at least three cut vertices of the graph. Then a vertex of C has degree two within C if and only if it is a cut vertex of the graph. Now the vertices of C which have degree four within C can be easily dropped and a terminal block can be enlarged with the free vertices and edges thus obtained. Then C will be a cycle and if it has more than three vertices, we can reduce

C until we get a triangle. Thus finally every block of the graph containing at least three cut vertices of the graph is a triangle. Now collect all the triangle-blocks together so that no block containing only two cut vertices of the graph separates two triangle-blocks. Next collect all the terminal chains at one place leaving exactly one block for each of the others. Now reduce the number of vertices in each of the terminal blocks of the non-exceptional chains to six. If the exceptional chain has more than two blocks, we increase the number of triangle-blocks, so we may take that the number of vertices in the exceptional chain (including the vertex by which the chain is attached to the rest of the graph) is at most twelve. Thus we have proved the following

<u>Theorem 1.18</u> If $n < 11$, then $A(n,2n,4) = 0$. Otherwise, let λ be the integer such that $6 \leq n - 7\lambda - 5 \leq 12$. Then $A(n,2n,4) = 2\lambda + 1 + \delta$ where

$$\delta = \begin{cases} 0 & \text{if } n - 7\lambda - 5 < 11 \\ 1 & \text{otherwise} \end{cases}$$

The following result can be deduced easily from theorem 1.18 : a connected regular graph of degree four on n vertices with (exactly) p cut vertices exists if and only if $0 \leq p \leq A(n, 2n, 4)$.

If $A'(n,m,d)$ denotes the maximum number of cut vertices in a (not necessarily connected) graph G on n vertices with m edges and with minimum degree $\geq d$, it can be easily proved that $A'(n,m,d) = A(n,m,d)$ provided $m \geq n-1$. This is because if G is not connected, choose a neutral edge (x,y). If there is an isolated vertex z in the graph, suppress the edge (x,y) and join x to z. Otherwise there exists an edge (z,t) in another component, then suppress the edges (x,y), (z,t) and join x to z and y to t. The new graph has fewer components than G and at least as many cut vertices as G.

Now we can easily prove the following result. Let p be the number of cut vertices in a graph on n vertices, with m edges and with minimum degree $\geq d$. Then the range of p is given by (assuming $n > 4$ to avoid some trivialities):

1. if $d = 0$ and $m < n - 1$, then $0 \leq p \leq m - 1$
2. if $d = 0$ and $m \geq n - 1$, then $0 \leq p \leq r(n,m)$
3. if $d = 1$ and $[\frac{n}{2}]^+ \leq m < n-1$, let $m - [\frac{n}{2}]^+ = \delta$.

Then

if n is even and $\delta = 1$, then $p = 2$

if n is even and $\delta \neq 1$, then $0 \leq p \leq 2\delta$

if n is odd, then $0 \leq p \leq 2\delta + 1$

4. if $d = 1$ and $m \geq n-1$, then $0 \leq p \leq r(n,m)$
5. if $d = 2$, then $0 \leq p \leq A(n,m,2)$
6. if $d = 3$, then $0 \leq p \leq A(n,m,3)$ and p is even if $m = \frac{3n}{2}$
7. if $d = 4$ and $m = 2n$, then $0 \leq p \leq A(n,2n,4)$.

Next we consider the determination of the value of $B(n,m,d)$. It is trivial to prove that $B(n,m,2)$ is $n-2$ or 0 according as $m = n-1$ or $m = n$.

To determine $B(n,m,3)$, we start, as before, with an arbitrary connected graph G on n vertices with m edges and with maximum degree ≤ 3. Then by a sequence of modifications we reduce the graph to one with $B(n,m,3)$ cut vertices such that at each stage the graph remains connected, with n vertices, m edges and with maximum degree ≤ 3. The following are the important stages of this reduction. First the graph is reduced until there are exactly two pieces with respect to any cut vertex and there is no block containing more than two cut vertices of the graph. Next the blocks are reduced until each of them, except possibly one, consist of one of the graphs of figure 1.4 or a block on five vertices with seven edges. Finally the graph is reduced to one of the graphs shown in figures 1.6 and 1.7.

If G has three pieces with respect to some cut vertex x, remove one of the pieces, then suppress x by homeomorphism, create a vertex x' by homeomorphism in a terminal block and attach the piece deleted from x at x'. Thus we reduce the graph until there are exactly two pieces with respect to any cut vertex. Suppose now that there is a block C containing at least three cut vertices of the graph and C is not a triangle. Then we may assume that C has a Hamiltonian cycle and two cut vertices x_1, x_2 of the graph are adjacent on this cycle. Now transfer the other piece at x_1 and the vertex x_1 to a terminal block. We reduce the graph thus until every block containing at least three cut vertices of the graph is a triangle. Now if there exists such a triangle-block C, suppress the other piece A at one of the vertices of C. If there is a neutral vertex in a terminal block with degree < 3, attach A there. Otherwise a terminal block B has $2k+1$ vertices and $3k+1$ edges, $k \geq 2$. Removing one vertex and two edges from B, get a neutral vertex in it with degree < 3 and attach A there. Then enlarge the triangle C to a block on four vertices with five edges suitably. Thus the graph is reduced until every block contains at most two cut vertices.

Suppose now that a middle block (i.e., a block containing two cut vertices of the graph) C has at least four vertices. If there is a vertex of C with degree two, drop it by homeomorphism and create a vertex by splitting a cut edge. If C has six or more vertices of which only two have degree two within C, then remove two vertices and three edges from C and enlarge a terminal block with them. Thus finally every middle block is one of the three graphs shown in figure 1.4. If the graph has two triangle-blocks, they can be combined as

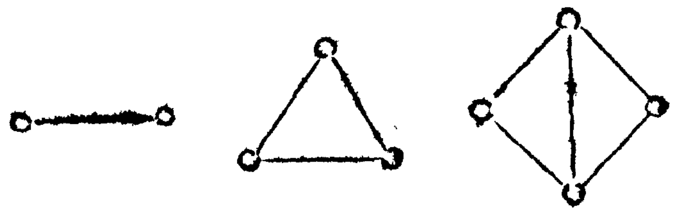

Figure 1.4

shown in figure 1.5.

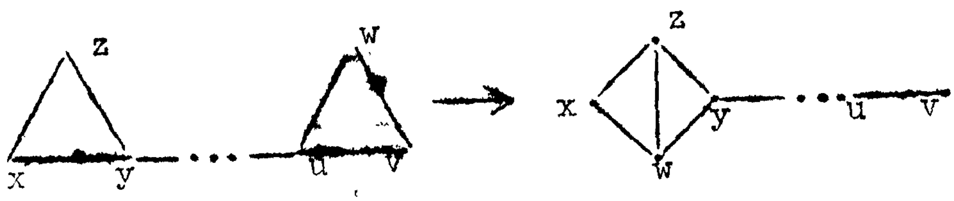

Figure 1.5

Now suppose that there is no pendant vertex. Then one of the terminal blocks, say T_1, can be reduced easily until it has five or less vertices. If now T_1 has a neutral vertex of degree two and is different from the graphs of figure 1.4, it can be reduced further.

<u>Case (i)</u> $m < [\frac{3n}{2}]$. Then if T_1 has five vertices and seven edges, remove one vertex and two edges from it suitably. Now if there is a neutral vertex of degree two in the graph, enlarge the block containing that vertex with the free vertex and the two free edges. Otherwise there are two cut vertices x and y in the graph, each with degree two. We now make x, y adjacent, then join the free vertex to x and y. Thus we reduce T_1 until it is one of the graphs shown in figure 1.4. Now reduce the other terminal block T_2 also until it is one of the graphs of figure 1.4 or all its vertices have degree three. If now T_2 has more than five vertices, then either there is a cut vertex of degree two or there is a block on three vertices or T_1 is an edge, so T_2 can be reduced further. Thus we finally reduce the graph to one of the three graphs shown in figure 1.6.

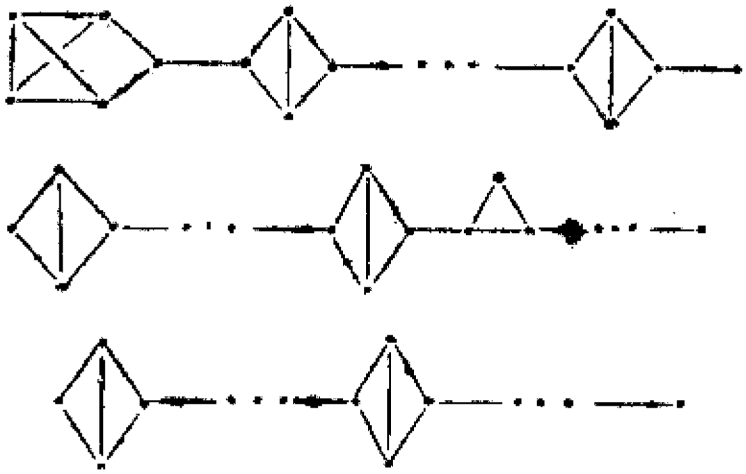

Figure 1.6

Case (ii) $m = [\frac{3n}{2}]$. Then the graph can easily be reduced until it is one of the four graphs shown in figure 1.7.

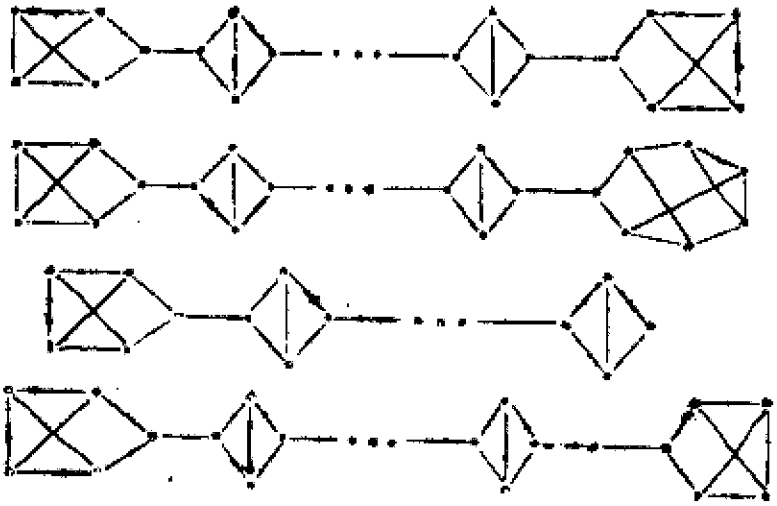

Figure 1.7

Thus we have proved the following

Theorem 1.19 $B(n,m,2)$ is $n-2$ or 0 according as $m = n-1$ or $m = n$. Also

$B(n, \frac{3n}{2}, 3) = \max\{0, 2t-4\}$ where $n = 4t-2$ or $4t$.

f $k \geq 1$, then

$$B(n, \frac{3n-k}{2}, 3) = \begin{cases} \max\{0, 2t+k-3\} & \text{if } n = 4t+k \\ \max\{0, 2t+k-4\} & \text{if } n = 4t+k-2 \end{cases}$$

If $B'(n,m,d)$ is the maximum number of cut vertices in a (not necessarily connected) graph G on n vertices with m edges and with maximum degree $\leq d$, then it can be easily proved that $B'(n,m,d) = B(n,m,d)$ provided $m \geq n-1$. Hence we have the following result.

Let p be the number of cut vertices in a graph on n vertices with m edges and with maximum degree $\leq d$. Then the range of p is given by (assuming $m \neq 3$ in 3 and 4 below):

1. if $d = 0$ or $d = 1$, then $p = 0$
2. if $m < n-2$ and $d = 2$, then $0 \leq p \leq m-1$
3. if $m = n-2$ and $d = 2$, then $0 \leq p \leq m-4$ or $m-2 \leq p \leq m-1$
4. if $m = n-1$ and $d = 2$, then $p = m-1$ or $0 \leq p \leq m-4$
5. if $m = n$ and $d = 2$, then $p = 0$

6. if $m < n-1$ and $d \geq 3$, then $0 \leq p \leq m-1$

7. if $n-1 \leq m < \frac{3n}{2}$ and $d=3$, then $0 \leq p \leq B(n,m,3)$

8. if $m = \frac{3n}{2}$ and $d=3$, then $0 \leq p \leq B(n,m,3)$ and p is even.

The reduction procedure used above for small values of does not seem to be useful for solving the general problem. However we make the following conjecture : if d is odd, then

$$B(n,\tfrac{nd}{2},d) = \max\{0, 2t-4\} \text{ if } (d+1)(t-1) < n \leq (d+1)t.$$

1.5 Cut edges with bounds on degrees

In this section we consider the determination of the range of the number of cut edges in a graph on n vertices with m edges in the following two cases :

(1) the minimum degree $\geq d$

(2) the maximum degree $\leq d$.

The problem is solved when $d \leq 3$ and is trivial if d is even and $m = \frac{nd}{2}$.

Let $E(n,m,d)$, $F(n,m,d)$ denote the maximum number of cut edges in the cases (1) and (2) respectively, under the additional restriction that the graph be connected.

First we shall consider case (1). From theorem 1.16, the following result can be easily deduced : a connected graph on n vertices with m edges, with minimum degree ≥ 2 and with exactly p cut edges exists if and only if $0 \leq p \leq E(n,m,2)$ where

$$E(n,m,2) = \begin{cases} r-1 & \text{if } r \geq 2,\ k \geq 2,\ \ell = 2 \\ r-2 & \text{if } r \geq 2,\ k \geq 2,\ \ell > 2 \\ 0 & \text{if } r \leq 1, \text{ or } k=1 \end{cases}$$

This is because, if a graph has $p \neq 0$ cut edges and the minimum degree ≥ 2, then it has at least $p + 1$ cut vertices.

To determine $E(n,m,3)$ we use a method similar to that used to prove theorem 1.17. So let G be an arbitrary connected graph on n vertices with m edges and with minimum degree ≥ 3. If there are two blocks, each on more than two vertices and with a common vertex, combine them into a single block. Now reduce the graph as in the proof of theorem 1.17 until with respect to any cut vertex x, either there are three pieces and the degree of x is three or there are two pieces. Next reduce all blocks containing at least three cut vertices of the graph until they are triangles. If such a triangle exists, shrink it to a vertex and enlarge a terminal block with the two free vertices and three free edges. If now a block C on more than two vertices separates two vertices x,y each of which is a cut vertex with three pieces, remove C to one end as shown in figure 1.8.

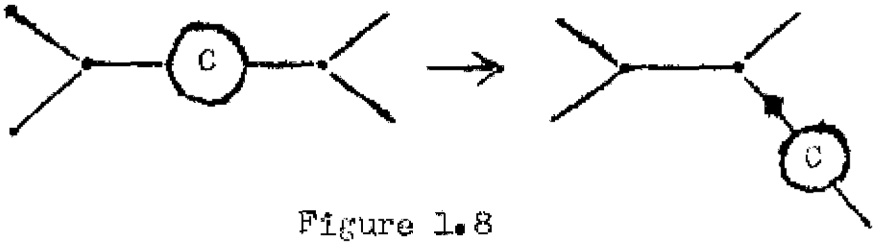

Figure 1.8

The graph can thus be reduced until it consists of a tree T such that all its non-pendant vertices have degree three, with ''terminal chains'' attached at its pendant vertices. Collect all the terminal chains at one place so that all of them except one consist of just one block. Let now $m = \frac{3n+k}{2}$ and $p = \min(k, \ell + 1)$ where ℓ is the number of vertices of the tree T with degree = 3. Then reduce p of the non-exceptional terminal chains to complete graphs on four vertices and the rest of the non-exceptional terminal chains to blocks on five vertices with seven edges.

If now the exceptional terminal chain has more than three blocks, it can be replaced by a graph of the type shown in figure 1.9. Here x is the vertex by which the exceptional chain is attached to the rest of the graph and A is a block on four or five vertices. In the final graph we thus obtain

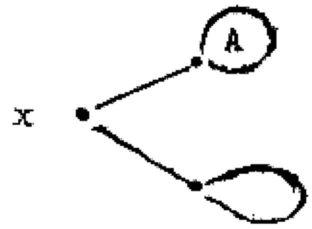

Figure 1.9

we may take that the exceptional chain has at most three blocks and cannot be replaced by a graph with a greater number of cut edges. If now n_2, m_2 are the number of vertices and the number of edges respectively in the exceptional chain, then obviously n_2, m_2 are given by (1.12), (1.13) and satisfy the conditions (1.14) and (1.15). Here ℓ (≥ 0) denotes the number of vertices x of the graph such that degree of $x = 3$ and there are three pieces with respect to x, and $p = \min(k, \ell + 1)$ where $m = \frac{3n+k}{2}$.

$$n_2 = n - \{\ell + 4p + 5(\ell + 1 - p)\} = n-6\ell + p-5 \qquad (1.12)$$

$$m_2 = m - (2\ell+1) - 6p - 7(\ell + 1 - p) = m - 9\ell + p-8 \qquad (1.13)$$

$$\frac{2+3(n_2-1)}{2} \leq m_2 \leq \binom{n_2}{2} \qquad (1.14)$$

$$n_2 \geq 4 \qquad . \qquad (1.15)$$

Further since the exceptional chain is ''extremal'', n_2 and m_2 satisfy one of the following three conditions.

$$n_2 < 9 \qquad (1.16)$$

$$n_2 = 9 \quad \text{and} \quad m_2 = 13 \qquad (1.17)$$

$$n_2 \geq 9 \quad \text{and} \quad m_2 > \binom{n_2-5}{2} + 8 \qquad (1.18)$$

Also the number of cut edges in the exceptional chain is given by

$$\varepsilon_{n_2,m_2} = \begin{cases} 0 & \text{if either } n_2 < 8 \\ & \quad \text{or } n_2 \geq 8 \text{ and } m_2 > \binom{n_2-4}{2}+7 \\ 1 & \text{otherwise.} \end{cases} \qquad (1.19)$$

Thus we have proved the following

Theorem 1.20 If a non-negative integer λ exists such that n_2, m_2 defined by (1.12) and (1.13) satisfy the conditions (1.14), (1.15) and one of the conditions (1.16) to (1.18), then $E(n,m,3) = 2/$ [illegible] ε_{n_2,m_2} where ε_{n_2,m_2} is given by (1.19). Otherwise $E(n,m,3) = 0$.

Now it is not difficult to prove that a connected graph on n vertices, with m edges, with minimum degree ≥ 3 and with exactly p cut edges exists if and only if $0 \leq p \leq E(n,m,3)$.

It is trivial to prove that $E(n,m,d) = 0$ if d is even and $m = \frac{nd}{2}$.

If $E'(n,m,d)$ is defined as the maximum number of cut edges in a (not necessarily connected) graph on n vertices with m edges and with minimum degree $\geq d$, then it can be easily proved that $E'(n,m,d) = E(n,m,d)$ provided $m \geq n-1$. We thus have the result : let p be the number of cut edges in a connected graph on n vertices with m edges and with minimum degree $\geq d$. Then the range of p is given by :

1. if $d = 0$ and $m \leq n-1$, then $0 \leq p \leq m-3$ or $p = m$
2. if $d = 0$ and $m > n-1$, then $0 \leq p \leq r(n,m)$
3. if $d = 1$ and $[\frac{n}{2}]^{+} \leq m \leq n-1$, then $n-m <$ [illegible] m
4. if $d = 1$ and $m > n-1$, then $0 \leq p \leq r(n,m)$
5. if $d = 2$, then $0 \leq p \leq E(n,m,2)$

6. if $d=3$, then $0 \leq p \leq E(n,m,3)$

7. if d is even and $m = \frac{nd}{2}$, then $p = 0$.

To prove that if a graph G has n vertices, m edges and minimum degree ≥ 1, then it has at least $n-m$ cut edges, let p be the number of cut edges in G. Then the number of neutral edges is $m-p$ and there are at most $m-p$ vertices of G incident to some neutral edge. Thus there are at least $n-m+p$ vertices of G covered by the cut edges, so $p \geq \frac{n-m+p}{2}$, i.e., $p \geq n-m$.

Next we consider the determination of the range of the number of cut edges in a graph on n vertices with m edges and with maximum degree $\leq d$. It is trivial to prove that a graph on n vertices with m edges, with maximum degree ≤ 2 and with exactly p cut edges exists if and only if $p = 0$ when $m = n$ and $0 \leq p \leq m-3$ or $p = m$ when $m \leq n-1$.

To determine the value of $F(n,m,3)$, we start with any connected graph G' on n vertices with m edges and with maximum degree ≤ 3. Then with respect to any cut vertex x, either there are three pieces (and degree of $x = 3$) or there are two pieces. If there is a block containing three or more cut vertices of the graph, it can be first reduced to a triangle and then shrunk to a vertex as in the proof of theorem 1.20. If there is either a block on at least three vertices or a cut vertex of degree two, separating two cut vertices x and y each with three pieces, it can be removed to one end.

Thus we reduce the graph until it consists of a tree T such that all its non-pendant vertices have degree three, with ''terminal chains'' attached at the pendant vertices of T. Now collect the terminal chains at one place so that each of the rest consists of just one block or is empty. Now the blocks of the non-exceptional terminal chains can be reduced easily until they are one of the two graphs shown in figure 1.10. If there is an edge as a terminal block, it can be shifted to the exceptional chain by homeomorphism. At this stage, each of the terminal chains, excepting one, either consists

Figure 1.10

of the graph G_2 of figure 1.10 or is empty. Now let λ be the number of cut vertices of the graph, each with three pieces. Also let $p = \min(\lambda + 1, [k])$ where $m = \frac{3n}{2} - k$. If a cut vertex with three pieces cannot be obtained from the terminal chain, then p of the non-exceptional terminal chains can be made empty. If now n_3, m_3 denote the number of vertices and the number of edges in the exceptional chain respectively, they are given by (1.20) and (1.21) and satisfy (1.22).

$$n_3 = n - 6\lambda + 4p - 5 \tag{1.20}$$

$$m_3 = m - 9\lambda + 7p - 8 \tag{1.21}$$

$$n_3 - 1 \leq m_3 \leq \min \left[\frac{3n_3-1}{2}, \binom{n_3}{2}\right] \tag{1.22}$$

If further the exceptional chain is extremal, then n_3, m_3 satisfy one of the conditions (1.23) to (1.27)

$$n_3 \leq 2 \tag{1.23}$$

$$n_3 = 3 \quad \text{and} \quad m_3 = 3 \tag{1.24}$$

$$n_3 = 4 \quad \text{and} \quad m_3 = 4 \tag{1.25}$$

$$n_3 = 5 \quad \text{and} \quad m_3 = 6 \tag{1.26}$$

$$4 \leq n_3 \leq 9 \quad \text{and} \quad m_3 = \left[\frac{3n_3 - 1}{2}\right] \tag{1.27}$$

Also the number of cut edges in the exceptional chain is given by

$$\lambda_{n_3,m_3} = \begin{cases} 0 & \text{if } n_3 = 1 \\ & \text{or } n_3 = 3 \\ & \text{or } n_3 = 4 \text{ and } m_3 = 5 \\ & \text{or } n_3 = 5 \text{ and } m_3 = 7 \\ & \text{or } n_3 = 7 \text{ and } m_3 = 10 \\ 1 & \text{otherwise.} \end{cases} \qquad (1.28)$$

Thus we have proved the

<u>Theorem 1.21</u> If a non-negative integer ℓ exists such that n_3, m_3 defined by (1.20) and (1.21) satisfy (1.22) and one of the conditions (1.23) to (1.27), then $F(n,m,3) = 2\ell + 1 + \lambda_{n_3,m_3}$ where λ_{n_3,m_3} is given by (1.28). Otherwise $F(n,m,3) = 0$.

Now it is not difficult to prove that a connected graph on n vertices, with m edges, with maximum degree ≤ 3 and with exactly q cut edges exists if and only if $q = n-1$ if $m = n-1$; $0 \leq q \leq F(n,m,3)$ if $m \geq n$, and q is even if $m = \frac{3n}{2}$.

When graphs, not necessarily connected, are considered, we have the following result : let p be the number of cut edges in a graph on n vertices, with m edges and with maximum degree $\leq d$. Then the range of p is

1. if $d = 1$, then $p = m$

2. if $d \geq 2$ and $m \leq n-1$, then $0 \leq p \leq m-3$ or $p = m$.

3. if d is even and $m = \frac{nd}{2}$, then $p = 0$.

4. if $d = 3$ and $m \geq n$, then $0 \leq p \leq F(n,m,3)$

   and p is even if $m = \frac{3n}{2}$.

The problem for general d has not been solved, but perhaps it is easier than the corresponding problem for cut vertices considered in section 1.4.

CHAPTER 2

LOCALLY RESTRICTED GRAPHS

The power (or the connectivity index) of a vertex x of a connected graph G is the number of connected components of $G-x$. We shall denote the power of a vertex x by $p(x)$ and by the power $p(G)$ of a graph G we mean the maximum power of a vertex of G.

In this chapter, we study the existence of locally restricted graphs, i.e., graphs with given powers and degrees. The maximum power of a connected graph on n vertices with m edges is also determined.

2.1 Graphs with given powers

In this section we obtain necessary and sufficient conditions for the existence of a connected graph with $p_1, p_2, \ldots, p_n$ as the powers of its vertices.

Lemma 2.1 A tree with $q_1, q_2, \ldots, q_n$ as the powers of its vertices exists if and only if $\sum_{i=1}^{n} q_i = 2(n-1)$. Also then any connected graph with $q_1, \ldots, q_n$ as the powers of its vertices is a tree.

Proof : It is evident that the power of a vertex x of a tree T coincides with the degree of x in T. So it is enough to prove that if $\sum_{i=1}^{n} q_i = 2(n-1)$, then a tree T with degrees $q_1, \ldots, q_n$ exists.

Take a vertex $a_{0,1}$. Then take q_1 new vertices $a_{1,1}, a_{1,2}, \ldots, a_{1,q_1}$ and join each of them to $a_{0,1}$. At the i-th stage, $i \geq 2$, take q_i-1 new vertices $a_{i,1}, a_{i,2}, \ldots, a_{i,q_i-1}$ and join each of them to $a_{i-1,1}$, provided $q_i-1 \geq 1$. Suppose i_0 is the largest integer i such that $q_i-1 \geq 1$. Then it can be easily shown that

$$1 + q_1 + (q_2-1) + \ldots + (q_{i_0}-1) = n$$

so that the above construction is possible and gives a tree T with $q_1, q_2, \ldots, q_n$ as the degrees of its vertices.

Theorem 2.2 Let $p_1, p_2, \ldots, p_n$ be positive integers. Then there exists a connected graph G on n vertices with $p_1, \ldots, p_n$ as the powers of its vertices if and only if

$$\sum_{i=1}^{n} p_i \leq 2(n-1). \qquad (2.1)$$

<u>Proof</u>: If $p_1, p_2, \ldots, p_n$ are the powers of the vertices of a connected graph G, take a spanning tree T of G. The power of any vertex in G is not greater than the power of the same vertex in T. Hence the inequality (2.1) follows from lemma 2.1.

Conversely, let $p_1, p_2, \ldots, p_n$ be positive integers satisfying the condition (2.1). Let $k = 2(n-1) - \sum_{i=1}^{n} p_i$. Then $0 \leq k \leq n-2$. Now without loss of generality, we may and do assume that $p_1 \geq p_2 \geq \cdots \geq p_n$. Define a new sequence $q_1, q_2, \ldots, q_n$ by

$$q_j = p_j + 1 \quad \text{for} \quad i = 1, \ldots, k$$

$$q_j = p_j \quad \text{for} \quad i = k+1, \ldots, n.$$

Then $\sum_{i=1}^{n} q_i = 2(n-1)$ and let T be a tree with $q_1, q_2, \ldots, q_n$ as the powers of its vertices, constructed by the method of lemma 2.1.

If $k = 0$, the proof of the theorem is complete, so let $k \geq 1$. Then it is obvious that $i_0 \geq k$. Let i_1 be the largest integer i such that $q_i - 1 \geq 2$. The case $p_1 = 1$ is trivial, so we take $i_1 \geq 1$. We consider two cases now.

<u>Case (i)</u>. $i_1 \geq k$. Then join $a_{i,1}$ to $a_{i,2}$ for $i=1,\ldots,k$.

<u>Case (ii)</u>: $i_1 < k$. Then join $a_{i,1}$ to $a_{i,2}$ for $i = 1,\ldots,i_1$ and join $a_{i,1}$ to $a_{i_1,2}$ for $i = i_1+1,\ldots,k$.

Now it can be easily verified that the resulting graph has $p_1,p_2,\ldots,p_n$ as the powers of its vertices. This completes the proof of the theorem.

<u>Theorem 2.3</u> Let $p_1, p_2,\ldots,p_n$ be positive integers and $m \geq n$. Then the following two conditions are necessary and sufficient for the existence of a connected graph G on n vertices with m edges and with $p_1,p_2,\ldots,p_n$ as the powers of its vertices

$$\sum_{i=1}^{n} p_i \leq 2(n-1) \tag{2.1}$$

$$m \leq \binom{k+2}{2} + n-k-2 \tag{2.2}$$

where $k = 2(n-1) - \sum_{i=1}^{n} p_i$.

<u>Proof</u>: The necessity of condition (2.1) was proved in theorem 2.2. To prove the necessity of (2.2), let G be a connected graph on n vertices with m edges such that $p_1,\ldots,p_n$ are the powers of the vertices of G. If t is

the number of blocks in G, it can be easily proved by induction on t that $\sum_{i=1}^{n} p_i = n+t-1$. Thus $k = n-t-1$.

Now from the proof of lemma 1.2, we have

$$m \leq \binom{n-t+1}{2} + t-1 = \binom{k+2}{2} + n-k-2.$$

To prove sufficiency, let conditions (2.1) and (2.2) be satisfied and let $p_1 \geq p_2 \geq \cdots \geq p_n$. Then construct a graph H with $p_1,\ldots,p_n$ as the powers of its vertices by the method of theorem 2.2. If $k = 1$, then $m = n$ and H has m edges. So let $k \geq 2$. We consider two cases.

<u>Case (i)</u>. $i_1 \geq k$. Then remove the edges incident to the vertices $a_{1,2}, a_{2,2},\ldots,a_{k-1,2}$ and join each of these vertices to $a_{k-1,1}$ and $a_{k,1}$. The powers of the vertices of the graph are not altered by this. Next replace the block on the $k+2$ vertices $a_{1,2}, a_{2,2},\ldots,a_{k-1,2}, a_{k,1}, a_{k,2}, a_{k-1,1}$ by an elementary cycle C on the same vertices. The graph H_1 thus obtained has n edges. Now if we write $m = n + \ell$ then by (2.2), $\ell \leq \binom{k+2}{2} - k - 2$, so ℓ new edges can be added to the cycle C of H_1.

<u>Case (ii)</u>. $i_1 < k$. The case $p_1 = 1$ is trivial, so let $i_1 \geq 1$. If $i_1 = 1$, then the $k+2$ vertices $a_{0,1}, a_{1,1},\ldots,a_{k,1}, a_{1,2}$ form a block in H. If $i_1 > 1$,

then remove the edges incident to the vertices $a_{1,2}, a_{2,2}, \dots,$ $a_{i_1-1,2}$ and join each of these vertices to $a_{k-1,1}$ and $a_{k,1}$. Then we get a block on the k+2 vertices $a_{1,2}, \dots, a_{i_1,2},$ $a_{i_1-1,1}, \dots, a_{k,1}$. Now this block can be replaced by a cycle and the construction completed as in case (i). This completes the proof of the theorem.

2.2 Maximum power of a graph

In this section we determine the maximum power of a connected graph on n vertices with m edges.

Theorem 2.4 The maximum power of a connected graph on n vertices with m edges is $r+1$ where $r = r(n,m)$ is given by (1.1).

Proof: Let G be any connected graph on n vertices with m edges. If t is the number of blocks in G, obviously $p(G) \leq t$. Now by theorem 1.3 it follows that $t-1 \leq r$. Hence $p(G) \leq r+1$. To construct a graph which attains the power $r+1$, take any biconnected graph on $n-r$ vertices with $m-r$ edges and attach a claw with r edges at one of its vertices. This completes the proof of the theorem.

The following result can be deduced easily from theorem 2.4 : a connected graph on n vertices with m edges and with power p exists if and only if $1 \leq p \leq r(n,m) + 1$ when $m \geq n$, and $p = n-1$ when $m = n-1$.

<u>Theorem 2.5</u> Let $r = r(n,m)$ be given by (1.1). Then the connected graphs on n vertices with m edges and with power $r+1$ are the following :

(1) a biconnected graph on $n-r$ vertices with $m-r$ edges to which a claw with r edges is attached at some vertex

(2) a complete graph on $n-r-1$ vertices to which a claw with $r-1$ edges and a triangle are attached at some vertex.

<u>Proof</u>: Let G be a connected graph on n vertices with m edges and with power $r+1$ attained by a vertex x. Then the $r+1$ pieces with respect to x are biconnected. Arranging them in the form of a chain, we get a graph with r cut vertices, hence its structure is given by theorem 1.8. Now the present theorem follows easily.

2.3 Graphs with given degrees

In this section we obtain necessary and sufficient conditions for the existence of a graph with given degrees $d_1, d_2, \ldots, d_n$. This problem was solved earlier by Tutte[19], Erdos and Gallai [6], and Hakimi [9]. But here we deduce the conditions from the theorems of Ryser, Gale and Fulkerson by simple considerations.

Let $d_1, d_2, \ldots, d_n$ be non-negative integers. Without loss of generality we may and do assume throughout this section that $d_1 \geq d_2 \geq \cdots \geq d_n$.

<u>Theorem 2.6</u> If loops are allowed, a graph with degrees $d_1, d_2, \ldots, d_n$ exists if and only if

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} d_i^* \quad \text{for} \quad k = 1, 2, \ldots, n-1 \tag{2.3}$$

where d_j^* is the number of d_i which are greater than or equal to j.

<u>Proof:</u> Necessity of the condition (2.3) can be verified easily.

To prove sufficiency, let $d_1, d_2, \ldots, d_n$ be non-negative integers satisfying (2.3). Then by a theorem of Gale [8] and Ryser [17], there exists a (0,1)-matrix $A = (a_{ij})$ with row sum vector and column sum vector equal to $(d_1, d_2, \ldots, d_n)$. We will now show that from **A** can be obtained a symmetric (0,1)-matrix B with the same row sums and column sums as **A**. We prove this by induction on n. The result is trivial for n=2, so assume the result for n-1 and let **A** be of order n. Suppose now that $a_{1i} \neq a_{i1}$ for some i. Then we assume that there exist indices i, j, $2 \leq i < j \leq n$ such that $a_{1i} = a_{j1} = 0$ and $a_{1j} = a_{i1} = 1$, for otherwise A^T can be considered instead of **A**. Since $d_i \geq d_j$, there exists an integer k, $2 \leq k \leq n$, such that $a_{ki} = 1$ and $a_{kj} = 0$. Now interchange 0 and 1 in the four cells (1,i), (1,j), (k,i), (k,j). In the new matrix, $a_{1i} = a_{i1}$ and $a_{1j}=a_{j1}$. Repeating this as long as necessary, we get a matrix with the same row sums and column sums as **A** and in which the first column is the transpose of the first row. Now the matrix obtained by deleting the first row and the first column can be replaced by a symmetric matrix by the induction hypothesis and we get the required matrix B. Evidently the matrix B is equivalent to a graph (loops allowed) with degrees $d_1, d_2, \ldots, d_n$ and this completes the proof of the theorem.

__Lemma 2.7__ Let $\sum_{i=1}^{n} d_i$ be even and let $A = (a_{ij})$ be a (0,1)-matrix with row sum vector and column sum vector equal to $(d_1, d_2, \ldots, d_n)$. Further let k be an integer such that $a_{kk} = 1$, $a_{ii} = 0$ if $i \neq k$, $a_{k,k+1} = a_{k+1,k} = 0$ and $d_k = d_{k+1} + 1$. Then there exists a (0,1)-matrix B with the same row sums and column sums as **A** and with trace 0.

__Proof:__ Suppose B does not exist. Then by a theorem of Fulkerson [7], there exists an integer i_0 such that $1 \leq i_0 \leq n$ and

$$\sum_{i=1}^{i_0} d_i > \sum_{i=1}^{i_0} d_i^{**} \qquad (2.4)$$

where d_j^{**} is the number of d_i such that $i < j$ and $d_i \geq j-1$ plus the number of d_i such that $i > j$ and $d_i \geq j$. Let row $M = (m_{ij})$ be the matrix with **1**'s in the first d_i places of the **i-th** row excepting that on the principal diagonal and with 0's elsewhere. Then the right side expression in (2.4) is the number of 1's in the first i_0 columns of M. Hence it follows that $i_0 = k$ and $a_{ij} = m_{ij}$ whenever $j \leq k$ and $(i,j) \neq (k,k)$. **Also** by (2.4), $a_{kj} = 1$ if $j \leq k$. Thus $d_k \geq k$, and $d_{k+1} \leq k-1$ since $m_{k+1,k} = 0$. So $d_k = k$, $d_{k+1} = k-1$ and

$$a_{ij} = \begin{cases} 1 & \text{if } i \leq k,\ j \leq k,\ i \neq j \\ 0 & \text{if } i \geq k+1,\ j \geq k+1. \end{cases}$$

Thus the total number of 1's in A is odd, a contradiction which proves the lemma.

<u>Theorem 2.8</u> A graph with degrees $d_1, d_2, \ldots, d_n$ exists if and only if

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} d_i^{**} \qquad \text{for } i = 1, \ldots, n \tag{2.5}$$

and

$$\sum_{i=1}^{n} d_i \text{ is even} \tag{2.6}$$

where d_j^{**} is the number of d_i such that $i < j$ and $d_i \geq j-1$ plus the number of d_i such that $i > j$ and $d_i \geq j$.

<u>Proof</u> : The necessity of the condition (2.5) was proved by Fulkerson [7] and the necessity of (2.6) is evident.

To prove sufficiency, let $d_1, \ldots, d_n$ satisfy the conditions (2.5) and (2.6). The result is trivial for $n=2$, so assume the result for $n-1$ and let $n \geq 3$. Then by Fulkerson's theorem [7] , there exists a (0,1)-matrix $E=(e_{ij})$ with row sum vector and column sum vector equal to $(d_1, d_2, \ldots, d_n)$ and with trace zero.

Suppose the corresponding elements of the first row and the first column of E differ in 2p places. If $p > 1$, we may assume that $e_{1i} = e_{j1} = e_{k1} = 0$, $e_{i1} = e_{1j} = e_{1k} = 1$, $e_{1\ell} = e_{\ell 1}$ for $\ell \leq i-1$, where $i < j < k$. If $r \neq k$ exists such that $e_{ri} = 1$ and $e_{rj} = 0$, then interchange 0 and 1 in the positions $(1,i)$, $(1,j)$, (r,i), (r,j). If such an r does not exist, but there exists $s \neq k$ such that $e_{si} = 1$ and $e_{sk} = 0$, then interchange 0 and 1 in the positions $(1,i)$, $(1,k)$, (s,i), (s,k). If r and s do not exist, then $e_{ik} = 0$ and $e_{ji} = e_{jk} = 1$, so we interchange 0 and 1 in the positions $(i,1)$, (i,k), $(j,1)$, (j,k). Thus the matrix is reduced until $p = 1$. Then let $e_{1i} = e_{j1} = 0$, $e_{1j} = e_{i1} = 1$ where $i < j$. If $t \neq j$ exists such that $e_{ti} = 1$ and $e_{tj} = 0$ then the interchange is obvious. Otherwise, $d_i = d_j$, $e_{ii} = e_{ij} = e_{jj} = 0$ and $e_{ji} = 1$. Now interchange 0 and 1 in the positions $(1,i)$, $(1,j)$, (j,i), (j,j). In the matrix F thus obtained, the first column is the transpose of the first row. Let G be the sub-matrix of F obtained by deleting the first row and the first column. The $(i-1)$th column sum of G is one more than the $(j-1)$th column sum of G. So by a permutation π of the rows and the same permutation π of the columns of G, we transform G to a matrix $\mathbf{A}$ of the form described in lemma 2.7. Then we get a (0,1)-matrix B with the same row sums and column sums as A and with trace zero. Now by the induction hypothesis there exists a symmetric (0,1)-matrix C with trace zero and with the same row sums and column sums as B. Now apply the inverse permutation of π to the rows and columns of C and attach the first row and the first column of F. Then we get a symmetric (0,1)-matrix with trace zero and with row sum vector and column sum vector equal to $(d_1, d_2, \ldots, d_n)$. This completes the proof of the theorem.

<u>Theorem 2.9</u> A connected graph with degrees $d_1, d_2, \ldots, d_n$ exists if and only if $d_1, d_2, \ldots, d_n$ satisfy conditions (2.5), (2.6) and the following two conditions

$$\sum_{i=1}^{n} d_i \geq 2(n-1) \tag{2.7}$$

$$d_n \geq 1. \tag{2.8}$$

<u>Proof:</u> Necessity of the conditions (2.7) and (2.8) is trivial. To prove sufficiency, let d_1, d_2, ..., d_n satisfy conditions (2.5) to (2.8). Then by theorem 2.8, there exists a graph G with degrees $d_1, d_2, \ldots, d_n$. If G is not connected, then edges (x,y) and (z,t) can be chosen from two distinct

components such that (x,y) belongs to a cycle and so is a neutral edge. Now suppress these two edges and join x to z and y to t. The resulting graph has fewer components and the same degrees. Repeating this process we finally obtain a connected graph with degrees $d_1, d_2, \dots, d_n$.

<u>Theorem 2.10</u> A biconnected graph with degrees $d_1, d_2, \dots, d_n$ exists if and only if conditions (2.5), (2.6) and the following two conditions are satisfied

$$d_1 \leq m - n + 2 \tag{2.9}$$

$$d_n \geq 2 \tag{2.10}$$

where $m = \frac{1}{2} \sum_{i=1}^{n} d_i$.

<u>Proof</u> : If G is a biconnected graph with degrees $d_1, d_2, \dots, d_n$ then (2.10) is evident. To prove (2.9) we observe that if x is a vertex with degree d_1, then $G-x$ has $m - d_1$ edges, $n-1$ vertices and is connected.

Conversely let $d_1, d_2, \dots, d_n$ satisfy the conditions (2.5), (2.6), (2.9) and (2.10). Then by theorem 2.9, there exists a connected graph H with degrees $d_1, d_2, \dots, d_n$.

Suppose H has a cut vertex x with degree d_1. Then by (2.9) and (2.10) it follows that at least one of the components of H-x has a cycle. Choose an edge (y,z) on this cycle and let (t,u) be a neutral edge in another piece with respect to x, such that $t \neq x$ and $u \neq x$. Now suppress the edges (y,z), (t,u) and join y to t and z to u. This decreases the power of x by unity and does not increase the power of any other vertex. By a repeated application of this procedure we get a graph in which every vertex has power unity, i.e., a biconnected graph. This completes the proof of the theorem.

Generalizing the problem of theorem 2.10, we may ask for necessary and sufficient conditions for the existence of a graph on n vertices with degree of the i-th vertex equal to d_i and power of the i-th vertex equal to p_i, $i = 1,2, \ldots, n$. But this problem seems to be much more difficult and is not yet solved.

CHAPTER 2

LINE GRAPHS

The line graph L(G) of a graph G is the graph whose vertices are the edges of G, two vertices of L(G) being joined by an edge if and only if the corresponding edges of G have a common vertex.

In this chapter we study some properties and characterizations of line graphs.

It is easy to see that L(G) is connected if and only if G is connected, provided G has no isolated vertices. So we consider only connected graphs in this chapter, for, if a graph is not connected, its components can be treated similarly.

Whitney [20] proved that an isomorphism between L(G) and L(H) induces an isomorphism between G and H if G has at least five vertices. From this it follows that G is uniquely determined by L(G) except when G is either a triangle or a claw on four vertices.

Krausz[13] has obtained the following characterization of line graphs.

<u>Theorem 3.1</u> A graph H is the line graph of some graph if and only if there exist complete subgraphs $C_1, C_2, \ldots, C_n$ such that

(1) every edge of H belongs to exactly one of $C_1, C_2, \ldots, C_n$

(2) every vertex of H belongs to exactly two of $C_1, C_2, \ldots, C_n$.

Further if $H = L(G)$ then G is isomorphic to the graph constructed as follows : take n vertices $v_1, v_2, \ldots, v_n$ and join v_i to v_j by an edge if and only if C_i and C_j have a common vertex in H.

It may be observed that condition (2) of theorem 3.1 can be replaced by :

(2') every vertex of H belongs to at most two of $C_1, C_2, \ldots, C_n$.

A triangle of a graph H is called <u>even</u> if every vertex of H is adjacent to either no vertex or two vertices of the triangle. Otherwise the triangle is <u>odd</u>.

We now determine the subgraphs $C_1, C_2, \ldots, C_n$ satisfying the two conditions of theorem 3.1 when H is a line graph.

Lemma 3.2 Let G be a graph on at least five vertices and let $C = \{u_1, u_2, \ldots, u_k\}$ form a complete subgraph of $L(G)$. Then $u_1, u_2, \ldots, u_k$ are all the edges of G incident to some vertex x (of G) if and only if one of the following three conditions is satisfied by C.

(1) C is a maximal complete subgraph of H and is odd if $k = 3$.

(2) $k = 2$, there is exactly one vertex v of H adjacent to both u_1 and u_2 and u_1, u_2, v form an even triangle.

(3) $k = 1$, u_1 and all its adjacent vertices form a complete subgraph of H which is odd if it is a triangle.

Further the vertex x is uniquely determined by C.

The proof of this lemma is a simple verification and is omitted.

Theorem 3.3 Let H be a graph on at least four vertices and different from those of figure 3.1. Let $C_1, C_2, \ldots, C_n$ be all the subgraphs of H satisfying one of the conditions (1) to (3) of lemma 3.2. Then H is a line graph if and only if $C_1, C_2, \ldots, C_n$ satisfy the two conditions of theorem 3.1.

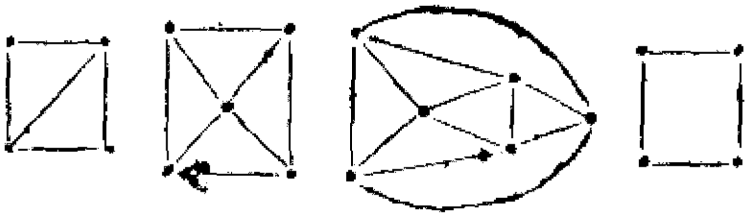

Figure 3.1

The proof of theorem 3.3 is similar to that of theorem 3.1 and is omitted. We note that the graphs of figure 3.1 are the line graphs of graphs on four vertices with at least four edges.

Harary and Ross [11] have given a method of determining all the maximal complete subgraphs, with at least three vertices, of a given graph. Using this and theorem 3.3, one can determine whether a given graph H is a line graph. We obtain below another method for determining whether H is a line graph.

<u>Lemma 3.4</u> A graph H is the line graph of a tree if and only if every block of H is a complete subgraph and no three blocks have a common vertex.

<u>Proof:</u> Let $H = L(G)$ where G is a tree. Then if vertices $u_1, u_2, \ldots, u_k$ of H form a cycle, the corresponding edges of G meet in a vertex since G has no cycle. Thus every block of H is complete. If (x,y) is an edge of G, all the edges of G incident to x are mutually adjacent

and all the edges of G incident to y are mutually adjacent. So there are at most two blocks of H containing any vertex of H.

Conversely let H be a graph such that each block is complete and no three blocks have a common vertex. Then by theorem 3.3 there exists a graph G such that $H = L(G)$. If edges $u_1, \ldots, u_k$ of G form a cycle in that order and $k \geq 4$, then the vertices $u_1, \ldots, u_k$ of H form an elementary cycle but u_1, u_3 are not adjacent, a contradiction. If G contains a cycle on three edges u_1, u_2, u_3 and G is not a triangle, then there exists an edge u of G adjacent to exactly two of the edges u_1, u_2, u_3. Then u_1, u_2, u_3, u form a biconnected subgraph, which is not complete, in H. This contradiction proves the lemma.

__Lemma 3.5__ Let G be a graph on at least five vertices and let u be an edge of G. In $L(G)$, let S_1 be the subgraph generated by the vertices other than u and let S_2 be the subgraph generated by the vertices adjacent to u. Then u is a neutral edge of G if and only if

(1) S_1 is connected

(2) the vertices of S_2 can be partitioned into two non-empty subgraphs C_1 and C_2 which satisfy one of the

conditions (1) to (3) of lemma 3.2 when considered as subgraphs of S_1.

Proof: Suppose that $u = (x_1, x_2)$ is a neutral edge of G. Let C_i be the set of all edges of G, other than u and incident to x_i, $i = 1,2$. Then S_1 is the line graph of G-u and conditions (1) and (2) are easily verified.

Conversely if conditions (1) and (2) are satisfied by a vertex u of L(G), then if u is an isthmus of G (an edge whose removal creates two components, each with at least two vertices), condition (1) is violated ; if u is a pendant edge of G, condition (2) is violated. So u is a neutral edge of G and this proves the lemma.

Let now $H = H_0$ be a graph on at least four vertices and different from those of figure 3.1. If there is a vertex u_i in H_i satisfying the conditions (1) and (2) of lemma 3.5, let $H_{i+1} = H_i - u_i$ (u_i may be chosen arbitrarily). Let H_k be the final graph thus obtained. If now H is a line graph, it is obvious that H_k is the line graph of a tree. Conversely let $H_k = L(G_k)$ where G_k is a tree. In H_{k-1}, let the vertices adjacent to u_{k-1} be partitioned into two sets C_1 and C_2 as in lemma 3.5. Then by lemma 3.2, C_1 and C_2 determine two vertices x_1 and x_2 of G_k. Since C_1 and C_2 are disjoint, x_1 and x_2 are not adjacent in G_k. If now G_{k-1} is the graph obtained from G_k by adding the new edge (x_1, x_2), then $H_{k-1} = L(G_{k-1})$. Proceeding similarly, we finally obtain a graph G_0 such that $H = L(G_0)$. Thus H is a line graph if and only if H_k is the line graph of a tree, conditions for which are given by lemma 3.4. This gives a method of determining whether a given graph H is a line graph of some graph.

<u>Theorem 3.6</u> $L(G)$ is regular if and only if either G is regular or G is bipartite and the vertices in the same part have equal degrees.

<u>Proof</u>: If part of the theorem follows from the fact that the degree of a vertex u of $L(G)$ is $d_1 + d_2 - 2$ where d_1 and d_2 are the degrees of the end vertices of the edge u of G.

To prove the converse, let $L(G)$ be regular. If $[x_1, x_2, x_3]$ is a path of length two in G, then degree of x_1 = degree of x_3. Hence it also follows that the degrees of two vertices x and y of G are equal whenever there is a path of even length joining x and y. Now the theorem follows easily.

<u>Theorem 3.7</u> A graph H is the line graph of a graph without triangles if and only if

(1) for any edge (u,v) of H, all the vertices adjacent to both u and v form a complete subgraph

(2) for any vertex u of H, all the vertices adjacent to u either form a complete subgraph or can be partitioned into two sets each of which forms a complete subgraph.

<u>Proof</u> : If $H = L(G)$ and G does not have any triangle, then (1) is satisfied since G has no triangles and (2) is satisfied since H is a line graph.

To prove the converse, let H satisfy the conditions (1) and (2). Let $C_1, C_2, \ldots, C_n$ be the subgraphs of H mentioned in theorem 3.3. Then obviously every edge of H belongs to at least one C_i. If an edge belongs to both C_i and C_j, then by (1), $C_i \cup C_j$ is complete, so $C_i = C_j$. Thus every edge of H belongs to exactly one C_i. Suppose next that u is a vertex of H such that all the vertices adjacent to u together with u form a complete subgraph C. Then u belongs to exactly two of $C_1, \ldots, C_n$ namely $\{u\}$ and C. Let now v be a vertex of H such that the vertices adjacent to v can be partitioned into two sets D_1 and D_2, each of them forming a complete subgraph. Let $E_i = \{v\} \cup D_i$, $i = 1,2$. If E_1 is an even triangle, then a vertex of D_2 must be joined to at least one of the vertices of D_1, and by (1), to both the vertices of D_1. Thus E_1 is not even, a contradiction. So E_1, E_2 are not even triangles and they are contained respectively in C_i and C_j for some i and j. Then it is easy to prove that v does not belong to C_k, $k \neq i, j$. Thus v belongs to exactly two of $C_1, \ldots, C_n$ and so H is a line graph, say $H = L(G)$. Now by (1) it follows that G does not contain any triangle and this proves the theorem.

It may be noted that in the statement of theorem 3.7, condition (2) can be replaced by the condition that H does not contain any claw on four vertices, but not by the condition that any vertex belongs to at most two blocks.

Now we will obtain a characterization of the line graph of the complete ℓ-partite graph $K_{n,n,\ldots,n}$. We assume throughout that $\ell \geq 2$, $n \geq 1$ and $k = (\ell-1)n > 3$.

For any two vertices x, y of a graph H, $\underline{\Delta(x,y)}$ denotes the number of vertices which are adjacent to both x and y.

<u>Theorem 3.8</u> A graph H is the line graph of the complete ℓ-partite graph $K_{n,n,\ldots,n}$ if and only if H satisfies the following four conditions .

(1) The number of vertices of H is $\binom{\ell}{2}n^2$.

(2) H is regular of degree $2(k-1)$.

(3) For any edge (u,v) of H, the subgraph generated by the vertices joined to both u and v is either a complete graph on $k-2$ vertices or the union of a complete graph on $k-2$ vertices and an isolated vertex.

(4) There is no complete subgraph on ℓ vertices, which is even if $\ell = 3$, such that all its edges (u,v) have $\Delta(u,v) = k-1$.

<u>Proof</u> : Necessity of the conditions (1) to (4) is easily verified.

To prove sufficiency, let H satisfy the conditions (1) to (4). From condition (3) it is evident that every edge of H belongs to exactly one clique on k vertices. Further every vertex belongs to exactly two such cliques by condition (2). So H is a line graph, say $H = L(G)$. Since the number of cliques of size k in H is ℓn from condition (1), it follows that G has ℓn vertices and is regular of degree k. If G has a complete subgraph on $\ell + 1$ vertices $x_1, x_2, \ldots, x_{\ell+1}$, then the vertices of H corresponding to the edges $u_i = (x_i, x_{i+1})$, $i = 1, 2, \ldots, \ell$, of G form a complete subgraph of H and $\Delta(u_i, u_j) = k-1$ for all i and j. This contradiction shows that G does not contain any complete subgraph on $\lambda + 1$ vertices. Now by Turan's theorem (see [15]) it follows that G is the complete λ-partite graph $K_{n,n,\ldots,n}$.

Theorem 3.9 Let $k = (\ell-1)n > 10$. Then a graph H is the line graph of the complete λ-partite graph $K_{n,n,\ldots,n}$ if and only if the following five conditions are satisfied by H.

(1) The number of vertices of H is $\binom{\lambda}{2}n^2$.

(2) H is regular of degree $2(k-1)$.

(3) For any edge (u,v) of H, $\Delta(u,v)$ is either $k-2$ or $k-1$.

(4) For any two non-adjacent vertices u and v of H, $\Delta(u,v) \leq 4$.

(5) The number of edges (u,v) of H such that $\Delta(u,v) = k-2$ is $\binom{\lambda}{2} n^2(n-1)$.

Proof : The necessity of the conditions (1) to (5) can be verified easily.

To prove sufficiency, let H satisfy the conditions (1) to (5) and let $k > 10$. Then it can be proved as in the proof of theorem 4.1 that every edge of H belongs to exactly one grand clique and every vertex of H belongs to exactly two grand cliques where a grand clique is a clique with at least $k-1$ vertices. Let now C_1 be a grand clique with $k+1$ vertices and C_2 a grand clique on $k-1$ vertices intersecting C_1 in a vertex u. Then by condition (3) it follows that every vertex of C_2-u is joined to at least one vertex of C_1-u but no vertex of C_1-u is joined to any vertex of C_2-u. This contradiction shows that every grand clique is of size k.

Now by theorem 3.1, H is a line graph, say $H = L(G)$. Then G has λn vertices and is regular of degree k. Let now (x,y) be any edge of G. Then the number of vertices of G not adjacent to y is $n-1$. Thus there are at most $n-1$ edges (x,z) of G such that y and z are not adjacent. Such an edge (x,z) of G gives an edge (u,v) of H with $\Delta(u,v) = k-2$ where $u = (x,y)$ and $v = (x,z)$. So by condition (5) it follows that the $n-1$ vertices $y_1,y_2,\ldots,y_{n-1}$ of G which are not adjacent to y are all adjacent to x. By the same argument, $y,y_1,y_2,\ldots,y_{n-1}$ form a maximal independent set. It is now evident that G is the complete λ-partite graph $K_{n,n,\ldots,n}$. This completes the proof of the theorem.

It may be observed that K is not assumed to be connected in theorems 3.8 and 3.9. Also the condition (5) of theorem 3.9 is not redundant since the line graph of any graph on λn vertices which is regular of degree $k = (\lambda-1)n$ satisfies the conditions (1) to (4).

When $\lambda=2$, it can be easily shown that theorem 3.9 remains valid whenever $k = n > 5$ if condition (4) is replaced by the condition that $\Delta(u,v) \leq 2$ for any two non-adjacent vertices u and v. Shrikhande [18] proved that the theorem is true whenever $n \neq 4$ and false for $n = 4$ if it is assumed that $\Delta(u,v) = 2$ for any two non-adjacent vertices u and v.

When $\lambda = 3$, it can be shown that theorem 3.9 remains valid whenever $k = 2n > 7$ (i.e., $n > 3$) if condition (4) is replaced by the condition that $\Delta(u,v) \leq 3$ for any two non-adjacent vertices u and v.

When $n = 1$, it can be shown that theorem 3.9 remains valid whenever $k = \lambda-1 > 7$. This result was obtained by Connor [3]. It is also known that the theorem is true whenever $\lambda \neq 8$ and false for $\lambda = 8$.

CHAPTER 4

SOME CHARACTERIZATIONS

In this chapter we obtain characterizations of the graph $G(\lambda, n, m)$ defined below and the line graph of a BIB design with $\lambda = 1$.

Let S be a set of vertices of a graph G and x another vertex. Then the subgraph generated by $S \cup x$ is called a claw if x is adjacent to every vertex of S and no two vertices of S are adjacent. The order of the claw is $|S|$ and x is called the vertex of the claw.

A clique is a complete subgraph.

If K, L are two sets of vertices of a graph G, a bridge between K and L is an edge with one end vertex in K-L and the other end vertex in L-K.

For any pair of vertices x, y, $\Delta(x,y)$ denotes the number of vertices simultaneously joined to x and y.

4.1 The graph $G(\lambda, n, m)$

In this section we obtain a characterization of the graph $G(\lambda, n, m)$ defined as follows : take λ disjoint sets $S_1, S_2, \ldots, S_\lambda$ each with n elements. The vertices of

$G(\lambda,n,m)$ are the unordered m-tuples of elements of $\bigcup_{i=1}^{\lambda} S_i$ where in any such m-tuple, at most one element of S_i can be present for $i = 1,2,\ldots,\lambda$. Two vertices of $G(\lambda,n,m)$ are joined by an edge if the corresponding m-tuples have an (m-1)-tuple in common.

Let $\lambda \geq m \geq 2$, $n \geq 1$ and $k = (\lambda-m+1)n$. If G is a $G(\lambda,n,m)$ graph then we will show that G satisfies the following six conditions.

(4.1) The number of vertices in G is $\binom{\lambda}{m}n^m$.

(4.2) For any edge (x,y), $\Delta(x,y)$ is either $k-2$ or $k+m-3$. An edge (x,y) will be said to be of type 1 or type 2 according as $\Delta(x,y) = k-2$ or $k+m-3$.

(4.3) G is regular of degree $m(k-1)$ and $m(n-1)$ of the edges incident to any vertex are of type 1.

(4.4) Let (x,y), (x,z) be edges and (y,z) not an edge. Let S be the subgraph generated by the vertices joined to both y and z. Then

$S = G_1$ if both (x,y) and (x,z) are of type 1;

$S = G_2$ if exactly one of (x,y), (x,z) is of type 1;

$S = G_2$ or $S = G_3$ if both (x,y) and (x,z) are of type 2, where G_1, G_2, G_3 are given in figure 4.1.

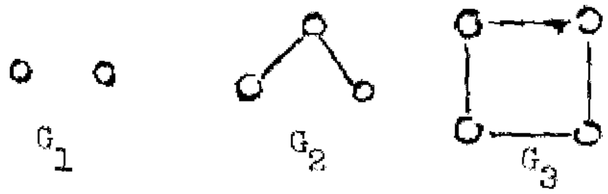

Figure 4.1

(4.6) There is no complete subgraph on more than $\max\{m+1, \lambda-m+1\}$ vertices with all edges of type 2.

(4.6) If (x,y), (x,z) are edges of type 1 and (y,z) is not an edge, then the number of vertices which are at distance 1 from y and at distance three from z is $(m-2)(k-1)$.

Let the elements of S_i be numbered $a_{1i}, a_{2i}, \dots, a_{ni}$. Then (4.1) is obvious. To prove (4.2), let without loss of generality, $x = (a_{11}, a_{12}, \dots, a_{1m})$ and $y = (a_{11}, a_{12}, \dots, a_{1,m-1}, a_{ij})$. If $j = m$ then the vertices adjacent to both x and y are $(a_{11}, a_{12}, \dots, a_{1,m-1}, a_{rs})$ where $1 \leq r \leq n$, $m \leq s \leq \lambda$ and $(r,s) \neq (1,m)$ or (i,m). If $j > m$, then the vertices adjacent to both x and y are

$$(a_{11}, a_{12}, \dots, a_{1,m-1}, a_{rs}), \quad 1 \leq r \leq n, \; m \leq \quad \leq \lambda,$$
$$(r,s) \neq (1,m) \text{ or } (i,j)$$

and

$$(a_{11}, \dots, a_{1,q-1}, a_{1,q+1}, \dots, a_{1,m-1}, a_{1m}, a_{ij}), \; 1 \leq q \leq m-1.$$

To prove (4.3), let $x = (a_{11}, a_{12}, \ldots, a_{1m})$. Then the vertices adjacent to x are

$$y = (a_{11}, \ldots, a_{1,q-1}, a_{ij}, a_{1,q+1}, \ldots, a_{1m}),\ 1 \leq q \leq m,$$

$$\text{either } j = m \text{ and } i \neq 1 \text{ or } j \geq m+1.$$

The edge (x, y) is of type 1 if and only if $j = m$ (and $i \neq 1$).

To prove (4.4), let $x = (a_{11}, a_{12}, \ldots, a_{1m})$ and $y = (a_{11}, \ldots, a_{1,m-1}, a_{ij})$. We consider two cases.

Case(i). $j = m$. Let $z = (a_{11}, \ldots, a_{1,q-1}, a_{rs}, a_{1,q+1}, \ldots, a_{1m})$, $1 \leq q \leq m-1$, either $s = q$ and $r \neq 1$ or $s \geq m + 1$.

First let $= q$ and $r \neq 1$. Then (x, y) and (x, z) are edges of type 1. Then the only vertex other than x which is adjacent to both y and z is $(a_{11}, \ldots, a_{1,q-1}, a_{rs}, a_{1,q+1}, \ldots, a_{1,m-1}, a_{im})$. Thus $S = G_1$. It is easy to see that two vertices are at distance d in G if and only if the corresponding m-tuples have exactly $m-d$ common elements. So the vertices which are at distance 1 from y and at distance three from z are $(a_{11}, \ldots, a_{1,h-1}, a_{fg}, a_{1,h+1}, \ldots, a_{1,m-1}, a_{im})$, $1 \leq h \leq m-1$, $h \neq q$, either $g = h$ and $f \neq 1$ or $g \geq m + 1$. This proves (4.6).

Next let $s \geq m+1$. Then (x,y) is of type 1 and (x,z) is of type 2. Then the vertices other than x and adjacent to both y and z are $(a_{11},\ldots,a_{1,m-1},a_{rs})$ and $(a_{11},\ldots,a_{1,q-1}, a_{rs}, a_{1,q+1},\ldots,a_{1,m-1},a_{1m})$, thus $S = G_2$.

<u>Case (ii).</u> $j \geq m+1$. Let $z = (a_{11},\ldots,a_{1,q-1}, a_{rs}, a_{1,q+1},\ldots,a_{1m})$, $1 \leq q \leq m-1$ and $s \geq m+1$. If $s = j$, then the vertices other than x which are adjacent to both y and z are $(a_{11},\ldots,a_{1,m-1},a_{rj})$ and $(a_{11},\ldots,a_{1,q-1},a_{1j},a_{1,q+1},\ldots,a_{1m})$, thus $S = G_2$. If $s \neq j$ then the vertices other than x which are adjacent to both y and z are $(a_{11},\ldots,a_{1,m-1}, a_{rs})$, $(a_{11},\ldots,a_{1,q-1},a_{rs},a_{1,q+1},\ldots,a_{1,m-1},a_{1j})$, $(a_{11},\ldots, a_{1,q-1}, a_{1j}, a_{1,q+1},\ldots,a_{1m})$, thus $S = G_3$. This completes the proof of (4.4).

Let $x = (a_{11},\ldots,a_{1m})$ and $y = (a_{11},\ldots,a_{1,m-1},a_{1,m+1})$. Then the vertices z such that (z,x) and (z,y) are edges of type 2 are $(a_{11},\ldots,a_{1,m-1},a_{1j})$, $j \geq m+2$ and $(a_{11},\ldots,a_{1,q-1},a_{1,q+1},\ldots,a_{1,m-1},a_{1m},a_{1,m+1})$, $1 \leq q \leq m-1$. Now it can be easily proved that if C is a clique containing x,y and such that all its edges are of type 2 then C has either $\lambda-m+1$ or $m+1$ vertices. Thus G satisfies (4.5).

<u>Theorem 4.1</u> If G is a graph satisfying the conditions (4.1) to (4.6), then $G = G(\lambda, n, m)$ provided $m \leq 5$ and

(4.7) $k > \max \{ 1 + \frac{3}{2}m(m+1),\ 2m^2-3m+5 \}$

<u>Proof</u> : If $m = 2$, the present theorem follows from theorem 3.9. So we assume the result for $m-1$ and prove it for m when $3 \leq m \leq 5$.

Let G be a graph satisfying conditions (4.1) to (4.5) and (4.7), and let $3 \leq m \leq 5$. We prove some properties of G in the following lemmas.

We call a clique K of G major if it contains at least $k-(m-1)^2$ vertices. A maximal major clique is called a grand clique. Then we have

<u>Lemma 4.2</u> G does not contain any claw of order $m+1$. Further every edge of G belongs to exactly one grand clique and every vertex of G belongs to exactly m grand cliques.

<u>Proof</u>: We first prove that G does not contain any claw of order $m+1$ by a method due to Bose [2] . Suppose the vertices $x, y_1, y_2, \dots, y_s$ form a claw with x as the vertex of the claw. Let $S = \{y_1, y_2, \dots, y_s\}$ and let Γ be the set of all vertices q such that q is adjacent to x and does not belong to S. Let $f(j)$ be the number of vertices

q in T such that q is adjacent to exactly j vertices of S, $j = 0,1,\ldots,s$. Then it can be proved as in [2] that

$$\sum_{j=0}^{s} f(j) = m(k-1)-s \tag{4.9}$$

$$\sum_{j=0}^{s} jf(j) = \sum_{i=1}^{s} \Delta(x, y_i) \tag{4.10}$$

$$\sum_{j=1}^{s} j(j-1)\, f(j) \leq 3s(s-1). \tag{4.11}$$

Considering (4.9) − (4.10) + $\frac{1}{2}$(4.11), we get

$$0 \leq m(k-1)-s-s(k-2) + \frac{3}{2}s(s-1).$$

If now we take $s = m+1$, we get $k \leq 1 + \frac{3}{2}m(m+1)$, a contradiction. Thus G does not contain any claw on $m+1$ vertices.

Considering (4.9) − (4.10), we get

$$f(0) - \sum_{j=1}^{s} (j-1)f(j) \geq m(k-1)-s-s(k+m-3). \tag{4.12}$$

Taking $s < m$ in (4.12), we get $f(0) > 0$. So any claw of order less than m can be extended to a claw of order m. Taking $s = m-1$ in (4.12), we get $f(0) \geq k-1-(m-1)^2$. So any claw of order $m-1$ can be extended to a claw of order m (with the same vertex) by using at least $k-1-(m-1)^2$ distinct vertices.

Let (x, y) be an edge of G. Then vertices $y_1, y_2, \ldots, y_{m-1}$ can be found such that $x, y, y_1, \ldots, y_{m-1}$ form a claw with x as the vertex. Let W be the set of all vertices w such that $x, w, y_1, \ldots, y_{m-1}$ form a claw with x as the vertex. Then $y \in W$ and $|W| \geq k-1-(m-1)^2$. Since G does not contain any claw of order $m+1$, W is complete. Now $W \cup x$ contains the edge (x, y) and is contained in a grand clique. Thus every edge belongs to at least one grand clique.

Suppose now that an edge (x, y) is contained in two distinct grand cliques K and L. Then vertices u and v can be found in K and L such that they are not adjacent. By (4.4), we have

$$|K \cap L| \leq \Delta(u, v) \leq 4.$$

Also by condition (4.2),

$$k+m-3 \geq \Delta(x, y) \geq |K \cup L| - 2.$$

Hence

$$2k-2(m-1)^2 \leq |K| + |L| = |K \cap L| + |K \cup L| \leq 4+k+m-1.$$

This contradicts (4.7) and proves that every edge of G belongs to exactly one grand clique.

We will denote by <x, y> the unique grand clique containing the edge (x, y).

Let $x, y_1, \ldots, y_m$ form a claw with x as the vertex and let T be defined as before. Let H_j^* be the set of all vertices q of T which are adjacent to y_j and not adjacent to y_i, $i \neq j$. Let $H_j = H_j^* \cup y_j \cup x$. If $f(j)$ is defined as before, $f(0) = 0$ since there is no claw of order $m+1$. Also from (4.9) and (4.10) we get

$$\sum_{j=1}^{m} (j-1) f(j) \leq m(m-1)$$

and so

$$f(1) \geq \sum_{j=1}^{m} f(j) - \sum_{j=1}^{m} (j-1) f(j) \geq m(k-m-1).$$

Now H_j^* and so H_j are complete subgraphs of G. Since $H_1^*, \ldots, H_m^*$ are disjoint,

$$|H_1^*| + \cdots + |H_m^*| = f(1) \geq m(k-m-1).$$

Let $K_j = \langle x, y_j \rangle$. If possible, let $|K_j| < |H_j|$. Since K_j is grand, H_j is contained in a grand clique K_j'. Thus K_j and K_j' are distinct grand cliques containing the edge (x, y_j), a contradiction. So $|K_j| \geq |H_j|$. Now

$$\sum_{j=1}^{m} |K_j-x| \geq \sum_{j=1}^{m} |H_j-x| = m + \sum_{j=1}^{m} |H_j^*| \geq m(k-m).$$

Suppose now that there is another grand clique K_{m+1} containing the vertex x. Then

$$m(k-1) \geq \sum_{j=1}^{m+1} |K_j-x| \geq m(k-m) + k - (m-1)^2 - 1.$$

Hence $k \leq 2m^2 - 3m + 2$, a contradiction. Thus every vertex of G belongs to exactly m grand cliques. This completes the proof of the lemma.

Consider the m grand cliques containing a vertex x. If one of these has $k+1$ or more vertices then by condition (4.2), all its edges are of type 2, a contradiction to condition (4.5). Since the degree of x is $m(k-1)$, it follows that every grand clique of G has exactly k vertices.

If K,L are two grand cliques of G we will write $(K,L) \in \lambda(G)$ if K and L have a common vertex. We will write $(K,L) \in \eta(G)$ if there is a one-one correspondence between the vertices of K and the vertices of L in such a way that a vertex of K and a vertex of L are adjacent if and only if they correspond. We also define a new graph thus : the vertices of G^* are the grand cliques of G and two vertices K and L of G^* are adjacent if and only if either $(K,L) \in \lambda(G)$ or $(K,L) \in \eta(G)$.

<u>Lemma 4.3</u> Let $K_1, K_2, \ldots, K_m$ be the grand cliques containing a vertex u. If v_1 is a vertex of K_1 such that (u, v_1) is of type 2, then v_1 is joined to exactly one vertex v_i of $K_i - u$ for $i = 2, \ldots, m$, and $v_1, v_2, \ldots, v_m$ form a clique. Further the m grand cliques containing the vertex v_1 are $K_1 < v_1, v_2>, \ldots, <v_1, v_m>$.

<u>Proof</u> : There exists a vertex w_i in K_i, $2 \leq i \leq m$, such that v_1 is not adjacent to w_i. Then by (4.4) it follows that v_1 is joined to at most one vertex of $K_i - u$, and by (4.2), v_1 is joined to exactly one vertex v_i of $K_i - u$.

If possible, let (v_i, v_j) be not an edge where $2 \leq i < j \leq m$. Then v_i is joined to a vertex v_j' of $K_j - u$ and v_j is joined to a vertex v_i' of $K_i - u$. Now u, v_1, v_i', v_j' are joined simultaneously to v_i and v_j contradicting (4.4).

The last part of the lemma follows from the fact that the grand cliques K_1, $\langle v_1, v_2 \rangle$, ..., $\langle v_1, v_m \rangle$ are all distinct.

Now from condition (4.3) and lemma 4.3 it also follows that if K is a grand clique containing a vertex u, then there are exactly $n-1$ vertices v in K such that (u,v) is of type 1.

We now consider the following condition (4.6)* : If two grand cliques K_1, K_2 of G have two bridges at least one of which is of type 1, then $(K_1,K_2) \in \eta(G)$.

Lemma 4.4 Let G satisfy condition (4.6)* also. Let (u_1,u_2) be an edge of type 1 and let K_1 be a grand clique containing u_1 but not u_2. Then there is exactly one grand clique K_2 containing u_2 such that $(K_1,K_2) \in \eta(G)$.

Proof : Let v_1 be a vertex of K_1 such that (u_1,v_1) is of type 1. Then by (4.4), v_1 is joined to exactly one vertex v_2 adjacent to u_2 and different from u_1. Taking $\langle u_2, v_2 \rangle$ as K_2, we have by (4.6)* that $(K_1,K_2) \in \eta(G)$. If $(K_1,K_3) \in \eta(G)$ for some other grand clique K_3 containing u_2, then $\Delta(u_2,v_1) > 2$, a contradiction which proves the lemma.

<u>Lemma 4.5</u> Let G satisfy condition (4.6)*. If vertices u_1, v_1, w_1 of G form a triangle and if (u_1, v_1) and (u_1, w_1) are of type 1, then (v_1, w_1) is also of type 1.

<u>Proof</u> : If possible let (v_1, w_1) be of type 2. By hypothesis, u_1, v_1 and w_1 belong to the same grand clique K_1. Let u_2 be a vertex not in K_1 such that (u_1, u_2) is an edge of type 1. Then by lemma 4.4, there is a grand clique K_2 containing u_2 such that $(K, K_2) \in \eta(G)$. Let v_2 and w_2 be the vertices of K_2 joined to v_1 and w_1 respectively. Since (v_1, w_1) is of type 2, w_1 is joined to a vertex x of $\langle v_1, v_2 \rangle$ other than v_1. Now by (4.4), x is joined to w_2. Since (w_1, w_2) is of type 1, w_1, w_2, x belong to the same grand clique. Thus $\langle v_1, v_2 \rangle$ and $\langle w_1, w_2 \rangle$ have a common vertex. Now by (4.6)*, $(\langle u_1, u_2 \rangle, \langle v_1, v_2 \rangle) \in \eta(G)$ and $(\langle u_1, u_2 \rangle, \langle w_1, w_2 \rangle) \in \eta(G)$. This is a contradiction to lemma 4.4 and proves lemma 4.5.

<u>Lemma 4.6</u> Let G satisfy condition (4.6)*. If $(K_1, K_2), (K_2, K_3) \in \eta(G)$ and there is a bridge (u_1, u_3) between K_1 and K_3, then $(K_1, K_3) \in \eta(G)$ and $(K_2, \langle u_1, u_3 \rangle) \in \lambda(G)$.

<u>Proof</u> : Let (u_1, u_3) be a bridge between K_1 and K_3, and let u_2 be the vertex of K_2 adjacent to u_1. If u_2

is not adjacent to u_3, let v_3 be the vertex of K_3 adjacent to u_2. Then K_3 and $< u_1, u_2 >$ have two bridges (u_1, u_3), (u_2, v_3) and (u_2, v_3) is of type 1. Hence $(K_3, < u_1, u_2 >) \varepsilon \eta(G)$, a contradiction to lemma (4.4). Thus u_2 is adjacent to u_3. Now by lemma 4.5, (u_1,u_3) is of type 1 and u_1,u_2,u_3 belong to the same grand clique. Let now w_2 be a vertex of K_2 such that (u_2, w_2) is of type 1 and let w_i be the vertex of K_i adjacent to w_2, i = 1, 3. Then (u_1,w_1) and (u_3,w_3) are also of type 1 and by lemma (4.4) it follows that w_1,w_2,w_3 belong to the same grand clique. Hence by (4.6)*, $(K_1,K_3) \varepsilon \eta(G)$ and this completes the proof of the lemma.

<u>Lemma 4.7</u> Let G satisfy conditions (4.1) to (4.5) and (4.7) with $m \leq 5$. Then condition (4.6) implies condition (4.6)*, and if $m \leq 4$, then condition (4.6)* implies condition (4.6).

<u>Proof</u>: Let G satisfy condition (4.6). Let K_1 and K_2 be two grand cliques with two bridges, one of which, say (u_1, u_2), is of type 1. Let $L_1 = <u_1,u_2>$, $L_2=K_2,L_3,\ldots,L_m$ be the grand cliques containing the vertex u_2. If v_1 is a vertex of K_1 such that (u_1, v_1) is of type 1, then by

(4.4), v_1 is joined to exactly one vertex other than u_2 of $\bigcup_{i=2}^{m} L_i$. Now by (4.6), this vertex belongs to L_2 and it easily follows that $(K_1, K_2) \in \eta(G)$.

To prove the converse, let G satisfy the condition (4.6)* and let $m \leq 4$. We will take $m = 4$, the case $m = 3$ is easier. Let K_i, $i = 1,2,3,4$, be the grand cliques containing a vertex u and let w be a vertex joined to vertices v_1 of K_1 and v_2 of K_2 by edges of type 1, where (u, v_1) and (u, v_2) are also of type 1. Then $(K_1, \langle w, v_2 \rangle) \in \eta(G)$ and $(K_2, \langle w, v_1 \rangle) \in \eta(G)$. Suppose now that there exists a vertex in $K_3 \cup K_4$, say x_3 of K_3, such that $d(w, x_3) \leq 2$. We consider two cases.

<u>Case (i)</u>. (u, x_3) is of type 1. Then let L_1, L_2 and L_4 be the grand cliques containing the vertex x_3 such that $(L_i, K_i) \in \eta(G)$ for $i = 1,2,4$. Then by lemma 4.6, w cannot be adjacent to any vertex of L_1 or L_2. This is a contradiction since by (4.4), w is joined to at least two vertices in distinct grand cliques containing x_3.

<u>Case (ii)</u>. (u, x_3) is of type 2. Then by lemma 4.3, x_3 is adjacent to a vertex x_i of K_i for $i = 1,2,4$. Now $d(w, x_3) \neq 1$, so let y be a vertex joined to both w and x_3.

If $y \in \langle x_3, x_1 \rangle$, then by lemma 4.4, $(\langle w, v_1 \rangle, \langle x_1, x_3 \rangle) \in \eta(G)$. So by lemma 4.6, $(K_2, \langle x_1, x_3 \rangle) \in \eta(G)$, a contradiction. Thus $y \notin \langle x_3, x_1 \rangle$ and similarly $y \notin \langle x_3, x_2 \rangle$. This is a contradiction since w is joined to at least two vertices in distinct grand cliques containing x_3.

Thus we have proved that every vertex of K_3 and K_4 (except u) is at distance 3 from w. This completes the proof of the lemma.

From now onwards we take G to be a graph satisfying conditions (4.1) to (4.7) with $3 \leq m \leq 5$.

<u>Lemma 4.8</u> If two grand cliques K_1 and K_2 of G have a bridge (u_1, u_2) of type 1, and there exist vertices $v_1 \in K_1 - u_1$ and $v_2 \in K_2 - u_2$ such that (u_1, v_1) is of type 1 and $d(v_1, v_2) \leq 2$, then $(K_1, K_2) \in \eta(G)$. If further (u_2, v_2) is of type 1 and (v_1, v_2) is not an edge, then there are exactly two vertices adjacent to both v_1 and v_2 and these belong to K_1 and K_2.

<u>**Proof**</u> : By lemma 4.4, there is a unique grand clique K containing the vertex u_2 such that $(K, K_1) \in \eta(G)$. If $K \neq K_2$, then by condition (4.6), $d(v_1, v_2) = 3$, a

contradiction. Thus $(K_1,K_2) \in \eta(G)$. If now (u_2,v_2) is of type 1 and (v_1,v_2) is not an edge, then v_1 is joined to a vertex x_2 of K_2, and v_2 is joined to a vertex x_1 of K_1. Then it follows from lemma 4.5 and condition (4.4) that $\Delta(v_1,v_2) = 2$ and the lemma is proved.

Since every vertex of G belongs to exactly m grand cliques and every grand clique has exactly k vertices, it follows that the number of vertices of G* is $\binom{\lambda}{m-1} n^{m-1}$.

Consider a grand clique K_1 of G. Let u be a vertex of K_1 and (u,v) an edge not in K_1 and of type 1. Then there is exactly one grand clique K containing the vertex v such that $(K_1, K) \in \eta(G)$. Now there are k vertices u in K_1 and for any fixed u there are (m-1)(n-1) vertices v. The k(m-1)(n-1) vertices v thus obtained are all distinct, for otherwise, a vertex outside K_1 will be joined to two vertices of K_1 by edges of type 1. If K is any grand clique such that $(K_1,K) \in \eta(G)$, then each of the k bridges taken as (u,v) gives rise to the same K. Thus the number of grand cliques K such that $(K_1,K) \in \eta(G)$ is (m-1) (n-1). Also the number of grand cliques interesting K_1 is k(m-1). Hence the degree of the vertex K_1 in G* is $(m-1)[(\lambda-m+2)n-1]$.

Let $(K_1,K_2) \varepsilon \eta(G)$. Then there are exactly k grand cliques intersecting both K_1 and K_2. There cannot be any grand clique K such that $(K, K_1) \varepsilon \lambda(G)$ and $(K, K_2) \varepsilon \eta(G)$. Let (u_1,u_2) be a bridge between K_1 and K_2. Take a vertex u_3 in the grand clique $< u_1, u_2 >$ such that (u_1,u_3) is of type 1. By lemma 4.4 there is a grand clique K_3 containing u_3 such that $(K_1K_3) \varepsilon \eta(G)$. Since K_2 and K_3 have a bridge (u_2,u_3), it follows by lemma 4.6 that $(K_2,K_3) \varepsilon \eta(G)$. This way we get $n-2$ grand cliques $K_3,K_4,\ldots,K_n$ such that (K_1,K_i), $(K_2,K_i) \varepsilon \eta(G)$ for $i = 3,4,\ldots,n$. Suppose now that K is a grand clique such that (K,K_1), $(K,K_2) \varepsilon \eta(G)$. Then by lemma 4.6, some vertex u of K is joined to u_1 and u_2, thus $u = u_i$ and $K = K_i$ for some $i = 3, 4,\ldots,n$. Thus $\Delta(K_1,K_2)$ in G^* is $k+n-2 = (\lambda-m+2)n-2$.

Let $(K_1,K_2) \varepsilon \lambda(G)$. Then there are exactly $k-1-(n-1) = (\lambda-m)n$ bridges between K_1 and K_2. Also there are $m-2$ other grand cliques containing the vertex common to K_1 and K_2. Thus the number of grand cliques intersecting both K_1 and K_2 is $(\lambda-m)n+m-2$. By lemma 4.4, there is no grand clique K such that (K,K_1), $(K,K_2) \varepsilon \eta(G)$. The number of grand cliques K such that $(K,K_1) \varepsilon \eta(G)$ and $(K,K_2) \varepsilon \lambda(G)$ is $n-1$. Thus $\Delta(K_1,K_2)$ in G^* is $(\lambda-m)n+m-2+2(n-1) = (\lambda-m+2)n + (m-1) - 3$.

Let K_1, K_2, K_3 be grand cliques of G such that $(K_1,K_2) \varepsilon \lambda(G)$, $(K_1,K_3) \varepsilon \eta(G)$ and (K_2,K_3) is not an edge in G^*. Let (u_1,u_3), (v_1,v_3) etc., be bridges between K_1 and K_3, and let u_1 be the vertex common to K_1 and K_2. By lemmas 4.4 and 4.6, $< u_1, u_3 >$ is the only grand clique intersecting both K_2 and K_3. Let now K be a grand clique of G such that $(K,K_2) \varepsilon \lambda(G)$ and $(K,K_3) \varepsilon \eta(G)$. If K intersects K_2 in a vertex u_2 other than u_1, then by lemma 4.6, u_1, u_2, u_3 belong to the same grand clique, a contradiction. Thus K intersects K_2 in u_1, so $K = K_1$. Let L be a grand clique intersecting K_3 and let $(L, K_2) \varepsilon \eta(G)$. Suppose that v_3 is the vertex common to L and K_3. Now through u_3 passes a grand clique K_4 such that $(K_2,K_4) \varepsilon \eta(G)$. Since L and K_4 have a bridge(v_3,u_3), by lemma 4.6, u_1, u_3, v_3 belong to the same grand clique, a contradiction. Thus $L = K_4$. It is evident that there is no grand clique K such that (K,K_2), $(K, K_3) \varepsilon \eta(G)$. Thus the vertices of G^* joined simultaneously to K_2 and K_3 are K_1, $< u_1, u_3 >$ and K_4, and it can be easily seen that the subgraph generated by these three vertices is G_2 of figure 4.1.

Let K_1, K_2, K_3 be grand cliques of G such that (K_1,K_2), $(K_1,K_3) \varepsilon \lambda(G)$ and (K_2,K_3) is not an edge in G^*.

Let K_1 meet the cliques K_2 and K_3 in u_2 and u_3 respectively. If (u_2,u_3) is of type 1, we get the situation described in the previous paragraph. So let (u_2,u_3) be of type 2. Then u_2 is joined to another vertex v_3 of K_3, and u_3 is joined to another vertex v_2 of K_2. Now by lemma 4.3, (v_2,v_3) is an edge. If there is another bridge (w_2,w_3) between K_2 and K_3, then another vertex x_2 of K_2 exists such that (x_2, w_3) is an edge. Now u_3, v_3, w_2, x_2 are adjacent simultaneously to v_2 and w_3 contradicting (4.4). Thus there are exactly four bridges between K_2 and K_3 and it can be easily verified that the grand cliques containing them form a cycle in G^*. It is evident that there is no grand clique K such that either $(K,K_2),(K,K_3) \in \eta(G)$ or $(K, K_2) \in \lambda(G)$ and $(K, K_3) \in \eta(G)$.

Let K_1,K_2,K_3 be grand cliques of G such that (K_1,K_2), $(K_1,K_3) \in \eta(G)$ and (K_2,K_3) is not an edge in G^*. Let (u_1,u_2), (v_1,v_2) etc., be bridges between K_1 and K_2, let (u_1,u_3), (v_1,v_3) etc. be bridges between K_1 and K_3. Let $K \neq K_1$ be any grand clique such that (K, K_2), $(K, K_3) \in \eta(G)$. Suppose that a vertex u of K is joined to u_2 of K_2 and v_3 of K_3. Then by lemma 4.8, $(\langle u_1,u_3\rangle,\langle u_2,u\rangle) \in \eta(G)$ and v_3

belongs to one of $\langle u_1, u_3 \rangle$ and $\langle u_2, u \rangle$, a contradiction. Thus if a vertex u of K is joined to u_2 of K_2, then u is joined to u_3 of K_3. Since $\Delta(u_2, u_3) = 2$, it follows that there is at most one grand clique $K \neq K_1$ such that $(K, K_2), (K, K_3) \in \eta(G)$. It is evident that there is no grand clique K such that either $(K,K_2), (K,K_3) \in \lambda(G)$ or $(K, K_2) \in \lambda(G)$ and $(K, K_3) \in \eta(G)$. Let now u_1, u_4 be the vertices joined to both u_2 and u_3; v_1, v_4 the vertices joined to both v_2 and v_3 etc. There is a grand clique U_4 containing the vertex u_4 such that $(K_2,U_4) \in \eta(G)$; there is a grand clique V_4 containing the vertex v_4 such that $(K_2,V_4) \in \eta(G)$ and so on. Suppose first that two of the grand cliques $U_4, V_4, \ldots$ are equal, say $U_4 = V_4$. Then there are two bridges of type 1 between U_4 and K_3, so $(U_4, K_3) \in \eta(G)$. Now it also follows that $U_4, V_4, \ldots$ are all equal. Suppose next that $U_4, V_4, \ldots$ are all distinct. Now

$$(\ell-m+2)n > 1 + \frac{3}{2}(m-1)m$$

and we have shown that G^* satisfies conditions (4.1) to (4.3) with m replaced by $m-1$. Also if K,L are vertices of G^* at distance two, then $\Delta(K,L) \leq 4$. So by the results of [2], G^* does not contain any claw of order m. Since

$k > m$, it follows that of the k grand cliques $U_4, V_4, \ldots,$ of G, two are joined by an edge in G^*. Since $(U_4, V_4) \notin \lambda(G)$, without loss of generality let $(U_4, V_4) \in \eta(G)$. If (u_2, x) is a bridge between K_2 and V_4, then by lemma 4.6, u_2, u_4 and x belong to the same grand clique. Now (u_4, x) is a bridge of type 1 between $< u_4, u_3 >$ and V_4, and $d(u_3, v_4) \leq 2$. So by lemma 4.8, $(<u_4, u_3>, V_4) \in \eta(G)$. But $(U_4, V_4) \in \eta(G)$, a contradiction. Thus K_1 and U_4 are the vertices of G^* adjacent to both K_2 and K_3 and it is easily seen that (K_1, U_4) is not an edge in G^*.

Let K_1 and K_2 be two grand cliques of G intersecting in a vertex u and let $K_3, \ldots, K_m$ be the other grand cliques through u. If (x_1, x_2) is a bridge between K_1 and K_2, then $< x_1, x_2 >$ cannot intersect K_i for $3 \leq i \leq m$. Now there are $\mathbb{K}-1-(n-1) = (\lambda-m)n$ vertices y in K_1-u such that (u, y) is of type 2. These can be partitioned into $\lambda - m$ sets $T_1, T_2, \ldots, T_{\lambda-m}$ such that two vertices of the same set are joined by an edge of type 1 and two vertices of different sets are joined by an edge of type 2. This is done by defining an equivalence relation as : $y \sim z$ if (y, z) is an edge of type 1. Now it is evident that if $y_1 \in T_i$, $z_1 \in T_j$, (y_1, y_2) and (z_1, z_2) are bridges between K_1 and

K_2, then $< y_1, y_2 >$ and $< z_1, z_2 >$ cannot intersect. Thus there are at most $\lambda - m$ bridges between K_1 and K_2 such that the grand cliques containing any two of them intersect. This proves that in G^* there is no clique on more than $\max \{ m + 2, \lambda - m + 2 \}$ vertices with all edges of type 2.

Let K_1, K_2, K_3 be vertices of G^* such that (K_1, K_2), (K_2, K_3) are edges of type 1 and (K_1, K_3) is not an edge. Let K_4 be the other vertex of G^* adjacent to both K_1 and K_3. Let (u_i, u_j), (v_i, v_j) etc., be bridges between K_i and K_j for $(i,j) = (1,2), (2,3), (3,4), (1,4)$. Let L be a vertex of G^* joined to both K_2 and K_3; let K be a vertex of G^* joined to K_1 but unjoined to K_2 and K_4. We will show that K and L cannot be joined in G^*.

<u>Case (i)</u>. $(K, K_1), (L, K_2) \in \lambda(G)$. Then $(L, K_3) \in \lambda(G)$ and let L be the grand clique containing the bridge (u_2, u_3) between K_2 and K_3. Now $K \neq < u_1, u_2 >$ and $K \neq <u_1, u_4>$. If $(K,L) \in \lambda(G)$, then K_1 and L have two bridges one of which is of type 1, so $(K_1, L) \in \eta(G)$, a contradiction. If $(K,L) \in \eta(G)$, let v_1 be the vertex common to K and K_1. Then v_1 is joined to a vertex of L and so $(K_1, L) \in \eta(G)$, a contradiction. Thus (K,L) cannot be an edge in G^* in this case.

Case (ii). $(K,K_1) \varepsilon \eta(G)$ and $(L,K_2) \varepsilon \lambda(G)$. As in case (i) above, let $L = \langle u_2, u_3 \rangle$. If $(K,L) \varepsilon \lambda(G)$, then the vertex common to K and L is joined to a vertex of K_1, so $(K_1,L) \varepsilon \eta(G)$, a contradiction. If $(K,L) \varepsilon \eta(G)$, then by lemma 4.6, $(K_1,L) \varepsilon \eta(G)$, a contradiction.

Case (iii). $(K,K_1) \varepsilon \lambda(G)$ and $(L, K_2) \varepsilon \eta(G)$. Let u be the vertex of L adjacent to u_2 of K_2 and u_3 of K_3. If $(K,L) \varepsilon \lambda(G)$, then by lemma 4.6, $(K_1,L) \varepsilon \eta(G)$ and (u_1,u) is an edge, a contradiction since $u \neq u_2$. If $(K,L) \varepsilon \eta(G)$, then the vertex common to K and K_1 is joined to some vertex of L, hence $(K_1,L) \varepsilon \eta(G)$ and (u_1,u) is an edge, a contradiction.

Case (iv). (K,K_1), $(L,K_2) \varepsilon \eta(G)$. If $(K,L) \varepsilon \lambda(G)$, then the vertex common to K and L is joined to some vertex of K_1, hence $(K_1,L) \varepsilon \eta(G)$, a contradiction. If $(K,L) \varepsilon \eta(G)$, let u be the vertex of L adjacent to u_2 of K_2 (and u_3 of K_3). Then there exists a vertex x of K such that x is adjacent to u_1 and u. Since $(\langle u_2, u_3 \rangle, \langle u_1, u_4 \rangle) \varepsilon \eta(G)$, $x \varepsilon \langle u_1, u_4 \rangle$. Hence by lemma 4.6, $(K, K_4) \varepsilon \eta(G)$, a contradiction.

This proves that G^* satisfies condition $(4.6)^*$.

Thus we have proved that G^* satisfies the conditions (4.1) to (4.5), (4.6)* and (4.7), with m replaced by $m-1$. Since $m-1 \leq 4$, by lemma 4.7, G^* satisfies (4.6). So by the induction hypothesis, $G^* = G(\lambda, n, m-1)$. Let $S_1, S_2, \ldots, S_\lambda$ be λ disjoint sets, each with n elements. Let $C_{m-1}(K)$ be the $(m-1)$-tuple of elements of $\bigcup_{i=1}^{\lambda} S_i$ corresponding to the vertex K of G^*. Let now u be a vertex of G and let $K_1, K_2, \ldots, K_m$ be the grand cliques of G containing u. Define

$$C_m(u) = C_{m-1}(K_1) \cup C_{m-1}(K_2).$$

Since (K_1, K_2) is an edge of type 2 in G^*, it follows that $C_m(u)$ is an m-tuple of elements of $\bigcup_{i=1}^{\lambda} S_i$ with at most one element from any S_i. If $m > 2$, we will now verify that $C_m(u)$ is uniquely defined. If $C_{m-1}(K_3) \not\subseteq C_m(u)$, then clearly $C_{m-1}(K_3) \supset C_{m-2} = C_{m-1}(K_1) \cap C_{m-1}(K_2)$. The number of $(m-1)$-tuples with at most one element from each S_i and containing the $(m-2)$-tuple C_{m-2} is $(\lambda - m + 2)\, n > m$. Thus there is a grand clique K of G not containing the vertex u such that $C_{m-2} \subseteq C_{m-1}(K)$. Then (K, K_i) is an edge in G^* for $i = 1,2,3$, a contradiction since $u \notin K$. Thus $C_{m-1}(K_3) \subseteq C_m(u)$ and it follows that the union of any two of the sets $C_{m-1}(K_i)$, $i = 1,2,\ldots,m$, equals the union of all of them. Since the number of $(m-1)$-tuples contained in $C_m(u)$ is exactly m, for any vertex u and any grand clique K of G, $C_{m-1}(K) \sqsubset C_m(u)$ if and only if $u \in K$.

Let now (u,v) be an edge of G and $K = \langle u, v \rangle$. Then $C_{m-1}(K) \sqsubset C_m(u)$ and $C_{m-1}(K) \sqsubset C_m(v)$. Thus the m-tuples $C_m(u)$ and $C_m(v)$ have a common $(m-1)$-tuple. Conversely if u and v are vertices of G such that $C_m(u)$ and $C_m(v)$ contain a common $(m-1)$-tuple C_{m-1}, then if K is the grand clique of G defined by $C_{m-1}(K) = C_{m-1}$, it follows that $u \in K$ and $v \in K$. Hence (u,v) is an edge of G. Thus $G = G(\lambda,n,m)$ and this completes the proof of theorem 4.1.

It may be noted that condition (4.5) of theorem 4.1 may be replaced by : there is no clique on $k+1$ vertices with all edges of type 2.

Corollary. If G is a graph satisfying the following conditions for $m \leq 5$, then $G = G(m,n,m)$.

(1) The number of vertices of G is n^m.

(2) G is regular of degree $m(n-1)$.

(3) For any edge (x,y) of G, $\Delta(x,y) = n-2$.

(4) If y and z are vertices of G at distance two, then there are exactly two vertices joined to both y and z; also the number of vertices which are at distance one from y and at distance three from z is $(m-2)(n-1)$.

(5) $n > 1 + \frac{m(m+1)}{2}$.

In this Corollary, a slight reduction in the hypothesis is made possible by the facts that there are edges of type 1 only and if $d(y,z) = 2$, then there are only two vertices joined to both y and z.

It is easily seen that $G(2,n,2)$ is the line graph of the complete bipartite graph $K_{n,n}$. It is also known to be equivalent to the graph of the L_2-association scheme and was first characterized by Shrikhande [18]. $G(3, n, 3)$, the cubic lattice graph was recently characterized by Laskar[14].

The graph $G(\Lambda, 1, m)$ was called the T_m-graph by Dowling. This was characterized for $m = 2$ by Connor; for $m=3$ by Bose and Laskar, and for general m by Dowling [4]. In Dowling's theorem, condition (4.5) of our theorem was not assumed and condition (4.4) was weakened to : $\Delta(y,z) \leq 4$ whenever $d(y,z) = 2$.

It may be remarked that theorem 4.1 will be proved for all m if it can be shown that when G satisfies conditions (4.1) to (4.7), then G^* satisfies condition (4.6) also with m replaced by $m-1$.

4.2 Line graph of a BIBD with $\lambda = 1$

In this section we obtain a characterization of the line graph of a BIB design with $\lambda = 1$.

A BIB design D with parameters b, v, r, k, λ consists of a set of v elements called treatments and a collection of b subsets called blocks such that each block contains k treatments, each treatment is contained in r blocks and every pair of treatments is contained in λ blocks.

The graph $H(D)$ of a BIB design D is the bipartite graph defined as follows. Take a vertex for each treatment and a vertex for each block. Join two vertices if and only if one is a treatment and the other is a block containing the treatment. The line graph $L(D)$ of D is by definition the line graph of $H(D)$.

Let $vr = bk$, $v-1 = r(k-1)$, $r \geq k \geq 2$. If $r = k$, the design reduces to a finite projective plane and a characterization of its line graph was obtained by Dowling and Laskar [5] . So let $r > k$.

If G is the line graph of a BIB design with parameters $b,v,r,k,1$, then the following four conditions are easily verified :

(1) The number of vertices in G is vr .

(2) G is regular of degree $r+k-2$.

(3) For any edge (x,y) of G, $\Delta(x,y)$ is either $r-2$ or $k-2$.

(4) If x,y are two vertices of G at distance two, $\Delta(x,y) = 1$.

In the following theorem we show that the above four properties characterize the line graph of a BIB design with $\lambda = 1$, provided $r - 2k + 1 < 0$.

<u>Theorem 4.2</u> If G is a graph satisfying the condition (1) to (4) above and if $r - 2k + 1 < 0$, then G is the line graph of a BIB design with parameters $b,v,r,k,1$.

<u>Proof:</u> Suppose that G contains a claw on the four vertices x, y_1, y_2, y_3 with x as the vertex of the claw. Let T be the set of all vertices of G other than y_1, y_2, y_3 which are adjacent to x. Let $f(j)$ be the number of vertices of T each of which is joined to exactly j of the vertices y_1, y_2, y_3. Then $f(2) = f(3) = 0$ by condition (4) and

$$f(0) + f(1) = |T| = r + k - 5,$$

$$f(1) = \sum_{i=1}^{3} \Delta(x, y_i) \geq 3(k-2).$$

Hence $r - 2k + 1 \geq 0$, a contradiction. Thus it follows that G does not contain any claw on four vertices.

Let (x, y_1) be any edge of G. Since $\Delta(x, y_1) < r+k-2$, there exists a vertex y_2 which is adjacent to x and is not adjacent to y_1. Let W_i be the set of all vertices joined to x and unjoined to $y_i (i = 1, 2)$. Then W_1, W_2 are complete subgraphs of G and by (4), they partition the vertices adjacent to x. If $\Delta(x, y_1) = r-2$, then $|W_1| = k-1$ and so $|W_2| = r-1$. By (3), it follows that no vertex of W_1 is joined to any vertex of W_2. The case $\Delta(x, y_1) = k-2$ is similar.

We call a maximal clique of G grand if it contains at least k vertices. By the results of the previous paragraph, every grand clique has either k or r vertices, every edge of G belongs to exactly one grand clique and every vertex of G belongs to exactly two grand cliques, one on k vertices and the other on r vertices. By (1) it follows that the number of grand cliques of size r is v and the number of grand cliques of size k is b. Now let H be the graph whose vertices are the $v+b$ grand cliques of G, two vertices of H being joined if and only if the corresponding cliques of G intersect. Since two grand cliques of the same size cannot intersect in G, H is bipartite. Let K be a grand clique of size r in G; $L_1, L_2, \ldots, L_r$ the grand cliques intersecting K, and $K, K_{i,1}, K_{i,2}, \ldots, K_{i,k-1}$ the grand cliques intersecting L_i. By condition (4), we see that $K_{ij} \neq K_{i',j'}$ whenever $(i,j) \neq (i', j')$. Since $v = 1+r(k-1)$, it follows that $K, K_{i,j}$ $(i = 1,2,\ldots,r;\ j = 1,2,\ldots,k-1)$ are all the grand cliques of size r. Thus any two grand cliques of size r are intersected by exactly one grand clique of size k. Taking the vertices of H corresponding to cliques of size r as treatments and the vertices of H corresponding to cliques of size k as blocks, it can be easily verified that

H is the graph of a BIB design D with parameters b,v,r,k,1. If we now make the correspondence

$$x \leftrightarrow (K, L)$$

between vertices of G and edges of H, where K,L are the grand cliques of G containing the vertex x, then it is obvious that G is the line graph of H. This completes the proof of the theorem.

Taking $r = k+1$, we get a characterization of the line graph of a finite affine plane. Another characterization of the line graph of a finite affine plane through the eigen values of its adjacency matrix has been obtained by Hoffman and Ray-Chaudhuri [12].

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