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OPTIMAL BLOCK DESIGNS IN ONE AND MULTI-WAY SETTINGS

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By

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Sunanda Bagchi

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Table 4.1.12. Chi-square values based on percent frequencies for intergroup differences with respect to axial triradius (d.f.3)

Dhangar Castes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1. AHIR	-	0.57	1.85	1.01	5.92	1.95	1.08	2.44	0.20	0.49	0.90	4.26	9.48*	0.30	1.45	1.20	3.71	1.25	0.35	0.56
2. DANGE		-	4.15	2.22	8.97*	1.86	3.14	4.50	1.32	1.93	1.87	3.60	2.46	0.26	2.63	2.29	5.41	3.14	0.36	0.67
3. GADHARI-DHENGAR			-	2.54	4.76	5.80	0.57	2.62	0.99	1.00	3.17	7.43	7.28	2.61	2.57	1.07	1.37	0.99	3.29	3.87
4. GADHARI-NIKHAR				-	4.92	3.56	1.03	0.47	1.27	1.22	0.65	8.33*	2.87	2.09	2.91	2.36	5.91	0.58	0.91	0.72
5. HANDE					-	5.46	2.92	4.44	4.63	3.32	3.43	9.00*	5.78	8.37*	2.83	8.76*	10.44*	5.04	7.81*	6.60
6. HATKAR						-	3.71	5.96	2.26	2.13	1.37	1.99	0.61	2.54	1.07	5.86	9.23*	5.22	2.42	1.74
7. KANNADE							-	1.22	0.52	0.28	1.20	6.71	4.35	2.23	1.49	1.72	3.41	0.49	21.4	2.15
8. KHATIK								-	2.39	2.11	1.71	11.54*	5.00	3.97	4.36	3.15	6.43	0.55	2.55	2.33
9. KHUTEKAR									-	0.12	1.06	4.18	3.30	0.72	0.95	1.13	2.93	1.07	1.03	1.24
10. KURMAR										-	0.79	4.40	3.04	1.34	0.62	1.77	3.61	1.08	1.48	1.45
11. LADSHE											-	5.38	1.09	2.07	1.24	3.59	7.17	1.73	1.18	0.59
12. MENDHE												-	4.54	3.83	2.43	7.45	8.60*	9.17*	5.52	5.46
13. SANGAR													-	3.61	2.36	7.11	11.82*	5.26	2.41	1.37
14. SHEGAR														-	2.37	1.10	3.36	2.24	0.50	1.07
15. TELANGI															-	3.96	5.67	3.25	2.77	2.37
16. THELLARI																-	1.11	1.17	7.78	2.81
17. UNNIKANKAN																	-	3.39	5.20	6.73
18. VARHADE																		-	1.73	1.99
19. ZENDE																			-	0.19
20. ZADE																				-

* Significant at 5% and below level.

CONTENTS

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CHAPTER 1

GENERAL INTRODUCTION AND SUMMARY

The study of optimality of block designs formally began with Wald (1943) proving a very important optimality property, designated as D-optimality of Latin Square Designs in a given physical set up. Next Ehrenfield (1955) proved the E-optimality property of Latin Square Designs. Since then, the work in the area has primarily consisted in evolving a number of useful optimality criteria for comparing block design in a reasonable set up given and characterizing and/or constructing designs satisfying the so called optimality criteria developed. But it was not until 1958, when the theory was given a proper and precise formulation with the problems defined in a systematic and standardized manner by Kiefer. Restricting to the class of connected block designs, Kiefer (1958, 1959) considered the general problem of estimation of a set of orthonormal treatment contrasts $\eta = P\tau$ and gave precise definitions of a number of standard optimality criteria including A-, D- and E-optimality, for measuring the performances of the least square estimates of these treatment contrasts from the design. Given the parameters b = number of blocks, k = constant block size and v = number of treatments, a BIBD, when it exists, was proved to be optimal with regard to all relevant optimality criteria defined by Kiefer (1958). At the same time Mote (1958) proved the E-optimality and Khirsagar (1958) the A- and D-optimality of the BIBDs independently in the same set up. Roy (1958) took up the problem of estimation of all elementary treatment contrasts and proved that the most efficient design for this problem for given b,k and v with regard to A-optimality criterion is a BIBD, when it exists. Kiefer (1975) defined a very general class of optimality criteria including the previously considered and widely used A-, D- and E-optimality criteria. He also generalised the BIBD's to define BBD's (which reduced to the former in the special case $k < v$) and proved their universal optimality property, i.e. their optimal property with regard to every optimality criterion belonging to the general class defined by him. The major part of this paper was devoted to the optimality study of designs in a two-way heterogeneity setting, which we discuss in a latter paragraph.

First result on the optimality of asymmetrical designs was established by Takeuchi (1961). Using a new technique, he proved the E-optimality of the GDD's with $\lambda_2 = \lambda_1 + 1$. Later, Conniffe and Stone (1974) derived an improved upper bound to the efficiency of a block design with regard to A-optimality criterion. Cheng (1978b) used a technique similar to that used by Conniffe and Stone (1974) to establish the optimality of the MBGDD's of type i.e. the GDD's with $m = 2$ and $\lambda_2 = \lambda_1 + 1$, with regard to a general class of optimality criteria termed by him ϕ_f- optimality criteria of type 1, which included the age old A-, D- and E-optimality criteria mentioned before. Linked Block Designs (LBD's) constitute an important class of useful asymmetrical designs. Sinha (1980) proved the ϕ_p- optimality of LBD's within a class $\mathcal{D}_r$ of proper, connected and equireplicate designs, given b,k, and v. Cheng (1980a) proved the E-optimality of LBD's and also of the duals of MBGDD's of type 1 within the class $\mathcal{D}$ of all proper and connected designs, given b,k and v.

Search for E-optimal designs in the asymmetrical case led research workers in this field to explore for such designs in the class of non-equireplicate designs. The recent contributions in this area include Cheng (1980a), Jacroux (1980a, 1980b), Constantine (1981). Cheng (1979a) also prepared a list of all D-optimal designs for $v = 4$ and the list was extended with a few significantly general results to $v \leq 6$ by Gaffke (1981).

As have been already stated, the two way setting in its generality was first considered by Kiefer (1975). In Kiefer's paper, both regular and nonregular settings were dealt with and optimality properties, viz, A-, D- and E-optimality properties of a class of symmetrical designs, termed Generalized Youden Designs (GYDs), were established within the class of proper and connected two-way designs given row size b_1, column size b_2 and v The constructional aspects of such designs were discussed by Kiefer (1975), Ruiz and Seiden (1974), Seiden and Wu (1978), Ash (1977) and several other authors. Cheng (1981a) defined a class of two way designs called Pseudo Youden Designs (PYDs) and proved their optimal behaviour in the special cases of two-way settings when the number of levels for both the heterogeneity directions were the same. Constructions of some series of such designs were also presented in the same paper and in Cheng (1981b).

The optimality problems of block designs under the assumption of multiway heterogeneity setting among the experimental units were handled by Cheng (1978a). Generalising the GYD's to the m-way setting, he defined Youden Hyperrectangles (YHR's) and proved their optimal behaviour very similar to those of the GYD's in the class of connected designs appropriate in the situation. The methods of construction for such designs were also discussed by Cheng (1979b). Cheng (1981a) extended PYD's to the m-way setting when the number of levels for each of the m heterogeneity directions was the same, and established that all the optimality properties of the YHR's were satisfied by the PYD's in the given context. The constructional aspects of the PYD's have been discussed in Cheng (1981a).

In all the papers mentioned so far, the model assumed is one of fixed effects, which means that the block effects in one or more heterogeneity directions and treatment effects included in the model are all fixed. While beginning with the investigation of how far these known optimality results hold under the changed context of mixed effects model, the present work also deals with the relevant optimality behaviour or efficiency of some known to be simple or easily constructible designs in the asymmetrical cases and handle the construction problems of a few series of block designs in multiway heterogeneity set up, whose optimality properties have been established. A chapterwise summary of the contents follows.

In Section 2.3 of Chapter 2, one-way heterogeneity situation is considered. Under mixed effects model, the fixed effects optimality properties of the BBD's given in Kiefer (1975) as well as those of the asymmetrical designs, viz., LBD's given in Sinha (1980) and Cheng (1980a), have been reestablished in Theorems 2.3.1.2, 2.3.2.2, 2.3.2.4. As regards the MBGDD's,

Theorem 2.3.2.6 states that the MBGDD's of type 1 are ϕ_f - optimal of type 1 within a class $\mathcal{D}_1$ of proper, connected and binary designs with given b,k and v under mixed effects model. So, Cheng's (1978b) result on ϕ_f - optimality of type 1 of MBGDD's of type 1 is yet to be established in general under mixed effects model. It may be noted that the same optimality property of the MBGDD's of type 1 within the restricted class $\mathcal{D}_r$ under mixed effects model is observed to hold by Khatri and Shah (1981).

In Section 2.4, the two-way heterogeneity setting is dealt with. In the two-way setting, two types of mixed effects model may arise, viz, one with exactly one of the effects, either row or column random. This model is termed, in the present work, mixed effects model of type 1 and the other with both effects random is termed mixed effects model of type 2. Under the mixed effects models of both the types, Kiefer's results known to be true under fixed effects model are observed to hold, as shown in Theorems

2.4.1.1, 2.4.2.2 and 2.4.2.6. Furthermore, in a particular situation under mixed effects model of type 2, a GYD turns out to be universally optimal, even in the nonregular setting (derived in Theorem 2.4.2.1). The optimality properties of the PYD's as shown in Cheng (1981a) are observed to hold in the same situations under mixed effects model of type 2, assuming the variances of both the row and column effects equal. The major part of the contents of Sections 2.3 and 2.4 has been published in Mukhopadhyay (1981b).

In Section 2.5, Cheng's (1978) multiway heterogeneity setting, where each one of all the possible level combinations of the heterogeneity directions or factors appears a constant number of times (≥ 1), has been extended to a balanced multiway heterogeneity setting. The latter setting does not necessarily involve all combinations of all the levels of the factors, but these present must make possible orthogonal estimation of the main effects of each factor. Thus the number of experimental units may be largely reduced in this so called balanced setting. With this setting, fixed effects model as well as mixed effects model with all factor effects random is studied. Generalising the YHR of Cheng (1978a), a design termed Balanced Youden Hyperrectangle (BYHR) is defined in the context of the present setting. Similarly, the PYD of Cheng (1981a) is extended to a Balanced Pseudo Youden Design (BPYD). All Cheng's (1978a, 1981a) results proved in the context of the complete setting with reference to the YHR's and PYD's have been found to hold for BYHR's and BPYD's respectively in the balanced multiway setting under the fixed effects and mixed effects models. As in the case of two-way setting, universal optimality has been proved for BYHR's under mixed effects model when the variances of factor effects satisfy a certain condition. The relevant results are given in Theorems 2.5.2.1, 2.5.3.1, 2.5.3.2, 2.5.3.6 and 2.5.3.7. The material of the Section 2.5 is primarily based on the paper by Mukhopadhyay, A.C. and Mukhopadhyay, Sunanda (1982).

Characterization of optimal designs in the asymmetrical case, particularly under mixed effects model, poses a relatively more difficult problem. But there are a large number of asymmetrical designs which have been in use for years for their nearly balanced structure as well as their algebraic simplicity. So it is interesting to note the relative performances of those designs in the absence of a symmetrical optimal design. The class of PBIBD's or their generalised version, viz PBBD's with two associate class association schemes is a class of such widely used block designs and more or less extensive tables for such designs are also available e.g. in Bose, Shrikhande and Clatworthy (1954) and Clatworthy (1956). Their relative performances were first studied by Sinha and Sinha (1969), with regard to A-optimality criterion. In Chapter 3, we extend the study further and prove that within the class $\mathcal{D}_p$ of all PBBD's with two associate class association schemes with given b,k and v, a GDD with $\lambda_2 = \lambda_1 + 1$ (not necessarily an MBGDD of type 1), if it exists, is A- and D-optimal under fixed effects model as well as under mixed effects model (Theorems 3.3.1.2 and 3.3.2.2). The results on A-optimality included in this chapter have been published in Mukhopadhyay (1981a)

In Chapter 4, extensions of an experiment are studied. The idea of extension of an experiment was introduced by Sinha and Sinha (1969). The essence of their method consists in making use of an experiment involving a larger number of treatments than required, retaining the experimental field unaltered. The latter is called an extended design. This design may be useful in a situation where no known optimum design exist with the given set of parameters but an optimal design with a larger value of v and the same values for b and k is available. Here we call the extension introduced in Sinha and Sinha (1969) an extension of type 1 and introduce another type of extension, called of type 2, in which each one of the additional treatments in the extended design is replaced by one of the original treatments, so that the new design involves the original set of treatments only. The basic optimal designs which have been made use of in the construction of extension types 1 and 2 are either (i) BBD's or (ii) GDD's with $\lambda_2 = \lambda_1 + 1$ with number of extra treatments $p = 1$ in the second case. The extended designs of types 1 and 2 obtained from a BBD are denoted by $d_1(B)$ and $d_2(B)$ respectively. The extended design of type 1 obtained from a GDD with $\lambda_2 = \lambda_1 + 1$ is denoted by $d_1(G)$. Two distinct extended designs of type 2 arise in the second case, viz., one with the extra treatment replaced by a treatment belonging to the same group, denoted by $d_2(G)$ and another with the extra treatment replaced by one belonging to a different group, denoted by $d'_2(G)$. The following observations are made.

(i) In both the cases of extensions considered the extended *design of type 2 is A and E-better than the corresponding* extended design of type 1. Moreover, $d_2(B)$ is also D-better than $d_1(B)$. These results are derived in theorems 4.3.3, 4.4.2.1 and 4.4.3.1.

(ii) Under certain conditions on the parameters, as given in theorems 4.4.1.1 and 4.4.1.2 the extended designs of type 2, viz $d_2(B)$, $d_2(G)$ and $d'_2(G)$ are E-optimal within a class $\mathcal{D}$ of proper and connected block designs with given b,k and v when $k < v$.

(iii) $d_2(G)$ is A-better than $d'_2(G)$, except when $m = 2$ and $\lambda_2 = 1$, $\lambda_1 = 0$, in which case they are equally efficient, as shown in theorem 4.4.3.2.

In Chapter 5, methods of constructions have been given for some series of such designs as have been proved to be optimal in Chapters 2 and 3. As several methods of constructions for one-way and two-way optimal designs are available in the literature, here attention is focussed mainly on designs in a balanced n-way setting. All the constructions here are based on finite geometries with the treatments identified with the points of $EG(2,s)$.

The following series of designs have been constructed. In each case s is a prime number or a prime power.

(a) A completely regular BYHC with parameters $(v = s^2, n = s^3, m = s+1, (b = s)^m)$.

(b) A nonregular BYHC with parameters $(v = s^2, n = (s^2+s)^2, m, (b = s^2+s)^m)$, $3 \leq m \leq t$, $t = \min(m'+2, s+1)$, when m' mutually orthogonal latin squares of order $s+1$ exist. It may be noted that Ruiz and Seiden (1974) constructed a GYD with the same values of v, n and b.

(c) (i) A BPYD $(v = s^2, n = s^2(s-1) + ts^3, m = s+1, (b = s)^m), t \geq 0$,

(ii) a BPYD $(v = s^2, n = s^2 + ts^3, m = s+1, (b = s)^m), t \geq 1$

and (iii) a BPYD $(v = s^2, n = (s^2+s)^2/4, m = 4, (b = s^2+s)^4)$

when $s \equiv 3 \pmod 4$. (It may be noted that (iii) adds two more factors to the PYD of Cheng (1981b) with no additional observations).

(d) Two asymmetrical designs in a balanced m-way setting with respect to all but one factor, say factor 1, the designs representing GDD's with $\lambda_2 = \lambda_1 + 1$ with respect to factor 1 and balanced with respect to each one of the remaining factors. The parameters of the designs are

(i) $v = s^2, n = s^3(s-1), m = s, b_1 = s^2, b_2 = s(s-1),$
$b_3 = \ldots = b_m = s$.

(ii) $v = s^2$, $n = s^3(s+1)$, $n = s$, $b_1 = s^2$, $b_2 = s(s+1)$,

$b_3 = b_4 = \dots = b_n = s$.

The materials of this chapter, pertaining to the constructions (a), (b), (c) and (d) are from joint works with Mukhopadhyay, A.C., the designs constructed in (a) and (b) being published as a technical report (Mukhopadhyay, A.C. and Mukhopadhyay, Sunanda (1981) and the designs entisted in (c) and (d) are still in the manuscript stage.

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CHAPTER 2

MIXED EFFECTS MODEL - ONE, TWO AND MULTIWAY SETTINGS

2·1. Introduction

In this chapter, we present optimality results in the context of block designs under the assumption of mixed effects additive model.

In the theory of block designs, the problem is to infer about a set of suitable linear functions of the effects of a few given treatments on the basis of the observations made on a number of experimental units given. These experimental units, in general, are heterogeneous in nature along one or more orthogonal directions and the effect of these heterogeneities have to be eliminated from the required treatment comparisons. Given a set of constraints, usually specifying a few parameters of a class of designs, our aim is to find out a design in the class, based on which one can obtain the best linear unbiased estimates of a given set of estimable linear functions of treatment effects with some optimum property and such a design is termed optimal in the sense specified by the optimality criterion used. In a given situation the optimal design with a specified optimality criterion certainly depends on the specific estimation problem as well as the model which fits appropriately into the set of experimental units.

In this chapter, we first consider one-way elimination of heterogeneity situation and then two-way and multiway heterogeneity situations, focussing our attention on a specific type of estimation problem commonly encountered in practice, viz the problem of estimation of an exhaustive set of orthonormal treatment contrasts, the model being assumed one of mixed effects.

2.2 Definitions and Notation

Let us suppose that we are given v treatments $1, 2, \ldots, v$ and n experimental units with necessary information about the nature of heterogeneity among them, viz., the number of heterogeneity directions an experimental unit is subjected to and the number of levels along each direction. In the problem of linear inference, our aim is to infer about a set of treatment contrasts $\underset{\sim}{\eta}(\ell \times 1) = L(\ell \times v)\, \underset{\sim}{\tau}(v \times 1)$, where $\tau = (\tau_1, \ldots, \tau_v)'$ is the vector of treatment effects. We are to assign the treatments to the experimental units in such a way that (i) each of the parametric functions η_i is estimable, where $\underset{\sim}{\eta} = (\eta_1, \ldots, \eta_\ell)'$ and (ii) the least square estimates of the η_i's based on the observations made on this arrangement (or design) satisfy some well defined optimality property.

For the problem considered in the present thesis, Rank $(L) = v - 1$ and in order that the requirement (i) is always satisfied, we shall be confined to the class of connected designs. The search for a design satisfying requirement (ii) among the connected designs obviously requires information about the actual form of η as well as the specific optimality criterion under consideration.

Following Kiefer (1958), we consider the problem of estimation of a set of $v-1$ orthonormal treatment contrasts, i.e.

$$\pi : \underline{\eta}(\overline{v-1} \times 1) = L(\overline{v-1} \times v)\, \underline{\tau}_{v \times 1} \qquad \ldots(2.2.1)$$

where $L = P(\overline{v-1} \times v)$ is such that

$$O = \begin{bmatrix} v^{-1/2}\, 1_v' \\ P \end{bmatrix}$$ is an orthogonal matrix.

For a given connected design d, the reduced normal equations for obtaining the least square estimates of the treatment contrasts based on d are written in the form

$C_d\, \underline{\tau}_d = \underline{Q}_d$, where the forms of the matrix C_d and the vector Q_d depend on the heterogeneity setting as well as the model assumed. [The structure of the C_d-matrix is crucial in the search for optimal designs and will be discussed in details in the following sections].

For the problem π considered here, $\widehat{\underline{\eta}}_d = P\widehat{\underline{\tau}}_d$ is the least square estimator of $\underline{\eta}$ based on the design d. Let V_d denote the dispersion matrix of $\widehat{\eta}_d$, which is of the form

$$V_d = (PC_dP')^{-1} \qquad \ldots(2.2.2)$$

as shown in Kiefer (1975).

Given a problem π and a class $\mathcal{D}$ of designs, our purpose is to find a design $d^* \varepsilon \mathcal{D}$ which optimises V_d with regard to some well defined optimality criterion defined on the class of possible C_d matrices. More specifically, an optimal design is defined in the following manner. (the word 'definition' is abbreviated here as 'DFN').

DFN 2.2.1. For a specified optimality functional ψ, a design $d^* \varepsilon \mathcal{D}$ is said to be ψ-optimal in $\mathcal{D}$ for the problem π, if

$$\psi(V_{d*}) \leq \psi(V_d) \ \forall \ d \ \varepsilon \ \mathcal{D} \ .$$

Let us now state the definitions of standard optimality criteria. It is clear from (2.2.2) that it is possible to write the optimality criterion as a function on the set of possible C_d-matrices, i.e. given the problem π, for every ψ, there exists a unique function ϕ defined on the class of C_d matrices such that

$$\psi(V_d) = \phi(C_d) \ .$$

Clearly it is convenient to work with ϕ, as pointed out in Kiefer (1975). Moreover, if ψ is orthogonal in variant, in the sense that $\psi(QV_dQ') = \psi(V_d)$ for every orthogonal matrix Q, we can also write ϕ as a function ϕ^* of the positive eigenvalues $\mu_{d1}, \ldots, \mu_{d(v-1)}$ of C_d, for the estimation problem π, since for this problem, the eigenvalues of V_d are nothing but the reciprocals of the positive eigenvalues of C_d. Hence, Kiefer (1975) considered a very general class of optimality functionals defined on a class $\mathcal{C}_d$ of matrices described as follows :

$\mathcal{C}_d$ is the class of all $v \times v$ positive semidefinite matrices with zero row sums and column sums. Obviously $\mathcal{C}_d$ includes every C_d-matrix whatever be the setting and/or model considered.

DFN 2.2.2. Φ is the class of real valued functionals ϕ defined on $\mathcal{C}_d$ with the following properties

(i) ϕ is convex ,

(ii) $\phi(b.C_d)$ is nonincreasing in the scalar $b \geq 0$,

(iii) for every given $C_d \in \mathcal{C}_d$, ϕ is invariant under each permutation of rows and the same on columns.

DFN 2.2.3. A design d^* is said to be universally optimal if it is ϕ-optimal with respect to every $\phi \in \Phi$.

Cheng (1978b) has considered a subclass of Φ, where the optimality functionals are defined on the positive eigen values $\mu_{d1},\ldots,\mu_{d(v-1)}$ of C_d.

DFN 2.2.4. An optimality criterion is said to be a ϕ_f-optimality criterion of type 1 if it is of the following form.

$$\phi_f(C_d) = \sum_{i=1}^{v-1} f(\mu_{di}),$$ where f is a real valued function defined on $[0, t_{\mathcal{D}}]$, $t_{\mathcal{D}} = \max_{d \in \mathcal{D}} \operatorname{tr} C_d$, where $\mathcal{D}$ is the class of designs under consideration, $\operatorname{tr} A$ denotes the trace of a matrix A, and f has the following properties :

(i) f is continuously differentiable on $(0, t_{\mathcal{D}})$ and f is strictly concave on $(0, t_{\mathcal{D}})$, i.e. $f' < 0$, $f'' > 0$ and $f''' < 0$ on $(0, t_{\mathcal{D}})$.

DFN 2.2.5. A generalised ϕ_f criterion of type 1 is defined as the pointwise limit of a sequence of the ϕ_f - optimality criteria of type 1.

An important subclass of the class of ϕ_f - optimality criteria of type 1 is the class of Φ_p - optimality criteria, given by

$\Phi_p(C_d) = (\phi_f(C_d))^{1/p}$ where f is given by $f(x) = x^{-p}$,

$0 < p < \infty$, ...(2.2.3)

with the limiting values

$$\Phi_o(C_d) = \prod_{i=1}^{v-1} \mu_{di}^{-1/(v-1)}$$

...(2.2.3a)

$$\text{and} \quad \Phi_\infty(C_d) = \max_{1 \le i \le v-1} \mu_{di}^{-1}$$

The most commonly used optimality criteria are

(i) A-optimality, i.e. Φ_1 - optimality, which means minimisation of $\operatorname{tr} V_d$.

(ii) D-optimality, i.e. Φ_o - optimality, which means minimisation of determinant of V_d.

(iii) E-optimality, i.e. Φ_∞ - optimality which means minimisation of the maximum eigen value of V_d.

Apart from the most commonly used problem π described in (2.2.1), various other estimation problems have been handled by research workers in this area. The problem of estimation of all elementary treatment contrasts considered by Roy (1958) is one of them. For this problem, A-optimality is the only meaningful criterion and that is equivalent to the A-Optimality criterion for the problem π. Hence, these two problems are equivalent, so long as one is concerned with the A-optimality criterion.

It may be safely assumed that the only estimation problem handled in the present chapter and the subsequent chapters is π.

Having formulated the problem in the general set up, let us now describe the different heterogeneity settings of block designs one by one.

DFN 2.2.6 If the experimental units are subjected to heterogeneity along one direction, then it is said to be one-way heterogeneity or simply one-way setting.

If there are b levels of this heterogeneity, then the experimental units are grouped into b blocks in such a way that the units belonging to the ith block is subjected to the effect of the ith level of the heterogeneity or ith block effect. The model assumed is of the following form.

$$\underline{Y}(n\times 1) = \mu.1_n + A'(n\times v)\,\underline{\tau}(v\times 1) + B'(n\times b)\underline{\beta}(b\times 1) + \underline{\varepsilon}(n\times 1), \qquad \ldots(2.2.5)$$

where μ is the general effect, $\underline{\tau}$ is the vector of treatment effects, $\underline{\beta}$ is the vector of block effects and $\underline{\varepsilon}$ is the vector of errors with $E(\underline{\varepsilon}) = 0$ and $D(\underline{\varepsilon}) = \sigma^2 I_n$, where $D(\underline{x})$ denotes the dispersion matrix of a random vector $\underline{x}$. The matrices A and B are as described below.

$$A = (a_{iu})_{\substack{1 \leq i \leq v,\\ 1 \leq u \leq n}} \text{ where } a_{iu} = 1 \text{ if the ith treatment}$$

is applied to the u-th experimental unit and 0 otherwise.

$$B = (b_{ju})_{\substack{1 \leq j \leq b,\\ 1 \leq u \leq n}} \text{ where } b_{ju} = 1 \text{ if u-th experimental}$$

unit belongs to the jth block and 0 otherwise.

DFN 2.2.7. In the one-way setting, the model is said to be one of mixed effects if $\underline{\tau}$ is fixed and $\underline{\beta}$ is random. In the case of mixed effects model, we assume $E(\underline{\beta}) = 0$ and $D(\underline{\beta}) = \sigma_1^2 I_b$

...(2.2.6)

We also use the notations $w = (\sigma^2)^{-1}$ and $\bar{w} = (\sigma + k\sigma_1^2)^{-1}$

...(2.2.7)

Let us now write down some standard characteristics of block designs.

DFN 2.2.8. $B.1_n = (k_1, \ldots, k_b)'$ is called the vector of block sizes and $A.1_n = (r_1, \ldots, r_v)'$ is called the vector of replication numbers.

DFN 2.2.9. A one-way design is said to be proper if the block sizes are equal, i.e. $k_1 = k_2, \ldots, = k_b = k$.

DFN 2.2.10. A block design is said to be equireplicate if all the treatments are equally replicated, i.e. $r_1 = r_2 = \ldots = r_v = r$.

DFN 2.2.11. The incidence matrix of a one-way design d is given by $N_d(v \times b) = AB'$ and so the (i,j)th element is n_{dij} = number of times the ith treatment occurs in the jth block.

For simplifying notation, subscript d may be removed from the relevant expressions defined, e.g., in N_d, n_{dij}, C_d, μ_{di}'s etc., when from the given context the expressions are self-evident and there is no scope for confusion.

It is clear that there is a one-one correspondence between a block design and its incidence matrix and so the following definition is meaningful.

DFN 2.2.12. The dual of a block design with incidence matrix N is the block design whose incidence matrix is the transpose of N, denoted by N'.

So the dual of a design with v treatments and b blocks will be one with b treatments and v blocks.

Next we list a few standard designs.

DFN 2.2.13. A proper and connected one-way design is said to be a Balanced Block Design (BBD) if its incidence matrix satisfy the following relations.

(i) $v|n$ and $r_i = \sum_{j=1}^{b} n_{ij} = n/v$, $i = 1,2,\ldots,v$

(ii) $|n_{ij} - k/v| < 1$, $j = 1,2,\ldots,b$, $i = 1,2,\ldots,v$,

(iii) $\lambda_{ii'} = \sum_{j=1}^{b} n_{ij}\, n_{i'j} = \lambda \;\forall\; i \neq i'$, $i,i' = 1,2,\ldots,v$.

A BBD with v = number of treatments, b = number of blocks, r = common replication for each treatment, k = block size and λ as given in (iii) will be denoted by BBD (v,b,r,k,λ).

The special case of a balanced block design when $k < v$ is known as the Balanced Incomplete Block Design (BIBD) which is incomplete in the sense that not all treatments are present in any block and its incidence matrix is binary, i.e. n_{ij} can take the values 0 and 1 for all i and j.

DFN 2.2.14. A Linked Block Design (LBD) is a design whose dual is a BBD.

DFN 2.2.15. Bose and Nair (1939) suggested the use of Partially Balanced Incomplete Block Designs (PBIBD's) which Cheng (1978) has generalised to Partially Balanced Block Designs (PBBD) as follows.

Given v symbols $1,2,\ldots,v$, a relation satisfying the following conditions is said to be an association scheme with m associate classes. $m \geq 2$.

1. Any two distinct symbols are either 1st, 2nd... or mth associates of each other, relation of association being symmetric.

2. Each symbol α has n_s s-th associates. The number n_s is independent of α.

3. If any two distinct symbols α and β are s-th associates then the number of symbols that are t-th associates of α and u-th associates of β is p^s_{tu} i.e. independent of the pair of s-th associates α and β (i.e. it depends only on s,t and u).

If we are given an association scheme with m classes $(m \geq 2)$ for the v symbols designating v treatments then a PBBD is a design such that its incidence matrix N satisfies relations (i) and (ii) of definition 2.13, together with the following relations,

(iii) $\lambda_{ii'} = \lambda_s$ if treatments i and i' are s-th associates,

(iv) there are at least two distinct λ_s's.

DFN 2.2.16. A Group Divisible Design (GDD) is a PBBD with two associate classes, when the association scheme is as given below:

$v = mn$ symbols are partioned into m group of n each and any two treatments are first associates if they belong to the same group and second associates if they belong to two different groups.

DFN 2.2.17. A Most Balanced GD Design (MBGDD) of type 1 is a GDD with $m = 2$ and $\lambda_2 = \lambda_1 + 1$.

A GDD with v = number of treatments, b = number of blocks, r = replication number, k = block size, λ_1, λ_2 as defined in (iii) of DFN 2.2.15 and m,n as defined in DFN 2.2.16 will be denoted by a GDD $(v,b,r,k,\lambda_1,\lambda_2,m,n)$.

The terms and concepts used in two-way heterogeneity setting are described in the following lines.

DFN 2.2.18. If the experimental units are subjected to heterogeneities in two perpendicular directions then it is said to be a two-way heterogeneity setting. In such a situation, the experimental units are arranged in a rectangular arrangement in such a way that the unit in the (i,j)th position is subjected to the ith level of the first heterogeneity direction which we may call row effect and jth level of the column effect. Let us suppose that the number of levels of the row effects is b_1 and that of column effects is b_2. Then the number of experimental units is $n = b_1 b_2$ (assuming no row-column combination is repeated and each row-column combination appears in the experiment).

An arrangement of v treatments in a $b_1 \times b_2$ array representing a two-way heterogeneity setting as described above is said to be a two-way design with parameters b_1, b_2 and v.

The model for the two-way heterogeneity setting is as follows.

$$\underline{Y}(n\times 1) = \mu.1_n + A'(n\times v)\underline{\tau}(v\times 1) + \sum_{i=1}^{2} B_i'(n\times b_i)\underline{\beta}_i(b_i\times 1) + \underline{\varepsilon}(n\times 1), \qquad \ldots(2.2.8)$$

where the scalar μ and the vectors $\underline{\tau}$ and $\underline{\varepsilon}$ are as in (2.2.5), $\underline{\beta}_1$, $\underline{\beta}_2$ represent the vectors of row effects and column effects respectively. The matrix A as in (2.2.5) and the matrices B_1, B_2 are as follows.

$B_i = (b_{ju}^{(i)})_{\substack{1 \leq j \leq b_i \\ 1 \leq u \leq n}}$ where $b_{ju}^{(1)} = 1$ if the u-th experimental unit belongs to the jth row and 0 otherwise and similarly $b_{ju}^{(2)} = 1$ if the uth experimental unit belongs to the jth column and 0 otherwise.

DFN 2.2.19. The model is said to be one of fixed effects, if $\underset{\sim}{\tau}$ is fixed and also both $\underline{\beta}_1$ and $\underline{\beta}_2$ are fixed. It is said to b mixed effects model of type 1 if $\underset{\sim}{\tau}$ is fixed, but exactly one c $\underline{\beta}_1$ and $\underline{\beta}_2$ (say $\underline{\beta}_1$) is random with $E(\underline{\beta}_1) = 0$, $D(\underline{\beta}_1) = \sigma_1^2 I_b$ and mixed effects model of type 2 if $\underset{\sim}{\tau}$ is fixed but both $\underline{\beta}_1$, are random with $E(\underline{\beta}_i) = 0$, $D(\underline{\beta}_i) = \sigma_i^2 I_{b_i}$, $i = 1,2$.

Let w be as in one-way setting and

$$\bar{w}_i = (\sigma^2 + nb_i^{-1}\sigma_i^2)^{-1} . \qquad \ldots(2.2.9)$$

Replication numbers for treatments and equireplicateness of a design are the same as those of an one-way design. A two-way design has two incidence matrices, viz,

$$\begin{aligned} N_{d1}(v \times b_1) &= AB_1' = \text{treatment row incidence matrix} \\ N_{d2}(v \times b_2) &= AB_2' = \text{treatment-column incidence matrix} \end{aligned} \qquad \ldots(2.$$

DFN 2.2.20. A two-way setting is said to be regular if $v|b_1$ c $v|b_2$.

We now define standard two-way designs whose optimality properties have been studied under mixed effects model.

DFN 2.2.21. An arrangement of v treatments into a rectangle of size $b_1 \times b_2$ such that the rows constitute the blocks of a BBD and similarly the columns also constitute the blocks of a BBD has been termed a Generalised Youden Design (GYD) by Kiefer (1975). In other words a GYD is a two-way design d with both the incidence matrices N_{d1} and N_{d2} as incidence matrices of BBD's.

In the special case of a regular setting, when v equals either b_1 or b_2, a GYD is called Youden Square Design (YSD).

DFN 2.2.22. In the case $b_1 = b_2 = b$, Cheng (1981) has defined a Pseudo Youden Design (PYD) as an arrangement of v treatments in a $b \times b$ square such that the rows and columns together constitute the blocks of a BBD.

If the experimental units are subjected to heterogeneities along more than two directions, then it is said to be a multiway heterogeneity setting. We call it an m-way setting if the number of heterogeneity directions is m and refer to the heterogeneity directions as factors.

Let us suppose that there are m factors and the number of levels of the ith factor is b_i, $i = 1,2,\ldots,m$. Then the number of all possible combinations of the levels of the factors is $b_1 b_2, \ldots, b_m$, which may be too large to be available for experimentation. For that we consider a 'balanced multiway setting', in which not all possible combinations of levels of the factors are necessarily present, but those present satisfy an orthogonality property with respect to each pair of factors, so that the main effects of each of the factors are orthogonally estimable. This is adequate since the effects of interactions between factors are usually assumed to be negligible, so that the orthogonal estimability of merely the main effects of the factors is sufficient. Let us now formally define the setting we consider.

We first define an orthogonal array with variable number of symbols.

DFN 2.2.23. Let A be an $n \times n$ array, the entries in the ith row of which are from the set S_i of $b_i \geq 2$ symbols. Let A satisfy the following properties.

Let a $d \times n$ submatrix of A be constructed, consisting of the rows $\{i_1, i_2, \ldots, i_d\}$ with $\{i_1, i_2, \ldots, i_d\} \subset \{1, 2, \ldots, n\}$. Then among the n $d \times 1$ column vectors of the submatrix, there occurs with equal frequency, say $\lambda(i_1, i_2, \ldots, i_d)$ each of the $\prod_{j=1}^{d} b_{i_j}$ $(d \times 1)$ column vectors which can be obtained by drawing the jth element of the column vector from the set S_{i_j} of b_{i_j} symbols, $j = 1, 2, \ldots, d$ and this property holds for every choice of an $d \times n$ submatrix of A. Such an array A is called an Orthogonal Array with Variable Number of Symbols (OAVS), with n constraints and of strength d. We shall refer to the frequency $\lambda(i_1, i_2, \ldots, i_d)$ as the index of the set of rows $\{i_1, \ldots, i_d\}$. Some authors prefer the use of the term OAVS to distinguish it from the symmetrical case, (i.e. when $b_1 = b_2 = \ldots = b_m$) and denote the array A by OAVS $(n, m, b_1, \ldots, b_m, d)$. In the case the OAVS has $b_1 = b_2 = \ldots = b_m = b$ which implies $\lambda(i_1, i_2, \ldots, i_d) = \lambda$, for every $d \times n$ submatrix of A, the array is called simply an orthogonal array of strength d and index λ and is denoted by $OA(n, m, b, d)$. In this latter case, $n = \lambda b^d$.

DFN 2.2.24. Let us consider an OAVS $(n, m, b_1, \ldots, b_m, 2)$, $m \geq 3$, the columns of the array being identified with experimental units and the i-th entry in the u-th column representing the level of the i-th factor at the u-th experimental unit, $1 \leq i \leq m$, $1 \leq u \leq n$. The setting thus represented by an OAVS of strength 2 is said to be a balanced multiway setting (here m-way setting) and for v given treatments is denoted by $M(v; n; m; b_1, \ldots, b_m)$. The hyper rectangular setting of Cheng (1978) where each factor combination occurs a constant number of times (> 0) among the experimental units, is a special case of the balanced multiway setting. The arrangement of v treatments in a balanced m-way setting $M(v; n; m; b_1, \ldots, b_m)$ may be represented by an $(m+1) \times n$ array, the first m rows of which constitutes an OAVS representing the set up of experimental units as described and the u-th element in the $(m+1)$th row denotes the treatment applied to the u-the experimental unit.

DFN 2.2.25. A setting $M(v; n; m; b_1, \ldots, b_m)$ is said to be regular with respect to the ith factor if $v | (n/b_i)$.

DFN 2.2.26. A setting $M(v;n;m;b_1,\ldots,b_m)$ is said to be completely regular if it is regular with respect to each factor and is said to be nonregular if it is not regular with respect to any factor, $m \geq 3$.

In the special case when $b_1 = b_2 = \ldots = b_m = b$, the n-way setting $M(v;n;m;b_1,\ldots,b_m)$ is represented by $M(v;n;n;b^n)$.

DFN 2.2.27. An arrangement of treatments in $M(v;n;m;b_1,\ldots,b_m)$ is said to form a Balanced Youden Hyperrectangle (BYHR) denoted by $(v,n,m,b_1,\ldots,b_m)$ if it satisfies the following conditions :

Let S_{ij} be the set of treatments applied to the experimental units for which the i-th factor is at level j, $j = 1,2,\ldots,b_i$. Then the sets $S_{i1}, S_{i2},\ldots,S_{ib_i}$ constitute the blocks of a BBD, for each $i = 1,2,\ldots,m$.

DFN 2.2.28. A BYHR with parameters $(v,n,m,b_1,\ldots,b_m)$ is called a Balanced Youden Hypercube denoted by BYHC (v,n,m,b^m), when $b_1 = b_2 = \ldots = b_m = b$.

DFN 2.2.29. In the special case $b_1 = b_2 = \ldots = b_n = b$, a design in the setting $M(v;n;m;b^m)$ is said to be a Balanced Pseudo Youden Design (BPYD) denoted by BPYD (v,n,m,b^m), if it satisfies the following conditions.

Let the sets S_{ij}, $j = 1,2,\ldots,b$, $i = 1,2,\ldots,n$ be as given in DFN 2.2.26. Then the sets S_{ij}, $j = 1,2,\ldots,b$, $i = 1,2,\ldots,n$ together constitute a BBD.

An n dimensional Pseudo Youden Design (PYD) defined in Cheng (1981a) can be looked upon as a particular case of the BPYD obtained in the complete hypercube setting, i.e. $M(v\,;n\,;m;b^m$ with $n = b^m$.

In a balanced multiway heterogeneity setting $M(v\,;n\,;m\,;$ $b_1,\ldots b_m)$, the model can be written as

$$\underline{Y}(n\times 1) = \mu.1_n + A'(n\times v)\underline{\tau}(v\times 1) + \sum_{i=1}^{m} B_i'(n\times b_i)\underline{\beta}_i(b_i\times 1) + \underline{\varepsilon}(n\times 1), \quad \ldots(2.2.11)$$

where the scalar μ, the vectors $\underline{\tau}$ and $\underline{\varepsilon}$ are as in (2.2.5) and $\underline{\beta}_i$ is the vector of the ith factor effects, $i = 1,2,\ldots,m$. The matrix A is as in (2.2.5) and the matrices B_i, $i = 1,2,\ldots,m$ are obtained from the OAVS $(n,m,b_1,\ldots,b_m, 2)$ in the following manner.

$$B_i = (b_{ju}^{(i)})_{\substack{1\leq j\leq b_i,\\ 1\leq u\leq n}} \text{ where } b_{ju}^{(i)} = 1 \text{ if the } (i,u)\text{th entry}$$

in the OAVS is j and 0 otherwise.

DFN 2.2.30. A block design d (one-way, two-way or multiway) is said to be binary with respect to factor i, if the incidence matrix N_{di} is a binary matrix, i.e. with entries 0 and 1. A design is said to be binary if it is binary with respect to all the factors.

2.3 Optimality Results in the One-way Heterogeneity Setting Under Mixed Effects Model

For the class $\mathcal{D}$ of proper and connected block designs with block size k and number of treatments v, the reduced normal equations (after the elimination of block effects) for the treatment effects $\tau_1, \ldots, \tau_v$ based on a design $d \in \mathcal{D}$ are given by Bose (1975) as

$$C_d^{(M)} \hat{\underline{\tau}} = \underline{Q}_d^{(M)}, \qquad \ldots(2.3.1)$$

where $C_d^{(M)} = w(D_r - k^{-1} N_d N_d') + \bar{w}(k^{-1} N_d N_d' - n^{-1} \underline{r}\,\underline{r}')$ and

$\underline{Q}_d^{(M)} = w(\underline{T} - k^{-1} N_d . \underline{B}) + \bar{w}(k^{-1} N_d . \underline{B} - n^{-1} G \underline{r})$

with $D_r = \text{diag}\,(r_1, \ldots, r_v)$,

$\underline{r} = (r_1, \ldots, r_v)'$

= vector of replications of treatments, N_d as defined in DFN 2.2.11,

$\underline{T}$ = vector of unadjusted treatment totals,

$\underline{B}$ = vector of unadjusted block totals, G is the grand total and w and $\bar{w}$ are as given in (2.2.7).

Further $E(\underline{Q}_d^{(M)}) = C_d^{(M)} \underline{\tau}$

and $D(\underline{Q}_d^{(M)}) = C_d^{(M)}$. $\qquad \ldots(2.3.2)$

Before going into the relevant optimality results, we state a result without proof in the form of a proposition from Kiefer (1975).

The result is an important tool in the search for universally optimal designs.

Proposition 2.3.1. (Kiefer (1975). Suppose a class $\mathcal{C} = \{C_d : d \in \mathcal{D}\}$ of nonnegative definite matrices with zero row and column sums contain a C_{d*} for which

(a) C_{d*} is completely symmetric in the sense that it is of the form

$$C_{d*} = \alpha I_v + \beta J_v, \text{ and}$$

(b) $$\text{tr } C_{d*} = \max_{d \in \mathcal{D}} \text{tr } C_d ,$$

then d^* is universally optimal in $\mathcal{D}$.

Block designs satisfying property (a) of position 2.3.1 were termed completely symmetric by Kiefer (1975), whereas a design for which the C_d matrix is not completely symmetric has been termed asymmetric.

3.1. Universal Optimality of BBD's.

Here we reestablish the universal optimality of BBDs within a class $\mathcal{D}$ of proper and connected block designs with given b, k and v, under mixed effects model.

The A-, D- and E-optimality properties of BIBDs within the class $\mathcal{D}$ defined above with $k < v$ are well known from the results of Kiefer (1958) and Roy (1958). More recently, the universal optimality of BBD's in general within the same class has been established by Kiefer (1975). All these results are based on the assumption of fixed effects model. They are valid under the assumption of mixed effects model too as we shall presently see. In order to verify Kiefer's (1975) result on universal optimality under mixed effects model, our task would be to simply verify conditions (a) and (b) of proposition 2.3.1 with d^* as a BBD, $C_d^{(M)}$ for C_d and $\mathcal{D}$ as defined in the begining of this section.

From (2.3.1), $C_d^{(M)}$ can be written as

$$C_d^{(M)} = (w - \bar{w})\, C_d^{(F)} + \bar{w}\, C_d^{(O)} \qquad \ldots(2.3.1.1)$$

where $C_d^{(F)} = D_r - k^{-1} N_d N_d'$ is the C_d matrix under the corresponding fixed effects model and

$$C_d^{(O)} = D_r - n^{-1}\, r r' .$$

Since $\operatorname{tr} C_d^{(O)} = (v-1)\bar{r} - \sum_{i=1}^{v} (r_i - \bar{r})^2$,

where $\bar{r} = (\sum_{i=1}^{v} r_i)/v = n/v$, it is clear that $\operatorname{tr} C_d^{(O)}$ is maximum when d is an equireplicate design.

Now the following lemma follows easily.

<u>Lemma 2.3.1.1</u>. If d^* is such that

(i) $|n_{ij} - k/v| < 1$

and (ii) $r_1 = r_2 = \ldots = r_v = \bar{r}$, then

$$\operatorname{tr} C_{d^*}^{(M)} = \max_{d \in \mathcal{D}} \operatorname{tr} C_d^{(M)} .$$

Now that $d^* =$ BBD satisfies condition (b) of proposition 2.3.1 is immediate. That a $d^* =$ BBD also satisfies condition (a) is clear from the expression for $C_d^{(M)}$. So we have the following result.

Theorem 2.3.1.2. A BBD, if it exists is universally optimal within the class $\mathcal{D}$ of all proper and connected block designs with given b,k and v under mixed effects model.

2.3.2. Optimal Asymmetrical Designs

A symmetrical design, as we know, is not always feasible for a given set of parameters b,k and v. For this reason, attempts have been made to obtain asymmetrical designs which are optimal in the absence of a symmetrical design. So far, the known optimal asymmetrical designs under fixed effects model are the LBD's and the MBGDD's. In this section we verify their optimal behaviour under mixed effects model.

Let us first consider the LBD's. Within the class of proper, connected and equireplicate block designs, the A-, D- and E-optimality of the LBD's were pointed out in Sinha (1971) and Shah, Raghavarao and Khatri (1976) under fixed effects model. The ϕ_p - optimality of the LBD's in such a setting has also been pointed out in Sinha (1980). We shall follow the approach of Sinha (1980) to establish the validity of the same result under mixed effects model.

Let $\mathcal{D}_r^*$ be the class of all proper, connected and equireplicate designs with given parameters b^* = number of blocks, k^* = block size, v^* = number of treatments, r^* = common replication for each treatment, where $b^* \leq v^*$.

Let $\mathcal{D}_r = \left\{ d : d \text{ is the dual of a } d^* \varepsilon \mathcal{D}_r^* \right\}$.

Then $\mathcal{D}_r$ refers to the class of all proper, connected and equireplicate designs with $b = v^*$, $k = r^*$, $v = b^*$ and $r = k^*$, where $b \geq v$.

Let $t = R(N_d)$ where N_d is the incidence matrix of d. Denote the positive eigenvalues of $N_d' N_d$ by $\mu_1, \mu_2, \ldots, \mu_{t-1}$, $\mu_t = rk$.

Then the positive eigenvalues of $C_d^{(M)}$ are $wr - \mu_i(w-\bar{w})/k$, $i = 1,2,\ldots,t-1$ and wr with multiplicity $v-t$ and those of $C_{d*}^{(M)}$ are $wk - \mu_i(w-\bar{w})/r$, $i = 1,2,\ldots,t-1$ and wk with multiplicity $b-t$. So, from DFN 2.2.5, we can write

$$\phi_p(C_d^{(M)}) = (wr)^{-1}(v-1)^{-1/p}\left\{ v-t+\sum_{i=1}^{t-1}(1-(w-\bar{w})\mu_i / (w.rk))^{-p} \right\}^{1/p} \qquad 0 < p < \infty$$

and similarly,

$$\phi_p(C_{d*}^{(M)}) = (wk)^{-1}(b-1)^{-1/p}\left\{ b-t+\sum_{i=1}^{t-1}(1-(w-\bar{w})\mu_i / (w.rk))^{-p} \right\}^{1/p} \qquad 0 < p < \infty .$$

From the relations between $\phi_p(C_d^{(M)})$ and $\phi_p(C_{d*}^{(M)})$, the following lemma is immediate.

Lemma 2.3.2.1. $d \in \mathcal{D}_r$ is ϕ_p-optimal

$\iff d^* \in \mathcal{D}_r^*$ is ϕ_p- optimal, for $0 < p < \infty$.

Now the following result is obtained by using theorem 2.3.1.2.

Theorem 2.3.2.2. An LBD, if it exists, is ϕ_p- optimal within the class of proper, connected and equireplicate designs with given b,k,v and r.

This theorem is observed to hold even when $p \longrightarrow \infty$, so that the E-optimality of the LBD's within $\mathcal{D}_r$ follows.

Cheng (1980b) has proved the E-optimality of the LBD's within $\mathcal{D}$, in the situations when $k < v$, under fixed effects model. This result can easily be extended to the mixed effects model. The analogue of theorem 2.2 of Cheng (1980b) in the context of mixed effects model is as written in the following lemma, the proof of which is immediate, following Cheng (1980b).

Lemma 2.3.2.3. For any nonequireplicate design d,

$$\mu_{d1} \leq (w-\bar{w})r_1(k-1)v/(k(v-1)) + \bar{w}(r_1 - r_1^2/n), \qquad \ldots(2.3.2.1)$$

where $r_1 \leq r_2, \ldots, \leq r_v$ and $r_1 \leq r-1$, $r = n/v$, and μ_{d1} is the minimum positive eigenvalue of $C_d^{(M)}$. Let d^* represent a BIBD with parameters $(b^*, k^*, v^*, r^*, \lambda^*)$ where $b^* = v$, $v^* = b$, $k^* = r$, $r^* = k$. Let $\bar{d}$ represent its dual, which is a LBD with parameters (b,k,v,r). Then, we have

$$\mu_{\bar{d}1} = (w-\bar{w})(r-(k-\lambda^*)/k) + \bar{w}r .$$

Clearly the coefficient of $w-\bar{w}$ in $\mu_{\bar{d}1}$, say

$$c_w = (r-1)(1+1/(b-1)) \geq r-1.$$

Again, since $k < v$, $v(k-1)/(k(v-1)) < 1$, so that the coefficient of $(w-\bar{w})$ in the R.H.S. of (2.3.2.1) is $< (r-1) \leq c_w$. Also $r \geq r_1 - r_1^2/n$, as $r_1 \leq r-1$.

$\therefore \mu_{\bar{d}1} \geq \mu_{d1} \; \forall \; w-\bar{w} \geq 0$ and $w \geq 0$ and hence the following optimality result is proved by virtue of lemma 2.3.2.3 and theorem 2.3.2.2.

Theorem 2.3.2.4. An LBD, if it exists is E-optimal within $\mathcal{D}$, under mixed effects model whenever $k < v$.

Let us now take up the MBGDDs.

Cheng (1978b) established the type 1 ψ_f-optimality of the MBGDDs of type 1 within the class $\mathcal{D}$ of proper and connected one-way designs under the assumption of fixed effects model.

(ψ_f-optimality of type 1 and MBGDD of type 1 are described in DFN 2.2.5 and DFN 2.2.17 respectively).

Shah and Khatri (1981) has considered mixed effects model and shown the type 1 ψ_f-optimality of the MBGDDs of type 1 within class of proper, connected and equireplicate designs, as an easy consequence of Cheng's (1978b) results.

We shall presently establish the ψ_f-optimality of type 1 of the MBGDDs within another important subclass of $\mathcal{D}$, viz $\mathcal{D}_1 =$

the class of all proper, connected and binary designs with given b,k and v, under the assumption of mixed effects model.

Let d^* denote an MBGDD of type 1 in $\mathcal{D}_1$.

Then in view of theorem 2.2 of Cheng (1978b) it is enough to verify the following conditions.

(i) $C_{d^*}^{(M)}$ has two positive eigenvalues μ and μ' $(\mu > \mu')$, the multiplicity of μ being 1

(ii) d^* maximises $\operatorname{tr} C_d^{(M)}$ over all $d \in \mathcal{D}_1$

(iii) $\operatorname{tr}(C_{d^*}^{(M)})^2 < (v-2)^{-1}(\operatorname{tr} C_{d^*}^{(M)})^2$

(iv) d^* maximises

$$\operatorname{tr} C_d^{(M)} - (v-1)^{1/2} / (v-2)^{1/2} \, P_{d^*}^{(M)} \quad \text{over all} \quad d \in \mathcal{D}_1 \qquad \ldots(2.3.2.2)$$

where $I_u^{(Q)} = \left\{ \operatorname{tr}(C_d^{(Q)})^2 - (v-1)^{-1} (\operatorname{tr} C_d^{Q})^2 \right\}^{1/2}$, with $Q = M$ or F.

Now for any equireplicate design d_0 with constant replication r, $C_{d_0}^{(0)} = r(I_v - v^{-1} J_v)$ and so the ith eigenvalue of $C_{d_0}^{(M)}$ is $\mu_i^{(M)} = Z . \mu_i^{(F)} + \bar{w} r$, where $\mu_i^{(F)}$ is the corresponding ith eigenvalue of $C_{d_0}^{(F)}$ and $Z = w - \bar{w}$, $i = 1,2,\ldots,v-1$.

Hence condition (i) holds for $C_{d^*}^{(M)}$ as it is known to hold for $C_{d^*}^{(F)}$ (from Cheng (1978b)) and d^* is an equireplicate design. Condition (ii) follows from the definition of an MBGDD of type 1 in view of Lemma 2.3.1.1.

For proving condition (iii) of (2.3.1.2), we note that for an equireplicate design d_o,

$$C_{d_o}^{(F)} C_{d_o}^{(0)} = r\, C_{d_o}^{(F)} \qquad \ldots(2.3.2.3)$$

$$\text{and} \quad C_{d_o}^{(0)})^2 = r\, C_{d_o}^{(0)} ,$$

since $C_d^{(F)}.1_v = C_d^{(0)}.1_v = 0$.

$$\text{Hence} \quad (P_{d_o}^{(M)})^2 = Z^2(P_{d_o}^{(F)})^2 + 2Z\bar{w}\left\{tr(C_{d_o}^{(F)} C_{d_o}^{(0)}) - (v-1)^{-1} tr\, C_{d_o}^{(F)} tr\, C_{d_o}^{(0)}\right\} + \bar{w}^2\left\{tr(C_{d_o}^{(0)})^2 - (v-1)^{-1}(tr\, C_{d_o}^{(0)})^2\right\}$$

$$= Z^2(P_{d_o}^{(F)})^2$$ since the coefficients of $Z\bar{w}$ and $\bar{w}^2$ vanish by virtue of (2.3.2.3).

In particular, $P_{d*}^{(M)} = Z.P_{d*}^{(F)}$ ...(2.3.2.4)

since d^* is an equireplicate design. So, by theorem 3.1 of Cheng (1978b) and the fact that $w - \bar{w} > 0$, condition (iii) follows.

So now we are left with the verification of condition (iv) of (2.3.2.2) which is equivalent to

$$tr\, C_{d*}^{(M)} - tr\, C_d^{(M)} \geq \left(\frac{v-1}{v-2}\right)^{1/2}(P_{d*}^{(M)} - P_d^{(M)}) \;\forall\, d \in \mathcal{D}_1 . \qquad \ldots(2.3.2.5)$$

Since $tr\, C_{d*}^{(0)} \geq tr\, C_d^{(0)} \;\forall\, d \in \mathcal{D}_1$, L.H.S. of (2.3.2.5) is $\geq Z(\text{tr } C_{d*}^{(F)} - \text{tr } C_{d}^{(F)})$.

So, in view of theorem 3.1 of Cheng (1978b), for proving (2.3.2.5), it is enough to show that

$$(P_d^{(M)})^2 \geq Z^2 (P_d^{(F)})^2 \quad \forall\, d \in \mathcal{D}_1. \qquad \ldots(2.3.2.6)$$

Now $$(P_d^{(M)})^2 = Z^2 (P_d^{(F)})^2 + 2Z\bar{w}\, Q_d + \bar{w}^2 R_d \qquad \ldots(2.3.2.7)$$

where $Q_d = \text{tr}(C_d^{(F)}\, C_d^{(0)}) - (v-1)^{-1} \text{ tr } C_d^{(F)} \text{ tr } C_d^{(0)}$

and $R_d = \text{tr}(C_d^{(0)})^2 - (v-1)^{-1}(\text{tr } C_d^{(0)})^2$.

Now $$R_d = \sum_{i=1}^{v-1} (\mu_i - \bar{\mu})^2$$

where μ_i is the ith eigenvalue of $C_d^{(0)}$

and $\bar{\mu} = \text{tr } C_d^{(0)} / (v-1)$

and so $R_d \geq 0$.

So, if we can show that $Q_d \geq 0$, then (2.3.2.6) will hold.

Now writing $C_d^{(F)} = (c_{dij})_{1 \leq i,j \leq v}$

and $C_d^{(0)} = (\bar{c}_{dij})_{1 \leq i,j \leq v}$, we have

$$Q_d = \sum_{i=1}^{v} c_{dii}\, \bar{c}_{dii} + \sum_{i \neq j} c_{dij} \bar{c}_{dij} - (v-1)^{-1} \text{ tr } C_d^{(F)} \text{ tr } C_d^{(0)}$$

$$= (k-1)k^{-1} \sum_{i=1}^{v} (r_i - \bar{r})\bar{c}_{dii} - v^{-1}(v-1)^{-1} \text{ tr}(C_d^{(F)}\, C_d^{(0)})$$

$$+ \sum_{i \neq j} c_{dij} \bar{c}_{dij} \qquad \ldots(2.3.2.8)$$

where $r = \sum_{i=1}^{v} r_i / v = n/v$.

Now since $C_d^{(F)} . \underline{1}_v = C_d^{(0)} . \underline{1}_v = 0$,

$\text{tr } C_d^{(F)} = -\sum_{i \neq j} c_{dij} = \sum_{i \neq j} \lambda_{ij} / k$, where λ_{ij}'s are as defined in DFN 2.2.13

and $\text{tr } C_d^{(0)} = -\sum_{i \neq j} \bar{c}_{dij} = \sum_{i \neq j} r_i r_j / n$.

So, (2.3.2.8) can be written as

$$(k-1)k^{-1} \sum_{i=1}^{v} (r_i - r) \bar{c}_{dii} + (kn)^{-1} \sum_{i \neq j} (\lambda_{ij} - \bar{\lambda}) r_i r_j . \qquad \ldots(2.3.2.9)$$

Now the matrix $\Lambda = (\lambda_{ij})_{1 \leq i,j \leq v}$, where $\lambda_{ii} = 0$, $i = 1,2,\ldots,v$, has all elements nonnegative. So we can apply the inequality of Atkinson, Walterson and Moran (1960) to obtain

$$v^2 \sum_i \sum_j \lambda_{ij} \lambda_{i.} \lambda_{.j} \geq \lambda^3_{..}$$

where $\lambda_{i.} = \sum_{j=1}^{v} \lambda_{ij} = r_i(k-1) = \lambda_{.i}$, $i = 1,2,\ldots,v$ and $\lambda_{..} = \sum_{i=1}^{v}$

Hence the second term of (2.3.2.9)

$$= (kn(k-1)^2)^{-1} \left[\sum_{i \neq j} \lambda_{ij} \lambda_{i.} \lambda_{.j} - \lambda_{..} / v(v-1) \sum_{i \neq j} \lambda_{i.} \lambda_{.j} \right]$$

$$\geq \lambda_{..} \left\{ kn(k-1)^2 v(v-1) \right\}^{-1} \sum_{i=1}^{v} (\lambda_{i.} - \lambda_{..}/v)^2 \geq 0 .$$

Again the first term of (2.3.2.9)

$$= n^{-1} \left\{ (v-2) r \sum_i (r_i - r)^2 - \sum_i (r_i - r)^3 \right\} \qquad \ldots(2.3.2.10)$$

on simplification.

If the design d is such that $r_i \leq (v-1)r$ for each $i = 1,2,\ldots,v$, then the quantity within second bracket of (2.3.2.10) is nonnegative and the result follows. So we assume that one of the r_i's (say r_1) is $> (v-1)r$.

Now for a given value of r_1 such that $(v-1)r < r_1 < vr$,

$$r_i - r < 0, \ i = 2,\ldots,v$$

and $$\sum_{i=2}^{v} (r_i - r) = -(r_1 - r).$$

So, we have $$\sum_{i=2}^{v} (r_i - r)^2 \geq (r_1 - r)^2/(v-1)$$

and $$-\sum_{i=2}^{v} (r_i - r)^3 = \sum_{i=2}^{v} (r - r_i)^3$$

$$\geq (r_1 - r)^3/(v-1)^2$$

Hence $$\sum_{i=1}^{v} (r_i - r)^2 \geq v(v-1)^{-1}(r_1 - r)^2$$

and $$-\sum_{i=1}^{v} (r_i - r)^3 \geq -v(v-2)(v-1)^{-2}(r_1 - r)^3$$

So, the first term of (2.3.2.9) is

$$\geq v(v-2).n^{-1}(v-1)^{-2}(r_1 - r)^2\left\{(v-1)r - (r_1 - r)\right\}$$

$$> 0.$$

Thus $Q_d \geq 0$ and so (2.3.2.6) holds and thus we arrive at the following result.

<u>Theorem 2.3.2.6</u>. *An MBGDD of type 1, if it exists is ψ_f-optimal of type 1 within the class $\mathcal{D}_1$ of all proper connected and binary design with given b,k and v under the assumption of mixed effects model.*

2.4 Optimality Results in a Two-way Heterogeneity Setting Under Mixed Effects Model.

Following Bose (1975) we obtain the reduced normal equations for the treatment effects based on a two-way design d with parameters b_1, b_2 and v (DFN 2.2.18).

$$C_d^{(M)} \hat{\underline{\tau}} = \underline{Q}_d^{(M)} \qquad \ldots(2.4.1)$$

with $C_d^{(M)} = w\, C_d^{(F)} + \bar{w}_1 (b_2^{-1} N_{d1} N'_{d1} - n^{-1} \underline{r}\,\underline{r}')$ ...(2.4.2)

if the mixed model is of type 1

and $C_d^{(M)} = w\, C_d^{(F)} + \sum_{u=1}^{2} \bar{w}_u\, n^{-1} (b_u N_{du} N'_{du} - \underline{r}\,\underline{r}')$ ...(2.4.3)

if the mixed model is of type 2,

where $C_d^{(F)}$ is the C_d-matrix for the same design d under fixed effects model and is given by

$$C_d^{(F)} = D_r - b_2^{-1} N_{d1} N'_{d1} - b_1^{-1} N_{d2} N'_{d2} + n^{-1} \underline{r}\,\underline{r}', \qquad \ldots(2.4.4)$$

D_r and $\underline{r}$ are as defined in (2.3.1) of section 2.3, N_{1d} and N_{2d} are as defined in (2.2.10), and $w, \bar{w}_1, \bar{w}_2$ are as defined in (2.2.9). Further, the relation (2.3.2) also hold here.

In the following sections we shall derive optimality results in the two-way setting under the two types of mixed effects models. Throughout the rest of this section the matrix $C_d^{(M)}$ will be assumed to be of the form (2.4.3) which is the most general in this case in the sense that the form (2.4.2) follows as a special case of (2.4.3) by writing $\bar{w}_2 = 0$.

The investigation of optimality results in the general two-way set-up was first made by Kiefer (1975) under the assumption of fixed effects model. He considered both regular and non-regular settings and obtained the following optimality results.

Theorem 2.4.1 (Kiefer (1975)). In regular settings the GYD, if it exists is universally optimal.

Theorem 2.4.2 (Kiefer (1975)). In the nonregular setting, a GYD if it exists is A- and E-optimal for $v \geq 4$ and D-optimal for $v \geq 6$.

We consider mixed effects models of the two types mentioned earlier and follow essentially with adequate manipulations, the approach of Kiefer (1975) in the new context, for the derivation of optimality results. We reestablish Kiefer's findings in the mixed effects model with one modification, viz when $w \leq \bar{w}_1 + \bar{w}_2$ for the mixed effects model of type 2, a GYD turns out to be universally optimal even in the nonregular setting.

2.4.1 Universal Optimality of a GYD in a Regular Setting

A regular setting is formally defined in DFN 2.2.20. Now the form of $C_d^{(M)}$ as given in (2.4.3) shows that it is completely symmetric for $d = d^* = GYD$.

Without loss of generality, we can assume $\bar{w}_1 \geq \bar{w}_2 \geq 0$, the case $\bar{w}_2 = 0$ corresponding to the mixed effects model of type 1, and then $C_d^{(M)}$ can be written as

$$C_d^{(M)} = (w - \bar{w}_1)\, C_d^{(F)} + E\,, \qquad \ldots(2.4.1.1)$$

where $E = (\bar{w}_1 - \bar{w}_2)\, C_d^{(2)} + \bar{w}_2\, C_d^{(0)}$,

$$C_d^{(2)} = D_r - b_2\, n^{-1}\, N_{d2}\, N_{d2}'$$

and $C_d^{(0)} = D_r - n^{-1}\, r\, r'$.

In (2.4.1.1), the matrix E is of the same form as the $C_d^{(M)}$ matrix in the one-way setting considered in (2.3.1.1) and d^* satisfying the conditions of lemma 2.3.1.1 maximises $\mathrm{tr}\, E$. Now the fact that d^* maximises $\mathrm{tr}\, C_d^{(M)}$ follows in view of Kiefer's (1975) comparable result in the context of fixed effect model. Hence, by proposition 2.3.1, we have

<u>**Theorem 2.4.1.1**</u>. A GYD, if it exists is universally optimal under mixed effects models of type 1 and type 2 within the class of proper and connected two-way designs with given b_1 , b_2 and v where v divides one of b_1 and b_2.

2.4.2 <u>Optimality of a GYD in a Nonregular Setting</u>.

From (2.4.1.1), $C_d^{(M)}$ can be expressed as

$$C_d^{(M)} = \sum_{i=1}^{2} (w - \bar{w}_i) C_d^{(i)} + (w - \bar{w}_1 - \bar{w}_2) C_d^{(0)}\,, \qquad \ldots(2.4.2.1)$$

where the matrices $C_d^{(i)}$, $i = 0,2$ are as defined in (2.4.1.1) and $C_d^{(1)} = D_r - b_1\, n^{-1}\, N_{d1}N_{d1}'$.

Now that $d^* = $ GYD maximises the trace of each $C_d^{(i)}$, $i = 0,1,2$ is clear and so in the case $w \leq \bar{w}_1 + \bar{w}_2$, d^* maximises $\operatorname{tr} C_d^{(M)}$. Hence by proposition 2.3.1, we have the following result.

Theorem 2.4.2.1. When the variances $\sigma^2, \sigma_1^2, \sigma_2^2$ satisfy the relation $w \leq \bar{w}_1 + \bar{w}_2$, a GYD, if it exists is universally optimal within the class of all proper and connected two-way designs with given b_1, b_2 and v, in the nonregular setting.

Remark - We observe that for the conditions $w \leq \bar{w}_1 + \bar{w}_2$ to be satisfied, both $\bar{w}_1$ and $\bar{w}_2$ have to be strictly positive and so this can arise only when the model is of type 2. Again under the model of type 2, $\bar{w}_1$ and $\bar{w}_2$ have to be sufficiently large, in order that $\bar{w}_1 + \bar{w}_2$ exceeds w. This means that σ_1^2 and σ_2^2 have to be sufficiently small, in comparison with σ^2. In particular if $b_1\sigma_1^2 < \sigma^2$ and $b_2\sigma_2^2 < \sigma^2$, then the condition $w < \bar{w}_1 + \bar{w}_2$ holds and so a GYD, if it exists in such a case is universally optimal.

We shall now explore the situation when $w \geq \bar{w}_1 + \bar{w}_2$. This can arise under mixed effects models of both the types and so the optimality results that follow are valid under each of them.

Following the notation of Kiefer (1975), we define

(i) $c^{(0)}(r) = r - r^2/n$

(ii) $c^{(u)}(r) = \max\limits_{\sum_{j=1}^{b_u} n_{ij}^{(u)} = r} \left\{ C_{ii}^{(u)} \right\}, \; u = 1,2$

where $C_{ii}^{(u)}$ is the ith diagonal element of $C_d^{(u)}$, $u = 1,2$ defined in (2.4.1.1) and (2.4.2.1), $n_{ij}^{(u)}$ being the (i,j)th element of N_{du},

(iii) $g^{(u)}(r) = n.c^{(u)}(r)$

(iv) $C(r) = \sum\limits_{u=1}^{2} (w - \bar{w}_u)c^{(u)}(r) - (w - \bar{w}_1 - \bar{w}_2)c^{(0)}(r)$

(v) $g(r) = n.C(r)$.

In all these expressions, the symbol r has been used to denote a scalar. Moreover, for simplifying notation the symbols have been used without the subscript i which would appear more natural. This is valid because the optimality criteria considered here are permutation invariant ((iii) of DFN 2.2.2) and so the subscript i has no role.

Now from (2.4.2.1), it is clear that

$$C(r) = \max \left\{ C_{d_{ii}}^{(M)} \right\} \text{sub to } \sum_{j=1}^{b_u} n_{ij}^{(u)} = r,\; u = 1,2,$$

and

$$g(r) = \sum_{u=1}^{2} (w - \bar{w}_u)(nr - b_u h(r, b_u)) - (w - \bar{w}_1 - \bar{w}_2)(nr - r^2) \quad \ldots(2.4.2.3)$$

where $h(r,b) = \min \sum_{j=1}^{b} n_{ij}^2$ sub to $\sum_{j=1}^{b} n_{ij} = r$

$$= r + (2r - b)[r/b] - b[r/b]^2$$ from Kiefer (1975).

If we write $s_u(r) = b_u(1 + 2[r/b_u])$

and $t_u(r) = b_u^2 [r/b_u](1 + [r/b_u])$, ...(2.4.2.3a)

then we have from (2.4.2.3), that

$$g(r) = \alpha + \beta r + \gamma r^2, \quad \text{...(2.4.2.4)}$$

where $\alpha = \sum_{u=1}^{2} (w - \bar{w}_u) t_u$,

$$\beta = \sum_{u=1}^{2} (w - \bar{w}_u)(n - s_u) - n(w - \bar{w}_1 - \bar{w}_2)$$

and $\gamma = w - \bar{w}_1 - \bar{w}_2$, writing t_u and s_u for $t_u(r)$ and $s_u(r)$ respectively.

So, $\Delta g(r) = g(r+1) - g(r)$

$$= \sum_{u=1}^{2} (w - \bar{w}_u)(n - s_u) - \gamma(n - 2r - 1). \quad \text{...(2.4.2.5)}$$

We shall now establish the E-optimality property of a GYD.

From the relevant result in section 3.2 of Kiefer (1975), it is clear that for the E-optimality of a GYD it is enough to show that

$$\max_H \min_i g(r_i) = g(\bar{r}),$$

where $H = \left\{ (r_1, \ldots, r_v) : r_i \right.$ is a nonnegative integer for each i, s.t. $\left. \sum_{i=1}^{v} r_i = v\bar{r} \right\}$.

This is satisfied if $\Delta g(r) \geq 0 \;\; \forall\, r \leq \bar{r}$. ...(2.4.2.6)

From (2.4.2.5), we have

$$\Delta g(r) \geq w(n - b_1 - b_2 - 2r + 1) + \sum_{u=1}^{2} \bar{w}_u (b_u - 1)$$

$$> w(n - b_1 - b_2 - 2\bar{r} + 1) \quad \text{for any } r < \bar{r},$$

since $b_u > 1$, $u = 1,2$

$$= w\left\{(b_1 - 1)(b_2 - 1) - 2b_1 b_2 / v\right\} > 0 \quad \forall\, v \geq 4.$$

Hence (2.4.2.6) is satisfied and we have the following result.

__Theorem 2.4.2.2__. A GYD, if it exists is E-optimal in the class $\mathcal{D}$ of theorem 2.4.2.1 in the nonregular setting, whenever $v \geq 4$.

Below is discussed the A- and D- optimality of a GYD in a nonregular setting.

Let us recall the definitions of step (b) of section 3.4 of Kiefer (1975).

$$\mathcal{N} = \{N : 0 \leq N \leq n = b_1 b_2,\; N = ib_1 \text{ or } ib_2, \text{ where } i \text{ is integral}\},$$

$$\mathcal{M} = \{m : m \in \mathcal{N},\; m \leq b_1 b_2 / 2\}.$$

$[C, D]$ is called an elementary internal whenever C and D are consecutive members of $\mathcal{N}$. Let $[C_0, D_0]$ be the elementary interval containing $\bar{r}$ and it is called the basic internal.

Now we observe that $\Delta g(r)$ satisfies all the properties e(i) through (v) of its fixed model analogue dealt at length in

Kiefer (1975) and so the conclusions drawn in the steps d and e of section 3.4 of Kiefer (1975) also hold. For the sake of completeness, the properties are enumerated below with a brief outline of proof, when it is necessary and the method differs from that of Kiefer (1975).

Let us recall that we are here considering the case $\gamma = w - \bar{w}_1 - \bar{w}_2 > 0$. The case $\gamma \leq 0$ has already been explored when a GYD satisfies universal optimality.

Lemma 2.4.2.3. $g(r)$ satisfies the following properties :

(i) $\Delta g(r)$ is linear in r and is increasing within each elementary interval $[C, D]$.

(ii) $\Delta g(r) \geq 0$ (i.e. $g(r)$ is increasing in r) $\forall\, r \leq D_0 - 1$.

(iii) $g(r)$ is symmetric about $n/2$.

(iv) If $c_1, c_2 \in \mathcal{N}$ with $c_1 < c_2$, then $\Delta g(c_1) \geq \Delta g(c_2)$.

(v) $g(r)$ is nondecreasing on $\mathcal{M}$ and nonincreasing on the remaining part of $\mathcal{N}$.

Proof. (i) follows from the expression for $g(r)$ in 2.4.2.5.

(ii) We shall show that $\Delta g(r) > \gamma \geq 0 \;\forall\, r \leq D_0 - 1$.

From (2.4.2.5),

$$\Delta g(r) - \gamma = \sum_{u=1}^{2} (w - \bar{w}_u)(n - s_u) - \gamma(n - 2r)$$

$$\geq w(n - b_1 - b_2 - 2r) + \sum_{u=1}^{2} \bar{w}_u b_u$$

$$> 0 \text{ for any } r < \bar{r} \;\; \forall\; v \geq 4.$$

That $\Delta g(r) - \gamma \geq 0$ for $r \in [\bar{r}, D_o - 1]$ follows from (..).

(iii) Since $g^{(0)}(r)$ is symmetric about $n/2$, we are to show that $g^{(i)}(r)$ is symmetric about $n/2$, $i = 1,2$.

But $g^{(i)}(r) = g^{(0)}(r) - b_i^2(f_i(r) - f_i^2(r))$

where $f_i(r) = r/b_i - [r/b_i]$, and so, it is enough to show that $\phi_i(r) = f_i(r)(1 - f_i(r))$ is symmetric about $b_1 b_2 / 2$. We shall consider $i = 1$. The other case will follow similarly.

Actually, we shall show that

$$f_1(b_1 b_2 / 2 + r) = 1 - f_1(b_1 b_2 / 2 - r).$$

<u>Case - 1</u>. b_2 is even.

Then $[(b_1 b_2 / 2 + r) / b_1] = b_2/2 + [r / b_1]$

and $[(b_1 b_2 / 2 - r) / b_1] = b_2 / 2 - [r / b_1] - 1$.

So, $f_1(b_1 b_2 / 2 + r) = f_1(r)$

and $f_1(b_1 b_2 / 2 - r) = 1 - f_1(r) = 1 - f_1(b_1 b_2 / 2 + r)$.

<u>Case - 2</u>. b_2 is odd and $f_1(r) \geq 1/2$.

Then $[(b_1 b_2 / 2 + r) / b_1] = [b_2 / 2] + [r / b_1] + 1$

and $[(b_1 b_2 / 2 - r) / b_1] = [b_2 / 2] - [r / b_1] - 1$

$\Rightarrow f_1(b_1 b_2 / 2 + r) = f_1(r) - 1/2$

and $f_1(b_1 b_2 / 2 - r) = 3/2 - f_1(r) = 1 - f_1(b_1 b_2 / 2 + r)$.

<u>Case - 3</u>. b_2 is odd and $f_1(r) < 1/2$.

Then $[(b_1 b_2/2 + r)/b_1] = [b_2/2] + [r/b_1]$

and $[(b_1 b_2/2 - r)/b_1] = [b_2/2] - [r/b_1]$

$\Rightarrow f_1(b_1 b_2/2 + r) = f_1(r) + 1/2$

and $f_1(b_1 b_2/2 - r) = 1/2 - f_1(r) = 1 - f(b_1 b_2/2 + r)$.

Hence the result.

(iv) It is enough to show this result in the case when c_1, c_2 are consecutive members of $\mathcal{N}^{o}$.

From (2.4.2.5), we can write

$$\Delta g(c_2) - \Delta g(c_1) = 2\gamma(c_2 - c_1) - 2\sum_{u=1}^{2}\left\{(w-\bar{w}_u)b_u([c_2/b_u] - [c_1/b_u])\right\}$$

$$= 2\left\{\gamma(c_2 - c_1) - \sum_{u=1}^{2}(w-\bar{w}_u)\delta u\right\}, \text{ say,}$$

where one of the δu's is 0 and the other is $\geq c_2 - c_1$.

$$\therefore \Delta g(c_2) - \Delta g(c_1) \leq 2(c_2 - c_1) \cdot \begin{cases} [(w - \bar{w}_1 - \bar{w}_2) - (w - \bar{w}_1)] \\ \text{or } [(w - \bar{w}_1 - \bar{w}_2) - (w - \bar{w}_2)] \end{cases}$$

$$\leq 0 \text{ as } c_2 > c_1 \text{ and } \bar{w}_1, \bar{w}_2 \geq 0.$$

(v) Proof is same as that in Keifer (1975).

Finally to make use of Kiefer's proof technique we are left with the verification of the concavity of $q(r)$ in the basic interval where $q(r) = -f(g(r))$ and $f(x) = x^{-1}$ in the case of

A-optimality and $f(x) = \log x$ in the case of D-optimality.

In other words, we are to show that

$$f(g(r+1)) + f(g(r-1)) - 2f(g(r)) \geq 0 \quad \text{for } C_o + 1 \leq r < D_o \qquad \ldots(2.4.2.7)$$

where f is either of the forms out lined.

Now $f(x) = x^{-1}$ implies that (2.4.2.7) is equivalent to

$$\Gamma_A(r) = 3\gamma^2 r^2 + 3\beta\gamma r + \beta^2 - \alpha\gamma - \gamma^2 \geq 0 \qquad \ldots(2.4.2.8)$$

and similarly $f(x) = \log x$ reduces (2.4.2.7) into

$$\Gamma_D(r) = 2\gamma^2 r^2 + 2\beta\gamma r + \beta^2 - 2\alpha\gamma - \gamma^2 \geq 0 . \qquad \ldots(2.4.2.9)$$

We observe that

$$\frac{d\,\Gamma_A(r)}{dr} = 3\gamma\,(\beta + 2\gamma r)$$

and $$\frac{d\,\Gamma_D(r)}{dr} = 2\gamma\,(\beta + 2\gamma r) .$$

Now $\beta + 2\gamma r = g(r) - \gamma > 0$ for $C_o + 1 \leq r < D_o$ follows from (ii) of lemma 2.4.2.2.

So, both the derivatives are positive and it is sufficient to show that

$$\Gamma_A(C_o + 1) \geq 0 \text{ and } \Gamma_D(C_o + 1) \geq 0 . \qquad \ldots(2.4.2.10)$$

Now if $\Gamma_A^{(F)}(r)$ and $\Gamma_D^{(F)}(r)$ denote the expressions analogous to $\Gamma_A(r)$ and $\Gamma_D(r)$ respectively under fixed effects which is a constant and hence $s_i(r)$ and $t_i(r)$, $i = 1,2$, are also independent of r for any r in the basic interval. We shall henceforth refer to them as simply s_i and t_i, respectively. As a consequence, α, β, $\alpha^{(F)}$ and $\beta^{(F)}$ are all independent of r for $C_o + 1 \leq r < D_o$.

Let us now indicate a few algebraic relations between the variables under consideration in the form of a lemma.

Lemma 2.4.2.4. For any integer $r \varepsilon (C_o, D_o)$ the following relations hold

(i) $s_i^2 = b_i^2 + 4t_i$, $i = 1,2$

(ii) $n - s_i > 0$, $i = 1,2$, $v \geq 4$

(iii) $r - 1 \geq b_1, b_2$

(iv) $\beta^{(F)} > 0$ for $v \geq 6$

and $\beta^{(F)} > -(b_1 + b_2)$ for $v = 4$

(v) $\beta^{(F)} + r > 0$ for $v \geq 4$.

Proof. (i) follows from the definitions of s_i and t_i as given in (2.4.2.3a)

(ii) $n - s_i \geq b_1 b_2 - b_i - 2\bar{r} > 0$ for $v \geq 4$, $i = 1,2$

(iii) follows from (2.4.2.15)

since $\bar{r} = b_1 b_2 / v \geq b_1, b_2$.

(iv) Because of (2.4.2.15), $s_i = b_i + 2b_i [\bar{r} / b_i]$

and hence $\beta^{(F)} = n - s_1 - s_2 > n - b_1 - b_2 - 4\bar{r} = z$ say.

Then $z = -(b_1 + b_2)$ when $v = 4$

and $z \geq b_1 b_2 / 3 - b_1 - b_2 > 0$ for $v \geq 6$.

(v) From (2.4.2.15), it follows that

$$\beta^{(F)} + r \geq \beta^{(F)} + b_u [\bar{r} / b_u] + 1$$

$$> n - b_1 - b_2 - 3b_u [\bar{r} / b_u] + 1 \quad \text{using } (2.4.2.14)$$

$$> \frac{1}{4} (b_1 - 4) b_2 - 4) \quad \text{for } v \geq 4$$

$$> 0.$$

Hence the lemma.

Now without loss of generality, we can assume that $C_o = i b_1$ where i is an integer. (It can be easily seen that all the future developments would go through when $C_o = ib_2$). Then for $r = C_o + 1$, some further relations hold as shown in the following lemma.

Lemma 2.4.2.5. For $r = C_o + 1$ and $C_o = ib_1$

(i) $[r / b_1] = (r - 1) / b_1$

(ii) $s_1 - 2r = b_1 - 2 > 0$

(iii) $t_1 = (r-1)(r + b_1 - 1)$

(iv) $2t_1 - s_1(b_1 + b_2) / 2 > 0$

(v) $r(n - s_1) - t_1 > 0$.

Proof. (i), (ii) and (iii) are immediate from the definitions of r, s_1 and t_1.

(iv) $2t_1 - s_1(b_1 + b_2)/2$

$$= 2(r-1)(r + b_1 - 1) - (b_1 + b_2)(r + b_1 - 1 + r - 1)/2$$

$$= (r + b_1 - 1)(r - 1 - b_1/2 - b_2/2)$$

$$+ (r-1)(r + b_1 - 1 - b_1/2 - b_2/2)$$

≥ 0 in view of (iii) of Lemma 2.4.2.4.

(v) $r(n - s_1) - t_1 > r\{n - b_1 - 2r + 2 - (r + b_1 - 1)\} + r + b_1 - 1$

$$= r(n - 3r - 2b_1 + 4) + b_1 - 1$$

> 0 for $v \geq 4$ and $b_1, b_2 \geq 6$.

Having the algebraic results of Lemmas 2.4.2.4 and 2.4.2.5 at hand, we now proceed to the proofs of (2.4.2.11), (2.4.2.12) and (2.4.2.13).

Let us express α and β in the following form

$$\alpha = (w - \bar{w}_2)\,\alpha^{(F)} + f_1(\bar{w}_1, \bar{w}_2)$$

where $f_1(\bar{w}_1, \bar{w}_2) = -(\bar{w}_1 - \bar{w}_2)t_1$. ...(2.4.2.16)

$$\beta = (w - \bar{w}_2)\,\beta^{(F)} + f_2(\bar{w}_1, \bar{w}_2),$$

where $f_2(\bar{w}_1, \bar{w}_2) = \bar{w}_1 s_1 + \bar{w}_2(n - s_1)$. ...(2.4.2.17)

Also, we write $\gamma = (w - \bar{w}_2) - \bar{w}_1$. ...(2.4.2.18)

From (2.4.2.17) and (2.4.2.18),

$$\beta + 2\gamma r = (w-\bar{w}_2)(\beta^{(F)} + 2r) + f_3(\bar{w}_1, \bar{w}_2)$$

where $f_3(\bar{w}_1, \bar{w}_2) = \bar{w}_1(s_1 - 2r) + \bar{w}_2(n - s_1)$

and $\beta + \gamma r = (w-\bar{w}_2)(\beta^{(F)} + r) + \bar{w}_1(s_1 - r) + \bar{w}_2(n - s_1) > 0$

for $C_0 + 1 \leq r < D_0$ and $v \geq 4$ in view of (ii) and (iv) of lemma 2.4.2.3 and (ii) of lemma 2.4.2.3 as $\bar{w}_1, \bar{w}_2$ and $w - \bar{w}_2$ are positive. This implies (2.4.2.11) as a direct consequence.

Let us now look at (2.4.2.12). We can write

$$\begin{aligned}\Gamma_D(r) &= \tfrac{1}{2}(\beta + 2\gamma r)^2 + \tfrac{1}{2}\beta^2 - 2\alpha\gamma - \gamma^2 \\ &= (w-\bar{w}_2)^2\left\{\tfrac{1}{2}(\beta^{(F)} + 2r)^2 + \tfrac{1}{2}(\beta^{(F)})^2 - 2\alpha^{(F)} - 1\right\} \\ &\quad + \tfrac{1}{2}(f_2(\bar{w}_1, \bar{w}_2))^2 + \tfrac{1}{2}(f_3(\bar{w}_1, \bar{w}_2))^2 \\ &\quad + (w-\bar{w}_2)(\beta^{(F)} + 2r)f_2(\bar{w}_1, \bar{w}_2) \\ &\quad + (w-\bar{w}_2)\beta^{(F)}f_3(\bar{w}_1, \bar{w}_2) \\ &\quad + 2(w-\bar{w}_2)\bar{w}_1\alpha^{(F)} - 2(w-\bar{w}_2)f_1(\bar{w}_1, \bar{w}_2) \\ &\quad + 2\bar{w}_1 f_1(\bar{w}_1, \bar{w}_2) + \bar{w}_1(\gamma + w - \bar{w}_2)\,.\end{aligned}$$

From this, omitting the last term in the expression which is positive, we have

$$\Gamma_D(r) - (w-\bar{w}_2)^2\,\Gamma_D^{(F)}(r) > F_1(\bar{w}_1, \bar{w}_2) + (w-\bar{w}_2)F_2(\bar{w}_1, \bar{w}_2)$$

where $F_1(\bar{w}_1, \bar{w}_2) = \tfrac{1}{2}(f_2(\bar{w}_1, \bar{w}_2))^2 + \tfrac{1}{2}(f_3(\bar{w}_1, \bar{w}_2))^2 + 2\bar{w}_1 f_1(\bar{w}_1, \bar{w}_2)$

and $F_2(\bar{w}_1, \bar{w}_2) = (\beta^{(F)} + 2r)f_2(\bar{w}_1, \bar{w}_2) + \beta^{(F)}f_3(\bar{w}_1, \bar{w}_2)$

$$+ 2\bar{w}_1 \alpha^{(F)} - 2f_1(\bar{w}_1, \bar{w}_2).$$

We shall now show that

$F_1(\bar{w}_1, \bar{w}_2) > 0$ and $F_2(\bar{w}_1, \bar{w}_2) > 0$, when $r = C_0 + 1$.
We observe as follows (throughout the rest of this subsection, r should be read as C_0+1).

Coefficient of $\bar{w}_1^2$ in $F_1(\bar{w}_1, \bar{w}_2)$

$$= (1/2)(s_1 - 2r)^2 + (1/2)s_1^2 - 2t_1 > 0,$$

by vitue of (i) of lemma 2.4.2.4.

Coefficient of $\bar{w}_1\bar{w}_2$ in $F_1(\bar{w}_1, \bar{w}_2)$

$$= s_1(n - s_1) + (s_1 - 2r)(n - s_1) + 2t_1$$

> 0 by (ii) of lemma 2.4.2.4 and (ii) of lemma 2.4.2.5.

Coefficient of $\bar{w}_2^2 = (n - s_1)^2 > 0$

Hence $F_1(\bar{w}_1, \bar{w}_2) > 0$.

Coefficient of $\bar{w}_2$ in $F_2(\bar{w}_1, \bar{w}_2)$

$$= 2\left\{(\beta^{(F)} + r)(n - s_1) - t_1\right\}$$

$> 2(r(n - s_1) - t_1)$ for $v \geq 6$ from (iv) of lemma 2.4.2.4.

Coefficient of $\bar{w}_1$ in $F_2(\bar{w}_1, \bar{w}_2)$

$$= (\beta^{(F)} + 2r)s_1 + \beta^{(F)}(s_1 - 2r) + 2(\alpha^{(F)} + t_1)$$

> 0 since each term is > 0.

So, $F_2(\bar{w}_1, \bar{w}_2) > 0$.

Hence (2.4.2.12) holds, as $w - \bar{w}_2 > 0$

(2.4.2.11) and (2.4.2.12) together imply D- and A-optimality of a GYD under mixed effects model of both the types whenever $v \geq 6$. For the proof of A-optimality in the case $v = 4$, we need to show (2.4.2.13).

$\Gamma_A(r)$ can be written as
$$\Gamma_A(r) = (3/4)(\beta + 2\gamma r)^2 + (1/4)\beta^2 - \alpha\gamma - \gamma^2 = (w - \bar{w}_2)^2 \Gamma_A^{(F)}(r) + F_3(\bar{w}_1, \bar{w}_2)$$
$$+ (w - \bar{w}_2) F_4(\bar{w}_1, \bar{w}_2) + \bar{w}_1(\gamma + w - \bar{w}_2),$$

where
$$F_3(\bar{w}_1, \bar{w}_2) = (3/4) f_2^2(\bar{w}_1, \bar{w}_2) + (1/4) f_3^2(\bar{w}_1, \bar{w}_2) + \bar{w}_1 f_1(\bar{w}_1, \bar{w}_2),$$

and
$$F_4(\bar{w}_1, \bar{w}_2) = (3/2)(\beta^{(F)} + 2r) f_2(\bar{w}_1, \bar{w}_2) + (1/2)\beta^{(F)} f_3(\bar{w}_1, \bar{w}_2) + \bar{w}_1 \alpha^{(F)} - f_1(\bar{w}_1, \bar{w}_2).$$

So, it is enough to show that

$$F_3(\bar{w}_1, \bar{w}_2) > 0 \text{ and } F_4(\bar{w}_1, \bar{w}_2) > 0.$$

As regards $F_3(\bar{w}_1, \bar{w}_2)$, in it,

the coefficient of $\bar{w}_1^2 = (3/4)(s_1 - 2r)^2 + (1/4)s_1^2 - t_1$

> 0 by (i) of lemma 2.4.2.4,

the coefficient of $\bar{w}_1 \bar{w}_2 = (3/2)(s_1 - 2r)(n - s_1) + (1/2)s_1(n - s_1) + 2t_1$

> 0 by (ii) of lemma 2.4.2.4 and (ii) of lemma 2.4.2.5, the coefficient of $\bar{w}_2^2 = (3/4)(n-s_1)^2 + (1/4)(n-s_1)^2 > 0$.

Hence $F_3(\bar{w}_1, \bar{w}_2) > 0$.

Similarly, as regards $F_4(\bar{w}_1, \bar{w}_2)$, in it

the coefficient of $\bar{w}_1 = (3/2)(\beta^{(F)} + 2r)(s_1 - 2r) + (1/2)\beta^{(F)}s_1 + 2t_1 +$

$> -(1/2)(b_1 + b_2)s_1 + 2t_1 + t_2$ for $v = 4$, by (ii) of lemma 2.4.2.5,

$> t_2$ because of (iv) of lemma 2.4.2.5,

> 0

The coefficient of $\bar{w}_2 = (n-s_1)\left\{(3/2)(\beta^{(F)} + 2r) + (1/2)\beta^{(F)}\right\} - t_1$

$= (2\beta^{(F)} + 3r)(n - s_1) - t_1$

$> r(n-s_1) - t_1$ for $v \geq 4$, using (v) of lemma 2.4.2.4

> 0 by (v) of lemma 2.4.2.4.

Hence $F_4(\bar{w}_1, \bar{w}_2) > 0$.

So, (2.4.2.13) is proved and we have in view of (2.4.2.11) and (2.4.2.12) the following optimality result.

Theorem 2.4.2.6. A GYD, if it exists, is A-optimal whenever $v \geq 4$ and D-optimal when $v \geqslant 6$. within the class of proper and connected two-way designs, with given b_1, b_2 and v under the assumption of mixed effects model with either or both of the row effects and column effects random and $w > \bar{w}_1 + \bar{w}_2$.

When $b_1 = b_2$, Cheng (1981) has shown that the matrix $C_d^{(F)}$ is completely symmetric if $N_d = [N_{d1} : N_{d2}]$ is the incidence matrix of a BBD, i.e. d is a PYD, and in such a situation, Cheng (1981) has shown that a PYD satisfies the optimality properties satisfied by a GYD.

When the model is one of mixed effects of type II, in the case $b_1 = b_2'$, if the additional condition $\bar{w}_1 = \bar{w}_2$, i.e. $\sigma_1^2 = \sigma_2^2$ is also satisfied (which is satisfied when the row effects and column effects belong to the same population), then the $C_d^{(M)}$ matrix is completely symmetric, if d is a PYD. So in this type of situations all the optimality properties of a GYD will be satisfied by a PYD.

2.5 Optimality Results in a Balanced m-way ($m > 2$) Setting.

A balanced m-way setting has been defined in section 2 (DFN 2.2.24). As stated there, it includes two different cases, viz., a balanced complete m-way setting which has been considered by Cheng (1978a) and balanced incomplete m-way setting where all the factor combinations are not present. We consider the general balanced setting under the assumptions of both fixed effects model and mixed effects model with all factor effects random.

In the general m-way setting, the C_d matrix under fixed effects model is of a similar form as Cheng (1978a) derived for the complete setting considered by him. Under mixed effects model, the corresponding C_d-matrix has been worked out. Optimality results in both the situations have been investigated and the main results are similar to those of Cheng (1978a), of course, with the difference in the setting. Moreover, when $w \leq \sum_{i=1}^{m} \bar{w}_i$, one can obtain a universally optimal design even in the nonregular case, the latter result being an extension of a similar result in the context of two-way setting.

2.5.1 The C_d-Matrices under Fixed Effects Model and Mixed Effects Model with All Factor Effects Random.

The matrices B_i's written for the ith factor in a balanced m-way setting given by an OAVS$(n,m,b_1,\ldots,b_m, 2)$ as stated in DFN 2.2.24 are observed to satisfy the following relations.

$$B_i B_i' = k_i I_{b_i}, \text{ where } k_i = n/b_i, \; i = 1,2,\ldots,m \quad \ldots(2.5.1.1)$$

$$\text{and } B_i B_{i'}' = n/(b_i b_{i'}) J_{b_i \times b_{i'}}, \; i \neq i', \; i,i' = 1,2,\ldots,m \quad \ldots(2.5.1.2)$$

Let Σ denote the dispersion matrix of $\underline{y}$ and so $\Sigma = \sigma^2 I_n$ in the case of fixed effects model. Let $N_{di}(v \times b_i)$ be the incidence matrix of the treatments with the levels of the ith factor, $i = 1,2,\ldots,m$ for a given design d. Then, we have

$$A.1_n = \underline{r}$$

$$B_i . 1_n = k_i . 1_{b_i}, \; i = 1, 2, \ldots, m \qquad \ldots(2.5.1.3)$$

$$A A' = D_r$$

and $A B_i' = N_{di}, \; i = 1, 2, \ldots, m$.

Moreover, if $\underline{T}$ denotes the vector of treatment totals and $\underline{R}_i$ denotes the vector of ith factor totals for each i, $i = 1, 2, \ldots, m$ and G denotes the grand total, then we have

$$\underline{T} = A \underline{Y}$$

$$\underline{R}_i = B_i \underline{Y} \qquad \ldots(2.5.1.4)$$

and $G = 1_n' \underline{Y}$

using the relations (2.5.1.1), (2.5.1.2), (2.5.1.3) , (2.5.1.4) and the expression for Σ, the set of reduced normal equations for treatment effects can be written in the following form under fixed effects model.

$$(D_r - \sum_{i=1}^{m} k_i^{-1} N_{di} N_{di}' + (m-1) n^{-1} \underline{r}\,\underline{r}') \hat{\underline{\tau}}$$

$$= \underline{T} - \sum_{i=1}^{m} k_i^{-1} N_{di} \underline{R}_i + (m-1) n^{-1} \underline{r}\, G$$

where k_i is as given in (2.5.1.1).

Hence the C_d-matrix for a given design d under fixed effects model is given by

$$C_d^{(F)} = \sum_{i=1}^{m} C_d^{(i)} - (m-1)\, C_d^{(0)} \qquad \ldots(2.5.1.5)$$

where $C_d^{(i)} = D_r - k_i^{-1} N_{di}N_{di}'$, $i = 1,2,\ldots,m$ and $C_d^{(0)}$ is as given in (2.3.1.1).

Under the assumption of mixed effects model with all the factor effects random, let the vector of ith factor effects $\underline{\beta}_i$ have $E(\underline{\beta}_i) = 0$, $D(\underline{\beta}_i) = \sigma_i^2 I_{b_i}$.

Then the dispersion matrix of y is

$$\Sigma = \sigma^2 I_n + \sum_{i=1}^{m} \sigma_i^2 B_i'B_i . \qquad \ldots(2.5.1.6)$$

Henceforth in multiway setting, by mixed effects model we mean the model in which all factor effects are random.

Let us write $K_i = B_i'B_i$, $i = 1,2,\ldots,m$.

Then $K_i J_{n\times n} = k_i J_{n\times n}$ and $K_i^2 = k_i K_i$, $i = 1,2,\ldots,m$

$\ldots(2.5.1.7)$

Moreover, $K_iK_{i'}' = n/(b_ib_{i'}) J_{n\times n}$, $i \neq i'$, $i,i' = 1,2,\ldots,m$.

$\ldots(2.5.1.8)$

From the relations (2.5.1.7) and (2.5.1.8), we obtain

$$\Sigma^{-1} = wI_n - \sum_{i=1}^{m} ((w - \bar{w}_i)/k_i)K_i + z.J_n \qquad \ldots(2.5.1.9)$$

where $w = (\sigma^2)^{-1}$, $\bar{w}_i = (\sigma^2 + k_i\sigma_i^2)^{-1}$ and z is a quantity involving w, $\bar{w}_i$ and k_i, $i = 1,2,\ldots,m$.

Now in this case, the reduced normal equations for the treatment effects are given by

$$(A\Sigma^{-1} A' - n^{-1} r 1_n'\Sigma^{-1} A')\,\hat{\underline{\tau}} = (A\Sigma^{-1} - n^{-1} 1_n'\Sigma^{-1})\,\underline{y} .$$

This can be simplified to the following form, using (2.5.1.9).

$$C_d^{(M)} \hat{\underline{\tau}} = Q_d^{(M)} \qquad \ldots(2.5.1.10)$$

where $C_d^{(M)} = wD_r - \sum_{i=1}^{m} (w-\bar{w}_i)k_i^{-1} N_{di}N_{di}'$

$$+ \left(\sum_{i=1}^{n} (w-\bar{w}_i) - w\right)n^{-1}\underline{r}\,\underline{r}'$$

and $Q_d^{(M)} = w\underline{T} - \sum_{i=1}^{m} (w-\bar{w}_i)k_i^{-1}N_{di}R_i + \left(\sum_{i=1}^{m} (w-\bar{w}_i) - w\right)n^{-1}\underline{r}\,G$.

2.5.2. Optimality Results in the Regular Setting

Regularity with respect to the various factors in an m-way setting is defined in DFN 2.2.25.

From (2.5.1.10) the matrix $C_d^{(M)}$ can be written as

$$C_d^{(M)} = \sum_{i=1}^{m} (w-\bar{w}_i)C_d^{(i)} - \left(\sum_{i=1}^{n} (w-\bar{w}_i) - w\right)C_d^{(0)} \qquad \ldots(2.5.2.1)$$

where $C_d^{(i)}$, $i = 0,1,\ldots,m$ are as defined in (2.5.1.6).

This together with (2.5.1.6) shows that both the matrices $C_d^{(F)}$ and $C_d^{(M)}$ are of the form

$$C_d^{(Q)} = \sum_{i=1}^{m} a_i^{(Q)}(C_d^{(i)} - C_d^{(0)}) + a_o^{(Q)}C_d^{(0)} \qquad \ldots(2.5.2.2)$$

where $Q = F$ or M, $a_i^{(F)} = 1$, $i = 0,1,\ldots,m$,

and $a_i^{(M)} = w-\bar{w}_i$, $i = 1,2,\ldots,n$,

$a_o^{(M)} = w$,

and so $a_i^{(Q)} > 0$, $Q = F$ or M, $i = 0,1,\ldots,m$. ...(2.5.2.3)

Now we know that for each i, $1 \leq i \leq m$,

$$C_d^{(i)} - C_d^{(0)} = -k_i^{-1}(N_{d_i}N_{d_i}' - b_i^{-1}\mathbf{r}\mathbf{r}').$$

In a setting regular with respect to the factor i, if $\bar{d}$ is a design which is balanced with respect to the factor i, then $C_d^{(i)} - C_d^{(0)}$ is a null matrix and i all other cases $C_d^{(i)} - C_d^{(0)}$ is a nonpositive definite matrix, $i = 1,2,\ldots,m$. So from (2.5.2.2), in view of (2.5.2.3), the following result which is analogous Theorem 3.1 of Cheng (1978a) is immediate.

<u>Theorem 2.5.2.1</u>. Let $M(v;n;m;b_1,\ldots,b_m)$ be a setting which is regular relative to a set of $m-m'$ factors, say $S = \{m'+1,\ldots,m\}$ with $m' < m$. Also let d^* be a design which is balanced relative to the factors in S and is ϕ-optimal (where ϕ is a nonincreasing criterion in the sense that $\phi(C) \leq \phi(D)$, whenever $C-D$ is nonnegative definite), when considered as a design in the setting $M(v;n;m';b_1,\ldots,b_{m'})$. Then d^* is also ϕ-optimal in the setting $M(v;n;m;b_1,\ldots,b_m)$.

An immediate consequence of this result, by virtue of theorem 3.1 of Cheng (1978b) regarding the ϕ_f-optimality of type 1 of the MBGDD's of type 1 is the following.

<u>Corollary 2.5.2.1.1</u>. If the setting $M(v;n;m;b_1,\ldots,b_m)$ is regular relative to the factors $2,3,\ldots,m$, then a design d, which is balanced relative to the ith factor for each

$i = 2,3,\ldots,m$ and is an MBGDD of type 1 in the one-way setting $M(v;n;1;b_1)$ is optimal wrt every ψ_f-optimality criterion of type 1 in the class of all proper and connected designs in the setting $M(v;n;m;b_1,\ldots,b_m)$ under fixed effects model.

Similarly under mixed effects model, in view of theorem 2.3.2.6 and a result of Khatri and Shah (1981) mentioned in section 2.3 in the context of optimal behaviour of the MBGDD's of type 1, we have the following corollary to theorem 2.5.2.1.

Corollary 2.5.2.1.2. With the same setting as in corollary 2.5.2.1.1, the design d^* of corollary 2.5.2.1.1 is optimal with respect to every ψ_f-optimality criterion of type 1 in the class of all proper, connected binary and/or equireplicate designs under the assumption of mixed effects model.

The following corollary follows trivially from the theorem 2.5.2.1.

Corollary 2.5.2.1.3. In a completely regular balanced m-way setting, a BYHR, if it exists is universally optimal within the class of all proper and connected designs under the assumption of fixed effects model as well as mixed effects model.

Remark. The corollary 2.5.2.1.3 implies the universal optimality of the F-hyperrectangles of Cheng (1980b) as a special case. These F-hyperrectangles, we may note, are generalisations of grace latin square designs and F-square designs, the optimality properties of which with regard to another class of optimality criteria, termed u-optimality in Kurochka (1971) have been proved in Kurochka (1972).

2.5.3 Optimality Results in a Non-regular Setting

Let us consider a nonregular setting $M(v;n;m;b_1,\ldots,b_m)$ and let $\mathcal{D}$ be the class of all proper and connected designs in the setting. Since $d^* = \mathrm{BYHR}(v,n,m,b_1,\ldots,b_m)$ (defined in DFN 2.2.27) maximises the trace of each $C_d^{(i)}$, $i = 0,1,\ldots,m$, within $\mathcal{D}$, the form of $C_d^{(M)}$ as written in (2.5.2.1) shows that in the case $(m-1)w \leq \sum_{i=1}^{m} \bar{w}_i$, d^* maximises the trace of $C_d^{(M)}$. Hence by proposition 2.3.1, we have the following result.

Theorem 2.5.3.1. A $\mathrm{BYHR}(v,n,m,b_1,\ldots,b_m)$ if it exists is universally optimal within $\mathcal{D}$ in the case of mixed effects model for which $(m-1)w \leq \sum_{i=1}^{m} \bar{w}_i$.

Remark. This result is a generalisation of theorem 2.4.2.1 in the multiway setting.

Henceforth, in the multiway set up, the mixed effects model with all factor effects random and the variances of the factor effects satisfying the relation $(m-1)w \leq \sum_{i=1}^{m} \bar{w}_i$ will be referred to as mixed effects model I. The other mixed effects model considered here, viz., with all factor effects random and

$(n-1)w > \sum_{i=1}^{n} \bar{w}_i$ will be referred to as mixed effects model II.

We shall now investigate the optimality properties of a BYHR under fixed model as well as mixed effects model II extending the proof techniques used in two-way setting to the present case of n-way setting.

Now $C_d^{(M)}$ is the general form of the C_d-matrix, which reduces to $C_d^{(F)}$ on substituting $\bar{w}_i = 0$, $i = 1,2,\ldots,n$. So for proving an optimality result, it may be sufficient to consider only $C_d^{(M)}$ and that is what is done for proving the E-optimality of a BYHR in the setting $M(v;n;m;b_1,\ldots,b_m)$. For A- and D-optimality, however, the proofs are more complicated, even though we assume $b_1 = b_2 = \ldots = b_m$. Therefore we consider fixed effects model first for proving the optimality results and then using these results prove the corresponding analogous results under the assumption of mixed effects model II.

We use the same notation as in section 2.4.2, with the modification that in (ii) and (iii) the superscript u in $n_{ij}^{(u)}$ can take values 1 through n.

Now it is clear that

$$C(r) = \text{Max}\left\{C_{ii}^{(M)}\right\} \text{ sub to } \sum_{j=1}^{b_u} n_{ij}^{(u)} = r,\ u = 1,\ldots,m$$

$\therefore\ g(r) = nC(r)$

$$= \sum_{u=1}^{m} (w - \bar{w})(nr - b_u h(r\,,\, b_u)) - ((m-1)\,w - \sum_{u=1}^{m} \bar{w}_u)(nr - r^2)$$

$$\ldots(2.5.3.1)$$

where h is as in (2.4.2.3).

So, $\Delta\, g(r) = g(r+1) - g(r)$

$$= \sum_{u=1}^{m} (w - \bar{w}_u)(n - b_u - 2b_u\, [r / b_u])$$

$$-((m-1)\,w - \sum_{u=1}^{m} \bar{w}_u)(n - 2r - 1)\,.$$

By the same argument as in section 3.2 of Kiefer (1975), a sufficient condition for the E-optimality of a BYHR $(v\,,n\,,m,$ $b_1,\ldots,b_m)$ is that

$$\Delta\, g(r) \geq 0 \quad \forall\, r < \bar{r}, \text{ where } \bar{r} = n / v\,.$$

It may be recalled that the same condition was used to prove the E-optimality of a GYD in a two-way setting.

$$\text{Here } \Delta\, g(r) \geq w(n - \sum_{i=1}^{m} b_i + m - 1 - 2r) + \sum_{i=1}^{m} \bar{w}_i (b_i - 1) \qquad \ldots(2.5.3.2)$$

$$\geq w(n - \sum_{i=1}^{m} b_i + m - 1 - 2r)\,,$$

since $\bar{w}_i \geq 0$ and $b_i > 1$, $i = 1,2,\ldots,m$.

For any $r < \bar{r} = n / v$, the coefficient of w, say C_w is

$$> n - 2n/v - \sum_{i=1}^{m} b_i + m - 1$$

$$> n / 2 - \sum_{i=1}^{m} b_i + m - 1 \quad \forall \quad v \geq 4\,.$$

Hence a sufficient condition for E-optimality of a BYHR is that

$$n/2 - \sum_{i=1}^{m} b_i + n - 1 \geq 0. \qquad \ldots(2.5.3.3)$$

Thus the following optimality result is proved.

<u>Theorem 2.5.3.2</u>. A BYHR $(v, n; m, b_1, \ldots, b_m)$, if it exists, is E-optimal within a class of proper and connected designs in the setting $M(v;n;m;b_1,\ldots,b_m)$ when $n \geq 2[\sum_{i=1}^{m} b_i - (m-1)]$, under fixed effects model as well as under mixed effects model II.

Let us now proceed to the proofs of A- and D-optimality of a Balanced Youden Hypercube in the setting $M(v; n; m; b^m)$.

We shall follow the same notation as used in case of two-way heterogeneity situation.

Here $\mathcal{N} = \{x : x$ is a multiple of $b\}$ and the notation $\mathcal{M}$, $[C, D]$, the basic interval $[C_0, D_0]$ are as in section 2.4.3.

From (2.5.3.2), $g(r)$ can be written as

$$g(r) = \alpha + \beta r + \gamma r^2 \qquad \ldots(2.5.3.4)$$

where $\alpha = b^2 [r/b] (1 + [r/b] \sum_{u=1}^{m} (w - \bar{w}_u)$,

$$\beta = (n - b - 2b[r/b]) \sum_{u=1}^{m} (w - \bar{w}_u) - n\gamma$$

and $\gamma = \sum_{u=1}^{m} (w - \bar{w}_u) - w = (m-1)w - \sum_{u=1}^{m} \bar{w}_u$.

With this form of $g(r)$ it is easy to verify again lemma 2.4.2.2, following the same lines of proof. So by the arguments in the steps (d) and (e) of Kiefer (1975), in order to establish A- and D-optimality of a BYHC, we are to show the concavity of $q(r) = -f(g(r))$, in the basic interval where $f(x) = x^{-1}$ in the case of A- optimality and $f(x) = \log x$ in the case of D-optimality. This is equivalent to showing

$$\Gamma_A(r) = 3\gamma^2 r^2 + 3\beta\gamma r + \beta^2 - \alpha r - \gamma^2 \geq 0 \quad \text{for} \quad C_0 + 1 \leq r < D_0 \qquad \ldots(2.5.3.5)$$

and $$\Gamma_D(r) = 2\gamma^2 r^2 + 2\beta\gamma r + \beta^2 - 2\alpha\gamma - \gamma^2 \geq 0 \quad \text{for} \quad C_0 + 1 \leq r < D_0 . \qquad \ldots(2.5.3.6)$$

Similar to the two-way setting, we have here

$$\frac{d\,\Gamma_A(r)}{dr} = 3\gamma(\beta + 2\gamma r) \quad \text{and} \quad \frac{d\,\Gamma_D(r)}{dr} = 2\gamma(\beta + 2\gamma r) .$$

Now, $$\beta + 2\gamma r = \Delta g(r) - \gamma$$
$$> w(n - mb - 2r) + b \sum_{u=1}^{m} \bar{w}_u , \text{ from } (2.5.3.2)$$
$$\geq w(n - mb - 2\bar{r}) \quad \forall \quad r < \bar{r}$$
$$> 0 \quad \text{whenever} \quad v \geq 4 \quad \text{and} \quad n/2 \geq mb . \qquad \ldots(2.5.3.7)$$

Hence by property (i) of $\Delta g(r)$ in lemma 2.4.2.2 it follows that $\beta + 2\gamma r > 0 \;\; \forall \;\; r \ni C_0 + 1 \leq r < D_0$,

if $v \geq 4$ and $n \geq 2mb$. ...(2.5.3.8)

Now we can assume n to be $\geq 2mb$, since in general m is not large. So, for the verification of A- and D-optimality of a BYHC, it is enough to show that

$$\Gamma_A(C_o + 1) \geq 0 \quad \text{and} \quad \Gamma_D(C_o + 1) \geq 0 \qquad \ldots(2.5.3.9)$$

as both $\Gamma_A(r)$ and $\Gamma_D(r)$ are monotone increasing for $r \in [C_o + 1, D_o]$.

All the results proved so far in this section are valid for both the models, viz the fixed effects model and the mixed effects model II. In the rest of the proof of A- and D-optimality, we shall treat the two models separately.

Let $\Gamma_A^{(F)}(r)$ and $\Gamma_A^{(M)}(r)$ denote the expressions for $\Gamma_A(r)$ under fixed effects model and mixed effects model II respectively and similarly $\Gamma_D^{(F)}(r)$ and $\Gamma_D^{(M)}(r)$ denote the corresponding expressions for $\Gamma_D(r)$.

Then we shall prove the following lemmas from which the optimality results will follow.

<u>Lemma 2.5.3.3</u>. $\Gamma_A^{(Q)}(r) - \Gamma_D^{(Q)}(r) \geq 0 \quad \forall \quad r < \bar{r}$

whenever $v \geq 2m + 2$, and $n \geq 2mb$, for $Q = M$ or F.

<u>Proof</u>. $\Gamma_A(r) - \Gamma_D(r) = \gamma r\,(\beta + \gamma r) \qquad \ldots(2.5.3.10)$

from (2.5.3.5) and (2.5.3.6).

Now in the case of mixed effects model II,

$$\beta + \gamma r \geq \sum_{u=1}^{m} (w - \bar{w}_u)(n - b - 2r) - (n - r)\gamma$$

$$= \sum_{u=1}^{m} \bar{w}_u(r + b) + w(n - mb - (m + 1)r)$$

$$> w(n - mb - (m + 1)r)$$

$$= wP \text{ say, where } P = n - mb - (m + 1)r$$

and similarly in the case of fixed effects model, $\beta + \gamma r > P$.

But $P \geq n - mb - (m + 1)n / v \quad \forall \quad r \leq \bar{r} = n / v$

$$\geq n / 2 - mb, \quad \text{for} \quad v \geq 2m + 2$$

$$\geq 0, \quad \text{for} \quad n \geq 2mb.$$

So, $\beta + \gamma r > 0$ for both the models and the lemma follows from (2.5.3.10) since $\gamma > 0$.

Lemma 2.5.3.4. $\Gamma_D^{(F)}(C_o + 1) > 0$ for $v \geq 2m + 2$ and $n - 4m^2 b + 4m(m - 1) \geqslant 0$.

Proof. Let $\alpha^{(F)}$, $\beta^{(F)}$ and $\gamma^{(F)}$ denote the expressions for α, β and γ under fixed effects model. Then,

$$\gamma^{(F)} = m - 1,$$

$$\beta^{(F)} = n - mb - 2mb\,[r / b]$$

and $\alpha^{(F)} = mb^2\,[r / b]\,([r / b] + 1)$.

For $r \in [C_o + 1, D_o - 1]$, we have

$$\beta^{(F)} = n - mb - 2m\,C_o \qquad \ldots(2.5.3.11)$$

and $\alpha^{(F)} = mC_o(C_o + b) \qquad \ldots(2.5.3.12)$

Now, $\Gamma_D(r) = (1/2)(\beta + 2\gamma r)^2 + (1/2)\beta^2 - 2\alpha\gamma - \gamma^2$ ...(2.5.3.13)

$$\therefore \quad \Gamma_D^{(F)}(C_o + 1) = (1/2)\left\{n - mb - 2C_o + 2(m-1)\right\}^2 + (1/2)(n - mb - 2mC_o)^2 - 2m(m-1)C_o(C_o + b) - (m-1)^2$$

$$= 2(m+1)C_o^2 - 2C_o\left\{n(m+1) - 2mb + 2(m-1)\right\} + (n - mb + m - 1)^2$$

$$= Q(C_o), \quad \text{say}.$$

The lemma will follow if we can show that $Q(C_o)$ is a decreasing function in C_o for $C_o < \bar{r}$ and $Q(n/(2m+2)) \geq 0$ for $v \geq 2m+2$ and $n - 4m^2b + 4m(m-1) \geq 0$.

$$\text{Now} \quad \frac{dQ(x)}{dx} = -(m+1)(n - 4x - 2b + 2) - 2(b-2)$$

$$< -(n - 4x - 2b + 2)$$

$$< \quad \text{for} \quad x \leq \bar{r} = n/v \leq n/(2m+2).$$

Again for $C_o = n/(2m+2)$,

$$Q(C_o) = n/(2m+2)\left[n - 4m^2b + 4m(m-1)\right] + (m - 1 - mb)^2$$

$$\geq 0 \quad \text{whenever} \quad n - 4m^2b + 4m(m-1) \geq 0.$$

Hence the lemma.

<u>Lemma 2.5.3.5</u>. $\Gamma_D^{(M)}(C_o + 1) \geq w^2\,\Gamma_D^{(F)}(C_o + 1) \quad \forall\, v \geq 4.$

<u>Proof</u>. Let $\xi = \sum_{i=1}^{m} \bar{w}_i$, then we can write

$$\gamma = w\gamma^{(F)} - \xi$$

$$\beta = w\beta^{(F)} + \xi\, b(1 + 2[r/b])$$

and $\alpha = w\alpha^{(F)} - \xi\, b^2 [r/b](1 + [r/b])$. From (2.5.3.13) we can write

$$\Gamma_D^{(M)}(C_o+1) = (1/2)\left\{ w\beta^{(F)} + 2\gamma^{(F)}(C_o+1) + \xi(b-2)\right\}^2$$
$$+ (1/2)\left\{ w\beta^{(F)} + \xi(b+2C_o)\right\}^2$$
$$- 2(w\gamma^{(F)} - \xi)\left\{ w\alpha^{(F)} - \xi\, C_o(C_o+b)\right\}$$
$$- (w\gamma^{(F)} - \xi)^2$$
$$= w^2\Gamma_D^{(F)}(C_o+1) + \xi^2\left\{(1/2)(b-2)^2 + (1/2)(b+2C_o)^2 - \right.$$
$$\left. - 2C_o(C_o+b)\right\} + w\xi\,\left[\beta^{(F)}(2b-2+2C_o)\right.$$
$$\left. + 2\gamma^{(F)}\left\{(b-2)(C_o+1) + C_o(C_o+b) + 1\right\} + 2\alpha^{(F)}\right]$$

or $\Gamma_D^{(M)}(C_o+1) - w^2\Gamma_D^{(F)}(C_o+1)$

$$= \xi^2\left[(1/2)(b-2)^2 + (1/2)b^2 - 1\right]$$
$$+ w\xi\left[2(n - mb - 2mC_o)(b + C_o - 1)\right.$$
$$\left. + 2(m-1)\left\{C_o^2 + 2(b-1)C_o + b - 1\right\} + 2mC_o(C_o+b)\right], \quad \ldots(2.5.3.14)$$

using (2.5.3.11) and (2.5.3.12).

Coefficient of ξ^2 in (2.5.3.14) is clearly > 0.

Coefficient of $w\xi$ is $= 2\left[(n-mb)(b+C_o-1) + 2mC_o + (m-1)\left\{C_o^2 + 2(b-1)C_o + b - 1\right\}\right]$

which is clearly > 0 and hence the lemma.

From the lemmas the following theorem is immediate.

Theorem 2.5.3.6. A Balanced Incomplete Youden Hypercube $(v; n; m; b^n)$ is A- and D-optimal under fixed effects model as well as under mixed effects model II whenever $v \geq 2n+2$ and $n - 4n^2b + 4n(n-1) \geq 0$.

Proof. It is enough to verify (2.5.3.9) as we have already noted. Now because of lemma 2.5.3.3, if we can show $\Gamma_D(C_o + 1) \geq 0$ for $v \geq 2n+2$ and $n - 4n^2b + 4n(n-1) \geq 0$, then the result follows. But that this is true follows from lemmas 2.5.3.4 and 2.5.3.5.

Hence the theorem.

Remark. The relevant optimality results of Cheng (1978a), viz theorems 4.1 and 5.1 follow as corollaries from theorems 2.5.3.2 and 2.5.3.6 in the present chapter.

In the same set up considered, it may be noted that $C_d^{(F)}$ is completely symmetric if $N_d = [N_{d1} | N_{d2}, \ldots, | N_{dn}]$ is the incidence matrix of a BBD, i.e. d is a BPYD (defined in DFN 2.2.29).

Similarly, when the model is one of mixed effects with all factor effects random, if the condition $\bar{w}_1 = \bar{w}_2 = \ldots = \bar{w}_n$, i.e. $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2$ is satisfied, then $C_d^{(M)}$ is completely symmetric if d is a BPYD. So, in these specific situations the same arguments as are used in deriving the results of this section would go through if BYHC's are replaced by the corresponding BPYD's and then the following result would hold.

Theorem 2.5.3.7. A BPYD (v,n,n,b^n) satisfies all the optimality properties of a BYHC (v,n,n,b^n) as shown in theorems 2.5.3.2 and 2.5.3.6 under (i) fixed effects model and as well as under (ii) mixed effects model with all factor effects random assuming all the factor effects have the same variance.

CHAPTER 3

RESTRICTED CLASS OF PBBD'S WITH TWO ASSOCIATE CLASSES

3·1. Introduction

In the preceding chapter, optimality results of block designs have been dealt with, particularly in the context of mixed effects model. It has been observed that whether we assume mixed effects or fixed effects model, optimality considerations become much more complex in the asymmetrical case, more so when the model is one of mixed effects. However, in such asymmetrical situations, PBBDs, specially those with two associate classes have been in use for the last four decades or more since their introduction, primarily because of their algebraic simplicity, elegant structural properties and their apparent closeness to the symmetrical designs which are known to satisfy all the practically meaningful optimality criteria evolved so far. So, in the present chapter an attempt is made to assess the relative efficiencies of designs within the class of PBBDs with two associate classes and find the optimal designs, if any within the class with regard to a few standard optimality criteria, viz., A- and D- optimality criteria.

We observe that given b, k and v, a GDD with $\lambda_2 = \lambda_1 + 1$ (not necessarily an MBGDD of type 1), if it exists is A- and

D- optimal within this restricted class of all connected PBBD's with two associate classes.

3.2 Preliminaries

For the description of the estimation problem, different optimality criteria and definitions of the different types of block designs we come across in this chapter, one is referred to Chapter 2.

A PBBD with m associate classes has been defined in DFN 2.2.15. Throughout this chapter we confine our attention to the class of all connected PBBD's with two associate classes with given b, k and v. This class of designs in denoted by $\mathcal{D}_p$. We consider the estimation problem π and derive the optimal designs within the class $\mathcal{D}_p$ with regard to A- and D- optimality criteria. From the definitions of A- and D- optimality (vide DFN 2.2.5) and relations (2.2.3), our task is to minimise

$$\phi_A = \sum_{i=1}^{v-1} \mu_{di}^{-1} \quad \text{and} \quad \phi_D = \prod_{i=1}^{v-1} \mu_{di}^{-1} \qquad \ldots(3.2.1)$$

where μ_{di} is the ith positive eigenvalue of the C-matrix of a PBBD with two associate classes.

The relations that exist among the parameters of a PBIBD is relatively well known. But as the expressions in the case of a general PBBD, though straightforward, are not easily found in the literature, we state them explicitily in the following lines.

From DFN 2.2.15, we note that the ith diagonal element of $N_d N_d'$ is given by

$$\sum_{j=1}^{b} n_{ij}^2 = h(r\,,\,b) = r + (2r - b)\,[k\,/\,v] - b\,[k\,/\,v]^2\,,$$
$$i = 1, 2, \ldots, v\,, \qquad \ldots(3.2.2)$$

we shall write the expression simply as h and note that $h = r$ when the design is binary, i.e. a PBIBD.

Now the relations among the parameters of a PBBD with two associate classes, viz, $v, b, r, k, \lambda_1, \lambda_2, n_1, n_2, p^i_{jk}, i, j, k = 1, 2$ are stated as follows.

(i) $v\,r = bk$

(ii) $n_1 + n_2 = v - 1$

(iii) $n_1\lambda_1 + n_2\lambda_2 = rk - h$

(iv) $p^i_{j\ell} = p^i_{\ell j}\,,\ i, j, \ell = 1, 2,$

(v) $n_i p^i_{j\ell} = n_j p^j_{i\ell}\,,\ i, j, \ell = 1, 2,$

(vi) $\sum_{\ell=1}^{2} p^i_{j\ell} = n_j - \delta_{ij}\,,\ i, j = 1, 2,$

where δ_{ij} is the kronecker δ. ...(3.2.3)

Under a mixed effects model, the C-matrix of a PBBD with two associate classes, is obviously of the form,

$$C_d = (w - \bar{w})(r\,I_v - k^{-1}\,N\,N') + \bar{w}r\,(I_v - v^{-1}\,J_v)\,,$$

from (2.3.1.1) of Chapter 2.

Now, following the lines of Bose and Mesner (1959) and using the relations in (3.2.3), one can easily obtain the eigenvalues of NN' as follows.

$e_0 = rk$ with multiplicity 1

and $e_i = h - z_i$ with multiplicity α_i, $i = 1, 2.$ ...(3.2.4)

where $z_i = (1/2)\left[(\lambda_1 + \lambda_2) + (\lambda_1 - \lambda_2)(-\delta + (-1)^i\sqrt{\Delta}\,)\right]$,

and $\alpha_i = (v-1)/2 + (-1)^i(2\sqrt{\Delta})^{-1}\left[n_1 - n_2 + \delta(n_1 + n_2)\right]$

with $\Delta = \delta^2 + 2\beta + 1$, $\beta = p_{12}^2 + p_{12}'$ and $\delta = p_{12}^2 - p_{12}'$

e_0 corresponds to the zero eigenvalue of C_d and the positive eigenvalues of C_d are given by $\mu_i = (w - \bar{w})(rk - h + z_i) + wr$ ...(3.2.5)

with multiplicity α_i, $i = 1, 2.$

Also μ_1 and μ_2 are related through the relation

$$\alpha_1\mu_1 + (v - 1 - \alpha_1)\mu_2 = \text{tr } C_d$$

$$= (w - \bar{w})(rk - h) + \bar{w}r \qquad ...(3.2.6)$$

3.3. Optimality Results in the Class $\mathcal{D}_p$.

Let us write $\theta = \lambda_1 - \lambda_2$.

Then using the relations (3.2.3), μ_i's and α_i's and hence the optimality functionals ψ_A and ψ_D can be expressed as the functions of three variables θ, n_1 and p_{12}^1, the other quantities like v, r, k, w and $\bar{w}$ being constants. Then our task is to optimise the functional ψ_A or ψ_D by varying the arguments θ, n_1 and p_{12}^1 within their feasible ranges.

Let us denote p_{12}^1 by p.

Then from (3.2.3), we have

$$p_{12}^2 = n_1 - (n_1 / n_2)p \quad \text{and so}$$

$$\delta = n_1 - p(v-1)/n_2 \quad \text{and} \quad \beta = \delta + 2p. \qquad \ldots(3.3.1)$$

Also from the relation (iii) of (3.2.3), we can write

$$\lambda_1 = (rk - h + n_2\theta)/(v-1)$$

$$\text{and} \quad \lambda_2 = (rk - h - n_1\theta)/(v-1). \qquad \ldots(3.3.2)$$

Then from (3.2.5), we can express μ_i in the following form

$$\mu_i = k^{-1}(w-\bar{w})\left\{v(rk-h)(v-1)^{-1} + \theta(f(p,n_1) + (-1)^i s)\right\} + \bar{w}r, \quad i = 1,2,$$

$$= c\,[a + \theta(f(p,n_1) + (-1)^i s)] \qquad \ldots(3.3.3)$$

where $f(p,n_1) = (1/2)\,[(n_2 - n_1 v)/(v-1) + p(v-1)/n_2]$,

$$s = \sqrt{\Delta}/2, \quad c = (w-\bar{w})/k$$

and $a = \{(w-\bar{w})k^{-1}v(rk-h)(v-1)^{-1} + \bar{w}r\}/c$.

Also $\alpha_i = (v-1)/2\,[1 - (-1)^i f(p,n_1)/s]$, $i = 1,2$. ...(3.3.4)

Again from (3.2.4) and (3.3.3), the following relations hold,

$$s^2 = (f(p,n_1))^2 + g(n_1), \qquad \ldots(3.3.5)$$

where $g(n_1) = n_1(v-1-n_1)v/(v-1)^2$.

This together with (3.3.4) implies

$$\alpha_2\mu_2 + \alpha_1\mu_1 = c\,[a(v-1) + \theta(v-1).2f(p,n_1)]. \qquad \ldots(3.3.6)$$

Also (3.3.4) implies in view of (3.3.5)

$$\mu_1\mu_2 = c^2 \left[a^2 + a\theta.2f(p,n_1) - \theta^2 g(n_1)\right]. \qquad \ldots(3.3.7)$$

From (3.2.1), $\psi_A = (\alpha_1\mu_2 + \alpha_2\mu_1) / \mu_1\mu_2$

$$= \frac{(v-1)(a + \theta.2f(p,n_1))}{c\left[a(a + \theta.2f(p,n_1)) - \theta^2 g(n_1)\right]},$$

and in view of (3.3.6) and (3.3.7)

$$\psi_A = (v-1)c^{-1}a^{-1}\left[1 + \theta^2 g(n_1) / Q(\theta,n_1,p)\right], \qquad \ldots(3.3.8)$$

where $Q(\theta,n_1,p) = a^2 + a\theta.2f(p,n_1) - \theta^2 g(n_1)$.

With this simple expression for ψ_A, we shall go ahead with the optimization problem with respect to the variables p, n_1 and θ in section 3.3.1.

The expression for ψ_D is not, however, so simple and optimization of ψ_D is algebraically more involved. We shall consider the optimization of ψ_D in section 3.3.2.

3.3.1. <u>A- optimality Criterion</u>

The expression for ψ_A is given in (3.3.8). It is a function of θ, n_1 and p and we first observe the behaviour of ψ_A as a function of the variables, one at a time, keeping the other two fixed. This will enable us to proceed with the actual optimization problem and finally derive the A- optimal design within $\mathcal{D}_p$.

First, let us treat ψ_A as a function of θ when the association scheme is assumed to be given, i.e. n_1 and p are given. Because of the restriction (iii) of (3.2.3), the admissible range for θ is given by

$$-(rk-h)/n_2 \leq \theta \leq (rk-h)/n_1 . \qquad \ldots(3.3.1.1)$$

Now whatever be the given association scheme, $Q(\theta, n_1, p)$ is a quadratic function in θ of the form

$Q(\theta) = A + B\theta - D\theta^2$ where $A = a^2$, $D = g(n_1)$ and $B = a.2f(p, n_1)$, the expressions for $g(n_1)$ and $f(p, n_1)$ are to be found in (3.3.3) and (3.3.5). It is found that A and D are always positive and the sign of B depends on the quantity $f(p, n_1)$, i.e. on the parameters of the association scheme.

In any case, $Q(\theta)$ has both the roots real and they are of opposite signs, lying beyond the admissible range for θ (given in (3.3.1.1), so that for θ in the admissible range,

(i) $Q(\theta) > 0$, which implies ψ_A is always finite.

We also abserve the following facts about ψ_A.

(ii) $\dfrac{d^2(\psi_A)}{d\theta^2} > 0$

(iii) $\psi_A(\theta) \gtrless \psi_A(-\theta)$ according as $B \gtrless 0$.

So, either $\psi_A(\theta) > \psi_A(-\theta) \quad \forall\, \theta$

or $\psi_A(\theta) = \psi_A(-\theta) \quad \forall\, \theta$

or $\psi_A(\theta) < \psi_A(-\theta) \quad \forall\, \theta$. $\qquad \ldots(3.3.1.2)$

Next let us see how ψ_A depends on the association scheme.

We know that $0 \leq p \leq \min(n_1 - 1, n_2)$ ...(3.3.1.3)

For a given θ and n_1 it is easy to see from (3.3.8) that

(i) in the case $\theta > 0$, ψ_A is a monotone decreasing function in p, which means that ψ_A is minimum when $p = \min(n_1 - 1, n_2)$

and

(ii) in the case $\theta < 0$, ψ_A is a monotone increasing function in p, which means that ψ_A is minimum when $p = 0$. ..(3.3.1.4)

Having observed the behaviour of ψ_A dealing with one variable at a time, we are now in a position to minimise ψ_A with respect to all the three variables θ, n_1 and p simultaneously.

Now from (3.3.8),

$$\psi_A = (v-1)a^{-1}c^{-1}\left[1 + (\phi_A(\theta,n_1,p) - 1)^{-1}\right] \qquad ...(3.3.1.5)$$

where $\phi_A(\theta,n_1,p) = a(a + \theta.2f(p,n_1)) / (\theta^2 g(n_1))$.

So, minimisation of ψ_A with respect to θ, n_1 and p is equivalent to the maximisation of $\phi(\theta,n_1,p)$ with respect to the same set of variables.

Before proceding further with the optimisation of $\phi(\theta,n_1,p)$, let us note the following facts.

Observation 1. Among the two cases, namely $\theta < 0$ and $\theta > 0$, it is enough to examine one of them, since a PBBD with $\theta > 0$, i.e. $\lambda_1 > \lambda_2$ and a two associate class association scheme can be looked upon as a PBBD with $\theta < 0$ (i.e $\lambda_2 > \lambda_1$) belonging to a different two associate class association scheme obtained uniquely from the first association scheme. The latter association scheme is defined as follows. Two treatments which are first (second) associates of one another in the former association scheme are second (first) associates of each other in the latter association scheme.

Observation 2. From the relation (3.3.2),

$$\lambda_2 = (rk - h) / (v-1) - \theta n_1 / (v-1) .$$

Now since λ_2 is a positive integer and $n_1 < v-1$, for $\theta = -j$, where j is a positive integer, we have

$$i + 1 \leq \lambda_2 \leq i + j , \qquad \ldots(3.3.1.6)$$

$$\text{and} \quad (1 - t) / j \leq n_1 / (v-1) \leq (j - t) / j , \qquad \ldots(3.3.1.7)$$

where $i = [(rk - h) / (v-1)]$

and $t = (rk - h) / (v-1) - i$.

Also $\lambda_1 = \lambda_2 + \theta = \lambda_2 - j$, so that from (3.3.1.6),

$$i - j + 1 \leq \lambda_1 \leq i .$$

$$\text{and as} \quad \lambda_1 \geq 0, \; 1 \leq j \leq i + 1 . \qquad \ldots(3.3.1.8)$$

In particular, for $\theta = -1$, λ_2 must be $i + 1$ and n_1 can take only one value, viz.,

$$n_1 = (v-1)(1-t) . \qquad \ldots(3.3.1.9)$$

Henceforth, we shall consider the case $\theta < 0$ only and enote θ by $-j$ where j is a positive integer.

Let $\theta_0^A(j,n_1) = a^{-1}\phi_A(\theta = -j, n_1, p = 0)$

$$= \frac{a - j(1 - (v+1)n_1 / (v-1)}{j^2 g(n_1)} \qquad \ldots(3.3.1.10)$$

[from (3.3.1.5)]

Then the property of ϕ_0^A shown in the following lemma is a crucial step in the optimization procedure.

<u>Lemma 3.3.1.1</u>. For a fixed integer j satisfying (3.3.1.8) ϕ_0^A as a function of n_1 has exactly one extremum within the range $0 < n_1 < v - 1$ and that is a minimum.

[Note that the minimum may not be within the feasible range of n_1].

<u>Proof</u>. Let $x = n_1 / (v-1)$ and $u(x) = \frac{a-j(1-(v+1)x)}{x(1-x)} = \phi_0^A(j,n_1)$ for a fixed integer j. ...(3.3.1.11)

So, from (3.3.1.10), it is enough to show that $u(x)$ has exactly one extremum in the range $0 < x < 1$ and that point is a minimum.

$$\text{Now } \frac{du(x)}{dx} = x^{-2}(1-x)^{-2}\, q(x), \qquad \ldots(3.3.1.12)$$

where $q(x) = j(v+1)x^2 + 2(a-j)x - (a-j)$

$\therefore\ \frac{du}{dx} = 0$, if and only if $q(x) = 0$.

But $q(x)$ has only one positive root, viz.,

$$x_1 = \frac{a-j}{((a-j)(a+vj))^{1/2} + a-j} \qquad \ldots(3.3.1.13)$$

x_1 is $< 1/2$, i.e. lies within the range $0 < x < 1$.

To prove that the point is a minimum, it is enough to show that

$$\frac{d^2 u(x)}{dx^2} > 0 \quad \text{at} \quad x = x_1 .$$

But
$$\frac{d^2 u}{dx^2} = 2x^{-3}(1-x)^{-3}\left[j(v+1)x^3 + (a-j)(1-3x+3x^2)\right]$$
$$> 0 \text{ at } x = x_1 ,$$

since (i) $a-j > v(i+t)-j$ from (3.3.3) and (3.3.1.7)

$$> 0$$

and (ii) $(1-3x_1+3x_1^2) = (1-x_1)(1-2x_1) + x_1^2 > 0$,

because $0 < x_1 < 1/2$.

Hence the lemma.

Now, from observation 1, we can write the problem as :

Maximise $\emptyset_A(\theta = -j, n_1, p)$ with respect to j, n_1 and p,

where j is a positive integer.

$$\max_{j,n_1,p} \emptyset_A(\theta = -j, n_1, p)$$

$$= \max_{j,n_1} \left\{ \max_{0 \leq p \leq \min(n_1-1,n_2)} \emptyset_A(\theta = j, n_1, p) \;\middle|\; \text{given } j \text{ and } n_1 \right\}$$

$$= \max_{j,n_1} \emptyset_c^A(j, n_1) \quad \text{from (ii) of (3.3.1.4)}$$

$$= \underset{1 \leq j \leq i+1}{\text{Max}} \left\{ \underset{\frac{1-t}{j} \leq \frac{n_1}{v-1} \leq \frac{j-t}{j}}{\text{Max}} \phi_o^A (j, n_1) \;\middle|\; \text{given } j \right\}$$

$$= \underset{1 \leq j \leq i+1}{\text{Max}} \text{Max}(\phi_{o1}^A (j), \phi_{o2}^A (j)) \qquad \ldots(3.3.1.14)$$

by lemma 3.3.1.1.

where $\phi_{o1}^A(j) = \phi_o^A(j, n_1 = (v-1)(1-t)/j)$

$$= (a-j+(v+1)(1-t))(1-t)^{-1}(j-1+t)^{-1} \qquad \ldots(3.3.1.15)$$

and $\phi_{o2}^A(j) = \phi_o^A(j, n_1 = (v-1)(j-t)/j)$

$$= (a-j+(v+1)(j-t))\,t^{-1}(j-t)^{-1}. \qquad \ldots(3.3.1.16$$

Now the expression in (3.3.1.14)

$$= \text{Max}\left\{ \underset{1 \leq j \leq i+1}{\text{Max}} \phi_{o1}^A(j), \underset{1 \leq j \leq i+1}{\text{Max}} \phi_{o2}^A(j) \right\}.$$

But it is clear from (3.3.1.15) and (3.3.1.16) that both $\phi_{o1}^A(j)$ $\phi_{o2}^A(j)$ are decreasing functions of j for $j \geq 1$, and so attain their respective minimum at $j = 1$. Again $\phi_{o1}^A(1) = \phi_{o2}^A(1)$ from (3.3.1.9).

Hence $\underset{j, n_1, p}{\text{Max}} \phi_A(\theta = -j, n_1, p)$

$$= \phi_A(\theta = -1, n_1 = (v-1)(1-t), p = 0).$$

This implies that ψ_A is minimum when $\theta = -1$, i.e. $\lambda_2 = \lambda_1 + 1$, $n_1 = (v-1)(1-t)$ and $p'_{12} = 0$, provided of course a PBBD exists with these parameters. The parameters of this PBBD implies that it is a GDD with $\lambda_2 = \lambda_1 + 1$ and $n = (v-1)(1-t)+1$. Hence we have the following optimality result.

Theorem 3.3.1.2. If a GDD with $\lambda_2 = \lambda_1 + 1$ exists, it is A- optimal under fixed effects and mixed effects model over all designs in $\mathcal{D}_p$. In particular, if $t = v/(2v-2)$, then the A-optimal PBBD with two associate class association scheme is precisely the MBGDD of type 1.

Remark. It may be recalled that although A-optimality (in fact ψ_f optimality of type 1) of MBGDD is known under fixed effects model within the wider class $\mathcal{D}$ of all block designs given b,k and v, the result could not be proved in general under mixed effects model. What we have been able to show in Chapter 2 is that under mixed effects model, the optimality holds, if we restrict to the class of designs which are either binary or equireplicate. Obviously, the class of designs $\mathcal{D}_r$ include $\mathcal{D}_p$ and hence theorem 3.3.1.2 does not prove any additional property of MBGDD's as such. But the theorem states that if we restrict to the class of designs $\mathcal{D}_p$, any GDD with $\lambda_2 = \lambda_1 + 1$ (not necessarily an MBGDD of type 1), if it exists is A- optimal.

3.3.2. D- optimality criterion

We first note from (3.2.1) that the minimisation of ψ_D is equivalent to the maximisation of

$$\phi_D(\theta, n_1, p) = \alpha_1 \log \mu_1 + \alpha_2 \log \mu_2 \qquad \ldots(3.3.2.1)$$

where α_i and μ_i, $i = 1,2$ are as given in (3.3.3) and (3.3.4).

For handling the problem of maximisation of ϕ_D as a function of θ, n_1 and p, we proceed along the some times as in the case of ϕ_A in section 3.3.1. We consider the case $\theta < 0$ only throughout this section and write $-j$ for θ where j is a positive integer satisfying (3.3.1.8). The feasible ranges for p and n_1 are given in (3.3.1.3) and (3.3.1.7) respectively.

In the first step we keep n_1 and j fixed, and maximise ϕ_D as a function of p.

Since each of $\alpha_1, \alpha_2, \mu_1, \mu_2$ are functions of p, we have from (3.3.2.1),

$$\frac{\partial \phi_D}{\partial p} = \frac{\partial \alpha_1}{\partial p} \log \mu_1 + \frac{\partial \alpha_2}{\partial p} \log \mu_2 + \frac{\alpha_1}{\mu_1}\frac{\partial \mu_1}{\partial p} + \frac{\alpha_2}{\mu_2}\frac{\partial \mu_2}{\partial p} . \qquad \ldots(3.3.2.2)$$

Now, from (3.3.3),

$$\frac{\partial f(p,n_1)}{\partial p} = (v-1)/(2n_2). \qquad \ldots(3.3.2.3)$$

Also, from (3.3.5),

$$\frac{\partial s}{\partial p} = \frac{f(p,n_1)}{s}\frac{\partial f(p,n_1)}{\partial p} \qquad \ldots(3.3.2.4)$$

$$\therefore \frac{\partial \mu_i}{\partial p} = (-j)(v-1)/(2n_2)\,[1 + (-1)^i f(p,n_1)/s], \quad i = 1,2$$

using (3.3.2.3) and (3.3.2.4)

$$= (-j/n_2)\alpha_{2-i}, \quad i = 1,2. \qquad \ldots(3.3.2.5)$$

Similarly, $\frac{\partial \alpha_i}{\partial p} = (-1)^{i+1}(v-1)^2 g(n_1) / (4n_2 s^3)$, $i = 1, 2$. ...(3.3.2.6)

Substituting (3.3.2.5) and (3.3.2.6) in (3.3.2.1), we obtain

$$\frac{\partial \phi_D}{\partial p} = (v-1)^2 / (4n_2 s^3) [\log (\mu_1 / \mu_2) g(n_1) + (-j / n_2)\alpha_1 \alpha_2 (\mu_1^{-1} + \mu_2^{-1})$$

$$= [(v-1)^2 g(n_1) / (4n_2 s^3)] z \text{ , using (3.3.4) and (3.3.5)}$$

where $z = \log (\mu_1 / \mu_2) - j\, s\, (\mu_1^{-1} + \mu_2^{-1})$.

Since $\frac{\partial \phi_D}{\partial p}$ is of the same sign as z, if we can show that $z < 0$ in the feasible range of p, then that will imply $\phi_D(j, n_1, p)$ is a decreasing function of p for any fixed value of j and n_1 and hence attains it's maximum at $p = 0$. On simplification,

$z = \log u + (2u)^{-1} - u/2$ where $u = \mu_1 / \mu_2$ and this u is a function of p taking values > 1, in view of (3.3.3) as we have assumed $\theta < 0$.

One can easily see that $z(u)$ considered as a function in the variable u för $u > 0$ is maximum at $u = 1$ and the maximum value is 0.

So, $z < 0$ so long as $u > 1$, i.e. $\mu_1 > \mu_2$ and hence the result.

In the next step, we study the behaviour of

$\phi_0^D(j, n_1) = \phi_D(\theta = -j, n_1, p = 0)$ as a function of n_1, keeping j fixed.

At $p=0$, $f(n_1, p) = (1/2)(1-(v+1)n_1/(v-1))$

and $s = (n_1+1)/2$.

So, $\mu_1 = c(a + n_1 v j/(v-1))$ and $\mu_2 = c(a - n_2 j/(v-1))$

...(3.3.2.7)

Similarly, $\alpha_1 = (v-1-n_1)/(n_1+1)$ ...(3.3.2.8)

and $\alpha_2 = n_1 v/(n_1+1)$

So, $\phi_0^D(j, n_1) = (n_1+1)^{-1}[(v-1-n_1)\log\mu_1 + n_1 v \log\mu_2]$

...(3.3.2.9)

where $\mu_1 . \mu_2$ are given by (3.3.2.7).

Differentiating partially with respect to n_1, keeping j fixed, we have on simplification

$$v^{-1}(n_1+1)^2 \frac{\partial\phi_0^D(j,n_1)}{\partial n_1}$$

$$= \log(\mu_2/\mu_1) + (v-1)^{-1}(\mu_1-\mu_2)(n_1\mu_2^{-1} + n_2\mu_1^{-1})$$

...(3.3.2.10)

Let $x = n_1/(v-1)$ and $y = (\mu_1-\mu_2)/\mu_1$. ...(3.3.2.11)

Then $$y = \frac{(n_1+1)j}{a + n_1 v j/(v-1)} = \frac{j(1+(v-1)x)}{a + j v x}.$$ ...(3.3.2.12)

The feasible range for x is given by (3.3.1.7) and that for y is given by $0 < y < 1$. ...(3.3.2.12)

Now substituting (3.3.2.11) and (3.3.2.12) in (3.3.2.10) we have

$$v^{-1}(n_1+1)^2 \frac{\partial \phi_0^D(j,n_1)}{\partial n_1}$$

$$= \log (1-y) + y + xy^2 / (1-y)$$

$$= q(x) \text{ say .} \qquad \ldots(3.3.2.14)$$

As in the case of A- optimality, we shall show that $\phi_0^D(j,n_1)$ as a function of n_1, given j has a unique extremum in the range $0 < n_1 < v-1$ and that is a minimum, so that the maximum value of $\phi_0^D(j,n_1)$ is attained at one of the extreme points of the feasible range for n_1 which is

$$(v-1)(1-t) / j \leq n_1 \leq (v-1)(j-t) / j .$$

The expression for $\frac{\partial \phi_0^D(j,n_1)}{\partial n_1}$ it self is quite complicated and so we shall not go into the derivation of the second derivative. Instead, we shall prove the result in a different way.

Our proof is based on the following observations written in the form of a lemma.

Lemma 3.3.2.1. The function $q(x)$ defined by (3.3.2.14) and (3.3.2.12) for $x > 0$ has the following properties

(i) $q(x) < 0$ for $x \leq 1/3$

(ii) $q(x) > 0$ for $x \geq 1/2$

(iii) $q(x)$ is a monotone increasing function of x in the range $1/3 < x < 1/2$.

Proof. Since $0 < y < 1$, $q(x)$ can be expanded as

$$q(x) = \sum_{\ell=2}^{n} (x - 1/\ell)y^{\ell} + R_n$$

where R_n in the remainder term and

$$R_n > (x-1/(n+1))y^{n+1} / (1-y) > 0$$

$\therefore$ $x \geq 1/2 \Longrightarrow q(x) > 0$, proving (ii).

Again $x \leq 1/3 \Longrightarrow q(x) < -y^2/6 + xy^4/(1-y)$

$$\leq \frac{y^2}{6(1-y)} (y+1)(2y-1) \qquad \ldots(3.3.2.15)$$

But since $a > v(i + v)$ from (3.3.4) and (3.3.1.7), $j(v+4) < 3a$ by (3.3.1.8) which implies that $y < 1/2$ and so

$$q(x) < 0 \quad \text{for} \quad x \leq 1/3, \quad \text{from } (3.3.2.15)$$

Hence (i) is proved.

For proving (iii) we note that $q(x)$ is continuous and differenciable with respect to x, $\forall\, x \in (0,1)$,

and $$\frac{dq(x)}{dx} = [1-(1-y)^{-1} + x(2y - y^2)(1-y)^{-2}] \frac{dy}{dx} + y^2/(1-y)$$

or $$y^{-1}(1-y)^2 \frac{dq(x)}{dx} = y(1-y) + [x(2-y) - (1-y)] \frac{dy}{dx}$$

Now $$\frac{dy}{dx} = \frac{j(a(v-1) - jv)}{(a + jvx)^2}, \text{ so that}$$

$$y^{-1}(1-y)^2 \frac{dq(x)}{dx} \cdot \frac{(a + jvx)^2}{j} = A + Bx + A'x^2 + B'x^3$$

$$= P(x) \quad \text{say}$$

where $A = -(a-j)(a(v-2) - jv)$,

$B = aj + a(a-j)(v-1) + jv(a-j) + (a(v-1) - jv)(2a - j)$

$A' = aj(v-1) + jv(v-1)(a-j) + jv + j(v+1)(a(v-1) - jv)$

and $B' = j^2 v(v-1)$

$\therefore\ P(x) > A + Bx\ ,\quad \because\ A', B' > 0$

$> a(v-1)\left\{ a(3x-1) + j(1-2x) \right\}$

> 0 whenever $\frac{1}{3} < x < \frac{1}{2}$.

Hence (iii) holds and the proof of the lemma is complete.

Now lemma 3.3.2.1 implies that $q(x)$ has at most one zero in the feasible range for x (given in 3.3.1.7) and if $q(x)$ has a zero in the feasible range, $q(x)$ is monotone increasing at that point. But this means by virtue of (3.3.3.14) that $\phi_0^D(j,n_1)$ has at most one extremum in the feasible range for n_1 and if such an extremum exists within the feasible range for n_1, then $\frac{\partial \phi_0^D(j,n_1)}{\partial n_1}$ is increasing at that point, so that the point is a minimum. Hence the maximum of $\phi_0^D(j,n_1)$ is attained at one of the end points of the feasible range for n_1 as was the case in the proof of A- optimality.

Therefore, by a similar argument as in the last section, we can write

$$\max_{\theta, n_1, p} \phi_D(\theta, n_1, p)$$

$$= \max \left\{ \max_{1 \leq j \leq i+1} \phi^D_{o1}(j), \max_{1 \leq j \leq i+1} \phi^D_{o2}(j) \right\}$$

where $\phi^D_{o1}(j) = \phi^D_o(j, n_1 = (v-1)(1-t)/j)$

and $\phi^D_{o2}(j) = \phi^D_o(j, n_1 = (v-1)(j-t)/j)$.

In the third and final step we shall show that both $\phi^D_{o1}(j)$ and $\phi^D_{o2}(j)$ are decreasing functions of j from which our optimality result will follow by an argument similar to that in the relevant part of the section 3.3.1.

From (3.3.2.7), at $n_1 = (v-1)(1-t)/j$,

$$\mu_1 = c(a + v(1-t)) \quad \text{and} \quad \mu_2 = c(a - (j - 1 + t)). \quad ..(3.3.2.16)$$

From (3.3.2.9), on simplification,

$$\phi^D_{o1}(j)/(v-1) = \log \mu_1 + v(1-t)\,[j + (v-1)(1-t)]^{-1} \log(\mu_2/\mu_1)$$

which is clearly a decreasing function of j from (3.3.2.16).

Similarly, at $n_1 = (v-1)(j-t)/j$,

$$\mu_1 = c(a+v(j-t)) \quad \text{and} \quad \mu_2 = c(a-t) \quad ...(3.3.2.17)$$

$\therefore$ From (3.3.2.9),

$$\phi^D_{o2}(j)/(v-1) = \log \mu_2 + t(vj - (v-1)t)^{-1} \log(\mu_1/\mu_2).$$

Let us denote $c(vj - (v-1)t)$ by $\alpha(j)$ which is $> 0 \ \forall j \geq 1$, as $t < 1$. Then, at $n_1 = (v-1)(j-t)/j$, $\mu_1 = \mu_2 + \alpha(j)$, from (3.3.2.17)

$$\therefore \left[\phi^D_{o2}(j) - \phi^D_{o2}(j+1) \;/\; t(v-1) \right] \alpha(j)\alpha(j+1)$$

$$= -cv \log\left(1 - \frac{\alpha(j)}{\mu_2 + \alpha(j)}\right) - \alpha(j)\log\left(1 + \frac{cv}{\mu_2 + \alpha(j)}\right), \quad \ldots(3.3.2.18)$$

on simplification.

But we know that for $|x| < 1$, $\log(1+x) < x$ and $\log(1-x) < -x$.

Hence the expression in (3.3.2.18) is

$$> c\{v\alpha(j)/\mu_1 - \alpha(j)v/\mu_1\} = 0.$$

This means that $\phi^D_{o2}(j) > \phi^D_{o2}(j+1) \;\forall\; j \geq 1$, i.e. $\phi^D_{oj}(j)$ is a decreasing function in j. So, we arrive at the following optimality result.

<u>Theorem 3.3.2.2</u>. A GDD with $\lambda_2 = \lambda_1 + 1$, if it exists, is D-optimal under fixed effects model over all designs in $\mathcal{D}_p$.

Incidentally, theorem 3.3.2.1 solves the maximal spanning tree problem of a certain restricted class of graphs. The graph theoretic aspect of the result is not being elaborated here and the relevant references in this connection are Cheng (1981c) and Gaffke (1982).

In situations when a GDD with $\lambda_2 = \lambda_1 + 1$ is nonexistent, the search for an optimal design even within the class $\mathcal{D}_p$ seems much more difficult as this involves solving, to begin with, the general problem of finding the least possible value of p which admits a feasible two associate class association scheme, for given θ and n_1. The problem is an open one and known to be hard.

$p = 0$ in the class $\mathcal{D}_p$ will necessarily imply that the design is a GDD and it is quite possible that the given parameters b,k and v precludes the feasibility of a GDD with $\lambda_2 = \lambda_1 + 1$ in the class $\mathcal{D}_p$. However, p is a nonnegative integer and given $\lambda_2 = \lambda_1 + 1$, n_1 remains fixed and the best possible value of p in this case must be 1, provided, of course, such a design exists in $\mathcal{D}_p$ with $p = 1$. This design when it exists is a prospective candidate for optimality test. By similar arguments as used in the proofs of the theorems 3.3.1.1 and 3.3.2.1, the A and D- optimality of the design in $\mathcal{D}_p$ with $\lambda_2 = \lambda_1 + 1$ and $p = 1$ can be checked by simply verifying the following conditions (It is assumed that the given parameters b,k and v preclude the existence of a GDD with $\lambda_2 = \lambda_1 + 1$ and a design with $p = 1$ and $\lambda_2 = \lambda_1 + 1$ exists).

$$\phi_Q(\theta = -1,\ n_1 = (v-1)(1-t),\ p = 1) = \phi^Q_{11}(1), \quad \text{say}$$

$$\geq \text{Max}(\phi^Q_{01}(2),\ \phi^Q_{02}(2)) \qquad \ldots(3.3.2.19)$$

where $Q = A$ or D.

We shall now consider two examples in each of which neither a BIBD, nor a GDD can exist for the given set of parameters b,k and v, but a design with $p = 1$ and $\lambda_2 = \lambda_1 + 1$ exists. In both the examples, the design is shown to be A- optimal within $\mathcal{D}_p$.

Example 1. Let us consider the set of parameters $b = 5$, $k = 3$ and $v = 5$.

As mentioned already, neither a BIBD, nor a GDD is feasible with the given set of parameters. But a design with cyclical association scheme exists, with parameters $r = 3$, $n_1 = n_2 = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $p_{12}^1 = 1$, as is obtained from the design 12 of p.66 of Clatworthy (1956), by interchanging the roles of first and second associates of the association scheme.

For this example,

$$\phi_{11}^A(1) = \phi_A(p = 1,\ \theta = -1,\ n_1 = (v-1)(1-t)) = a^2 / (5t(1-t))$$

where $a \geq 15/2$ and $t = 1/2$.

Also $\phi_{o1}^A(2) = (a^2 - 6at + 4a) / (5(1-t^2))$

and $\phi_{o2}^A(2) = (a^2 - 6at + 10a) / (5t(2-t))$.

Hence condition (3.3.1.9) is satisfied with $Q = A$.

By the same technique the D- optimality of the design could not be established.

Example 2. $b = 15$, $k = 3$, $v = 5$.

Here again neither a BIBD nor a GDD is feasible, but a cyclic design with parameters $r = 9$, $n_1 = 2$, $p_{12}^1 = 1$, $\lambda_1 = 4$, $\lambda_2 = 5$, can be obtained from the corresponding design in Clatworthy, (1956) by interchanging the roles of the first and second associates of the association scheme.

Here $a \geq 45/2$, $t = 1/2$.

Since v is the same as that of example 1, the general expressions with a and t for $\phi^A_{11}(1)$, $\phi^A_{01}(2)$ and $\phi^A_{02}(2)$ are the same as those given in example 1 and it is easy to see that condition (3.3.2.19) is satisfied with $Q = A$, so that the design is A-optimal within $\mathcal{D}_p$.

However, condition (3.3.2.19) is not satisfied with $Q = D$, so that the D- optimality of the design is not established.

Remark 1. The optimality results obtained in the present chapter in the context of one-way heterogeneity setting, restricted to the class $\mathcal{D}_p$ can be easily extended to two-way heterogeneity settings, provided we restrict to a class of designs similar to $\mathcal{D}_p$, defined as follows.

Let $\mathcal{D}_p^{(1)}$ be the class of two-way designs with given b_1, b_2 and v, such that the columns constitute the blocks of a BBD and the rows constitute the blocks of a PBBD with two associate class association scheme. Then the C-matrix of a design d belonging to $\mathcal{D}_p^{(1)}$ under mixed effects model is given by

$$C_d^{(M)} = (w - \bar{w})(r.I_v - b_2^{-1} N_1 N_1') + a_o(I_v - v^{-1} J_v)$$

where $a_o = (w - \bar{w}_2)b_2(b_1 - 1)/(v - 1) - (w - \bar{w}_1 - \bar{w}_2)r$, and N_1 is the incidence matrix of the PBBD with two associate class association scheme (i.e. giving the incidence of treatments in rows).

So, the eigenvalues of $C_d^{(M)}$ are of the same form as (3.3.3) with the expressions for a and c as follows.

$$c = (w - \tilde{w}_1) / b_2 \qquad \ldots(3.3.2.20)$$

and $\quad a = v(rb_2 - h(r, b_1))(v-1)^{-1} + a_0 / c$

where $h(r, b_1)$ is as defined in (3.2.2).

The multiplicities of α_1 and α_2 are also the same as given in (3.3.4).

Now it can be trivially checked that the proofs of the theorems 3.3.1.1 and 3.3.2.1 would go through with the values of a and c given in (3.3.2.20) and the theorem remain still valid in the present context. In other words, we obtain the following optimality result.

<u>Theorem 3.3.2.2</u>. A two-way design with the rows constituting the blocks of a GDD with $\lambda_2 = \lambda_1 + 1$ and the columns constituting the blocks of a BBD, if it exists is A- and D- optimal over all designs in $\mathcal{D}_p^{(1)}$.

<u>Remark 2</u>. It the two-way setting is regular with respect to one of the factors, say, the second, then because of theorem 2.5.3.6 of Chapter 2, the theorem 3.3.2.2 holds over a wider class of designs, each of which is a PBBD with two associate classes with

respect to factor 1 i.e. rows, but no restriction is imposed on the designs with respect to other factors, i.e. columns. More generally, in a multiway setting which is regular with respect to all but one factor, a similar optimality result will hold as stated below.

Theorem 3.3.2.3. Let us consider a balanced m-way setting regular with respect to the factors 2,3,...,m. Let $\mathcal{D}_p^{(m)}$ be the class of designs where blocks with respect to the first factor constitute a PBBD with two associate classes (no restriction is imposed on the other factors). Then a design d^* which is balanced with respect to the factors 2,3,...,m and which has N_{d*} (= incidence matrix of treatments with respect to the levels of the first factor), representing the incidence matrix of a GDD with $\lambda_2 = \lambda_1 + 1$, is A- and D- optimal within $\mathcal{D}_p^{(m)}$, $m \geq 3$.

N.B. Just before submitting the thesis, I came across the paper of Cheng (1981c) who has proved the optimality of the GD Designs with $\lambda_2 = \lambda_1 + 1$ and the examples of PBIB Design with cyclical association scheme for $v = 5$ considered in the chapter under any generalised ϕ_f criterion of type 1. So, all the results in the present chapter follow from Cheng's (1981c) Theorem 3.1. However, the results of this chapter were derived independently the part of which regarding A-optimality has already been published (Mukhopadhyay (1981)) and the approach is different from Cheng's.

CHAPTER 4

EXTENSIONS OF EXPERIMENTS

4.1 Introduction

In the preceding two chapters optimality properties have been studied of standard block designs, given b,k and v. The given set of parameters may often not permit the existence of such optimal designs. In the present chapter attention is focussed on such situations in the one-way heterogeneity setting and study the behaviour of what is termed 'extended designs' with regard to A-, D- and E- optimality criteria.

The concept of extension of an experiment is due to Sinha and Sinha (1969). Their method of extension involves the addition of a few (say p) extra treatments to the given set of v treatments with the other parameters (b and k) remaining unaltered, such that with this addition the new set of parameters offers an available optimal block design, say for instance a BBD. Here we introduce another type of extension which we call extension of type 2 (the extension of Sinha and Sinha being called of type 1) where the available optimum block design with a larger set of treatments is converted into a design involving the original v treatments only.

In the present chapter, the optimal designs considered as the basis from which such extensions are obtained are (i) either

BBD or (ii) a GDD with $\lambda_2 = \lambda_1 + 1$ with the number of extra treatments $p = 1$ in the latter case.

In both the cases, the extended designs of type 2 turn out to be E-optimal under certain conditions on the parameters of the design, for fixed effects model. No such straight forward result seems to hold in the case of mixed effects model and hence mixed effects model have been excluded from such considerations here.

The behaviour of the extended designs with regard to A- and D- optimality criteria (only A- optimality criterion for the asymmetrical extended designs) have also been studied and a lower bound to their efficiencies have been calculated.

4.2 Preliminaries

For the definitions of the optimality criteria and the designs considered in the present chapter one is referred to Chapter 2. Most of the symbols have also been introduced in Chapter 2. There are, however, a few symbols and algebraic results that have not been mentioned elsewhere and we present them in this section for completeness.

4.2.1 Averaged Version of the C-matrix and its Properties

DFN 4.2.1.1. Let A be a $v \times v$ nonnegative definite matrix with zero row sums and column sums and let $\mathcal{P} = \{P_i\}_{i=1}^{m}$ be a collection of $v \times v$ permutation matrices. Then the matrix $\bar{A} = \frac{1}{m} \sum_{i=1}^{m} P_i' A P_i$ is said to be the averaged version of A over $\mathcal{P}$.

Here we shall consider the averaged version of the matrix C_d, the C- matrix of a given design d over two classes of permutation matrices as described below.

(i) Let $T_1 = \{i_1, i_2, \ldots, i_u\}$, $u < v$ ...(4.2.1.1)

be a subset of the set of all treatments $T = \{1, 2, \ldots, v\}$. Then we consider the collection $\mathcal{P}^{(1)}$ of all possible $u!$ permutation matrices involving the rows and columns indexed by T_1 (while retaining the remaining $v-u$ rows and columns of C_d unaltered) and call the averaged version of C_d over $\mathcal{P}^{(1)}$ the averaged version (i) over T_1.

(ii) Let T_1 be as defined in (i) and let

$$T_2 = T - T_1 \qquad \ldots(4.2.1.2)$$

We consider the class $\mathcal{P}^{(2)}$ of all possible $u!\,(v-u)!$ permutation matrices permuting the rows and columns indexed by T_1 among themselves and also the rows and columns indexed by T_2 among themselves. This average of C_d over $\mathcal{P}^{(2)}$ will be called the averaged version (ii) over T_1 and T_2. Usually the type of the averaged version considered will be clear from the context and there will be no ambiguity and so the averaged version of C_d of either type is denoted simply by $\bar{C}_d$.

Next we write down a few relevant results of Constantine (1981) without proof in lemma 4.2.2.1.

Lemma 4.2.2.1. Whatever be the set of permutations, the averaged version $\bar{C}$ of a C-matrix C satisfy the following properties.

(a) $\text{tr}\ \bar{C} = \text{tr}\ C$

(b) $\mu_1(\bar{C}) \geq \mu_1(C)$

where $0 = \mu_0(C) \leq \mu_1(C) \ldots \leq \mu_{v-1}(C)$

denote the eigenvalues of the matrix C of order v.

Moreover, the averaged version (ii) of C_d over the sets of treatments $T_1 = \{1, 2, \ldots, u\}$ and $T_2 = T - T_1$, $u < v$ is of the following form.

$$\bar{C}_d = \left[\begin{array}{c|c} (\bar{a}_1 + \bar{\alpha}_1)I_u - \bar{\alpha}_1 J_u & -\bar{\beta}\, J_{u \times \overline{v-u}} \\ \hline -\bar{\beta}\, J_{\overline{v-u} \times u} & (\bar{a}_2 + \bar{\alpha}_2)I_{\overline{v-u}} - \bar{\alpha}_2\, J_{\overline{v-u}} \end{array}\right] \quad \ldots(4.2.1.3)$$

and hence $\bar{C}_d$ has the following positive eigenvalues.

(i) $\bar{a}_1 + \bar{\alpha}_1$ with multiplicity $u-1$

(ii) $\bar{a}_2 + \bar{\alpha}_2$ with multiplicity $v-u-1$

(iii) $v\bar{\beta}$ with multiplicity 1.

Since $\bar{C}_d \cdot 1_v = 0$, the parameter, $\bar{a}_i, \bar{\alpha}_i, i = 1, 2$ and $\bar{\beta}$ satisfy the following relations.

$$\bar{a}_1 = (u-1)\,\bar{\alpha}_1 + (v-u)\,\bar{\beta}$$

$$\bar{a}_2 = (v-u-1)\,\bar{\alpha}_2 + u\bar{\beta} \quad \ldots(4.2.1.4)$$

4.2.2. The Variances and Covariances of the Elementary Treatment Contrasts based on a GDD, under Fixed Effects Model

In Chapter 3, the parametric relations of a PBBD with two associate classes and the eigenvalues of the C_d - matrix have been studied. The C_d - matrix of a GDD d with parameters $(v,b,r,k,\lambda_1, \lambda_2, m,n)$ under fixed effects model is of the following form

$$C_d = \begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & & \\ B & B & \cdots & A \end{bmatrix} \qquad \ldots(4.2.2.1)$$

where $A = (a_o - b_1)I_n + b_1 J_n$

and $B = b_2 J_n$

with $a_o = (rk - h)/k$ and $b_i = -\lambda_i / k$, $i = 1,2$, ...(4.2.2.2)

h as given in (3.2.2).

The eigenvalue of C_d can now be obtained easily from the structure of C_d in (4.2.2.1) as

(i) $a_o - b_1$ with multiplicity $m(n-1)$

(ii) $a_o + (n-1)b_1 - nb_2$ with multiplicity $m-1$ ...(4.2.2.3)

Now let $\hat{\underline{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_v)'$ be a solution of the reduced normal equations $C_d \hat{\tau} = Q_d$, where C_d is as given in (4.2.2.1). Then the variances and covariances of $\hat{\tau}_i$'s are given in (3.1.11) of Sinha and Sinha (1969) for the case $k < v$. In the general situation, the variances and covariances can easily be seen to be of the following form

$$V(\hat{\tau}_i) = \sigma^2 a_o^{-1}, \quad i = 1,2,\ldots,v$$

$$\text{Cov}(\hat{\tau}_i, \hat{\tau}_{i'}) = \sigma^2 a_o^{-1} c_j / k, \quad i \neq i' \qquad \ldots(4.2.2.4)$$

where $j = 1$ if i,i' belong to the same group and $j = 2$ if they belong to different groups,

$$c_1 = -kb_1(a_o - b_1)^{-1}$$

$$c_2 = (k/v)\left[1 + (n-1)(b_1 - b_2)b_1 b_2^{-1}(a_o - b_1)^{-1}\right]. \qquad \ldots(4.2.2.5)$$

Hence the variances and covariances of the estimate of the elementary treatment contrasts are as given below

$$V(\hat{\tau}_i - \hat{\tau}_{i'}) = 2t_j \qquad \ldots(4.2.2.6)$$

where $j = 1$ if i,i' belongs to the same group,

$j = 2$ otherwise

and $t_j = 2\sigma^2 a_o(1 - c_j/k)$, $j = 1,2$.

$\text{Cov}(\hat{\tau}_i - \hat{\tau}_j, \hat{\tau}_{i'} - \hat{\tau}_{j'})$

$= 2(t_2 - t_1)$ if i, i', j, j' are all distinct and i,i' belong to one group whereas j,j' belong to another group,

$= 2(t_1 - t_2)$ if i,i',j,j' are distinct and i,i' belong to one group and i',j belong to another,

$= 2t_2 - t_1$, if $i = i'$, $j \neq j'$ and j, j' belong to one group which is different from the group to which i belongs,

$= t_2$ if $i = i'$ and i, j, j' belong to three different groups,

$= t_1$ if $i = i' \neq j \neq j'$ and at least one of j and j' belongs to the same group as i,

$= 0$ in all other cases. ...(4.2.2.7)

4.3 Extension of an Experiment and Two Types of Extensions

Let us consider the problem of designing an experiment in the one-way heterogeneity setting with the given parameters b, k and v which preclude the existence of a commonly known optimum design, e.g. a BBD or a GDD with $\lambda_2 = \lambda_1 + 1$, but an optimum design is known with the same values for b and k but a larger value of v. It is this type of situation when extension of an experiment is conceivable.

DFN 4.3.1. (Sinha and Sinha (1969)) An experiment E' is defined as an extension of type 1 of another experiment E, if the set of treatments in E' includes these of E as a proper subset and the experimental field is the same in both the experiments (i.e. b and k remain same in both E and E').

We define an extension of type 2 in the following manner.

DFN 4.3.2. As in DFN 4.3.1, let E be an experiment involving the set of treatments $T = \{1, 2, \ldots, v\}$ and E', an experiment involving a set of $v + p$ treatments $(p < v)$,

$$T' = \{1, 2, \ldots, v, v+1, \ldots, v+p\},$$

the experimental field being same in both the cases.

Let $S_1 = \{i_1, \ldots, i_p\} \subset T$. Then E'', an extended experiment of type 2 is defined as the experiment obtained by replacing the treatment $v + j$ by the i_j, $j = 1, 2, \ldots, p$ in E'.

We will usually denote the available optimal design corresponding to the experiment E' by d, its extension of type 1 by d_1 and its extension of type 2 by d_2.

Here we consider situations when the available optimum designs E' in DFN's 4.3.1 and 4.3.2 are (i) BBDs or (ii) GDDs with $\lambda_2 = \lambda_1 + 1$. (It is to be noted that the aforementioned GDD's are known to be E-optimal designs in general under fixed effects model and within the class of PBBD's with two associate class association scheme are proved to be A- and D- optimal both under fixed and mixed effects models in Chapter 3). Let the extended designs of type 1 in the two situations be denoted by $d_1(B)$ and $d_1(G)$ respectively.

Now we note that given an extended design of type 1, an extended design of type 2 is not unique and the structure of its C-matrix depends on the choice of S_1, unless of course the original optimal design is a symmetrical one, i.e. a BBD. Hence we can (and do) choose S_1 to be $\left\{v-p+1,\ldots,v\right\}$ ...(4.3.1) in the symmetrical case without any loss of generality and denote such an extended experiment by $d_2(B)$.

When the optimal design we start with is a GDD with parameters $(v^* = v+p, b, r^*, k, \lambda_1, \lambda_2 = \lambda_1 + 1, m, n)$, for different possible choices of the pairs of treatments assumed identical, viz., $v+j$ and i_j, $j = 1,2,\ldots,p$, a number of cases arise depending on whether the identical treatments ($v+j$ and i_j) belong to the same group or different groups. In this type of extensions, we have confined our attention to the situation $p = 1$. Such a situation may arise particularly when bk is not divisible by v but is divisible by $v+1$. If we denote the m groups of treatments by $G_1, G_2, \ldots, G_m$ where $G_i = \left\{(i-1)n+1,\ldots,i\,n\right\}$ then there are precisely two distinct extended designs of type 2 (upto renumbering of treatments) corresponding to the two choices for i_1, viz.,

$$\text{(i)}\quad i_1 = v \in G_m \quad \text{and} \quad \text{(ii)}\quad i_1 = (m-1)n \in G_{m-1}\ . \qquad \ldots(4.3.2)$$

We denote these designs by $d_2(G)$ and $d_2'(G)$ respectively.

The behaviour of these extended designs with regard to D- and A- optimality criteria have been studied in the next sections. In the present section we note a general property of the extended designs, which is satisfied by any extended design obtained from an optimal design E′ as the basis, under the assumption of both fixed and mixed effects model.

The property of the extended designs will follow from the algebraic result proved below.

Lemma 4.3.1. Let C_1 be the C-matrix of a connected design with given parameters b, k and $v+q$ $(q \geq 1)$ under the assumption of fixed effects or mixed effects model. Let C_2 be a $v \times v$ matrix obtained from C_1 by adding each of the rows (columns) of it indexed by $v+1, v+2, \ldots, v+q$ to the vth row (column). Then $\mu_1(C_2) \geq \mu_1(C_1)$, where the notation $\mu_1(C)$ is as defined in lemma 4.2.2.1.

Proof. Let $\bar{C}_1$ be the averaged version (i) of C_1 over the set of treatments $T_1 = \{v, v+1, \ldots, v+q\}$. Then, in view of lemma 4.2.2.1, it is sufficient to show that

$$\mu_1(C_2) \geqslant \mu_1(\bar{C}_1) \,. \qquad \ldots(4.3.3)$$

Connectedness of C_1 implies that $\bar{C}_1$ is connected and therefore C_2 is also connected.

For any connected C-matrix of order t, it is well known that for any $t \times 1$ vector α, s.t. $\alpha' 1_t = 0$, $\alpha' C^- \alpha$ is invariant for all choices of the g-inverse C^- of C.

Moreover, $(\mu_1(C))^{-1} = \underset{\alpha\,:\,\alpha' 1_t = 0}{\text{Max}} (\alpha' C^- \alpha)/(\alpha'\alpha)$.

So, (4.3.3) will follow if we can prove the inequality

$$\underset{\alpha_{v\times 1}\,:\,\alpha' 1_v = 0}{\text{Max}} \left\{(\alpha' C_2^- \alpha)/(\alpha'\alpha)\right\} \leq \underset{\beta_{(v+q)\times 1}\,:\,\beta' 1_{v+q}=0}{\text{Max}} \left\{(\beta' \bar{C}_1^- \beta)/(\beta'\beta)\right\}. \qquad \ldots(4.3.4)$$

This inequality will follow from lemma 4.3.2 proved below.

Lemma 4.3.2. For any nonnull $v \times 1$ vector α such that $\alpha' 1_v = 0$, there exists a $(v+q) \times 1$ vector β with $\beta' 1_{v+q} = 0$ such that

$$(\alpha' C_2^- \alpha)/(\alpha'\alpha) \leq (\beta' \bar{C}_1^- \beta)/(\beta'\beta),$$

where the matrices $\bar{C}_1$ and C_2 are as defined in lemma 4.3.1.

Proof. Let $\bar{C}_1 \hat{\tau} = Q_1$ ...(4.3.5)

be a consistent set of equations, where $\hat{\tau}' = (\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_{v+q})$, $Q_1' = (Q_1^*, Q_2^*, \ldots, Q_{v+q}^*)$ and $D(Q_1) = \bar{C}_1$. Let us consider another set of quations

$$C_2 \hat{\hat{\tau}} = Q_2 \qquad \ldots(4.3.6)$$

where $\hat{\hat{\tau}}' = (\hat{\hat{\tau}}_1, \hat{\hat{\tau}}_2, \ldots, \hat{\hat{\tau}}_v)$,

$$Q_2' = (Q_1^*, Q_2^*, \ldots, Q_{v-1}^*, Q_v^* + Q_{v+1}^* + \ldots + Q_{v+q}^*).$$

$\bar{C}_1$ and C_2 is as defined in lemma 4.3.1.

Then, one can show that the set of equations (4.3.6) is also consistent and $D(Q_2) = C_2$.

Let $\hat{\tau}^{(1)}$ and $\hat{\hat{\tau}}^{(2)}$ be two sets of solutions of (4.3.5) and (4.3.6) respectively.

Then we show that $\mathrm{Var}(\alpha' \hat{\hat{\tau}}^{(2)}) = \alpha' C_2^- \alpha$ and $\mathrm{Var}(\beta' \hat{\tau}^{(1)}) = \beta' \bar{C}_1^- \beta$ for all α and β satisfying the conditions of the lemma and the expressions are invariant for all choices of the g-inverses.

Given the structures of $\bar{C}_1$ and C_2, it is obvious that if $\hat{\tau}^{(1)} = (\hat{\tau}_1^*,\ldots,\hat{\tau}_{v+q}^*)'$ is a solution of (4.3.5), we can choose

$$\hat{\hat{\tau}}^{(2)} = (\hat{\hat{\tau}}_1^*,\ldots,\hat{\hat{\tau}}_v^*)'$$

as a solution of (4.3.6)

where $\hat{\hat{\tau}}_j^* = \hat{\tau}_j^*$, $j = 1,2,\ldots,v-1$ and $\hat{\hat{\tau}}_v^* = \sum_{j=0}^{q} \hat{\tau}_{v+j}^*/q$.

So, given a vector $\alpha = (\alpha_1,\ldots,\alpha_v)'$ with $\alpha' 1_v = 0$, we can choose $\beta = (\beta_1, \beta_2,\ldots,\beta_{v+q})'$

where $\beta_j = \alpha_j$, $j = 1,2,\ldots,v-1$ and $\beta_v = \beta_{v+1} = \ldots = \beta_{v+q} = \alpha_v/q$.

Then, $\beta' 1_{v+q} = 0$ and $\alpha' \hat{\hat{\tau}}^{(2)} = \beta' \hat{\tau}^{(1)}$.

So, $\mathrm{Var}(\alpha' \hat{\hat{\tau}}^{(2)}) = \mathrm{Var}(\beta' \hat{\tau}^{(1)})$.

Again $\beta'\beta \leq \alpha'\alpha$ and hence the lemma is proved.

This implies the following property of extended designs. The method imployed in the lemmas 4.3.1 and 4.3.2 can be applied to C-matrices repeatedly for disjoint sets of treatments combined together in the resulting C_2 matrix and thus is proved the following theorem.

<u>Theorem 4.3.3</u>. Let d_2 be extended designs of type 2 obtained from a certain design d in the same way as described in DFN 4.3.2. Then we have

$$\mu_1(C_{d_2}) \geq \mu_1(C_d),$$

where C_d denotes the C-matrix of a design μ under either fixed effects or mixed effects model.

<u>Remark</u>. Comparison of the extended designs d_1 and d_2, in general, has not been possible. But, considering the normalised contrasts and their variances ab initio, when the original design d is either a BBD or a GDD with $\lambda_2 = \lambda_1 + 1$, one can trivially prove that $d_2(B)$ is E-better than $d_1(B)$, also each of $d_2(G)$ and $d_2'(G)$ is E-better than $d_1(G)$.

4.4. <u>The Behaviour of the Extended Designs with regard to A-, D- and E- optimality Criteria</u>.

Let us first consider the extended designs obtained from a BBD with parameters $(v^*, b, r^*, k, \lambda^*)$.

$$C_{d_1(B)} = a_o^* \left[I_{v^*} - (v^*)^{-1} J_{v^*} \right] \qquad \ldots(4.4.1)$$

where $a_o^* = (r^*k - h^* + \lambda^*)/k$

whith $h^* = r^* + (2r^* - b)\left[k/v^*\right] - b\left[k/v^*\right]^2$.

With the form of S_1 as chosen in (4.3.1) we obtain $C_{d_2(B)}$ as $C_{d_2(B)} = a_o^* H$, with

$$H = \left[\begin{array}{c|c} I_{v-p} - (v^*)^{-1} J_{\overline{v-p}} & -2(v^*)^{-1} J_{\overline{v-p}\times p} \\ \hline -2(v^*)^{-1} J_{p\times\overline{v-p}} & 2 I_p - 4(v^*)^{-1} J_p \end{array}\right] \qquad \ldots(4.4.2)$$

So, the positive eigenvalues of $C_{d_2(B)}$ are as follows.

(i) a_o^* with multiplicity $v-p-1$

(ii) $2a_o^*$ with multiplicity $p-1$

(iii) $(2v/v^*)a_o^*$ with multiplicity 1 ...(4.4.3)

using these facts we study the behaviour of $d_2(B)$ with regard to E- optimality in section 4.4.1 and A- and D- optimality in 4.4.2.

Let us now consider the situations when the extended designs are based on a GDD with parameters $(v^* = v+1, b, r^*, k, \lambda_1, \lambda_2 = \lambda_1 + 1, n, n)$. The matrix $C_{d_1(G)}$ is of the same form as C_d in (4.2.2.1) and its elements are obtained by modifying those of C_d in accordance with the parameters of $d_1(G)$, viz., by replacing a_o by $a_1 = (r^*k - h^*)/k$ ..(4.4.4)

where h^* is as given in (4.4.1). Let $a_1' = a_1 - b_1$ ...(4.4.4a)

From the values of i as chosen in (4.3.2) the matrices $C_{d_2(G)}$ and $C_{d_2'(G)}$ are determined as follows

$$C_{d_2(G)} = \left[\begin{array}{cccc|c} A & B & \cdots & B & B^* \\ B & A & \cdots & B & B^* \\ \vdots & \vdots & & \vdots & \vdots \\ B & B & \cdots & A & B^* \\ \hline (B^*)' & (B^*)' & \cdots & (B^*)' & A^* \end{array}\right] \qquad ...(4.4.5)$$

where the submatrices A and B are same as the corresponding submatrices of $C_{d_1(G)}$ and the submatrices A^* and B^* are as described below

$$A^{*} = \left[\begin{array}{c|c} a_1' I_{n-2} + b_1 J_{n-2} & b_1^{*}\, 1_{n-2} \\ \hline b_1^{*}\, 1_{n-2}' & a_1^{*} \end{array}\right]$$

$$B^{*} = \left[\begin{array}{c|c} b_2\, J_{b\times \overline{n-2}} & b_2^{*}\, 1_n \end{array}\right],$$

where $a_1^{*} = 2(a_1 + b_1)$, $b_i^{*} = 2b_i$, $i = 1,2$, a_1 as given in (4.4.4) and b_1, b_2 as in (4.2.2.2).

Similarly, $C_{d_2'(G)}$ is of the following form

$$C_{d_2'(G)} = \left[\begin{array}{cccc|c} A & B & \cdots & B & B^{**} \\ B & A & \cdots & B & B^{**} \\ \vdots & \vdots & & \vdots & \vdots \\ B & B & \cdots & A & B^{**} \\ \hline (B^{**})' & (B^{**})' & \cdots & (B^{**})' & A^{**} \end{array}\right] \qquad \ldots(4.4.6)$$

with the submatrices A and B same as these of $C_{d_1(G)}$ and A^{**} and B^{**} are as described below

$$A^{**} = \left[\begin{array}{c|c|c} a_1' I_{n-1} + b_1 J_{n-1} & b_1^{**} 1_{n-1} & b_2 J_{n-1} \\ \hline b_1^{**}\, 1_{n-1}' & a_1^{**} & b_1^{**}\, 1_{n-1}' \\ \hline b_2\, J_{n-1} & b_1^{**}\, 1_{n-1} & a_1' J_{n-1} + b_1 J_{n-1} \end{array}\right]$$

and $$B^{**} = \left[\begin{array}{c|c|c} b_2 J_{n\times\overline{n-1}} & b_2^{**} 1_n & b_2 J_{n\times\overline{n-1}} \end{array}\right] \qquad \ldots(4.4.7)$$

where a_1, a_1', b_1 and b_2 are as given in (4.2.2.2) and (4.4.4a), $a_1^{**} = 2(a_1 + b_2)$, $b_1^{**} = b_1 + b_2$, $b_2^{**} = 2b_2$.

The determination of all the eigenvalues of the matrices $C_{d_2(G)}$ and $C_{d_2'(G)}$ is not straightforward and have not been attempted. The minimum eigenvalues of both of them, however, have been determined with the help of lemma 4.3.1 in section 4.4.1 and hence the conditions for E- optimality of the designs $d_2(G)$ and $d_2'(G)$ have been derived. The expressions for their efficiencies with respect to the A- optimality criterion have also been derived in section 4.4.2 using a different approach, but the D-optimality criterion has not been considered at all for these designs.

4.4.1. E-optimality of the Extended Designs of Type 2 under Fixed Effects Model

From (4.4.3), we observe that

$$\mu_1(C_{d_2(B)}) = \mu_1(C_{d_1(B)}) = a_o^* \; . \qquad \ldots(4.4.1.1)$$

Now we shall prove the following optimality property of $d_2(B)$.

Theorem 4.4.1.1. The extended design $d_2(B)$ is E-optimal over the class $\mathcal{D}$ of all proper and connected block design with given b,k and v, under fixed effects model provided the parameters satisfy the following conditions.

(a) $k < v$

(b) $(p+i-1)r^* < v-i-1$ for some integer $i \geq 2$

and (c) $\lambda^* \leq v(i-1)/(i\,p)$ for the same integer i in (b).

Proof. Let d be any design in $\mathcal{D}$ with parameters $b,k,v,$ $r_{d1} \leq r_{d2} \ldots \leq r_{dv}$, $\lambda_{djj'}$, $j \neq j'$, $j,j' = 1,2,\ldots,v$. First we note that because of condition (b), there are at least i treatments with replications $\leq r^*$. For if possible, let us suppose there are at most $i-1$ treatments replicated r^* times or less. Then we have

$$(v-i+1)(r^*+1) \leq bk = (v+p)r^*$$

or $(p+i-1)r^* \geq v-i+1$, contradicting (b).

Let $T_1 = \{1,2,\ldots,i\}$. Now $\mathcal{D}$ can be partitioned into two disjoint subclasses $\mathcal{D}_1$ and $\mathcal{D}_2$ as follows.

$$\mathcal{D}_1 = \{d : d \in \mathcal{D} \text{ and there exists } j,j';\, j \neq j' \text{ such that } \lambda_{djj'} \leq \lambda^*\}.$$

$$\mathcal{D}_2 = \{d : d \in \mathcal{D} \text{ and } \forall\ j \neq j',\ \lambda_{djj'} \geq \lambda^*+1\}.$$

Let us consider $\mathcal{D}_1$ first. Renumbering if necessary, we can make $j=1$ and $j'=2$. Then applying inequality (3.4) of Constantine (1981) we have for $d \in \mathcal{D}_1$, $\mu_1(C_d) \leq a_o^*$.

Hence, $d_2(B)$ is E- better than each design in $\mathcal{D}_1$ by (4.4.1.1).

Let us now look at a design d in $\mathcal{D}_2$. Let $T_2 = T - T_1$. Averaging C_d over the sets of treatments T_1 and T_2 as in the average version (ii) of DFN 4.2.1.1, we obtain $\overline{C}_d$ which is of the same form as $\bar{C}_d^{(2)}$ in (4.2.1.3) with $u = i$,

$$\bar{a}_i = |T_i|^{-1} \sum_{j \in T_i} (kr_{dj} - h_j)/k, \quad i = 1,2$$

$$\bar{\alpha}_i = |T_i|^{-1} \sum_{\substack{j \neq j' \\ j,j' \in T_i}} \lambda_{djj'}/k, \quad i = 1,2$$

and $$\bar{\beta} = (i(v-i))^{-1} \sum_{j \in T_1} \sum_{j' \in T_2} \lambda_{djj'}/k \qquad \ldots(4.4.1.2)$$

where $h_j = \sum_{t=1}^{b} n_{jt}^2$, n_{jt} being the (j,t)th element of N_α.

Since, $h_j \geq r_{dj}$, $j = 1,2,\ldots,v$,

we have $\bar{a}_1 \leq r^*(k-1)/k$

also $\bar{\alpha}_1 \geq (\lambda^* + 1)/k$ $\ldots(4.4.1.3)$

Now using (4.4.1.2) and (4.4.1.3) we have

$$(v-i)\bar{\beta} \leq [r^*(k-1) - (i-1)(\lambda^*+1)]/k .$$

But from lemma 4.2.2.1, $v\bar{\beta}$ is an eigenvalue of $\bar{C}_d$.

$$\mu_{d_1}(\bar{C}_d) \leq v\bar{\beta}$$
$$\leq v(v-i)^{-1}\left\{r^*(k-1) - (i-1)(\lambda^*+1)\right\}/k .$$

So, in view of (4.4.1.1), a sufficient condition for d_2 being E-better than any design d in $\mathcal{D}_2$ is that

$$v(v-i)^{-1}\left\{r^*(k-1) - (i-1)(\lambda^*+1)\right\} \leq r^*(k-1) + \lambda^*$$

which reduces to condition (c) on simplification using condition (a). Hence the result.

Let us now study the extended designs $d_2(G)$ and $d_2'(G)$. In order to determine the minimum eigenvalues of $C_{d_2(G)}$ and $C_{d_2'(G)}$, we note that since $\lambda_2 = \lambda_1 + 1$, a_1' is the minimum eigenvalue of the matrix C_d as given in (4.2.2.2). Again the forms of $C_{d_2(G)}$ and $C_{d_2'(G)}$ in (4.4.4) and (4.4.5) shows that a_1' is an eigenvalue of both the matrices. But this means in view of lemma 4.3.1 that

$$\mu_1(C_{d_2(G)}) = \mu_1(C_{d_2'(G)}) = \mu_1(C_d), \qquad \ldots(4.4.1.4)$$

where d refers to the original GD Designs from which the extensions are derived. We are now in a position to prove the following optimality result.

<u>Theorem 4.4.1.2</u>. Both the designs $d_2(G)$ and $d_2'(G)$ are E-optimal over the class $\mathcal{D}$ of all proper and connected block designs with given b,k and v, under fixed effects model, provided the parameters satisfy the condition (a'), (b') and (c'), where (a') is the same as the condition (a) of theorem 4.4.1.1, (b') is obtained from (b) of the same theorem by substituting $p=1$ and (c') is

(c') $\lambda_2 \leq n-1-(v-i)/i$ for the same i in (b').

<u>Proof</u>. Let d be any design in $\mathcal{D}$ with parameters b,k,v, $r_{d1} \leq r_{d2} \cdots \leq r_{dv}$, $\lambda_{djj'}$, $j \neq j', j, j' = 1, \ldots, v$.

Then, following the same arguments as in the proof of theorem 4.4.1.1, we establish by virtue of condition (b') the existence of $T_1 \subset T$, $|T_1| = i$ with $r_{dj} \leq r^* \; \forall \; j \in T_1$ and thus partition into two disjoint subclasses $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$, where

$$\mathcal{D}_1^* = \left\{ d : d \in \mathcal{D}, \text{ there exist } j, j' \in T_1, j \neq j' \text{ such that } \lambda_{djj'} \leq \lambda_1 \right\}$$

and $$\mathcal{D}_2^* = \left\{ d : d \in \mathcal{D} \text{ and } \lambda_{djj'} \geq \lambda_2 \; \forall \; j \neq j', \; j, j' \in T_1 \right\}.$$

By the same argument as in the proof of theorem 4.4.1.1, we can show that for any $d \in \mathcal{D}_1^*$, $\mu_1(C_d) \leq a_1'$ and hence both $d_2(G)$ and $d_2'(G)$ are E-better than d by virtue of (4.4.1.4).

For any design $d \in \mathcal{D}_2^*$, we follow the same lines as in the proof of theorem 4.4.1.1 and obtain that

$$\mu_1(C_d) \leq v(v-1)^{-1} \left\{ r^*(k-1) - (i-1)\lambda_2 \right\} / k \qquad \ldots(4.4.1.5)$$

A sufficient condition for the right hand side of the inequality (4.4.1.5) to be $\leq a_1'$ is that $i r^* (k-1) \leq i(v-1)\lambda_2 - (v-i)$ and this reduces to condition (c') on simplification, using condition (a').

4.4.2. The Behaviour of $d_1(B)$ and $d_2(B)$ with regard to A- and D- optimality Criteria

Let us use the notation A_d and D_d for $\phi_1(C_d)$ and $\phi_0(C_d)$ (the notation defined in (2.2.3a) of Chapter 2), respectively. Then, from (4.4.1) we have

$$A_{d_1(B)} = (v-1)(a_o^*)^{-1}$$

and $$D_{d_1(B)} = (a_o^*)^{-(v-1)} \qquad \ldots(4.4.2.1)$$

Similarly, from (4.4.3),

$$A_{d_2(B)} = (v-1)(a_o^*)^{-1}(1-p/2v)$$

$$= A_{d_1(B)}{}^{(1-p/2v)} \qquad \ldots(4.4.2.2)$$

and $$D_{d_2(B)} = (a_o^*)^{-(v-1)}\ v^*/(2^p v)$$

$$= D_{d_1(B)}\ v^*/(2^p v)\ . \qquad \ldots(4.4.2.2a)$$

The following result is now immediate in view of (4.4.2.1) and (4.4.2.2) and (4.4.2.2a).

Theorem 4.4.2.1. For any integer $p \geq 1$, the extended design $d_2(B)$ is more efficient than $d_1(B)$ with regard to both A- and D- optimality criteria.

Next we shall calculate the efficiency of $d_2(B)$ as compared to the hypothetical best design for a given set of parameters b,k and v with regard to A- and D- optimality criteria.

Given a problem π, an optimality criterion ϕ and a set of parameters b,k and v, the efficiency of a design is to be assessed by comparing it against the optimal design, when it exists and can be found out. When such an optimal design with a given structure cannot be worked out (because, for instance the parameters of the design do not permit the existence of such a structured design), the best possible value of the criterion can be assessed by writing down the expression for it in the case of the hypothetically best design, without carring for the integrality restrictions of the parameters, even though such a design cannot obviously exist. Then we compare the value of the criterion obtained for a given existing design under the conditions against its hypothetical best value, without car ing for whether the so-called best design exists or can exist. The efficiency of a given design is thus measured as the ratio of the criterion value for the hypothetically best design to its value for the given design. The efficiency thus measured always provides actually a lower bound to the same. While considering the efficiency with regard to A- and D- optimality criteria, the hypothetical best design d_o for given b,k and v refers to the BBD, i.e. d_o has the C-matrix same as that of a BBD with the given parameters, disregarding whether the parameters derived for the design are feasible or not.

Clearly $A_{d_o} = (v-1)\, a_o^{-1}$ and $D_{d_o} = a_o^{-(v-1)}$ ...(4.4.2.3)

where $a_o = (v/v-1)(rk-h)/k$ with $r = bk/v$ (not necessarily an integer) and h as given in (3.2.2) of Chapter 3. Hence the efficiencies of $d_2(B)$ are given by

$$e_A = A_{d_o} / A_{d_2(B)} \quad \text{and} \quad e_D = D_{d_o} / D_{d_2(B)} \qquad \ldots(4.4.2.4)$$

where the expressions for A_{d_i}, $i = 0,2$ and D_{d_i}, $i = 0,2$ are given in (4.4.2.2), (4.4.2.a) and (4.4.2.3).

Next a few examples will be presented where the extended designs are better than existing GDD's, feasible for the given set of parameters.

The positive eigenvalues of the C-matrix of a GDD d are given in (4.2.2.3). From them the expressions for A_d and D_d are calculated for a GDD $(v,b,r,k,\lambda_1, \lambda_2, m,n)$ as follows.

$$A_d = k\left\{v\lambda_2 - n(\lambda_2-\lambda_1)\right\}^{-1}\left\{(v-1)-(m-1)(\lambda_2-\lambda_1)/(n\lambda_2)\right\} \quad \ldots(4.4.2.5)$$

and $$D_d = k^{-(v-1)}\left\{v\lambda_2 - n(\lambda_2-\lambda_1)\right\}^{-m(n-1)}(v\lambda_2)^{-(m-1)} \quad \ldots(4.4.2.6)$$

Example 1. Sinha and Sinha (1969) have observed that given $b = as(s^2+1)$, $k=s$, $v=s^2$, where s is a prime power and a is a positive integer, each one of the series of extened designs of type 1 obtained from the BIBD $(s^2+1,\ as(s^2+1),\ as^2, s, a(s-1))$ (this design can be constructed by slightly modifying the BIBD representing the lines of EG (2,s) as blocks) is A- and D- better than the corresponding existing GDD with parameters $(b=as(s^2+1)$, $k=s$, $v=s^2$, $r=a(s^2+1)$, $m=n=s$, $\lambda_1 = a(s^2-s+1)$, $\lambda_2 = a)$ (the existence of this GDD series is ensured by corollary 8.6.3.1 of Raghavarao (1971)), for any integer $a \geq 1$ and any prime power $s \geq 3$. Hence it follows from theorem 4.4.2.1 that the corresponding extended design of type 2 is still better than the existing GDD.

There are examples where the extended design of type 1, viz $d_1(B)$ is less efficient than the existing GDD, but $d_2(B)$ is more efficient than the same GDD, as is illustrated below.

Example 2. Let us consider the problem when $b = as(s^2-1)$, $k = s^2(s^2-1) + s$ and $v = s^2-1$, where s is a prime power ≥ 2 and a is a positive integer. A GDD d exists with the given values for b,k and v and other parameters as follows $r = as^2 + as^3(s^2-1)$, $n = s+1$, $n = s-1$, $\lambda_1 = 2as^2(s^2-1) + as^5(s^2-1)$, $\lambda_2 = \lambda_1 + as$ (the existence follows from the corollary 8.6.2.1 of Raghavarao (1971)). Also, a BBD exists with the same values for b,k and $v^* = v+1$, $r^* = a(s^2-1) + as(s^2-1)^2$, $\lambda^* = a(s-1) + 2a(s^2-1)^2 + as(s^2-1)^3$ (obtained from BIBD representing the lines of EG(2,s) as blocks). Then we have from (4.4.2.5),

$$A_d = k \ (as^2(s-1))^{-1} \left\{1 + 2(s^2-1) + s^3(s^2-1)\right\}^{-1} \Big[r^2 - 2 - s(s+1)^{-1}\left\{1 + 2s(s^2-1) + s^4(s^2-1)\right\}^{-1}\Big] \quad \ldots(4.4.2.7)$$

and similarly from (4.4.2.6),

$$D_d = k^{s^2-2}(as(s-1))^{-(s^2-2)} \ s^{-(s^2-s-2)}(s+1)^{-s} \Big[1 + 2(s^2-1) + s^3(s^2-1)^{-s^2-s-2)} \left\{1 + 2s(s^2-1) + s^4(s^2-1)\right\}^{-s}\Big] \quad \ldots(4.4.2.8)$$

Also, from (4.4.2.1),

$$A_{d_1}(B) = k(s^2-2)\Big[a(s-1)s^2\left\{1+2(s^2-1)(s+1) + s(s+1)(s^2-1)^2\right\}\Big]^{-1} \quad \ldots(4.4.2.9)$$

and $D_{d_1}(B) = k^{s^2-2}\Big[as^2(s-1)\left\{1 + 2(s+1)(s^2-1) + s(s+1)(s^2-1)^2\right\}\Big]^{-(s^2-2)}$ ...(4.4.2.10)

The expressions for $A_{d_2(B)}$ and $D_{d_2(B)}$ can be obtained from the expressions for $A_{d_1(B)}$ and $D_{d_1(B)}$ by using the relations (4.4.2.2).

Comparing these expressions against those for A_d and D_d as given in (4.4.2.7) and (4.4.2.8), we have

$$A_{d_2(B)} < A_d < A_{d_1(B)} \quad \forall \quad s \geq 2 \quad \text{and} \quad a \geq 1$$

although $D_d < D_{d_2(B)} < D_{d_1(B)} \quad \forall \quad s \geq 2$ and $a \geq 1$.

4.4.3. The behaviour $d_2(G)$ and $d_2'(G)$ with regard to A- optimality criterion

In the absence of BBD's, the best possible designs one looks for in practical situations are usually the class of GDD's with $\lambda_2 = \lambda_1 + 1$. It may be recalled that these GDD's have been proved to be A- and D- optimal within a restricted class of designs in Chapter 3. So, in the present section, we assume the parent design d to be a GDD with $\lambda_2 = \lambda_1 + 1$ and an extension of type 2 is obtained from it by assuming exactly two treatments in it to be identical, i.e. the designs $d_2(G)$ and $d_2'(G)$ introduced in section 4.3. We calculate the lower bounds to the efficiencies of these designs with regard to A- optimality criterion as indicated in section 4.4.2.

Let us first consider the design $d_2(G)$.

Let $\underline{\hat{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_{v+1})'$ be a solution of the normal equations $C_d \underline{\hat{\tau}} = \underline{Q_d}$.

Then $\hat{\hat{\tau}} = (\hat{\hat{\tau}}_1, \ldots, \hat{\hat{\tau}}_v)$, with $\hat{\hat{\tau}}_j = \hat{\tau}_j$, $j = 1, 2, \ldots, v-1$

and $\hat{\hat{\tau}}_v = (\hat{\tau}_v + \hat{\tau}_{v+1})/2$ is a solution of the normal equations $C_{d_2(G)} \underline{\hat{\hat{\tau}}} = \underline{Q_{d_2(G)}}$.

$$\therefore \hat{\hat{\tau}}_j - \hat{\hat{\tau}}_{j'} = \hat{\tau}_j - \hat{\tau}_{j'}, \; j \neq j', \; j, j' = 1, 2, \ldots, v-1$$

$$\text{and } \hat{\hat{\tau}}_v - \hat{\hat{\tau}}_j = (1/2)\left[(\hat{\tau}_v - \hat{\tau}_{j'}) + (\hat{\tau}_{v+1} - \hat{\tau}_j)\right], j = 1, 2, \ldots, v-1 \qquad \ldots(4.4.3.1)$$

Let $\bar{V}_2$ denote the average variance of all elementary treatment contrasts based on $d_2(G)$.

$$\text{Then} \quad \bar{V}_2 = \binom{v}{2}^{-1} \sum_{\substack{j < j' \\ j,j'=1}}^{v} V(\hat{\hat{\tau}}_j - \hat{\hat{\tau}}_{j'})$$

$$\text{or} \quad \binom{v}{2} \bar{V}_2 = \sum_{j=1}^{v-1} V(\hat{\hat{\tau}}_v - \hat{\tau}_j) + (1/2) \sum_{\substack{j \neq j' \\ j,j'=1}}^{v-1} V(\hat{\tau}_j - \hat{\tau}_{j'}) \qquad \ldots(4.4.3.2)$$

using (4.4.3.1).

$$\text{First term of } (4.4.3.2) = (1/4) \sum_{j=1}^{v-1} \left\{ V(\hat{\tau}_v - \hat{\tau}_j) + V(\hat{\tau}_{v+1} - \hat{\tau}_j) + 2\,\mathrm{Cov}(\hat{\tau}_v - \hat{\tau}_j, \hat{\tau}_{v+1} - \hat{\tau}_j) \right\}$$

Second term of (4.4.3.2)

$$= (1/2) \sum_{j=1}^{v-1} \sum_{\substack{j' \neq j, \\ j'=1}}^{v+1} \left[V(\hat{\tau}_j - \hat{\tau}_{j'}) - V(\hat{\tau}_j - \hat{\tau}_v) - V(\hat{\tau}_j - \hat{\tau}_{v+1}) \right]$$

$$\binom{v}{2} \bar{V}_2 = (1/2) \sum_{j=1}^{v-1} \sum_{\substack{j' \neq j \\ j'=1}}^{v+1} V(\hat{\tau}_j - \hat{\tau}_{j'})$$

$$- (1/4) \sum_{j=1}^{v-1} \Big\{ V(\hat{\tau}_j - \hat{\tau}_v) + V(\hat{\tau}_j - \hat{\tau}_{v+1})$$

$$- 2\, Cov(\hat{\tau}_v - \hat{\tau}_j , \hat{\tau}_{v+1} - \hat{\tau}_j) \Big\}$$

$$= (v-1) \Big\{ (n-3/2)t_1 + (m-1)nt_2 \Big\} \qquad \ldots(4.4.3.3)$$

using (4.2.2.6) and (4.2.2.7), on simplification.

Let us now derive the expression for $\bar{V}_2'$, the average variance of all elementary treatment contrasts based on $C_{d_2'(G)}$.

Let $\hat{\hat{\tau}} = (\hat{\hat{\tau}}_1, \ldots, \hat{\hat{\tau}}_v)'$ be a solution of $C_{d_2'(G)} \hat{\hat{\tau}} = Q_{d_2(G)}$. The solution $\hat{\hat{\tau}}$ can be obtained from the solution $\hat{\tau}$ as follows : The form of $C_{d_2'(G)}$ as given in (4.4.6) shows that we can assume

$$\hat{\hat{\tau}}_{(m-1)n} = (\hat{\tau}_{(m-1)n} + \hat{\tau}_{v+1})/2 \quad \text{and}$$

$$\hat{\hat{\tau}}_j = \hat{\tau}_j , \; j \in G_i , \; i = 1,2,\ldots,m-2 .$$

Then $\hat{\hat{\tau}}_j$, $j \in G_{m-1}, G_m$, $j \neq (m-1)n, v+1$ can be obtained from the set of linear equations $M \hat{\hat{\tau}} = L$, where

$$M = \left[\begin{array}{c|c} (a_o - b_1)I_{n-1} + b_1 J_{\overline{n-1}\times\overline{n-1}} & b_2 J_{\overline{n-1}\times\overline{n-1}} \\ \hline b_2 J_{\overline{n-1}\times\overline{n-1}} & (a_o - b_1)I_{n-1} + b_1 J_{\overline{n-1}\times\overline{n-1}} \end{array}\right]$$

and $L = (L_{(m-2)n+1}, \ldots, L_{(m-1)n-1}, L_{(m-1)n+1}, \ldots, L_{mn-1})$

with $L_\alpha = (a_o - b_1)\,\widehat{\tau}_\alpha + b_1 \sum_{j \varepsilon G_i} \widehat{\tau}_j + b_2 \sum_{j \varepsilon G_{i'}} \widehat{\tau}_j$

$$+ \varepsilon(b_1 - b_2)(\widehat{\tau}_{(m-1)n} - \widehat{\tau}_{v+1})/2,$$

$\alpha \varepsilon G_i$, $\varepsilon = 1$ if $i = m-1$

$= -1$ if $i = m$,

$i, i' = m-1, m$.

The expressions for a_o, b_1 and b_2 are given in section 4.2.

So $$M^{-1} = \left[\begin{array}{c|c} pI_{n-1} + qJ_{n-1} & sJ_{n-1} \\ \hline sJ_{n-1} & pI_{n-1} + qJ_{n-1} \end{array}\right]$$

where $p = (a_o - b_1)^{-1}$,

$$q = (a_o - b_1)^{-1} \frac{b_1(a_o + \overline{n-2}b_1) - (n-1)b_2^2}{(a_o + \overline{n-2}b_1)^2 - (n-1)^2 b_2^2}$$

and $s = -b_2 \left\{(a_o + \overline{n-2}b_1)^2 - (n-1)^2 b_2^2\right\}^{-1}$.

Thus, we have

$$\widehat{\widehat{\tau}}_\alpha = pL_\alpha + q \sum_{j \varepsilon G_i} L_j + s \sum_{j \varepsilon G_{i'}} \widehat{\tau}_j$$

$$= \widehat{\tau}_\alpha + g_i(\widehat{\tau}_{(m-1)n} - \widehat{\tau}_{v+1})/2 \qquad \ldots(4.4.3.4)$$

where $g_i = (b_i - b_{i'})\left\{a_o - b_1 + (n-1)(b_1 - b_2)\right\}^{-1}$,

$\alpha \varepsilon G_{i'}$, $i' \neq i$, $i, i' = m-1, m$.

Finally, proceeding as in $\overline{V}_2$ we have the expression for $\overline{V}_2'$ given by

$$\binom{v}{2}\overline{V}_2' = (n-1)(v-1-2vg_1)t_1 + \left\{(n-1)v(1+g_1)^2 + (m-2)n(v-1/2)\right\}t_2 . \qquad \ldots(4.4.3.5)$$

Now the average variance of all elementary treatment contrasts based on $d_1(G)$ (which is the same design d with the treatment $v+1 = mn$ omitted from all considerations) is given by

$$\binom{v}{2}\overline{V}_1 = \sum_{\substack{j,j'=1\\ j<j'}}^{v} V(\widehat{\tau}_j - \widehat{\tau}_{j'}) = (v-1)\left\{(n-1)t_1 + (m-1)nt_2\right\}.$$

Hence from (4.4.3. 3) it is clear that $\overline{V}_2 < \overline{V}_1$.

Again, $$\binom{v}{2}(\overline{V}_2' - \overline{V}_1) = -2v(n-1)g_1t_1 + \left\{v(n-1)(g_1^2 + 2g_1) - (v-1)/2\right\}t_2 < -v(n-1)g_1\left\{2t_1 - (g_1+2)t_2\right\} . \qquad \ldots(4.4.3.6)$$

Now from (4.2.2.5) and (4.2.2.6) we have

$$t_2 = ((v+1)\lambda_2 - (\lambda_2 - \lambda_1))\left\{(v+1)\lambda_2\right\}^{-1} t_1 . \qquad \ldots(4.4.3.7)$$

But from (4.4.3. 4) we have

$$g_1 = (\lambda_2 - \lambda_1)\left\{(v+1)\lambda_2 - (\lambda_2 - \lambda_1)\right\}^{-1}, \qquad \ldots(4.4.3.8)$$

so that from (4.2.3. 7), since $\lambda_2 - \lambda_1 = 1$,

$$t_2 = g_1^{-1}((v+1)\lambda_2)^{-1}\ t_1 \qquad \ldots(4.4.3.9)$$

$$\therefore \quad 2t_1 - (g_1 + 2)t_2 = ((v+1)\lambda_2)^{-1}\ t_1 > 0 .$$

This implies that the expression in (4.4.3.6) is negative and hence $\bar{V}_2' < \bar{V}_1$. So, we have the following result.

<u>Theorem 4.4.3.1</u>. Both the designs $d_2(G)$ and $d_2'(G)$ are more efficient than $d_1(G)$ with regard to A- optimality criterion.

Let us now compare the design $d_2(G)$ and $d_2'(G)$. From (4.4.3.3) and (4.4.3.5), we have

$$\binom{v}{2}(\bar{V}_2 - \bar{V}_2') = -(v-1) / 2(t_1 - t_2) + v(n-1)g_1 \left\{ 2t_1 - (g_1 + 2)t_2 \right\}$$

$$= ((v+1)\lambda_2)^{-1}t_1 \left[-(v-1) / 2 + v(n-1) / ((v+1)\lambda_2 - 1) \right] \qquad \ldots(4.4.3.10)$$

using (4.4.3.8) and (4.4.3.9).

Now $n - 1 = (v+1)/m - 1 \leq (v-1)/2$,

'=' when $m = 2$.

$\therefore$ The quantity within the third bracket in the expression (4.4.3.10) $\leq (v-1)/2 \ \left[v / ((v+1)\lambda_2 - 1) - 1 \right]$

< 0 for all $\lambda_2 > 1$

$= 0$ for $\lambda_2 = 1$.

So, we have the following result.

<u>Theorem 4.4.3.2</u>. With respect to the A- optimality criterion, the design $d_2(G)$ is more efficient than $d'_2(G)$, so long as $m > 2$ and/or $\lambda_2 > 1$. When $m = 2$ and $\lambda_2 = 1$, the two designs are equally efficient.

Next we shall calculate the efficiency of $d_2(G)$ with respect to the hypothetical best design, i.e. the hypothetical BBD with the given value of b,k and v, with regard to A-optimality criterion.

From (4.4.3. 3), $\overline{V}_2 = (2/v)\left\{(n-3/2)t_1 + (m-1)nt_2\right\}$

where t_1, t_2 are as given in (4.2.2.6).

Again the average variance of the hypothetical BBD is given by

$$\overline{V}_o = 2\sigma^2((v / v-1)a_o)^{-1}$$
$$= (2 / v)\sigma^2(v-1)a_o^{-1},$$

where a_o is as given in (4.2.2.2).

Efficiency of $d_2(G)$ is given by

$$e_A(G) = \overline{V}_o / \overline{V}_2$$
$$= \sigma^2 a_o^{-1}\left\{(n-3/2)t_1 + (m-1)nt_2\right\}^{-1}$$
$$= (v-1) / 2\left\{(n-3/2)(1-c_1/k) + (m-1)n(1-c_2/k)\right\}^{-1} \quad ...(4.4.3.11)$$

<u>Example</u>. Let us consider the problem when the given set of parameters are $b = 4$, $k = 2$ and $v = 3$. No feasible BBD or a GDD

with $\lambda_2 = \lambda_1 + 1$ exist with this set of parameters, but an MBGDD exists with the same values of b and k and $v^* = 4$, $r^* = 2$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = n = 2$.

So, we consider the extended design $d_2(G)$ based on this MBGDD of type 1.

Here $c_1 = 0$ and $c_2 = 1/2$, so that the efficiency of $d_2(G)$ is obtained from (4.4.3.11) as $e_A(G) = 1/2$.

CHAPTER 5

CONSTRUCTION OF OPTIMAL DESIGNS

5.1 Introduction

In the present chapter we discuss the constructional aspects of some series of designs which belong to one or more classes of optimal designs established in Chapter 2 and 3.

Several methods of construction for optimal one-way designs, symmetrical as well as asymmetrical, are available in the literature. In the two-way setting, under fixed effects and mixed effects models, the GYD's and the PYD's are optimal in the appropriate contexts. Constructions of some series of GYD's are given by Kiefer (1975), Ruiz and Seiden (1974), Seiden and Wu (1978), Ash (1977) and several other authors. An infinite series of PYD's has been constructed by Cheng (1981b). In the asymmetrical two-way settings, in general, no optimal designs are known except in the regular settings. Also, many of the known optimal asymmetrical one-way designs may be appropriately modified to obtain optimal asymmetrical two-way designs in the regular setting.

Let us now have a look at the multi way heterogeneity situations. Constructions of some series of optimal designs in the complete multiway setting of Cheng (1978a), vize., the YHR's, YHC's and PYD's in the appropriate contexts have been discussed by Cheng (1979, 1981a). Construction of an optimal design in a general balanced multiway setting defined in section 2.2, particularly when the setting is incomplete, appears to be more involved and construction of some series of such optimal designs in the balanced incomplete multiway setting has been attempted in the present chapter.

The section 5.2 contains the preliminaries and notation required for subsequent development of the constructions. In section 5.3, we present methods of construction for BYHR's. Along with a series of completely regular BYHR's, the series of square configuration known from Ruiz and Seiden (1974) to represent GYD's has been modified accordingly to represent a corresponding series of BYHC's in a nonregular balanced m-way setting, where $3 \leq m \leq s+1$, with no additional observations required. In section 5.4, constructional aspects of BPYD's have been discussed and three series of BPYD's have been constructed. Finally, in section 5.5, balanced m-way settings have been dealt with, the setting being regular with respect to all but one direction, and constructions of a few series of asymmetrical optimal designs have been presented in such situations.

5.2. Definitions and Notation

A balanced m-way setting has been defined in DFN 2.2.24 of Chapter 2. Optimal designs in this setting, viz., the BYHR's and BPYD's and other designs we construct in the present chapter have also been defined in section 2.2. For the purpose of construction of m-way optimal designs we shall make use of certain concepts we introduce in the present section.

Let us consider an $m \times n$ matrix A and let $I(m) = \{1, 2, \ldots, m\}$ and $I(n) = \{1, 2, \ldots, n\}$ be respectively the index sets for the rows and columns of A.

DFN 5.2.1. A set of ℓ positions $\{(i_1, j_1), \ldots, (i_\ell, j_\ell)\}$ $i_p \in I(m)$, $j_p \in I(n)$, $p = 1, 2, \ldots, \ell$ is said to be a transversal of length ℓ or simply an ℓ-tranversal of A, denoted by $\ell - \sigma$, if it intersects each row a constant number of times and each column a constant number of times. In other words, the complete collection $(i_1, i_2, \ldots, i_\ell)$ consists of all elements of $I(m)$, each repeated a constant number of times (actually ℓ / m), and similarly the complete collection $(j_1, j_2, \ldots, j_\ell)$ consists of all elements of $I(n)$, each repeated a constant number of times (actually ℓ / n).

The set of elements of A positioned on an ℓ-transversal $\ell - \sigma = \{(i_1, j_1), \ldots, (i_\ell, j_\ell)\}$ will be denoted by $(\ell - \sigma)(A)$. In other words, $(\ell - \sigma)(A)$ will denote the set of elements

$$\{a_{i_1, j_1}, a_{i_2, j_2}, \ldots, a_{i_\ell, j_\ell}\}$$

where $A = (a_{ij})_{\substack{i \in I(m) \\ j \in I(n)}}$.

DFN 5.2.2. Two ℓ-transversals $\ell\text{-}\sigma$ and $\ell\text{-}\sigma^1$ are said to be parallel if they have no position in common.

DFN 5.2.3. A set of pairwise parallel ℓ-transversals of A whose union comprises all positions of A is said to be a complete set of parallel ℓ-transversals and is denoted by CSPT (ℓ). The cardinality of such a set is clearly $k = mn/\ell$.

DFN 5.2.4. Let $\ell\text{-}\Sigma = \{\ell\text{-}\sigma^1, \ell\text{-}\sigma^2, \ldots, \ell\text{-}\sigma^k\}$ and $\bar{\ell}\text{-}\bar{\Sigma} = \{\bar{\ell}\text{-}\bar{\sigma}^1, \bar{\ell}\text{-}\bar{\sigma}^2, \ldots, \bar{\ell}\text{-}\bar{\sigma}^{\bar{k}}\}$ represent a CSPT (ℓ) and a CSPT ($\bar{\ell}$) respectively of A, where ℓ and $\bar{\ell}$ may or may not be equal. They are said to be mutually orthogonal, if for every pair $h, \bar{h}$, $h = 1, 2, \ldots, k$, $\bar{h} = 1, 2, \ldots, \bar{k}$, the ℓ-transversal $\ell\text{-}\sigma^h$ and the $\bar{\ell}$-transversal $\bar{\ell}\text{-}\bar{\sigma}^{\bar{h}}$ intersect in a constant number $\lambda(\ell, \bar{\ell})$ of positions.

In most of the constructions presented in the following sections, it has been found to be convenient to think of an ℓ-transversal, say, of A as the union of some k-transversals, say, of a few of the submatrices of A chosen suitably. So we introduce the following notation.

Let A be an $mk \times nk$ matrix partitioned into square matrices A_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, each of order k. For each pair i,j, $1 \leq i \leq m$, $1 \leq j \leq n$, let $\ell\text{-}\pi_{ij}$ be a t-transversal of A_{ij}, and let $t\text{-}\Pi$ denote an $m \times n$ matrix with (i,j)th element as the set $t\text{-}\pi_{ij}$. Also let $\ell\text{-}\sigma = \{(i_1, j_1), \ldots, (i_\ell, j_\ell)\}$ be an

ℓ-transversal of an $m \times n$ matrix. Since each position of the submatrix A_{ij} may be regarded as a position of A, the followi expression

$$\phi = \bigcup_{k=1}^{\ell} (t - \pi_{i_k, j_k}) \qquad \ldots(5.2.1)$$

represents a transversal of A. We call it a composite ℓt-transversal of A with components $\ell - \sigma$ and $t - \prod$ and denote it by

$$(\ell - \sigma) \cdot (t - \prod). \qquad \ldots(5.2.2)$$

Now that definitions and notation in the general set up have been presented, let us enlist below the simplified terminology and notation to be used in this chapter in special cases.

DFN 5.2.5. (a) An n-tranversal $n - \sigma$ of a square matrix $\bar{A}$ of order n will be called simply a transversal and will be denoted simply by σ. Similarly, a CSPT(n) of $\bar{A}$, viz., $n - \Sigma = \{\sigma^1, \ldots, \sigma^n\}$ (we note that the cardinality of a CSPT(n) of $\bar{A}$ is n) will be simply called a CSPT and will be denoted by only Σ. Again from DFN 5.2.4, it follows that two CSPT's $\Sigma = \{\sigma^1, \sigma^2, \ldots, \sigma^n\}$ and $\bar{\Sigma} = \{\bar{\sigma}^1, \ldots, \bar{\sigma}^n\}$ of $\bar{A}$ are orthogonal to each other when for each pair $h, \bar{h}$, $1 \leq h, \bar{h} \leq n$, the transversals σ^h and $\bar{\sigma}^h$ intersect at exactly one position.

(b) We observe that each permutation $\pi = (\pi(1), \ldots, \pi(n))$ of I(n) defines a transversal $\{(i, \pi(i)), i \in I(n)\}$ of $\bar{A}$ and conversely, each transversal of $\bar{A}$, say $\{(1, \pi(1)), \ldots, (n, \pi(n))\}$ defines a suitable permutation $\pi = (\pi(1), \ldots, \pi(n))$ of I(n).

Hence there is a 1-1 correspondence between the permutation of $I(n)$ and the transversals of a square matrix of order n and without any loss of generality we shall denote the transversal of $\bar{A}$ by the corresponding permutation π of $I(n)$.

(c) For a square matrix $\bar{A}$ of order nk, the composite transversal ψ with components σ and Π, where σ is a transversal of a square matrix of order n and Π represents the matrix $k-\Pi$ of (5.2.2), where each of the submatrices of $\bar{A}$, say $\bar{A}_{ij}$, $1 \leq i,j \leq n$, is of order k is given by

$$\psi = \sigma . \Pi = \bigcup_{j=1}^{n} \pi_{(j,\sigma(j))} \qquad \ldots(5.2.4)$$

(d) The set of elements of A positioned on a transversal of $\bar{A}$ is denoted by $\pi(\bar{A})$ ...(5.2.5)

We may note that the definition of a transversal given here is concerned only with the positions of a square or a rectangle and differs from the usual definition of a transversal used in connection with a latin square, where besides the positions, the symbols of the latin square occupying those positions are also important in defining the transversal. Moreover, the CSPT's of a square matrix of order n defined here have the following correspondence with the latin squares of order n as spelt out in L(i) through L(iii).

L(i) The set of positions occupied by each symbol in a latin square of order n is a transversal of a square matrix of order n

L(ii) The two transversals determined by the positions occupied by two distinct symbols of a latin square are parallel, so that each latin square of order n correspond to a CSPT of a square matrix of order n.

L(iii) Two mutually orthogonal latin squares of order n correspond to two mutually orthogonal CSPT's of a square matrix of order n, the correspondence being as described in L(i) and L(ii) above.

Let us recall the treatment arrangements in a balanced m-way setting $M(v;n;m;b_1,\dots,b_n)$ (refer DFN 2.2.24). The arrangements can be represented by an $(m+1)\times n$ array, the first m rows of which constitute an OAVS $(n, m, b_1,\dots,b_n, 2)$ and the $(m+1)$th row gives the treatments allotted to the experimental units identified by the levels of the n factors given by the first m rows of the array in order. The j-th block corresponding to the i-th factor is represented by the set of experimental units $S_{ij} = \{(x_1,\dots,x_m) : x_i = j\}$, $j = 1,2,\dots,b_i$, $i = 1,2,\dots,n$.

Let us consider the situations when there exist at least one pair $i \neq i'$, $i,i' = 1,2,\dots,n$, such that the index of the pair of rows (i,i') is $\lambda(i,i') = 1$. In such a situation, the allotment of treatments to the experimental units may be represented in a manner described below.

Without loss of generality, we can assume $i = 1$, and $i' = 2$. Then we consider a $b_1 \times b_2$ matrix D and identify the (i,j)th position of D with the experimental unit $(x_1,\ldots,x_n)$ where $x_1 = i$ and $x_2 = j$. Since $\lambda(1,2) = 1$ for the OAVS, there is exactly one such experimental unit and thus there is a 1 - 1 correspondence between the experimental units and the positions of the matrix D determined by the first two elements x_1 and x_2 in the experimental units.

The entry in a position of D will denote the treatment applied to the corresponding experimental unit. Moreover, by the construction of D, we have the following structural properties P(i) through P(iii) of D.

P(i). The set of experimental units $(x_1,\ldots,x_n)$ with $x_i = j$, for $i = 3,4,\ldots,n$ corresponds to a k_i - transversal of D, with $k_i = n/b_i$, $j = 1,\ldots,b_i$.

P(ii). The collection of b_i sets of experimental units $\{(x_1,\ldots,x_n) : x_i = j\}$, $j = 1,2,\ldots,b_i$ constitute a CSPT (k_i) of D, denoted by, say $S_i(k_i)$, $i = 3,4,\ldots,n$.

P(iii). For each pair $i \neq i'$, $i,i' = 3,\ldots,n$, $S_i(k_i)$ and $S_{i'}(k_{i'})$ as defined in P(ii) are mutually orthogonal.

In most of the constructions presented in this chapter, transversals of a square matrix of order s have been utilized, where s is a prime power. So, we shall now list a set of s-1 pairwise orthogonal CSPT's of a square matrix of order s.

Let $F = \{\alpha_0 = 0, \alpha_1,\ldots,\alpha_{s-1}\}$ be the elements of GF(s).

Let us index the rows, columns and transversals of any given CSPT of a square matrix of order s in accordance with the elements of F and list them in that specified order for all future references. Now we define the following. Henceforth s will stand for a prime or a prime power.

DFN 5.2.6. A CSPT of type t of a square matrix of order s (with the rows and columns indexed by F) is defined as

$$\Phi_t = \left\{ \phi_t^{\alpha_h} ,\ h = 0,1,\ldots,s-1 \right\}$$

where the transversal $\phi_t^{\alpha_h}$ is given by the set of s positions

$$\phi_t^{\alpha_h} = \left\{ (\alpha_p , \alpha_q) :\ \alpha_p + \alpha_q . \alpha_t = \alpha_h ,\ p,q = 0,1,\ldots,s-1 \right\},$$
$$t = 1,2,\ldots,s-1 .$$

5.3. Construction of Balanced Youden Hyperrectangle

In section 5.3.1, completely regular balanced n-way settings have been considered and in section 5.3.2, a series of BYHC'shas been constructed in a nonregular n-way setting with $3 \leq n \leq s+1$.

5.3.1. A Series of BYHR's in Completely Regular Multiway Settings

We may recall that a completely regular BYHR, if it exists in a balanced n-way setting $M(v\,;\,n\,;\,m\,;\,b_1,\ldots,b_n)$ is universally optimal, as shown in corollary 2.5.1 of Chapter 2.

Now a completely regular BYHR with parameters $(v, n, m, b_1, \ldots, b_m)$ may be represented by an OAVS$(n, m+1, b_1, \ldots, b_m, b_{m+1} = v, 2)$. Here we shall construct an OAVS$(s^3, s+2, s, \ldots, s, s^2, 2)$ where s is a prime power. This represents a BYHC with parameters $(v = s^2, n = s^3, m = s+1, b = s)$.

Let $F = \{\alpha_0 = 0, \alpha_1, \ldots, \alpha_{s-1} = -1\}$ be the elements of GF(s). Let F_2 denote the set of s^2 ordered pairs with elements from F. Let us now consider the s^2 $(s+1)$-tuples $(x_1, \ldots, x_{s+1})$ where for (x_1, x_2) we choose all the s^2 possible ordered pairs from F_2 and

$$x_{i+2} = x_1 + \alpha_i x_2, \quad i = 1, 2, \ldots, s-1 \qquad \ldots(5.3.1.1)$$

These $(s+1)$-tuples constitute the columns of an OA$(s^2, s+1, s, 2)$ of index 1, which we denote by $\bar{A}$ say.

Next we add one more row to $\bar{A}$ with elements of F_2 as symbols, in the following manner. We add the ordered pair $\binom{x_1}{x_2}$ to the column $(x_1, x_2, \ldots, x_{s+1})'$ of $\bar{A}$. We call this augmented array A.

Now for each $i = 1, 2, \ldots, s-1$, we consruct an array A_i from A_o in the following way. Let $(y^o_{1u}, y^o_{2u}, \ldots, y^o_{s+2,u})'$ denote the u-th column of A_o and $(y^i_{1u}, \ldots, y^i_{s+2,u})$ denote the u-th column of A_i, $i = 1, 2, \ldots, s-1$. Then, for each $i = 1, 2, \ldots, s-1$ and each $u = 1, 2, \ldots, s^2$,

$$y^i_{ju} = y^o_{ju}, \quad j = 1, 2, \ldots, s,$$

$$y^{-}_{s+1,u} = y^{o}_{s+1,u} + \alpha_i ,$$

$$y^{i}_{s+2,u} = y^{o}_{s+2,u} + \binom{\alpha_i}{\alpha_i}, \qquad \ldots(5.3.1.2)$$

Given an element α of F, let $\beta_{ij}(\alpha)$ denote the following set of elements of F_2.

$$\left\{ \beta_{ij}(\alpha) = y^{i}_{s+2,u} : u \text{ is such that } y^{i}_{ju} = \alpha \right\},$$

$$j = 1,2,\ldots,s+1,\ i = 0,1,\ldots,s-1 \qquad \ldots(5.3.1.3)$$

Then from the construction of A_i's the following lemma is immediate.

Lemma 5.3.1.1. For each element α of F and for each $i = 0,1,\ldots,s-1$, the sets $\beta_{ij}(\alpha)$, $j = 1,2,\ldots,s+1$ satisfy the following properties :

(i) $\beta_{i1}(\alpha) = \left\{ \binom{x_1}{x_2} : x_1 = \alpha + \alpha_i ,\ x_2 \in F \right\}$

(ii) $\beta_{i2}(\alpha) = \left\{ \binom{x_1}{x_2} : x_2 = \alpha + \alpha_i ,\ x_1 \in F \right\}$

(iii) $\beta_{ij}(\alpha) = \left\{ \binom{x_1}{x_2} : x_1 + \alpha_{j-2}x_2 = \alpha + \alpha_i(1 + \alpha_{j-2}),\ x_1, x_2 \in F \right\}$, $j = 3,4,\ldots,s$.

(iv) $\beta_{i,s+1}(\alpha) = \left\{ \binom{x_1}{x_2} : x_1 - x_2 = \alpha - \alpha_i ,\ x_1, x_2 \in F \right\}$.

Now let us consider the array $A = [A_0 \ldots A_{s-1}]$. The first $s+1$ rows of if represent an OAVS $(s^3, s+1, s, 2)$ as follows from the construction. So, lemma 5.3.1.1 implies the following theorem as an immediate consequence.

Theorem 5.3.1.2. The array $A = [A_0 A_1 \ldots A_{s-1}]$ represents an OAVS$(s^3, s+2, s, \ldots, s, s^2, 2)$ and hence a completely regular BYHC $(s^2, s^3, s+1, s^{s+1})$.

Many other constructions for orthogonal arrays with variable number of symbols are available in the literature. There are other simple methods of constructing the OAVS of theorem 5.3.1 and more factors with s levels can also be accommodated to the design. But the method of construction expounded above yields a particular structure of the design, which is exploited in constructing two series of PYD's in section 5.4. For a general information on the OAVS's of strength 2, one is referred to Rao (1973), Adhikari and Das (1980) and Mukhopadhyay (1980a, 1980b).

5.3.2. Construction of BYHC in Nonregular Balanced m-way Setting

In this section we construct a BYHC $(s^2, (s^2+s)^2, m, (s^2+s)^m)$, where s is a prime power and $3 \leq m \leq t$, $t = \min(m'+2, s+1)$, m' being the number of mutually orthogonal latin squares (MOLS) of order $s+1$. The OAVS representing the balanced m-way setting has the same number of symbols in each of the rows and it is of index 1. So, we use a $b \times b$ matrix D to represent the design constructed, in which blocking with respect to different factors and their structures are indicated in P(i) through P(iii) in section 5.2.

As mentioned earlier, this construction essentially consists in modifying the treatment arrangement of a GYD of Ruiz and Seiden (1974) in such a way that it can represent the multi-way design D. Now in view of the structural properties P(i), P(ii) and P(iii) of D, we observe that this is possible if the modification keeps the structures of the rows and columns of the GYD unaltered, but makes it possible to choose $m-2$ mutually orthogonal CSPT's of D satisfying some desirable property.

Before going to the details of construction, let us develop a few general results regarding the properties of ℓ-transversals, which provide the basic principles underlying the constructions discussed in this section as well as in the following sections.

<u>Lemma 5.2.1</u>. Let A be an $mk \times nk$ matrix partitioned into square submatrices A_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ each of order k. Let $\ell-\sigma$ and $\ell-\sigma'$ be two ℓ-transversals of an $m \times n$ matrix. Let $t-\Pi$ and $t-\Pi'$ be two $m \times n$ matrices with their (i,j)th elements as the set represented by the t-transversal $t-\pi_{ij}$ and $t-\pi'_{ij}$ respectively $1 \leq i \leq m$, $1 \leq j \leq n$. Then we have the following result.

(i) If either $\ell-\sigma$ and $\ell-\sigma'$ are parallel or else $t-\pi_{ij}$ and $t-\pi'_{ij}$ are parallel for each pair i,j, $1 \leq i \leq m$, $1 \leq j \leq n$, then the ℓt-transversals

$\phi = (\lambda - \sigma).(t - \Pi)$ and

$\phi' = (\lambda - \sigma').(t - \Pi')$

are parallel, where the meaning of the notation is as introduced in (5.2.1) and (5.2.2).

(ii) Let $\Sigma = \{\lambda - \sigma', \ldots, \lambda - \sigma^p\}$ be a CSPT(λ) of an $m \times n$ matrix and for each pair i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, let $\{t - \pi^h_{ij}, h = 1, 2, \ldots, q\}$ be a CSPT (t) of A_{ij}. Let $t - \Pi^h$ denote the $m \times n$ matrix with the (i,j)th element as the set represented by $t - \pi^h_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Then the set $\{\psi_{u,h}, 1 \leq u \leq p, 1 \leq h \leq q\}$, with $\psi_{u,h} = (\lambda - \sigma^u).(t - \Pi^h)$ is a CSPT ($t\lambda$) of A.

Proof of this lemma is easy and hence omitted.

We now write down another lemma without proof.

Lemma 5.3.2.2. Let $\lambda - \Sigma = \{\lambda - \sigma^1, \ldots, \lambda - \sigma^p\}$ be a CSPT(λ) and $\bar{\lambda} - \bar{\Sigma} = \{\bar{\lambda} - \bar{\sigma}^1, \ldots, \bar{\lambda} - \bar{\sigma}^{\bar{p}}\}$ be a CSPT($\bar{\lambda}$) of an $m \times n$ matrix and they are orthogonal. For each pair i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, let $\{t - \pi^h_{ij}, h = 1, 2, \ldots, q\}$ be a CSPT(t) and $\{\bar{t} - \pi^{\bar{h}}_{ij}, \bar{h} = 1, 2, \ldots, \bar{q}\}$ be a CSPT($\bar{t}$) of A_{ij}, orthogonal to each other. Then the CSPT ($t\lambda$) $= \{\psi_{u,h}, 1 \leq u \leq p, 1 \leq h \leq q\}$ with $\psi_{u,h} = (\lambda - \sigma^u).(t - \Pi^h)$ and the CSPT($\bar{t}\,\bar{\lambda}$) $= \{\bar{\psi}_{\bar{u},\bar{h}}, 1 \leq \bar{u} \leq \bar{p}, 1 \leq \bar{h} \leq \bar{q}\}$

with $\bar{\psi}_{\bar{u},\bar{h}} = (\bar{\lambda} - \bar{\sigma}^{\bar{u}}).(\bar{t} - \bar{\Pi}^{\bar{h}})$ are mutually orthogonal.

Let us now take up the construction of a series of BYHC's with parameters (s^2, n, n, b^n), where $b = s^2 + s$, $n = (s^2 + s)^2$, $n \geq 3$ and the maximum value that n can take will depend on the number of mutually orthogonal CSPT's of a square matrix of order $s + 1$. More specifically in the situations when a set of n' mutually orthogonal CSPT's of order $s + 1$ are known, n can take any value $\leq \min(n' + 2, s + 1)$ ...(5.3.2.1)

We now illustrate the construction with $n = 3$, using any given CSPT of a square matrix of order $s + 1$, say

$$\Sigma = \left\{\sigma^0, \sigma^1, \ldots, \sigma^s\right\} \qquad \ldots(5.3.2.2)$$

where the positions $(0,1)$ and $(1,0)$ of the square matrix are **not** contained in the same σ^i, $0 \leq i \leq s$. [The rows and columns of a square matrix of order $s + 1$ are numbered $0, 1, \ldots, s$].

Ruiz and Seiden (1974) constructed a GYD represented by a square matrix G of order $s^2 + s$ where s is a prime power, the rows of which constitute the blocks of a BBD with parameters $b = k = s^2 + s$, $v = s^2$, $r = (s + 1)^2$, $\lambda = s^2 + 3s + 3$ and columns also constitute the blocks of a BBD with the same parameters. This BBD will be referred to as d_0 in the present section. The s^2 treatments are identified with ordered pairs of elements of $GF(s)$, i.e. the points of $EG(2,s)$. In each block, (row or column of G), $s^2 - s$ treatments occur once each and each of the remaining s treatments occurs twice. The latter treatments occuring twice as points of $EG(2,s)$ constitute a line of $EG(2,s)$

and in this sense each row (also each column) of G corresponds to a distinct line of $EG(2,s)$.

Our object is the following. Let $I(s^2+s) = \{1,2,\ldots,s^2+s\}$ and $D = (d_{ij})_{i,j \in I(s^2+s)}$. Our D should satisfy all the properties of G of Ruiz and Seiden (1974). In addition, we want to construct a CSPT of D, say represented by the set of s^2+s transversals $\Gamma = \{\gamma^1,\ldots,\gamma^{s^2+s}\}$ with the help of the given CSPT Σ of a square matrix of order $s+1$ in such a manner that the properties $M(i)$ and $M(ii)$ given below hold.

$M(i)$. For each h, $1 \leq h \leq s^2+s$, $L_h = \gamma^h(D)$ (the meaning of this notation is as stated in (5.2.5)) contains all s^2 treatments among which each of the s treatments constituting a line of $EG(2,s)$ occurs twice and each of the other treatments occurs once; and moreover $M(ii)$: For each distinct h, L_h corresponds to a distinct line of $EG(2,s)$.

Our construction will be completed in two steps.

Step 1. Partition D into $(s+1)^2$ square submatrices $(D_{ij})_{0 \leq i,j \leq s}$, each of order s such that the s^2 treatments, i.e. the s^2 points of $EG(2,s)$ occurs exactly once in D_{ij}, for each pair i,j, $0 \leq i,j \leq s$. (For any square matrix of order s, the rows as well as columns are indexed by the elements of $F = GF(s)$ and for the representation of its transversals, we use the notation introduced in DFN 5.2.6).

Step 2. For each pair i,j, $0 \leq i,j \leq s$, construct a CSPT $\prod_{ij} = \pi^{\alpha}_{ij}, \alpha \varepsilon F$ of D_{ij} in such a manner that the following condition C holds.

Condition C : For each pair α, β, $\alpha \varepsilon F$, $0 \leq \beta \leq s$, we obtain the composite transversals

$$\psi_{\beta ; \alpha} = \sigma^{\beta} \cdot \prod^{\alpha} \qquad \ldots(5.3.2.3)$$

where σ^{β} is the β-th transversal of the given CSPT, Σ as stated in (5.3.2.2) and $\prod^{\alpha}$ is a matrix with (i,j)th element as the set of positions represented by the transversal π^{α}_{ij} of D_{ij} [the notation used in (5.3.2.3) is as explained in (5.2.4)]. Then the set $\left\{\psi_{\beta ; \alpha}, \alpha \varepsilon F, 0 \leq \beta \leq s\right\}$ which may be regarded as a CSPT Γ of D should satisfy the properties M(i) and M(ii)

In order to carry out the steps, we now look at the details of the structure of G, constructed by Ruiz and Seiden (1974).

$$G = \begin{bmatrix} X & \sigma Y' \\ Y & L \end{bmatrix}, \qquad \ldots(5.3.2.4)$$

where σ is a permutation of points of EG(2,s), s.t. $\sigma(x,y) = (y,x)$, L is a latin square of order s^2

$$\text{and} \qquad Y = \begin{bmatrix} Y_0 \xi_0 \\ Y_1 \xi_1 \\ \vdots \\ Y_{s-1} \xi_{s-1} \end{bmatrix}$$

where the components are as described below.

$\xi_o = I_s, \xi_1, \ldots, \xi_{s-1}$ are permutation matrices such that if ξ_{iu} represents the α_u-th column of ξ_i, then $\xi_{ou}, \xi_{1u}, \ldots, \xi_{s-1,u}$ consists of all distinct columns of I_s for each $u = 0, 1, \ldots, s-1$; $X, Y_o, \ldots, Y_{s-1}$ are square matrices of order s with the following structure (We recall that the rows and columns of each submatrix of order s are indexed by F). The (α_p, α_q)th element of X is the point $(x = \alpha_p,\ y = \alpha_q)$ of $EG(2,s)$, $p,q = 0, 1, \ldots, s-1$. The (α_p, α_q)th element of Y_i is the point $(x = \alpha_q, y = \alpha_i x + \alpha_p)$ of $EG(2,s)$, $p,q = 0,1,\ldots,s-1$, $i = 0,1,\ldots,s-1$.

To begin with, our design matrix D will be of the same form as G, with the components suitably modified.

Step 1. We shall consider permutation matrices ξ_i, $i = 0,1,\ldots, s-1$ of a particular structure as described below.

We index the columns of I_s and also ξ_i's as already stated, by the elements of F. Then the α_u-th column of ξ_i is the $(\alpha_u + \alpha_i)$th column of I_s, $u = 0,1, \ldots, s-1$, $i = 0,1,\ldots,s-1$.

Now we choose D to be of the form :

$$D = \begin{bmatrix} D_{oo} & E_{o1} \\ E_{10} & E_{11} \end{bmatrix}, \qquad \ldots(5.3.2.5)$$

where $D_{oo} = X$, $E_{o1} = [D_{o1}, \ldots, D_{os}]$,

$$E_{10} = \begin{bmatrix} D_{10} \\ \vdots \\ D_{so} \end{bmatrix}, \quad \text{with } D_{oi} = \sigma(Y_i\ \xi_i),$$

$D_{io} = Y_i \xi_i$, $i = 1,2,\ldots,s$ and $E_{11} = ((D_{ij}))_{1 \leq i,j \leq s}$ is an $s^2 \times s^2$ latin square chosen suitably.

So it is clear that the submatrices D_{oo}, $D_{o1}, \ldots, D_{os}$, $D_{10}, \ldots, D_{so}$ satisfy the requirements of step 1.

Again it is possible to construct a latin square of order s^2 with elements as points of $EG(2,s)$ which can be partitioned into s^2 square submatrices satisfying the requirements of Step 1. One way of doing this would be to choose the initial submatrix D_{11} identical with D_{o1} and obtaining the other submatrices by permuting the rows and columns of D_{11} suitably, e.g., choosing

$$D_{ij} = \xi'_{j-1} D_{11} \xi_{i-1}, \ 1 \leq i,j \leq s. \qquad \ldots(5.3.2.6)$$

So we assume E_{11} to be of the above structure and thus complete step 1.

Before we go to the next step, let us notice the following facts.

Without any loss of generality, we can assume

$$\sigma^h(0) = h, \ 0 \leq h \leq s, \qquad \ldots(5.3.2.7)$$

Also, from the assumption already made about Σ, we have in view of (5.3.2.7) that $\sigma^1(1) \neq 0$. ...(5.3.2.8)

<u>Step 2</u>. Our next step is to find suitable transversals of the submatrices. For each of the submatrices D_{ij}, $0 \leq i,j \leq s$, we shall choose one of the CSPT's Φ_t, $t = 1,2,\ldots,s-1$, defined in DFN 5.2.6. In order to determine the suitable value of t for

a given submatrix D_{ij}, we need to note the characterisation of the set of elements positioned on a transversal of the matrix D_{ij}. This will follow from a general result written below in the form of a lemma.

<u>Lemma 5.3.2.3</u>. Let M be a square matrix of order s with the points of $EG(2,s)$ as elements and let $\bar{M} = \xi_j' M \xi_i$, where ξ_i's are permutation matrices as defined in step 1. Let Φ_t be the CSPT of type t as defined in DFN 5.2.6. Then the elements of M positioned on the α_h-th transversal of this CSPT, denoted by $\phi_t^{\alpha_h}(\bar{M})$ satisfy the following algebraic relations for each h, $0 \leq h \leq s-1$, for the following given structures of M.

(i) (α_p, α_q)th element of M is given by the point (x,y) of $EG(2,s)$, where for given m and n, $\alpha_m x + y = \alpha_p$, $x + \alpha_n y = \alpha_q$, $p,q = 0,1,\ldots,s-1$, $m,n = 0,1,\ldots,s-1$.

Then $$\phi_t^{\alpha_h}(M) = \Big\{(x,y) : (\alpha_m + \alpha_t)x + y(1 + \alpha_t . \alpha_n) = \alpha_h - \alpha_t . \alpha_i - \alpha_j\Big\}$$

(ii) (α_p, α_q)th element of M is given by the point (x,y) of $EG(2,s)$, where for given m and n

$$x + \alpha_n y = \alpha_p,\ y + \alpha_m x = \alpha_q,\ p,q = 0,1,\ldots,s-1,$$

$m,n = 0,1,\ldots,s-1$.

Then $$\phi_t^{\alpha_h}(\bar{M}) = \Big\{(x,y) : (1 + \alpha_t\alpha_m)x + (\alpha_n + \alpha_t)y = \alpha_h - \alpha_t . \alpha_i - \alpha_j\Big\}.$$

<u>Proof</u>. We observe that from the definition of $\bar{M}$ and ξ_i's, the

(α_p, α_q)th element of $\overline{M}$ is the same as the $(\alpha_p - \alpha_j, \alpha_q - \alpha_i)$ element of M.

So, we have in case (i)

$$\phi_t^{\alpha_h}(\overline{M}) = \Big\{(x,y) : \alpha_m x + y = \alpha_p - \alpha_j,\ x + \alpha_n y = \alpha_q - \alpha_i, \text{ with } \alpha_p + \alpha_q \cdot \alpha_t = \alpha_h,\ p,q = 0,1,\dots,s-1\Big\}$$

$$= \Big\{(x,y) : \alpha_m x + y = \alpha_h - \alpha_t \cdot \alpha_q - \alpha_j,\ x + \alpha_n y = \alpha_q - \alpha_i,\ q = 0,1,\dots,s-1\Big\}$$

$$= \Big\{(x,y) : (\alpha_m + \alpha_t)x + (1 + \alpha_t \cdot \alpha_n)y = \alpha_h - \alpha_t \cdot \alpha_i - \alpha_j .$$

Similarly, in case (ii)

$$\phi_t^{\alpha_h}(\overline{M}) = \Big\{(x,y) : x + \alpha_n y = \alpha_p - \alpha_j,\ \alpha_m x + y = \alpha_q - \alpha_i, \text{ where } \alpha_p + \alpha_t \cdot \alpha_q = \alpha_h,\ p,q = 0,1,\dots,s-1\Big\}$$

$$= \Big\{(x,y) : (1 + \alpha_t \cdot \alpha_m)x + (\alpha_n + \alpha_t)y = \alpha_h - \alpha_t \cdot \alpha_i - \alpha_j\Big\} .$$

Corollary 5.3.2.3.1. From the structures of the submatrices D_{ij}, $0 \le i,j \le s$, as given in (5.3.2.5) and (5.3.2.6) we obtain the following characterizations of their transversals.

(i) $\phi_t^{\alpha_h}(D_{oo}) = \Big\{(x,y) : x + \alpha_t y = \alpha_h\Big\}$, $0 \le h \le s-1$

(ii) $\phi_t^{\alpha_h}(D_{io}) = \Big\{(x,y) : y + (\alpha_t - \alpha_{i-1})x = \alpha_h - \alpha_t \cdot \alpha_{i-1}\Big\}$

$0 \le h \le s-1,\ 1 \le i \le s$

(iii) $\phi_t^{\alpha_h}(D_{oj}) = \left\{(x,y) : \alpha_t x + (1-\alpha_t \cdot \alpha_{j-1})y = \alpha_h - \alpha_{j-1}\right\}$

$0 \leq h \leq s-1, \ 1 \leq j \leq s$

(iv) $\phi_t^{\alpha_h}(D_{ij}) = \left\{(x,y) : y + \alpha_t x = \alpha_h - \alpha_t \cdot \alpha_{i-1} - \alpha_{j-1}\right\}$

$0 \leq h \leq s-1, \ 1 \leq i,j \leq s.$

Now for each pair $i,j, 1 \leq i,j \leq s$, we define a function $t(i,j)$ taking integral values from 1 to s-1 and a function $g_{ij}(\alpha_h)$ from F onto itself, i.e. $g_{ij}(\alpha_h), h = 0,1,\ldots,s-1 = F.$ Our task is to find the above functions in such a way that by taking

$$\pi_{ij}^{\alpha_h} = \phi_{t(i,j)}^{g_{ij}(\alpha_h)}, \quad \begin{array}{l} 0 \leq h \leq s-1 \\ 1 \leq i,j \leq s \end{array} \qquad \ldots(5.3.2.9)$$

the condition C can be satisfied.

The following theorem gives a set of suitable values for $t(i,j)$'s and g_{ij}'s based on the given CSPT, Σ of a square matrix of order $s+1$.

<u>Theorem 5.3.2.4</u>. Let T be the set of all treatments, i.e. the set of all points of EG(2,s). Writing $k = (\sigma^1)^{-1}(0)$ ($k \neq 1$ by virtue of (5,3.2.7) and (5.3.2.8)), let us choose $t(i,j)$'s and g_{ij}'s as follows :

T(i) : $t(0,j) = t_o, 2 \leq j \leq s$, where t_o can take any integral value from 1 through s-1

T(ii) : $t(0,0)$ is given by $\alpha_{t(0,0)} = \alpha_{t_o}^{-1}$

T(iii) : $t(0,1) = t_1$, where $\alpha_{t_1} = -\alpha_{k-1}$,

T(iv) : $t(k,0) = k-1$,

T(v) : $t(i,0) = t_i$, $1 \leq i \leq s$, $i \neq k$

where $\alpha_{t_i} = \alpha_{i-1} - \alpha_{k-1}$

T(vi) : $t(i,j) = t_1$, where α_{t_1} is as given in T(iii) ;

and,

G(i) : $g_{oo}(\alpha_h) = \alpha_h \cdot \alpha_{t_o}^{-1}$

G(ii) : $g_{o1}(\alpha_h) = \alpha_h + \alpha_{k-1}^2$

G(iii) : $g_{oj}(\alpha_h) = \alpha_h + \alpha_{j-1}$, $2 \leq j \leq s$

G(iv) : $g_{io}(\alpha_h) = \alpha_h + \alpha_{i-1}^2$, $1 \leq i \leq s$

G(v) : $g_{ij}(\alpha_h) = \alpha_h + \alpha_{j-1}$, $1 \leq i,j \leq s$.

Then, for each $h = 0,1,\ldots,s-1$,

$$\psi_{p,\alpha_h}(D) = T \; \underline{\textcircled{U}} \; L(\alpha_{h,p}), \quad p = 0,1,\ldots,s ,$$

where ψ_{p,α_h} is as described in (5.3.2.3), $L(\alpha_{h,p})$ is a set of s points lying in a line of EG(2,s) given by

$$L(\alpha_{h,p}) = \begin{cases} \{(x,y) : \alpha_{t_o} x + y = \alpha_h\}, & \text{when } p = 0, \\ \{(x,y) : y = \alpha_h\}, & \text{when } p = 1, \\ \{(x,y) : \alpha_{t_o} x + (1 - \alpha_{t_o} \cdot \alpha_{p-1}) y = \alpha_h\}, & \text{when } p \geq 2 \end{cases}$$

[The symbol $(\underline{U})$ in theorem 5.3.2.4 is defined in the following manner :

For two collections of elements P_1 and P_2, $P = P_1 (\underline{U}) P_2$ indicates a collection of elements, where x, and element appears in P, $p = p_1 + p_2$ times when x appears in P_1, p_1 times and in P_2, p_2 times].

Proof. From corollary 5.3.2.3.1, using relations T(i) through T(vi) we obtain the following.

For any fixed integer h, $0 \leq h \leq s-1$, writing $\bar{h}$ for α_h, we have

(a) $\phi_t^{\bar{h}} (D_{oo}) = \left\{ (x,y) : \alpha_{t_o} x + y = \bar{h} \cdot \alpha_{t_o} \right\}$

(b) $\phi_t^{\bar{h}} (D_{o1}) = \left\{ (x,y) : y = \alpha_{k-1} x + \bar{h} \right\}$

(c) $\phi_t^{\bar{h}} (D_{oj}) = \left\{ (x,y) : \alpha_{t_o} x + (1 - \alpha_{t_o} \cdot \alpha_{j-1}) y = \bar{h} - \alpha_{j-1} \right\}$, $2 \leq j \leq s$

(d) $\phi_t^{\bar{h}} (D_{ko}) = \left\{ (x,y) : y = \bar{h} - \alpha_{k-1}^2 \right\}$

(e) $\phi_t^{\bar{h}} (D_{io}) = \left\{ (x,y) : y = \alpha_{k-1} x + \bar{h} - \alpha_{i-1}^2 + \alpha_{i-1} \cdot \alpha_{k-1} \right\}$, $1 \leq i \leq s,\ i \neq k$

(f) $\phi_t^{\bar{h}} (D_{ij}) = \left\{ (x,y) : y = \alpha_{k-1} x + \bar{h} + \alpha_{i-1} \cdot \alpha_{k-1} - \alpha_{j-1} \right\}$, $1 \leq i,j \leq s$.

Now using relations G(i) through G(v'), we obtain in view of (5.3.2.9) that (a') $\pi_{oo}^{\bar{h}}\ (D_{oo}) = \{(x,y) : y + \alpha_{t_o} x = \bar{h}\}$

(b') $\pi_{o1}^{\bar{h}}\ (D_{o1}) = \{(x,y) : y = \alpha_{k-1} x + \bar{h} + \alpha_{k-1}^2\}$

(c') $\pi_{oj}^{\bar{h}}\ (D_{oj}) = \{(x,y) : (1 - \alpha_{t_o} \cdot\cdot \alpha_{j-1})y + \alpha_{t_o} x = \bar{h}\}$,

$j = 2, \ldots, s$

(d') $\pi_{ko}^{\bar{h}}\ (D_{ko}) = \{(x,y) : y = \bar{h}\}$

(e') $\pi_{io}^{\bar{h}}\ (D_{io}) = \{(x,y) : y = \alpha_{k-1} x + \bar{h} + \alpha_{i-1} \cdot \alpha_{k-1}\}$

$1 \leq i \leq s,\ i \neq k$

(f') $\pi_{ij}^{\bar{h}}\ (D_{ij}) = \{(x,y) : y = \alpha_{k-1} x + \bar{h} + \alpha_{i-1} \cdot \alpha_{k-1}\}$

$1 \leq i,j \leq s$.

For a fixed h, $0 \leq h \leq s-1$, writing $\bar{h}$ for α_h,

$$Q_p(\bar{h}) = \underset{i=o}{(\underline{U})}\ Q_{ip}(\bar{h}),$$

where $Q_{ip}(\bar{h}) = \pi_{i\sigma^{p}(i)}^{\bar{h}}(D_{i,\sigma^{p}(i)})$.

So, we are to show that

$$Q_p(\bar{h}) = L(\bar{h}, p)\ (\underline{U})\ T \quad \text{for each} \quad p = 0, 1, \ldots, s.$$

We shall actually show that

$$\left.\begin{array}{ll} \text{(i)} & Q_{op}(\bar{h}) = L(\bar{h}, p) \\ \text{(ii)} & \underset{i=1}{\overset{s}{(\underline{U})}}\ Q_{ip}(\bar{h}) = T \end{array}\right\} \quad \text{for} \quad p \neq 1 \qquad \ldots(5.3.2.10)$$

and (i) $Q_{kp}(\bar{h}) = L(\bar{h},p)$ } for $p = 1$ ...(5.3.2.11)

(ii) $\overset{s}{\underset{\substack{i=0\\ i\neq k}}{(\bar{U})}} \; Q_{ip}(\bar{h}) = T$

Since, $Q_{op}(\bar{h}) = \pi^{\bar{h}}_{op}(D_{op})$ by (5.3.2.7), (a') and (c') imply (i) of (5.3.2.10).

Again, $Q_{k1}(\bar{h}) = \pi^{\bar{h}}_{ko}(D_{ko})$, so that (5.3.2.11) follows from (d').

Now, when $p = 1$,

$$\overset{s}{\underset{\substack{i=0\\ i\neq k}}{(\bar{U})}} \; Q_{ip}(\bar{h}) = \overset{s}{\underset{\substack{i=0\\ i\neq k}}{(\bar{U})}} \; \pi^{\bar{h}}_{i,\sigma^p(i)} \, (D_{i,\sigma^p(i)})$$

$$= \overset{s}{\underset{\substack{i=1\\ i\neq k}}{(\bar{U})}} \left\{(x,y) : y = \alpha_{k-1} x + \bar{h} + \alpha_{i-1} \, \alpha_{k-1}\right\}$$

$$(\bar{U}) \left\{(x,y) : y = \alpha_{k-1} + \bar{h} + \alpha^2_{k-1}\right\} \text{ using (b') and (f')}$$

$= T$, proving (ii) of (5.3.2.11).

Similarly, when $p = 0$,

$$\overset{s}{\underset{i=1}{(\bar{U})}} \; Q_{ip}(\bar{h}) = \overset{s}{\underset{i=1}{(\bar{U})}} \; \pi^{\bar{h}}_{i,\sigma^o(i)} \, (D_{i,\sigma^o(i)})$$

$$= \overset{s}{\underset{i=1}{(\bar{U})}} \left\{(x,y) : y = \alpha_{k-1} x + \bar{h} + \alpha_{i-1} \cdot \alpha_{k-1}\right\}$$

$= T$, proving (ii) of (5.3.2.10) for $p = 0$.

Also, when $p \geq 2$,

$$\underset{i=1}{\overset{s}{(\underline{U})}} Q_{ip}(\bar{h}) = \underset{\substack{i=1 \\ i \neq m_p}}{\overset{s}{(\underline{U})}} \pi^{\bar{h}}_{i,\sigma^p(i)} (D_{i,\sigma^p(i)}) (\underline{U}) \pi^{\bar{h}}_{m_p,o} (D_{m_p,o}),$$

where $m_p = (\sigma^p)^{-1}(0)$

$$= \underset{i=1}{\overset{s}{(\underline{U})}} \left\{ (x,y) : y = \alpha_{k-1} x + \bar{h} + \alpha_{i-1} \cdot \alpha_{k-1} \right\}$$

using (e') and (f')

$= T$, and so (5.3.2.13) is proved for $p \geq 2$.

Hence the theorem.

We now illustrate the theorem with an example, where the given CSPT Σ is given by $\Sigma = \left\{ \sigma^p, p = 0,1,\ldots,s \right\}$, where $\sigma^p(i) = (p+i) \bmod (s+1)$, so that $k = s$.

<u>Example</u>. $s = 3$, GF(s) = class of residues mod 3.

The treatments are identified with the points EG(2,3) viz., (0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2).

In the method of construction described, the matrix D in the partitioned form will be as follows.

$$D = \begin{bmatrix}
0 & 01 & 02 & 00 & 10 & 20 & 22 & 02 & 12 & 21 & 01 & 11 \\
10 & 11 & 12 & 01 & 11 & 21 & 00 & 10 & 20 & 12 & 22 & 02 \\
20 & 21 & 22 & 02 & 12 & 22 & 11 & 21 & 01 & 00 & 10 & 20 \\
00 & 10 & 20 & 00 & 10 & 20 & 02 & 12 & 22 & 01 & 11 & 21 \\
01 & 11 & 21 & 01 & 11 & 21 & 00 & 10 & 20 & 02 & 12 & 22 \\
02 & 12 & 22 & 02 & 12 & 22 & 01 & 11 & 21 & 00 & 10 & 20 \\
22 & 00 & 11 & 20 & 00 & 10 & 22 & 02 & 12 & 21 & 01 & 11 \\
20 & 01 & 12 & 21 & 01 & 11 & 20 & 00 & 10 & 22 & 02 & 12 \\
21 & 02 & 10 & 22 & 02 & 02 & 21 & 01 & 11 & 20 & 00 & 10 \\
12 & 21 & 00 & 10 & 20 & 00 & 12 & 22 & 02 & 11 & 21 & 01 \\
10 & 22 & 01 & 11 & 21 & 01 & 10 & 20 & 00 & 12 & 22 & 02 \\
11 & 20 & 02 & 12 & 22 & 02 & 11 & 21 & 01 & 10 & 20 & 00
\end{bmatrix}$$

The rows (also the columns) of D give a BBD with parameters $v = 0$, $b = k = 12$, $r = 16$, $\lambda = 21$. The values of $t(i,j)$'s and $g_{ij}(h)$'s needed in the construction of transversals of the partitions are obtained from theorem 5.3.2.4 as follows.

(i) $t(0,j) = 1, \quad j = 0,2,3$

(ii) $t(0,1) = 1$

(iii) $t(i,0) = 1, \quad 1 \leq i \leq 2$

(iv) $t(3,0) = 2$

(v) $t(i,j) = 1, \quad 1 \leq i,j \leq 2.$

(i) $g_{00}(h) = h$

(ii) $g_{01}(h) = h + 1$

(iii) $g_{oj}(h) = (h+j-1) \pmod 3, \quad 2 \leq j \leq 3$

(iv) $g_{io}(h) = (h+(i-1)^2 \pmod 3, \quad 1 \leq i \leq 3$

(v) $g_{ij}(h) = (h+j-1) \pmod 3, \ 1 \leq i,j \leq 3, \quad 0 \leq h \leq 2.$

The positions specified by the transversals belonging to the sets Φ_1 and Φ_2 are shown in the following figures, the positions specified by ϕ_t^h being marked by h.

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Φ_1

	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Φ_2

Thus a complete set of parallel transversals of D which will identify the third required dimension is specified. Let us now write down the block compositions $\psi_{p,h}$, $h = 0,1,2$, $p = 0,1,2,3$.

$\psi_{p,h}(D)$	Blocks				Corresponding $L_{h,p}$
$\psi_{0,0}(D)$	00,21,12 ;	00,12,21 ;	20,11,02 ;	10,22,01	y = 2x
$\psi_{0,1}(D)$	10,01,22 ;	01,10,22 ;	21,00,12 ;	11,02,20	y = 2x+1
$\psi_{0,2}(D)$	20,11,02 ;	02,11,20 ;	22,01,10 ;	12,21,00	y = 2x+2
$\psi_{1,0}(D)$	01,10,22 ;	00,12,21 ;	20,02,11 ;	10,20,00	y = 0
$\psi_{1,1}(D)$	02,11,20 ;	01,10,22 ;	21,00,12 ;	11,21,01	y = 1
$\psi_{1,2}(D)$	00,12,21 ;	02,11,20 ;	22,01,10 ;	12,22,02	y = 2
$\psi_{2,0}(D)$	00,02,01 ;	00,12,21 ;	20,02,11 ;	10,22,01	x = 0
$\psi_{2,1}(D)$	11,10,12 ;	01,10,22 ;	21,00,12 ;	11,20,02	x = 1
$\psi_{2,2}(D)$	22,21,20,	02,11,20 ;	22,01,10 ;	12,21,00	x = 2
$\psi_{3,0}(D)$	00,22,11 ;	00,12,21 ;	20,02,11 ;	10,22,01	y = x
$\psi_{3,1}(D)$	12,01,20 ;	01,10,22 ;	21,00,12 ;	11,20,02	y = x+1
$\psi_{3,2}(D)$	21,10,02 ;	02,11,20 ;	22,01,10 ;	12,21,00	y = x+2

So the transversals of D, constituting the third orthogonal dimension again give a BBD with same parameters as the set of rows and the set of columns of D.

With the preceding theorem, the construction of a BYHC$(s^2, (s^2+s)^2, m, (s^2+s)^m)$ with $m = 3$ is complete. We shall now extend this method to higher value of m, without altering the other parameters.

The structure of the CSPT Φ_t defined in DFN 5.2.6 of a square matrix of order s shows that two CSPT's Φ_t and $\Phi_{t'}$ are mutually orthogonal if and only if $t \neq t'$. Making use of this fact and lemma 5.3.2.2, we can easily prove the following lemma.

<u>Lemma 5.3.2.5</u>. Let $\Sigma = \{\sigma^0, \ldots, \sigma^s\}$ and $\bar{\Sigma} = \{\sigma^0, \ldots, \bar{\sigma}^s\}$ *represent two mutually orthogonal* CSPT's of a square matrix of order $s+1$. For each pair i, j, $0 \leq i, j \leq s$,

Let $\Pi_{ij} = \Phi_{t(i,j)}$ and $\bar{\Pi}_{ij} = \Phi_{\bar{t}(i,j)}$

be two CSPT's of the square submatrix D_{ij} of order s, where $t(i,j)$ and $\bar{t}(i,j)$ are integers such that $1 \leq t(i,j), \bar{t}(i,j) \leq s-1$. Then the composite CSPT's of $D = (D_{ij})$ defined by

$$\psi = \{\psi_{\beta,\alpha},\ \alpha \in F', 0 \leq \beta \leq s\} \text{ with } \psi_{\beta,\alpha} = \sigma^\beta . \Pi^\alpha$$

and $$\bar{\psi} = \{\bar{\psi}_{\beta,\alpha},\ \alpha \in F, 0 \leq \beta \leq s\} \text{ with } \bar{\psi}_{\beta,\alpha} = \bar{\sigma}^\beta . \Pi^\alpha$$

vide (5.2.4), are orthogonal if $t(i,j) \neq \bar{t}(i,j)$ for each pair i, j, $0 \leq i, j \leq s$.

Let us now assume that we are given m' mutually orthogonal CSPT's $\Sigma_j = \{\sigma_j^0, \sigma_j^1, \ldots, \sigma_j^s\}$, $j = 1, 2, \ldots, m'$ of a square matrix of order $s+1$, each satisfying the condition (5.3.2.8), where m' is assumed to be $\leq s-1$.

Now, let $k_j = (\sigma_j^1)^{-1}(0)$, $j = 1,2,\ldots,m'$,

so that $k_{j'} \neq k_j$ for $j' \neq j$, $j,j' = 1,2,\ldots,m'$, since the CSPT's Σ_j's are mutually orthogonal and also $k_j \neq 1$, $j = 1,2,\ldots,m'$ ($\because$ each Σ_j satisfies condition (5.3.2.8)). Then the following theorem describes the method of constructing a BYHC$(s^2, (s^2+s)^2, m, (s^2+s)^m)$ with $m = m'+2$.

<u>Theorem 5.3.2.6</u>. Let $\{t_{oj}, j = 1,2,\ldots,m'\}$ be a collection of distinct integers from the set of integers $\{1,2,\ldots,s-1\}$. (This is possible since $m' \leq s-1$). Then, for each $j = 1,2,\ldots,m'$, starting with the CSPT, Σ_j and applying the methods of theorem 5.3.2.4, substituting t_{oj} for t_o and k_j for k in the formulae for $t(i,j)$'s and g_{ij}'s, a CSPT of D (the matrix of order s^2+s considered in theorem 5.3.2.4) is constructed, say represented by the set of transversals $\psi^j = \{\psi^j_{\beta,\alpha}, \alpha \in F, 0 \leq \beta \leq s\}$. These CSPT's ψ^j, $j = 1,2,\ldots,m'$ satisfy the following properties

(i) The collection of the sets of treatments $\{\psi^j_{\beta,\alpha}(D), \alpha \in F, 0 \leq \beta \leq s\}$ constitute the blocks of the BBD d_o for each j, $j = 1,2,\ldots,m'$.

(ii) For each pair $j \neq j'$, $j,j' = 1,2,\ldots,m'$, the CSPT's ψ_j and $\psi_{j'}$ are mutually orthogonal.

<u>Proof</u>. (i) follows from theorem 5.3.2.4, starting with the different CSPT's Σ_j's of the square matrix of order $s+1$,

$j = 1, 2, \ldots, m'$, and (ii) follows from lemma 5.3.2.5.

This theorem implies that the matrix D in this situation represents a BYHC $(s^2, (s^2+s), m, (s^2+s)^m)$, with $m = m' + 2$.

Finally, we note that given any set of mutually orthogonal CSPT's $\Sigma_1, \Sigma_2, \ldots, \Sigma_{m'}$, $m' \leq s-1$ of a square matrix of order $s+1$, it is always possible to renumber the rows of the matrix in such a way that the condition (5.3.2.8) is satisfied by each of the CSPT's and thus a BYHC $(s^2, (s^2+s)^2, m, (s^2+s)^m)$ can be constructed with the help of theorem 5.3.2.6 where $m = m' + 2 \leq s+1$.

It is to be noted that the maximum number of factors that can be accommodated by this construction procedure cannot exceed $s+1$. This is because the t_{oj}'s corresponding to the distinct t_j's have to be all distinct and different from zero.

We shall denote this BYHC $(s^2, (s^2+s)^2, m, (s^2+s)^m)$ by simply d_o^m. This BYHC d_o^m can easily be modified into a BYHR $(v = s^2, n = (s^2+s)^2, m, b_1 = b_2 = \ldots = b_q = s^2+s, b_{q+1} = \ldots = b_m = s+1$ in a balanced m-way setting, regular with respect to the factors $q+1, q+2, \ldots, m$, where $0 \leq q \leq m$, as shown below.

The BYHC d_o^m, the construction procedure for which has been explained in the theorems 5.3.2.4 and 5.3.2.6 can be conveniently represented by an $\overline{m+1} \times n$ array, say, A_{m+1} with the first m rows representing the m factors which correspond to the rows, columns and the mutually orthogonal CSPT's of D, used in the construction of the design. The first n rows of A_{n+1} represents an $OA(n,n, s^2+s,2)$ of index 1, which we call $\bar{A}_n$ and the last row designate the treatments applied to the experimental units denoted by the respective columns of $\bar{A}_n$. The correspodence between the square matrix D and the array A_{n+1} is as described in P(i) through P(iii) in section 5.2. So the first row of A_{n+1} corresponds to the row number for the different cells of D, the second row of A_{n+1} correspond to the columns numbers of the different cells of D and similarly the $(j+2)$th row of A_{n+1} represent the jth CSPT of D, using s^2+s distinct symbols for the s^2+s distinct transversals of the set. Let us recall the representation of D given in (5.3.2.5) and represent the (α,α')th cell of D_{ij}, by $((\alpha,i),(\alpha',j))$, $\alpha,\alpha' = \alpha_0,\alpha,\ldots,\alpha_{s-1}$, $i,j = 0,1,\ldots,s$, i.e. the rows (also columns) of D are conveniently numbered as (α,β), $\alpha \in F$, $0 \leq \beta \leq s$. Then the transversal $\psi^j_{\beta,\alpha}$ belonging to the jth CSPT of D as defined in theorem 5.3.2.6 will also be specified by the ordered pair (α,β), $\alpha \in F$, $0 \leq \beta \leq s$, $j = 1,2,\ldots,n-2$. So, now the symbols occurring in each row of $\bar{A}_n$ are the ordered pairs (α,β), $\alpha \in F$, $0 \leq \beta \leq s$. Now given an integer $q < n$, we construct an array B_{n+1} from A_{n+1} by replacing the symbol (α,β) in the $(q+j)$th row of A_{n+1} by β for all $\alpha \in F$ and $0 \leq \beta \leq s$, $j = 1,2,\ldots,n-q$. The rows numbered, $1,2,\ldots,q$ in the array are left unaltered. Clearly, the array B_{n+1} is a BYHR with parameters $(v = s^2, n = (s^2+s)^2, n, b_1 = b_2 = \ldots = b_q = s^2+s$, $b_{q+1} = \ldots = b_n = s+1)$. Let us denote this derived design by $d_0^q(n)$.

This balanced m-way setting is regular with respect to the factors $q+1,\ldots,m$.

Remark 1. It is easy to see that in the above construction, for the factors with number of levels $s+1$, one need not go into the constructions of the composite CSPT's ϕ^j, $j = q+1,\ldots,m$. Given m' mutually orthogonal CSPT's Σ_j, $j=1,2,\ldots,m'$ of a square matrix of order $s+1$, one can construct a BYHC $(s^2, (s^2+s)^2, q, (s^2+s)^q)$, making use of the CSPT's Σ_j, $j = 1,2,\ldots,q-2$, following the method of theorem 5.3.2.6. This design which we may denote by d_o^q may also be represented by an $\overline{q+1}\times n$ array, say, A_{q+1} as explained earlier (with m replaced by q). Let us denote the first q rows of A_{q+1}, which correspond to the rows, columns and the CSPT's Σ_j, $j=1,2,\ldots,q-2$ of D, by $\bar{A}_q$. Now this array $\bar{A}_q$ can be extended to an $m\times n$ array $\bar{B}_m$ with the help of the remaining CSPT's, Σ_j, $j = q-1,\ldots,m-2$ in a very simple manner. With the help of Σ_j, a composite $s^2(s+1)$-transversal of D can be constructed by choosing always the s^2-transversal of the submatrix D_{ik} as the complete matrix D_{ik} for $0\leq i,\ k\leq s$. Then, we can write β for all (α,β), $\alpha\in F$, $0\leq\beta\leq s$ as done earlier. Hence, the BYHR $(v=s^2, n=(s^2+s)^2$, $b_1=b_2=\ldots=b_q=s^2+s$, $b_{q+1}=\ldots=b_m=s+1)$ denoted by B_{m+1} is obtained by adding the last row of A_{q+1} designating the treatment arrangements, to the $m\times n$ array $\bar{B}_m$. Let this BYHR regular with respect to the last m-q factors be denoted by $d_o^q(m)$. If a complete set of mutually orthogonal CSPT's of a square matrix of order $s+1$ exists, by the above method we can construct a BYHR $(s^2, (s^2+s)^2, m, b_1 = b_2 = \dots = b_q = s^2+s, b_{q+1} = \dots = b_m = s+1)$, where $q < m \leq s+2$.

Remark 2. We recall from theorems 2.5.3.2 and 2.5.3.6 of Chapter 2 that a BYHC (v,n,m,b^m) is E-optimal if $n \geq 2mb-(m-1)$ and A and D-optimal when $n \geq 4m^2b - 4m(m-1)$. So, it follows that the BYHC d_o^m is always E- optimal within a class $\mathcal{D}$ of proper and connected designs in a balanced m-way setting $M(s^2; (s^2+s)^2; m; b_1 = b_2 = \dots = b_m = s^2+s)$ $2 \leq m \leq \text{Min}(m'+2, s+1)$, where there are m' mutually, orthogonal CSPT's of a square matrix of order $s+1$. Moreover, d_o is A- and D- optimal, whenever

$$m \leq (s^2+s)^{1/2}/2.$$

Now the BYHR $d_o^q(m)$ is a design in a balanced m-way setting regular with respect to $m-q$ factors, so that by theorem 2.5.3.1, of Chapter 2, its optimal behaviour in the given setting is implied by the optimal behaviour of the corresponding q-factor design, viz the BYHC $(s^2; (s^2+s)^2, q, (b = s^2+s)^q)$ in the q-way setting $M(v; n; m; b_1 = b_2 = \dots = b_q = s^2+s)$.

So, $d_o^q(m)$ always E - optimal and A- and D- optimal, whenever

$$q \leq (s^2+s)^{1/2}/2.$$

5.4. Construction of Balanced Pseudo Youden Designs

The definition of a BPYD is given in DFN 2.2.29 and its optimality properties have been established in theorem 2.5.3.7. In this section we first give a method of construction for two series of BPYD's with parameters respectively

(i) $(v = s^2, n = ts^3 + s^2(s-1), m = s+1, (b = s)^m)$, $t \geq 0$

and (ii) $(v = s^2, n = ts^3 + s^2, m = s+1, (b = s)^m)$, $t \geq 1$,

where s is a prime power. Then the series of PYD's constructed by Cheng (1981b) is modified into a series of BPYD's in a balanced 4-way setting.

5.4.1. BPYD's with parameters (i) $(v = s^2, n = ts^3 + s^2(s-1), m = s+1, (b = s)^m)$, $t \geq 0$ and (ii) $(v = s^2, n = ts^3 + s^2, m = s+1, (b = s)^m)$, $t \geq 1$.

Let us recall the array A constructed in section 5.3.1 which represents a completely regular BYHC with parameters $(v = s^2, n = s^3, m = s+1, b^m = s^m)$.

It is obvious from the construction procedure described there that for any given integer k, $0 \leq k \leq s-1$, A_k constitutes a BPYD $(v = s^2, n = s^2, m = s+1, (b = s)^m)$. But A_k cannot be used as a design, since the number of experimental units has to be reasonably large, as pointed out while establishing the optimality properties of these designs. Moreover, in this particular case, all the $s^2 - 1$ degrees of freedom carried by the s^2 experimental units in A_k are accounted for by the $s+1$ factors and no degree of freedom is left for the treatment effects. But we can construct connected BPYDs from the array A as follows :

$$\text{Let} \quad \bar{B}_k = [A_0 \; A_1 \; \ldots \; A_{k-1} \; A_{k+1} \; \ldots \; A_{s-1}]. \qquad \ldots(5.4.1.1)$$

Then for any $t \geq 0$, we construct the following arrays

$$B_k = [\underbrace{A \; A \; \ldots \; A}_{t \text{ times}} : \bar{B}_k],$$ where $\bar{B}_k$ is as given in (5.4.1.1) and

$$C_k = [\underbrace{A \; A \; \ldots \; A}_{t \text{ times}} : A_k].$$

Here $t \geq 0$ in B_k and $t \geq 1$ in C_k

clearly B_k represents a connected BPYD with parameters $(v = s^2,\ n = t s^3 + s^2(s-1),\ n = s+1, (b = s))$ for all $t \geq 0$ when $s \geq 3$ and $t \geq 1$ when $s = 2$, the parameters of the corresponding BBD (DFN 2.2.13) being $(s^2,\ s^2 + s,\ (s+1)(ts + s-1),\ t s^2 + s(s-1),\ s^2 - s-1 + 2t(s+1)(\quad s-1) + s(s+1)t^2)$. Similarly, C_k represents a connected BPYD with parameters $(v = s^2,\ n = t s^3 + s^2,\ n = s+1, (b = s))$ for all $t \geq 1$, the parameters of the corresponding BBD being $(s^2,\ s^2 + s,\ (s+1)(ts + 1),\ t s^2 + s,\ s(s+1)t^2 + 2(s+1)\,t \qquad + 1)$.

Now from theorem 2.5.3.7, we know that a BPYD (v,n,n,b) is E - optimal if $n/2 \geq nb - n + 1$ and it is A- and D- optimal if $n \geq 4m^2b - 4m(m-1)$. Therefore, it follows that the BPYD B_k constructed above is E-optimal for any integer $t \geq 0$ when $s \geq 3$. For $s = 2$, t must be ≥ 1 for E-optimality of B_k. The same design is A- and D- optimal also, for (i) $t \geq 4$ when $s \geq 5$, (ii) $t \geq 5$ when $s \geq 3$, (iii) $t \geq 6$ when $s = 2$.

Similarly, C_k is E-optimal for any integer $t \geq 1$ and $s \geq 2$. The same design is A- and D- optimal for (i) $t \geq 5$, when $s \geq 3$ and (ii) $t \geq 6$, when $s = 2$.

5.4.2. Construction of a BPYD $(v = s^2,\ n = (s^2 + s)^2/4,\ m = 4,\ b = (s^2 + s)/2)^4)$ where s is a prime power and $s \equiv 3 \pmod 4$

We shall modify the PYD constructed by Cheng (1981b) so as to represent a BPYD with four factors.

Cheng (1981b) has constructed a PYD with parameters $(v = s^2,\ n = (s^2 + s)^2/4,\ m = 2,\ b = (s^2 + s)/2)$, where s is a prime power of the form $3 \pmod 4$. This design is represented by a square matrix of order $(s^2 + s)/2$, the rows and columns of which together constitute the blocks of a BIBD $(s^2,\ s^2 + s,\ (s+1)^2/2,\ (s^2 + s)/2,\ (s+1)(s+2)/4)$. We shall refer to the BIBD with these parameters as d_1.

Here we shall construct a square matrix D of order $(s^2 + s)/2$ with a structure similar to but not identical with the matrix D of Cheng (1981b), where the rows, the columns and the transversals of the mutually orthogonal CSPT's of D together constitute a design in which each block of d_1 appears exactly twice.

To construct the matrix, we proceed along the following steps.

Step 1. Construct a $(s^2+s)/2 \times (s^2+s)$ matrix B with the points of EG(2,s) as elements with the help of the symmetrical BIBD (s,s,(s+1) / 2, (s+1) / 2, (s+1) / 4) exactly in the manner Cheng (1981b) constructed the matrix D. Then partition the matrix B as follows $B = [B_1 B_2 B_3 B_4]$, where each B_i is a $(s^2+s)/2 \times (s^2+s)/4$ matrix.

Step 2. Let (i_1, i_2, i_3, i_4) be a permutation of (1,2,3,4). Construct a square matrix D of order $(s^2+s)/2$ in such a way that its columns together constitute the collection of all the columns of the submatrix B_{i_1}, each occurring twice and its rows together constitute the collection of the columns of another submatrix B_{i_2} of B, each occurring twice.

Step 3. Construct two mutually orthogonal CSPT's $\{\psi_h, 1 \le h \le (s^2+s)/2\}$ and $\{\bar{\psi}_h, 1 \le h \le (s^2+s)/2\}$ of D such that the collection of sets $\psi_h(D)$, $1 \le h \le (s^2+s)/2$ constitute the columns of B_{i_3}, each occurring twice and similarly the sets $\bar{\psi}_h(D)$, $1 \le h \le (s^2+s)/2$ constitute the columns of B_{i_4} each occurring twice. Let us now carry out the steps.

<u>Construction in step 1</u>. Although the block structure of the BIBD, d_1 is given in Cheng (1981b), we describe it briefly here for the sake of completeness in a manner which would make it convenient for us to construct the required CSPT's later.

Let $F = \left\{ \alpha_o = 0,\ \alpha_1 = \alpha^0 = 1,\ \alpha_1 = \alpha, \ldots, \alpha_{s-1} = \alpha^{s-2} \right\}$ ...(5.4.2.1)

be the elements of GF(s), where α is a primitive element of GF(s). Henceforth the rows, the columns and the transversals of a CSPT of a square matrix of order s will be indexed in accordance with the elements of F, as ordered in (5.4.2.1). Then we list the lines of EG(2,s) as follows. For each i, $0 \leq i \leq s$, the ith pencil of parallel lines of EG(2,s) is represented by a square matrix P_i of order s with each column representing a distinct line of the pencil with the s points in the line appearing in same order. More specifically, the α_t - th column of P_j consists of the s points on the line

$$L_j(x,y) = \alpha_t \text{ of EG}(2,s), \qquad \ldots(5.4.2.2)$$

where $L_j(x,y)$ is a suitable linear function of x and y. The functions $L_j(x,y)$'s can be chosen in many ways. One such feasible choice for L_j's is described below.

For $j = 0,1,\ldots,\ell$, $L_j(x,y) = x + \alpha_j y$, $L_{\ell+1}(x,y) = y$

and for $j = \ell+2,\ldots,s$, $L_j(x,y) = y + \alpha_{j-\ell} x$, ...(5.4.2.2a)

where $\ell = (s-1)/2$. It is obvious that P_j's, $j = 0,1,\ldots,s$

constitute all the (s+1) distinct pencils of EG(2,s). Now let

$$B_{ij} = P_j \xi_{2i}, \; i = 0,1,\ldots,\ell \;, \; j = 0,1,\ldots,s \qquad \ldots(5.4.2.3)$$

where ξ_i's are permutation matrices as described in 5.3.2.4. Then the columns of the matrix

$$B = (B_{ij})_{\substack{0 \le i \le \ell \\ 0 \le j \le s}} \qquad \ldots(5.4.2.4)$$

represent the blocks of the BIBD d_1. The matrix B given in (5.4.2.4) in essentially same as the matrix D of Cheng (1981b). The reasons for choosing B_{ij}'s in the manner stated in (5.4.2.3) is that $(\alpha_0, \alpha_2, \alpha_4, \ldots, \alpha_{s-1})$ constitutes a difference set for the BIBD (s,s,(s+1) / 2, (s+1) / 2, (s+1) / 4) which when written in the form of a Youden Square forms a basis for the construction of the BIBD d_1, both here as well as in Cheng (1981b).

Now we complete step 1 of our construction with the following partitioning of the matrix B.

$$B = [\, B_1 \; B_2 \; B_3 \; B_4 \,] \qquad \ldots(5.4.2.5)$$

where $B_t = (B_{ij})_{\substack{0 \le i \le \ell \\ (t-1)(s+1)/4 \le j \le t(s+1)/4-1}}, \; t = 1,2,3,4$

<u>Construction in Step 2</u>. Before going to the construction of our design matrix D, let us note the following fact written in the form of a lemma.

<u>Lemma 5.4.2.1</u>. Given any two distinct pencils P_i and P_j of

EG(2,s), one can construct a square matrix Q of order s with the points of EG(2,s) as elements such that the α_q-th column of it constitutes the α_q-th line of the pencil P_i and the α_p-th row constitute the α_p-th line of P_j, $p,q = 0,1,\ldots,s-1$ (Recall that the rows and columns of a square matrix of order s are indexed by the elements of GF(s)).

Clearly, in Q each point of EG(2,s) occurs exactly once.

Let us choose $i_1 = 1$, $i_2 = 3$, $i_3 = 2$, $i_4 = 4$. Now we construct our design matrix D as

$$D = (D_{ij})_{0 \leq i,j \leq \ell} \qquad \ldots(5.4.2.6)$$

where the submatrices are obtained as follows. For each pair i,j, $0 \leq i,j \leq \ell$, let $u(i)$ and $w(j)$ be functions of i and j respectively, chosen in such a way that the set of points of EG(2,s) contained in the α_q-th column of D_{ij} is the same as the line constituted by the α_q-th column of $B_{i,w(j)}$ and similarly, the α_p-th row of D_{ij} is the same as the line constituted by the α_p-th row of $B_{j,u(i)}$, $0 \leq p,q \leq s-1$. We note that such an arrangement is possible, provided $u(i) \neq w(j)$.

We choose the following functional forms for $u(i)$ and $w(j)$ so that the required row-column structure of D is obtained.

$$\begin{aligned} w(j) &= j \pmod{(s+1)/4} \\ u(i) &= i \pmod{(s+1)/4} + (s+1)/2 \end{aligned} \qquad \ldots(5.4.2.7)$$

<u>Construction in Step 3</u>. Except for the cases $\ell+1=2$ and 6, i.e. $s=3$ and 11, there always exist two mutually orthogonal latin squares of order $\ell+1$, i.e. two mutually orthogonal CSPT's of a square matrix of order $\ell+1$.

We denote them by $\Sigma=\{\sigma^0, \sigma^1, \ldots, \sigma^\ell\}$ and $\bar{\Sigma}=\{\bar{\sigma}^0, \ldots, \bar{\sigma}^\ell\}$ respectively.

Now we shall find for each pair i, j, $0 \leq i, j \leq \ell$, two mutually orthogonal CSPT's of the submatrix D_{ij}, viz $\pi_{ij}=\{\pi_{ij}^\alpha, \alpha \in F\}$ and $\bar{\pi}_{ij}=\{\bar{\pi}_{ij}^\alpha, \alpha \in F\}$ such that the composite CSPT's

$$\psi=\{\psi_{\beta,\alpha}, 0 \leq \beta \leq \ell, \alpha \in F\} \text{ where } \psi_{\beta,\alpha}=\sigma^\beta . \prod^\alpha$$

and $$\bar{\psi}=\{\bar{\psi}_{\beta,\alpha}, 0 \leq \beta \leq \ell, \alpha \in F\}, \text{ where } \bar{\psi}_{\beta,\alpha}=\bar{\sigma}^\beta . \bar{\prod}^\alpha ; \qquad \ldots(5.4.2.8)$$

(the notation used here are the same as those given in (5.2.4) of section 5.2), satisfy the required property stated in step 3, with $i_3=2$ and $i_4=4$.

This can be done, if the CSPT's π_{ij}'s and $\bar{\pi}_{ij}$'s are chosen in such a way that the following condition is satisfied.

<u>Condition D</u>. For each $\alpha \in F$, each $\beta \in I(\ell)=\{0,1,\ldots,\ell\}$ and $i \in I(\ell)$, let $L_{i,\beta}^{(\alpha)}=\pi^\alpha_{i,\sigma^\beta(i)}(D_{i,\sigma^\beta(i)})$

and $$\bar{L}_{i,\beta}^{(\alpha)}=\bar{\pi}^\alpha_{i,\bar{\sigma}^\beta(i)}(D_{i,\bar{\sigma}^\beta(i)}) . \qquad \ldots(5.4.2.9)$$

Then $L_{i,\beta}^{(\alpha)}$ and $\bar{L}_{i,\beta}^{(\alpha)}$ are the same as the sets of points of EG(2,s) contained in the α-th column of the submatrices $B_{i,f(\beta)}$ and $B_{i,\bar{f}(\beta)}$ respectively,

where $f(\beta) = \beta \pmod{(s+1)/4} + (s+1)/4$

and $\bar{f}(\beta) = \beta \pmod{(s+1)/4} + 3(s+1)/4 \quad 0 \leq \beta \leq \ell$.

The structures of the CSPT's ϕ_t, $1 \leq t \leq s-1$ as defined in DFN 5.2.6 and the matrix of the pencils of EG(2,s) imply the following lemma.

<u>Lemma 5.4.2.2</u>. Consider the matrix Q constructed in lemma 5.4.2.1, where rows (columns) constitute the lines of $P_i(P_j)$, i and j given, $i \neq j$. Given any two distinct pencils $P_{i'}$, $P_{j'}$ of EG(2,s), $i' \neq i,j$, $j' \neq i,j$, $i' \neq j'$, it is always possible to find integers t and t', $1 \leq t, t' \leq s-1$, $t \neq t'$ and functions g and g' from F onto itself, such that the set of points of EG(2,s) given by $\phi_t^{g(\alpha_h)}(Q)$ constitute the α_h-th line of $P_{i'}$, and similarly the set $\phi_{t'}^{g'(\alpha_h)}(Q)$ constitutes the α_h-th line of $P_{j'}$, $h = 0,1,\ldots,s-1$. Moreover as $t \neq t'$, the CSPT's ϕ_t and $\phi_{t'}$ are mutually orthogonal.

The lemma 5.4.2.2 makes it clear that it is possible to choose, for each pair i,j, $0 \leq i,j \leq \ell$, integers $t(i,j)$ and $\bar{t}(i,j)$, $1 \leq t(i,j), \bar{t}(i,j) \leq s-1$, $t(i,j) \neq \bar{t}(i,j)$ and functions g_{ij} and $\bar{g}_{ij}$ from F onto itself, so that if we take

$$\pi_{ij}^{\alpha} = \phi_{t(i,j)}^{g_{ij}(\alpha)} \quad \text{and} \quad \bar{\pi}_{ij}^{\alpha} = \phi_{\bar{t}(i,j)}^{\bar{g}_{ij}(\alpha)} \qquad \ldots(5.4.2.10)$$

then the condition D in the construction in step 3 will hold. The choices for $t(i,j)$, $\bar{t}(i,j)$ of course will depend on the CSPT's Σ and $\bar{\Sigma}$. Once Σ and $\bar{\Sigma}$ are chosen, these functions can easily be worked out by observing the structures of the sub-matrices D_{ij}'s and using lemma 5.3.2.3.

Two mutually orthogonal CSPT's of order $\lambda + 1$ do not exist when $\lambda = 1$ or 5 and hence, the method fails for the corresponding values of s, viz., $s = 3$ and 11. So, the methods works for all $s \geq 7$ but $s \neq 11$. As an illustration, an example of a BPYD of the series when $s = 7$ is provided here.

Here 3 is a primitive element of GF(7) and the arrangement of the elements of GF(7) in accordance with (5.4.2.1) is given by

$$F = GF(7) = \left\{ \alpha_0 = 0,\ \alpha_1 = 1,\ \alpha_2 = 3,\ \alpha_3 = 2,\ \alpha_4 = 6,\ \alpha_5 = 4,\ \alpha_6 = 5 \right\}.$$

Then from (5.4.2.2) and (5.4.2.2a), the lines of EG(2,s) constituted by the α_h-th column of the matrices P_j is given as follows, $j = 0,1,2,\ldots,s$, $h = 0,1,\ldots,6$.

$$P_0 : x = \alpha_h$$

$$P_1 : x + y = \alpha_h$$

$$P_2 : x + 3y = \alpha_h$$

$$P_3 : x + 2y = \alpha_h$$

$$P_4 : y = \alpha_h$$

P_5 : $y + 3x = \alpha_h$

P_6 : $y + 2x = \alpha_h$

P_7 : $y + 6x = \alpha_h$

Now using (5.4.2.3) and (5.4.2.4), the matrix B representing the BIBD d_1 as columns can easily be constructed. Then the design matrix D can be constructed by following the instructions stated after (5.4.2.6). In the following table the structures of the submatrices D_{ij}'s of D are shown. In the (i,j)th entry of the table is given the (α_p, α_q)th element of D_{ij}, p, $q = 0,1,\ldots,6$, $i,j = 0,1,\delta,\delta+1$, where $\{0,1,\delta,\delta+1\} = GF(4)$, $\delta^2 = \delta + 1$.

	0	1	δ	$\delta + 1$
0	$x = \alpha_q$, $y = \alpha_p$	$x + y = \alpha_q$, $y = \alpha_p + 3$	$x = \alpha_q$ $y = \alpha_p + 6$	$x + y = \alpha_q$ $y = \alpha_p + 5$
1	$x = \alpha_q + 3$ $y + 3x = \alpha_p$	$x + y = \alpha_q + 3$ $y + 3x = \alpha_p + 3$	$x = \alpha_q + 3$ $y + 3x = \alpha_p + 6$	$x + y = \alpha_q + 3$ $y + 3x = \alpha_p + 5$
δ	$x = \alpha_q + 6$ $y = \alpha_p$	$x + y = \alpha_q + 6$ $y = \alpha_p + 3$	$x = \alpha_q + 6$ $y = \alpha_p + 6$	$x + y = \alpha_q + 6$ $y = \alpha_p + 5$
$\delta+1$	$x = \alpha_q + 5$ $y + 3x = \alpha_p$	$x = y = \alpha_q + 5$ $y + 3x = \alpha_p + 3$	$x = \alpha_q + 5$ $y + 3x = \alpha_p + 6$	$x + y = \alpha_q + 5$ $y + 3x = \alpha_p + 5$

Next we choose the composite CSPT's ψ and $\bar{\psi}$ of step 3, based on two given CSPT's Σ and $\bar{\Sigma}$ of a square matrix of order 4 as defined in (5.4.2.8). Now if we index the rows, columns and transversals of CSPT of a square matrix of order 4 in accordance with the elements of $GF(4) = \{0, 1, \delta, \delta + 1\}$, then it is easy to see that the following pair of sets represent two mutually orthogonal CSPT's of a square matrix of order 4.

$$\Sigma = \{\sigma^j,\ j = 0, 1, \delta, \delta + 1\},\ \text{with}\ \sigma^j(i) = i + j,\ i = 0, 1, \delta, \delta + 1$$

and $\bar{\Sigma} = \{\bar{\sigma}^j,\ j = 0, 1, \delta, \delta+1\}$, with $\bar{\sigma}^j(i) = i\delta + j$, $i = 0, 1, \delta, \delta+1$.

Based on Σ and $\bar{\Sigma}$ thus chosen, we shall now choose $t(i,j)$'s, $g_{ij}(\alpha)$'s, $\bar{t}(i,j)$'s and $\bar{g}_{ij}(\alpha)$'s such that the CSPT's π_{ij}'s and π_{ij}'s of the submatrices D_{ij}'s as described in (5.4.2.10) satisfy condition D of the construction in step 3. In the (i,j)th entry of the following table, the value of $\alpha_{t(i,j)}$ and the functional form of $g_{ij}(\alpha)$ are written, for each pair (i,j), $i,j = 0, 1, \delta, \delta + 1$,

	0	1	δ	$\delta+1$
0	$\alpha_t = 5$ $g(\alpha) = 5\alpha$	$\alpha_t = 1$ $g(\alpha) = \alpha + 4$	$\alpha_t = 5$ $g(\alpha) = 5\alpha + 1$	$\alpha_t = 1$ $g(\alpha) = \alpha + 2$
1	$\alpha_t = 1$ $g(\alpha) = 4\alpha$	$\alpha_t = 3$ $g(\alpha) = 6\alpha + 6$	$\alpha_t = 1$ $g(\alpha) = 4\alpha + 3$	$\alpha_t = 3$ $g(\alpha) = 6\alpha + 4$
δ	$\alpha_t = 5$ $g(\alpha) = 5\alpha$	$\alpha_t = 1$ $g(\alpha) = \alpha + 4$	$\alpha_t = 5$ $g(\alpha) = 5\alpha + 1$	$\alpha_t = 1$ $g(\alpha) = \alpha + 2$
$\delta+1$	$\alpha_t = 1$ $g(\alpha) = 4\alpha + 1$	$\alpha_t = 3$ $g(\alpha) = 6\alpha+5$	$\alpha_t = 1$ $g(\alpha) = 4\alpha + 2$	$\alpha_t = 3$ $g(\alpha) = 6\alpha + 3$

In the next table, the values of $\alpha_{\bar{t}(i,j)}$'s and the functional forms of $\bar{g}_{ij}(\alpha)$'s are shown in a similar fashion.

$\alpha_{\bar{t}} = 2$ $\bar{g}(\alpha) = \alpha$	$\alpha_{\bar{t}} = 3$ $\bar{g}(\alpha) = 4\alpha + 4$	$\alpha_{\bar{t}} = 2$ $\bar{g}(\alpha) = \alpha + 1$	$\alpha_{\bar{t}} = 2$ $\bar{g}(\alpha) = 4\alpha + 2$
$\alpha_{\bar{t}} = 6$ $\bar{g}(\alpha) = \alpha + 6$	$\alpha_{\bar{t}} = 5$ $\bar{g}(\alpha) = 6\alpha$	$\alpha_{\bar{t}} = 6$ $\bar{g}(\alpha) = \alpha$	$\alpha_{\bar{t}} = 5$ $\bar{g}(\alpha) = 6\alpha + 5$
$\alpha_{\bar{t}} = 6$ $\bar{g}(\alpha) = \alpha + 5$	$\alpha_{\bar{t}} = 5$ $\bar{g}(\alpha) = 6\alpha + 3$	$\alpha_{\bar{t}} = 6$ $\bar{g}(\alpha) = \alpha + 6$	$\alpha_{\bar{t}} = 5$ $\bar{g}(\alpha) = 6\alpha + 1$
$\alpha_{\bar{t}} = 3$ $\bar{g}(\alpha) = \alpha + 4$	$\alpha_{\bar{t}} = 1$ $\bar{g}(\alpha) = 2\alpha + 2$	$\alpha_{\bar{t}} = 3$ $\bar{g}(\alpha) = \alpha + 5$	$\alpha_{\bar{t}} = 1$ $\bar{g}(\alpha) = 2\alpha$

Thus our construction of a BPYD (49,784,4,28) is complete.

5.5. Construction of Asymmetrical Optimal Designs

In this section we construct two asymmetrical designs in the balanced m-way settings, regular with respect to all but one factor.

5.5.1. The setting $M(s^2; s^3(s-1); s; s^2, s(s-1), s, \ldots, s)$

Let us start with a GDD, $\bar{D}$ with parameters $v = s^2$, $b = s^2$, $r = k = s$, $m = n = s$, $\lambda_1 = 0$, $\lambda_2 = 1$. The treatments of this design may be identified with the points of EG(2,s) and the blocks with the lines of EG(2,s) so that the design is expressed in the following form.

$$\bar{D} = \begin{bmatrix} \bar{D}_0 \\ \bar{D}_1 \\ \vdots \\ \bar{D}_{s-1} \end{bmatrix} \qquad \ldots(5.5.1.1)$$

where $\bar{D}_i$ is an $s \times s$ matrix with the rows and columns indexed by the elements of F and the (α_p, α_q)th element of $\bar{D}_i$ is given by the point $(x = \alpha_q - \alpha_i, y = \alpha_i x + \alpha_p)$, $i = 0, 1, \ldots, s-1$. We note that the columns of $\bar{D}_i$ constitute the lines $x = \alpha_q - \alpha_i$, $q = 0, 1, \ldots, s-1$, so that each column of $\bar{D}$ contains each treatment exactly once. Again the rows of $\bar{D}_i$ represent the line of EG(2,s) of the form : $y = \alpha_i x + \alpha_p$, $p = 0, 1, \ldots, s-1$; $i = 0, 1, \ldots, s-1$; so that if the ith group of treatments is given by $\{(x,y) : x = \alpha_i\}$, $i = 0, 1, \ldots, s-1$, then the rows of $\bar{D}$ constitute a GDD with parameters as stated. By theorem 3.3.2.2, $\bar{D}$ is A- and D- optimal within the class $\mathcal{D}_p^{(1)}$ of two-way designs. We shall now construct another design D, which is some sort of a complement of $\bar{D}$ and show that D can be regarded as a design in a balanced m-way setting with $m \geq 3$ and D is A- and D- optimal within $\mathcal{D}_p^{(m)}$ of m-way designs as considered in theorem 3.3.2.3.

Let us construct D as follows

$$D = (D_{ij})_{\substack{0 \le i \le s-1 \\ 1 \le j \le s-1}} \qquad \ldots(5.5.1.2)$$

where the (α_p, α_q)th element of the submatrix D_{ij} is the point $(x = \alpha_q - \alpha_i,\ y = \alpha_i x + \alpha_p - \alpha_j)$ of EG(2,s). The design represented by the matrix $[\bar{D} : D]$ has in each of its rows, each treatment replicated exactly once. So, in this sense D can be looked upon as some sort of a complement of $\bar{D}$.

Our aim is to choose $m-2$ mutually orthogonal CSPT $(s^2(s-1))$'s of D, say,

$$\psi_i = \left\{ \psi_i^\alpha ,\ \alpha \in F \right\},\ i = 1,2,\ldots,m-2 \text{ such that } \psi_i^\alpha(D)$$

consists of all points of EG(2,s) each occurring s-1 times.

First we find out one of them, say ψ_1. To do this, we shall follow the steps similar to those in section 5.3.2. Step 1 is already completed here. Also we notice that for a matrix of order $s \times \overline{s-1}$, the only possible λ-transversal is the one with $\lambda = s(s-1)$, since s and s-1 are relatively prime. So here

$$\Sigma = \left\{ \sigma \right\}, \qquad \ldots(5.5.1.3)$$

where $\sigma = \left\{ (i,j),\ 0 \le i \le s-1,\ 1 \le j \le s-1 \right\}$. In the next and final step, our task is to choose a suitable CSPT $\pi_{ij} = \left\{ \pi_{ij}^\alpha ,\ \alpha \in F \right\}$ of the submatrix D_{ij}, $j = 1,2,\ldots,s-1$, $i = 0,1,\ldots,s-1$, in such a way that $\psi_1^\alpha = \sigma . \prod^\alpha$, $\alpha \in F$ is the required CSPT $(s^2(s-1)$ of D.

Let T be the set of all treatments. Then the following condition C_1 is a set of sufficient conditions for ψ_1 being the required CSPT $(s^2(s-1))$ of D.

Condition C_1: $\underset{j=1}{\overset{s-1}{(\underline{\overline{U}})}} \pi^{\alpha}_{ij} (D_{ij}) (\underline{\overline{U}}) \pi^{\alpha}_{oi}(D_{oi}) = T$, $\alpha \varepsilon F$, $i = 1,2,\ldots,s-1$.

The symbol $(\underline{\overline{U}})$ is defined in theorem 5.3.2.4. In the present context $(\underline{\overline{U}})$ reduces to the simple union of sets as the sets happen to be disjoint. As in the preceding sections, we shall choose one of the CSPT's Φ_t, $t = 1,2,\ldots,s-1$ for each of the submatrices D_{ij}'s.

From the structures of the submatrices, applying lemma 5.3.2.1, we have

$$\phi_t^{\alpha_h} (D_{ij}) = \left\{(x,y) : (1 - \alpha_i\alpha_t)x - \alpha_t y = \alpha_h - \alpha_t \cdot \alpha_i - \alpha_j\right\}$$

$$h = 0,1,\ldots,s-1, \quad t = 1,2,\ldots,s-1 . \qquad \ldots(5.5.1.4)$$

For each pair i,j, $0 \leq i \leq s-1$, $1 \leq j \leq s-1$, let $t(i,j)$ be an integer taking values 1 through $s-1$ and $g_{ij}(\alpha)$ be a function defined on F onto itself. Our task is to find $t(i,j)$ and $g_{ij}^{(\alpha)}$'s in such a way that if we choose

$$\pi^{\alpha}_{ij} = \phi^{g_{ij}(\alpha)}_{t(i,j)}, \quad \alpha \varepsilon F, \; 0 \leq i \leq s-1, \; 1 \leq j \leq s-1,$$

then the condition C_1 will be satisfied. In other words, we are to choose $t(i,j)$'s and $g_{ij}^{(\alpha)}$'s such that the following condition will hold.

$$\overset{s-1}{\underset{j=1}{(\overline{U})}} \; \phi^{g_{ij}(\alpha_h)}_{t(i,j)}(D_{ij}) \; (\overline{U}) \; \phi^{g_{oj}(\alpha_h)}_{t(o,i)}(D_{oi}) = T,$$

$$i = 1,2,\ldots,s-1. \qquad \ldots(5.5.1.5)$$

In view of (5.5.1.3), let us choose the $t(i,j)$'s as :

$$t(i,j) = t(i),\ j = 1,2,\ldots,s-1,\ i = 1,2,\ldots,s-1,$$

$$t(o,i) = \bar{t}(i),\ i = 1,2,\ldots,s-1, \qquad \ldots(5.5.1.6)$$

where the relation between $t(i)$ and $\bar{t}(i)$ are given by

$$\alpha^{-1}_{\bar{t}(i)} = \alpha^{-1}_{t(i)} - \alpha_i , \qquad \ldots(5.5.1.7)$$

so that for any given i, $t(i)$ can take any one of the $s-2$ integral values

$$\left\{t(i) = 1,2,\ldots,s-1,\ \alpha_{t(i)} \neq \alpha_i^{-1}\right\}. \qquad \ldots(5.5.1.8)$$

For a given set of values for $t(i)$'s, $i = 1,2,\ldots,s-1$, $g_{ij}(\alpha_h)$, $0 \leq i \leq s-1$, $1 \leq j \leq s-1$ may be chosen as follows

$$g_{ij}(\alpha_h) = \alpha_h,\ \ 1 \leq i,\ j \leq s-1,$$

$$g_{oi}(\alpha_h) = \alpha_{\bar{t}(i)} \cdot \alpha^{-1}_{t(i)} \alpha_h + \alpha_i(1 - \alpha_{\bar{t}(i)}). \qquad \ldots(5.5.1.9)$$

Now from the definition of Φ_t as given in DFN 5.2.6, it can be easily observed that for a set of $t(i)$'s satisfying condition (5.5.1.8), if we choose $g_{ij}(\alpha)$'s as given in (5.5.1.9), the resulting CSPT $(s^2(s-1)) = \phi_1$ satisfies condition C_1. It is known that the CSPT Φ_t of a square matrix of order s is orthogonal to the CSPT $\Phi_{t'}$, if and only if $t \neq t'$.

Hence, let us assume that the following two sets T_1 and T_2 are given

$$T_1 = \{t(1),\ t(2), \ldots, t(s-1)\}$$

$$\text{and} \quad T_1' = \{t'(1),\ t'(2), \ldots, t'(s-1)\},$$

where both $t(i)$ and $t'(i)$ for each i satisy condition (5.5.1.8) and $t(i) \neq t'(i)$, $i = 1,2,\ldots,s-1$. Then, by the procedure described two CSPT $(s^2(s-1))$'s can be constructed which are mutually orthogonal, the set T_1 used for $t(i)$'s in the construction of the first CSPT $(s^2(s-1))$ and the set T_2 used for $t(i)$'s in the second case.

Now, for any given i, $t(i)$ can assume any one of the values 1 through $s-1$ with the restriction $\alpha_{t(i)} \neq \alpha_i^{-1}$. So, there are $s-2$ possible values for every $t(i)$, given i, $i = 1,2,\ldots,s-1$. Let $I = \{1,2,\ldots,s-1\}$ and $I_i = I - j$ where $\alpha_j = \alpha_i^{-1}$, $i = 1,2,\ldots,s-1$.

Choose $\quad t(i,j) = t_p(i) \in I_i, \quad 1 \leq i, j \leq s-1,$

$$t(o,j) = \bar{t}_p(i), \quad \text{given by (5.5.1.7)}$$

and $t_p(i) \neq t_{p'}(i)$ for $p \neq p'$.

For each $p = 1,2,\ldots,s-2$, the p-th CSPT $(s^2(s-1))$ is of the form $\psi_p = \psi_p^\alpha, \alpha \in F$, where

$$\psi_p^\alpha = \underset{i=0}{\overset{s-1}{(\underline{U})}} \ \underset{j=1}{\overset{s-1}{(\underline{U})}} \ \phi^{g_{ij}(\alpha)}_{t_p(i,j)},$$ with $g_{ij}(\alpha)$'s determined for the respective CSPT $(s^2(s-1))$'s from the relations (5.5.1.9).

Thus, from the construction procedure it is clear that $s-2$ such CSPT $(s^2(s-1))$'s, viz., ψ_p's, $p = 1,2,\ldots,s-2$ can be constructed.

In the resulting design so constructed the maximum number of factors that can be accommodated is $s-2+2 = s$. The first factor represented by rows has s^2 levels. The second factor represented by columns has $s(s-1)$ levels. The $(p+2)$th factor represented by the CSPT $(s^2(s-1))$, ψ_p has s levels, $p = 1,2,\ldots,s-2$. The design is obviously regular and balanced with respect to all the factors excluding 1. That it is a GDD with $\lambda_2 = \lambda_1 + 1$ with respect to factor 1 has already been stated.

5.5.2. <u>The setting $M(s^2 ; s^3(s+1) ; s ; s(s+1), s,\ldots,s)$</u>.

Let us consider the matrix $D^* = [\,\bar{D} \mid L\,]$, where L is the latin square constructed in step 1 of section 5.3.2 and $\bar{D}$ is the GDD with $\lambda_2 = \lambda_1 + 1$, used as the basis for the construction of the design in section 5.5.1. D^* is written in the form

$$D^* = (D_{ij})_{\substack{0 \leq i \leq s-1, \\ 0 \leq j \leq s}} \qquad \ldots(5.5.2.1)$$

where the (α_p, α_q)th element of the submatrix D_{i0} is the point $(x = \alpha_q - \alpha_i,\ y = \alpha_i x + \alpha_p)$ of $EG(2,s)$, and the (α_p, α_q)th element of D_{ij} is the point $(x = \alpha_q - \alpha_i,\ y = \alpha_p - \alpha_j)$ of $EG(2,s)$, $p,q = 0,1,\ldots,s-1$, $i = 0,1,\ldots,s-1$, $j = 0,1,\ldots,s$.

Clearly, the rows of D constitute the blocks of a symmetrical GDD with parameters $b = v = s^2$, $r = k = s^2 + s$, $\lambda_1 = s^2 + 2s$, $\lambda_2 = (s+1)^2$, $m = n = s$. Also each column of D consists of all the s^2 points of $EG(2,s)$.

We shall now choose $s-2$ mutually orthogonal CSPT $(s^2(s+1))$'s of D^*, viz., $\left\{\psi_u^\alpha, \alpha \in F\right\}$, $u = 1,2,\ldots,s-2$, such that for each $\alpha \in F$, $u = 1,2,\ldots,s-2$, the set $\psi_u^\alpha(D)$ consists of all the points of $EG(2,s)$ each occurring $s+1$ times, so that D^* may be regarded as an optimal s-factor design over the class of designs $\mathcal{D}_p^{(s)}$.

Now the only possible ℓ-transversal for an $s \times (s+1)$ matrix is an $s(s+1)$-transversal. So, the CSPT$(s(s+1))$ of an $s \times (s+1)$ matrix we begin with is, say

$$\Sigma = \left\{\sigma\right\}, \text{ with } \sigma = \left\{(i,j),\ 0 \leq i \leq s-1,\ 0 \leq j \leq s\right\} \qquad \ldots(5.5.2.2)$$

For every (i,j), $0 \leq i \leq s-1$, $0 \leq j \leq s$, let $\prod_{ij} = \left\{\pi_{ij}^\alpha, \alpha \in F\right\}$ be a CSPT of D_{ij}. Then a composite CSPT $(s^2(s+1))$ of D^* can be constructed denoted by $\psi = \left\{\psi^\alpha, \alpha \in F\right\}$, where $\psi^\alpha = \sigma \cdot \prod^\alpha$, $\alpha \in F$, $\psi^\alpha(D^*)$ is a collection of the points of $EG(2,s)$ where each point occurs exactly $(s+1)$ times. Now a sufficient condition for this to happen is given by

condition C_2: (i) $\displaystyle\underset{j=0}{\overset{s-1}{(\underline{U})}}\ \pi_{ij}^\alpha\ (D_{ij}) = T$, $\alpha \in F$, $0 \leq i \leq s-1$,

(ii) $\displaystyle\underset{i=0}{\overset{s-1}{(\underline{U})}}\ \pi_{is}^\alpha\ (D_{is}) = T$, $\alpha \in F$.

As in the preceding section 5.5.1, making use of the CSPT's ϕ_t, $1 \le t \le s-1$ for a square matrix of order s in the same manner, we can get hold of a suitable set of Π_{ij}'s by appropriately finding out the $g_{ij}(\alpha)$'s and $t(i,j)$'s, the expressions used in this chapter throughout so far. And by exactly the same arguments as in section 5.5.1, it is possible to find $s-2$ mutually orthogonal CSPT $(s^2(s+1))$'s of D^* satisfying property C_2. The number of factors thus accommodated is s. The first factor represented by rows has s^2 levels. The second factor represented by columns has $s(s+1)$ levels. All the other $(s-2)$ factors represented by the CSPT $(s^2(s+1))$'s have s levels. The exact method of construction for the $(s-2)$ mutually orthogonal CSPT $(s^2(s+1))$'s derived are presented hereunder for completeness. The derivation follows the same arguments as in section 5.5.1 and is not repeated here.

The construction of the p-th CSPT $(s^2(s+1))$ of the mutually orthogonal set is as follows.

Let $T(0) = \{1, 2, \ldots, s-1\}$

$T(i) = T(0) - i \quad , \ 1 \le i \le s-1$

$t_p(i,0) = t_p(i) \in T(i), \ i = 0, 1, \ldots, s-1$

$t_p(i,j) = \bar{t}_p(i), \ j = 1, 2, \ldots, s-1, \ i = 0, 1, \ldots, s-1$

$t_p(i,s) = t_p(s) \in T(0), \ \ i = 0, 1, \ldots, s-1,$

where $^{\alpha}\bar{t}_{p}(i) = {}^{\alpha}t_p(i) - {}^{\alpha}i$.

The corresponding $g_{ij}(\alpha)$'s which happen to be the same for all p's are given as

$$g_{ij}(\alpha) = \alpha\,, \quad j = 1,2,\ldots,s,\ i = 0,1,\ldots,s-1$$

$$g_{i0}(\alpha) = \alpha + \alpha_i^2\,, \quad i = 0,1,\ldots,s-1\,.$$

From the construction procedure outlined it is apparent that p can assume (s-2) possible values and hence there can be s-2 mutually orthogonal CSPT $(s^2(s+1))$'s satisfying property C_2.

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