Quantum stochastic flows: Trotter product formula, dilations and quantum Brownian motion

Biswarup Das

INDIAN STATISTICAL INSTITUTE, KOLKATA
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Indian Statistical Institute
203, B.T. Road, Kolkata, India.

In the memories of my father and grandfather

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Contents

Notations

$\mathbb{C}$	The set of complex numbers
$\mathbb{R}$	The set of real numbers
$I\!N$	The set of natural numbers
$\mathbf{Z}$	The set of integers
$M_n(\mathbb{C})$	The set of $n \times n$ matrices over complex numbers
$\mathbf{I}_n$	The identity matrix of order n
$\langle\ A\ \rangle_{\mathbb{C}}$	The span of elements from the set A over complex numbers
$\mathcal{B}(\mathcal{H})$	The von-Neumann algebra of bounded operators on a Hilbert space $\mathcal{H}$
$\mathcal{K}(\mathcal{H})$	The C^*-algebra of compact operators on $\mathcal{H}$
$L^2(\phi)$	The G.N.S Hilbert space associated with a state or weight ϕ on a C^* or von-Neumann algebra
$C^\infty(M)$	The space of smooth functions over a manifold M
$\mathcal{M}(\mathcal{A})$	The multiplier algebra of a C^*-algebra $\mathcal{A}$
$\mathcal{L}(E, F)$	The space of adjointable maps between Hilbert modules E and F
$\mathcal{L}(E)$	The space of adjointable maps from a Hilbert module E to itself
$G_1 \rtimes G_2$	The semidirect product of two groups G_1 and G_2
ev_x	Evaluation at x
id	The identity map
$\mathbb{T}$	The circle group
$\mathbb{T}^n$	The n-torus
$Lin(V_1, V_2)$	The set of linear, possibly densely defined maps between Banach spaces V_1 and V_2

$Dom(R)$ or $D(R)$	The domain of a map R
$Ran(R)$	The range of a map R

Chapter 0

Introduction

Motivated by the major role played by probabilistic models in many areas of science, several quantum (i.e. non-commutative) generalizations of classical probability have been attempted by a number of mathematicians. The pioneering works of K.R. Parthasarathy, L. Accardi, R.L. Hudson, P.A. Meyer and others led to the development of one such non-commutative model called 'quantum probability' which has a very rich theory of quantum stochastic calculus a la Hudson and Parthasarathy. Within the framework of quantum stochastic calculus, the 'grand design' that engages us is the canonical construction and study of $*$-homomorphic flows $(j_t)_{t\geq 0}$ on a given C^* or von-Neumann algebra of observables, say $\mathcal{A}$, where $j_t : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_0)))$ satisfies a differential equation of the form

$$dj_t(\cdot) = j_t(\theta^\mu_\nu(\cdot))\Lambda^\nu_\mu(dt),$$

$\Lambda^\nu_\mu(dt)$ being the well-known quantum stochastic integrators in the Fock-space $\Gamma(L^2(\mathbb{R}_+, k_0))$ (see for example [52]), with k_0 being a Hilbert space called the noise space. The vacuum expectation semigroup of the flow, generated by $\theta^0_0(\cdot)$, is a contractive semigroup of completely positive maps on the said algebra. In the realm of classical probability, such semigroups are typically the expectation semigroups associated with Markov processes. Of particular importance are the so called heat semigroups on Riemannian manifolds, which are the expectation semigroups associated with manifold-valued Brownian motions. The quantum analogue of 'dilation problem', i.e. to construct a Markov process from its expectation semigroup, is very interesting and important in quantum probability too.

There is an interesting confluence of Riemannian geometry and classical stochastic process, under the framework of 'stochastic geometry'. In particular there are interesting connections between the geometry of a Riemannian manifold and the probabilistic information obtained from a Brownian motion taking value in the manifold. For example, the geometric invariants of the manifold such as mean curvature, volume etc. can be obtained from the asymptotic analysis of exit time of the Brownian motion from balls of small volume.

It is therefore natural to explore the possibility of extending this philosophy to the quantum set-up, i.e. the possibility of connecting quantum stochastic calculus with non-commutative geometry (a la Connes), leading to a development of 'quantum stochastic geometry'. As Brownian motions on manifolds are Markov processes with *unbounded* generators, it is important for pursuing this programme to have a reasonable theory of quantum stochastic calculus with *unbounded* coefficients.

In this thesis, we shall begin by studying different aspects of quantum stochastic calculus with unbounded coefficients, and in the end, we shall try to connect the theory with non-commutative geometry.

In chapter 1, we discuss the basic defintions and results e.g. C^* and von-Neumann algebras, quantum isometry group, compact quantum group, quantum stochastic calculus, quantum dynamical semigroup, quantum stop-time etc, that we will be using in this thesis.

The first difficulty in dealing with quantum stochastic calculus with unbounded coefficients is the absence of a convenient method for proving the homomorphic property of the quantum stochastic flow. Using the quantum Ito formula, ona can easily write algebraic relations which are necessary for a quantum stochastic flow $(j_t)_{t\geq 0}$ to be homomorphic. But it is not known whether such algebraic conditions are also sufficient. We will prove in 2 that such algebraic conditions will be sufficient, if we furthermore assume the existence of a faithful, semifinite trace on the underlying (C^* or von-Neumann) algebra, the analyticity of the vacuum expectation semigroup and a suitable L^1-bound for j_t namely $|\langle j_t(x)uf^{\otimes^m}, vg^{\otimes^n}\rangle| = O(e^{\beta t})\|x\|_1$ for some $\beta > 0$. The crucial aspect of this proof is implementation of an inductive procedure on the Ito formula which is more natural in the set-up of quantum stochastic flows with unbounded coefficients than the usual iterative procedure. In this chapter, we also prove a Trotter-Kato product formula for quantum stochastic flows. Given two quantum stochastic flows, say $j_t^{(1)}$ and $j_t^{(2)}$, with noise spaces k_1 and k_2 respectively, we form their 'Trotter product' $\phi_t^{(n)}(\cdot) \in \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_1 \oplus k_2)))$ and give several sufficient conditions for its convergence in the weak as well as strong operator topology.

In Chapter 3 we investigate the problem of dilating semigroups of completely positive (CP for short) maps with unbounded generators. We employ the results of Chapter 2 to generalize a previously known dilation result (due to Goswami, Sahu and Sinha) of an important class of CP semigroups with unbounded generator, constructed on uniformly hyper-finite (UHF for short) algebras, which arise in many physical context. In this chapter, we also prove the existence of dilation of arbitrary symmetric (with respect to the canonical trace) quantum dynamical semigroup on type-I factors, which are implemented by unitary cocycles satisfying Hudson-Parthasarathy type equations.

The main theme of Chapter 4 is to explore the possibility of connecting quantum probability with non-commutative geometry through the theory of quantum isometry group in the sense of [27], in the spirit of classical stochastic geometry. We formulate and study various aspects of quantum Brownian motion including an analytic counterpart of Schürmann type (see [61]) construction, its behaviour with respect to Rieffel deformation etc. We also give some explicit computations of generators of quantum Brownian motion on some well-known quantum isometry groups and their homogeneous spaces. Finally we formulate a general principle of quantum exit-time asymptotics and as a case study, we explicitly compute these asymptotics on non-commutative 2-torus, and try to give possible interpretations of mean curvature, intrinsic dimension, extrinsic dimension etc.

Chapter 1

Preliminaries

1.1 Operator algebras and Hilbert modules

We presume the reader's familiarity with the theory of operator algebras and Hilbert modules. However, for the sake of completeness, we give a sketchy review of some basic definitions and facts and refer to [66, 18] for the details. Throughout this thesis, all algebras will be over $\mathbb{C}$ unless otherwise mentioned.

1.1.1 C^*-algebras

Definition 1.1.1. *A complex $*$-algebra $\mathcal{A}$, equipped with a C^*-norm, i.e. $\|x^*x\| = \|x\|^2$, is called a pre-C^*-algebra. Furthermore, if $\mathcal{A}$ is complete with respect to the C^*-norm, then it is called a C^*-algebra.*

A C^*-algebra is called unital or non-unital depending upon the existence of identity element on it. The following result completely characterises the commutative C^*-algebras:

Theorem 1.1.2. *(Gelfand-Naimark) Let $\mathcal{A}$ be a commutative C^*-algebra. Then there exists a locally compact Hausdorff space X such that $\mathcal{A}$ is isometrically isomorphic to the C^*-algebra $C_0(X)$. Moreover if $\mathcal{A}$ is unital, then X is compact.*

Any non-unital C^*-algebra can be isometricaly embedded as a non-degenerate two sided ideal in a unital C^*-algebra canonically. The multiplier algebra of a possibly non-unital C^* algebra, denoted by $\mathcal{M}(\mathcal{A})$ is the largest C^* algebra containing $\mathcal{A}$ as a non-degenerate two sided ideal. Suppose that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, the embedding being non-degenerate. Then $\mathcal{M}(\mathcal{A}) = \{a \in \mathcal{B}(\mathcal{H}) : \; ax, xa \in \mathcal{A} \; \forall \; x \in \mathcal{A}\}$. There is a canonical locally convex topology on $\mathcal{M}(\mathcal{A})$ called the *strict topology*, which is generated by the family of seminorms $\{\| \cdot \|_a, \| \cdot \|^a\}_{a \in \mathcal{A}}$, where for $x \in \mathcal{B}(\mathcal{H}), a \in \mathcal{A}$,

$\|x\|_a := \|ax\|$ and $\|x\|^a := \|xa\|$. For the rest of the section, we will consider unital C^*-algebras. Every C^*-algebra has an approximate identity, i.e. a net $(E_\lambda)_{\lambda\in\Lambda} \in \mathcal{A}$ such that $\lim_{\lambda\in\Lambda} E_\lambda A = \lim_{\lambda\in\Lambda} AE_\lambda = A$.

For $x \in \mathcal{A}$, the spectrum of x denoted by $\sigma(x)$ is the complement of the set $\{\lambda \in \mathbb{C} : (x-\lambda)^{-1} \in \mathcal{A}\}$. An element in a C^* algebra $\mathcal{A}$ is called self-adjoint if $x^* = x$, normal if $x^*x = xx^*$, unitary if $x^* = x^{-1}$, projection if $x^* = x = x^2$ and positive if $x = y^*y$ for some $y \in \mathcal{A}$. For any element $x \in \mathcal{A}$, there is a holomorphic functional calculus for x which sends any function $f \in H(\Omega)$, where Ω is any open set containing $\sigma(x)$ to $f(x) \in \mathcal{A}$, defined by $f(x) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{\omega - x} d\omega$, where Γ is any closed curve surrounding the spectrum. If x is normal, holomorphic functional calculus is a special case of the continuous functional calculus, which sends any function $f \in C(\sigma(x))$ to $f(x) \in C^*(x)$, such that $f \to f(x)$ is an isometric $*$-homomorphism between $C(\sigma(x))$ and $C^*(x)$, where $C^*(x)$ is the C^*-algebra generated by x.

1.1.2 von-Neumann algebras

We recall that for a Hilbert space $\mathcal{H}$, **the strong operator topology (SOT), the weak operator topology (WOT) and the ultraweak topology** are the locally convex topologies given by family of seminorms $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ respectively, where $\mathcal{F}_1 := \{p_\xi |\ \xi \in \mathcal{H}\}$, $\mathcal{F}_2 := \{p_{\xi,\eta} |\ \xi, \eta \in \mathcal{H}\}$, $\mathcal{F}_3 := \{p_\rho |\ \rho$ is a trace class operator$\}$ and $p_\xi(x) := \|x\xi\|$, $p_{\xi,\eta}(x) = \langle \xi, x\eta\rangle$, $p_\rho(x) = Tr(\rho x)$, for $x \in \mathcal{B}(\mathcal{H})$, where Tr is the usual trace on $\mathcal{B}(\mathcal{H})$. SOT convergence, as well as ultraweak convergence are stronger than WOT convergence. However, on bounded subsets of $\mathcal{B}(\mathcal{H})$, WOT convergence and ultraweak convergence coincide.

For any subset $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$, let $\mathcal{B}'$ denote the commutant of $\mathcal{B}$ in $\mathcal{B}(\mathcal{H})$. A $*$-subalgebra $\mathfrak{N} \subseteq \mathcal{B}(\mathcal{H})$ is called a von-Neumann algebra, if $\mathfrak{N} = \mathfrak{N}''(:= (\mathfrak{N}')')$, which is equivalent to the fact that $\mathfrak{N}$ is closed in any of the above three topologies. It is worthwhile to mention that in general, given a $*$ subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, which is closed in any of the above three topologies, we need not have $\mathcal{A} = \mathcal{A}''$. It will be so if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is non-degenerate, so that in particular, unital algebras will satisfy this. Henceforth all embeddings $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ will be taken to be non-degenerate. A von-Neumann algebra $\mathfrak{N}$ is closed under L^∞ functional calculus i.e. given a normal element $x \in \mathfrak{N}$ and a bounded borel measurable function f, $f(x) \in \mathfrak{N}$. As a result, it follows that a von-Neumann algebra $\mathfrak{N}$ contains enough projections and unitaries i.e. it is the SOT closure of the $*$-algebra generated by all projections (unitaries) in $\mathfrak{N}$. Furthermore, any self adjoint element of a von-Neumann algebra can be written as a difference of two positive elements and any element in a von-Neumann algebra can be written as sum of four unitary elements in it.

On the positive cone $\mathfrak{N}_+$ of a von-Neumann algebra $\mathfrak{N}$, there exists a partial order denoted by " $\leq$ " defined as: For $a, b \in \mathfrak{N}_+$, $a \leq b$ if $b - a \geq 0$. Let $\Phi : \mathfrak{N}_1 \to \mathfrak{N}_2$ be a positive map between von-Neumann algebras $\mathfrak{N}_1$ and $\mathfrak{N}_2$. We will call Φ normal, if the following happens:

if $(x_\alpha)_\alpha$ is an increasing (in the partial order defined above) net of positive elements in $\mathfrak{N}_1$, we have $sup_\alpha \Phi(x_\alpha) = \Phi(sup_\alpha x_\alpha)$.

It is known that a positive linear map is normal if and only if it is ultraweakly continuous. In view of this, we will call a bounded map between two von-Neumann algebras normal if it is continuous in the ultraweak topology.

Proposition 1.1.3. *[62, 66, 18] A functional ϕ on a von-Neumann algebra $\mathfrak{N} \subseteq \mathcal{B}(\mathcal{H})$ is normal if and only if there exists a trace class operator ρ on $\mathcal{H}$ such that $\phi(x) = Tr(\rho x)$.*

It is known that if a normal functional ϕ is $*$ preserving, then $\phi = \phi_+ - \phi_-$, where each of ϕ_+ and ϕ_- are positive normal functionals.

Proposition 1.1.4. *[62, 66] Given a normal homomorphism $\pi : \mathfrak{N}(\subseteq \mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$, there exists a pair (Γ, k) where k is a Hilbert space and Γ is a partial isometry from $\mathcal{K}$ to $\mathcal{H} \otimes k$ such that $\pi(x) = \Gamma^*(x \otimes 1_k)\Gamma$, and the projection $\Gamma\Gamma^*$ commutes with $x \otimes 1_k$ for all $x \in \mathfrak{N}$. Moreover if π is unital, Γ is an isometry. In case $\mathcal{H}$ is separable, one can choose k to be seperable as well.*

A closed densely defined linear operator B on $\mathcal{H}$, with the polar decomposition $B = V|B|$ is affiliated to the von-Neumann algebra $\mathfrak{N}$ if $V \in \mathfrak{N}$ and $f(|B|) \in \mathfrak{N}$ for any bounded borel measurable function $f : \mathbb{R}_+ \to \mathbb{R}$.

Given a faithful, semifinite, normal trace τ on a von-Neumann algebra $\mathfrak{N}$, there exists a notion of non-commutative L^p-spaces. For $1 \le p < \infty$ and $x \in Dom(\tau)$, $\|x\|_p := (\tau(|x|^p))^{\frac{1}{p}}$, where $|x| := \sqrt{x^*x}$, defines a norm on $Dom(\tau)$. The closure of $Dom(\tau)$ under this norm is denoted by $L^p(\tau)$. It shares many natural similar properties with the L^p-spaces of measures. We denote $\mathfrak{N}$ by $L^\infty(\tau)$.

We conclude this section with examples of traces which are semifinite but not finite.

(a) Consider $L^\infty(\mathbb{R})$ with respect to the usual Lebesgue measure on $\mathbb{R}$. Integration with respect to Lebesgue measure is an example of a trace on $L^\infty(\mathbb{R})$ which is semifinite but not finite e.g. the function $f(x) = \frac{1}{x}$, $x \in [1, +\infty)$ and zero elsewhere, is in $L^\infty(\mathbb{R})$ but not integrable.

(b) Let $\mathcal{H}$ be a seperable Hilbert space. It is a well-known fact that $B(\mathcal{H})$ has a unique (upto a constant) faithful normal trace. This trace is not finite e.g. Consider the following compact operator:$\sum_{i=1}^\infty \frac{1}{i}|e_i >< e_i|$, where $\{e_i\}_{i=1}^\infty$ is an orhtonormal basis for $\mathcal{H}$ and $|e_i >< e_i|$ is the rank one projection on $\mathcal{H}$, defined by $|e_i >< e_i|(x) := \langle e_i, x\rangle e_i$, This compact operator is not of trace class as $\sum_{i=1}^\infty \frac{1}{i} = \infty$. Thus the trace in $B(\mathcal{H})$ is indeed semifinite.

1.2 Tensor product of Banach spaces

Here we collect a few facts about the minimal tensor product and the projective tensor product of Banach spaces and algebras (see [66]) which is an important technical tool.

For two Banach spaces E_1, E_2, there are generally many possible choices of a 'cross-norm' $\|\cdot\|$, on $E_1 \otimes_{alg} E_2$ i.e. one which satisfies $\|\xi \otimes \eta\| = \|\xi\|\|\eta\|$. The smallest and the largest of such norms are called the injective (or minimal) and projective (or maximal) norm respectively.

The projective norm $\|\cdot\|_\gamma$ is given by

$$\|X\|_\gamma = \inf \sum_i \|x_i\|\|y_i\|,$$

where infimum is taken over all possible expressions of X of the form $X = \sum_{i=1}^n x_i \otimes y_i$. The completion of $E_1 \otimes_{alg} E_2$ under this norm is called the projective tensor product of E_1 and E_2 and is denoted by $E_1 \otimes_\gamma E_2$.

It is easy to see that a linear functional ϕ on $E_1 \otimes_{\text{alg}} E_2$ (equivalently, a bilinear functional ϕ on $E_1 \times E_2$) extends to a bounded linear functional on $E_1 \otimes_\gamma E_2$ if and only if there is some constant C such that $|\phi(x \otimes y)| \leq C\|x\|\|y\|$ for all $x \in E_1, y \in E_2$.

Lemma 1.2.1. *Suppose $T_j \in \mathcal{B}(E_j, F_j)$ where E_j, F_j, for $j = 1, 2$ are Banach spaces. Then $T_1 \otimes_{alg} T_2$ extends to a bounded operator*

$$T_1 \otimes_\gamma T_2 : E_1 \otimes_\gamma E_2 \longrightarrow F_1 \otimes_\gamma F_2$$

with bound

$$\|T_1 \otimes_\gamma T_2\| \leq \|T_1\|\|T_2\|.$$

Proof. The proof is an easy consequence of the estimate:

$$\begin{aligned} &\|(T_1 \otimes_{alg} T_2)(\sum_{i=1}^k x_i \otimes y_i)\|_\gamma \leq \sum_{i=1}^k \|T_1(x_i) \otimes T_2(y_i)\|_\gamma \\ &\leq \sum_{i=1}^k \|T_1(x_i)\|_{F_1}\|T_2(y_i)\|_{F_2} \leq \|T_1\|\|T_2\| \sum_{i=1}^k \|x_i\|_{E_1}\|y_i\|_{E_2}. \end{aligned} \tag{1.1}$$

□

Lemma 1.2.2. *Suppose T_t and S_t are two C_0 semigroups of bounded operators on E_1 & E_2 with generators L_1 and L_2 respectively. Then $T_t \otimes_\gamma S_t$ becomes a C_0 semigroup of operators on $E_1 \otimes_\gamma E_2$ whose generator is the closed extension of the operator $L_1 \otimes_{alg} 1 + 1 \otimes_{alg} L_2$, defined on $Dom(L_1) \otimes_{alg} Dom(L_2)$ in the space $E_1 \otimes_\gamma E_2$.*

Proof. Since $(T_t \otimes_\gamma S_t) \circ (T_s \otimes_\gamma S_s) = (T_{t+s} \otimes_\gamma S_{t+s})$ (in $E_1 \otimes_{alg} E_2$), both sides being continuous in $E_1 \otimes_{alg} E_2$, and as $E_1 \otimes_{alg} E_2$ is dense in $E_1 \otimes_\gamma E_2$, the above identity extends by Lemma (1.2.1) to $E_1 \otimes_\gamma E_2$ leading to the semigroup property

$$(T_t \otimes_\gamma S_t) \circ (T_s \otimes_\gamma S_s) = (T_{t+s} \otimes_\gamma S_{t+s}).$$

Also similar reasoning gives us $(T_t \otimes_\gamma 1) \circ (1 \otimes_\gamma S_t) = T_t \otimes_\gamma S_t$ and thus the strong continuity of $T_t \otimes_\gamma 1$ as well as of $1 \otimes_\gamma S_t$ as a function of t yields the strong continuity of $T_t \otimes_\gamma S_t$. Hence $(T_t \otimes_\gamma S_t)_{t\geq 0}$ is a C_0 semigroup. Moreover $T_t \otimes_\gamma S_t$ keeps $Dom(L_1) \otimes_{alg} Dom(L_2)$ invariant, which is dense in $E_1 \otimes_\gamma E_2$. Thus $Dom(L_1) \otimes_{alg} Dom(L_2)$ is a core for the generator of $T_t \otimes_\gamma S_t$ (see [19]) which is the closure of the operator $L_1 \otimes_{alg} 1 + 1 \otimes_{alg} L_2$ denoted by $L_1 \otimes_\gamma 1 + 1 \otimes_\gamma L_2$ □

We state without proof the following corollary:

Corollary 1.2.3. *The operator $L_1 \otimes_{alg} I$ defined on $Dom(L_1) \otimes_{alg} E_2$ is closable in $E_1 \otimes_\gamma E_2$. Similar results hold for $I \otimes_{alg} L_2$ on $E_1 \otimes_{alg} Dom(L_2)$. We denote the respective closures by $L_1 \otimes_\gamma I$ and $I \otimes_\gamma L_2$.*

When $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H}_2)$ are two C^*-algebras with the embeddings being non-degenerate, one will be naturally interested in those cross-norms on $\mathcal{A} \otimes_{alg} \mathcal{B}$ which also have the C^* property i.e. $\|X^*X\| = \|X\|^2$ for all $X \in \mathcal{A} \otimes_{alg} \mathcal{B}$. The smallest C^* cross-norm on $\mathcal{A} \otimes_{alg} \mathcal{B}$ is called the injective C^*-norm on $\mathcal{A} \otimes_{alg} \mathcal{B}$ and the completion under it will be denoted by $\mathcal{A} \otimes \mathcal{B}$. This is the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with respect to the C^* norm inherited from $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Remark 1.2.4. *When $\mathcal{A}$ and $\mathcal{B}$ are von-Neumann algebras, we do* **NOT** *view them merely as C^* algebras and thus their canonical tensor product which we will denote again by $\mathcal{A} \otimes \mathcal{B}$ will be the von-Neumann algebra obtained by taking the closure of the algebra $\mathcal{A} \otimes_{alg} \mathcal{B}$ with respect to the strong operator topology inherited from $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.*

At this point, we fix the convention that if $\mathcal{A}, \mathcal{B}$ are vector spaces or algebras without any underlying topology, $\mathcal{A} \otimes \mathcal{B}$ will mean the algebraic tensor product. On the other hand if $\mathcal{A}, \mathcal{B}$ are C^* (von-Neumann) algebras, $\mathcal{A} \otimes \mathcal{B}$ will denote injective (von-Neumann algebra) tensor product.

1.2.1 Positive maps and states on C^*-algebras

Definition 1.2.5. *Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{B}$ be a C^*-algebra. A linear map $T : \mathcal{A} \to \mathcal{B}$ is called a positive map, if $T(x^*x) \geq 0$. It will be called completely positive map (CP for short), if $T \otimes I_n : \mathcal{A} \otimes M_n(\mathbb{C}) \to \mathcal{B} \otimes M_n(\mathbb{C})$ is positive for all n.*

Definition 1.2.6. *Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{B}$ be a $*$-subalgebra of $\mathcal{A}$. A map $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ is called Conditionally Completely positive (CCP for short) if we have*

$$\sum_{i=1}^{n} b_i^* \mathcal{L}(a_i^* a_j) b_j \geq 0$$

for $a_i, b_i \in \mathcal{A},\ i = 1, 2, ...n$, whenever $\sum_{i=1}^{n} a_i b_i = 0$.

Positive maps are real i.e. $T(x^*) = T(x)^*$. It is a well-known fact [64] that a map $T : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ are C^*-algebras, is CP if and only if it is of the form $V^*\pi(x)V$, where $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a C^*-representation, $\mathcal{H}, \mathcal{K}$ being Hilbert spaces. Furthermore, if any of the C^*-algebras $\mathcal{A}$ or $\mathcal{B}$ be abelian, then any positive map $T : \mathcal{A} \to \mathcal{B}$ will also be a CP map. For a CP map say T from $\mathcal{A}$ to $\mathcal{B}$, $\mathcal{A}, \mathcal{B}$ being unital $*$-algebras, we have the following analogue of Cauchy-Schwartz inequality:

$$T(x)^*T(x) \leq \|T(1)\| T(x^*x)$$

for all $x \in \mathcal{A}$. A CP map between two C^*-algebras $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ is called non-degenerate if for $\xi \in \mathcal{H}$, $T(x)\xi = 0$ for all $x \in \mathcal{A}$ implies $\xi = 0$.

Henceforth, $\mathcal{A}$ will be a C^*-algebra, unless mentioned otherwise. In the above discussion, if $\mathcal{B} = \mathbb{C}$, then T will be called a positive functional. It is known that if ϕ is a positive functional on $\mathcal{A}$, then $\|\phi\| = \phi(1)$. Moreover, any Hahn-Banach extension of a positive functional defined on a unital C^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$, to $\mathcal{A}$, will be a positive functional on $\mathcal{A}$. A positive functinal ϕ on $\mathcal{A}$ will be called a state if $\phi(1) = 1$. A state ϕ on $\mathcal{A}$ will be called a trace if $\phi(ab) = \phi(ba)$ for all $a, b \in \mathcal{A}$ (e.g. the usual normalized trace on $M_n(\mathbb{C})$ is a state) and faithful if $\phi(x) = 0 \Rightarrow x = 0$ for all $x \in \mathcal{A}$ such that $x \geq 0$. Faithful states may not exist in general but if $\mathcal{A}$ is seperable, it is possible to construct a faithful state on it. Every state ϕ on a C^*-algebra $\mathcal{A}$ gives rise to a triple $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ (called a GNS triple), where $\mathcal{H}_\phi$ is a Hilbert space, $\pi_\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$ is a C^*-representation and ξ_ϕ is a cyclic vector for $\pi_\phi(\mathcal{A})$, so that we have $\phi(a) = \langle \xi_\phi, \pi_\phi(a)\xi_\phi \rangle$. Such triples are

unique upto unitary equivalence. Let $\mathcal{S}(\mathcal{A})$ denote the collection of all states on $\mathcal{A}$. The representation $\widetilde{\pi} := \oplus_{\phi \in \mathcal{S}(\mathcal{A})} \pi_\phi$ is called the universal represenation of $\mathcal{A}$ and the Hilbert space $\widetilde{\mathcal{H}} := \oplus_{\phi \in \mathcal{S}(\mathcal{A})} \mathcal{H}_\phi$ is called the universal Hilbert space of $\mathcal{A}$. $\widetilde{\pi}$ is a non-degenerate embedding of $\mathcal{A}$ in $\mathcal{B}(\widetilde{\mathcal{H}})$. The von-Neumann algebra $\widetilde{\mathcal{A}} := \widetilde{\pi}(\mathcal{A})''$ is called the universal enveloping von-Neumann algebra of $\mathcal{A}$. It has the following universal proerty:
If $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ be any representation of $\mathcal{A}$ in the Hilbert space $\mathcal{K}$, then there exists a unique normal $*$-homomorphism $\phi : \widetilde{\mathcal{A}} \to \pi(\mathcal{A})''$ such that $\phi \circ \widetilde{\pi} = \pi$. Thus in particular, by virtue of Stinespring's theorem, any CP map $T : \mathcal{A} \to \mathcal{B}$ lifts to a normal CP map, again denoted by T such that $T : \widetilde{\mathcal{A}} \to \mathcal{B}$, for any von-Neumann algebra $\mathcal{B}$.

The extreme points of $\mathcal{S}(\mathcal{A})$ are called pure states. If $\mathcal{A}$ is abelian, the pure states are the complex valued homomorphisms of $\mathcal{A}$. Note that in general pure states on $\mathcal{A}$ need not be multiplicative. For a pure state ϕ, the GNS representation π_ϕ associated with ϕ is irreducible. Conversely, given an irreducible representation π of $\mathcal{A}$, there exists a pure state ϕ such that $\pi = \pi_\phi$. The pure states (equivalently the irreducible representations of $\mathcal{A}$) are point separating on $\mathcal{A}$, i.e. given a $x \in \mathcal{A}$, $x \neq 0$, we can get a pure state ϕ (equivalently an irreducible representation π of $\mathcal{A}$) such that $\phi(a) \neq 0$ ($\pi(a) \neq 0$.) For a normal element $x \in \mathcal{A}$, we have $\sigma(x) = \{\phi(a) : \phi$ is a pure state on $\mathcal{A}\}$.

For a Hilbert space $\mathcal{H}$, let $\mathcal{K}(\mathcal{H})$ denote the non-unital C^* algebra of compact operators on $\mathcal{H}$. It is known that $\mathcal{K}(\mathcal{H})$ is simple and any irreducible representation of $\mathcal{K}(\mathcal{H})$ is unitarily equivalent to the identity representation, so that any non-degenerate representation of $\mathcal{K}(\mathcal{H})$ can be extended to a normal representation of $\mathcal{B}(\mathcal{H})$.

1.2.2 Hilbert C^*- modules

Given a $*$-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, a semi-Hilbert $\mathcal{A}$-module E is a right $\mathcal{A}$-module equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}$ satisfying $\langle x, y \rangle^* = \langle y, x \rangle$, $\langle x, ya \rangle = \langle x, y \rangle a$ and $\langle x, x \rangle \geq 0$ for $x, y \in E$ and $a \in \mathcal{A}$. A semi-Hilbert module E is called a pre-Hilbert module if $\langle x, x \rangle = 0$ if and only if $x = 0$; and it is called a Hilbert C^*-module if furthermore E is complete in the norm $x \to \|\langle x, x \rangle\|^{\frac{1}{2}}$ where $\| \cdot \|$ is the C^* norm of $\mathcal{A}$.

Let E, F be two Hilbert $\mathcal{A}$ modules. We say that a $\mathbb{C}$-linear map $L : E \to F$ is adjointable if there exists a $\mathbb{C}$-linear map $L^* : F \to E$ such that $\langle L(x), y \rangle = \langle x, L^*(y) \rangle$. We call L^* the adjoint of L. The set of all adjointable maps between E and F is denoted by $\mathcal{L}(E, F)$. In case $E = F$, we write $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$. For an adjointable map L, both L and L^* are automatically $\mathcal{A}$ linear and norm bounded maps between Banach spaces. We say that $L \in \mathcal{L}(E, F)$ is an isometry if $\langle L(x), L(y) \rangle = \langle x, y \rangle$. It is called unitary if it is an isometry and $Ran(L) = F$. On $\mathcal{L}(E, F)$, we may define a norm by $\|L\| := sup_{x \in E, \ \|x\| \leq 1} \|L(x)\|$, which becomes a C^* norm for $\mathcal{L}(E)$.

The simplest example of Hilbert $\mathcal{A}$ modules are the so called trivial $\mathcal{A}$ modules of the form $\mathcal{H} \otimes \mathcal{A}$, where $\mathcal{H}$ is a Hilbert space with an $\mathcal{A}$ valued sesquilinear form defined on $\mathcal{H} \otimes_{alg} \mathcal{A}$ by: $\langle \xi \otimes a, \xi' \otimes a' \rangle = \langle \xi, \xi' \rangle a^* a'$. The completion of $\mathcal{H} \otimes_{alg} \mathcal{A}$ with respect to this pre Hilbert module structure is a Hilbert $\mathcal{A}$ module and is denoted by $\mathcal{H} \otimes \mathcal{A}$.

We state the following proposition, whose proof being straightforward is omitted. For details, we refer to [36]

Proposition 1.2.7. *1. Suppose that $\mathcal{A}$ is a C^* algebra and $\mathcal{H}$ is a Hilbert space such that $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a non-degenerate CP map, which is continuous with respect to norm topology on $\mathcal{A}$ and strong operator topology on $\mathcal{B}(\mathcal{H})$. Then ϕ can be extended to a strictly continuous (i.e. continuous with respect to the strict topology on $\mathcal{M}(\mathcal{A})$) CP map (which we again denote by ϕ) from $\mathcal{M}(\mathcal{A})$ to $\mathcal{B}(\mathcal{H})$.*

2. For a Hilbert space k', define $\hat{\phi} := \phi \otimes id_{\mathcal{K}(k')} : \mathcal{A} \otimes_{alg} \mathcal{K}(k') \to \mathcal{B}(\mathcal{H}) \otimes_{alg} \mathcal{K}(k')$. Then $\hat{\phi}$ extends to a strictly continuous map (which we again denote by $\hat{\phi}$) from $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}(k'))$ ($\cong \mathcal{L}(\mathcal{A} \otimes k')$) to $\mathcal{B}(\mathcal{H} \otimes k')$.

Let $\mathcal{A}$ be a C^* algebra and $\mathcal{H}, k'$ be Hilbert spaces.

Lemma 1.2.8. *Suppose that $(\phi_t)_{t \geq 0}$ is a family of CP contractive maps, $\phi_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Assume that $\lim_{t \to s} \phi_t(x) = \phi_s(x)$ in the strong operator topology of $\mathcal{B}(\mathcal{H})$. Define $\hat{\phi}_t : \phi_t \otimes id_{\ k'} : \mathcal{A} \otimes_{alg} \ k' \to \mathcal{B}(\mathcal{H}) \otimes_{alg} \ k'$. Then*

1. *$\hat{\phi}_t$ extends to a contractive map (which we again denote by $\hat{\phi}_t$) from the C^* Hilbert module $\mathcal{A} \otimes k'$ to the Banach space $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes k')$.*

2. *For $X \in \mathcal{A} \otimes k'$, $t \to \hat{\phi}_t(X)$ is continuous with respect to the strong operator topology of $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes k')$.*

Proof. Proof of (1):
Let $(e_\alpha)_{\alpha \in \mathfrak{J}}$ be an orthonormal basis of k'. The Hilbert C^* module $\mathcal{A} \otimes k'$ consists of elements of the form $X = \sum_\alpha x_\alpha \otimes e_\alpha$, $x_\alpha \in \mathcal{A}$ such that $\sum_\alpha x_\alpha^* x_\alpha$ is convergent in the norm of $\mathcal{A}$. Let $X_F := \sum_{\alpha \in F} x_\alpha \otimes e_\alpha$, where $F \subseteq \mathfrak{J}$ is a finite set. By definition, $\hat{\phi}_t(X_F) = \sum_{\alpha \in F} \phi_t(x_\alpha) \otimes e_\alpha$. For $\xi \in \mathcal{H}$, using complete positivity of the maps ϕ_t, we have $\|\hat{\phi}_t(X_{F'} - X_F)\xi\|^2_{\mathcal{H} \otimes k'} \leq \langle \phi_t(\sum_{\alpha \in F' \cap F^c} x_\alpha^* x_\alpha)\xi, \xi \rangle \leq \|\xi\|^2 \| \sum_{\alpha \in F' \cap F^c} x_\alpha^* x_\alpha \|$, for finite sets F, F' such that $F \subseteq F'$. This goes to zero as the sets F, F' increase to $\mathfrak{J}$. This proves (1).

Proof of (2):
Let $t \geq 0$ and $\xi \in \mathcal{H}$. Fix $X \in \mathcal{A} \otimes k'$ such that $X = \sum_\alpha x_\alpha \otimes e_\alpha$ and as before,

let $X_F := \sum_{\alpha \in F} x_\alpha \otimes e_\alpha$, where $F \subseteq \mathfrak{I}$ is a finite subset. Choose F so that $\|X - X_F\|_{\mathcal{A}\otimes k'} < \frac{\epsilon}{3}$. We have

$$\begin{aligned}&\hat{\phi}_t(X) - \hat{\phi}_s(X)\\ &= \hat{\phi}_t(X - X_F) - \hat{\phi}_s(X - X_F) + \hat{\phi}_t(X_F) - \hat{\phi}_s(X_F).\end{aligned}$$

Since $\hat{\phi}_t$ is contractive, $\|\hat{\phi}_t(X - X_F)\|_{\mathcal{A}\otimes k'} < \frac{\epsilon}{3}$ and $\|\hat{\phi}_s(X - X_F)\|_{\mathcal{A}\otimes k'} < \frac{\epsilon}{3}$. As $X_F \in \mathcal{A} \otimes_{alg} k'$, by our hypothesis, there exists $\delta > 0$, small, depending on $t, \xi \in \mathcal{H}$, such that for all s with $|t - s| < \delta$, $\|[\hat{\phi}_t(X_F) - \hat{\phi}_s(X_F)]\xi\|_{\mathcal{H}\otimes k'} < \frac{\epsilon}{3}$. Thus we have $\|[\hat{\phi}_t(X) - \hat{\phi}_s(X)]\xi\|_{\mathcal{H}\otimes k'} < \epsilon$. This proves (2). □

1.2.3 Hilbert von-Neumann modules

If $\mathfrak{N} \subseteq B(h)$ is a non-degenrate von-Neumann algebra for some Hilbert space h, a right Hilbert $\mathfrak{N}$ module E is called a Hilbert von-Neumann module, if it is equipped with the weakest possible locally convex topology such that the map $\xi(\in E) \to \langle \xi, \xi \rangle^{\frac{1}{2}}(\in \mathfrak{N})$ is continuous (with respect to ultraweak topology on $\mathfrak{N}$) and E is complete in this topology.

Lemma 1.2.9. *[62] Any element X of the von-Neumann Hilbert module $\mathcal{H} \otimes \mathfrak{N}$ can be written as $X = \sum_{\alpha \in J} y_\alpha \otimes x_\alpha$, where $(y_\alpha)_{\alpha \in J}$ is an orthonormal basis of $\mathcal{H}$ and $x_\alpha \in \mathfrak{N}$. The above sum possibly over an uncountable index set J makes sense in the usual way: it is strongly convergent and for all $u \in h$, there exists an at most countable subset J_u of J such that $Xu = \sum_{\alpha \in J_u} y_\alpha \otimes x_\alpha u$. Moreover, once $(y_\alpha)_{\alpha \in J}$ is fixed, $x'_\alpha s$ are uniquely determined by X.*

Lemma 1.2.10. *[62] Let $\mathfrak{N} \subseteq \mathcal{B}(h)$ and $X \in \mathcal{B}(h, h \otimes \mathcal{H})$. Then X belongs to $\mathcal{H} \otimes \mathfrak{N}$ if and only if $\langle \gamma, X \rangle \in \mathfrak{N}$ for all γ in some dense subset $\mathcal{D}$ of $\mathcal{H}$.*

1.3 Some general theory of Semigroups on Banach spaces

Here we collect a few facts about semigroup of operators on locally convex spaces. For details we refer to [69, 19].

Definition 1.3.1. *Let $\mathcal{X}$ be a locally convex space. A one parameter family of bounded linear operators on $\mathcal{X}$, say $(T_t)_{t \geq 0}$, is called a semigroup of operators if it satisfies $T_t \cdot T_s = T_{t+s}$ for all $t, s \geq 0$, and $T_0 = I$. The semigroup is called C_0 semigroup or strongly continuous semigroup if we have $\lim_{t \downarrow 0} T_t(a) = a$ for all $a \in \mathcal{X}$.*

For a C_0 semigroup of oeprators $(T_t)_{t\geq 0}$, we define a linear operator $\mathcal{L}$ on $\mathcal{X}$ as follows:

$Dom(\mathcal{L}) := \{x \in \mathcal{X}| \ \lim_{t\downarrow 0} \frac{T_t(x)-x}{t}$ exists$\}$ and for $x \in Dom(\mathcal{L})$, $\mathcal{L}(x) := \lim_{t\downarrow 0} \frac{T_t(x)-x}{t}$. It is known that $Dom(\mathcal{L})$ is dense in $\mathcal{X}$ and the operator $\mathcal{L}$ becomes a closed densely defined operator. $\mathcal{L}$ is called the infinitesimal generator of $(T_t)_{t\geq 0}$. If $\mathcal{X}$ is a Banach space, a C_0 semigroup $(T_t)_{t\geq 0}$ on $\mathcal{X}$ is quasi-bounded, i.e. there exist constants $M \geq 0$ and $\beta \geq 0$ such that $\|T_t\| \leq Me^{\beta t}$ for all $t \geq 0$. We use the notation $\mathcal{G}(M,\beta)$ to denote the collection of all C_0 semigroups which are quasi-bounded with constants M,β. We will call $(T_t)_{t\geq 0} \in \mathcal{G}(M,\beta)$ contractive, if $M = 1$. For $(T_t)_{t\geq 0} \in \mathcal{G}(M,\beta)$, the set $\{\lambda \in \mathbb{C}| \ Re(\lambda) > \beta\}$ is contained in the resolvent of $\mathcal{L}$, such that $(\lambda - \mathcal{L})^{-1} = \int_0^\infty dte^{-\lambda t}T_t$, for $Re(\lambda) > \beta$. For $\lambda \in \mathbb{C}$ with $Re(\lambda) > \beta$, we have $Ran(\lambda - \mathcal{L}) = \mathcal{X}$. It is known [19] that a dense subspace $\mathcal{D} \subseteq Dom(\mathcal{L})$ is a core for $\mathcal{L}$ if $T_t(\mathcal{D}) \subseteq \mathcal{D}$ for all $t \geq 0$.

The following theorem due to Hille and Yosida completely characterizes the generators of C_0 semigroup of operators on Banach spaces:

Theorem 1.3.2. *(Hille-Yosida theorem) Let $(\mathcal{L}, Dom(\mathcal{L}))$ be a densely defined closed linear operator on a Banach space $\mathcal{X}$. Then $(\mathcal{L}, Dom(\mathcal{L}))$ is the generator of a quasi-bounded C_0 semigroup $(T_t)_{t\geq 0} \in \mathcal{G}(M,\beta)$ if and only if $\mathcal{L}$ satisfies*

$$\|(\lambda - \mathcal{L})^{-1}\| \leq \frac{M}{Re(\lambda) - \beta}, \ \textit{for some } \lambda \in \mathbb{C} \textit{ with } Re(\lambda) > \beta.$$

Let $\mathcal{H}(\omega,\beta)$ denote the set of all densely defined, closed linear map A on the Banach space $\mathcal{X}$, which has the property that for every $\epsilon > 0$, there exists a positive constant M_ϵ such that for all complex number ξ with $Re(\xi) > 0$ and $|arg(\xi)| \leq \frac{\pi}{2}+\omega-\epsilon$, the operator $(A-\beta-\epsilon)$ has a bounded inverse and $\|(A-\beta-\epsilon)^{-1}\| \leq \frac{M_\epsilon}{|\xi|}$. The semigroups generated by elements of $\mathcal{H}(\omega,\beta)$ for some ω,β are called holomorphic (analytic) semigroups.

We give an example of a semigroup which belongs to $\mathcal{G}(1,0)$ but is not a holomorphic semigroup:

Consider the Ornstein-Uhlenbeck operator given by:

$$\mathcal{L} = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \sum_{i,j=1}^{d} b_{ij} y_j \frac{\partial}{\partial x_i},$$

where $y_i \in \mathbb{R}, \ i = 1(1)d$ and $B := (b_{ij})$ is a non-zero self-adjoint matrix. Then the corresponding Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$ on $C_b(\mathbb{R}^d)$ (the C^* algebra of bounded continuous functions on $\mathbb{R}^d$) belongs to $\mathcal{G}(1,0)$ but it is not holomorphic (see [57]).

On the otherhand, the semigroup generated by the unbounded operator $\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ generates an analytic semigroup on $C_b(\mathbb{R}^d)$. It is infact the so-called 'Heat semigroup'.

Let $\mathcal{L}$ and A be two operators with same domain space $\mathcal{X}$ with $Dom(\mathcal{L}) \subseteq Dom(A)$. Then the operator A is said to be relatively bounded with respect to $\mathcal{L}$ if there exist non-negetive constants a and b such that

$$\|Ax\| \leq a\|x\| + b\|\mathcal{L}x\| \text{ for all } x \in Dom(\mathcal{L}).$$

The infimum of all possible constants b in the above inequality is called the bound of A relative to $\mathcal{L}$.

Proposition 1.3.3. *[19, 34] Given $A \in \mathcal{H}(\omega, \beta)$ and $\epsilon > 0$, there are positive constants γ, δ such that whenever B is A-bounded and $\|Bu\| \leq a\|u\| + b\|Au\|$ for all $u \in Dom(A)$, with $a < \delta$, $b < \delta$, then we have $A + B \in \mathcal{H}(\omega - \epsilon, \gamma)$.*

Proposition 1.3.4. *(Trotter product formula) Let A, B, Z be densely defined closed operators on a Banach space $\mathcal{X}$. Suppose that A generates $(T_t)_{t\geq 0}$, B generates $(S_t)_{t\geq 0}$ and Z generates $(P_t)_{t\geq 0}$, where each of these semigroups belongs to $\mathcal{G}(1, \beta)$. Furthermore, assume that there is a core $\mathcal{D}$ for Z such that $\mathcal{D} \subseteq Dom(A) \cap Dom(B)$ and $Z = A + B$ on $\mathcal{D}$. Then we have*

$$(T_{\frac{t}{n}} \cdot S_{\frac{t}{n}})^n(a) \to P_t(a) \textit{ for all } a \in \mathcal{X}.$$

1.4 Compact Quantum groups

In this section, we collect a few facts about compact quantum groups. For details, we refer to [46, 35, 9].

1.4.1 Hopf algebras

We recall that an associative algebra with an unit is a vector space $\mathcal{A}$ over $\mathbb{C}$, equipped with two linear maps $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ called multiplication and $\eta : \mathbb{C} \to \mathcal{A}$ called unit, such that $m\circ(m\otimes id) = m\circ(id\otimes m)$ and $m\circ(\eta\otimes id) = m\circ(id\otimes\eta) = id$. Dualizing this, we get the following definition:

Definition 1.4.1. *A coalgebra $\mathcal{A}$ is a vector space over $\mathbb{C}$ equipped with two linear maps $\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ called comultiplication and $\epsilon : \mathcal{A} \to \mathbb{C}$ called counit, such that:*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta,$$
$$(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta.$$

Definition 1.4.2. *Let $(\mathcal{A}, \Delta_{\mathcal{A}}, \epsilon_{\mathcal{A}})$ and $(\mathcal{B}, \Delta_{\mathcal{B}}, \epsilon_{\mathcal{B}})$ be two coalgebras. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called a cohomomorphism if we have*

$$\Delta_{\mathcal{B}} \circ \phi = (\phi \otimes \phi) \circ \Delta_{\mathcal{A}},$$

$$\epsilon_{\mathcal{A}} = \epsilon_{\mathcal{B}} \circ \phi.$$

Let $\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the flip map given by $\sigma(a \otimes b) = b \otimes a$.

Definition 1.4.3. *A coalgebra is called cocommutative if $\sigma \circ \Delta = \Delta$.*

Definition 1.4.4. *A linear subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a subcoalgebra if $\Delta(\mathcal{B}) \subseteq \mathcal{B}\otimes\mathcal{B}$.*

Sweedler notation: We introduce the so called Sweedler notation for comultiplication. Let $\mathcal{A}$ be a coalgebra and $a \in \mathcal{A}$. Then $\Delta(a)$ in $\mathcal{A} \otimes_{alg} \mathcal{A}$ is a finite sum, namely $\Delta(a) = \sum_i a_{1i} \otimes a_{2i}$, where for each i, a_{1i} and a_{2i} belong to $\mathcal{A}$. However such representation is not unique. For notational simplicity, we will write $\Delta(a) = a_{(1)} \otimes a_{(2)}$, where the subscripts 1 and 2 refer to the corresponding tensor factors.

Definition 1.4.5. *A vector space $\mathcal{A}$ is called a bialgebra if it is an algebra as well as a coalgebra along with the condition that $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and $\epsilon : \mathcal{A} \to \mathbb{C}$ are homomorphisms (or equivalently $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\eta : \mathbb{C} \to \mathcal{A}$ are cohomomorphisms.)*

Definition 1.4.6. *A bialgebra is called a Hopf algebra if there exist a linear map $\kappa : \mathcal{A} \to \mathcal{A}$, called the antipode or coinverse, satisfying $m \circ (\kappa \otimes id) \circ \Delta = \eta \circ \epsilon = m \circ (id \otimes \kappa) \circ \Delta$.*

Definition 1.4.7. *A Hopf $*$-algebra is a Hopf algebra $(\mathcal{A}, \Delta, \kappa, \epsilon)$ equipped with an involution $*$ such that $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a $*$ homomorphism.*

It is known that in a Hopf $*$-algebra $\mathcal{A}$, one has $\epsilon(a^*) = \overline{\epsilon(a)}$ and $\kappa((\kappa(a^*))^*) = a$ for all $a \in \mathcal{A}$.

1.4.2 Compact Quantum Groups: basic definitions and examples

Definition 1.4.8. *A* **compact quantum group** *(CQG for short) is a pair $(\mathcal{Q}, \Delta)$, where $\mathcal{Q}$ is a seperable unital C^*-algebra and $\Delta : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$ ($\otimes$ refers to injective tensor product) is a $*$ homomorphism such that:*

- $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$,
- *each of the sets $\Delta(\mathcal{Q})(\mathcal{Q} \otimes 1)$ and $\Delta(\mathcal{Q})(1 \otimes \mathcal{Q})$ is total in $\mathcal{Q} \otimes \mathcal{Q}$.*

It is well-known (see [46]) that there is a canonical dense $*$-subalgebra $\mathcal{Q}_0$ of $\mathcal{Q}$, consisting of the matrix elements of inequivalent unitary (co)-representation (to be defined shortly) of $\mathcal{Q}$ such that $\mathcal{Q}_0$ is a Hopf $*$-algebra. On $(\mathcal{Q}, \Delta)$, there exists a unique state h called the Haar state, satisfying $(h \otimes id) \circ \Delta(a) = (id \otimes h) \circ \Delta(a) = h(a)1$, for all $a \in \mathcal{Q}$. h is faithful on $\mathcal{Q}_0$. The Haar state is tracial if and only if $\kappa^2 = id$.

Definition 1.4.9. *Let $(\mathcal{Q}_1, \Delta_1)$ and $(\mathcal{Q}_2, \Delta_2)$ be two compact quantum groups. A CQG morphism $\pi : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a unital C^* homomorphism such that $(\pi \otimes \pi) \circ \Delta_1 = \Delta_2 \circ \pi$.*

It follows that in such a case, $\pi((\mathcal{Q}_1)_0) \subseteq (\mathcal{Q}_2)_0$, $\pi \circ \kappa_1 = \kappa_2 \circ \pi$ and $\epsilon_2 \circ \pi = \epsilon_1$, where $\epsilon_1, \kappa_1, \epsilon_2, \kappa_2$ are the counit and coinverse associated with $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively.

Suppose $(\mathcal{Q}, \Delta)$ is a CQG. A CQG $(\mathcal{A}, \Delta_{\mathcal{A}})$ is called a quantum subgroup of $\mathcal{Q}$ if there exists a surjective CQG morphism $\pi : \mathcal{Q} \to \mathcal{A}$.

Corepresentation of a compact quantum group: For a map $X \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we will use the notation $X_{(12)}$ to denote the operator $X \otimes I_{\mathcal{H}_3}$ and the notation $X_{(13)}$ to denote the operator $\Sigma_{23} X_{(12)} \Sigma_{23}$, where $\Sigma_{23} \in U(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$ is the flip between $\mathcal{H}_2$ and $\mathcal{H}_3$.

Definition 1.4.10. *A map $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{Q}$, where $\mathcal{H}$ is a Hilbert space, is called a unitary (co)representation of the CQG $\mathcal{Q}$ on the Hilbert space $\mathcal{H}$, if $\widetilde{U} \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{Q})$ defined by $\widetilde{U}(\xi \otimes b) := U(\xi)(1 \otimes b)$ for $\xi \in \mathcal{H}$ and $b \in \mathcal{Q}$, is a unitary operator which further satisfies $(id_{\mathcal{H}} \otimes \Delta)\widetilde{U} = \widetilde{U}_{(12)}\widetilde{U}_{(13)}$.*

If dimension of $\mathcal{H}$ is $n < \infty$, we may alternatively represent U by the $\mathcal{Q}$-valued $n \times n$ invertible matrix $[\langle Ue_i, e_j \rangle]_{i,j}$, where $\{e_k\}_{k=1}^{n}$ is an orthonormal basis for $\mathcal{H}$. We will call n the dimension of the representation U. Henceforth, we will drop the adjective "co" from the word corepresentation.

By G.N.S construction, $\mathcal{Q} \subseteq \mathcal{B}(L^2(h))$. Then Δ viewed as $\Delta : L^2(h) \to L^2(h) \otimes \mathcal{Q}$ becomes a unitary representaion (say U) such that $\Delta(x) = \widetilde{U}(x \otimes 1)\widetilde{U}^*$. Moreover, $\mathcal{Q}_0$ is the linear span of the matrix coefficients of all finite dimensional unitary inequivalent representations (see [9, 46]). Furthermore, $L^2(h) = \oplus_\pi \mathcal{H}_\pi$ and $\mathcal{Q}_0 (\subseteq L^2(h)) = \oplus_\pi^{alg} \mathcal{H}_\pi$ as a vector space, where $\mathcal{H}_\pi$ is of dimension $d_\pi^2 < \infty$, obtained from the decomposition of Δ (viewed as U) into finite dimensional irreducibles π of dimension d_π by the Peter-Weyl theory for CQG[46].

We cite few examples of CQG:

1. Consider $C(G)$, the algebra of continuous functions on a compact group G. Define $\Delta : C(G) \to C(G) \otimes C(G)$ by $\Delta(f)(g, h) = f(gh)$ where $f \in C(G)$, $g, h \in G$. Then $(C(G), \Delta)$ is a CQG. Note that here the CQG is cocommutative.

2. An important non-commutative example of a CQG is $SU_q(2)$ whose description is as follows:

   As a C^* algebra, it is the universal C^* algebra generated by two elements α, γ satisfying

   $\alpha^*\alpha + \gamma^*\gamma = 1$, $\alpha^*\alpha + q^2\gamma\gamma^* = 1$;
   $\alpha\gamma = q\gamma\alpha$, $\alpha\gamma^* = q\gamma^*\alpha$, $\gamma\gamma^* = \gamma^*\gamma$.
   The comultiplication is given by

   $$\Delta(\alpha) := \alpha \otimes \alpha - q\gamma^* \otimes \gamma,$$

   $$\Delta(\gamma) := \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Action of a compact quantum group on a C^*-algebra: We say that a CQG $(\mathcal{Q}, \Delta)$ (co)-acts on a unital C^*-algebra $\mathcal{B}$, if there is a unital C^* homomorphism (called a coaction) $\alpha : \mathcal{B} \to \mathcal{B} \otimes \mathcal{Q}$, satisfying the following:

1. $(\alpha \otimes id) \circ \alpha = (id \otimes \Delta) \circ \alpha$,

2. the linear span of $\alpha(\mathcal{B})(1 \otimes \mathcal{Q})$ is dense in $\mathcal{B} \otimes \mathcal{Q}$.

Henceforth, we will drop the adjective "co" from the word coaction.

It has been shown in [58] that (2) is equivalent to the existence of a dense $*$-subalgebra $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $\alpha(\mathcal{B}_0) \subseteq \mathcal{B}_0 \otimes_{alg} \mathcal{Q}_0$. We say that an action α is faithful, if there is no proper Woronowicz C^*-subalgebra (see [9],[46]) $\mathcal{Q}_1$ of $\mathcal{Q}$ such that α is a C^* action of $\mathcal{Q}_1$ on $\mathcal{B}$. We refer the reader to [9] and the references therein for details of C^* action.

For a CQG $(\mathcal{Q}, \Delta)$, denote by $Irr_{\mathcal{Q}}$, the index set of inequivalent, unitary irreducible representations of $\mathcal{Q}$ and let u^γ be a representation of $\mathcal{Q}$ of dimension d_γ, for $\gamma \in Irr_{\mathcal{Q}}$. We will call a vector subspace $V \subseteq \mathcal{B}$ a subspace corresonding to u^γ if

- $dimV = d_\gamma$,
- $\alpha(e_i) = \sum_{k=1}^{d_\gamma} e_k \otimes u^\gamma_{ki}$, for some orthonormal basis $\{e_j\}_{j=1}^{d_\gamma}$ of V.

Proposition 1.4.11. *[58] Let α be an action of a CQG $(\mathcal{Q}, \Delta)$ on a C^*-algebra $\mathcal{B}$. Then there exists vector subspaces $\{W_\gamma\}_{\gamma \in Irr_{\mathcal{Q}}}$ of $\mathcal{B}$ such that*

1. *$\mathcal{B} = \overline{\oplus_{\gamma \in Irr_{\mathcal{Q}}} W_\gamma}$*
2. *For each $\gamma \in Irr_{\mathcal{Q}}$, there exists a set I_γ and vector subspaces $W_{\gamma i}$, $i \in I_\gamma$, such that*
   - a. *$W_\gamma = \oplus_{i \in I_\gamma} W_{\gamma i}$.*
   - b. *$W_{\gamma i}$ corresponds to u^γ for each $i \in I_\gamma$.*
3. *Each vector subspace $V \subseteq \mathcal{B}$ corresponding to u^γ is contained in W_γ.*
4. *The cardinal number of I_γ does not depend on the choice of $\{W_{\gamma i}\}_{i \in I_\gamma}$. It is denoted by c_γ and called the multiplicity of u^γ in the spectrum of α.*

Definition 1.4.12. *[58] Suppose a CQG $(\mathcal{Q}, \Delta)$ acts on a C^*-algebra $\mathcal{B}$. Then $\mathcal{B}$ is called*

1. **A quotient of $(\mathcal{Q}, \Delta)$ by a quantum subgroup $(S, \Delta|_S)$** *if:*
   - *a) $\mathcal{B}$ is C^*-isomorphic to the algebra $\mathcal{C} := \{x \in \mathcal{Q} : (\pi \otimes id)\Delta(x) = 1 \otimes x\}$,*
   - *b) the action α is given by $\alpha := \Delta|_{\mathcal{C}}$,*

   where π is the CQG morphism from $\mathcal{Q}$ to S.
2. **Embeddable**, *if there exists a faithful C^*-homomorphism $\psi : \mathcal{B} \to \mathcal{Q}$ such that*
   $\Delta \circ \psi = (\psi \otimes id) \circ \alpha$.

3. **Homogeneous** *if the multiplicity of the trivial representation of $\mathcal{Q}$ in the spectrum of α be* 1 *(see [58]).*

Henceforth, we will refer to a C^*-algebra $\mathcal{B}$ on which a CQG acts, as a quantum space. It can be easily shown that a quantum space is homogeneous if and only if the corresponding action of the CQG is ergodic (i.e. $\alpha(x) = x \otimes I$ implies x is a scalar multiple of the identity of $\mathcal{B}$.

We observe the following fact, the proof of which is trivial and hence omitted.

Lemma 1.4.13. *The action is ergodic if and only if the quantum space is homogeneous.*

Proposition 1.4.14. *[58] Let α be the action of a CQG $(\mathcal{Q}, \Delta)$ on a C^*-algebra $\mathcal{B}$. Then*

a) $(\mathcal{B}, \alpha)$ is quotient $\Rightarrow$ $(\mathcal{B}, \alpha)$ is embeddable $\Rightarrow$ $(\mathcal{B}, \alpha)$ is homogeneous.

b) In the classical case, $(\mathcal{B}, \alpha)$ is quotient $\Longleftrightarrow$ $(\mathcal{B}, \alpha)$ is embeddable $\Longleftrightarrow$ $(\mathcal{B}, \alpha)$ is homogeneous.

We refer the reader to [58] for more discussions on these three types of quantum spaces.

1.4.3 Rieffel Deformation

Let $\theta = ((\theta_{kl}))$ be a skew symmetric matrix of order n. We denote by $C^*(\mathbb{T}^n_\theta)$ the universal C^*-algebra generated by n unitaries $(U_1, U_2, ...U_n)$ satisfying $U_kU_l = e^{2\pi\theta_{kl}}U_lU_k$, for $k \neq l$. If $\theta_{kl} = \theta_0$ for $k < l$, where $\theta_0 \in \mathbb{R}$, we will denote the corresponding universal C^*-algebra by $C^*(\mathbb{T}^n_{\theta_0})$ and $\mathcal{W}$ will denote the $*$-subalgebra generated by unitaries $U_1, U_2, ...U_n$.

Let $\mathcal{A}$ be a unital C^*-algebra on which there is a strongly continuous $*$-automorphic action σ of $\mathbb{T}^n$. Denote by τ the natural action of $\mathbb{T}^n$ on $C^*(\mathbb{T}^n_\theta)$ given on the generators $(U_i)_{i=1}^n$ by $\tau(\overline{z})U_i = z_iU_i$, for $i = 1, 2, ...n$, where $\overline{z} = (z_1, z_2, ...z_n) \in \mathbb{T}^n$. Let τ^{-1} denote the inverse action $s \to \tau_{-s}$.

Definition 1.4.15. *The fixed point algebra of $\mathcal{A} \otimes C^*(\mathbb{T}^n_\theta)$, under the action $(\sigma \times \tau^{-1})$, i.e. $(\mathcal{A} \otimes C^*(\mathbb{T}^n_\theta))^{\sigma\times\tau^{-1}}$, is called the Rieffel deformation of $\mathcal{A}$ under the action σ of $\mathbb{T}^n$, and is denoted by $\mathcal{A}_\theta$.*

In Rieffel's original approach (see [60]), the Rieffel deformation $\mathcal{A}_\theta$ of the C^*-algebra $\mathcal{A}$ was given by completing (with respect to a suitable norm) the algebra obtained from $\mathcal{A}^\infty$ (see [60, 9]) with respect to a new (twisted) multiplication $\times_\theta$ gieven by:

$$a \times_\theta b := \int_{u\in\mathbb{T}^n}\int_{v\in\mathbb{T}^n} \alpha_{\theta u}(a)\alpha_v(b)e(u.v)dudv,$$

where the integral is an oscillatory integral [9, 60] and $e(u.v) := e^{2\pi i\langle u,v\rangle}$ [9, 60].

There is a natural isomorphism between $(\mathcal{A}_\theta)_{-\theta}$ and $\mathcal{A}$, given by the identification of $\mathcal{A}$ with the subalgebra of $((\mathcal{A} \otimes C^*(\mathbb{T}^n_\theta))^{\sigma\times\tau^{-1}} \otimes C^*(\mathbb{T}^n_{-\theta}))^{(\sigma\otimes id)\times\tau^{-1}}$ generated by elements of the form $a_{\overline{p}} \otimes U^{\overline{p}} \otimes (U')^{\overline{p}}$, where $\overline{p} = (p_1, p_2, ...p_n) \in \mathbf{Z}^n$, $U^{\overline{p}} := U_1^{p_1}U_2^{p_2}...U_n^{p_n}$, $(U')^{\overline{p}} := (U_1')^{p_1}(U_2')^{p_2}....(U_n')^{p_n}$, $U_1', U_2', ...U_n'$ being the generators of $C^*(\mathbb{T}^n_{-\theta})$ and $a_{\overline{p}}$ belongs to the spectral subspace of the action σ corresponding to the character $\overline{p}$.

Let $(\mathcal{Q}, \Delta)$ be a CQG and assume that there exists a surjective CQG morphism $\pi : \mathcal{Q} \to C(\mathbb{T}^n)$ which identifies $C(\mathbb{T}^n)$ as a quantum subgroup of $\mathcal{Q}$. For $s \in \mathbb{T}^n$, let $\Omega(s)$ denote the state defined by $\Omega(s) := ev_s \circ \pi$, where ev_s denotes evaluation at s. Define an action of $\mathbb{T}^{2n}$ on $\mathcal{Q}$ by $(s, u) \to \chi_{(s,u)}$, where $\chi_{(s,u)} := (\Omega(s) \otimes id) \circ \Delta \circ (id \otimes \Omega(-u)) \circ \Delta$. It has been shown in [67] that the Rieffel deformation $\mathcal{Q}_{\widetilde{\theta},-\widetilde{\theta}}$ of $\mathcal{Q}$ with respect to $\widetilde{\theta} := \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ can be given a unique CQG structure such that the coalgebra structure of the Hopf$*$ algebra of $\mathcal{Q}_{\theta,-\theta}$ is isomorphic with that of the canonical Hopf$*$ algebra of $\mathcal{Q}$.

1.5 Quantum Isometry group

We begin by defining spectral triple (also called spectral data). We shall refer the reader to [16] and [9] for details.

Definition 1.5.1. *An odd spectral triple or spectral data is a triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ where $\mathcal{H}$ is a separable Hilbert space, $\mathcal{A}^\infty$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$,(not necessarily norm closed) and D is a self adjoint (typically unbounded) operator such that for all a in $\mathcal{A}^\infty$, the operator $[D, a]$ has a bounded extension. Such a spectral triple is also called an odd spectral triple. If in addition, we have γ in $\mathcal{B}(\mathcal{H})$ satisfying $\gamma = \gamma = \gamma^1$, $D\gamma = \gamma D$ and $[a, \gamma] = 0$ for all a in $\mathcal{A}^\infty$, then we say that the quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, \gamma)$ is an even spectral triple. The operator D is called the Dirac operator corresponding to the spectral triple.*

Since in the classical case, the Dirac operator has compact resolvent if the manifold is compact, we say that the spectral triple is of compact type if $\mathcal{A}^\infty$ is unital and D has compact resolvent. A spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ will be called Θ summable if e^{-tD^2} is a trace class operator $(t > 0)$. Next we discuss the notion of Hilbert space of k-forms in non-commutative geometry.

Proposition 1.5.2. *Given an algebra $\mathcal{B}$, there is a (unique upto isomorphism)$\mathcal{B}-\mathcal{B}$ bimodule $\Omega^1(\mathcal{B})$ and a derivation $\delta : \mathcal{B} \to \Omega^1(\mathcal{B})$, satisfying the following properties:*

1. *$\Omega^1(\mathcal{B})$ is spanned as a vector space by elements of the form $a\delta(b)$ with a, b belonging to $\mathcal{B}$;*

2. *for any $\mathcal{B}-\mathcal{B}$ bimodule E and a derivation $d : \mathcal{B} \to E$, there is a unique $\mathcal{B}-\mathcal{B}$ linear map $\eta : \Omega^1(\mathcal{B}) \to E$ such that $d = \eta \circ \delta$.*

The bimodule $\Omega^1(\mathcal{B})$ is called the space of universal 1-forms on $\mathcal{B}$ and δ is called the universal derivation. Given a Θ-summable spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, it is possible to define an inner product structure on $\Omega^0(\mathcal{A}^\infty) \equiv \mathcal{A}^\infty$ and $\Omega^1(\mathcal{A}^\infty)$. The corresponding Hilbert spaces are denoted by $\mathcal{H}_D^0$ and $\mathcal{H}_D^1$ respectively. $\mathcal{H}_D^0$ and $\mathcal{H}_D^1$ are called the Hilbert space of zero and one forms respectively (see[9]).

We now define quantum isometry group. Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a Θ-summable spectral triple which is admissible in the sense that it satisfies the regularity conditions (i)-(v) as given in [27, pages 9-10].
Let $\mathcal{L} := -d_D^* d_D$, which is a densely defined self-adjoint operator on $\mathcal{H}_0$ and is called the Laplacian of the spectral triple. We will denote by $\mathbb{Q}'^{,\mathcal{L}}$ the category whose objects are triplets (S, Δ, α) where (S, Δ) is a CQG acting smoothly and isometrically on the given noncommutative manifold, with α being the corresponding action.

Proposition 1.5.3. *[27] For any admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, the category $\mathbb{Q}'^{,\mathcal{L}}$ has a universal object denoted by $(QISO^{\mathcal{L}}, \alpha_0)$. Moreover, $QISO^{\mathcal{L}}$ has a coproduct Δ_0 such that $(QISO^{\mathcal{L}}, \Delta_0)$ is a CQG and $(QISO^{\mathcal{L}}, \Delta_0, \alpha_0)$ is a universal object in the category $\mathbb{Q}'^{,\mathcal{L}}$. The action α_0 is faithful.*

The reader may see [27] and [9] for further details of $QISO^{\mathcal{L}}$. We now give some examples of quantum isometry groups.

1. **non-commutative 2-tori:** The non-commutative 2-tori $C^*(\mathbb{T}^2_\theta)$ is the universal C^*-algebra generated by a pair of unitaries U, V such that $UV = e^{2\pi i\theta} VU$ i.e. Rieffel deformation of $C(\mathbb{T}^2)$ with respect to $\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$. The C^* algebra underlying the quantum isometry group of the standard spectral triple on $C^*(\mathbb{T}^2_\theta)$ (see [16]) is given by
$\oplus_{i=1}^4 (C(\mathbb{T}^2) \oplus C^*(\mathbb{T}^2_\theta))$ (see [10]). Let U_{k1}, U_{k2} be the generators of $C(\mathbb{T}^2)$ for odd k and $C^*(\mathbb{T}^2_\theta)$ for even k, $k = 1, 2, ...8$. Define
$$M = \begin{pmatrix} A_1 & A_2 & C_1^* & C_2^* \\ B_1 & B_2 & D_1^* & D_2^* \\ C_1 & C_2 & A_1^* & A_2^* \\ D_1 & D_2 & B_1^* & B_2^* \end{pmatrix},$$
where $A_1 = U_{11} + U_{41},\ A_2 = U_{62} + U_{72},\ B_1 = U_{52} + U_{61},\ B_2 = U_{12} + U_{22},$ $C_1 = U_{21} + U_{31},\ C_2 = U_{51} + U_{82},\ D_1 = U_{71} + U_{81},\ D_2 = U_{32} + U_{42}$. Then the coproduct Δ and the counit ϵ are given by $\Delta(M_{ij}) = \sum_{k=1}^4 M_{ik} \otimes M_{kj}$, $\epsilon(M_{ij}) = \delta_{ij}$. The action of the QISO on $C^*(\mathbb{T}^2_\theta)$, say α, is given by
$$\alpha(U) = U \otimes (U_{11} + U_{41}) + V \otimes (U_{52} + U_{61}) + U^{-1} \otimes (U_{21} + U_{31}) + V^{-1} \otimes (U_{71} + U_{81}),$$
$$\alpha(V) = U \otimes (U_{62} + U_{72}) + V \otimes (U_{12} + U_{22}) + U^{-1} \otimes (U_{51} + U_{82}) + V^{-1} \otimes (U_{32} + U_{42}).$$

2. **The θ deformed sphere S_θ^{2n-1}:** The non-commutative manifold S_θ^{2n-1}, for a skew symmetric matrix θ is the universal C^*-algebra generated by $2n$ elements $\{z^\mu, \overline{z}^\mu\}_{\mu=1,2,..2n}$, satisfying the relations:

   - $(z^\mu)^* = \overline{z}^\mu$;
   - $z^\mu z^\nu = e^{2\pi i\theta_{\mu\nu}} z^\nu z^\mu$, $\overline{z}^\mu z^\nu = e^{2\pi i\theta_{\nu\mu}} z^\nu \overline{z}^\mu$;
   - $\sum_{\mu=1}^{2n} z^\mu \overline{z}^\mu = 1$.

   The quantum isometry group of the spectral triples on S_θ^{2n-1}, as described in [16, 17] is $O_\theta(2n)$ whose CQG structure is described as follows: It is generated by $(a_\nu^\mu, b_\nu^\mu)_{\mu,\nu=1,2,...2n}$, satisfying:

   (a) $a_\nu^\mu a_\rho^\tau = \lambda_{\mu\tau}\lambda_{\rho\nu} a_\rho^\tau a_\nu^\mu$, $a_\nu^\mu a_\rho^{*\tau} = \lambda^{\tau\mu}\lambda_{\nu\rho} a_\rho^{*\tau} a_\nu^\mu$,

   (b) $a_\nu^\mu b_\rho^\tau = \lambda_{\mu\tau}\lambda_{\rho\nu} b_\rho^\tau a_\nu^\mu$, $a_\nu^\mu b_\rho^{*\tau} = \lambda^{\tau\mu}\lambda_{\nu\rho} b_\rho^{*\tau} a_\nu^\mu$,

   (c) $b_\nu^\mu b_\rho^\tau = \lambda_{\mu\tau}\lambda_{\rho\nu} b_\rho^\tau b_\nu^\mu$, $b_\nu^\mu b_\rho^{*\tau} = \lambda^{\tau\mu}\lambda_{\nu\rho} b_\rho^{*\tau} b_\nu^\mu$,

   (d) $\sum_{\mu=1}^{2n}(a_\alpha^{*\mu} a_\beta^\mu + b_\alpha^\mu b_\beta^{*\mu}) = \delta_{\alpha\beta}1$, $\sum_{\mu=1}^{2n}(a_\alpha^{*\mu} b_\beta^\mu + b_\alpha^\mu a_\beta^{*\mu}) = 0$,

   The coproduct Δ is given by $\Delta(a_\nu^\mu) = \sum_{\lambda=1}^{2n}[a_\lambda^\mu \otimes a_\nu^\lambda + b_\lambda^\mu \otimes b_\nu^{*\lambda}]$,
   $\Delta(b_\nu^\mu) = \sum_{\lambda=1}^{2n}[a_\lambda^\mu \otimes b_\nu^\lambda + b_\lambda^\mu \otimes a_\nu^{*\lambda}]$; and the counit ϵ is given by $\epsilon(a_\nu^\mu) = \delta_{\mu\nu}$, $\epsilon(b_\nu^\mu) = 0$.

   The action of the QISO on S_θ^{2n-1}, say α, is given by

   $$\alpha(z^\mu) = \sum_\nu (z^\nu \otimes a_\nu^\mu + \overline{z}^\nu \otimes b_\nu^\mu),\ \alpha(\overline{z}^\mu) = \sum_\nu (\overline{z}^\nu \otimes \overline{a}_\nu^\mu + z^\nu \otimes \overline{b}_\nu^\mu).$$

3. **The free sphere S_+^{2n-1}:** The free sphere denoted by S_+^{2n-1} is defined as the universal C^* algebra generated by elements $\{x_i\}_{i=1}^{2n-1}$ satisfying $x_i = x_i^*$ and $\sum_{i=1}^{2n-1} x_i^2 = 1$. Consider the spectral triples as described in Theorem 6.4 in page 13 of [6]. It has been shown (see [6]) that the quantum isometry group associated to this spectral triple is the free orthogonal group $C^*(O_+(n))$ which is described as the universal C^*-algebra generated by $4n^2$ elements $\{x_{ij}\}_{i,j=1}^{2n}$ satisfying

   a. $x_{ij} = x_{ij}^*$ for $i, j = 1, 2, ...2n$;

   b. $\sum_{k=1}^{2n} x_{ki}x_{kj} = \delta_{ij}\mathbf{1}$, $\sum_{k=1}^{2n} x_{ik}x_{jk} = \delta_{ij}\mathbf{1}$.

For more examples, we refer the reader to [9].

1.6 Quantum Dynamical Semigroup

Definition 1.6.1. *A one parameter C_0 semigroup $(T_t)_{\geq 0}$ of CP maps on a C^* algebra $\mathcal{A}$ is called a quantum dynamical semigroup (QDS for short). For a von-Neumann algebra $\mathcal{A}$, a QDS is a one parameter C_0 (with respect to the ultraweak topology) semigroup of normal CP maps $(T_t)_{t\geq 0}$ on $\mathcal{A}$.*

In either case, the semigroup will be called conservative if $T_t(1) = 1 \ \forall t \geq 0$.

Any bounded CCP map $\mathcal{L}$ on a C^* or von-Neumann algebra gives rise to a QDS $(e^{t\mathcal{L}})_{t\geq 0}$. Conversely, the generator of a uniformly continuous QDS is a bounded CCP map. The generators of uniformly continuous QDS are characterised by the following Theorem:

Theorem 1.6.2 (Christensen-Evans)**.** *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a unital von-Neumann algebra and $(T_t)_{t\geq 0}$ be a uniformly continuous QDS. Suppose that $\mathcal{L}$ be its ultraweakly continuous generator. Then there is a quintuple $(\rho, \mathcal{K}, \alpha, \mathcal{H}, R)$ where ρ is a unital normal $*$-representation of $\mathcal{A}$ in a Hilbert space $\mathcal{K}$ and a ρ-derivation $\alpha : \mathcal{A} \to \mathcal{B}(\mathcal{H}, \mathcal{K})$ (i.e. $\alpha(xy) = \alpha(x)y + \rho(x)\alpha(y)$) such that the set $\mathcal{D} := \{\alpha(x)u : \ x \in \mathcal{A}, \ u \in \mathcal{H}\}$ is total in $\mathcal{K}$, H is a self-adjoint element of $\mathcal{A}$ and $R \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\alpha(x) = Rx - \rho(x)R$, and $\mathcal{L}(x) = R^*\rho(x)R - \frac{1}{2}(R^*R - \mathcal{L}(1))x - \frac{1}{2}x(R^*R - \mathcal{L}(1)) + i[H, x]$ for all $x \in \mathcal{A}$. Furthermore, $\mathcal{L}$ satisfies the following algebraic identity, called the cocycle property (or cocycle relation) with α as coboundary, i.e.*

$$\mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y) + x^*\mathcal{L}(1)y = \alpha(x)^*\alpha(y).$$

Moreover, R can be chosen from the ultraweak closure of $sp\{\alpha(x)y : \ x, y \in \mathcal{A}\}$ and hence in particular $R^*\rho(x)R \in \mathcal{A}$.

1.6.1 Minimal quantum dynamical semigroups associated to form generators

We briefly state the theory of minimal quantum dynamical semigroup associated with a given form generator. For details, we refer to [62].

Suppose that h and $\mathcal{K}$ are two Hilbert spaces, $\mathcal{A} \subseteq \mathcal{B}(h)$ is a von-Neumann algebra, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a normal, unital, $*$-representation;$(P_t)_{t\geq 0}$ is a C_0-contraction semigroup on h with generator G; and $R : h \to \mathcal{K}$ is a closed, densely defined, linear (possibly unbounded) map. Formally we introduce a map $\mathcal{L}$ by $\mathcal{L}(x) = R^*\pi(x)R + xG + G^*x$, $x \in \mathcal{A}$; where G is the generator of $(P_t)_{t\geq 0}$. Let us make the following assumptions on G and R :

(Ai) G is affiliated to $\mathcal{A}$ and $R^*\pi(x)R$ is affiliated to $\mathcal{A}$, for all $x \in \mathcal{A}$.

(Aii) $Dom(G) \subseteq Dom(R)$ and for all $u, v \in Dom(G)$, $\langle Ru, Rv\rangle + \langle u, Gv\rangle + \langle Gu, v\rangle = 0$.

We define an equivalence relation on $B_1(h)$ as follows:
For $\rho_1, \rho_2 \in B_1(h)$, $\rho_1 \sim \rho_2$ if $tr(\rho_1 x) = tr(\rho_2 x)$ for all $x \in \mathcal{A}$. Then we have $\mathcal{A}_* \cong B_1(h)/\sim$ (see [62]).

It is known [62] that there exists an ultraweakly continuous semigroup $(T_t^{min})_{t\geq 0}$ on $\mathcal{A}$ such that the generator of its predual semigroup $(T_{*t}^{min})_{t\geq 0}$ extends the map $[\rho] \in B_1(h)/\sim \mapsto \pi_*([R\rho R^*]) + [G\rho] + [\rho G^*]$, where $\rho = (1-G)^{-1}\sigma(1-G^*)^{-1}$ for $\sigma \in B_1(h)$ and π_* is the predual map of π. Moreover, if $(T_t)_{t\geq 0}$ be another QDS on $\mathcal{A}$, such that the generator of its predual semigroup $(T_{*t})_{t\geq 0}$ (which is a semigroup on $\mathcal{A}_*$) extends the map decribed above, then we have $T_{*t}^{min}(\rho) \leq T_{*t}(\rho)$ for $\rho \in \mathcal{A}_*$ such that ρ is a positive functional on $\mathcal{A}$. The semigroup $(T_t^{min})_{t\geq 0}$ is called the minimal semigroup.

If the minimal semigroup is conservative, i.e. $T_t^{min}(1) = 1$ for all $t \geq 0$, then we have the following "Feller condition"[62]:

$$\{x \in \mathcal{A}: \ \langle Ru, \pi(x)Rv\rangle + \langle Gu, xv\rangle + \langle u, Gxv\rangle = \lambda\langle u, xv\rangle \ \forall \ u, v \in Dom(G)\} = 0$$

for some $\lambda > 0$.

Suppose that h and k_0 are Hilbert spaces.

Proposition 1.6.3. *Let $R : Dom(R) \to h \otimes k_0$ be a densely defined closed operator with $Dom(R) \subseteq h$. Furthermore, let G be a densely defined operator on h such that G generates a C_0 semigroup in h. Moreover, we have $Dom(G), Dom(G^*) \subseteq Dom(R)$ and $\langle Ru, Rv\rangle + \langle u, Gv\rangle + \langle Gu, v\rangle = 0$.*

Then the minimal semigroup on $\mathcal{B}(h)$, generated by the form generator by $R^*(x \otimes id_{k_0})R + xG + G^*x$ is conservative.

Proof. Observe that by our hypothesis, both the assumptions **(Ai)** and **(Aii)** hold for the maps R and G. Thus by the discussion above, there is a minimal semigroup say $(T_t^{min})_{t\geq 0}$ on $\mathcal{B}(h)$, whose form generator $\mathcal{L}^{min}$ on a certain dense subspace is of the form $R^*(x \otimes 1_{k_0})R + xG + G^*x$. We prove that $(T_t^{min})_{t\geq 0}$ is conservative:

Let $\mathcal{D} \subseteq h$ be the subspace such that for $x \in \mathcal{D}$, $L(x) := R^*(x \otimes 1_{k_0})R + xG + G^*x \in \mathcal{B}(h)$. Note that $1 := 1_{\mathcal{B}(h)} \in \mathcal{D}$. Let $(T^{min}{}_{*,t})_{t\geq 0}$ be the predual semigroup of $(T_t^{min})_{t\geq 0}$. It is known (see chapter 3 of [62]) that for $\sigma \in B_1(h)$ ($B_1(h)$ is the space of trace class operators on h), the linear span of operators ρ of the form $\rho = (1-G)^{-1}\sigma(1-G^*)^{-1}$, denoted by $\mathcal{B}$, belongs to $Dom(\mathcal{L}_*^{min})$, $\mathcal{L}_*^{min}$ being the generator of $(T_{*,t}^{min})_{t\geq 0}$. Moreover we have $\mathcal{L}_*^{min}(\rho) = R\rho R^* + G\rho + \rho G^*$ for $\rho \in \mathcal{B}$, and $\mathcal{B}$ is a core for $\mathcal{L}_*^{min}$. Now for $a \in \mathcal{D}, \rho \in \mathcal{B}$, $tr(L(a)\rho) = tr(a\mathcal{L}_*^{min}(\rho))$. Since $\mathcal{B}$ is a core for $\mathcal{L}_*^{min}$, we have $tr(L(a)\rho) = tr(a\mathcal{L}_*^{min}(\rho))$ for all $\rho \in Dom(\mathcal{L}_*^{min})$. Observe that for $\rho \in Dom(\mathcal{L}_*^{min})$,

$$\begin{aligned} tr\left(\frac{T_t^{min}(a) - a}{t}\rho\right) &= tr\left(a\left(\frac{T_{*,t}^{min}(\rho) - \rho}{t}\right)\right) = tr\left(a\mathcal{L}_*^{min}\left(t^{-1}\int_0^t T_{*,s}^{min}(\rho)ds\right)\right) \\ &= tr\left(L(a)\left(t^{-1}\int_0^t T_{*,s}^{min}(\rho)ds\right)\right); \end{aligned} \tag{1.2}$$

which proves that $a \in Dom(\mathcal{L}^{min})$ and by continuity, $\mathcal{L}^{min}(a) = L(a)$, for all $a \in \mathcal{D}$. Now $L(1) = 0$ which implies that $\mathcal{L}^{min}(1) = 0$, i.e. $(T_t^{min})_{t\geq 0}$ is conservative. □

1.6.2 Symmetric quantum dynamical semigroup

Definition 1.6.4. *Let $\mathcal{A}$ be a C^* or von-Neumann algebra, equipped with a faithful, semifinite and lower-semicontinuous (normal if $\mathcal{A}$ is a von-Neumann algebra) trace τ. A QDS $(T_t)_{t\geq 0}$ on $\mathcal{A}$ is called a symmetric QDS if we have $\tau(T_t(x)y) = \tau(xT_t(y))$ for $x, y \in Dom(\tau)$.*

Let $L^2(\tau)$ denote the non-commutative L^2 space as described in subsection 1.0.2. Then it is well known (see [62]) that $(T_t)_{t\geq 0}$ extends to a C_0 semigroup of self-adjoint operators (denoted again by $(T_t)_{t\geq 0}$) on $L^2(\tau)$. We will use the notation $\mathcal{L}$ to denote the norm (ultraweak) generator of $(T_t)_{t\geq 0}$ whereas $\mathcal{L}_2$ will denote the L^2-generator of $(T_t)_{t\geq 0}$.

It can be shown (see [62]) that $-\mathcal{L}_2$ is a densely defined positive operator on $L^2(\tau)$. The densely defined sesquilinear form $\mathcal{E}$ on $L^2(\tau)$, given by

$\mathcal{E}(x, y) = \langle(-\mathcal{L}_2)^{\frac{1}{2}}x, (-\mathcal{L}_2)^{\frac{1}{2}}y\rangle$, $x, y \in Dom((-\mathcal{L}_2)^{\frac{1}{2}})$, is called the Dirichlet form associated with the symmetric QDS $(T_t)_{t\geq 0}$. If $\mathcal{A}$ is a C^*-algebra, it is a well known fact (see [20, 21]) that $\mathcal{B} := \mathcal{A} \cap Dom((-\mathcal{L}_2)^{\frac{1}{2}})$ is a dense $*$-subalgebra of $\mathcal{A}$.

Let $\mathcal{E}$ be the Dirichlet form associated with the symmetric QDS $(T_t)_{t\geq 0}$ and let $Dom(\mathcal{E})$ denote the domain of the Dirichlet from.

We now state without proof a list of propositions, which we will be using in the next chapters.

Proposition 1.6.5. *[15]*

1. *For any $n \in I\!N \setminus 0$, $a_1, a_2, ...a_n \in \mathcal{B}$, the matrix*

$$[\frac{(-\mathcal{L}_2)}{I + \epsilon(-\mathcal{L}_2)}(a_j)^* a_i + a_j^* \frac{(-\mathcal{L}_2)}{I + \epsilon(-\mathcal{L}_2)}(a_i) - \frac{(-\mathcal{L}_2)}{I + \epsilon(-\mathcal{L}_2)}(a_j^* a_i)]_{i,j=1,...n}$$

is positive in $M_n(\mathcal{A})$;

2. *for a, b, c and $d \in \mathcal{B}$, the limit*

$$lim_{\epsilon \Downarrow 0} \frac{1}{2} \tau(d^* \frac{(-\mathcal{L}_2)}{I + \epsilon(-\mathcal{L}_2)}(c)^* ab + d^* c^* \frac{(-\mathcal{L}_2)}{I + \epsilon(-\mathcal{L}_2)}(a)b - d^* \frac{(-\mathcal{L}_2)}{I + \epsilon(-\mathcal{L}_2)}(c^* a)b)$$

exists in $\mathbb{C}$ and equals

$$\frac{1}{2}\mathcal{E}(c, abd^*) + \mathcal{E}(cbd^*, a) - \mathcal{E}(db^*, c^* a);$$

3. *the sesquilinear form, linear in the right-hand side and conjugate linear in the left-hand one, on the algebraic tensor product $\mathcal{B} \otimes_{alg} \mathcal{B}$ which, to $c \otimes d$ and $a \otimes b$, associates the limit above, is positive.*

Denote by $\mathcal{K}$, the Hilbert space obtained by separation and completion of the $\mathcal{B} \otimes_{alg} \mathcal{B}$ with respect to the seminorm provided by the positive sesquilinear form described in the above lemma.

Proposition 1.6.6. *[62] Let $\mathcal{B} := \mathcal{A} \cap Dom(\mathcal{E})$. Furthermore, assume that there exists a norm dense $*$-subalgebra $\mathcal{A}_0 \subseteq Dom(\mathcal{L}) \cap Dom(\mathcal{L}_2)$, which is a core for the norm generator $\mathcal{L}$ on one hand and a form core for $\mathcal{E}$ on the other. Suppose that $\mathcal{L}(\mathcal{A}_0) \subseteq \mathcal{A}_0$. Then the following conditions hold:*

1. *The Hilbert space $\mathcal{K}$ is equipped with an $\mathcal{A} - \mathcal{A}$ bimodule structure, in which the right action is denoted by $(a, \xi) \mapsto \xi a$, $\xi \in \mathcal{K}$, $a \in \mathcal{A}$ and the left action by $(a, \xi) \mapsto \pi(a)\xi$, $\xi \in \mathcal{K}$, $a \in \mathcal{A}$.*

2. *There is a densely defined closable linear map δ_0 from $\mathcal{A}$ into $\mathcal{K}$ such that $\mathcal{B} = Dom(\delta_0)$, and δ_0 is a bimodule derivation, that is, $\delta_0(ab) = \delta_0(a)b + \pi(a)\delta_0(b)$ for all $a, b \in \mathcal{B}$.*

3. *For $a \in \mathcal{A}_0$, $b \in \mathcal{B}$, $\|\delta_0(a)b\|_{\mathcal{K}} \leq C_a \|b\|_2$, where $\| \cdot \|_{\mathcal{K}}$ denotes the Hilbert space norm of $\mathcal{K}$, and C_a is a constant depending only on a.*

4. *Let $\delta(\cdot) := \sqrt{2}\delta_0(\cdot)$. The triple $(\mathcal{L}, \pi, \delta)$ satisfy the following cocycle relation:*

$$\delta(a)^* \pi(b) \delta(c) = \mathcal{L}(a^* bc) - \mathcal{L}(a^* b)c - a^* \mathcal{L}(bc) + a^* \mathcal{L}(b)c,$$

for $a, b, c \in \mathcal{A}_0$.

5. *$\mathcal{K}$ is the closed linear span of $\{\delta(a)b : \ a, b \in \mathcal{A}_0\}$.*

6. *π extends to a normal $*$-homomorphism on $\mathcal{A}''$.*

Proposition 1.6.7. *[62] Let $R : L^2(\tau) \to \mathcal{K}$ be defined as follows:*

$$Dom(R) = \mathcal{A}_0, \;\; Rx := \sqrt{2}\delta_0(x).$$

Then R has a densely defined adjoint R^*, whose domain contains the linear span of the vectors $\delta(x)y$ for $x, y \in \mathcal{A}_0$ and $R^*(\delta(x)y) = x\mathcal{L}(y) - \mathcal{L}(x)y - \mathcal{L}(xy)$.

We denote the closure of R by the same notation R. For $x, y \in \mathcal{A}_0$,

$$\mathcal{L}(x)y = (R^*\pi(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R)(y).$$

Furthermore,

$$\delta(x)y = (Rx - \pi(x)R)(y),$$

for $x, y \in \mathcal{A}_0$,

$$\mathcal{L}_2 = -\frac{1}{2}R^*R.$$

1.7 Quantum Stochastic Calculus

In this section we review the basics of the coordinatized as well as the cordinate free versions of the quantum stochastic calculus. For details, we refer to [52, 62].

1.7.1 Symmetric Fock space

Let $\mathcal{H}$ be a Hilbert space. For $n \in I\!N$, let $\mathcal{H}^{\otimes^n} := \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \cdots \otimes \mathcal{H}}_{\text{n copies}}$, the usual tensor product of Hilbert spaces and $\mathcal{H}^{\otimes^0} := \mathbb{C}$. Then the Hilbert space

$$\Gamma_{fr}(\mathcal{H}) := \oplus_{i=0}^{\infty}\mathcal{H}^{\otimes^i}$$

is called the free Fock space over $\mathcal{H}$. Let S_n denote the permutation group of n-elements. Define a projection operator $S^{(n)}$ on $\mathcal{B}(\mathcal{H}^{\otimes^n})$ by defining its values on the simple tensors as:

$$S^{(n)}(g_1 \otimes g_2 \otimes \cdots \otimes g_n) := \frac{1}{n!}\sum_{\sigma \in S_n} g_{\sigma^{-1}(1)} \otimes g_{\sigma^{-1}(2)} \otimes \cdots \otimes g_{\sigma^{-1}(n)}.$$

Note that $\mathbf{P} := \oplus_{n=0}^{\infty} S^{(n)}$ is a projection in $\mathcal{B}(\Gamma_{fr}(\mathcal{H}))$.

Definition 1.7.1. *The range of the projection $\mathbf{P}$ namely, $Ran(\mathbf{P})$ is defined as the symmetric Fock space over $\mathcal{H}$ and is denoted by $\Gamma(\mathcal{H})$.*

Let $f \in \mathcal{H}$ and suppose that $f^{(n)} := f \otimes f \otimes f \otimes \cdots f \in \mathcal{H}^{\otimes^n}$. The vector $e(f)$ defined by $e(f) := \oplus_{n \geq 0} \frac{1}{\sqrt{n!}} f^{(n)}$, is called an exponential vector. The exponential vector $e(0) = 1 \oplus 0 \oplus 0 \oplus \cdots$ is called the vacuum vector. It is well known that the set of exponential vectors is total in $\mathcal{H}$. Also note that for $f, g \in \mathcal{H}$, we have $\langle e(f), e(g) \rangle = e^{\langle f,g \rangle}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Consider the symmetric Fock space over $\mathcal{H} \oplus \mathcal{K}$, denoted by $\Gamma(\mathcal{H} \oplus \mathcal{K})$. Let $U : \Gamma(\mathcal{H} \oplus \mathcal{K}) \to \Gamma(\mathcal{H}) \otimes \Gamma(\mathcal{K})$ be the map defined by $U(e(f \oplus g)) = e(f) \otimes e(g)$ and extending by linearity and continuity to $\Gamma(\mathcal{H} \oplus \mathcal{K})$, where $f \in \mathcal{H},\ g \in \mathcal{K}$. Since U is an inner product preserving map on a total set of vectors which sends a total subset of the domain into a total subset of the range, it extends to a unitary operator. Thus $\Gamma(\mathcal{H} \oplus \mathcal{K}) \cong \Gamma(\mathcal{H}) \otimes \Gamma(\mathcal{K})$.

Let k_0 be a Hilbert space. Let $L^2(\mathbb{R}_+, k_0) := L^2(\mathbb{R}_+) \otimes k_0$. We will use the notations $\Gamma_{t]}, \Gamma_{(s,t]}$ and $\Gamma_{[t}$ to denote the Hilbert spaces $\Gamma(L^2([0,t], k_0))$, $\Gamma(L^2((s,t], k_0))$ and $\Gamma(L^2([t, +\infty), k_0))$ respectively. Since we have $L^2(\mathbb{R}_+) = L^2([0,s]) \oplus L^2((s,t]) \oplus L^2([t, +\infty)$ for $s < t$, it follows that $\Gamma(L^2(\mathbb{R}_+, k_0)) = \Gamma_{s]} \otimes \Gamma_{(s,t]} \otimes \Gamma_{[t}$. Note that since multiplication by χ_A, for any borel set A, is a projection in $L^2(\mathbb{R}_+, k_0)$, denoting $\chi_{[0,s]} f$, $\chi_{(s,t]} f$ and $\chi_{[t,+\infty)} f$ by $f_{s]}, f_{(s,t]}$ and $f_{[t}$ respectively, for $f \in L^2(\mathbb{R}_+, k_0)$, we have $e(f) = e(f_{s]}) \otimes e(f_{(s,t]}) \otimes e(f_{[t})$ and $\Omega = \Omega_{s]} \otimes \Omega_{(s,t]} \otimes \Omega_{[t}$, where $\Omega, \Omega_{s]}, \Omega_{(s,t]}$ and $\Omega_{[t}$ denotes respectively the vacuum vectors of $\Gamma, \Gamma_{s]}, \Gamma_{(s,t]}$ and $\Gamma_{[t}$. We define the three basic operators (creation, annihilation and number or conservation) over $\Gamma(\mathcal{H})$ as follows:

Let $f, g \in \mathcal{H},\ H \in \mathcal{B}(\mathcal{H})$. On the finite particle level, we have

- Creation: $a^\dagger(f) g^{\otimes^n} := \sum_{r=0}^{n} \frac{1}{\sqrt{n+1}} g^{\otimes^r} \otimes f \otimes g^{\otimes^{(n-r)}}$;
- Annihilation: $a(f) g^{\otimes^n} := \sqrt{n} \langle f, g \rangle g^{\otimes^{(n-1)}}$;
- Number: $\lambda(H) g^{\otimes^n} := \sum_{r=0}^{(n-1)} g^{\otimes^r} \otimes Hg \otimes g^{\otimes^{(n-r)}}$.

We note that each of these operators is closable and the closures will be denoted by the same symbols.

Let $\mathcal{E}(\mathcal{H})$ denote the subspace spanned by the exponential vectors over $\mathcal{H}$. This subspace belongs to the domain of each of the above three operators and we have:

- $a(f) e(g) = \langle f, g \rangle e(g)$;
- $a^\dagger(f) e(g) = \frac{d}{dt}|_{t=0} e(g + tf)$;
- $\langle e(f), \lambda(H) e(g) \rangle = \langle f, Hg \rangle \langle e(f), e(g) \rangle$;
- Annihilation and creation operators are adjoint of one another over the exponential vectors i.e. $\langle e(g_1), a^\dagger(f) e(g_2) \rangle = \langle a(f) e(g_1), e(g_2) \rangle = \langle f, g_1 \rangle \langle e(g_1), e(g_2) \rangle$.

1.7.2 Quantum Stochastic Calculus: The cordinate formalism

Let h, k_0 be Hilbert spaces. Let $\mathcal{D}$ and $\mathcal{M}$ be two dense subspaces of h and $L^2(\mathbb{R}_+, k_0)$ respectively.

Definition 1.7.2. *A family of operators $\{V_t\}_{t\geq 0}$ on $h \otimes \Gamma$ is said to be a $(\mathcal{D}, \mathcal{M})$-adapted process if:*

1. *$\mathcal{D} \otimes \mathcal{M} \subseteq Dom(V_t)$ for all $t \geq 0$,*
2. *For $u \in \mathcal{D},\ f \in \mathcal{M}$, we have $V_t(ue(f)) \in h \otimes \Gamma_{t]}$ and $V_t(ue(f)) = V_t(ue(f_{t]})) \otimes e(f_{[t})$.*

It is said to be regular if for $u \in h,\ f \in \mathcal{M}$, the map $t \to V_t(ue(f))$ is a continuous $h \otimes \Gamma$ valued function.

Let $\{e_j\}_{j\geq 0}$ be an orthonormal basis for k_0. Then fundamental processes associated with this basis, namely $\{\Lambda_\mu^\nu\}_{\mu,\nu\geq 0}$ are defined as follows:

$$\Lambda_\nu^\mu(t) := \begin{cases} t\mathbf{1} \text{ for } (\mu,\nu) = (0,0), \\ a(\chi_{[0,t]} \otimes e_j), \text{ for } (\mu,\nu) = (j,0), \\ a^\dagger(\chi_{[0,t]} \otimes e_i), \text{ for } (\mu,\nu) = (o,i), \\ \lambda(M_{[0,t]} \otimes |e_i\rangle\langle e_j|), \text{ for } (\mu,\nu) = (i,j), \end{cases}$$

where $M_{[0,t]}$ denotes multiplication operator on $L^2(\mathbb{R}_+)$ by $\chi_{[0,t]}$. The quantum Ito formula is given by

$$\Lambda_\nu^\mu(dt)\Lambda_\eta^\xi(dt) = \widehat{\delta}_\nu^\xi \Lambda_\eta^\mu(dt),\ \forall \mu,\nu,\xi,\eta \geq 0,$$

where $\widehat{\delta}_\nu^\mu := \begin{cases} 0 \text{ for } \mu = 0 \text{ or } \nu = 0, \\ = \delta_\nu^\mu \text{ otherwise,} \end{cases}$

where δ_ν^μ is the usual kronecker delta symbol.

Suppose that $\{X(t)\}_{t\geq 0}$ and $\{L_\nu^\mu(t)\}_{\mu,\nu,t\geq 0}$ are $\mathcal{D} \otimes \mathcal{E}(\mathcal{M})$ adapted, regular processes. We say that the process $X(t)$ satisfies a Hudson-Parthasarathy (HP) type quantum stochastic differential equation with initial value $id_{h\otimes\Gamma}$, symbolically:

$$\begin{aligned} dX(t) &= X(t)L_\nu^\mu(t)\Lambda_\mu^\nu(dt), \\ X(0) &= id_{h\otimes\Gamma}, \end{aligned}$$

if we have the following weak equation:

$$\langle X(t)ue(f), ve(g)\rangle = \\ \langle ue(f), ve(g)\rangle + \sum_{\mu,\nu} \int_0^t ds \langle X(s)L_\nu^\mu(s)ue(f), ve(g)\rangle g^\mu(s) f_\nu(s)$$

for $u, v \in \mathcal{D}$ and $f, g \in \mathcal{M}$, where $f^i(s) := \langle e_i, f(s)\rangle$ and $f_i(s) := \overline{f^i(s)}$, $f_0(s) = f^0(s) = 1$.

Let $\mathcal{A}$ be a unital C^* or von-Neumann algebra. A family of maps $\{j_t\}_{t\geq 0}$ from $\mathcal{A}$ to $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$ is said to be adapted to the filtration $\{\mathcal{A}'' \otimes \mathcal{B}(\Gamma_{t]})\}_{t\geq 0}$ if $j_t(x) \in \mathcal{A}'' \otimes \mathcal{B}(\Gamma_{t]}) \otimes I_{\Gamma_{[t}}$ for all $x \in \mathcal{A}$, $t \geq 0$.

Definition 1.7.3. *We say that a family of completely positive maps $(j_t)_{t\geq 0}$ adapted to $\{\mathcal{A}'' \otimes \mathcal{B}(\Gamma_{t]})\}_{t\geq 0}$, is a completely positive, contractive (CPC for short) flow with a noise or multiplicity space k_0 (Hilbert space) and structure maps (θ^μ_ν) belonging to $Lin(\mathcal{A}, \mathcal{A})$, where $\mu, \nu \in \{0\} \cup \{1, 2, ...dimk_0\}$, if the following holds:*

(i) There is a dense $*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ (norm dense for C^* algebra and ultraweakly dense for von-Neumann algebra) such that $\mathcal{A}_0$ is contained in the domain of all the maps θ^μ_ν, and j_t is normal if $\mathcal{A}$ is a von-Neumann algebra. Furthermore for $ue(f) \in h \otimes \Gamma$, the map $t \to j_t(x)ue(f)$ is continuous.

(ii) The family $\{j_t(x)\}_{t\geq 0}$ satisfy a weak q.s.d.e. of the form: for $u, v \in \mathcal{H},\ f, g \in L^2(\mathbb{R}_+, k_0)$ and $x \in \mathcal{A}_0$:

$$\begin{aligned} &\langle j_t(x)ue(f), ve(g)\rangle \\ &\quad = \langle\ xue(f), ve(g)\rangle + \sum_{\mu,\nu} \int_o^t ds \langle\ j_s(\theta^\mu_\nu(x))ue(f), ve(g)\rangle g^\mu(s) f_\nu(s) \end{aligned} \tag{1.3}$$

or symbolically,

$$j_t(x) = x + \sum_{\mu,\nu} \int_0^t j_s(\theta^\mu_\nu(x))\Lambda^\nu_\mu(ds) \tag{1.4}$$

where $f^i(s) = \langle e_i, f(s)\rangle$, $f_i(s) = \overline{f^i(s)}$, $f_0(s) = f^0(s) = 1$, $\{e_i\}_{i=1}^{dim\, k_0}$ being an orthonormal basis for the noise space k_0 with respect to which the structure maps $\theta^i_j((i, j) \neq (0, 0))$ are given, and where $\Lambda^\mu_\nu(s)$ are the fundamental integrators as described above.

The CPC flow $(j_t)_{t\geq 0}$ is called a $*$-homomorphic quantum stochastic flow if each j_t is a $*$ homomorphism. Such $*$-homomorphic flows are often refered to as Evans-Hudson flows.

A necessary condition for a CPC flow j_t to be $*$-homomorphic is the following algebraic relations among the structure maps:
For $x \in \mathcal{A}_0$,

$$\theta^\mu_\nu(xy) = \theta^\mu_\nu(x)y + x\theta^\mu_\nu(y) + \sum_{i=1}^{dim k_0} \theta^\mu_i(x)\theta^i_\nu(y), \quad \theta^\mu_\nu(x)^* = \theta^\nu_\mu(x^*). \tag{1.5}$$

Often it is convenient to associate a matrix, called the structure matrix, with the structure maps θ^μ_ν as follows:

$$\begin{pmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \sigma \end{pmatrix},$$

where $\sigma := \sum_{i,j} \theta^i_j(x) \otimes |e_j >< e_i|$, $\delta(x) := \sum_i \theta^i_0(x) \otimes e_i$, $\delta^\dagger(x) := \delta(x^*)^*$, and $\mathcal{L}(x) = \theta^0_0(x)$, for $x \in \mathcal{A}_0$.

1.7.3 Cordinate free quantum stochastic calculus

In this subsection, we briefly review the cordinate free formalism of the quantum stochastic calculus, as developed in [62].

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. Suppose that $R \in Lin(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ and $T \in Lin(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$. Given $f \in \mathcal{H}_2$, we define an operator $\langle f, R\rangle \in Lin(\mathcal{H}_1, \mathcal{H}_1)$ such that $Dom(\langle f, R\rangle) = Dom(R)$ and for $u \in Dom(R)$, $\langle f, R\rangle u$ is the unique vector in $\mathcal{H}_1$ determined by $\langle\langle f, R\rangle u, v\rangle = \langle Ru, v \otimes f\rangle$, for all $v \in \mathcal{H}_1$.

For $f, g \in \mathcal{H}_2$, we define another operator $\langle f, T_g\rangle \in Lin(\mathcal{H}_1, \mathcal{H}_1)$ as follows: $Dom(\langle f, T_g\rangle) = \{u |\ u \otimes f \in Dom(T) \text{ for } f \in \mathcal{H}_2\}$, and for $u \in Dom(\langle f, T_g\rangle)$, $\langle f, T_g\rangle u$ is the unique vector in $\mathcal{H}_1$ determined by $\langle\langle f, T_g\rangle u, v\rangle = \langle T(u \otimes g), v \otimes f\rangle$ for all $v \in \mathcal{H}_1$. Likewise, we may define the operator $T_f \in Lin(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ by $\langle T_f u, v \otimes g\rangle := \langle T(u \otimes f), v \otimes g\rangle$, for u such that $u \otimes f \in Dom(T)$.

Recall the symmetrization operator from the free Fock space $\Gamma_{fr}(\mathcal{H}_2)$ to the symmetric Fock space $\Gamma(\mathcal{H}_2)$, as defined in the subsection 1.7.1. We define the creation operator $a^\dagger(R)$ which will act on linear span of vectors of the form $ug^{\otimes^n}$ for $u \in Dom(R)$ and $g \in \mathcal{H}_2$ as follows:
$a^\dagger(R)ug^{\otimes^n} := \frac{1}{\sqrt{n+1}}(1_{\mathcal{H}_1} \otimes S)((Ru) \otimes f^{\otimes^n})$. It is easy to observe that $\sum_{n\geq 0} \frac{1}{n!}\|a^\dagger(R)uf^{\otimes^n}\|^2 < \infty$, as a result, we may define $a^\dagger(R)ue(f) := \oplus_{n\geq 0} \frac{1}{\sqrt{(n!)}} a^\dagger(R)uf^{\otimes^n}$.

We define annihilation $a(R)$ as $a(R)ue(f) := \langle R, f\rangle ue(f)$, where $\langle R, f\rangle := \langle f, R\rangle^*$, assuming that it exists. The number operator, denoted by $\Lambda(T)$ is defined by $\Lambda(T)ue(f) := a^\dagger(T_f)ue(f)$.

Now we use these definitions to define the three fundamental processes as follows: Let $\mathcal{H}_1 = h$, $\mathcal{H}_2 := k_0$, Let $\mathcal{K}_{[t} := L^2([t, +\infty), k_0)$. Then we have:

1. Creation process: Suppose that $\Delta \subseteq [t, +\infty)$. Define $R^\Delta_t \in Lin(h \otimes \Gamma_{t]}, h \otimes \Gamma_{t]} \otimes \mathcal{K}_{[t})$ by:
$$R^\Delta_t(ue(f_{t]})) := P(1_h \otimes \chi_\Delta(Ru) \otimes e(f_{t]})),$$

where P is the unitary flip sending $h \otimes \mathcal{K}_{[t} \otimes \Gamma_{t]}$ to $h \otimes \Gamma_{t]} \otimes \mathcal{K}_{[t}$. Then we define the creation process as $a_R^\dagger(\Delta) := a^\dagger(R_t^\Delta)$.

2. Annihilation process: We define the annihilation process as: $a_R(\Delta)ue(f) := ((\int_\Delta \langle R, f(s)\rangle)ue(f_{t]}))e(f_{[t})$.
3. Number process: Define $\widehat{T} \in Lin(L^2([t,+\infty), h \otimes k_0), L^2([t,+\infty), h \otimes k_0))$ by $(\widehat{T}(uf_{[t)}))(s) := T(uf(s))$. Define $T^\Delta_{f_{[t}} \in Lin(h \otimes \Gamma_{t]}, h \otimes \Gamma_{t]} \otimes \mathcal{K}_{[t})$ by $T^\Delta_{f_{[t}} ue(f_{t]}) := P((1_h \otimes \widehat{\chi_\Delta})\widehat{T}(uf_{[t)}e(f_{t]}))$. Now we define the number process $\Lambda_T(\Delta)$ as $\Lambda_T(\Delta)ue(f) := a^\dagger(T^\Delta_{f_{[t}})ue(f)$.

1.7.4 Hudson-Parthasarathy (HP) type equation in cordinate free form

Let $R, S \in Lin(h, h \otimes k_0)$, and $T \in Lin(h \otimes k_0, h \otimes k_0)$. Suppose that $\mathcal{D}$ and $\mathcal{M}$ are dense subsets of h and Γ respectively. Let $(X_t)_{t\geq 0}, (E_t)_{t\geq 0}, (F_t)_{t\geq 0}, (G_t)_{t\geq 0}$ and $(H_t)_{t\geq 0}$ be $(\mathcal{D}, \mathcal{M})$ adapted, regular processes. We will symbolically write

$$\begin{aligned} X_t &= \int_0^t (E_s \Lambda_T(dt) + F_s a_R(ds) + G_s a_S^\dagger(ds) + H_s ds) \\ X_0 &= id_{h\otimes\Gamma} \end{aligned}$$

to mean that X_t satisfies an equation of the form:

$$\begin{aligned} &\langle X_t ve(g), ue(f)\rangle = \\ &\qquad \int_0^t ds\ \langle \{\langle f(s), E_s PT_{g(s)}\rangle + F_s\langle R, g(s)\rangle + \\ &\qquad G_s\langle f(s), S\rangle + H_s\} ve(g), ue(f)\rangle. \end{aligned}$$

We will call a process $(X_t)_{t\geq 0}$ admissible, if we have

$$sup_{0\leq s\leq t} \|X_t ue(f)\| \leq \|r_t^f u\|,$$

for $u \in \mathcal{D}$, where r_t^f is a densely defined closable operator.

Suppose that $(X_t)_{t\geq 0}$, $(E_t)_{t\geq 0}$, $(F_t)_{t\geq 0}$, $(G_t)_{t\geq 0}$, $(H_t)_{t\geq 0}$, $(X'_t)_{t\geq 0}$, $(E'_t)_{t\geq 0}$, $(F'_t)_{t\geq 0}$, $(G'_t)_{t\geq 0}$, $(H'_t)_{t\geq 0}$, are $(\mathcal{D}, \mathcal{M})$-adapted, regular processes. Furthermore, assume that all these processes are also admissible in the above sense.

Assume that

$$X_t = \int_0^t\ (E_s \Lambda_T(ds) + F_s a_R(ds) + G_s a_S^\dagger(ds) + H_s ds),$$

$$X'_t = \int_0^t (E'_s \Lambda_T(ds) + F'_s a_R(ds) + G'_s a_S^\dagger(ds) + H'_s ds).$$

Assume that $\mathcal{D} \subseteq \cap_{\xi \in \mathcal{M}} Dom(\langle R, \xi\rangle) \cap Dom(\langle R', \xi\rangle)$. Then for $u, v \in \mathcal{D}$ and $f, g \in L^2(\mathbb{R}_+, k_0)$ such that $\int_0^t ds\ \|f(s)\|^4 < \infty$ and $\int_0^t ds\ \|g(s)\|^4 < \infty$, we have the quantum Ito formula given by:

$$\begin{aligned}
&\langle X_t ve(g), X'_t ue(f)\rangle \\
&\int_0^t ds\ [\langle X_s ve(g), \{\langle g(s), E'_s PT'_{f(s)}\rangle + F'_s\langle R', f(s)\rangle + G'_s\langle g(s), S'\rangle + H'_s\} ue(f)\rangle] \\
&+ \int_0^t ds\ [\langle \{\langle f(s), E_s PT_{g(s)}\rangle + F_s\langle R, g(s)\rangle + G_s\langle f(s), S\rangle + H_s\} ve(g), X'_s ue(f)\rangle] \\
&+ \int_0^t ds\ [\langle E_s PT_{g(s)}(ve(g)), E'_s PT'_{f(s)}(ue(f))\rangle + \langle E_s PT_{g(s)}(ve(g)), G'_s PS'(ue(f))\rangle \\
&+ \langle G_s PS(ve(g)), E'_s PT'_{f(s)}(ue(f))\rangle + \langle G_s PS(ve(g)), G'_s PS'(ue(f))\rangle].
\end{aligned} \tag{1.6}$$

We will say that X_t satisfies a Hudson-Parthasarathy (HP for short) quantum stochastic differential equation with coefficients (R, S, T, A) and initial condition $X_0 = id_{h\otimes\Gamma}$ if

$$\begin{aligned}
X_t &= \int_0^t X_s \circ (a_R(dt) + a_S^\dagger(dt) + \Lambda_T(dt) + Adt), \\
X_0 &= id_{h\otimes\Gamma},
\end{aligned} \tag{1.7}$$

where $R, S \in Lin(h, h\otimes k_0)$, $T \in Lin(h\otimes k_0, h\otimes k_0)$, $A \in Lin(h, h)$. We will call the matrix $Z := \begin{pmatrix} A & R^* \\ S & T \end{pmatrix}$, the coefficient matrix associated with the above HP type equation.

An HP equation with bounded coefficients always admits a unique solution i.e. if R, S, T, A are bounded, then there exists a unique $(X_t)_{t\geq 0}$ satisfying equation (1.7). The solution will be contractive i.e. X_t will be contractive for each $t \geq 0$ if and only if $Z + Z^* + Z\widehat{Q}Z^* \leq 0$ and the solution will be a co-isometry if $Z + Z^* + Z\widehat{Q}Z^* = 0$, where $\widehat{Q} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

We now briefly state few facts about HP equation with unbounded coefficients. For details we refer to [62, 49, 50].

Fix dense subspaces $\mathcal{D}_0 \subseteq h$ and $\mathcal{V}_0 \subseteq k_0$, and for the quadruple (R, S, T, A), assume that $\mathcal{D}_0 \subseteq Dom(R) \cap Dom(S) \cap Dom(A)$ and $\mathcal{D}_0 \otimes \mathcal{V}_0 \subseteq Dom(T)$. Furthermore assume that $\mathcal{D}_0 \subseteq \cap_{\xi\in\mathcal{V}_0} Dom(\langle R, \xi\rangle)$. Suppose that there exists a sequence of operators $(Z^{(n)})_n \in \mathcal{B}(h\otimes\widehat{k_0})$, where $\widehat{k_0} := \mathbb{C}\oplus k_0$, such that $Z^{(n)} + Z^{(n)*} + Z^{(n)}\widehat{Q}Z^{(n)*} \leq 0$, satisfying:

$\lim_{n\to\infty}\langle\widehat{\xi}, Z^{(n)}_{\widehat{\eta}}\rangle u = \langle\widehat{\xi}, Z_{\widehat{\eta}}\rangle u,$

$sup_{n\geq 1}\|Z^{(n)}_{\widehat{\eta}} u\| < \infty$, for $\widehat{\eta} \in \widehat{k}_0$.

Theorem 1.7.4. *[50, 49, 62] Let (R, S, T, A) satisfy the conditions stated above. Then the HP equation*

$$dV_t = V_t(a_R(dt) + a^{\dagger}_S(dt) + \Lambda_T(dt) + Adt),\ V_0 = id_{h\otimes\Gamma}$$

admits a contractive, (h, Γ) adapted regular solution.

Let $\mathcal{L}_0(x)$ be the bilinear form on $\mathcal{D}_0$ given by:

$$\langle v, \mathcal{L}_0(x)u\rangle := \langle v, xAu\rangle + \langle Av, xu\rangle + \langle Rv, xRu\rangle.$$

and for $X \in \mathcal{B}(h\otimes\Gamma)$, $\xi, \eta \in \mathbb{C}\oplus\mathcal{V}_0$, let $\mathcal{L}^{\xi}_{\eta}(X)$ denote the bilinear form on $\mathcal{D}_0\otimes\Gamma$ given by:

$$\langle v\psi, X\langle\xi, Z_\eta\rangle u\psi'\rangle + \langle\langle\xi, Z_\eta\rangle v\psi, Xu\psi'\rangle + \langle\widehat{Q}Z_\xi v\psi, X\widehat{Q}Z_\eta u\psi'\rangle,$$

where $u, v \in \mathcal{D}_0$ and $\psi, \psi' \in \Gamma$ and extend linearly. Let $\beta_\lambda := \{x \in \mathcal{B}(h) : \langle v, \mathcal{L}_0(x)u\rangle = \lambda\langle v, xu\rangle$ for all $u, v \in \mathcal{D}_0\}$.

Theorem 1.7.5. *[50, 49, 62]*

I *The solution of the HP equation in Theorem 1.7.4 will be isometric, if we have*

$\mathcal{L}^{\xi}_{\eta}(I) = 0$ *for all* $\xi, \eta \in \mathbb{C}\oplus\mathcal{V}_0$,

$\beta_\lambda = \{0\}$ *for some* λ.

II *If furthermore there exists dense subspaces $\widetilde{\mathcal{D}_0} \subseteq h$ and $\widetilde{\mathcal{V}_0} \subseteq k_0$, such that $\widetilde{\mathcal{D}_0}\otimes\widetilde{\mathcal{V}_0} \subseteq Dom(Z^*)$ and the following conditions hold:*

$\lim_{n\to\infty}\langle\widehat{\xi}, Z^{*(n)}_{\widehat{\eta}}\rangle u = \langle\widehat{\xi}, Z^{*}_{\widehat{\eta}}\rangle u$, *for all* $\xi, \eta \in \widetilde{\mathcal{V}_0}$, $u \in \widetilde{\mathcal{D}_0}$;

$sup_{n\geq 1}\|Z^{(n)*}_{\widehat{\eta}} u\| < \infty$ *for all* $\eta \in \widetilde{\mathcal{V}_0}$, $u \in \widetilde{\mathcal{D}_0}$;

$\widetilde{\mathcal{L}}^{\xi}_{\eta}(I) = 0$ *for all* $\xi, \eta \in \mathbb{C}\oplus\widetilde{\mathcal{V}_0}$;

$\widetilde{\beta}_\lambda = \{0\}$ *for some* $\lambda > 0$;

where the definitions of $\widetilde{\mathcal{L}}^{\xi}_{\eta}$ and $\widetilde{\beta}_\lambda$ are similar to the definitions of $\mathcal{L}^{\xi}_{\eta}$ and β_λ, with the replacement of Z by Z^*, $\mathcal{D}_0$ by $\widetilde{\mathcal{D}_0}$ and $\mathcal{V}_0$ by $\widetilde{\mathcal{V}_0}$.

Then the solution in Theorem 1.7.4 will be a co-isometry.

III *If both the conditions in* **I** *and* **II** *hold, then the solution will be unitary.*

Theorem 1.7.6. *Let $R : Dom(R) \to h \otimes k_0$ be a densely defined closed operator with $Dom(R) \subseteq h$, for Hilbert spaces h, k_0. Suppose there exists a dense subspace $\mathcal{W}_0 \subseteq k_0$, such that $u \otimes \xi \in Dom(R^*)$ for $u \in Dom(R), \xi \in \mathcal{W}_0$. Let H be a densely defined self adjoint operator on h such that $iH - \frac{1}{2}R^*R$ $(= G)$ as well as $-iH - \frac{1}{2}R^*R$ $(= G^*)$ generate C_0 semigroups in h. Furthermore, suppose that both of $Dom(G)$ and $Dom(G^*)$ are contained in $Dom(R)$. Then the QSDE:*

$$\begin{gathered} dU_t = U_t \circ (a_R^\dagger(dt) - a_R(dt) + (iH - \frac{1}{2}R^*R)dt) \\ U_0 = id; \end{gathered} \tag{1.8}$$

has a unique solution, which is unitary.

Proof. Let $Z = \begin{pmatrix} iH - \frac{1}{2}R^*R & -R^* \\ R & 0 \end{pmatrix}$. Suppose $H = u|H|,\ R = v|R|$ be the polar decomposition of H and R respectively.
Put $A^{(n)} := iu(1 + \frac{|H|}{n})^{-1}|H| - \frac{1}{2}R^{(n)*}R^{(n)}$, where $R^{(n)} := R(1 + \frac{|R|}{n})^{-1}$, and $Z^{(n)} = \begin{pmatrix} A^{(n)} & -R^{(n)*} \\ R^{(n)} & 0 \end{pmatrix}$. Then it can be verified that all the conditions of Theorem 1.7.4 hold. Thus the above equation has a contractive solution $U_t, t \geq 0$. Now observe that in the notation of Theorem 1.7.5, $\mathcal{L}^\gamma_\eta(I) = 0$, for all $\gamma, \eta \in \mathbb{C} \oplus \mathcal{W}_0$. We will prove that $\beta_\lambda = \{0\}$. Formally define $L(x) = R^*(x \otimes 1_{k_0})R + xG + G^*x$, where $G = iH - \frac{1}{2}R^*R$. Then the conditions of Theorem 1.6.3 hold, so that the minimal semigroup associated with the form generator is conservative. Thus by condition (v) of Theorem 3.2.16 of [62], $\beta_\lambda = \{0\}$. The same set of arguments hold for $G = -iH - \frac{1}{2}R^*R$, which implies that $\widetilde{\beta}_\lambda = \{0\}$ (in the notation of Theorem 1.7.5). Moreover $\widetilde{\mathcal{L}}^\gamma_\eta(I) = 0$. Thus all the conditions of Theorem 1.7.5 hold, which proves that the solution is unitary. The uniqueness follows from the results in [50, 49]. □

1.7.5 Evans-Hudson (EH) type equation in cordinate free form

Let $\mathcal{A}$ be a C^* or von-Neumann algebra. Let us assume that we are given the linear maps $(\mathcal{L}, \delta, \sigma)$ (called structure maps, defined on a dense $*$ subalgebra $\mathcal{A}_0$ of $\mathcal{A}$), where $\mathcal{L} \in Lin(\mathcal{A}, \mathcal{A})$, $\delta \in Lin(\mathcal{A}, \mathcal{A} \otimes k_0)$, $\sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(k_0))$.

We now define the four fundamental processes as follows:
Let $\Delta \subseteq [t, +\infty)$. All the basic four processes that we will define, are well-defined on $\mathcal{A}_0 \otimes \mathcal{E}(\mathcal{K})$ and takes values in $\mathcal{A} \otimes \Gamma$ (see [62]). We define them as:

Annihilation: $(a_\delta(\Delta) (\sum_{i=1}^n x_i \otimes e(f_i)))\, u = \sum_{i=1}^n a_{\delta(x_i^*)}(\Delta)(ue(f_i))$, Creation: $\left(a_\delta^\dagger(\Delta)\left(\sum_{i=1}^n x_i \otimes e(f_i)\right)\right) u = \sum_{i=1}^n a_{\delta(x_i)}^\dagger(\Delta)(ue(f_i))$,

Number: $(\Lambda_\sigma(\Delta)\left(\sum_{i=1}^n x_i \otimes e(f_i)\right)) u = \sum_{i=1}^n \Lambda_{\sigma(x_i)}(\Delta)(ue(f_i))$,

Time: $(\mathcal{I}_\mathcal{L}(\Delta)\left(\sum_{i=1}^n x_i \otimes e(f_i)\right)) u = \sum_{i=1}^n |\Delta|((\mathcal{L}(x_i)u) \otimes e(f_i))$, where $|\Delta|$ is the Lebesgue measure of Δ.

Definition 1.7.7. *Suppose that $\mathcal{A}$ is a C^* algebra. Let $j_t : \mathcal{A}_0 \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma)$ be a family of CP maps. Suppose that $J_t : \mathcal{A}_0 \otimes \mathcal{E}(\mathcal{K}) \to \mathcal{A}'' \otimes \Gamma$ be defined by $J_t(x \otimes e(f))u := j_t(x)(ue(f))$, where $ue(f) := u \otimes e(f)$. We say that j_t is a CPC flow satisfying an Evans-Hudson type quantum stochastic differential equation with structure maps $(\mathcal{L}, \delta, \sigma)$ and initial condition $j_0(x) := x \otimes 1_\Gamma$, symbolically*

$$\begin{aligned} dJ_t &= J_t \circ (a_\delta(dt) + a_\delta^\dagger(dt) + \Lambda_\sigma(dt) + \mathcal{I}_\mathcal{L} dt) \\ J_0 &= id_{\mathcal{A}''\otimes\Gamma}, \end{aligned} \tag{1.9}$$

if

1. *For $ue(f) \in h \otimes \Gamma$, the map $t \to j_t(x)ue(f)$ is continuous for all $x \in \mathcal{A}$.*
2. *j_t satisfies the following equation on $\mathcal{A}_0$:*

$$\begin{aligned} \langle j_t(x)ve(g), ue(f)\rangle = \langle xve(g), ue(f)\rangle \\ \int_0^t ds\ \langle\{\langle f(s), \sigma(x)_{g(s)}\rangle + \delta^\dagger(x)_{g(s)} + \\ \langle f(s), \delta\rangle + \mathcal{L}(x)\}ve(g), ue(f)\rangle, \end{aligned} \tag{1.10}$$

for $x \in \mathcal{A}_0$.

Remark 1.7.8. *If $\mathcal{A}$ is a von-Neumann algebra, then a family of completely positive contractive maps $(j_t)_{t\geq 0}$ from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{B}(\Gamma)$ will be called a CPC flow if it satisfies all the hypotheses of definition 1.7.7 along with the additional hypothesis that for each $t \geq 0$, the map j_t is normal.*

The matrix $\Theta := \begin{pmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \sigma \end{pmatrix}$, where $\delta^\dagger(x) := (\delta(x^*))^*$, is called the structure or coefficient matrix associated with the above equation.

Note that for $x \in \mathcal{A}$, $(j_t(x))_{t\geq 0}$ is an operator valued process, which is $(h, \mathcal{E}(L^2(\mathbb{R}_+, k_0)))$-adapted, regular and admissible (being contractive). This, together with Proposition

1.2.7, Lemma 1.2.8 and equation (1.6) yields the following quantum Ito formula for map valued processes (see [62]):

$$
\begin{aligned}
&\langle j_t(x)ue(f), j_t(y)ve(g)\rangle \\
&= \langle xue(f), yve(g)\rangle + \int_0^t ds\ \{j_s(\mathcal{L}(x) + \langle g(s), \delta(x)\rangle \\
&+ \delta^\dagger(x)_{f(s)} + \langle g(s), \sigma(x)_{f(s)}\rangle ue(f), j_s(y)ve(g)\rangle \\
&+ \langle j_s(x)ue(f), j_s(\mathcal{L}(y) + \langle f(s), \delta(y)\rangle + \delta^\dagger(y)_{g(s)} \\
&+ \langle f(s), \sigma(y)_{g(s)}\rangle)ve(g)\rangle \\
&+ \langle \hat{j}_s(\delta(x) + \sigma(x)_{f(s)})ue(f), \hat{j}_s(\delta(y) + \sigma(y)_{g(s)})ve(g)\rangle\},
\end{aligned}
\tag{1.11}
$$

where $\hat{j}_t := j_t \otimes id_{\Gamma_{fr}}$, as in Lemma 1.2.8.

The natural necessary algebraic conditions for such a CPC flow j_t to be $*$-homomorphic are the following, which are cordinate free versions of equation (1.5) discussed previously:

1. $\sigma(x) := \pi(x) - x \otimes 1_{k_0}$, where $\pi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(k_0)$ is a normal $*$ representation;
2. δ is a π derivation i.e. $\delta(xy) = \delta(x)y + \pi(x)\delta(y)$ for $x, y \in Dom(\delta)$.
3. $\mathcal{L}(x^*) = \mathcal{L}(x)^*$ for all $x \in \mathcal{A}_0$.
4. $$\delta(x)^*\delta(y) = \mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y),$$
   for all $x, y \in \mathcal{A}_0$.

It is known [52, 62] that if the structure maps $\mathcal{L}, \delta$ are bounded, then there exists a unique process $(j_t)_{t\geq 0}$ consisting of normal $*$-homomorphisms, which satisfies equation (1.10) with the initial condition $j_0(x) = x \otimes id_\Gamma$. However, such existence are not known in general (except some special cases e.g. [62, 30, 28, 25]), if the structure maps are unbounded.

1.7.6 Time-shift and cocycle property

Define $\theta_t : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ by $\theta_t(f)(s) := f(s-t)\chi_{[t,+\infty)}(s)$. This is known as the time-shift operator. We take the second quantization of the time-shift i.e. $\Gamma(\theta_t)$ which is defined on the exponential vectors as $\Gamma(\theta_t)(e(f)) := e(\theta_t(f))$. Note that $\Gamma(\theta_t)$ maps Γ isometrically onto $\Gamma_{[t}$.

Proposition 1.7.9. *[62] Suppose that $(V_t)_{t\geq 0}$ is a bounded, adapted, regular solution of an HP equation of the form (1.7) such that the coefficients R, S, T, A are all bounded. Then we have*

$$V_{t+s} = V_s[\Gamma(\theta_s)V_t\Gamma(\theta_s^*)]$$

for all $t, s \geq 0$.

As a result of this, the family of maps $V_t^{\xi,\eta} : h \to h$ defined by $V_t^{\xi,\eta} := \langle e(\chi_{[0,t]}\xi), V_{t_{e(\chi_{[0,t]}\eta)}} \rangle,\ t \geq 0$, are C_0 semigroups.

Proposition 1.7.10. *Suppose that* $(j_t)_{t\geq 0}$ *is a CPC flow, which satisfies an EH equation of the from (1.9) such that the structure maps* $\mathcal{L}, \delta, \sigma$ *are all bounded. Then we have*

$$j_{t+s}(\cdot) = j_s(\Gamma(\theta_s) j_t(\cdot) \Gamma(\theta_s^*)).$$

Such a property is usually referred to as cocycle property of the CPC flow $(j_t)_{t\geq 0}$.

As a result, if $j_t(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{B}(\Gamma)\ \forall\ t \geq 0$, where $\mathcal{A}$ is a C^*-algebra, the map $j_t^{\xi,\eta}(\cdot) := \langle e(\xi\chi_{[0,t]}), j_t(\cdot) e(\eta\chi_{[0,t]}) \rangle$ is a C_0 (with respect to the C^*-norm) semigroup on $\mathcal{A}$. If on the other hand, $\mathcal{A}$ is a von-Neumann algebra, then $j_t^{\xi,\eta}(\cdot) := \langle e(\xi\chi_{[0,t]}), j_t(\cdot) e(\eta\chi_{[0,t]}) \rangle$ is a C_0 (with respect to the ultraweak topology) semigroup on $\mathcal{A}$. The semigroup $(j_t^{0,0})_{t\geq 0}$ is called the vacuum expectation semigroup, associated with the CPC cocycle $(j_t)_{t\geq 0}$.

It is not known in general that if $(j_t)_{t\geq 0}$ is a CPC flow which satisfies an EH equation of the form (1.9) with unbounded structure maps, then whether $(j_t)_{t\geq 0}$ is a cocycle. However, in this case, if we assume that $(j_t)_{t\geq 0}$ is a cocycle and $(j_t^{0,0})_{t\geq 0}$ is a C_0 semigroup, we have the following lemma:

Lemma 1.7.11. *For a CPC cocycle flow* j_t, $j_t^{c,d}(x)$ *defined by* $j_t^{c,d}(x) := \langle e(c1_{[0,t]}), j_t(x) e(d1_{[0,t]}) \rangle$ *for* $x \in \mathcal{A}$, *is a* C_0 *semigroup on* $\mathcal{A}$. *Furthermore, the restriction of the generator of* $j_t^{c,d}(x)$ *to* $\mathcal{A}_0$ *is*
$\mathcal{L} + \langle c, \delta \rangle + \delta_d^\dagger + \langle c, \sigma_d \rangle + \langle c, d \rangle id$.

Proof. Let $s < t$, the semigroup property follows as:

$$\begin{aligned} j_{s+t}^{c,d}(x) &= \langle e(c1_{[0,s+t]}), j_{s+t}(x) e(d1_{[0,s+t]}) \rangle \\ &= \langle e(c1_{[0,s]}) \otimes e(c1_{[s,s+t]}), j_s \circ \Gamma(\theta_s) j_t(x) \Gamma(\theta_s^*) e(d1_{[0,s]}) \otimes e(d1_{[s,s+t]}) \rangle \\ &= j_s^{c,d}(\langle e(c1_{[s,s+t]}), \Gamma(\theta_s) j_t(x) \Gamma(\theta_s^*) e(d1_{[s,s+t]}) \rangle) \\ &= j_s^{c,d}(\langle e(c1_{[0,t]}), j_t(x) e(d1_{[0,t]}) \rangle) = j_s^{c,d} \circ j_t^{c,d}(x). \end{aligned} \tag{1.12}$$

C_0 property can be proved as follows: For a vector $c \in k_0$, we write c_t for $c1_{[0,t]}$.

$$\begin{aligned}
&|\langle ue(c_t), j_t(x)ve(d_t)\rangle - \langle u, xv\rangle| \\
&\leq |\langle u[e(c_t) - e(0)], j_t(x)ve(0)\rangle| + |\langle ue(0), j_t(x)ve(0)\rangle - \langle u, xv\rangle| \\
&+ |\langle ue(c_t), j_t(x)v[e(d_t) - e(0)]\rangle| \\
&\leq \|u\|\|v\|\|x\| \left\{ e^{\frac{\|d\|^2 t}{2}} \sqrt{e^{t\|c\|^2} - 1} + e^{\frac{\|c\|^2 t}{2}} \sqrt{e^{t\|d\|^2} - 1} \right\} + \\
&|\langle u, (j_t^{0,0}(x) - x)v\rangle|
\end{aligned} \tag{1.13}$$

If $j_t^{0,0}$ is C_0 in the norm topology of $\mathcal{A}$ i.e. if $\mathcal{A}$ is a C^* algebra, then the above estimates implies that $\langle u, j_t^{c,d}(x)v\rangle \rightarrow \langle u, xv\rangle$, uniformly over the unit ball of h. On the other hand, if $j_t^{0,0}$ is C_0 in the ultraweak topology of $\mathcal{A}$, in case $\mathcal{A}$ is a von-Neumann algebra, then it follows from the above estimates that $\langle u, j_t^{c,d}(x)v\rangle \rightarrow \langle u, xv\rangle$ for a given $u, v \in h$.

Now for $x \in \mathcal{A}_0$ *and* $u, v \in h$, we have

$$\begin{aligned}
&\langle ue(c1_{[0,s]}), j_s(x)ve(d1_{[0,s]})\rangle = \langle u, xv\rangle e^{\langle c,d\rangle s} \\
&+ \int_0^s d\tau \langle ue(c1_{[0,s]}), j_\tau(\mathcal{L}(x) + \langle c, \delta(x)\rangle + \delta^\dagger(x)_d + \langle c, \sigma(x)_d\rangle)ve(d1_{[0,s]})\rangle
\end{aligned}$$

i.e.

$$\langle u, e^{-s\langle c,d\rangle} j_s^{c,d}(x)v\rangle = \langle u, xv\rangle + \int_0^s d\tau \langle u, e^{-\tau\langle c,d\rangle} j_\tau^{c,d}(\mathcal{L}(x) + \langle c, \delta(x)\rangle + \delta^\dagger(x)_d + \langle c, \sigma(x)_d\rangle)v\rangle, \tag{1.14}$$

and from this the conclusion follows. □

1.7.7 Quantum stop time

There are many formulations of the concept of quantum stop times. We refer to [54],[5],[7] for details of such formulations. We briefly describe the two notions of quantum stop time:

Definition 1.7.12 (Parthasarathy,Sinha,Attal)**.** *Suppose that $(\mathcal{H}_{t]})_{t\geq 0}$ be a family of Hilbert spaces such that for $s \leq t$, $\mathcal{H}_{s]} \subseteq \mathcal{H}_{t]} \subseteq \mathcal{H}$ for some Hilbert space $\mathcal{H}$ (called a filtration). A quantum stop time adapted to this filtration is a resolution of identity $E : [0, +\infty] \rightarrow P(\mathcal{H})$, $P(\mathcal{H})$ being the set of projections on $\mathcal{H}$, such that $E([0, t]) \in P(\mathcal{H}_{t]})$.*

Another formulation of the concept is:

Definition 1.7.13. *[7][Barnett] Let $(\mathfrak{A}_t)_{t\geq 0}$ be an increasing family of von-Neumann algebras (called a filtration). A quantum random time or stop time adapted to the filtration $(\mathfrak{A}_t)_{t\geq 0}$ is an increasing family of projections $(E_t)_{t\geq 0},\ E_\infty = I$ such that E_t is a projection in $\mathfrak{A}_t$ and $E_s \leq E_t$ whenever $0 \leq s \leq t < +\infty$ and furthermore, for $t \geq s$, $E_t \downarrow E_s$ as $t \downarrow s$.*

Chapter 2

A new proof of homomorphism for flows

2.1 Set-up

Let $\mathcal{A}$ be a C^* or von-Neumann algebra, equipped with a semifinite, faithful, lower-semicontinuous (also normal in case $\mathcal{A}$ is a von-Neumann algebra) trace τ, and let $\mathcal{A}_0$ be a dense $*$-subalgebra of $\mathcal{A}$ as described in definition 1.7.3 in chapter 1, which is also dense in $h(\equiv L^2(\mathcal{A},\tau))$ in the L^2- topology. Assume that $(j_t)_{t\geq 0}$ is a CPC flow as described in subsection 1.7.2 and let $(T_t)_{t\geq 0}$ be the vacuum expectation semigroup of j_t, i.e. $\langle u, T_t(x)v\rangle := \langle ue(0), j_t(x)ve(0)\rangle = \langle u, j_t^{0,0}(x)v\rangle$ for $u,v \in h,\ x \in \mathcal{A}$. We assume that the vacuum expectation semigroup $(j_t^{0,0})_{t\geq 0}$ is a C_0 (in the norm or ultraweak topology according as $\mathcal{A}$ is C^* or von-Neumann algebra) semigroup. Furthermore, we make the following assumptions:

A(i) For each $t \geq 0$, T_t extends to a bounded operator (again denoted by T_t,) on the Hilbert space h such that $(T_t)_{t\geq 0}$ is a *contractive, analytic C_0-semigroup of operators in the Hilbert space h. We shall denote by $\mathcal{L}_2$ the generator of $((T_t)_{t\geq 0})$ in h.*

A(ii) *Suppose that $\mathcal{A}_0 \subseteq Dom(\mathcal{L}) \cap Dom(\mathcal{L}_2)$, and that T_t leaves $\mathcal{A}_0$ invariant.*

A(iii) *The map π defined by*

$$\pi(x) = \sigma(x) + x \otimes 1_{k_0},$$

is a $*$-homomorphism (normal if $\mathcal{A}$ is a von-Neumann algebra) from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{B}(k_0)$, and the map δ is a well defined π-derivation belonging to $Lin(\mathcal{A}_0, \mathcal{A} \otimes$

k_0), *i.e.*
$\delta(xy) = \delta(x)y + \pi(x)\delta(y)$, *for* $x, y \in \mathcal{A}_0$.

A(iv) *For* x, y *in* $\mathcal{A}_0$, *the following second order cocycle relation holds:*

$$\delta(x)^*\delta(y) = \mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y). \tag{2.1}$$

A(v) *For* $x \in \mathcal{A}_0$, $\mathcal{L}(x^*x) \in \mathcal{A} \cap L^1(\tau)$ *and* $\tau(\mathcal{L}(x^*x)) \leq 0$ *(a kind of weak dissipativity).*

Remark 2.1.1. **A(ii)** *implies* $\mathcal{A}_0$ *is a core for both* $\mathcal{L}$ *and* $\mathcal{L}_2$ *(see section 1.3 of chapter 1). Furthermore observe that because of analyticity in* **A(i)**, **A(iv)** *and* **A(v)**, *the real part of the operator* $(-2\mathcal{L}_2)$ *exists as an operator and is non-negative (see pages 322 and 336 of [34]).*

Moreover it also follows from these assumptions that $\delta(x) \in h \otimes k_0$ *for* $x \in \mathcal{A}_0$. *this is because for* $x \in \mathcal{A}_0$, *we have*

$$\begin{aligned}
\tau(\delta(x)^*\delta(x)) &= \tau(\mathcal{L}(x^*x) - x^*\mathcal{L}(x) - \mathcal{L}(x)^*x) \\
&= \tau(\mathcal{L}(x^*x)) + \langle Re(-2\mathcal{L}_2)x, x\rangle_{L^2(\tau)} \\
&\leq \langle Re(-2\mathcal{L}_2)x, x\rangle_{L^2(\tau)} \text{ since } \tau(\mathcal{L}(x^*x)) \leq 0 \\
&< +\infty.
\end{aligned}$$

Lemma 2.1.2. *If* $(T_t)_{t\geq 0}$ *is symmetric with respect to* τ, *then* **A(i)** *follows. Also if we assume furthermore that* T_t *is conservative and* **A(ii)** *is valid, then* **A(v)** *also follows.*

Proof. On $\mathcal{A}_0$, we have the cocycle identity given in assumption **A(iv)** by proposition 1.6.6 in chapter 1. Using assumption **A(ii)**, we may apply the same arguments as given in pages 66 – 67 of [62] to conclude that

$$\begin{aligned}
&\tau(\delta(x)^*\delta(x)) = sup_n \tau(e_n\delta(x)^*\delta(x)e_n) \\
&= 2sup_n \|\frac{\delta(x)}{\sqrt{2}}e_n\|^2_{h\otimes k_0} \leq 2sup_n\|e_n\|^2_\infty \mathcal{E}(x) \\
&< \infty
\end{aligned} \tag{2.2}$$

(where $\mathcal{E}$ is the Dirichlet form associated with $(T_t)_{t\geq 0}$ and $(e_n)_n$ is an approximate identity from $\mathcal{A}_\tau$.)

This proves that $\delta(x)^*\delta(x) \in L^1(\tau),\ x \in \mathcal{A}_0$. Since on $\mathcal{A}_0$, we have the cocycle identity given by **A(iv)** and $\mathcal{L}(x^*)x$ as well as $x^*\mathcal{L}(x)$ belongs to $L^1(\tau)$, it follows that $\mathcal{L}(x^*x)$ belongs to $L^1(\tau)$.

The fact that $\tau(T_t(x)y) = \tau(xT_t(y))$ for all $x, y \in \mathcal{A}$ such that $x, y \geq 0$, implies that for $x \in \mathcal{A}_0,\ y \in Dom(\mathcal{L})$, such that $x, y \geq 0$, we have $\tau(\mathcal{L}(x)y) = \tau(x\mathcal{L}(y))$. Taking $y = 1$, we have $\tau(\mathcal{L}(x)) = 0$ for all $x \in \mathcal{A}_0$ with $x \geq 0$ (since $\mathcal{L}(1) = 0$). This implies assumption **A(v)**.

□

Remark 2.1.3. *Consider a typical diffusion process in $\mathbb{R}$ whose generator is of the form:*

$$\mathcal{L} = \frac{1}{2}\frac{d}{dx}a^2(x)\frac{d}{dx} + b(x)\frac{d}{dx}.$$

The coefficients a and b are assumed to be smooth and a is assumed to be non-vanishing everywhere. Consider a change of variable $y \equiv \phi(x) = \int_0^x ds e^{\int_0^s dt \frac{2b(t)}{a^2(t)}} + C$ (where C is a constant). It can be seen that $\mathcal{L}$ is symmetric with respect to the trace τ' given by $\tau'(f) = \int f(x)\phi'(x)dx$. Thus the assumption of symmetry can accommodate the semigroup corresponding to an arbitrary one-dimensional diffusion with smooth coefficients.

Remark 2.1.4. *On the other hand, some of the most common QDS arising in classical probability for which the assumptions* **A(i)**- **A(v)** *hold, cannot be made symmetric even by a change of measure on the underlying function algebra. For example, consider the semigroup of the standard Poisson process on $\mathbf{Z}_+$, realized on the commutative von-Neumann algebra $l^\infty(\mathbf{Z}_+)$, equipped with the trace given by the counting measure. Here, the generator $\mathcal{L}_2$ is the bounded operator $l - I$, where l denotes the unilateral shift operator on $l^2(\mathbf{Z}_+)$. It is straightforward to see that for $\phi \in l^\infty(\mathbf{Z}_+)$, $\phi \geq 0$, $\tau((l - I)(\phi)) \leq 0$. However, if $\mathcal{L}$ is symmetric with respect to some measure μ, say given by a sequence $\{p_i = \mu(\{i\})\}$ of nonnegative numbers, then the symmetry condition will imply (since $\mathcal{L}$ has $1 \in l^\infty$ in its domain) that $\sum_{i\geq 0}(l - I)(\phi(i))p_i = 0$ for all $\phi \in l^\infty(\mathbf{Z}_+)$. From this, one can easily conclude that $p_i = 0$ for all i. That is, there does not exist any faithful positive trace on l^∞ for which $\mathcal{L}$ is symmetric.*

Remark 2.1.5. *Let E be a Banach space, $F : \mathbb{R} \to E$ be a strongly measurable map and let μ be a measure on $\mathbb{R}$. Suppose that the integrals $\int_{\mathbb{R}} d\mu(t)F(t)$*

and $\int_{\mathbb{R}} d\mu(t)T(F(t))$ *exist, where* T *is a closed densely defined operator in* E*. Then* $\int_{\mathbb{R}} d\mu(t)F(t) \in Dom(T)$ *and*

$$T(\int_{\mathbb{R}} d\mu(t)F(t)) = \int_{\mathbb{R}} d\mu(t)T(F(t)).$$

Let us denote by Γ_{fr} the free fock space over k_0. Let $\hat{\mathcal{L}} = \overline{\hat{\mathcal{L}}_2 \otimes_\gamma 1 + 1 \otimes_\gamma \hat{\mathcal{L}}_2}$, where $\hat{\mathcal{L}}_2(\cdot) := \mathcal{L}_2 \otimes id_{\Gamma_{fr}}$, $C = (-2Re(\mathcal{L}_2))^{\frac{1}{2}}$, $\hat{C} := C \otimes id_{\Gamma_{fr}}$ and let

$$\hat{C} \otimes_\gamma \hat{C} := (\hat{C} \otimes_\gamma 1) \circ (1 \otimes_\gamma \hat{C}) = (1 \otimes_\gamma \hat{C}) \circ (\hat{C} \otimes_\gamma 1) \text{ in } (h \otimes \Gamma_{fr}) \otimes_\gamma (h \otimes \Gamma_{fr}).$$

Observe that $\hat{\mathcal{L}}_2$ is the pregenerator of the semigroup $(\hat{T}_t)_{t \geq 0}$ on $h \otimes \Gamma_{fr}$, where $\hat{T}_t(\cdot) := T_t(\cdot) \otimes id_{\Gamma_{fr}}$. Note that by our hypothesis, $\hat{T}_t(\mathcal{A}_0 \otimes_{alg} \Gamma_{fr}) \subseteq \mathcal{A}_0 \otimes_{alg} \Gamma_{fr}$, which implies that $\mathcal{A}_0 \otimes \Gamma_{fr}$ is a core for $\hat{\mathcal{L}}_2$.

Moreover we set $\mathcal{F} := (\mathcal{A}_0 \otimes_{alg} \Gamma_{fr}) \otimes_{alg} (\mathcal{A}_0 \otimes_{alg} \Gamma_{fr})$, and $\mathcal{Y} := \{(\lambda - \hat{\mathcal{L}})^{-1}(X \otimes Y) |\ X, Y \in \mathcal{A}_0 \otimes_{alg} \Gamma_{fr}\}$.

The following lemma will be useful in proving the main result of this section:

Lemma 2.1.6. *Let* E *be a Banach space, and let* A *and* B *belong to* $Lin(E, E)$ *with dense domains* $Dom(A)$ *and* $Dom(B)$ *respectively. Suppose there is a total set* $D \subset Dom(A) \cap Dom(B)$ *with the properties :*

(i) $A(D)$ *is total in* E*,* **(ii)** $\|B(x)\| < \|A(x)\|$ *for all* $x \in D$*.*

Then $(A + B)(D)$ *is also total in* E*.*

Proof. First note that if $A(D) \subseteq (A+B)(D)$, then $F \equiv span\{(A+B)(D)\}$ is dense in E. Therefore without loss of generality we suppose $\overline{F} \neq E$, or that there is a non-zero y_0 in $A(D)$, such that $y_0 \notin (A + B)(D)$, and let $y_0 = A(x_0)$, for some $x_0 \in D$. Then by Hahn-Banach theorem, there exists $\Lambda \in E^*$, the topological dual of E, such that $\|\Lambda\| = 1$, $|\Lambda(y_0)| = \|y_0\|$ as well as $\Lambda((A+B)(D)) = 0$. Then $\|y_0\| = |\Lambda(A(x_0))|$ and $|\Lambda(A(x_0))| = |\Lambda(B(x_0))|$.
But $|\Lambda(B(x_0))| \leq \|B(x_0)\| < \|A(x_0)\| = \|y_0\|$ which leads to a contradiction. Therefore $\overline{F} = E$. □

2.2 Proof of homomorphism property.

The next two lemmas set the stage for the application of lemma 2.1.6 to our problem, leading to the main result of this section, viz. the proof of the homomorphism property of j_t under an additional hypothesis.

Lemma 2.2.1. *For $X \in \mathcal{A}_0 \otimes_{alg} \Gamma_{fr}$, $X \neq 0$,*

$$\int_0^\infty e^{-\lambda t} dt \|\hat{C}(\hat{T}_t(X))\|^2 < \|X\|^2.$$

Proof. For X in $\mathcal{A}_0 \otimes_{alg} \Gamma_{fr}$,

$$\frac{d}{dt}\|\hat{T}_t(X)\|^2 = \langle \hat{\mathcal{L}}_2(\hat{T}_t(X)), \hat{T}_t(X)\rangle_{h\otimes\Gamma_{fr}} + \langle \hat{T}_t(X), \hat{\mathcal{L}}_2(\hat{T}_t(X))\rangle_{h\otimes\Gamma_{fr}} = -\|\hat{C}\circ\hat{T}_t(X)\|^2. \tag{2.3}$$

We get for $\lambda > 0$,

$$\begin{aligned} \int_0^\infty e^{-\lambda t}\|\hat{C}(\hat{T}_t(X))\|^2 &= -\int_0^\infty e^{-\lambda t}\frac{d}{dt}\|\hat{T}_t(X)\|^2 \\ &= \left\{\|X\|^2 - \lambda\int_0^\infty e^{-\lambda t}\|\hat{T}_t(X)\|^2\right\}. \end{aligned} \tag{2.4}$$

Thus if $\lambda \int_0^\infty dt\ e^{-\lambda t}\|\hat{T}_t(X)\|^2 = 0$, for some $\lambda > 0$, then we have $\|\hat{T}_t(X)\| = 0$ for almost all t and by virtue of the continuity of $\|\hat{T}_t(X)\|$ as a function of t, this leads to the contradiction $X = 0$. Thus $\lambda \int_0^\infty dt\ e^{-\lambda t}\|\hat{T}_t(X)\|^2 > 0$ for all $\lambda > 0$ which gives us the required strict inequality. □

Lemma 2.2.2. $\|(\hat{C} \otimes_\gamma \hat{C})(X)\|_\gamma \leq \|(\lambda - \hat{\mathcal{L}})(X)\|_\gamma$ *for all X in $Dom(\hat{\mathcal{L}})$ and we have strict inequality if X is in $\mathcal{Y}$.*

Proof. Let $X \in \mathcal{F}$, such that $X = \sum_{i=1}^k X_i \otimes Y_i$. It is obvious that $(1 \otimes_\gamma \hat{C})(\mathcal{F}) \subset Dom((\hat{C} \otimes_\gamma 1))$. So using lemma 2.2.1,

$$\begin{aligned} &\int_0^\infty dt\ e^{-\lambda t}\|\hat{C} \otimes_\gamma \hat{C}(\hat{T}_t \otimes_\gamma \hat{T}_t)(X)\|_\gamma \\ &= \int_0^\infty dt\ e^{-\lambda t}\|\sum_{i=1}^k \hat{C}(\hat{T}_t(X_i)) \otimes \hat{C}(\hat{T}_t(Y_i))\|_\gamma \\ &\leq \sum_{i=1}^k (\int_0^\infty dt\ e^{-\lambda t}\|\hat{C}(\hat{T}_t(X_i))\|^2)^{\frac{1}{2}} (\int_0^\infty dt\ e^{-\lambda t}\|\hat{C}(\hat{T}_t(Y_i))\|^2)^{\frac{1}{2}} < \sum_{i=1}^k \|X_i\|\|Y_i\|. \end{aligned} \tag{2.5}$$

Equation(2.5) and Remark 2.1.5 together yields:

$$\|(\hat{C} \otimes_\gamma \hat{C})((\lambda - \hat{\mathcal{L}})^{-1}(X))\|_\gamma \leq \|X\|_\gamma.$$

Thus $(\hat{C} \otimes_\gamma \hat{C})(\lambda - \hat{\mathcal{L}})^{-1}$ is a contraction. As a consequence, $\hat{C} \otimes_\gamma \hat{C}$ extends to $Dom(\hat{\mathcal{L}})$ and we have the required inequality. Now with $X = x \otimes y$, where

$x, y \in \mathcal{A}_0 \otimes_{alg} \Gamma_{fr}$, the above equations give

$$\begin{aligned} &\|(\hat{C} \otimes_\gamma \hat{C})((\lambda - \hat{\mathcal{L}})^{-1}(x \otimes y))\|_\gamma < \|x\|\|y\| = \|x \otimes y\|_\gamma \\ \text{or} \quad &\|(\hat{C} \otimes_\gamma \hat{C})(Y)\|_\gamma < \|(\lambda - \hat{\mathcal{L}})(Y)\|_\gamma \quad \text{for } Y \in \mathcal{Y}. \end{aligned} \tag{2.6}$$

□

The assumptions **A(iv)** and **A(v)** lead to

$$\|\delta(x)\|^2_{h\otimes k_0} = \tau(\delta(x)^*\delta(x)) \leq \|C(x)\|^2_h \leq \|(C + \epsilon)(x)\|^2_h, \tag{2.7}$$

for all x in $\mathcal{A}_0$, $\epsilon > 0$. This implies $\delta(\cdot)$ can be extended to $Dom(C)$. Now define a map B belonging to $Lin\left((Dom(C) \otimes_{alg} \Gamma_{fr}) \otimes_{alg} (Dom(C) \otimes_{alg} \Gamma_{fr}), (h \otimes \Gamma_{fr}) \otimes_\gamma (h \otimes \Gamma_{fr})\right)$ by

$$B(X \otimes Y) = (\delta \otimes id_{\Gamma_{fr}})(X) \otimes (\delta \otimes id_{\Gamma_{fr}})(Y),$$

and extend linearly. This operator is well-defined, which can be proved as follows: using (2.7)

$$\begin{aligned} &\|\delta \otimes id_{\Gamma_{fr}}(X)\|\|\delta \otimes id_{\Gamma_{fr}}(Y)\| \leq \|(\hat{C} + \epsilon)(X)\|\|(\hat{C} + \epsilon)(Y)\| \\ &< \infty. \end{aligned} \tag{2.8}$$

Let $\hat{\delta} := \delta \otimes id_{\Gamma_{fr}}$.

Thus we have

$$\|B\{(\hat{C} + \epsilon)^{-1} \otimes_\gamma (\hat{C} + \epsilon)^{-1}\}(X \otimes Y)\|_\gamma \leq \|X \otimes Y\|_\gamma, \tag{2.9}$$

for $X, Y \in Dom(C) \otimes_{alg} \Gamma_{fr}$, which implies that $B\{(\hat{C} + \epsilon)^{-1} \otimes_\gamma (\hat{C} + \epsilon)^{-1}\}$ extends to a contraction from $h \otimes \Gamma_{fr} \otimes_\gamma h \otimes \Gamma_{fr}$ to itself and hence

$$\|B(X)\|_\gamma \leq \|\{(\hat{C} + \epsilon) \otimes_\gamma (\hat{C} + \epsilon)\}(X)\|_\gamma \tag{2.10}$$

for all $X \in (Dom(C) \otimes_{alg} \Gamma_{fr}) \otimes_{alg} (Dom(C) \otimes_{alg} \Gamma_{fr})$. Letting $\epsilon \to 0$, we see that

$$\|B(X)\|_\gamma \leq \|(\hat{C} \otimes_\gamma \hat{C})(X)\|_\gamma \tag{2.11}$$

for all X in $(Dom(C) \otimes_{alg} \Gamma_{fr}) \otimes_{alg} (Dom(C) \otimes_{alg} \Gamma_{fr})$. By Lemma 2.2.2, $\hat{C} \otimes_\gamma \hat{C}$ extends to $Dom(\hat{\mathcal{L}})$ and thus we can also extend B to $Dom(\hat{\mathcal{L}})$. So we have

$$\|B(X)\| \leq \|(\hat{C} \otimes_\gamma \hat{C})(X)\|_\gamma \leq \|(\lambda - \hat{\mathcal{L}})(X)\|_\gamma \text{ for all } X \in Dom(\hat{\mathcal{L}}). \tag{2.12}$$

Now $span\{\mathcal{Y}\} \subseteq Dom(\hat{\mathcal{L}})$, and in particular for Y in $\mathcal{Y}$,

$$\|B(Y)\|_\gamma \leq \|(\hat{C} \otimes_\gamma \hat{C})(Y)\|_\gamma < \|(\lambda - \hat{\mathcal{L}})(Y)\|_\gamma. \tag{2.13}$$

□

Theorem 2.2.3. *Assume that the flow j_t satisfies* **A(i)** - **A(v)** *and suppose that the following hold:*

A(vi) *There exists a total subset $\mathcal{W}$ of $L^2(\mathbb{R}_+, k_0)$, such that for f, g in $\mathcal{W}$, $x \in \mathcal{A} \cap L^1(\tau)$ and u,v in $L^\infty(\tau) \cap L^2(\tau)$, we have:*

$$sup\ _{0 \leq s \leq t} |\langle uf^{\otimes^m}, j_t(x) vg^{\otimes^n} \rangle| \leq C(u, v, f, g, m, n, t) \|x\|_1, \tag{2.14}$$

such that for fixed u,v,f,g,m,n, $C(u, v, f, g, m, n, t) = O(e^{\beta t})$ for some $\beta \geq 0$.

Then j_t is a $*$-homomorphism.

Proof. For f, g in $\mathcal{W}$, the flow equation 1.10 of subsection 1.7.5 in chapter 1, leads to :

$$\begin{aligned} &\langle j_t(x) ue(f), ve(g) \rangle = \langle xue(f), ve(g) \rangle + \\ &\int_o^t ds\ \langle j_s(\mathcal{L}(x) + \langle g(s), \delta(x) \rangle + \delta^\dagger(x)_{f(s)} + \langle g(s), \sigma(x)_{f(s)} \rangle) ue(f), ve(g) \rangle. \end{aligned} \tag{2.15}$$

Recall that using the quantum Ito formula (equation (1.11)) we get:

$$\begin{aligned} &\langle j_t(x) ue(f), j_t(y) ve(g) \rangle \\ &= \langle xue(f), yve(g) \rangle \\ &\quad + \int_0^t ds\ [\langle j_s(\mathcal{L}(x) + \langle g(s), \delta(x) \rangle + \delta^\dagger(x)_{f(s)} + \langle g(s), \sigma(x)_{f(s)} \rangle) ue(f), j_s(y) ve(g) \rangle \\ &\quad + \langle j_s(x) ue(f), j_s(\mathcal{L}(y) + \langle f(s), \delta(y) \rangle + \delta^\dagger(y)_{g(s)} + \langle f(s), \sigma(y)_{g(s)} \rangle) ve(g) \rangle \\ &\quad + \langle\ \hat{j}_s(\delta(x) + \sigma(x)_{f(s)}) ue(f), \hat{j}_s(\delta(y) + \sigma(y)_{g(s)}) ve(g) \rangle_{h \otimes \Gamma \otimes k_0}], \end{aligned} \tag{2.16}$$

where $\hat{j}_s$ is the extension of the map $j_s \otimes id_{\Gamma_{fr}}$ to the Hilbert module $\mathcal{A} \otimes \Gamma_{fr}$, which makes sense because of Lemma 1.2.8.

For fixed u, v in $\mathcal{A} \cap h$, f, g in $\mathcal{W}$, we define for each $t \geq 0$,
$\phi_t : (\mathcal{A}_0 \otimes_{alg} \Gamma_{fr}) \times (\mathcal{A}_0 \otimes_{alg} \Gamma_{fr}) \to \mathbb{C}$ by

$$\phi_t(X, Y) := \langle\ \hat{j}_t(X) ue(f), \hat{j}_t(Y) ve(g) \rangle_{h \otimes \Gamma \otimes \Gamma_{fr}} - \langle\ j_t(\langle Y, X \rangle) ue(f), ve(g) \rangle_{h \otimes \Gamma}. \tag{2.17}$$

Using (2.15) and (2.16), we get:

$$\begin{aligned} \phi_t(X, Y) = &\int_0^t ds\ [\phi_s(\hat{\mathcal{L}}(X), Y) + \phi_s(X, \hat{\mathcal{L}}(Y)) + \phi_s(\hat{\delta}(X), \hat{\delta}(Y)) \\ &+ \phi_s(\hat{\delta}^\dagger(X)_{f(s)}, Y) + \phi_s(\langle g(s), \hat{\delta}(X) \rangle, Y) + \phi_s(\langle g(s), \hat{\sigma}(X)_{f(s)} \rangle, Y) \\ &+ \phi_s(X, \hat{\delta}^\dagger(Y)_{g(s)}) + \phi_s(X, \langle f(s), \hat{\delta}(Y) \rangle) + \phi_s(X, \langle f(s), \hat{\sigma}(Y)_{g(s)} \rangle) \\ &+ \phi_s(\hat{\delta}(X), \hat{\sigma}(Y)_{g(s)}) + \phi_s(\hat{\sigma}(X)_{f(s)}, \hat{\delta}(Y)) + \phi_s(\hat{\sigma}(X)_{f(s)}, \hat{\sigma}(Y)_{g(s)})] \end{aligned} \tag{2.18}$$

Next we follow the ideas indicated in the pages 178-181 in [62] and define for m,n in $I\!N \cup 0$,

$$\phi_t^{m,n}(X,Y) := \frac{1}{(m!n!)^{\frac{1}{2}}}[\langle\ \hat{j}_t(X)uf^{\otimes^m}, \hat{j}_t(Y)vg^{\otimes^n}\rangle\ - \langle\ j_t(\langle Y,X\rangle)uf^{\otimes^m}, vg^{\otimes^n}\rangle\ ]$$
$$= \frac{1}{m!n!}\frac{\partial^m}{\partial\rho^m}\frac{\partial^n}{\partial\eta^n}\{\langle\hat{j}_t(X)ue(\rho f), \hat{j}_t(Y)ve(\eta g)\rangle - \langle j_t(\langle Y,X\rangle)ue(\rho f), ve(\eta g)\rangle\}|_{\rho,\eta=0}, \tag{2.19}$$

for $X, Y \in \mathcal{A}_0 \otimes_{alg} \Gamma_{fr}$.

Let $\phi_t^{(\rho,\eta)}(X,Y) := \langle\hat{j}_t(X)ue(\rho f), \hat{j}_t(Y)ve(\eta g)\rangle - \langle j_t(\langle Y,X\rangle)ue(\rho f), ve(\eta g)\rangle$. Observe that $\phi_t^{m,n}(X,Y) = \frac{1}{m!}\frac{1}{n!}\frac{\partial^m}{\partial\rho^m}\frac{\partial^n}{\partial\eta^n}\phi_t^{(\rho,\eta)}(X,Y)|_{\rho=0,\ \eta=0}$.

Then in terms of $\phi_t^{(\rho,\eta)}(X,Y)$, equation (2.18) becomes:

$$\begin{aligned}\phi_t^{(\rho,\eta)}(X,Y) = &\int_0^t ds\ [\phi_s^{(\rho,\eta)}(\hat{\mathcal{L}}(X),Y) + \phi_s^{(\rho,\eta)}(X,\hat{\mathcal{L}}(Y)) + \phi_s^{(\rho,\eta)}(\hat{\delta}(X),\hat{\delta}(Y))\\
&+ \phi_s^{(\rho,\eta)}(\hat{\delta}^\dagger(X)_{\rho f(s)},Y) + \phi_s^{(\rho,\eta)}(\langle\eta g(s),\hat{\delta}(X)\rangle,Y) + \phi_s^{(\rho,\eta)}(\langle\eta g(s),\hat{\sigma}(X)_{\rho f(s)}\rangle,Y)\\
&+ \phi_s^{(\rho,\eta)}(X,\hat{\delta}^\dagger(Y)_{\eta g(s)}) + \phi_s^{(\rho,\eta)}(X,\langle\rho f(s),\hat{\delta}(Y)\rangle) + \phi_s^{(\rho,\eta)}(X,\langle\rho f(s),\hat{\sigma}(Y)_{\eta g(s)}\rangle)\\
&+ \phi_s^{(\rho,\eta)}(\hat{\delta}(X),\hat{\sigma}(Y)_{\eta g(s)}) + \phi_s^{(\rho,\eta)}(\hat{\sigma}(X)_{\rho f(s)},\hat{\delta}(Y)) + \phi_s^{(\rho,\eta)}(\hat{\sigma}(X)_{\rho f(s)},\hat{\sigma}(Y)_{\eta g(s)})].\end{aligned} \tag{2.20}$$

On simplification, this becomes

$$\begin{aligned}\phi_t^{(\rho,\eta)}(X,Y) = &\int_0^t ds\ [\phi_s^{(\rho,\eta)}(\hat{\mathcal{L}}(X),Y) + \phi_s^{(\rho,\eta)}(X,\hat{\mathcal{L}}(Y)) + \phi_s^{(\rho,\eta)}(\hat{\delta}(X),\hat{\delta}(Y))\\
&+ \rho\phi_s^{(\rho,\eta)}(\hat{\delta}^\dagger(X)_{f(s)},Y) + \eta\phi_s^{(\rho,\eta)}(\langle g(s),\hat{\delta}(X)\rangle,Y) + \rho\eta\phi_s^{(\rho,\eta)}(\langle g(s),\hat{\sigma}(X)_{f(s)}\rangle,Y)\\
&+ \eta\phi_s^{(\rho,\eta)}(X,\hat{\delta}^\dagger(Y)_{g(s)}) + \rho\phi_s^{(\rho,\eta)}(X,\langle f(s),\hat{\delta}(Y)\rangle) + \rho\eta\phi_s^{(\rho,\eta)}(X,\langle f(s),\hat{\sigma}(Y)_{g(s)}\rangle)\\
&+ \eta\phi_s^{(\rho,\eta)}(\hat{\delta}(X),\hat{\sigma}(Y)_{g(s)}) + \rho\phi_s^{(\rho,\eta)}(\hat{\sigma}(X)_{f(s)},\hat{\delta}(Y)) + \rho\eta\phi_s^{(\rho,\eta)}(\hat{\sigma}(X)_{f(s)},\hat{\sigma}(Y)_{g(s)})].\end{aligned} \tag{2.21}$$

Observe that except the first three terms on the right hand side of equation (2.21), all the other terms are either of the form $\rho\phi_s^{(\rho,\eta)}(X_1,X_2)$ or $\eta\phi_s^{(\rho,\eta)}(X_1,X_2)$ or $\rho\eta\phi_s^{(\rho,\eta)}(X_1,X_2)$, for $X_1, X_2 \in \mathcal{A}\otimes\Gamma_{fr}$. Now we have

$$\begin{aligned}
&\frac{1}{m!n!}\frac{\partial^m\partial^n}{\partial\rho^m\partial\eta^n}\rho\phi_s^{(\rho,\eta)}(X_1,X_2)\\
&=\frac{1}{m!n!}\frac{\partial^{m-1}}{\partial\rho^{m-1}}[\rho\frac{\partial\partial^n}{\partial\rho\partial\eta^n}\phi_s^{(\rho,\eta)}(X_1,X_2)+\frac{\partial^n}{\partial\eta^n}\phi_s^{(\rho,\eta)}(X_1,X_2)]\\
&=\frac{1}{m!n!}\frac{\partial^{m-2}}{\partial\rho^{m-2}}[2\frac{\partial\partial^n}{\partial\rho\partial\eta^n}\phi_s^{(\rho,\eta)}(X_1,X_2)+\rho\frac{\partial^2\partial^n}{\partial\rho^2\partial\eta^n}\phi_s^{(\rho,\eta)}(X_1,X_2)]\\
&.\\
&.\\
&.\\
&.\\
&=\frac{1}{m!n!}[m\frac{\partial^{m-1}\partial^n}{\partial\rho^{m-1}\partial\eta^n}\phi_s^{(\rho,\eta)}(X_1,X_2)+\rho\frac{\partial^m\partial^n}{\partial\rho^m\partial\eta^n}\phi_s^{(\rho,\eta)}(X_1,X_2)]
\end{aligned}\tag{2.22}$$

So that we have $\frac{1}{m!n!}\frac{\partial^m\partial^n}{\partial\rho^m\partial\eta^n}\rho\phi_s^{(\rho,\eta)}(X_1,X_2)|_{\rho=0,\eta=0}=\frac{1}{(m-1)!n!}\frac{\partial^{m-1}\partial^n}{\partial\rho^{m-1}\partial\eta^n}\phi_s^{(\rho,\eta)}(X_1,X_2)|_{\rho=0,\eta=0}=\phi_s^{m-1,n}(X_1,X_2)$.

Similarly we have $\frac{1}{m!n!}\frac{\partial^m\partial^n}{\partial\rho^m\partial\eta^n}\eta\phi_s^{(\rho,\eta)}(X_1,X_2)|_{\rho=0,\eta=0}=\frac{1}{(m)!n-1!}\frac{\partial^m\partial^n-1}{\partial\rho^m\partial\eta^n-1}\phi_s^{(\rho,\eta)}(X_1,X_2)|_{\rho=0,\eta=0}=\phi_s^{m,n-1}(X_1,X_2)$.

$$\begin{aligned}
&\frac{1}{m!n!}\frac{\partial^m\partial^n}{\partial\rho^m\partial\eta^n}\rho\eta\phi_s^{(\rho,\eta)}(X_1,X_2)\\
&=\frac{1}{m!(n-1)!}\frac{\partial^m}{\partial\rho^m}\rho[\frac{\partial^{n-1}}{\partial\eta^{n-1}}\phi_s^{(\rho,\eta)}(X_1,X_2)+\eta\frac{\partial^n}{\partial\eta^n}\phi_s^{(\rho,\eta)}(X_1,X_2)].
\end{aligned}\tag{2.23}$$

Thus

$$\frac{1}{m!n!}\frac{\partial^m\partial^n}{\partial\rho^m\partial\eta^n}\rho\eta\phi_s^{(\rho,\eta)}(X_1,X_2)|_{\eta=0}=\frac{1}{m!(n-1)!}\frac{\partial^m}{\partial\rho^m}\rho\{\frac{\partial^{n-1}}{\partial\eta^{n-1}}|_{\eta=0}\phi_s^{(\rho,\eta)}(X_1,X_2)\}.\tag{2.24}$$

Taking $f(\rho):=\{\frac{\partial^{n-1}}{\partial\eta^{n-1}}|_{\eta=0}\phi_s^{(\rho,\eta)}(X_1,X_2)\}$, proceeding as before, we see that $\frac{\partial^m}{\partial\rho^m}\rho f(\rho)=m\frac{\partial^{m-1}}{\partial\rho^{m-1}}f(\rho)+\rho\frac{\partial^m}{\partial\rho^m}f(\rho)$, so that we have $\frac{\partial^m\partial^n}{\partial\rho^m\partial\eta^n}\rho f(\rho)|_{\rho=0}=m\frac{\partial^{m-1}}{\partial\rho^{m-1}}f(\rho)|_{\rho=0}$.
Thus we have $\frac{1}{m!n!}\frac{\partial^m\partial^n}{\partial\rho^m\partial\eta^n}\{\rho\eta\phi_s^{(\rho,\eta)}(X_1,X_2)\}|_{\rho=0,\eta=0}=\frac{1}{(m-1)!(n-1)!}\frac{\partial^{m-1}\partial^{n-1}}{\partial\rho^{m-1}\partial\eta^{n-1}}\phi_s^{(\rho,\eta)}(X_1,X_2)|_{\rho=0,\eta=0}$.

So we get a recursive integral relation amongst $\phi_t^{m,n}(X,Y)$ as follows:

$$\begin{aligned}\phi_t^{m,n}(X,Y) = \int_0^t ds\ [&\phi_s^{m,n}(\hat{\mathcal{L}}(X),Y) + \phi_s^{m,n}(X,\hat{\mathcal{L}}(Y)) + \phi_s^{m,n}(\hat{\delta}(X),\hat{\delta}(Y))\\
&+ \phi_s^{m-1,n}(\hat{\delta}^\dagger(X)_{f(s)},Y) + \phi_s^{m,n-1}(\langle g(s),\hat{\delta}(X)\rangle,Y) + \phi_s^{m-1,n-1}(\langle g(s),\hat{\sigma}(X)_{f(s)}\rangle,Y)\\
&+ \phi_s^{m,n-1}(X,\hat{\delta}^\dagger(Y)_{g(s)}) + \phi_s^{m-1,n}(X,\langle f(s),\hat{\delta}(Y)\rangle) + \phi_s^{m-1,n-1}(X,\langle f(s),\hat{\sigma}(Y)_{g(s)}\rangle)\\
&+ \phi_s^{m,n-1}(\hat{\delta}(X),\hat{\sigma}(Y)_{g(s)}) + \phi_s^{m-1,n}(\hat{\sigma}(X)_{f(s)},\hat{\delta}(Y)) + \phi_s^{m-1,n-1}(\hat{\sigma}(X)_{f(s)},\hat{\sigma}(Y)_{g(s)})]\end{aligned} \tag{2.25}$$

where $\phi_t^{-1,n}(X,Y) := \phi_t^{m,-1}(X,Y) := 0$ for all m,n and X,Y.

We set in (2.25), $m = n = 0$ to get

$$\phi_t^{0,0}(X,Y) = \int_0^t ds\{\phi_s^{0,0}(\hat{\mathcal{L}}(X),Y) + \phi_s^{0,0}(X,\hat{\mathcal{L}}(Y)) + \phi_s^{0,0}(\hat{\delta}(X),\hat{\delta}(Y))\} \tag{2.26}$$

and if we can show that the hypothesis of this theorem and (2.26) imply that $\phi_t^{0,0}(X,Y) = 0$, then we can embark on our induction hypothesis as

$$\phi_t^{k,l}(X,Y) = 0 \text{ for } k + l \leq m + n - 1.$$

Under the induction hypothesis, (2.25) reduces to

$$\phi_t^{m,n}(X,Y) = \int_0^t ds[\phi_s^{m,n}(\hat{\mathcal{L}}(X),Y) + \phi_s^{m,n}(X,\hat{\mathcal{L}}(Y)) + \phi_s^{m,n}(\hat{\delta}(X),\hat{\delta}(Y))] \tag{2.27}$$

for $X,Y \in (\mathcal{A}_0 \otimes_{alg} \Gamma_{fr})$, which is an equation similar to (2.26) leading to $\phi_t^{m,n}(X,Y) = 0$, as earlier and this will complete the induction process. Thus it only remains to show that the assumptions of this theorem lead to a trivial solution of equation of the type (2.26). Omitting the indices m,n, define a map ψ_t belonging to $Lin((\mathcal{A}_0 \otimes_{alg} \Gamma_{fr}) \otimes_{alg} (\mathcal{A}_0 \otimes_{alg} \Gamma_{fr}), \mathbb{C})$ by:

$$\psi_t(X \otimes Y) = \phi_t^{m,n}(X,Y),$$

and extend linearly. Thus equation (2.26) leads to:

$$\psi_t(X) = \int_0^t ds[\psi_s((\hat{\mathcal{L}} \otimes 1 + 1 \otimes \hat{\mathcal{L}} + \hat{\delta} \otimes \hat{\delta})(X))], \text{ for X in } \mathcal{F}. \tag{2.28}$$

The complete positivity of the map j_t implies that

$$\langle j_t(x)\xi, j_t(x)\xi\rangle \leq \langle j_t(x^*x)\xi, \xi\rangle \tag{2.29}$$

for $\xi \in h \otimes \Gamma$ and hence by **A(vi)**, we get that

$$\begin{aligned} |\langle j_t(x)uf^{\otimes^m}, j_t(y)vg^{\otimes^n}\rangle| &\leq (|\langle j_t(x^*x)uf^{\otimes^m}, uf^{\otimes^m}\rangle\langle j_t(y^*y)vg^{\otimes^n}, vg^{\otimes^n}\rangle|)^{\frac{1}{2}} \\ &\leq (C(u,u,f,f,m,m,t)C(v,v,g,g,n,n,t))^{\frac{1}{2}}\|x\|_2\|y\|_2 \quad (2.30) \\ &= O(e^{\beta t})\|x\|_2\|y\|_2. \end{aligned}$$

Furthermore, by assumption **A(vi)** we have

$$|\langle j_t(y^*x)uf^{\otimes^m}, vg^{\otimes^n}\rangle| \leq C(u,v,f,g,m,n)\|y^*x\|_1 \leq O(e^{\beta t})\|x\|_2\|y\|_2. \tag{2.31}$$

Thus we have $|\phi_t(x,y)| \leq O(e^{\beta t})\|x\|_2\|y\|_2$, for $x, y \in \mathcal{A}_0 \otimes \Omega$, Ω being the vacuum vector in Γ_{fr}. Now since $\mathcal{A}_0$ is dense in h, $\phi_t : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathbb{C}$ extends to a bounded sesquilinear form on h. Thus there exists an operator $M \in \mathcal{B}(h)$ such that $\phi_t(x,y) = \langle Mx, y\rangle_h$ for $x, y \in \mathcal{A}_0$, $\|M\| = O(e^{\beta t})$. ϕ_t can be extended to a sesquilinear form on $h \otimes \Gamma_{fr}$ as follows:
Observe that $\hat{M} := M \otimes id_{\Gamma_{fr}} \in \mathcal{B}(h \otimes \Gamma_{fr})$. So for $X, Y \in h \otimes \Gamma_{fr}$, $(X,Y) \to \langle \hat{M}X, Y\rangle_{h\otimes\Gamma_{fr}}$ defines a bounded sesquilinear form on $\mathcal{A}_0 \otimes_{alg} \Gamma_{fr}$ and hence can be extended to a bounded sesquilinear form on $h \otimes \Gamma_{fr}$. Note that for $X, Y \in \mathcal{A}_0 \otimes_{alg} \Gamma_{fr}$, $\langle \hat{M}X, Y\rangle = \phi_t(X,Y)$, so that we have $|\phi_t(X,Y)| \leq O(e^{\beta t})\|X\|_{h\otimes\Gamma_{fr}}\|Y\|_{h\otimes\Gamma_{fr}}$ for $X, Y \in \mathcal{A}_0 \otimes_{alg} \Gamma_{fr}$.

This yields

$$|\psi_t(X)| \leq O(e^{\beta t})\|X\|_\gamma, \text{ for } X \in \mathcal{F}, \tag{2.32}$$

which proves (by virtue of denseness of $\mathcal{F}$ in $(h \otimes \Gamma_{fr}) \otimes_\gamma (h \otimes \Gamma_{fr})$) that ψ_t extends as a bounded map from $(h \otimes \Gamma_{fr}) \otimes_\gamma (h \otimes \Gamma_{fr})$ to $\mathbb{C}$. If we let $G = \hat{\mathcal{L}} + B$, then for $X \in \mathcal{F}$, the equation (2.28) becomes:

$$\psi_t(X) = \int_0^t \psi_s(G(X))ds.$$

Note that by (2.32), $\int_0^\infty dt\ e^{-\lambda t}|\psi_t(X)| < \infty$ for $\lambda \geq \beta$ and thus

$$\int_0^\infty dt\ e^{-\lambda t}\psi_t(X) = \int_0^\infty dt\ e^{-\lambda t}\int_0^t ds\psi_s(G(X)),$$

which on an integration by parts leads to

$$\int_0^\infty dt\ e^{-\lambda t}\psi_t((G-\lambda)(X)) = 0, \text{ for } X \in \mathcal{F}. \tag{2.33}$$

Now for $Y \in \langle\mathcal{Y}\rangle_{\mathbb{C}}$, let $\{X_n \in \mathcal{F}\}$ be a sequence such that $G(X_n)$ goes to $G(Y)$ (this happens because of the fact that $\mathcal{F}$ is a core for $\hat{\mathcal{L}}$ and the inequality in (2.13)).

Thus we have
$$\int_0^\infty dt\ e^{-\lambda t}\psi_t((G-\lambda)(Y))=0,$$
by an application of the dominated convergence theorem. Lemma 2.1.6 with $A=(\hat{\mathcal{L}}-\lambda)$, $D=\mathcal{Y}$, and the inequality (2.13) together yields the denseness of $(G-\lambda)(span\{\mathcal{Y}\})$ follows. Therefore the last equation and (2.32) lead to
$$\int_0^\infty dt\ e^{-\lambda t}\psi_t(X)=0 \text{ for all } X\in h\otimes_\gamma h, \text{ for } \lambda>\beta.$$
This implies that $\psi_t(X)=0$ which in turn proves that $\phi_t^{m,n}(X,Y)=0$ for $X,Y\in\mathcal{A}_0\otimes_{alg}\Gamma_{fr}$, $t\geq 0$, and hence taking $X=x\otimes\Omega$, $Y=y\otimes\Omega$ for $x,y\mathcal{A}_0$, Ω being the vacuum vector in Γ_{fr}, we get the required result. □

Corollary 2.2.4. *Suppose the trace τ on the algebra is finite. Assume* **A(i)** *through* **A(v)***, and replace the assumption of analyticity in condition* **A(i)** *by the following: $\mathcal{A}_0\subseteq Dom(\mathcal{L}_2)\cap Dom(\mathcal{L}_2^*)$. Then the conclusion of Theorem 2.2.3 remains valid.*

Proof. Define a symmetric form $q(x,y)=-\langle\mathcal{L}_2(x),y\rangle-\langle x,\mathcal{L}_2(y)\rangle$ for all $x,y\in\mathcal{A}_0$, with domain $Dom(q)=\mathcal{A}_0$. This form is non-negative by **A(iv)** and **A(v)**. Since $q(x,y)=\langle x,(-\mathcal{L}_2-\mathcal{L}_2^*)y\rangle$, $\forall x,y\in Dom(q)$, the standard proof for the Friedrich extension (see [59], vol-II, page-177) is valid and we get a positive self-adjoint operator Z with $Dom(q)\subseteq Dom(Z)$ such that $q(x,y)=\langle x,Z(y)\rangle$. Set $C=Z^{\frac{1}{2}}$. Observe that by the form extension and hypothesis **A(v)**, we have
$$\frac{d}{dt}\|T_t(x)\|^2=\langle\mathcal{L}_2(T_t(x)),T_t(x)\rangle+\langle T_t(x),\mathcal{L}_2(T_t(x))\rangle=-\|C\circ T_t(x)\|^2. \tag{2.34}$$
Thus the rest of the proofs of Lemma 2.2.2 and of Theorem 2.2.3 remains valid. □

As an application to this method, we prove the Trotter product formula for quantum stochstic flows.

2.3 The Weak Trotter Product Formulae for q.s.d.e. with unbounded coefficients.

Definition 2.3.1. *The time shift operator θ_t, $\theta_t\colon L^2(\mathbb{R}_+)\to L^2([t,\infty))$ is defined as*
$$\begin{aligned}\theta_t(f)(s)&=0 && if\ s<t\\ &=f(s-t) && if\ s\geq t\ .\end{aligned} \tag{2.35}$$

Let $\Gamma(\theta_t)$ denotes its second quantization, that is $\Gamma(\theta_t)(e(g)) = e(\theta_t(g))$, for g in $L^2(\mathbb{R}_+, k_0)$ and extended linearly as an isometry on whole $\Gamma(L^2(\mathbb{R}_+, k_0))$. For $X \in \mathcal{A} \otimes \mathcal{B}(\Gamma_{[r,s]})$,

$$\Gamma(\theta_t)(X \otimes I_{\Gamma^s})\Gamma(\theta_t^*) \;=\; P_{12}(|\Omega_t >< \Omega_t| \otimes 1_{\Gamma^t_{r+t}} \otimes \hat{X} \otimes I_{\Gamma^{t+s}})P_{12}^*,$$

where $P_{12} : \Gamma_t \otimes h \otimes \Gamma^t \longrightarrow h \otimes \Gamma_t \otimes \Gamma^t (\cong h \otimes \Gamma)$ is the unitary flip between first and second tensor components. Let $\xi_t : \mathcal{B}(h \otimes \Gamma_s^r) \longrightarrow \mathcal{B}(h \otimes \Gamma_{t+s}^{t+r})$ be given by :

$$\xi_t(X) = \hat{X}.$$

Definition 2.3.2. *A CPC flow j_t is called a cocycle if*

$$j_{s+t}(x) = j_s \circ \xi_s \circ j_t(x), \;\; for \;\, x \in \mathcal{A}.$$

Henceforth, all the CPC flows considered are assumed to be cocycles. Note that by virtue of Lemma 1.7.11, the maps $j_t^{\xi,\eta}(\cdot) := \langle e(\xi\chi_{[0,t]}), j_t(\cdot)e(\eta\chi_{[0,t]})\rangle$ for $\xi, \eta \in k_0$ are C_0 semigroups.

Lemma 2.3.3. *Suppose the CPC flow $(j_t)_{t\geq 0}$ satisfies* **A(i)**-**A(v)** *and that for $x \in \mathcal{A} \cap L^1(\tau)$,*

$$\|j_t^{c,d}(x)\|_1 \leq exp(tM)\|x\|_1 \tag{2.36}$$

for c, d in k_0, where M depends only on $\|c\|, \|d\|$. Then the condition **A(vi)** *and hence the conclusion of Theorem 2.2.3 holds.*

Proof. Note that for a partition $0 = s_0 < s_1 < s_2 < .... < s_n = t$, and for functions of the form $f = \sum_j 1_{[s_{j-1},s_j]}c_j,\;\; g = \sum_j 1_{[s_{j-1},s_j]}d_j$, $(c_j, d_j \in k_j \;for\; j = 1, 2,)$ we get using the cocycle property of $j_t(\cdot)$ and (2.36) that

$$\|\langle e(f), j_t(x)e(g)\rangle\|_1 \leq exp(tM)\|x\|_1,$$

where $M = max_j(M_j)$, where each M_j depends only on $\|c_j\|$ and $\|d_j\|$. Let $\Lambda(z) := \langle ue(\bar{z}f), j_t(x)ve(g)\rangle$ and hence $|\langle ue(\bar{z}f), j_t(x)ve(g)\rangle| \leq exp(tM)\|x\|_1$, for $|z| = 1$. Clearly Λ is entire in z since $\{z \to e(zf)\}$ is strongly entire, and by considering a unit disc around zero and applying Cauchy's estimate for this function ,we obtain

$$(m!)^{\frac{1}{2}}|\langle uf^{\otimes^m}, j_t(x)ve(g)\rangle| \leq \|u^*v\|_\infty m! exp(tM)\|x\|_1, \tag{2.37}$$

for $u, v \in \mathcal{A} \cap L^2(\tau)$ and $x \in \mathcal{A} \cap L^1(\tau)$. Doing a similar calculation to the function $\beta(z) := \langle uf^{\otimes^m}, j_t(x)ve(zg)\rangle$, we get:

$$|\langle uf^{\otimes^m}, j_t(x)vg^{\otimes^n}\rangle| \leq \|u^*v\|_\infty (m!n!)^{\frac{1}{2}} exp(tM))\|x\|_1, \tag{2.38}$$

which proves that the CPC flow j_t satisfies $\mathbf{A(vi)}$, if we take
$C(f,g,m,n,t) := \|u^*v\|_\infty (m!n!)^{\frac{1}{2}} exp(tM)$. □

Corollary 2.3.4. *For a CPC flow $(j_t)_{t\geq 0}$ on a type-I von-Neumann algebra with atomic centre, the conditions* $\mathbf{A(i)}$ *through* $\mathbf{A(v)}$ *imply* $\mathbf{A(vi)}$ *and hence also imply that j_t is a $*$ homomorphism.*

Proof. Observe that in a type-I algebra with atomic centre, we have for $x \in L^1(\tau)$,

$$\|x\|_\infty \leq \|x\|_1.$$

As j_t is a contractive flow, we have that for $x \in L^1(\tau)$,

$$\begin{aligned} &\sup_{0\leq s\leq t} |\langle uf^{\otimes^m}, j_t(x)vg^{\otimes^n}\rangle| \\ &\leq \|x\|_\infty \|f^{\otimes^m}\|\|g^{\otimes^n}\|\|u\|_2\|v\|_2 \\ &\leq \|x\|_1 \|f^{\otimes^m}\|\|g^{\otimes^n}\|\|u\|_2\|v\|_2, \end{aligned} \tag{2.39}$$

from which the required estimate $\mathbf{A(vi)}$ follows. □

Let $\mathcal{A}$ be a C^* or von-Neumann algebra which is equipped with a faithful, semifinite and lower-semicontinuous trace τ. Suppose we are given two quantum stochastic flows

$$j_t^{(1)} : \mathcal{A} \longrightarrow \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_1)))$$

and

$$j_t^{(2)} : \mathcal{A} \longrightarrow \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_2))),$$

which satisfy two quantum stochastic differential equations of the type (1.4) with coefficients $(\mathcal{L}^{(1)}, \delta^{(1)}, \sigma^{(1)})$ and $(\mathcal{L}^{(2)}, \delta^{(2)}, \sigma^{(2)})$ respectively. *In the following, we assume that the hypothesis in the definition (2.1) is true for both sets of structure maps with the same $\mathcal{A}_0$.* Let $\Gamma_1 := \Gamma(L^2(\mathbb{R}_+, k_1))$ and $\Gamma_2 := \Gamma(L^2(\mathbb{R}_+, k_2))$. For $c^{(j)}, d^{(j)} \in k_j,\ j = 1, 2$, define $j_t^{c^{(j)}, d^{(j)}} = j_t^{(j)\ c^{(j)}, d^{(j)}}$. We now define the Trotter product of these two flows:

For $x \in \mathcal{A}$, define $\eta_t : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{B}(\Gamma_1 \otimes \Gamma_2)$ by :

$$\eta_t(x) = (j_t^{(1)} \otimes id_{\mathcal{B}(\Gamma_2)}) \circ j_t^{(2)}(x). \tag{2.40}$$

Take a dyadic partition of the whole real line $\mathbb{R}$ and consider the part of the partition in $[s,t]$ for large n, described in the picture below:

$$----\Big|_{[2^n s]\cdot 2^{-n}}---\Big[s--\Big|_{([2^n s]+1)\cdot 2^{-n}}----------\Big|_{[2^n t]\cdot 2^{-n}}--t\Big]--\Big|_{([2^n t]+1)\cdot 2^{-n}}---\ ,$$

where $[t] =$ *integer* $\leq t$ for real t.

Definition 2.3.5. *Set*

$$\phi^{(n)}_{[s,t]} = [(\xi_s \circ \eta_{([2^n s]+1)2^{-n}})] \circ \left\{ \prod_{j=[2^n s]+1}^{[2^n t]-1} \left(\xi_{j.2^{-n}} \circ \eta_{2^{-n}} \otimes 1_{B\left(\Gamma^{j.2^{-n}}_{(j+1).2^{-n}}\right)} \right) \right\} \circ \tag{2.41}$$
$$[(\xi_{[2^n t].2^{-n}} \circ \eta_{t-[2^n t].2^{-n}})].$$

Set $\phi^{(n)}_t := \phi^{(n)}_{[0,t]}$. The map $\phi^{(n)}_t$ will be called the n-fold Trotter product of the flows $j^{(1)}_t$ and $j^{(2)}_t$.

Clearly this map $\phi^{(n)}_{[s,t]}$ is a $*$-homomorphism for each n and being compositions of cocycles, $\phi^{(n)}_t$ itself is a cocycle. Let $(e_i)_{i\geq 1}$ be an orthonormal basis for k_1 and $(l_i)_{i\geq 1}$ be an orthonormal basis for k_2 so that the set $\mathcal{G} = \{(\lambda e_i, 0), (0, \beta l_j) | \lambda, \beta \in \mathbb{C}\}_{i,j\geq 1}$ is total in $k_1 \oplus k_2$. Let $\mathcal{M}$ be the set of step functions f supported over intervals with dyadic end points and taking values in $\mathcal{G}$.

It is known [63] that $\{e(f) | f \in \mathcal{M}\}$ is total in $\Gamma(L^2(\mathbb{R}_+, k_1 \oplus k_2))$.
Now we state the weak version of the *Random Trotter Product Formula for quantum stochastic flows, leaving the strong one for the next section.*

Theorem 2.3.6. The (weak) Trotter product formula-I :
Suppose $\mathcal{A}$ is a C^*-algebra and that for each c_j, d_j belonging to $k_j, j = 1, 2$, the closure of the operator $\sum_{j=1}^2 \left(\mathcal{L}^{(j)} + \langle c_j, \delta^{(j)} \rangle + \delta^{\dagger(j)}_{d_j} + \langle c_j, \sigma_{d_j} \rangle + \langle c_j, d_j \rangle \right)$ generates a C_0 contractive semigroup in $\mathcal{A}$.

Then $\phi^{(n)}_t(x)$ as defined above converges in the weak operator topology of $h \otimes \Gamma^1 \otimes \Gamma^2$ to $j_t(x)$ where j_t is another CPC flow satisfying a q.s.d.e. with structure matrix

$$\begin{pmatrix} \mathcal{L}^{(1)} + \mathcal{L}^{(2)} & \delta^{\dagger(1)} & \delta^{\dagger(2)} \\ \delta^{(1)} & \sigma^{(1)} & 0 \\ \delta^{(2)} & 0 & \sigma^{(2)} \end{pmatrix}.$$

Proof of Theorem 2.3.6 :

Since $\|\phi_t^{(n)}(x)\|_\infty \leq \|x\|_\infty$, it is enough to prove the weak convergence of this sequence of maps for u, v belonging to some dense subset of $L^2(\tau)$ and f, g belonging to some total subset $L^2(\mathbb{R}_+, k_1 \oplus k_2)$. Let $f \in \mathcal{M}$ be of the form :

$$\begin{aligned} f(x) &= 0 && if\ x < \frac{[2^n s]}{2^n}\ or\ x > \frac{[2^n t]+1}{2^n} \\ &= c_0^{(1)} \oplus c_0^{(2)} && if\ x \in [s, \frac{[2^n s]+1}{2^n}) \\ &= c_j^{(1)} \oplus c_j^{(2)} && if\ x \in [\frac{[2^n s]+j}{2^n}, \frac{[2^n s]+j+1}{2^n}) \\ &\text{where } c_j^{(1)} \oplus c_j^{(2)} \in \mathcal{G} \text{ for } j = 1, 2, ... \end{aligned} \tag{2.42}$$

Similarly, let g be of the form (2.42) with $c_j^{(1)} \oplus c_j^{(2)}$ replaced by $d_j^{(1)} \oplus d_j^{(2)}$. For an interval $[a, b] \subseteq [\frac{[2^n s]+j}{2^n}, \frac{[2^n s]+j+1}{2^n})$, let

$$\mathbf{\Sigma}_{[s,t]}^{(p)} = (j_{\frac{[2^p s]+1}{2^p}-s}^{c_j^{(1)},d_j^{(1)}} \circ j_{\frac{[2^p s]+1}{2^p}-s}^{c_j^{(2)},d_j^{(2)}}) \circ (j_{\frac{1}{2^p}}^{c_j^{(1)},d_j^{(1)}} \circ j_{\frac{1}{2^p}}^{c_j^{(2)},d_j^{(2)}})^{[2^p t]-[2^p s]-1} \circ (j_{t-\frac{[2^p t]}{2^p}}^{c_j^{(1)},d_j^{(1)}} \circ j_{t-\frac{[2^p t]}{2^p}}^{c_j^{(2)},d_j^{(2)}}).$$

For m, sufficiently larger than n, for $x \in \mathcal{A}$ and considering f and g as above, we have:

$$\begin{aligned} &\langle\ \phi_t^m(x)ue(f), ve(g)\rangle \\ &= \langle \mathbf{\Sigma}_{[s,\frac{[2^n s]+1}{2^n}]}^{(m)} \circ \mathbf{\Sigma}_{[\frac{[2^n s]+1}{2^n},\frac{[2^n s]+2}{2^n}]}^{(m)} \circ ...\mathbf{\Sigma}_{[\frac{[2^n s]+j}{2^n},\frac{[2^n s]+j+1}{2^n}]}^{(m)} \circ ...\mathbf{\Sigma}_{[t,\frac{[2^n t]}{2^n}]}^{(m)}](x)u, v\rangle \end{aligned} \tag{2.43}$$

So it is enough to prove the strong convergence of the operators $\mathbf{\Sigma}_{[a,b]}^{(p)}(x)$ for a single interval $[a, b]$. So let $c = c^{(1)} \oplus c^{(2)}$ and $d = d^{(1)} \oplus d^{(2)}$ belong to $\mathcal{G}$. Now for $f = (c^{(1)} \oplus c^{(2)})\chi_{[a,b]}$, $g = (d^{(1)} \oplus d^{(2)})\chi_{[a,b]}$, we have from (2.41) that

$$\begin{aligned} &\langle\ \mathbf{\Sigma}_{[a,b]}^{(m)}(x)ue(f), ve(g)\rangle \\ &= \langle\ (j_{\frac{[2^n a]+1}{2^n}-a}^{c^{(1)},d^{(1)}} \circ j_{\frac{[2^n a]+1}{2^n}-a}^{c^{(2)},d^{(2)}}) \circ \left\{ j_{\frac{1}{2^n}}^{c^{(1)},d^{(1)}} \circ j_{\frac{1}{2^n}}^{c^{(2)},d^{(2)}} \right\}^{[2^n b]-[2^n a]-1} \circ (j_{b-\frac{[2^n b]}{2^n}}^{c^{(1)},d^{(1)}} \circ j_{b-\frac{[2^n b]}{2^n}}^{c^{(2)},d^{(2)}})(x)u, v\rangle \\ &\equiv \langle Q_1 \circ \{Q_2\}^{[2^n b]-[2^n a]-1} \circ Q_3(x)u, v\rangle. \end{aligned} \tag{2.44}$$

We note that the semigroups $j_t^{c^{(j)},d^{(j)}}$ (discussed in page 14) are C_0 semigroups for $j = 1, 2$, and that $\frac{[nt]}{n} \longrightarrow t$ as $n \longrightarrow \infty$. Thus the maps Q_1 *and* Q_3 strongly converge to $1_{\mathcal{A}}$. As for Q_2, we get

$$\begin{aligned} Q_2^{[2^n b]-[2^n a]-1} &= (j_{\frac{1}{2^n}}^{c^{(1)},d^{(1)}} \circ j_{\frac{1}{2^n}}^{c^{(2)},d^{(2)}})^{[2^n b]-[2^n a]-1} \\ &= ((j_{\frac{1}{2^n}}^{c^{(1)},d^{(1)}} \circ j_{\frac{1}{2^n}}^{c^{(2)},d^{(2)}})^{2^n})^{\frac{[2^n b]-[2^n a]-1}{2^n}} \end{aligned} \tag{2.45}$$

which converges strongly by the *Trotter product formula for semigroups on Banach spaces*, since the generator of $(j_t^{c^{(l)},d^{(l)}})_{t\geq 0}$ restricted to $\mathcal{A}_0$ is $\mathcal{L}^{(l)} + \langle c_l, \delta^{(l)}\rangle + \delta_{d_l}^{\dagger(l)} + \langle c_j, \sigma_{d_j}\rangle + \langle c_l, d_l\rangle id$, for $l = 1,2$ and by the assumption of the theorem, the closure of $\sum_{j=1}^2 \left(\mathcal{L}^{(j)} + \langle c_j, \delta^{(j)}\rangle + \delta_{d_j}^{\dagger(j)} + \langle c_j, \sigma_{d_j}\rangle + \langle c_j, d_j\rangle id\right)$ generates a C_0 contractive semigroup in $\mathcal{A}$.

On $\mathcal{A}_0$, the semigroup $j_t^{c,d}$ satisfies the following:

$$j_t^{c,d}(x) - j_0^{c,d}(x) = \int_0^t ds j_s^{c,d} \circ (\mathcal{L}^{(1)}(x) + \mathcal{L}^{(2)}(x) + \left\langle c^{(1)} \oplus c^{(2)}, \delta^{(1)} \oplus \delta^{(2)} \right\rangle (x) +$$
$$(\delta^{\dagger(1)} \oplus \delta^{\dagger(2)})_{d^{(1)} \oplus d^{(2)}}(x) + \langle\ (c^{(1)} \oplus c^{(2)}), (\sigma^{(1)} \oplus \sigma^{(2)})_{d^{(1)} \oplus d^{(2)}} \rangle\ (x) + \langle c^{(1)} \oplus c^{(2)}, d^{(1)} \oplus d^{(2)} \rangle (x)). \tag{2.46}$$

Thus there exists a contractive map $j_{[s,t]} : \mathcal{A}_0 \longrightarrow \mathcal{A}'' \otimes \mathcal{B}(\Gamma_{[s,t]})$ such that $\phi_{[s,t]}^{(n)}(x)$ converges in the weak operator topology to $j_{[s,t]}(x)$ in $h \otimes \Gamma$. Clearly by density of $\mathcal{A}_0$ in $\mathcal{A}$, $j_{[s,t]}$ extends to the whole of $\mathcal{A}$. Thus j_t satisfies a weak q.s.d.e. in $h \otimes \Gamma_1 \otimes \Gamma_2 \cong h \otimes \Gamma(L^2(\mathbb{R}_+, k))$, with the structure matrix :

$$\begin{pmatrix} \mathcal{L}^{(1)} + \mathcal{L}^{(2)} & \delta^{\dagger(1)} & \delta^{\dagger(2)} \\ \delta^{(1)} & \sigma^{(1)} & 0 \\ \delta^{(2)} & 0 & \sigma^{(2)} \end{pmatrix},$$

where $k := k_1 \oplus k_2$ being the noise space.

It is clear that j_t is a cocycle and since j_t is contractive, j_t also satisfies the strong q.s.d.e. with the above structure matrix. □

Theorem 2.3.7. The (Weak) Trotter product formula-II :
Let $\mathcal{A}$ be a C^* or von-Neumann algebra, and τ be a trace on it. Furthermore assume that:

(a) in the structure matrices associated with $j_t^{(1)}$ and $j_t^{(2)}$, $\sigma^{(j)} = 0$ for $j = 1,2$,

(b) the closure of $\mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)}$ generates a C_0, contractive, analytic semigroup in $L^2(\tau)$.

Then $\phi_t^{(n)}(x)$ as defined above converges in the weak operator topology of $h \otimes \Gamma^1 \otimes \Gamma^2$ to $j_t(x)$ for all x in $\mathcal{A}$, where j_t is a CPC flow satisfying the q.s.d.e. with structure matrix

$$\begin{pmatrix} \mathcal{L}^{(1)} + \mathcal{L}^{(2)} & \delta^{\dagger(1)} & \delta^{\dagger(2)} \\ \delta^{(1)} & 0 & 0 \\ \delta^{(2)} & 0 & 0 \end{pmatrix}.$$

Proof of Theorem 2.3.7 : Set $\theta_0^{i,(1)}(x) := \langle e_i, \delta^{(1)} \rangle(x)$, $\theta_0^{i,(2)}(x) := \langle l_i, \delta^{(2)} \rangle(x)$, for $i \geq 1$. For $x \in \mathcal{A}_0$ and every positive integer n, we have

$$
\begin{aligned}
\|\delta^{(j)}(x)\|_2^2 = \sum_i \|\theta_0^{i,(j)}(x)\|_2 \leq 2\|\mathcal{L}_2^{(j)}(x)\|_2\|x\|_2 & \\
& \leq 2\frac{1}{\sqrt{2}\ n}\|\mathcal{L}_2^{(j)}(x)\|_2\frac{1}{\sqrt{2}}n\|x\|_2 \\
& \leq \left\{\frac{1}{\sqrt{2}}(\frac{1}{n}\|(\mathcal{L}_2^{(j)}(x))\|_2 + n\|x\|_2)\right\}^2 \\
& \text{for } j = 1,2\ .
\end{aligned}
\tag{2.47}
$$

Thus the operators $\theta_0^{i,(j)}$ are relatively bounded with respect to $\mathcal{L}_2^{(j)}$ with arbitrarily small bound. Similar calculations hold for $\theta_i^{0,(j)}(x)(\equiv \theta_0^{i,(j)}(x)^*)$, $j = 1, 2$. Since $\mathcal{L}_2^{(j)}$ are the pre-generators of contractive analytic semigroups in $L^2(\tau)$, we see that the operators

$$\theta_0^{i,(j)} + \theta_k^{0,(j)} + \mathcal{L}_2^{(j)},$$

for $j = 1, 2$ are pre-generators of C_0 semigroups (see [34] Theorem 2.4 and Corollary 2.5, p 497-498). A similar proof as above yields that for $x \in \mathcal{A}_0$, $c, d \in \mathcal{G}$,

$$\|\langle\ c, \delta^{(1)} \oplus \delta^{(2)} \rangle\ (x)\|_2 \leq \left\{(\frac{1}{\sqrt{2}\ n}\|(\mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)})(x)\|_2) + \frac{n}{\sqrt{2}}\|x\|_2\right\}\|c\|,$$

$$\|\langle \delta^{(1)} \oplus \delta^{(2)}, d \rangle\ (x)\|_2 \leq \left\{(\frac{1}{\sqrt{2}\ n}\|(\mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)})(x)\|_2) + \frac{n}{\sqrt{2}}\|x\|_2\right\}\|d\|.$$

Thus because of the hypothesis that $\mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)}$ generates an analytic C_0 semigroups in $L^2(\tau)$, we see that for $c, d \in \mathcal{G}$, the operator

$$\langle\ c, \delta^{(1)} \oplus \delta^{(2)} \rangle\ + \langle \delta^{(1)} \oplus \delta^{(2)}, d \rangle + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + \langle c, d \rangle$$

generates a C_0 semigroup in $L^2(\tau)$. The rest of the proof proceeds as that of the Theorem 2.3.6. □

Remark 2.3.8. *As can be noticed in the proof of the theorems 2.3.6 and 2.3.7, the convergence of $\phi_t^{(m)}(x)$ is actually in a topology stronger than the weak operator topology in $h \otimes \Gamma_1 \otimes \Gamma_2$; it is in the product topology of strong operator topology in h with weak operator topology in $\Gamma_1 \otimes \Gamma_2$.*

Proposition 2.3.9. *Let $j_t^{(k)}$ $(k = 1, 2)$ be two CPC cocycle flows satisfying all the assumptions for Theorem 2.3.6 such that the weak limit of the Trotter product $\phi_t^{(n)}$ exists. Assume furthermore that $j_t^{(k)}$ satisfy (2.36) for each k. Then j_t also satisfies (2.36).*

Proof. Let $\Delta_0 = [s, \frac{[2^n s]+1}{2^n}),\ \Delta_j = [\frac{[2^n s]+j}{2^n}, \frac{[2^n s]+j+1}{2^n})$ *for* $1 \leq j \leq [2^n t] - [2^n s] - 1$ and $\Delta' = [\frac{[2^n t]}{2^n}, t)$. Let $\chi_0 = 1_{\Delta_0}, \chi_j = 1_{\Delta_j}$ *and* $\chi' = 1_{\Delta'}$. Then for $c = c_1 \oplus c_2, d = d_1 \oplus d_2$ *and* $x \in L^1 \cap \mathcal{A}$, define $\eta_\tau^{c,d}(x) = \langle e(c\chi_{[0,\tau]}), \eta_\tau(x) e(d\chi_{[0,\tau]})\rangle$.

$$\begin{aligned}
&\langle e(c1_{[s,t]}), \phi^{(n)}_{[s,t]}(x) e(d1_{[s,t]})\rangle \\
&= \langle e(c\chi_0) \otimes_j e(c\chi_j) \otimes e(c\chi'), \phi^{(n)}_{[s,t]}(x) e(d\chi_0) \otimes_j e(d\chi_j) \otimes e(d\chi')\rangle \\
&= \eta^{c,d}_{([2^n s]+1)2^{-n}-s} \circ (\eta^{c,d}_{2^{-n}})^{[2^n t]-[2^n s]-1} \circ \eta^{c,d}_{t-[2^n t]2^{-n}}.
\end{aligned} \tag{2.48}$$

Now $\eta_\tau^{c,d}(x) = j_\tau^{c_1,d_1} \circ j_\tau^{c_2,d_2}(x)$. Thus

$$\begin{aligned}
&\|\eta_\tau^{c,d}(x)\|_1 \leq e^{\tau M_1} \|j_\tau^{c_2,d_2}(x)\|_1 \\
&\leq e^{\tau(M_1+M_2)} \|x\|_1 \text{ (where } M_j \text{ depends on } \|c_j\|, \|d_j\| \text{ for } j = 1, 2.)
\end{aligned} \tag{2.49}$$

Thus

$$\|\langle e(c1_{[s,t]}), \phi^{(n)}_{[s,t]}(x) e(d1_{[s,t]})\rangle\|_1 \leq e^{(t-s)(M_1+M_2)} \|x\|_1$$

and from this the conclusion follows. □

2.4 The Strong Trotter product formula.

The theorems 2.3.6 and 2.3.7 have established that $\phi_t^{(n)}$ converges weakly to j_t (a CPC cocycle flow) on $h \otimes \Gamma_1 \otimes \Gamma_2 \cong h \otimes \Gamma$. Clearly since $\phi_t^{(n)}$ is a $*$-homomorphism from $\mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma)$, the above convergence is strong if and only if j_t itself is a $*$-homomorphism. The next theorem, using the crucial results of Lemma 2.3.3 and Proposition 2.3.9, exactly does that.

Theorem 2.4.1. The (strong) Trotter product formula-III :

(i) *Suppose $\mathcal{A}$ is a C^* algebra. Let $j_t^{(1)}$ and $j_t^{(2)}$ be two quantum stochastic flows satisfying the condition of Theorem 2.3.6. Suppose furthermore that these two flows satisfy the following:*

- *(a) For $x \in \mathcal{A} \cap L^1(\tau)$, $j_t^{c(j),d(j)}(x)$ satisfies (2.36) or* **A(vi)** *for $c_j, d_j \in k_j$, $j = 1, 2$;*
- *(b) $\tau(\mathcal{L}^{(j)}(x^*x)) \leq 0$ for $j = 1, 2$;*
- *(c) each of the semigroups generated by $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ as well as their Trotter product limit have analytic $L^2(\tau)$ extensions as semigroups.*

Then $\phi_t^{(n)}(x)$ as defined above converges in the strong operator topology of $h \otimes \Gamma_1 \otimes \Gamma_2$ to $j_t(x)$ where j_t is another quantum stochastic flow which satisfy a q.s.d.e. with the structure matrix

$$\begin{pmatrix} \mathcal{L}^{(1)} + \mathcal{L}^{(2)} & \delta^{\dagger(1)} & \delta^{\dagger(2)} \\ \delta^{(1)} & \sigma^{(1)} & 0 \\ \delta^{(2)} & 0 & \sigma^{(2)} \end{pmatrix}.$$

(ii) *Let $\mathcal{A}$ be a von-Neumann algebra. Suppose the two quantum stochastic flows $(j_t^{(j)})_{t \geq 0}$, for $j = 1, 2$, satisfy the conditions of theorem 2.3.7 and conditions (a) and (b) of part(i) of the statement above. Then the same conclusion as in part(i) above holds.*

Proof of Theorem 2.4.1 :
It suffices to prove that the limiting CPC flow j_t is a $*$-homomorphism. Condition (2.36) implies **A(vi)** and thus by Proposition 2.3.9 and Theorem 2.2.3, j_t is a $*-$homomorphism. □

Remark 2.4.2. *Assumption (c) of Theorem 2.4.1 can be replaced by the assumption that each of the maps $\mathcal{L}_2^{(1)}$ and $\mathcal{L}_2^{(2)}$ satisfy the condition of Corollary 2.2.4.*

2.5 Applications.

2.5.1 Construction of classical and non-commutative stochastic processes

We shall now illustrate how to construct various multidimensional processes as random Trotter-Kato limits of the corresponding "marginals". Our examples will include Brownian Motion on Lie groups and random walk on discrete groups.

Let G be a second countable locally compact group. Let $\{\phi_i\}_{i=1}^{\infty}$ constitute a countable family of functions from $C_0(G)$ which separates points on G. Define a metric ρ as follows:

$$\rho(g, g') := \sum_{n=1}^{\infty} \{ \frac{|\phi_n(g) - \phi_n(g')|}{2^n(1 + |\phi_n(g) - \phi_n(g')|)} \}. \tag{2.50}$$

It can be shown that this metric gives the same topology on G and also that G is complete, and thus in particular, G is a Polish group.
Let $\mathcal{A}$ be a C^* algebra equipped with a faithful, semifinite, lower-semicontinuous trace τ. As before, we imbed $\mathcal{A}$ in $\mathcal{B}(h)$, where $h = L^2(\tau)$, and extend τ as a normal

semifinite trace on $\mathcal{A}''$. Assume furthermore that there is a strongly continuous, $*$-automorphic G-action α_g on $\mathcal{A}$ which is also τ-preserving ie $\tau(\alpha_g(a)) = \tau(a)$. This allows us to extend α_g to a unitary operator U_g on $L^2(\tau)$, and we extend α to $\mathcal{A}''$ as a normal $*$ automorphism given by $\alpha_g(\cdot) = U_g \cdot U_g^*$.

Lemma 2.5.1. *Let $(X_n)_n$ be a G-valued random variable on some space $(\Omega, \mathcal{F}, P)$, and suppose that for all ψ in $L^2(G)$ and for all ϕ in $C_0(G)$,*

$$\int_G dg \int_\Omega dP(\omega)|\psi(g)|^2|\phi(g.X_n) - \phi(g.X_m)|^2 \longrightarrow 0$$

as $n, m \longrightarrow \infty$, where dg is the left-invariant Haar measure on G. Then there exists a random variable $X : \Omega \longrightarrow G$ such that $X_n \longrightarrow X$ in probability.

Proof. We choose and fix some ψ in $L^2(G)$ with $\|\psi\|_2 = 1$, and let $d\mathbb{P}(\omega, g) := dP(\omega) \otimes |\psi(g)|^2 dg$. Since $\int_G dg|\psi(g)|^2 \int_\Omega dP(\omega)|\phi_i(g.X_n) - \phi_i(g.X_m)|^2 \to 0$, for every i, it follows by setting $Y_n(g, \omega) = g.X_n(\omega)$, and using the dominated convergence theorem for $g \in G$ $\omega \in \Omega$, that for every $\epsilon > 0$,

$$\mathbb{P}(\rho(Y_n, Y_m) \geq \epsilon) \leq \frac{1}{\epsilon^2} \mathbb{E}^{\mathbb{P}}(\rho(Y_n, Y_m)) \to 0,$$

as $m, n \longrightarrow \infty$. Thus there is a G-valued random variable Y, defined on Ω, such that

$$Y_n \xrightarrow{\mathbb{P}} Y.$$

So

$$\mathbb{P}(\rho(Y_n, Y) > \epsilon) = \int_G dg|\psi(g)|^2 P(\rho(g.X_n, Y) \geq \epsilon) \equiv \int_G dg|\psi(g)|^2 f_n(g) \longrightarrow 0,$$

where $f_n(g) = P(d(g.X_n, Y) \geq \epsilon)$. By Egoroff's theorem, there exists a measurable set say Δ of positive Haar-measure such that for all g in Δ, we have $f_n(g) \longrightarrow 0$, and the proof of the theorem is complete by taking $X(\omega) = g^{-1}Y(g, \omega)$, for any fixed g in Δ. $\square$

Lemma 2.5.2. *Let $(g_t)_{t \geq 0}$ be a G valued Levy process, defined on some probability space $(\Omega, \mathcal{F}, P)$. Define $j_t : \mathcal{A}'' \to L^\infty(\Omega, \mathcal{A}'') \subseteq \mathcal{B}(L^2(\tau) \otimes L^2(\Omega))$, by $j_t(x)(\omega) := \alpha_{g_t(\omega)}(x)$, and*
$T_t(x) = \mathbb{E}(\alpha_{g_t(\omega)}(x))$.

(i) Then for $f, g \in L^2(\mathbb{R}_+),\ x \in \mathcal{A} \cap L^1(\tau)$, we have

$$\begin{aligned} &\tau\left(|\langle e(f), j_t(x)e(g)\rangle|\right) \\ &\leq \|e(f)\|_2 \|e(g)\|_2 \|x\|_1. \end{aligned} \tag{2.51}$$

(ii) $(T_t)_{t\geq 0}$ is a normal QDS on $\mathcal{A}''$, and if its restriction on $\mathcal{A}$ leaves $\mathcal{A}$ invariant, it is a QDS on $\mathcal{A}$ with respect to the norm-topology. Furthermore, if g_t and $(g_t)^{-1}$ have the same distribution for each t, then the semigroup is τ-symmetric.

Proof. To prove (i), it suffices to show the inequality for positive $x \in \mathcal{A}$. For such x, we have

$$\begin{aligned} &\tau\left(|\langle e(f), j_t(x)e(g)\rangle|\right) \\ &\leq \mathbb{E}\tau\left(\left|exp[\int_0^\infty (\overline{f}+g)d\omega - \frac{1}{2}\int_0^\infty (\overline{f}^2 + g^2)dt] j_t(x)(\omega)\right|\right) \end{aligned} \tag{2.52}$$

from which the result follows. (ii) From the defining property of Levy processes, the semigroup property of T_t follows; while the normality of T_t is a consequence of the fact that j_t is implemented by an automorphism of $\mathcal{A}''$. Moreover, if g_t and g_t^{-1} have the same distribution, we have

$$\begin{aligned} &\tau(T_t(a)b) = \tau[\mathbb{E}\{\alpha_{g_t(\omega)}(a)b\}] \\ &= \tau[\mathbb{E}\{\alpha_{g_t(\omega)}(a\alpha_{g_t(\omega)^{-1}}(b))\}] = \mathbb{E}[\tau\{\alpha_{g_t(\omega)}(a\alpha_{g_t(\omega)^{-1}}(b))\}] \\ &= \mathbb{E}[\tau\{(a\alpha_{g_t(\omega)^{-1}}(b))\}] = \tau[\mathbb{E}\{(a\alpha_{g_t(\omega)^{-1}}(b))\}] \\ &= \tau[a\mathbb{E}\{\alpha_{g_t(\omega)}(b)\}] = \tau(aT_t(b)). \end{aligned} \tag{2.53}$$

□

After these two lemmas, we now give a few concrete examples .

(A) Classical and non-commutative Brownian motion: Assume now that G be a compact Lie group (of dimension k) acting smoothly on a C^* algebra $\mathcal{A}$ and τ is a lower semicontinuous, faithful, finite trace on $\mathcal{A}$. Denote by $\mathcal{A}^\infty$, the $*$-subalgebra of $\mathcal{A}$ generated by elements x such that $g \to \alpha_g(x)$ is norm-smooth, where $g \to \alpha_g$ is the group action. Let $\{\chi_\ell\}_{\ell=1}^k$ be a basis for the Lie algebra of G and let G_ℓ be the one-parameter subgroup $\exp(t\chi_l),\ t \in \mathbb{R}$. Define
$j_t^{(\ell)} : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(L^2(W^{(l)}))(\cong \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+))))$, by

$j_t^{(\ell)}(x)(\omega) := \alpha_{\exp(W_t^{(\ell)}(\omega)\chi_l)}(x)$, where $W_t^{(\ell)}$ is the standard Brownian motion on $\mathbb{R}$. Then by (i) of Lemma 2.5.2, replacing G by G_ℓ, estimate (2.36) or equivalently the condition (a) of
Theorem 2.4.1-(i) is verified for $j_t^{(\ell)}$. Furthermore, since $W_t^{(\ell)}$ and $-W_t^{(\ell)}$ have the same distribution, (ii) of Lemma 2.5.2 applies. Combining this with Remark 2.1.2 for $\mathcal{A}_0 = \mathcal{A}^\infty$, we see that condition (b) of Theorem 2.4.1-(i) holds. For applying Theorem 2.4.1-(i), we now need to check only that $\sum_{\ell=1}^k \mathcal{L}_2^{(\ell)}$ is the pregenerator of a C_0 semigroup. For this we proceed as follows:
As α_g is a trace preserving automorphism, it extends to a unitary operator U_g, in the Hilbert space $L^2(\tau)$. Let δ_ℓ be the norm generator of the automorphism group $(\alpha_{\exp(t\chi_\ell)})_{t\in\mathbb{R}}$. Then δ_ℓ extends to an unbounded, densely defined skew-adjoint operator in $L^2(\tau)$, which generates the unitary group $(U_{\exp(t\chi_\ell)})_{t\in\mathbb{R}}$. By an abuse of notation, we again denote this extension by δ_ℓ. Note that $\mathcal{L}_2^{(\ell)} = \frac{1}{2}\delta_\ell^2$, on $\mathcal{A}^\infty$, for all ℓ. Thus $\sum_{\ell=1}^k \mathcal{L}_2^{(\ell)} = \frac{1}{2}\sum_{\ell=1}^k \delta_\ell^2$ is a densely defined, negative and symmetric operator. By Nelson's analytic vector theorem for the representations of the Lie algebra [51, Theorem 3, p. 591] and [59, Theorem X.39], $\sum_{\ell=1}^k \mathcal{L}_2^{(\ell)}$ is essentially selfadjoint in $L^2(\tau)$, and hence its' closure generates a C_0 contraction semigroup. Thus condition (c) of Theorem 2.4.1-(i) holds. So Theorem 2.4.1-(i) applies. Specializing this to the case when $\mathcal{A} = C(G)$, and by Lemma 2.5.1, we get the convergence in probability of the following sequence of random variables:

$$X_t^{(n)} := \prod_{i=1}^{k} \prod_{l=0}^{[2^n t]} \exp((W^{(i)}_{\frac{[2^n l]+1}{2^n}} - W^{(i)}_{\frac{l}{2^n}})\chi_i), \tag{2.54}$$

where the limiting random variable is clearly a Brownian motion on G, giving a result similar to that in [53].

In case of $G = \mathbb{T}^2$ and $\mathcal{A}$ the irrational-rotational C^* algebra $\mathcal{A}_\theta$ (see page 254 of [62]), the quantum Brownian motion described in page-275 of [62] can be constructed using the method described here.

(B) Random walk in discrete group: Let G be a discrete, finitely generated group, generated by a symmetric set of torsion free generators, say $\{g_1, g_2, ...g_{2k}\}$, let e be the identity element of G and $g_1 g_{k+1} = e$, and α_g for each g be the automorphism obtained by the action of G on itself. Take $\mathcal{A} = C_0(G)$, τ to be the trace with respect to the counting measure. Consider 2k mutually independent Poisson-processes $(N_t^{(i)})_{t\geq 0}$, $i = 1, \ldots, 2k$, on $I\!N \cup \{0\}$, with intensity parameter $(\lambda_i)_{i=1}^{2k}$, respectively. Let $Z_t^{(i)} := N_t^{(i)} - N_t^{(k+i)}$, $i = 1, 2, \ldots, k$. Define $j_t^{(l)} : \mathcal{A} \to L^\infty(G) \otimes \mathcal{B}(L^2(N_t^{(l)}, N_t^{(k+l)}))$, $(l = 1, 2, ...k,)$

by $j_t^{(l)}(\phi)(\omega) = \alpha_{Z_t^{(l)}(\omega)}(\phi)$. Since the generator $\mathcal{L}^{(l)}$ of the vacuum semigroup associated with $j_t^{(l)}$ is bounded, so are the other structure maps. This implies that all the hypotheses of the Theorem 2.3.6 are satisfied and we do not need the homomorphism theorem of section 3 because homomorphism property follows from the fact that j_t, the limiting flow satisfies a q.s.d.e. with bounded structure maps [22]. So by Lemma 2.5.1, we have convergence in probability of the following sequence of random variables

$$X_t^{(n)} := \prod_{l=0}^{[2^n t]} \prod_{i=1}^{k} \mathcal{G}^{(i)}_{\frac{l+1}{2^n}} (\mathcal{G}^{(i)}_{\frac{l}{2^n}})^{-1},$$

where $\mathcal{G}_t^{(l)}(\omega) := g_l^{Z_t^{(l)}(\omega)}$. The limit is a random variable X_t which is a time homogeneous continuous time simple random walk.

Chapter 3

Dilation of quantum dynamical semigroups: Some new results

Dilation of quantum dynamical semigroups (QDS for short) using quantum stochastic calculus is one of the most interesting and important problem of Quantum Probability (see [1, 62, 52, 32]). It is known that a QDS with bounded generator always admits Hudson-Parthasarathy (HP for short) dilation (see [62, 45]). Construction of such dilation amounts to solving quantum stochastic differential equation (QSDE) with bounded coefficients, and prescribed initial values and proving the unitarity of the solution. Such unitary solution always exists as long as the coefficients are bounded [62, 45]. For a QDS with unbounded generator, no such results are known in general. However, certain sufficient conditions on the unbounded operator coefficients for e.g. [62, p.174],[25, 14, 50, 49, 2] are known using which one can solve QSDE with unbounded coefficients. Using these techniques, the authors of [62] proved the existence of Hudson-Parthasarathy dilation of symmetric QDS which are covariant with respect to the action of a Lie group [62, Theorem 8.1.23]. The key fact that allowed them to construct such dilations is the existence of a "nice" dense subspace within the domain of the adjoints of the coefficients. Such subspaces may not exist in general. In this chapter in section 1.1 we will show that in context of $\mathcal{B}(\mathcal{H})$, symmetry with respect to the canonical trace is sufficient to ensure the existence HP dilation of a QDS and hence the additional assumption of covariance is not required. Besides in section 1.2, we shall apply the techniques developed in chapter 2 to obtain EH dilation of a large class of interesting QDS appearing in mathematical physics literature, generalizing results of [29]. In chapter 4, another dilation result (HP type) will bepresented in section 4.3.

3.1 HP dilation of symmetric quantum dynamical semigroup on type-I factor

3.1.1 Notations and terminologies

HP dilation of a quantum dynamical semigroup

Definition 3.1.1. *A Hudson-Parthasarathy dilation (HP dilation for short) of a QDS $(T_t)_{t\geq 0}$ on a C^* or von-Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is given by a family $(U_t)_{t\geq 0}$ of unitary operators acting on $h \otimes \Gamma$, such that the following holds:*

(i) U_t satisfies a QSDE of the form (1.7) with initial condition $U_0 = I$.

(ii) For all $u, v \in h,\ x \in \mathcal{A}$,

$$\langle ve(0), U_t(x \otimes I)U_t^* ue(0)\rangle = \langle v, T_t(x)u\rangle.$$

It is known that QDS with bounded generator always admits HP dilation (see [62, 45]). Some partial results are also known for QDS with unbounded generator (see [62, 14, 2]).

The main goal of this paper is to prove the following theorem:

Theorem 3.1.2. *Suppose $(T_t)_{t\geq 0}$ is a conservative, symmetric QDS on $\mathcal{B}(\mathcal{H})$ (symmetric with respect to the canonical trace), with ultraweak generator $\mathcal{L}$. Then $(T_t)_{t\geq 0}$ always admits an HP dilation.*

Before proving Theorem 3.1.2, we recall some facts about unbounded derivations in the next section. We refer the reader to [11] for more discussions on the topic.

3.1.2 Unbounded derivations.

For a Hilbert space $\mathcal{H}$, let $K(\mathcal{H})$ denote the space of compact operators on $\mathcal{H}$. A derivation $\delta \in Lin(\mathcal{A}, \mathcal{A})$, where $\mathcal{A}$ is a $*$-algebra, is called symmetric if $\delta(A^*) = \delta(A)^*$.

Proposition 3.1.3. *[11, p.238] Let δ be a symmetric derivation defined on a $*$-subalgebra $\mathcal{D}$ of the bounded operators in a Hilbert space $\mathcal{H}$. Let $\Omega \in \mathcal{H}$ be a unit vector, cyclic for $\mathcal{D}$ in $\mathcal{H}$ and denote the corresponding state by ω (i.e. $\omega(x) = \langle \Omega, x\Omega\rangle$). Suppose we have $|\omega(\delta(A))| \leq L\{\omega(A^*A) + \omega(AA^*)\}^{\frac{1}{2}}$ for some constant L. Then there exists a symmetric operator H on $\mathcal{H}$ such that*

$$Dom(H) = Dom(\delta)\Omega,$$
$$\delta(A)\psi = i[H, A]\psi \ for\ \psi \in \mathcal{H};$$

where $[H, x] := Hx - xH$.

Lemma 3.1.4. *Let* $(\mathbf{A})_p, I\!\!B \in M_n(\mathbb{C})$*, such that* $\mathbf{A}_p \to I\!\!B$ *as* $p \to \infty$*. Suppose that* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $I\!\!B$ *with multiplicity* m*. Then given a small neighbourhood* U *of* λ*, there exists* $p_0(U) \in I\!\!N$ *such that for all* $p \geq p_0(U)$*,* $\mathbf{A}_p$ *will have* m *eigenvalues (including the multiplicities) in the neighbourhood* U*.*

Proof. Observe that $|det(\mathbf{A} - z\mathbf{I}_n) - det(I\!\!B - z\mathbf{I}_n)| \to 0$. The result now follows by an application of Schwarz's theorem. □

Lemma 3.1.5. *Let* δ *be a symmetric derivation on* $\mathcal{B}(\mathcal{H})$*, such that* $Dom(\delta)$ *is dense in the weak operator topology. Assume that* $Dom(\delta)$ *is closed under holomorphic functional calculus and* $Dom(\delta) \cap K(\mathcal{H}) \neq \{0\}$*. Then* $Dom(\delta)$ *contains a rank-one projection.*

Proof. The proof is an adaptation of the arguments given in [11]:

Suppose that $B\ (\neq 0)\ \in Dom(\delta) \cap K(\mathcal{H})$ and $C = B^*B$. Choose a non-zero eigenvalue λ of C. Note that λ is an isolated point of the spectrum of C, since $C \in K(\mathcal{H})$. Let E_λ be the finite rank spectral projection onto the eigenspace of λ. Then

$$\begin{aligned} E_\lambda &= \frac{1}{2\pi i\lambda} \int_\Gamma d\gamma\ C(\gamma - C)^{-1} \\ &= \frac{1}{\lambda} C \frac{1}{2\pi i} \int_\Gamma d\gamma (\gamma - C)^{-1}, \end{aligned}$$

where Γ is a closed curve in an open neighbourhood say V, such that $V \cap \sigma(C) = \{\lambda\}$ and λ is in the interior of γ. Let U be another neighbourhood such that $U \cap V = \phi$ and $(\sigma(C) - \{\lambda\}) \subseteq U$. Thus $E_\lambda = \frac{1}{\lambda} C f(C)$, where $f(\cdot)$ is the function $f(z) = 1,\ z \in V$ and $f(z) = 0,\ z \in U$. So $f(\cdot)$ is holomorphic in a neighbourhood of $\sigma(C)$. Thus $E_\lambda \in Dom(\delta)$, since by the hypothesis, $Dom(\delta)$ is closed under holomorphic functional calculus. Now choose a rank one projection P such that $E_\lambda P E_\lambda = P$. Get $A_n \in Dom(\delta)$ such that $A_n = A_n^*$ and $A_n \stackrel{SOT}{\to} P$. We have $E_\lambda A_n E_\lambda \stackrel{SOT}{\to} E_\lambda P E_\lambda$ which implies that $\|E_\lambda A_n E_\lambda - E_\lambda P E_\lambda\| \to 0$ since the C^*-algebra $E_\lambda \mathcal{B}(\mathcal{H}) E_\lambda$ is finite dimensional. Notice that 1 is a simple eigenvalue of $E_\lambda P E_\lambda$, since P is a rank one projection. Thus for large n, $E_\lambda A_n E_\lambda$ has a simple eigenvalue in a neighbourhood around 1, by lemma 3.1.4. Fix a large n. Let $\rho (\neq 0)$ be the simple eigenvalue of $E_\lambda A_n E_\lambda$ in a neighbourhood of 1. Let E be the rank one projection onto the 1 dimensional eigenspace of the eigenvalue ρ. Let γ be a closed curve in a neighbourhood say W, such that $W \cap \sigma(E_\lambda A_n E_\lambda) = \{\rho\}$ and as before, suppose that γ encloses the point ρ. Note that $E_\lambda A_n E_\lambda \in Dom(\delta)$. As before, consider a function $g(\cdot)$, such that $g(z) = 1,\ z \in W$ and zero in another neighbourhood which is disjoint from W and contains $\sigma(E_\lambda A_n E_\lambda) - \{\rho\}$. Thus $g(\cdot)$ is holomorphic in a neighbourhood of $\sigma(E_\lambda A_n E_\lambda)$. We have $E = \frac{1}{\rho} E_\lambda A_n E_\lambda g(E_\lambda A_n E_\lambda)$, which proves that $E \in Dom(\delta)$, by virtue of our hypothesis that $Dom(\delta)$ is closed under holomorphic functional calculus. □

Lemma 3.1.6. *Let δ be a symmetric derivation satisfying the hypotheses of Lemma 3.1.5. Then there exists a symmetric operator H on $\mathcal{H}$ such that $Dom(H) := Dom(\delta)\Omega$ and $\delta(x) = i[H, x]$ for all $x \in Dom(\delta)$, for some $\Omega \in \mathcal{H},\ \|\Omega\| = 1$.*

Proof. Let E be the finite rank projection as obtained in Lemma 3.1.5. Suppose that $\Omega \in Ran(E)$ such that $\|\Omega\| = 1$. Let $\omega(x) = \langle \Omega, x\Omega \rangle$. Then

$$\begin{aligned} |\omega(\delta(A))| &= |\omega(E\delta(A)E)| \\ &\leq |\omega(\delta(EAE))| + |\omega(\delta(E)A)| + |\omega(A\delta(E))| \\ &\leq 3\|\delta(E)\|\ [\omega(A^*A) + \omega(AA^*)]^{\frac{1}{2}}; \end{aligned}$$

so that by Proposition 3.1.3, we have the required result. □

Proposition 3.1.7. *[59] If H is a densely defined symmetric operator on $\mathcal{H}$ such that $dim(H - iI)^\perp \neq dim(H + iI)^\perp$. Then there exists a Hilbert space $\widehat{\mathcal{H}} \supseteq \mathcal{H}$ and a self-adjoint operator K acting on $\widehat{\mathcal{H}}$ such that $K|_{\mathcal{H}} = H$ and we have the integral representation*

$$\langle Hu, v \rangle = \int_{-\infty}^{\infty} t\ d\langle F_t u, v \rangle;$$

for $u \in Dom(H)$, $v \in \mathcal{H}$; where F_t is the generalized resolution of identity.

3.1.3 Existence of HP-dilation

Let $tr_{\mathcal{H}}$ denote the canonical trace of $\mathcal{B}(\mathcal{H})$. Observe that in this case, the Hilbert space $L^2(tr_{\mathcal{H}})$ is identified with the space of Hilbert-Schmidt operators on $\mathcal{H}$ which we denote by $B_2(\mathcal{H})$. Let $\mathcal{L}_2$ denote the L^2-extension of $\mathcal{L}$ and $\mathcal{B}$ denote the associated Dirichlet algebra. Clearly $\mathcal{B} = Dom((-\mathcal{L}_2)^{\frac{1}{2}})$. Note that with respect to the C^*-subalgebra $K(\mathcal{H})$, the Dirichlet form associated with the semigroup is a C^*-Dirichlet form since the $*$-subalgebra $\mathcal{B}$ is norm dense in $K(\mathcal{H})$. So the set of results in [15, p.84-p.89, p.91, p.96.] gives the following:

- There exists a $K(\mathcal{H})-K(\mathcal{H})$ Hilbert-bi-module $\mathcal{K}$ and a π-derivation $\delta_0 : \mathcal{B} \to \mathcal{K}$ such that
  $\langle \delta_0(x), \delta_0(y)e\rangle_{\mathcal{K}} = \lim_{\epsilon \to 0} tr_{\mathcal{H}} \left(K_\epsilon(x,y)e\right),$
  where $K_\epsilon(x,y) = \mathcal{L}_2(1-\epsilon\mathcal{L}_2)^{-1}(x^*y) - \mathcal{L}_2(1-\epsilon\mathcal{L}_2)^{-1}(x^*)y - x^*\mathcal{L}_2(1-\epsilon\mathcal{L}_2)^{-1}(y),$

  for $x, y \in \mathcal{B}, e \in K(\mathcal{H})$ where π is the left action of $K(\mathcal{H})$ on $\mathcal{K}$.

- δ_0 viewed as an element of $Lin(L^2(tr_{\mathcal{H}}), \mathcal{K}))$ denoted by R_0, has domain $\mathcal{B}$ such that
  $$\langle R_0(x), R_0(y)\rangle_{\mathcal{K}} = -tr_{\mathcal{H}}(2\mathcal{L}(x^*)y), \text{ for } x \in Dom(\mathcal{L}_2) \subseteq \mathcal{B}, \tag{3.1}$$
  so that vectors of $\mathcal{K}$ of the form $\delta_0(x)$ for $x \in Dom(\mathcal{L}_2)$ belongs to $Dom(R_0^*)$ and hence
  $$\mathcal{L}_2 = -\frac{1}{2}R_0^*R_0|_{Dom(\mathcal{L}_2)}. \tag{3.2}$$
  We also have $\delta_0(x)y = (R_0x - \pi(x)R_0)y$ for $x, y \in \mathcal{B}$.

Without loss of generality, we may suppose that $\pi : K(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a non-degenerate C^*-representation. Thus it is equal to a direct sum of irreducibles which are unitarily equivalent to the identity representation. So π extends to $\mathcal{B}(\mathcal{H})$ as a unital normal $*$-representation, which we again denote by π. Now by GNS construction with respect to $tr_{\mathcal{H}}$, $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}(L^2(tr_{\mathcal{H}}))$. Thus there exists an isometry $\Sigma : \mathcal{K} \to L^2(tr_{\mathcal{H}}) \otimes k_0$, for some separable Hilbert space k_0 such that $\pi(x) = \Sigma^*(x \otimes 1_{k_0})\Sigma$ and $\Sigma\Sigma^*$ commutes with $(x \otimes 1_{k_0})$ (by proposition 1.1.4 in chapter 1). Then $\delta := \Sigma\delta_0$ satisfies $\delta(xy) = \delta(x)y + (x \otimes 1_{k_0})\delta(y)$. Moreover, equations (3.1) and (3.2) hold with R_0 replaced by $R := \Sigma R_0$ and we have the identity $\delta(x)y = (Rx - (x \otimes 1_{k_0})R)y$ for $x, y \in \mathcal{B}$. Note that here $\delta : \mathcal{B} \to \mathcal{B}(\mathcal{H}) \otimes k_0$, since we have $L^2(tr_{\mathcal{H}}) \equiv B_2(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ and in this case $\|\cdot\|_\infty \leq \|\cdot\|_2$.

Let $\mathcal{V}_0 := \{\sum_{i=1}^k \lambda_i e_i : \ (e_i)_i$ is an orthonormal basis for k_0 and $\lambda_i \in \mathbb{C}\}$.

We make an easy observation at this point:

Lemma 3.1.8. *Suppose that $(\mathcal{T}_t)_{t\geq 0}$ is an ultraweakly continuous C_0 (in the ultraweak topology) contractive semigroup on $\mathcal{B}(\mathcal{H})$, with generator C. Let $\mathcal{Z} \subseteq \mathcal{B}(\mathcal{H})$ be a subspace of $\mathcal{B}(\mathcal{H})$ which is closed with respect to a locally convex topology (LCT for short) given by a family of seminorms, say $(p_\alpha)_\alpha$. Suppose that $(\mathcal{T}_t)_{t\geq 0}$ restricted to $\mathcal{Z}$ becomes a C_0 (with respect to the LCT described above) semigroup, with generator say $\widetilde{C}$. Then if $x \in Dom(C)$ such that $C(x) \in \mathcal{Z}$, then $x \in Dom(\widetilde{C})$ and $\widetilde{C}(x) = C(x)$.*

Proof. Let $x \in Dom(C) \cap Dom(\widetilde{C})$. Since $(\mathcal{T}_t)_{t\geq 0}$ is a C_0 semigroup with respect to the ultraweak topology of $\mathcal{B}(\mathcal{H})$, we have $\mathcal{T}_t(x) - x = \int_0^t ds\ \mathcal{T}_s(C(x))$, where the integral in convergent in the ultraweak topology. Moreover, as $\mathcal{T}_t$ is a contraction for each $t \geq 0$, $\int_0^t ds\ \mathcal{T}_s(C(x)) \in \mathcal{B}(\mathcal{H})$. But we have $C(x) \in \mathcal{Z}$ and as $(\mathcal{T}_t)_{t\geq 0}$ restricted to $\mathcal{Z}$ is a C_0 semigroup with respect to the LCT described in the hypothesis, the integral also converges in this LCT, which implies that $\frac{\mathcal{T}_t(x)-x}{t}$ converges in this LCT to $C(x)$. Thus we have the required result. □

Lemma 3.1.9. *The $*$-subalgebra $\mathcal{B} \subseteq Dom(\langle \xi, R\rangle^*)$ for $\xi \in \mathcal{V}_0$.*

Proof. Note that $\delta : \mathcal{B} \to \mathcal{B}(\mathcal{H}) \otimes k_0$ is a derivation satisfying the identity $\delta(xy) = \delta(x)y + (x \otimes 1_{k_0})\delta(y)$ for $x, y \in \mathcal{B}$. Let us define $\theta_0^i(\cdot) := \langle e_i, \delta(\cdot)\rangle$. Since δ is a derivation, it follows that $\theta_0^i(\cdot)$ is a derivation and $Dom(\theta_0^i) = \mathcal{B}$, for each $i \in I\!N$. We prove that $tr_{\mathcal{H}}(\theta_0^i(xy)) = 0$ for all $x, y \in \mathcal{B}, i \in I\!N$, which will imply the result.

Fix an $i \in I\!N$. Recall that in our case, $\mathcal{B} = Dom((-\mathcal{L}_2)^{\frac{1}{2}}) \subseteq B_2(\mathcal{H})$. Let us define two new derivations $\delta_1 := \frac{\theta_0^i + \theta_0^{i\dagger}}{2}$ and $\delta_2 := \frac{\theta_0^i - \theta_0^{i\dagger}}{2i}$, where $\theta_0^{i\dagger}(x) = (\theta_0^i(x^*))^*$. Then we have $\delta = \delta_1 + i\delta_2$ and $Dom(\delta_1) = Dom(\delta_2) = \mathcal{B}$. Moreover, δ_1 and δ_2 are symmetric derivations. The results in page.103 of [15] shows that given a C^*-Dirichlet form, the associated Dirichlet algebra is closed under C^1 functional calculus. Hence $\mathcal{B}$ is closed under C^1 functional calculus and thus it is closed under holomorphic functional calculus. So by Lemma 3.1.5, $Dom(\delta_1)$ contains a finite rank operator. Hence by Lemma 3.1.6, $\delta_1(x) = i[T, x]$ for some symmetric operator T acting on $\mathcal{H}$ and we have $Dom(T) := Dom(\delta_1)\Omega$, where $\Omega \in \mathcal{H}$ is cyclic for $Dom(\delta_1)$. Now suppose $dim(T - iI)^{\perp} \neq dim(T + iI)^{\perp}$. Let K denotes the self-adjoint extension of T as described in Proposition 3.1.7, so that $K = K^*$. Let $P : \widehat{\mathcal{H}} \to \mathcal{H}$ be the orthogonal projection. Let $\widehat{\mathcal{H}}$ be decomposed in the basis of P i.e. $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}^{\perp}$. With respect to this decomposition, an operator $S \in \mathcal{B}(\widehat{\mathcal{H}})$ can be viewed as a matrix $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$, where $S_{11} \in \mathcal{B}(\mathcal{H})$, $S_{12} \in \mathcal{B}(\mathcal{H}^{\perp}, \mathcal{H})$, $S_{21} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^{\perp})$ and $S_{22} \in \mathcal{B}(\mathcal{H}^{\perp})$. Moreover, if $tr_{\widehat{\mathcal{H}}}$, $tr_{\mathcal{H}}$ and $tr_{\mathcal{H}^{\perp}}$ denote the canonical traces of the operator algebras $\mathcal{B}(\widehat{\mathcal{H}})$, $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H}^{\perp})$ respectively, then we have $tr_{\widehat{\mathcal{H}}}(S) = tr_{\mathcal{H}}(S_{11}) + tr_{\mathcal{H}^{\perp}}(S_{22})$. Consider the ultraweakly continuous C_0 automorphism group $(\alpha_t)_{t\in\mathbb{R}}$ defined by $\alpha_t(X) = e^{itK} X e^{-itK}$ for $X \in \mathcal{B}(\widehat{\mathcal{H}})$. Let A denote the generator of the semigroup $(\alpha_t)_{t\geq 0}$. Then we have $A(X) = i[K, X]$, for $X \in Dom(A)$. Now note that $tr_{\widehat{\mathcal{H}}}(\alpha_t(x)) = tr_{\widehat{\mathcal{H}}}(x)$ for $x \geq 0$. Thus $(\alpha_t)_{t\in\mathbb{R}}$ restricted to $L^2(tr_{\widehat{\mathcal{H}}})$ becomes a contractive group of unitary operators on $L^2(tr_{\widehat{\mathcal{H}}})$, which we denote by $(\mathcal{U}_t)_{t\in\mathbb{R}}$. Let $\mathcal{P} := \langle |u\rangle\langle v| : u, v \in Dom(K)\rangle_{\mathbb{C}}$. Then it follows that $\lim_{t\to 0} \mathcal{U}_t(X) \xrightarrow{L^2(tr_{\widehat{\mathcal{H}}})} X$, for all $X \in \mathcal{P}$. Moreover, as $\mathcal{P}$ is dense in $L^2(tr_{\widehat{\mathcal{H}}})$ and $\mathcal{U}_t$ is a contraction operator on $L^2(tr_{\widehat{\mathcal{H}}})$ for each

$t \in \mathbb{R}$, it follows that $(\mathcal{U}_t)_{t\geq 0}$ is a C_0 semigroup of operators in $L^2(tr_{\widehat{\mathcal{H}}})$. Let its generator be denoted by $\widetilde{A}$. Note that $\widetilde{A}$ is also a derivation. It is easy to see that $L^2(tr_{\widehat{\mathcal{H}}}) = L^2(tr_{\mathcal{H}}) \oplus L^2(tr_{\mathcal{H}^\perp})$. Now $\mathcal{B} \in L^2(tr_{\mathcal{H}})$ and $A(x) = \delta_1(x) \in L^2(tr_{\mathcal{H}})$ for $x \in \mathcal{B}$ and thus by Lemma 3.1.8, we have $\mathcal{B} \subseteq Dom(\widetilde{A})$ and $A(x) = \widetilde{A}(x) = \delta_1(x)$. Furthermore, we have $tr_{\widehat{\mathcal{H}}}(A(XY)) = 0$ for $X, Y \in Dom(\widetilde{A})$. So we have

$$tr_{\widehat{\mathcal{H}}}(A(XY)) = tr_{\mathcal{H}}(A(XY)) = tr_{\mathcal{H}}(\delta_1(XY)) = 0,$$

for all $X, Y \in \mathcal{B}$. Likewise, one may prove $tr_{\mathcal{H}}(\delta_2(XY)) = 0$ for $X, Y \in \mathcal{B}$. Thus we have $tr_{\mathcal{H}}(\theta_0^i(xy)) = 0$ for $x, y \in \mathcal{B}$. Observe that if the deficiency indices of T, i.e. the numbers $dim(T - iI)^\perp$ and $dim(T + iI)^\perp$ are equal, then T has a self-adjoint extension which belongs to $Lin(\mathcal{H}, \mathcal{H})$. Then we may repeat the same argument as above and reach the same conclusion. Hence the lemma is proved. □

Lemma 3.1.10. *$Dom(\mathcal{L}_2)$ is a $*$-subalgebra.*

Proof. The QDS $(T_t)_{t\geq 0}$ is $*$-preserving i.e. $T_t(x^*) = (T_t(x))^*$ for each $t \geq 0$ and x belonging to $\mathcal{B}(\mathcal{H})$. Thus $Dom(\mathcal{L}_2)$ is a $*$-closed subspace. We prove that $\theta_0^{i*} \in Lin(L^2(tr_{\mathcal{H}}), L^2(tr_{\mathcal{H}}))$ is a derivation and $\theta_0^{i*}(x) = -\theta_i^0(x),\ \forall\ x \in \mathcal{B}$, where $\theta_i^0 := (\theta_0^i)^\dagger$ as follows:

It also follows immediately by using $tr_{\mathcal{H}}(\theta_0^i(xy)) = 0$ for $x, y \in \mathcal{B}$ that $\theta_0^{i*}(\cdot)|_{\mathcal{B}} = \theta_i^0(\cdot)$. Let $x \in Dom(\theta_0^{*i})$. Now note that $\theta_0^i(x) = \delta_1(x) + i\ \delta_2(x),\ x \in \mathcal{B}$, where $\delta_l(\cdot)$ is a symmetric derivation for each $l = 1, 2$. Let $\delta(x) = i[T_1, x]$ and $\delta_2(x) = i[T_2, x]$, where T_1, T_2 are the symmetric operators obtained by lemma 3.1.6. Since $\mathcal{B} \subseteq Dom(\delta_1) \cap Dom(\delta_2)$, following the proof of lemma 3.1.5, we see that we can select a common vector $\Omega \in \mathcal{H}$ such that $\mathcal{B}\Omega = Dom(T_1) = Dom(T_2)$. Now note that $\theta_0^i(x) = [T, x]$, where $T := iT_1 - T_2$. It is enough to prove that $\theta_0^{*i}(x) = [S, x]$, where $S = -iT_1 - T_2$. We prove this as follows:

Let $\mathcal{D} := \langle |u_1\rangle\langle u_2| :\ u_1, u_2 \in \mathcal{B}\Omega\rangle_{\mathbb{C}} \subseteq Dom(\delta_1) \cap Dom(\delta_2) \cap L^2(tr_{\mathcal{H}})$. Moreover $\mathcal{B}\Omega$ is dense in $\mathcal{H}$. Since $\langle x, \theta_0^i(y)\rangle = \langle \theta_0^{*i}(x), y\rangle$ for all $y \in Dom(\theta_0^i) \cap L^2(tr_\mathcal{H})$, in particular for $y \in \mathcal{D}$, it follows that $[S, x] \in \mathcal{B}(\mathcal{H})$ for $x \in Dom(\theta_0^{*i})$, S as described above and hence proved.

Now we have $Dom(\mathcal{L}_2) \subseteq \cap_{i\geq 1} Dom(\theta_0^{i*}\theta_0^i)$ and $\sum_{i\geq 1} \|(\theta_0^{i*}\theta_0^i)x\|_2 < \infty$ for $x \in Dom(\mathcal{L}_2)$. The fact that $\theta_0^i(Dom(\mathcal{L}_2)) \subseteq Dom(\theta_0^{i*})$ implies that if $x, y \in Dom(\mathcal{L}_2)$, then xy belongs to $Dom(\theta_0^{i*}\theta_0^i)$ i.e. $xy \in Dom(\theta_0^{i*}\theta_0^i)$ for each i. To prove that $xy \in Dom(\mathcal{L}_2)$, we just need to show that $\sum_{i\geq 1} \|(\theta_0^{i*}\theta_0^i)xy\|_2 < \infty$. Now for each i, we have

$$
\begin{aligned}
(\theta_0^{i*}\theta_0^i)xy &= \theta_0^{i*}(\theta_0^i(x)y + x\theta_0^i(y)), \\
&= \theta_0^{i*}\theta_0^i(x)y + x\theta_0^{i*}\theta_0^i(y) + \theta_0^i(x)\theta_0^{i*}(y) + \theta_0^{i*}(x)\theta_0^i(y); \\
&= \theta_0^{i*}\theta_0^i(x)y + x\theta_0^{i*}\theta_0^i(y) - \theta_0^i(x)\theta_i^0(y) - \theta_i^0(x)\theta_0^i(y), \text{ since } x, y \in Dom(\mathcal{L}_2) \subseteq \mathcal{B}.
\end{aligned} \tag{3.3}
$$

Observe that $\sum_{i\geq 1} \|(\theta_0^{i*}\theta_0^i)(x)\|_2 < \infty$ and $\sum_{i\geq 1} \|(\theta_0^{i*}\theta_0^i)(y)\|_2 < \infty$ since $x, y \in Dom(\mathcal{L}_2)$. Now

$$
\begin{aligned}
\|\theta_0^i(x)\theta_i^0(y)\|_2 &= \sqrt{tr_{\mathcal{H}}((\theta_i^0(y))^*(\theta_0^i(x))^*\theta_0^i(x)\theta_i^0(y))} \\
&\leq \|\theta_i^0(y)\|_2 \sqrt{tr((\theta_0^i(x))^*\theta_0^i(x))} \text{ since } \|\cdot\|_\infty \leq \|\cdot\|_2;
\end{aligned}
$$

so by an application of the Cauchy-Schwartz inequality, we have $\sum_{i\geq 1} \|\theta_0^i(x)\theta_i^0(y)\|_2 < \infty$ Similarly it can be proved that $\sum_{i\geq 1} \|\theta_0^i(y)\theta_i^0(x)\|_2 < \infty$. So we have $\sum_{i\geq 1} \|(\theta_i^0\theta_0^i)xy\|_2^2 < \infty$ which proves the lemma. □

Note that $Dom(\mathcal{L}_2)$ becomes a $*$-subalgebra which is ultraweakly dense in $\mathcal{B}(\mathcal{H})$ as well as dense in $L^2(tr_{\mathcal{H}})$ (i.e. in the norm $\|\cdot\|_2$). $Dom(\mathcal{L}_2)$ is also a core for the Dirichlet form $\mathcal{E}(\cdot,\cdot)$. Furthermore we have $T_t(Dom(\mathcal{L}_2)) \subseteq Dom(\mathcal{L}_2)$. Thus it is also a core for $\mathcal{L}$. Moreover, since $Dom(\mathcal{L}_2)$ is an algebra, we have

$$
\begin{aligned}
\delta(x)^*\delta(y) &= \mathcal{L}(x^*y) + \mathcal{L}(x^*)y - x^*\mathcal{L}(y), \\
\mathcal{L}(x) &= R^*(x \otimes 1_{k_0})R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R,
\end{aligned}
$$

for $x, y \in Dom(\mathcal{L}_2)$ (by proposition 1.6.6). We now return to the proof of the main Theorem.

Theorem 3.1.11. *Suppose $(T_t)_{t\geq 0}$ is a conservative QDS on $\mathcal{B}(\mathcal{H})$ which is symmetric with respect to the canonical trace on $\mathcal{B}(\mathcal{H})$. Let $\mathcal{L}$ be the ultraweak generator of $(T_t)_{t\geq 0}$ and $\mathcal{L}_2$ be the generator of the L^2 extension of $(T_t)_{t\geq 0}$. Then $(T_t)_{t\geq 0}$ always admits HP dilation.*

Proof. Consider the following QSDE:

$$
\frac{dV_t}{dt} = V_t \circ (a_\delta^\dagger(dt) - a_\delta(dt) - \frac{1}{2}R^*Rdt), \tag{3.4}
$$

with the initial condition $V_0 = id$. We will prove that there exists an unitary co-cycle $(U_t)_{t\geq 0}$ which is a solution for the above QSDE. The coefficient matrix associated with the above QSDE is $\mathcal{Z} = \begin{pmatrix} -\frac{1}{2}R^*R & -R^* \\ R & 0 \end{pmatrix}$.

Let $G_n = (1 - \frac{\mathcal{L}_2}{n})^{-1}$, $\mathcal{Z}^{(n)} = \begin{pmatrix} -\frac{1}{2}G_n R^* R G_n & -G_n R^* \\ R G_n & 0 \end{pmatrix}$ and $(e_i)_{i \in I\!N}$ be an orthonormal basis for k_0. For $\xi \in \mathcal{V}_0$, suppose $\hat{\xi} := 1 \oplus \xi$. We first prove that for $\omega \in Dom(\mathcal{L}_2)$, $sup_{n \geq 1} \|\mathcal{Z}^{(n)}_{\hat{\xi}} \omega\|^2 < \infty$. We have

$$\begin{aligned}
\|RG_n\omega\| &= \langle RG_n\omega, RG_n \rangle \\
&= \langle \omega, G_n^*(-2\mathcal{L}_2)G_n\omega \rangle \\
&= \langle \omega, (-2\mathcal{L}_2)^{\frac{1}{2}} G_n^* G_n (-2\mathcal{L}_2)^{\frac{1}{2}} \omega \rangle \\
&= \|G_n(-2\mathcal{L}_2)^{\frac{1}{2}}\omega\|^2 \\
&\leq \|(-2\mathcal{L}_2)^{\frac{1}{2}}\omega\|^2.
\end{aligned}$$

By Lemma 3.1.9 that $\omega\xi := \omega \otimes \xi \in Dom(R^*)$. Thus

$$\begin{aligned}
\|\mathcal{Z}^{(n)}_{\hat{\xi}}\omega\|^2 &= \| - \frac{1}{2} G_n R^* R G_n \omega + G_n R^*(\omega\xi)\|^2 + \|RG_n\omega\|^2 \\
&\leq 2\|G_n^2(-2\mathcal{L}_2)\omega\|^2 + 2\|G_n R^*(\omega\xi)\|^2 + \|RG_n\omega\|^2 \\
&\leq 2\|(-2\mathcal{L}_2)\omega\|^2 + \|(-2\mathcal{L}_2)^{\frac{1}{2}}\omega\|^2 + 2\|R^*(\omega\xi)\|^2;
\end{aligned}$$

which implies that $sup_{n\geq 1}\|\mathcal{Z}^{(n)}_{\hat{\xi}}\omega\| < \infty$. We next prove the following:

$$\lim_{n\to\infty} \langle \hat{\eta}, \mathcal{Z}^{(n)}_{\hat{\xi}}\omega \rangle = \langle \hat{\eta}, \mathcal{Z}_{\hat{\xi}}\omega \rangle, \tag{3.5}$$

for $\omega \in Dom(\mathcal{L}_2)$, $\eta, \xi \in \mathcal{V}_0$. We have

- $\lim_{n\to\infty} -\frac{1}{2}G_n R^* R G_n \omega = -\frac{1}{2}R^* R\omega$,
- $\lim_{n\to\infty} RG_n\omega = R\omega$;

for $\omega \in Dom(\mathcal{L}_2)$. Existence of the limit in (3.5) now follows from the above two limits. Thus by Theorem 1.7.4, there exists a contractive cocycle $(U_t)_{t\geq 0}$ satisfying the QSDE in (3.4). We will prove that the coefficients associated to the QSDE in (3.4), satisfy the hypotheses of Theorem 1.7.5. Hence it will follow that $(U_t)_{t\geq 0}$ is an unitary cocycle, which will give the required HP dilation of the semigroup $(T_t)_{t\geq 0}$.

Since the coefficient matrix is of the form $Z = \begin{pmatrix} -\frac{1}{2}R^*R & R^* \\ R & 0 \end{pmatrix}$, hypotheses (i) and (ii) of Theorem 1.7.5 will hold for $\mathcal{Z}$, once we prove that the minimal QDS associated with the map

$$\mathcal{L}(x) = R^*(x \otimes 1_{k_0})R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R$$

for $x \in Dom(\mathcal{L}_2)$, is conservative (see the discussion before propostion 1.5.3 in chapter 1).

Let $(\widetilde{T}_t)_{t\geq 0}$ denote the minimal semigroup associated with the above map and suppose $\tilde{\mathcal{L}}$ be its generator. We claim that $Dom(\mathcal{L}_2) \subseteq Dom(\tilde{\mathcal{L}})$. Fix any $a \in Dom(\mathcal{L}_2)$. Let $\mathcal{D}$ denote the linear span of operators of the form $(1+R^*R)^{-1}\sigma(1+R^*R)^{-1}$ for σ belonging to $B_1(L^2(tr_{\mathcal{H}}))$. Let tr denote the canonical trace of $\mathcal{B}(L^2(tr_{\mathcal{H}}))$. Using explicit forms of $\mathcal{L}$ and $\tilde{\mathcal{L}}$, we see that $tr(\mathcal{L}(a)\rho) = tr(a\widetilde{\mathcal{L}}_*(\rho))$ for $\rho \in \mathcal{D}$, where $\widetilde{\mathcal{L}}_*$ denote the generator of the predual semigroup of $(\widetilde{T}_t)_{t\geq 0}$. It is known (by Lemma 3.2.5 in p.42 of [62]) that $\mathcal{D}$ is a core for $\widetilde{\mathcal{L}}$. So we have $tr(\mathcal{L}(a)\rho) = tr(a\widetilde{\mathcal{L}}_*(\rho))$ for all $\rho \in Dom(\widetilde{\mathcal{L}}_*)$. Following the proof of Lemma 8.1.22 in p.204 of [62], we have $\widetilde{\mathcal{L}}(a) = \mathcal{L}(a)$. This implies that $Dom(\mathcal{L}_2) \subseteq Dom(\widetilde{\mathcal{L}})$ and as $Dom(\mathcal{L}_2)$ is a core for $\mathcal{L}$, we have $Dom(\mathcal{L}) \subseteq Dom(\widetilde{\mathcal{L}})$ and $\widetilde{\mathcal{L}}(a) = \mathcal{L}(a)$ for all $a \in Dom(\mathcal{L})$. Now the symmetric QDS $(T_t)_{t\geq 0}$ is conservative. Thus we have $1 \in Dom(\mathcal{L})$ and $\mathcal{L}(1) = 0$ which implies that $\widetilde{\mathcal{L}}(1) = 0$. Thus the minimal semigroup $(\widetilde{T}_t)_{t\geq 0}$ is conservative. Hence by Theorem 7.2.3 in p.179 of [62], the cocycle $(U_t)_{t\geq 0}$ is unitary, which completes the proof. □

3.2 Dilation of a class of quantum dynamical semigroup on U.H.F algebras

We recall here that a quantum stochastic flow j_t is called a quantum stochastic dilation of the associated vacuum semigroup $T_t := j_t^{0,0}$. In this section, we want to apply the results obtained in chapter 2, to construct quantum stochastic dilation (in the sense of [28]), to a class of QDS on uniformly hyperfinite (UHF for short) algebras. Let $\mathcal{A}$ be the UHF C^* algebra generated by the infinite tensor product of finite dimensional matrix algebras $M_N(\mathbb{C})$, ie the C^*-completion of $\otimes_{j\in\mathbf{Z}^d} M_N(\mathbb{C})$ where N and d are two fixed positive integers. The unique normalized trace tr on $\mathcal{A}$ is given by $tr(x) = \frac{1}{N^n}Tr(x)$ for $x \in M_{N^n}(\mathbb{C})$. For a simple tensor $a \in \mathcal{A}$, let $a_{(j)}$ be the j^{th} component of a. We define support of a to be the set:

$$\{j \in \mathbf{Z}^d |\ a_{(j)} \neq 1\}.$$

For a general element $a = \sum_{n=1}^{\infty} c_n a_n$, we define support of a to be

$$\cup_{n\geq 1} supp(a_n).$$

Let $\mathcal{A}_{loc}$ be the $*$-algebra generated by finitely supported simple tensors in $\mathcal{A}$. Clearly $\mathcal{A}_{loc}$ is dense in $\mathcal{A}$. For $k \in \mathbf{Z}^d$, the translation τ_k on $\mathcal{A}$ is an automorphism determined by $\tau_k(x_{(j)}) = x_{(j+k)}$.

Note that $M_N(\mathbb{C})$ is generated by a pair of non-commutative representatives of the finite discrete group $\mathbf{Z}_N = \{0, 1, 2, ...N-1\}$ such that $U^N = V^N = 1 \in M_N(\mathbb{C})$ and $UV = \omega VU$ where ω is the N^{th} root of unity. Using this, we get a projective unitary representation of $\mathcal{G} = \prod_{j \in \mathbf{Z}^d} G$ where $G = \mathbf{Z}_N \times \mathbf{Z}_N$, in $L^2(tr)$ given by

$$\mathcal{G} \ni g \to U_g = \prod_{j \in \mathbf{Z}^d} U^{(j)^{\alpha_j}} V^{(j)^{\beta_j}} \in \mathcal{A},$$

where $g = \prod_{j \in \mathbf{Z}^d} (\alpha_j, \beta_j)$, $\alpha_j, \beta_j \in \mathbf{Z}_N$. For a given CP map ψ on $\mathcal{A}$, formally we define the Linbladian $\mathcal{L} = \sum_{k \in \mathbf{Z}^d} \mathcal{L}_k$, where $\mathcal{L}_k(x) = \tau_k \mathcal{L}_0(\tau_{-k} x),\ x \in \mathcal{A}_{loc}$, with $\mathcal{L}_0(x) = -\frac{1}{2}\{\psi(1)x + x\psi(1)\} + \psi(x)$. Consider the Linbladian $\mathcal{L}$ for the CP map $\psi(x) = \sum_{l=1}^{p} r^{(l)*} x r^{(l)},\ x \in \mathcal{A}$, where each $r^{(l)}$ belongs to a suitable class. For $g^{(j)} = (\alpha_j, \beta_j) \in G,\ j \in \mathbf{Z}^d$, we set $W_{j,g^{(j)}} = U^{(j)\alpha_j} V^{(j)\beta_j} \in \mathcal{A}_{loc}$. Next let $\|x\|_1 = \sum_{j,g} \|W_{j,g^{(j)}} x W^*_{j,g^{(j)}} - x\|$ and let $\mathcal{C}^1(\mathcal{A}) = \{x \in \mathcal{A} : \|x\|_1 < \infty\}$. Matsui ([47]) proved that the Lindbladian $\mathcal{L}$ is well-defined on $\mathcal{C}^1(\mathcal{A})$, is closable, the closure generates a QDS (to be denoted by T_t^{ψ}) on $\mathcal{A}$ and $\mathcal{C}^1(\mathcal{A})$ is invariant under T_t^{ψ}.

In [28], the authors considered the problem of constructing quantum stochastic dilation of such QDS. However they constructed the associated quantum stochastic process for only those semigroups for which the CP map ψ is of the form: $\psi(x) = r^* x r$, when $r = \sum_{g \in \prod_{j \in \mathbf{Z}^d} \mathbf{Z}_N} c_g W_g$, $W_g = \prod_{j \in \mathbf{Z}^d} (U^a V^b)^{\alpha_j}$ for $g = \prod_{j \in \mathbf{Z}^d} \alpha_j$, $\sum_g |c_g||g|^2 < \infty$ and fixed $a, b \in \mathbf{Z}_N$; where $|g| := \#\{j | (\alpha_j, \beta_j) \neq (0,0)\}$. Here we will generalize this result. First of all we will prove a result by considering ψ of the form given earlier, viz. $\psi(x) = \sum_{m=1}^{p} r^{(m)*} x r^{(m)}$ with the following *assumptions*:

(i) There exists E-H dilation say $j_t^{(m)}$ for the QDS $T_t^{(m)}$ corresponding to the CP map $\psi^{(m)}(x) = r^{(m)} x r^{(m)*}$ for each $m = 1, 2, ..p$,

(ii) $r^{(m)} \in \mathcal{A}_{loc}$ and

(iii) $[r^{(m)}, r^{(m)*}] \leq 0$, for $m = 1, 2, ...p$.

Before proving the main theorem of this section, we first prove few lemmae:

Lemma 3.2.1. *Suppose $(T_t)_{t \geq 0}$ is a QDS on a C^*-algebra $\mathcal{A}$ and let τ be a finite trace on it. Furthermore, let $\tau(\mathcal{L}(y)) \leq 0$ for all $y \geq 0,\ y \in \mathcal{A}_0$, $\mathcal{L}$ being the norm generator of $(T_t)_{t \geq 0}$. Then $(T_t)_{t \geq 0}$ has contractive L^1-extension.*

Proof. In what follows, $L^1_{\mathbb{R}}$ denotes the real Banach space, obtained by taking the L^1 closure of the real Banach space of self-adjoint elements of $\mathcal{A}$, whereas L^1 denotes the complex Banach space obtained by taking the L^1 closure of $\mathcal{A}$. Let $f(t) = \tau(T_t(y)),\ y \geq 0$ and $y \in \mathcal{A}_0$. Then $f'(t) = \tau(\mathcal{L}(T_t(y))) \leq 0$. Thus $f(t)$ is monotone decreasing and so $f(t) \leq f(0)$, i.e.

$$\|T_t(y)\|_1 \leq \|y\|_1 \ , y \geq 0,\ y \in \mathcal{A}_0. \tag{3.6}$$

Let $y \in \mathcal{A},\ y \geq 0$, so that $y = x^*x$, for some $x \in \mathcal{A}$. Since $\mathcal{A}_0$ is dense in $\mathcal{A}$, there is a sequence $(x_n)_n \in \mathcal{A}_0$, such that $x_n \to x$ in $\|\cdot\|_\infty$, leading to the conclusion that $x_n^*x_n \to x^*x$ in $\|\cdot\|_\infty$. Since τ is finite, $\|\cdot\|_1 \leq \|\cdot\|_\infty$, and a positive element of $\mathcal{A}$ can be approximated by a sequence of positive elements from $\mathcal{A}_0$ in $\|\cdot\|_\infty$. Thus the same result follows in $\|\cdot\|_1$, which proves that the inequality (3.6) extends to all the positive elements of $\mathcal{A}$. Since every self-adjoint x in $\mathcal{A}$ can be decomposed as $x = x_+ - x_-$ such that $|x| = x_+ + x_-$, we have

$$\|T_t(x)\|_1 = \|T_t(x_+) - T_t(x_-)\|_1 \leq \|T_t(x_+)\|_1 + \|T_t(x_-)\|_1 \leq \|x_+\|_1 + \|x_-\|_1 = \|x\|_1. \tag{3.7}$$

Thus T_t extends as a contractive map on the real Banach space $L^1_{\mathbb{R}}$. We denote this extension by T_t^{sa}. We consider its complexification $T'_t : L^1 \to L^1$ given by $T'_t(x) = T_t^{\mathrm{sa}}(Re(x)) + iT_t^{\mathrm{sa}}(Im(x))$.
As $\|Re(x)\|_1 \leq 2\|x\|_1$ and $\|Im(x)\|_1 \leq 2\|x\|_1$, T'_t is a bounded (not necessarily contractive) map on L^1. It follows that the dual map
$T_t^{\prime,*} : L^\infty \to L^\infty$ is a weak-$*$ continuous map (i.e. ultraweakly continuous in this case). Moreover observe that for positive $x \in \mathcal{A}$ and positive $y \in \mathcal{A} \cap L^1 = \mathcal{A}$, we have, using the positivity of T_t on $\mathcal{A}$, that

$$\tau(T_t^{\prime,*}(x)y) = \tau(xT'_t(y)) = \tau(xT_t(y)) \geq 0, \tag{3.8}$$

hence $T_t^{\prime,*}(x) \geq 0$, i.e. $T_t^{\prime,*}$ is a positive map, which implies,

$$\begin{aligned} \|T_t^{\prime,*}\| = \|T_t^{\prime,*}(1)\|_\infty = \|T_t^{\mathrm{sa},*}(1)\|_\infty &= sup_{\|\rho\|_1 \leq 1, \rho \in L^1_{\mathbb{R}}} |\tau(T_t^{\mathrm{sa},*}(1)\rho)| \\ &= sup_{\|\rho\|_1 \leq 1, \rho \in L^1_{\mathbb{R}}} |\tau(T_t^{\mathrm{sa}}(\rho))| \leq 1. \end{aligned} \tag{3.9}$$

Thus $T_t^{\prime,*}$ is contractive on L^∞, and hence so is its predual T'_t on L^1. The semigroup property and strong continuity of $(T_t)_{t\geq 0}$ on L^1 follows from the similar properties of $T_t = T'_t|_{L^\infty}$ with respect to the L^∞-norm and the fact that $\|\cdot\|_1 \leq \|\cdot\|_\infty$. $\square$

Lemma 3.2.2. *Let $\mathcal{L}^{(m)}$ be the generator of the QDS $(T_t^{(m)})_{t\geq 0}$ corresponding to the CP map $\psi^{(m)}(x) := r^{(m)*}xr^{(m)}$, with $r^{(m)}$ satisfying conditions (ii) and (iii) above. Then the QDS $(T_t^{(m)})_{t\geq 0}$ extends to $L^1(tr)$ as a contractive C_0 semigroup.*

Proof. For simplicity, we will drop the index m. Let $r_k := \tau_k(r)$ and let $\mathcal{A}_0 := \mathcal{C}^1(\mathcal{A})$. Suppose $y \in \mathcal{A}_0$ and $y \geq 0$. A simple computation yields $tr(\mathcal{L}(y)) \leq 0$. Thus by Lemma 3.2.1, $(T_t^{(m)})_{t\geq 0}$ has contractive L^1 extension.

□

Lemma 3.2.3. *Each of the QDS $(T_t^{(m)})_{t\geq 0}$ for $m = 1, 2, ..p$, has L^2-extensions.*

Proof. Let $y \in \mathcal{A}_0$. Then using contractivity of T_t and the result of lemma 3.2.2, we have that

$$\|T_t(y)\|_2^2 = tr(T_t(y^*)T_t(y)) \leq tr(T_t(y^*y)) \leq tr(y^*y) = \|y\|_2^2, \tag{3.10}$$

since $tr(\mathcal{L}(x)) \leq 0$ for $x \geq 0,\ x \in \mathcal{A}_0$. The conclusion now follows, since $\mathcal{A}_0$ is dense in $L^2(tr)$ and $\|\cdot\|_2 \leq \|\cdot\|_\infty$. □

Lemma 3.2.4. *Let $(j_t^{(m)})_{t\geq 0}$ be the quantum stochastic dilation of the QDS generated by the Lindbladian $\mathcal{L}^{(m)}$ corresponding to the CP map $\psi^{(m)}(x) = r^{(m)*}xr^{(m)}$ ($j_t^{(m)}$ exists by assumption (i)). Then $j_t^{(m)}$ satisfies the conditions (a),(b) of Theorem 2.4.1 and the condition of Remark 2.4.2.*

Proof. Observe that $[r^{(m)}, r^{(m)*}] \leq 0$ implies condition (b) of Theorem 2.4.1. Let $\delta_j^{(m)}(x) := [x, \tau_j(r^{(m)})]$, $x \in \mathcal{A}_0$. Then

$$\|\delta_j^{(m)}(x)\|_1 \leq 2\|\tau_j(r^{(m)})\|_\infty \|x\|_1. \tag{3.11}$$

Thus $\delta_j^{(m)}$ extends to a bounded operator on L^1. Similar result holds for $\delta_j^{\dagger,(m)}$. So we obtain condition (a) of Theorem 2.4.1 for the quantum stochastic flow $j_t^{(m)}$. Condition (b) of Theorem 2.4.1 holds for each of the flows $j_t^{(m)}$ as $[r^{(m)}, r^{(m)*}] \leq 0$. The condition of remark 2.4.2 also holds which can be shown as follows: Since the computations are identical for different $m's$, we drop the index m and see that formally
$\mathcal{L}_2^*(x) = \frac{1}{2}\sum_{k\in\mathbf{Z}^d}\{r_k[x, r_k^*] + [r_k, x]r_k^* + x[r_k, r_k^*] + [r_k, r_k^*]x\}$. Let $x \in \mathcal{A}_0 (= C^1(\mathcal{A}))$. Then we have:

$$\|\mathcal{L}_2^*(x)\| \leq \frac{\|r\|}{2}\sum_{k\in\mathbf{Z}^d}\{\|\delta_k^\dagger(x)\| + \|\delta_k(x)\|\} + \|x\|_\infty \|\sum_{k\in\mathbf{Z}^d}[r_k, r_k^*]\|. \tag{3.12}$$

But $\sum_{k\in\mathbf{Z}^d}\{\|\delta_k^\dagger(x)\|+\|\delta_k(x)\|\}<\infty$ (see [28] and [47])and thus it suffices to show the convergence of the third series in (3.12). For this we proceed as follows: As $r=\sum_{g\in\mathcal{G}} c_g \prod_{j\in\mathbf{Z}^d} U^{(j)\alpha_j}V^{(j)\beta_j}$,

$$\begin{aligned}\sum_{k\in\mathbf{Z}^d}\tau_k\{[r,r^*]\} &= \sum_{k\in\mathbf{Z}^d}\sum_{g,h\in\mathcal{G}} c_g\overline{c_h}\tau_k\{[\prod_{j\in\mathbf{Z}^d} U^{(j)\alpha_j}V^{(j)\beta_j},\prod_{j\in\mathbf{Z}^d} V^{(j),-\beta'_j}U^{(j),-\alpha'_j}]\}\\ &= \sum_{k\in\mathbf{Z}^d}\sum_{g,h\in\mathcal{G}} c_g\overline{c_h}[\prod_{j\in\mathbf{Z}^d} U^{(j+k)\alpha_j}V^{(j+k)\beta_j},\prod_{j\in\mathbf{Z}^d} V^{(j+k),-\beta'_j}U^{(j+k),-\alpha'_j}]\\ &= \sum_{k\in\mathbf{Z}^d}\sum_{g,h\in\mathcal{G}} c_g\overline{c_h}[\prod_{j\in\mathbf{Z}^d} U^{(j)\alpha_{j-k}}V^{(j)\beta_{j-k}},\prod_{j\in\mathbf{Z}^d} V^{(j),-\beta'_{j-k}}U^{(j),-\alpha'_{j-k}}].\end{aligned} \tag{3.13}$$

Since $\alpha_{j-k}=\beta_{j-k}=\alpha'_{j-k}=\beta'_{j-k}=0\in\mathbf{Z}_N$ for $|k|\geq M$ by assumption (ii), $[r_k,r_k^*]=0$ for such k. Thus the series is actually finite and hence $\|\mathcal{L}_2^*(x)\|<\infty$, i.e. $\mathcal{A}_0\subseteq Dom(\mathcal{L}_2)\cap Dom(\mathcal{L}_2^*)$. □

Remark 3.2.5. *Note that if we assume normality for each* $r^{(m)}, m=1,2,....p$, *then we may drop the assumption that* $r^{(m)}\in\mathcal{A}_{loc}$*. This is because then* $[r_k,r_k^*]=\tau_k\{[r,r^*]\}=0$.

Now we prove the main theorem of this section.

Theorem 3.2.6. *Assume the hypotheses of Lemma 3.2.4. Then the QSDE:*

$$dj_t(x)=\sum_{j\in\mathbf{Z}^d}\sum_{m=1}^{p} j_t(\delta_j^{\dagger\,(m)}(x))da_j^{(m)}(t)+\sum_{j\in\mathbf{Z}^d}\sum_{m=1}^{p} j_t(\delta_j^{(m)}(x))da_j^{\dagger\,(m)}(t)+j_t(\mathcal{L}^{(m)})dt,$$
$$j_t(1)=1,\ t\geq 0, \tag{3.14}$$

admits a $*$*-homomorphic unique solution* j_t*, where* j_t *is the E-H dilation of the vacuum semigroup* $(T_t)_{t\geq 0}$ *which is the CP semigroup corresponding to the CP map* $\psi(\cdot):=\sum_{m=1}^{p} r^{(m)*}xr^{(m)}$.

Proof. Let $j_t^{(m)}$, $m=1,2,...p$, be the $*$ homomorphic quantum stochastic flow with the structure maps $(\mathcal{L}^{(m)},\delta^{(m)},\delta^{\dagger\,(m)})$ respectively. By Lemma 3.2.4, we are in the set up for applying
Theorem 2.4.1-(i) and hence the present theorem follows. □

Corollary 3.2.7. *Let* $r^{(m)} = \sum_{g \in \mathbf{Z}_N} c_g W_g$ *for each* m*, where* $W_g = \prod_{j \in \mathbf{Z}^d} (U^a V^b)^{\alpha_j}$ *for* $g = \prod_{j \in \mathbf{Z}^d} \alpha_j$ *(as in [28]). Then the hypothesis of Theorem 3.2.6 are satisfied and the same conclusion follows, which generalizes the dilation result obtained in [28].*

Proof. Since $r^{(m)} = \sum_{g \in \mathbf{Z}_N} c_g W_g$, assumption (i) is satisfied (by the dilation result in [28]). It can be verified that $r^{(m)}$ is normal for each m. Thus by Remark 3.2.5, all the hypotheses of Theorem 3.2.6 are verified as well, and hence the result. □

Chapter 4

Quantum Brownian motion on non-commutative manifolds

4.1 Interplay between geometry and probability: stochastic geometry

There is a very interesting confluence of Riemannian geometry and probability theory in the domain of (classical) stochatistic geometry. The role of the Brownian motion on a Riemannian manifold cannot be over-estimated in this context; in fact, classical stochastic geometry is almost synonimous with the analysis of Brownian motion on manifolds. Since the inception of the quantum or noncommutative analogues of Riemannian geometry and the theory of stochastic processes few dacades ago, in the name of noncommutative geometry (a la Connes) and quantum probability respectively, it has been a natural problem to explore the possibility of interaction and confluence of them. However, there is not really much work in this direction yet. In [62], some case-studies have been made but no general theory was really formulated. The aim of the present paper is to formulate at least some general principle of quantum stochastic geometry using a quantum analogue of Brownian motion on homogeneous spaces.

The first problem in this context is a suitable noncommutative generalization of Brownian motion, or somewhat more generaly, quantum diffusion or Gaussian processes on manifolds. In the theory of Hudson-Parthasarathy quantum stochastic analysis, a quantum stochastic flow is thought of as (quantum) diffusion or Gaussian if its quantum stochastic flow equation does not have any 'Poisson' or 'number' coefficients (see [52], [62] and references therein for details). An important question in this context is to characterize the quantum dynamical semigroups which arise as the vacuum expectation semigrooups of quantum Gaussian processes or quantum Brownian motions. In the classical case, such criteria formulated in terms of the 'locality' of the generator are quite well-known. However, there is no such intrinsic characterization in the general noncommutative framework, except a few partial results, e.g. [62, p.156-160], valid only for type I algebras.

On the other hand, in the algebraic theory of quantum Levy processes a la Schürmann et al, there are simple and easily verifiable necessary and sufficient conditions for a quantum Levy process on a bialgebra to be of Gaussian type. This means, in some sense, we have a better understanding of quantum Gaussian processes on quantum groups. On the other hand, for any Riemannian manifold M, the group of Riemannian isometries $ISO(M)$ is a Lie group, and Gaussian processes or Brownian motions on the group of isometries induces similar processes on the manifold. For a compact Riemannian manifold the canonical Brownian motion generated by the (Hodge) Laplacian arises in this way from a bi-invariant Brownian motion on $ISO(M)$. Moreover, whenever $ISO(M)$ acts transitively on M, i.e. when M is a homogeneous space for $ISO(M)$, any covariant Brownian motion does arise from a bi-invariant Brownian motion on $ISO(M)$. All these facts suggest that an extension of the framework of Schürmann et al to quantum homogeneous spaces is called for, and this is indeed one of the objectives of the present article. We also treat these concepts from an analytical viewpoint, realizing quantum Gaussian processes and quantum Brownian motions as bounded operator valued quantum stochastic flows. We then make use of the quantum isometry groups (recently developed by the second author and his collaborators, see, e.g. [27, 9, 6, 10]) of noncommutative manifolds described by spectral triples and define (and study) quantum Gaussian process or quantum Borwnian motion on those noncommutative manifolds which which are 'quantum homogeneous spaces' for their quantum isometry groups.

For constructing interesting noncommutative examples, we investigate the problem of 'deforming' quantum Gaussian processes in the framwork of Rieffel ([60]), and prove in particular that any bi-invariant quantum Gaussian process can indeed be deformed. This has helped us to explicitly describe all the Gaussian generators for certain interesting noncommutative manifolds. Finally, using our formulation of quantum Brownian motion on noncommutative manifolds, we propose an analogue of the classical results about the asymptotics of exit time of Brownian motion from a ball of small volume (see, for example,[56]). We carry it out explicitly for noncommutative two-torus, and obtain quite remarkable results. The asymptotic behaviour in fact differs sharply from the commutative torus, and resembles the asymptotics of a one-dimensional manifold, which is perhaps in agreement with the fact that the noncommutative two-torus is a model for the 'leaf space' of the Kronecker foliation, and this 'leaf space' is locally (i.e. restricted to a foliation chart) is one dimensional.

4.2 Technical preliminaries

4.2.1 Brownian motion on classical manifolds and Lie-groups

Let M be a compact Riemannian manifold of dimension d, equipped with the Riemannian metric $\langle \cdot, \cdot \rangle$. Let $Exp_x : T_xM \to M$ denote the Riemannian exponential map, given by $Exp_x(v) = \gamma(1)$, where $v \in T_xM$ and
$\gamma : [0, 1] \to M$ is the geodesic such that $\gamma(0) = x, \gamma'(0) = v$. The Laplace-Beltrami operator Δ on M is defined by:

$$\Delta f(x) := \sum_{i=1}^{d} \frac{d^2}{dt^2} f\left(Exp_x(tY_i)\right)|_{t=0} \ , \tag{4.1}$$

where $f \in C^2(M)$ and $\{Y_i\}_{i=1}^d$ is a set of complete orthonormal basis of T_xM. This definition is independent of the choice of orthonormal basis of T_xM. If $x_1, x_2, ...x_d$ be a local chart at x, then writing ∂_i for $\frac{\partial}{\partial x_i}$, Δ can be written as:

$$\Delta f(x) = \sum_{i,j=1}^{d} g^{ij}(x)\partial_j \partial_k f(x) - \sum_{i,j,k=1}^{d} g^{jk}(x)\Gamma^i_{jk}(x)\partial_i f(x), \tag{4.2}$$

where $(g^{jk}) = (g_{jk})^{-1}$, $g_{jk}(x) := \langle \partial_j, \partial_k \rangle_x$, and Γ^i_{jk} are the Christoffel symbols.

Definition 4.2.1. *The Hodge Laplacian on $C^\infty(M)$ is the elliptic differential operator defined in terms of local coordinates $(x_1, x_2, ...x_n)$ as:*

$$\Delta_0 f = -\frac{1}{\sqrt{det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(g^{ij}\sqrt{det(g)}\frac{\partial}{\partial x_i} f\right),$$

where $f \in C^\infty(M)$ and $g \equiv ((g_{ij}))$.

It may be noted that the Hodge Laplacian on M and the Laplace-Beltrami operator both has similar second order terms and in case $M = \mathbb{R}^d$, they coincide, except for the sign.

It is well known that a standard d-dimensional Brownian motion on $\mathbb{R}^d$ has the Hodge Laplacian as its generator. An M valued Markov process $X_t^m : (\Sigma, \mathcal{F}, P) \to M$ will be called a diffusion process starting at $m \in M$ if $X_0^m = m$ and the generator of the process, say L, when restricted to $C_c^\infty(M)$ will be a second order elliptic differential operator i.e.

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_i \partial_j f(x) + \sum_{i=1}^{d} b_i(x)\partial_i f(x),$$

where $((a_{ij}(\cdot)))$ is a nonsingular positive definite matrix. We will sometimes use the term *Gaussian process* for such a Markov process. The diffusion process will be called a Riemannian Brownian motion, if L restricted to $C_c^\infty(M)$ is the Hodge Laplacian restricted to the $C^\infty(M)$.

Remark 4.2.2. *It may be noted that the standard text books e.g. [65, 37] refer to a Markov process as a Riemannian Brownian motion if its generator is a Laplace-Belatrami operator with the negetive sign. We differ from this usual convention. However our convention will agree with the usual convention in context of symmetric spaces as will be explained later.*

The Markov semigroup associated with standard Brownian motion, given by $(T_t f)(m) = \mathbb{E}(f(X_t^m))$ is called the heat-semigroup. The Brownian motion gives a "stochastic dilation" of the heat semigroup.

Diffusion processes on classical manifolds are important objects of study as many geometrical invariants can be obtained by analyzing the exit time of the motion from suitably chosen bounded domains. For example,

Proposition 4.2.3. *[56] Consider a hypersurface $M \subseteq \mathbb{R}^d$ with the Brownian motion process X_t^m starting at m. Let $T_\varepsilon = inf\{t > 0 : \|X_t^m - m\| = \varepsilon\}$ be the exit time of the motion from an extrinsic ball of radius ε around m. Then we have*

$$\mathbb{E}_m(T_\varepsilon) = \varepsilon^2/2(d-1) + \varepsilon^4 H^2/8(d+1) + O(\varepsilon^5),$$

where H is the mean curvature of M.

Proposition 4.2.4. *[38] Let M be an n-dimensional Riemannian manifold with the distance function $d(\cdot,\cdot)$, and X_t^x be the Brownian motion starting at $x \in M$. Let $\rho_t := d(x, X_t^x)$ (known as the radial part of X_t^x). Let T_ϵ be the first exit time of X_t^x from a ball of radius ϵ around x, ϵ being fixed. Then*

$$\mathbb{E}(\rho^2_{t\wedge T_\epsilon}) = nt - \frac{1}{6}S(x)t^2 + o(t^2),$$

where $S(x)$ is the scalar curvature at x.

We shall need a slightly modified version of the asymptotics described by Proposition 4.2.3, using the expression obtained in [31], of the volume of a small extrinsic ball as described below:

Let $V_m(\epsilon)$ denote the ball of radius ϵ around $m \in M$. Let n be the intrinsic dimension of the manifold. Then we have

$$V_m(\epsilon) = \frac{\alpha_n \epsilon^n}{n}\left(1 - K_1\epsilon^2 + K_2\epsilon^4 + O(\epsilon^6)\right)_m, \tag{4.3}$$

where $\alpha_n := 2\Gamma(\frac{1}{2})^n\Gamma(\frac{n}{2})^{-1}$ and K_1, K_2 are constants depending on the manifold.

The intrinsic dimension n of the hypersurface M is obtained from $I\!E(\tau_\epsilon)$ as the unique integer n satisfying $\lim_{\epsilon\to 0} \frac{I\!E(\tau_\epsilon)}{V_\epsilon^{\frac{2}{m}}} = \begin{cases} \infty \text{ if } m < n-\epsilon \text{ for } 0<\epsilon<1, \\ \neq 0 \text{ if } m \neq n, \\ = 0 \text{ if } m > n. \end{cases}$

Observe that $\frac{V(\epsilon)^{\frac{2}{n}}}{\epsilon^2} \to (\frac{\alpha_n}{n})^{\frac{2}{n}}$ and $\frac{V(\epsilon)^{\frac{4}{n}}}{\epsilon^4} \to (\frac{\alpha_n}{n})^{\frac{4}{n}}$ as $\epsilon \to 0^+$. So the asymptotic expression of Proposition 4.2.3 can be recast as

$$I\!E(\tau_\epsilon) = \frac{1}{2(d-1)}(\frac{V(\epsilon)n}{\alpha_n})^{\frac{2}{n}} + \frac{H^2}{8(d+1)}(\frac{V(\epsilon)n}{\alpha_n})^{\frac{4}{n}} + O(V(\epsilon)^{\frac{5}{n}}).$$

In particular, we get the extrinsic dimension d and the mean curvature H by the following formulae:

$$d = \frac{1}{2}(1 + \lim_{\epsilon\to 0} \frac{1}{I\!E(\tau_\epsilon)}(\frac{nV(\epsilon)}{\alpha_n})^{\frac{2}{n}}), \tag{4.4}$$

$$H^2 = 8(d+1)(\frac{\alpha_n}{n})^{\frac{4}{n}} \lim_{\epsilon\to 0} \frac{I\!E(\tau_\epsilon) - \frac{1}{2(d-1)}(\frac{nV(\epsilon)}{\alpha_n})^{\frac{2}{n}}}{V(\epsilon)^{\frac{4}{n}}}. \tag{4.5}$$

If there is a Lie group G which has a left (right) action on M, then it is natural to study the diffusion processes $X_t \equiv \{X_t^m, m \in M\}$ which are left (right) invariant in the sense that $g \cdot X_t^m = X_t^{g\cdot m}$ ($X_t^m \cdot g = X_t^{m\cdot g}$) almost everywhere (with respect to the Wiener measure) for all $g \in G, m \in M$. In particular, if $M = G$, we shall call X_t^e (where e is the identity element of G) the canonical left (right) invariant diffusion process, and we will usually drop the adjective left or right. For such a diffusion process, the generator $L = \sum_i A_i X_i + \frac{1}{2}\sum_{i,j} B_{ij} X_i X_j$, where $(B_{ij})_{i,j}$ is a non-negetive definite matrix and $\{X_1, ...X_d\}$ is a basis of the Lie-algebra $\mathcal{G}$. The diffusion process defined above is called bi-invariant if it is both left and right invariant. We also note that such processes constitute a special class of the so-called *Levy processes* on groups [37] i.e. a stochastic process which has almost surely cadlag paths, left (right) independent increments and left (right) stationary increments (see [37] for details).

Proposition 4.2.5. *([33]) A necessary and sufficient condition for a diffusion process in a Lie group G to be bi-invariant is the following:*

$$A_j C_{kj}^l = 0,\ B_{ij} C_{kj}^l + B_{jl} C_{kj}^i = 0 \ \ (1 \le i, k, l \le d),$$

where C_{kj}^l are the Cartan coefficients of G. In particular, the Gaussian processes in Lie-groups with abelian Lie-algebras will be bi-invariant.

If M is a symmetric space (i.e. the isometry group G acts transitively on M), it is interesting to study the diffusion processes on M which are covariant i.e. $\alpha_g \circ L = L \circ \alpha_g$ for all $g \in G$, where L is the generator of the diffusion process and $\alpha : G \times M \to M$ is the action of G on M.

Proposition 4.2.6. *[37] Let G be a Lie group and let K be a compact subgroup. If g_t is a right K invariant left Levy process in G with $g_0 = e$, then its one point motion from $o = eK$ in $M = G/K$ is a G invariant Feller process in M. Conversely, if x_t is a G invariant Feller process in M with $x_0 = o$, then there is a right K invariant left Levy process g_t in G with $g_0 = e$ such that its one-point motion in M from o is identical to the process x_t in distribution.*

Suppose that G is compact. The proof of Proposition 4.2.6, as in [37] then implies that any covariant diffusion process x_t on M can be realized as restriction of a corresponding right K invariant diffusion process on G.

Algebraic Theory of Levy processes on involutive bialgebras

We refer the reader to [26] and [61] for the basics of the algebraic theory of Levy processes on involutive bialgebras, which we briefly review here.

Definition 4.2.7. *Let $\mathcal{B}$ be an involutive bialgebra with coproduct Δ. A quantum stochastic process $(l_{st})_{0\leq s\leq t}$ on $\mathcal{B}$ over some quantum probability space $(\mathcal{A}, \Phi)$ (i.e. $\mathcal{A}$ is a unital $*$-algebra and Φ is a positive functional such that $\Phi(1) = 1$) is called a Levy process, if the following four conditions are satisfied:*

1. *(increment property) We have $l_{rs} * l_{st} = l_{rt}$ for all $0 \leq r \leq s \leq t$, $l_{tt} = 1 \circ \epsilon$ for all $t \geq 0$, where $l_{rs} * l_{st} := m_{\mathcal{A}} \circ (l_{rs} \otimes l_{st}) \circ \Delta$.*
2. *(independence of increments) The family $(l_{st})_{0\leq s\leq t}$ is independent, i.e. the quantum random variables $l_{s_1t_1}, l_{s_2t_2}, ...l_{s_nt_n}$ are independent for all $n \in \mathbb{N}$ and all $0 \leq s_1 \leq t_1 \leq ....t_n$.*
3. *(Stationarity of increments) The marginal distribution $\phi_{st} := \Phi \circ l_{st}$ of j_{st} depends only on the difference $t - s$.*
4. *(Weak continuity) The quantum random variables l_{st} converge to l_{ss} in distribution for $t \to s$.*

Define $l_t := l_{0t}$.

Due to stationarity of increments, it is meaningful to define the marginal distributions of $(l_{st})_{0\leq s\leq t}$ by $\phi_{t-s} = \Phi \circ l_{st}$.

Lemma 4.2.8. *([26]). The marginal distributions $(\phi_t)_{t\geq 0}$ form a convolution semigroup of states on $\mathcal{B}$ i.e. they satisfy*

1. *$\phi_0 = \epsilon$, $\phi_t * \phi_s = \phi_{t+s}$ for all $s, t \geq 0$, and $\lim_{t\to 0} \phi_t(b) = \epsilon(b)$ for all $b \in \mathcal{B}$.*

2. *$\phi_t(1) = 1$ and $\phi_t(b^*b) \geq 0$ for all $t \geq 0$ and all $b \in \mathcal{B}$.*

This convolution semigroup characterizes a Levy process on an involutive bialgebra.

Definition 4.2.9. *A functional $l : \mathcal{B} \to \mathbb{C}$ is called conditionally completely positive (CCP for short) functional if $l(b^*b) \geq 0$ whenever $\epsilon(b) = 0$.*

The generator of the above convolution semigroup of states is a CCP functional on the bialgebra $\mathcal{B}$.

Proposition 4.2.10. *(**Schoenberg correspondence**)[26] Let $\mathcal{B}$ be an involutive bialgebra, $(\phi_t)_{t\geq 0}$ a convolution semigroup of linear functionals on $\mathcal{B}$ and l be its generator, i.e. $l(a) = \frac{d}{dt}|_{t=0}\phi_t(a)$. Then the following are equivalent:*

1. *$(\phi_t)_{t\geq 0}$ is a convolution semigroup of states.*

2. *$l : \mathcal{B} \to \mathbb{C}$ satisfies $l(1) = 0$, it is hermitian and CCP.*

Next we define Schürmann triple on $\mathcal{B}$.

Definition 4.2.11. *Let $\mathcal{B}$ be a unital $*$-algebra equipped with a unital hermitian character ϵ. A Schürmann triple on $(\mathcal{B}, \epsilon)$ is a triple (ρ, η, l) consisting of*

1. *a unital $*$-representation $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{D})$ of $\mathcal{B}$ on some pre-Hilbert space D,*

2. *a $\rho - \epsilon - 1$-cocycle $\eta : \mathcal{B} \to \mathcal{D}$, i.e. a linear map $\eta : \mathcal{B} \to D$ such that*
$$\eta(ab) = \rho(a)\eta(b) + \eta(a)\epsilon(b)$$
for all $a, b \in \mathcal{B}$,

3. *and a hermitian linear functional $l : \mathcal{B} \to \mathbb{C}$ that satisfies*
$$l(ab) = l(a)\epsilon(b) + \epsilon(a)l(b) + \langle \eta(a^*), \eta(b) \rangle$$
for all $b \in \mathcal{B}$.

A Schürmann triple is called surjective if the cocycle η is surjective. Upto unitary equivalence, we have a one-to-one correspondence between Levy processes on $\mathcal{B}$, convolution semigroup of states on $\mathcal{B}$ and surjective Schürmann triples on $\mathcal{B}$. Choosing an orthonormal basis $(e_i)_i$ of $\mathcal{D}$, we can write η as $\eta(\cdot) = \sum_i \eta_i(\cdot)e_i$. The $\{\eta_i\}_i$ will be called the 'cordinate' of the cocycle η.

We will denote by $\mathcal{V}_{\mathcal{A}}$, the vector space of ϵ-derivations on a bialgebra $\mathcal{A}_0$, i.e. for $\mathcal{V}_{\mathcal{A}}$ consists of all maps $\eta : \mathcal{A}_0 \to \mathbb{C}$, such that $\eta(ab) = \eta(a)\epsilon(b) + \epsilon(a)\eta(b)$.

Lemma 4.2.12. *Let l be the generator of a Gaussian process on $\mathcal{A}_0$. Suppose that (l, η, ϵ) be the surjective Schürmann triple associated to l. Let $d := dim\mathcal{V}_{\mathcal{A}}$. Then there can be atmost d cordinates of η.*

Proof. Let $(\eta_i)_i$ be the cordinates of η. Observe that η_i is an ϵ-derivation for all i. It is enough to prove that $\{\eta_i\}_i$ is a linearly independent set. Suppose that $\sum_{i=1}^k \lambda_i \eta_i(a) = 0$, for all $a \in \mathcal{A}_0$. This implies that $\langle \eta(a), \sum_{i=1}^k \lambda_i e_i \rangle = 0$, for all $a \in \mathcal{A}_0$, where $(e_i)_i$ is an orthonormal basis for k_0, the associated noise space. Since $\{\eta(a) : a \in \mathcal{A}_0\}$ is total in k_0, we have $\sum_{i=1}^k \lambda_i e_i = 0$ which implies that $\lambda_i = 0$ for $i = 1, 2, ...k$. Hence proved. □

Proposition 4.2.13. *[26] For a generator l of a Levy process, the following are equivalent:*

1. $l|_{K^3} = 0$, $K = ker\epsilon$,
2. $l(b^*b) = 0$ *for all* $b \in K^2$,
3. $l(abc) = l(ab)\epsilon(c) - \epsilon(ab)l(c) + l(bc)\epsilon(a) - \epsilon(bc)l(a) + l(ac)\epsilon(b) - \epsilon(ac)l(b)$,
4. $\rho|_K = 0$, *for any surjective Schürmann triple,*
5. $\rho = \epsilon 1$ *for any surjective Schürmann triple i.e. the process is "Gaussian",*
6. $\eta|_{K^2} = 0$ *for any Schürmann triple,*
7. $\eta(ab) = \eta(a)\epsilon(b) + \epsilon(a)\eta(b)$ *for any Schürmann triple.*

A generator l satisfying any of the above conditions is called a *Gaussian generator* or the generator of a Gaussian process.

Definition 4.2.14. *For a map $\mathcal{P} : \mathcal{A}_0 \to \mathcal{B}$, where $\mathcal{B}$ is a $*$-algebra, defined $\widetilde{\mathcal{P}} : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ by $\widetilde{\mathcal{P}} := (id \otimes \mathcal{P}) \circ \Delta$. For two such maps $\mathcal{P}_1, \mathcal{P}_2$, define $\mathcal{P}_1 * \mathcal{P}_2 := m_{\mathcal{B}} \circ (\mathcal{P}_1 \otimes \mathcal{P}_2) \circ \Delta$, where $m_{\mathcal{B}}$ denotes the multiplication in $\mathcal{B}$.*

It follows that $(id_{\mathcal{A}_0} \otimes m_{\mathcal{B}}) \circ (\widetilde{\mathcal{P}_1} \otimes id_{\mathcal{B}}) \circ \widetilde{\mathcal{P}_2} = \widetilde{\mathcal{P}_1 * \mathcal{P}_2}$.

Definition 4.2.15. *We will call a Gaussian Levy process with surjective Schürmann triple (l, η, ϵ), the algebraic Quantum Brownian motion (QBM for short) if span of the maps $\{\eta_i\}_i$ is the whole of $\mathcal{V}_{\mathcal{A}}$, where η_i are the 'cordinates' of the cocycle of the unique (upto unitary equivalence) surjective Schürmann triple.*

It is known [61] that the following weak stochastic equation

$$\langle l_t(x)e(f), e(g)\rangle = \epsilon(x)\langle e(f), e(g)\rangle + \int_0^t d\tau \langle \{l_\tau * (l + \langle g(\tau), \eta\rangle + \eta^\dagger_{f(\tau)} + \langle g(\tau), (\rho - \epsilon)_{f(\tau)}\rangle)\}(x)e(f), e(g)\rangle, \tag{4.6}$$

which can be symbolically written as

$$dl_t = l_t * (dA_t^\dagger \circ \eta + d\Lambda_t \circ (\rho - \epsilon) + dA_t \circ \eta^\dagger + ldt)$$

with the initial conditions

$$l_0 = \epsilon 1$$

has a unique solution $(l_{st})_{0 \leq s \leq t}$ such that l_{st} is an algebraic levy process on $\mathcal{A}_0$. Then using this algebraic quantum stochastic differential equation, it can be proved that $j_t = \widetilde{l_t}$ satisfies an EH type equation as defined in subsection 1.7.5 with $\delta = \widetilde{\eta}$, $\mathcal{L} = \widetilde{l}$, $\sigma = \widetilde{\rho - \epsilon}$. However, it is not clear whether $j_t(x) \in \mathcal{A}_0 \otimes_{alg} \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_0)))$. We shall prove later that at least for Gaussian generators, this will be the case i.e. $j_t(x)$ is bounded.

4.3 Quantum Brownian motion on non-commutative manifolds

4.3.1 Analytic construction of Quantum Brownian motion

Let $(\mathcal{Q}, \Delta)$ be a CQG, $\mathcal{Q}_0$ be the corresponding Hopf$*$ algebra and h be the Haar state on $\mathcal{Q}$. Let $\mathcal{Q}_0 := \oplus \mathcal{H}_\pi$ be the decomposition by Peter-Weyl theory as described in section 1.3.2 of chapter 1.

Theorem 4.3.1. *Let $(T_t)_{t \geq 0}$ be a QDS on $\mathcal{Q}$ such that it is left covariant in the sense that*
$(id \otimes T_t) \circ \Delta = \Delta \circ T_t$. Let $\mathcal{L}$ be the generator of $(T_t)_{t \geq 0}$. Then there exist a CCP functional l on $\mathcal{Q}_0$ such that $\widetilde{l} = \mathcal{L}$.

Proof. The generator $\widetilde{\mathcal{L}}$ is CCP in the sense that $\partial\mathcal{L}(x, y) = \mathcal{L}(x^* y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y)$ is a CP kernel (see [62]). The left covariance condition implies that for each $t \geq 0$, T_t as well as $\mathcal{L}$ keep each of the spaces $\mathcal{H}_\pi$ invariant. Consequently $\mathcal{L}(\mathcal{Q}_0) \subseteq \mathcal{Q}_0$, so that it makes sense to define $l = \epsilon \circ \mathcal{L}$. Moreover for $x, y \in \mathcal{Q}_0$, $\epsilon \circ \partial\mathcal{L}(x, y) = l\left((x - \epsilon(x))(y - \epsilon(y))\right)$, so that l is CCP in the sense of definition 4.2.9 of chapter 1. Hence our claim is proved. □

We shall prove the converse of Theorem 4.3.1 for the Gaussian generators. For this, we need a few preparatory lemmae.

Lemma 4.3.2. *In Sweedler notation,* $h(a_{(1)}b)a_{(2)} = h(ab_{(1)})\kappa(b_{(2)})$, *for* $a, b \in \mathcal{Q}_0$.

Proof.

$$\begin{aligned} h(ab_{(1)})\kappa(b_{(2)}) &= ((h \otimes 1) \circ \Delta)\,(ab_{(1)})\kappa(b_{(2)}) \\ &= (h \otimes id)\,\big(\Delta(ab_{(1)})(id \otimes \kappa)(b_{(2)})\big) \\ &= (h \otimes id)\{\Delta(a)\Delta(b_{(1)})(id \otimes \kappa(b_{(2)}))\} \\ &= (h \otimes id)\,[\Delta(a)\{(id \otimes m_{\mathcal{Q}})(\Delta \otimes id)(id \otimes \kappa)\Delta(b)\}] \\ &= (h \otimes id)\,[\Delta(a)\{(id \otimes m_{\mathcal{Q}})(id \otimes id \otimes \kappa)(\Delta \otimes id)\Delta(b)\}] \qquad (4.7) \\ &= (h \otimes id)\,[\Delta(a)\{(id \otimes m_{\mathcal{Q}})(id \otimes id \otimes \kappa)(id \otimes \Delta)\Delta(b)\}] \\ &= (h \otimes id)\,[\Delta(a)\{(id \otimes m_{\mathcal{Q}} \circ (id \otimes \kappa)\Delta)\Delta(b)\}] \\ &= (h \otimes id)\,[\Delta(a)\{(id \otimes \epsilon)\Delta(b)\}] = (h \otimes id)\,[\Delta(a)(b \otimes 1)] \\ &= h(a_{(1)}b)a_{(2)}. \end{aligned}$$

□

Corollary 4.3.3. *For any functional* $\mathcal{P} : \mathcal{Q}_0 \to \mathbb{C}$, $h\left(\widetilde{\mathcal{P}}(a)b\right) = h\left(a(\widetilde{\mathcal{P} \circ \kappa})(b)\right)$.

Proof.

$$\begin{aligned} h(\widetilde{\mathcal{P}}(a)b) &= (h \otimes id)\,[(id \otimes \mathcal{P})\Delta(a)(b \otimes 1)] \\ &= (id \otimes \mathcal{P})\,[(h \otimes id)(\Delta(a)(b \otimes 1))] \\ &= (id \otimes \mathcal{P})\,\big[h(ab_{(1)})\kappa(b_{(2)})\big] \qquad (4.8) \\ &= h(ab_{(1)})\mathcal{P}(\kappa(b_{(2)})) = h(ab_{(1)}\mathcal{P}(\kappa(b_{(2)}))) \\ &= h(a(id \otimes \mathcal{P})(id \otimes \kappa)\Delta(b)) = h(a(\widetilde{\mathcal{P} \circ \kappa})(b)). \end{aligned}$$

□

Lemma 4.3.4. *Let* $\eta : \mathcal{Q}_0 \to \mathbb{C}$ *be an* ϵ*-derivation. Put* $\delta := (id \otimes \eta) \circ \Delta$. *Then* $h(\delta(a)) = 0$ *for all* $a \in \mathcal{Q}_0$.

Proof.

$$\begin{aligned} h(\delta(a)) &= (h \otimes id)(id \otimes \eta) \circ \Delta(a) \\ &= (id \otimes \eta)(h \otimes id) \circ \Delta(a) \\ &= \eta(h(a)1_{\mathcal{Q}}) \\ &= h(a)\eta(1_{\mathcal{Q}}) = 0 \text{ for all } a \in \mathcal{Q}_0, \end{aligned} \qquad (4.9)$$

where we have used the fact that $(h \otimes id) \circ \Delta(a) = (id \otimes h) \circ \Delta(a) = h(a) 1_{\mathcal{Q}}$. $\square$

Let (l, η, ϵ) be the surjective Schürmann triple for l, so that on $\mathcal{Q}_0$, we have $l(a^* b) - \epsilon(a^*) l(b) - l(a^*)\epsilon(b) = \langle \eta(a), \eta(b) \rangle$. We recall that $\eta : \mathcal{Q}_0 \to k_0$, for some Hilbert space k_0 so that $\eta(a) = \sum_i \eta_i(a) e_i$, $(e_i)_i$ being an orthonormal basis for k_0 and $\eta_i : \mathcal{Q}_0 \to \mathbb{C}$ being an ϵ-derivation for each i. Define $\theta_0^i := (id \otimes \eta_i) \circ \Delta$ for each i. Observe that
$\| \sum_i \theta_0^i(x)^* \theta_0^i(x) \| \leq \| x_{(1)}^* x_{(1)} \| \ | \sum_i \overline{\eta_i(x_{(2)})} \eta_i(x_{(2)}) | < \| x_{(1)} \|^2 \| \eta(x_{(2)}) \|^2 < \infty$, so that $\delta := \sum_i \theta_0^i \otimes e_i = (id \otimes \eta) \circ \Delta$ is a derivation from $\mathcal{Q}_0$ to $\mathcal{Q} \otimes k_0$. Now $\mathcal{L}$ is a densely defined operator with $D(\mathcal{L}) = \mathcal{Q}_0 \subseteq L^2(h)$. By Corollary 4.3.3, $h(\mathcal{L}(a^*)b) = h(a^* \widetilde{l \circ \kappa}(b))$ i.e. $\langle \mathcal{L}(a), b \rangle_{L^2(h)} = \langle a, \widetilde{l \circ \kappa}(b) \rangle_{L^2(h)}$. Thus $\mathcal{L}$ has an adjoint which is also densely defined. Thus $\mathcal{L}$ is $L^2(h)$-closable, and we denote its closure by the same notation $\mathcal{L}$. Note that a linear map $S : \mathcal{Q}_0 \to \mathcal{Q}_0$ is left covariant i.e. $(id \otimes S)\Delta = \Delta \circ S$ if and only if $S(\mathcal{H}_\pi) \subseteq \mathcal{H}_\pi$ for all π. In such a case, we will denote by S_π the map $S|_{\mathcal{H}_\pi}$. Since $\delta : \mathcal{Q}_0 \to \mathcal{Q}_0 \otimes_{alg} k_0$ and h is faithful on $\mathcal{Q}_0$, we have $h(\delta(x)^* \delta(x)) < \infty$ which implies that $\delta(x) \in L^2(h) \otimes k_0$ for $x \in \mathcal{Q}_0$.

Lemma 4.3.5. *Let $l : \mathcal{Q}_0 \to \mathbb{C}$ be a CCP functional and (l, η, ϵ) be the surjective Schürmann triple associated with it. Then $\mathcal{L} = \widetilde{l}$ on $\mathcal{Q}_0$ has Christinsen-Evans form (see Theorem 1.5.2 in chapter 1) i.e.*

$$\mathcal{L}(x) = R^*(x \otimes 1_{k_0})R - \frac{1}{2}R^* R x - x\frac{1}{2}R^* R + i[T, x],$$

for densely defined closable operators R and T, with $T^* = T$.

Proof. Let $R := \delta : \mathcal{Q}_0 \left(\subseteq L^2(h)\right) \to L^2(h) \otimes k_0$, where $\delta := (id \otimes \eta) \circ \Delta$.
For $x \in \mathcal{Q}_0$, consider the quadratic forms

$$\langle \Phi(x) y, y' \rangle_{L^2(h)} = h\left(\mathcal{L}(y^* x^* y') - \mathcal{L}(y^* x^*) y' - y^* \mathcal{L}(x^* y') + y^* \mathcal{L}(x^*) y'\right); \tag{4.10}$$

$$\langle \mathcal{L}(x) y, y' \rangle_{L^2(h)} = h(y^* \mathcal{L}(x^*) y') \tag{4.11}$$

and

$$\frac{1}{2}\langle [\mathcal{L} - \widetilde{l \circ \kappa}, x] y, y' \rangle_{L^2(h)} = \frac{1}{2} h(\mathcal{L}(y^* x^*) y' - y^* x^* \mathcal{L}(y') - \mathcal{L}(y^*) x^* y' + y^* \mathcal{L}(x^* y')), \tag{4.12}$$

where $\Phi(x) = R^*(x \otimes 1_{k_0})R - \frac{1}{2}R^* R x - x\frac{1}{2}R^* R$. Observe that by subtracting (4.11) from (4.10) and adding (4.12) to it, we get zero. So by taking
$T = \frac{1}{2i}(\mathcal{L} - \widetilde{l \circ \kappa})$ on $\mathcal{Q}_0$, we get $\langle \mathcal{L}(x) y, y' \rangle = \langle (\Phi(x) + i[T, x]) \, y, y' \rangle$ for $x \in \mathcal{Q}_0$. Note that T is covariant, hence we have $T = \oplus_\pi T_\pi$ and since each $\mathcal{H}_\pi$ is finite dimensional and $T_\pi^* = T_\pi$ by corollary 4.3.3, we have that T has a self-adjoint extension on $L^2(h)$ which is the L^2-closure of T in $\mathcal{Q}_0$. $\square$

We are now in a position to prove the converse of Theorem 4.3.6 for Gaussian generators, which gives a left covariant QDS on $\mathcal{Q}$ and a left covariant Gaussian generator on $\mathcal{Q}_0$.

Theorem 4.3.6. *Given a Gaussian CCP functional l on $\mathcal{Q}_0$, there is a unique covariant QDS on $\mathcal{Q}$ such that its generator is an extension of $\widetilde{l}$.*

Proof. Note that in the notation of Lemma 4.3.5, we have $R^*R = \widetilde{l} + \widetilde{l \circ \kappa}$, $T = \frac{1}{2i}(\widetilde{l} - \widetilde{l \circ \kappa})$ and hence $G := iT - \frac{1}{2}R^*R = -\widetilde{l \circ \kappa}$. Hence $(id \otimes G) \circ \Delta = \Delta \circ G$. So each G_π generates a semigroup in $\mathcal{H}_\pi$ say T_t^π which is contractive, since the generator is of the form $iT_\pi - \frac{1}{2}(R^*R)_\pi$, with $T_\pi^* = T_\pi$. Take $S_t := \oplus_\pi T_t^\pi$, which is a C_0, contractive semigroup in $L^2(h)$. There exists a minimal semigroup $(T_t)_{t \geq 0}$ on $\mathcal{B}(L^2(h))$, such that its generator, say $\mathcal{L}^{min}$, is of the form given in Lemma 4.3.5 when restricted to a suitable dense domain (see subsection 1.5.1 of chapter 1). Now following the arguments used in proving Theorem 1.7.6, we can conclude that $\mathcal{L}^{min} = \widetilde{l}$ on $\mathcal{Q}_0$. Thus $\mathcal{L}^{min}(\mathcal{H}_\pi) \subseteq \mathcal{H}_\pi$. Furthermore, each $\mathcal{H}_\pi$ being finite dimensional, $T_t(x) = e^{t\mathcal{L}_\pi^{min}}(x) = \sum_n \frac{t^n}{n!}(\mathcal{L}_\pi^{min})^n(x)$, which converges in the norm for $x \in \mathcal{H}_\pi$. Thus in particular we see that $T_t(\mathcal{H}_\pi) \subseteq \mathcal{H}_\pi$ for all π and all $t \geq 0$ i.e. $(id \otimes T_t) \circ \Delta = \Delta \circ T_t$. □

Theorem 4.3.7. *The QDS generated by a Gaussian generator l as in Theorem 4.3.6, always admits an E-H dilation which is implemented by unitary cocycles.*

Proof. We will apply Theorem 1.7.6 with $H = T$. Let $\mathcal{V}_0 = \mathcal{Q}_0$ and $\mathcal{W}_0 = \langle e_i | i = 1, 2, 3...\rangle_{\mathbb{C}}$, where $(e_i)_i$ is an orthonormal basis for k_0. Observe that by Lemma 4.3.4, $R^* = -\sum_i \theta_i^0 \otimes \langle e_i|$. Thus $u \otimes \xi \in D(R^*)$ for all $u \in \mathcal{V}_0$ and $\xi \in \mathcal{W}_0$. The proof of Theorem 4.3.6 implies that $G := iT - \frac{1}{2}R^*R$ generates a C_0 contractive semigroup in $L^2(h)$. Noting that G^* is an extension of $-\widetilde{l}$, using arguments as in Theorem 4.3.6, we can prove that G^* generates a C_0 contractive semigroup in $L^2(h)$. Thus all the conditions of Theorem 1.7.6 hold, and we get unitary cocycles $(U_t)_{t \geq 0}$ satisfying an H-P equation. Then $j_t : \mathcal{B}(L^2(h)) \to \mathcal{B}(L^2(h)) \otimes \mathcal{B}(\Gamma)$ defined by $j_t(x) := U_t(x \otimes 1_\Gamma)U_t^*$, is a $*$-homomorphic EH flow on $\mathcal{B}(L^2(h))$, satisfying the stochastic differential equation:

$$\begin{aligned} dj_t =& j_t \circ \left(a_{\delta^\dagger}(dt) - a_\delta^\dagger(dt) + \mathcal{L}dt \right) \\ j_0 =& id, \end{aligned} \tag{4.13}$$

on $\mathcal{Q}_0$, where $\delta(x) = (id \otimes \eta) \circ \Delta(x) = [R, x]$ for $x \in \mathcal{Q}_0$. We need to show that $j_t(\mathcal{Q}_0) \subseteq \mathcal{Q}'' \otimes \mathcal{B}(\Gamma)$ i.e. $\langle e(f), j_t(x)e(g)\rangle \in \mathcal{Q}''$ for $f, g \in \Gamma$ and $x \in \mathcal{Q}_0$.

Let l_t be the algebraic Levy process associated with l, satisfying equation (4.6) with $\rho = \epsilon$. For $x \in \mathcal{Q}_0$ and ξ, ξ' belonging to k_0, let $T_t^{\xi,\xi'}$ and $\phi_t^{\xi,\xi'}$ denote the maps $\langle e(\chi_{[0,t]}\xi), j_t(\cdot)_{e(\chi_{[0,t]}\xi')}\rangle$ and $\langle e(\chi_{[0,t]}\xi), l_t(\cdot)e(\chi_{[0,t]}\xi')\rangle$ repectively. We claim that for all $x \in \mathcal{Q}_0$, $T_t^{\xi,\xi'}(x) = \widetilde{\phi_t^{\xi,\xi'}}(x)$ which will be shown towards the end of the proof. Let $\mathcal{D}$ denote the linear span of elements of the form $e(f)$ where f is a step function taking values in $(e_i)_i$. By the theorems in [63, 55], $\mathcal{D}$ is dense in Γ. Consider the step functions $f = \sum_i^k a_i \chi_{[t_{i-1},t_i]}$ and $g = \sum_i^k b_i \chi_{[t_{i-1},t_i]}$, where $t_0 = 0, t_k = t$, and a_i, b_i belong to $\{e_i : i \in I\!N\}$. Then note that for $x \in \mathcal{Q}_0$,

$$\begin{aligned}
\langle e(f), j_t(x)_{e(g)}\rangle &= T_{t_1-t_0}^{a_1,b_1} \circ T_{t_2-t_1}^{a_2,b_2} \circ ......T_{t-t_{k-1}}^{a_k,b_k}(x) \\
&= \widetilde{\phi_{t_1-t_0}^{a_1,b_1}} \circ \widetilde{\phi_{t_2-t_1}^{a_2,b_2}} \circ ......\widetilde{\phi_{t-t_{k-1}}^{a_k,b_k}}(x) \\
&= \widetilde{A}(x) \\
&= \langle e(f), \widetilde{l}_t(x)e(g)\rangle \ \in \mathcal{Q}_0,
\end{aligned} \tag{4.14}$$

where $A(x) = (\phi_{t_1-t_0}^{a_1,b_1} * \phi_{t_2-t_1}^{a_2,b_2} * ......\phi_{t-t_{k-1}}^{a_k,b_k})(x)$. Since $\mathcal{D}$ is total in Γ, this implies that the map $\langle e(f), j_t(x)e(g)\rangle \in \mathcal{Q}''$ for all $f, g \in \Gamma$ $x \in \mathcal{Q}_0$.

The proof of the theorem will be complete once we show that for $x \in \mathcal{Q}_0$, $\xi, \nu \in k_0$, we have $T_t^{\xi,\nu}(x) = \widetilde{\phi_t^{\xi,\nu}}(x)$. This can be achieved as follows:

Fix an $x \in \mathcal{Q}_0$. From the cocycle property, it follows that $T_t^{\xi\nu}$ is a C_0-semigroup on $\mathcal{B}(L^2(h))$ and $\phi_t^{\xi,\nu}$ is a convolution semigroup of states on $\mathcal{Q}_0$. Since l_t and j_t satisfy equations (4.6) and (4.13) respectively, it follows that the generator of the convolution semigroup $(\phi_t^{\xi,\nu})_{t\geq 0}$ is $L = l + \langle \xi, \eta\rangle + \eta_\nu^\dagger$ and the generator of the semigroup $(T_t^{\xi\nu})_t$ is $\widetilde{L}$. By the fundamental Theorem of coalgebra (see [26]), there is a finite dimensional coalgebra say C_x containing x. It follows that $\widetilde{L}(C_x) \subseteq C_x$. Note that C_x being finite dimensional, the map $\widetilde{L} : C_x \to C_x$ is bounded with $\|\widetilde{L}\| = M_x$(say), where M_x depends on x. Now

$$\begin{aligned}
T_t^{\xi,\nu}(x) &= x + \int_0^t T_s^{\xi,\nu}(\widetilde{L}(x))ds \\
&= x + t\widetilde{L}(x) + \int_{s_1=0}^t \int_{s_2=0}^{s_1} T_{s_2}^{\xi,\nu}(\widetilde{L}^2(x))ds \\
&= x + t\widetilde{L}(x) + \frac{t^2}{2!}\widetilde{L}^2(x) + \frac{t^3}{3!}\widetilde{L}^3(x) + ... + \int_{s_1=0}^t \int_{s_2=0}^{s_1} \int_{s_3=0}^{s_2} .... \int_{s_n=0}^{s_{n-1}} T_{s_n}^{\xi,\nu}(\widetilde{L}^n(x))ds;
\end{aligned} \tag{4.15}$$

Now $\| \int_{s_n=0}^{s_{n-1}} T_{s_n}^{\xi,\nu}(\widetilde{L}^n(x))ds\| \leq e^{t\langle \xi,\nu\rangle}\frac{t^n(M_x)^n\|x\|}{n!} \to 0$ as $n \to \infty$. Thus

$$\begin{aligned}
T_t^{\xi,\nu}(x) &= x + t\widetilde{L}(x) + \frac{t^2}{2!}\widetilde{L}^2(x) + ...\\
&= \widetilde{\epsilon}(x) + t\widetilde{L}(x) + \frac{t^2}{2!}\widetilde{(L*L)}(x) + \frac{t^3}{3!}\widetilde{(L*L*L)}(x) + ...\\
&= \widetilde{\phi_t^{\xi,\nu}}(x),\ where\\
&= \phi_t^{\xi,\nu}(x) = \left(\epsilon + tL + \frac{t^2}{2!}(L*L) + \frac{t^3}{3!}(L*L*L) + ...\right)(x).
\end{aligned} \tag{4.16}$$

□

We will call j_t a Quantum Gaussian process on $\mathcal{Q}$. If l generates the algebraic QBM (as defined after Proposition 4.2.13), then we will call j_t the Quantum Brownian motion (QBM for short) on $\mathcal{Q}$.

Remark 4.3.8. *If $l = l \circ \kappa$, we will call the above QBM symmetric. This is because under the given condition, $(T_t)_{t\geq 0}$ generated by $\mathcal{L}$ becomes a symmetric QDS i.e. $h(T_t(x)y) = h(xT_t(y))$.*

The following result, which is probably well-known, demonstrates the equivalence of the quantum and classical definitions of Gaussian processes on compact Lie-groups.

Theorem 4.3.9. *Let G be a compact Lie-group. Then a generator of a quantum Gaussian process (QBM) on $\mathcal{Q} = C(G)$ is also the generator of a classical Gaussian process (QBM) and vice-versa.*

Proof. Let l be the given generator and let $\mathcal{L} := \widetilde{l}$, as before. Observe that the semigroup $(T_t)_{t\geq 0}$ associated with the map $\mathcal{L}$ is covariant with respect to left action of the group. Moreover, $(T_t)_{t\geq 0}$ is a Feller semigroup. Thus by Theorem 2.1 in page 42 of [37], we see that $C^\infty(G) \subseteq D(\mathcal{L})$. Now on $C(G)$, there is a canonical locally convex topology generated by the seminorms $\|f\|_n := \sum_{i_1,i_2,...i_k:k\leq n} \|\partial_{i_1}\partial_{i_2}...\partial_{i_k}(f)\|$, where ∂_{i_l} is the generator of the one-parameter group $L_{exp(tX_{i_l})}$, such that $C^\infty(G)$ is complete and $\mathcal{Q}_0$ is dense in $C^\infty(G)$ in this topology (see [62] and references therein). Now as $\mathcal{L}$ is closable in the norm topology, it is closable in this locally convex topology and hence (by the closed graph theorem) continuous as a map from $(C^\infty(G), \{\|\cdot\|_n\}_n) \to (C(G), \|\cdot\|_\infty)$. From this, and using the fact that $\mathcal{L}$ commutes with L_g $\forall g$, it can be shown along the lines of Lemma 8.1.9 in page 193 of [62] that $\mathcal{L}(f) \in C^\infty(G)$. Moreover, we can extend the identity $\mathcal{L}(abc) =$

$\mathcal{L}(ab)c - ab\mathcal{L}(c) + a\mathcal{L}(bc) - \mathcal{L}(a)bc + \mathcal{L}(ac)b - ac\mathcal{L}(b)$ for all $a, b, c \in C^\infty(G)$ by continuity. Thus $\mathcal{L}$ is a local operator. Now by the main theorem in [69], this implies that $\mathcal{L}$ is a second order elliptic differential operator, and hence generator of a classical Gaussian process.

On the other hand, given a generator $\mathcal{L}$ of a classical Gaussian process, $(id \otimes \mathcal{L})\Delta = \Delta \circ \mathcal{L}$ implies that in particular, $\mathcal{L}(\mathcal{Q}_0) \subseteq \mathcal{Q}_0$. Moreover, it can be verified that $\mathcal{L}$ satisfies the identity $\mathcal{L}(abc) = \mathcal{L}(ab)c - ab\mathcal{L}(c) + a\mathcal{L}(bc) - \mathcal{L}(a)bc + \mathcal{L}(ac)b - ac\mathcal{L}(b)$ for $a, b, c \in C^\infty(G)$ and hence in $\mathcal{Q}_0$. Thus $\mathcal{L}$ is the generator of a quantum Gaussian process as well. □

4.3.2 Quantum Brownian motion on quantum spaces

Suppose that G is a compact Lie-group, with Lie-algebra $\mathfrak{g}$, of dimension n. There exists an $Ad(G)$-invariant inner product in $\mathfrak{g}$ which induces a bi-invariant Riemannian metric in G. Suppose that G acts transitively on a manifold M. Then as manifolds, $M \cong H/G$, for some closed subgroup $H \subseteq G$ and as the innerproduct on $\mathfrak{g}$ is in particular $Ad(H)$-invariant, it induces a G-invariant Riemannian metric on M. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie-algebras of G and H respectively. It is a well-known fact (see [37]) that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is a subspace such that $Ad(H)\mathfrak{p} \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$. Let $\{\mathcal{X}_i\}_{i=1}^d$ be a basis of $\mathfrak{g}$ such that $\{\mathcal{X}_i\}_{i=1}^d$ is a basis for $\mathfrak{p}$ and $\{\mathcal{X}_i\}_{i=d+1}^n$ is a basis for $\mathfrak{h}$. Let $\pi : G \to H/G$ be the quotient map given by $\pi(g) = Hg$, for $g \in G$. It follows that if $f \in C(H/G)$, $\mathcal{X}_i(f \circ \pi) \equiv 0$ for all $i = d+1, ...n$. The Laplace-Beltrami operator on M is thus given by

$$\frac{1}{2}\Delta_{H/G} f(x) = \frac{1}{2}\sum_{i=1}^{d} \mathcal{X}_i^2 (f \circ \pi)(g),$$

where $f \in C^\infty(M)$ and $x = Hg$, or in other words, if $\{W_t^{(i)}\}_{i=1}^d$ denote the standard Brownian motion in $\mathbb{R}^d$, the standard covariant Brownian motion on $M(\cong H/G)$, starting at m is given by $\mathcal{B}_t^m := m.\mathcal{B}_t$ where $\mathcal{B}_t := exp(\sum_{i=1}^d W_t^{(i)} \mathcal{X}_i)$ and exp denotes the exponential map of the Lie group G. Thus the probability space of the covariant Brownian motion $\mathcal{B}_t^m$ can be identified with the probability space of d-dimensional Wiener measure. Now suppose M is a compact Riemannian manifold such that the isometry group of M, say G, acts transitively on M. The above discussion applies to M and it may be noted in particular that in this case, the Laplace-Beltrami operator on M coincides with the Hodge-Laplacian on M restricted to $C^\infty(M)$, except for signs (see [37]). It follows from Proposition 2.8 in page 51 of [37] and the discussions preceeding it that a Riemannian Brownian motion on a compact Riemannian manifold M is induced by a bi-invariant Brownian motion on G, the isometry group of M, if G acts transitively on M. Furthermore by Proposition 1.4.12, it follows that if G acts transitively on M, then the action is ergodic i.e. $C(M)$ is homogeneous. Motivated by this, we may define a Quantum Brownian motion on a quantum space as follows:

Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple satisfying the conditions stated in section 1.4. Let $(\mathcal{Q}, \Delta)$ denote the quantum isometry group as obtained in Proposition 1.5.3, α being the action. Suppose that $(\mathcal{Q}, \Delta)$ acts ergodically on $\mathcal{A}^\infty$, **i.e. the quantum space** $\mathcal{A} := \overline{\mathcal{A}^\infty}^{\|\cdot\|_\infty}$ **is homogeneous**. Let $l : \mathcal{Q}_0 \to \mathbb{C}$ be the generator of a bi-invariant quantum gaussian process $j_t(\cdot)$ on $\mathcal{Q}$ i.e. $(l \otimes id) \circ \Delta = (id \otimes l) \circ \Delta$ on $\mathcal{Q}_0$.

Define the process $k_t := (id \otimes l_t) \circ \Delta : \mathcal{A}_0 \to \mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_0)))$ on $\mathcal{A}_0$. Since α is an ergodic action, it is known that there exists an α-invariant state τ on $\mathcal{A}$ (see [62]). Moreover, in the notation of Proposition 1.4.11, τ is faithful on $\mathcal{A}_0 := \oplus_{\gamma \in Irr_{\mathcal{Q}}} \left(\oplus_{i \in I_\gamma} W_{\gamma i}\right)$ and as a Hilbert space, $L^2(\tau) := \oplus_{\gamma \in Irr_{\mathcal{Q}}} \left(\oplus_{i \in I_\gamma} W_{\gamma i}\right)$.

Theorem 4.3.10. *There exists a unitary cocycle $(U_t)_{t \geq 0} \in \mathcal{B}(L^2(\tau) \otimes \Gamma)$ satisfying an HP equation, where $\Gamma := \Gamma(L^2(\mathbb{R}_+, k_0))$ such that $k_t(x) = U_t(x \otimes id_\Gamma)U_t^*$ for $x \in \mathcal{A}_0$. Thus k_t extends to a bounded map from $\mathcal{A}$ to $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$. Moreover, k_t satisfies an EH equation with coefficients $(\mathcal{L}_{\mathcal{A}}, \delta, \delta^\dagger)$, where $\mathcal{L}_{\mathcal{A}} := (id \otimes l) \circ \alpha$, $\delta := (id \otimes \eta) \circ \alpha$, and initial condition $j_0 = id$.*

Proof. Observe that $(\alpha \otimes id) \circ \alpha = (id \otimes \Delta) \circ \alpha$. Hence, proceeding as in subsection 4.3.1, with H_π replaced by W_γ for $\gamma \in Irr_{\mathcal{Q}}$, and $L^2(h)$ replaced by $L^2(\tau)$, we get the existence of a unitary cocycle $(U_t)_{t \geq 0}$ satisfying an HP equation with coefficient matrix $\begin{pmatrix} iT - \frac{1}{2}R^*R & R^* \\ R & 0 \end{pmatrix}$, with the initial condition $U_0 = I$, where T, R are the closed extensions of $\frac{1}{2i}(\mathcal{L}_{\mathcal{A}} - (id \otimes (l \circ \kappa)) \circ \alpha)$ and $(id \otimes \eta) \circ \alpha$ respectively. Now, proceeding as in Theorem 4.3.7, we get our result. □

Definition 4.3.11. *A generator of a covariant quantum Gaussian process (QBM) on the non-commutative manifold $\mathcal{A}$ is defined as a map of the form $l_{\mathcal{A}} := (id \otimes l) \circ \alpha$, where l is the generator of some bi-invariant quantum Gaussian process (QBM) on $\mathcal{Q}$. In such a case, the EH flow k_t obtained in Theorem 4.3.10 will be called covariant quantum-Gaussian process (QBM) with the generator $\mathcal{L}_{\mathcal{A}}$.*

We will usually drop the adjective 'covariant'. Observe that

$$\begin{aligned}
(\mathcal{L}_{\mathcal{A}} \otimes id_{\mathcal{Q}})\alpha &= (id_{\mathcal{A}} \otimes l \otimes id_{\mathcal{Q}})(\alpha \otimes id_{\mathcal{Q}})\alpha \\
&= (id_{\mathcal{A}} \otimes l \otimes id_{\mathcal{Q}})(id_{\mathcal{A}} \otimes \Delta)\alpha \\
&= (id_{\mathcal{A}} \otimes (l \otimes id_{\mathcal{Q}})\Delta)\alpha \\
&= (id_{\mathcal{A}} \otimes (id_{\mathcal{Q}} \otimes l)\Delta)\alpha \text{ (since } (l \otimes id)\Delta = (id \otimes l)\Delta) \\
&= (\alpha \otimes l)\alpha = \alpha \circ \mathcal{L}_{\mathcal{A}}.
\end{aligned} \tag{4.17}$$

It is not clear whether the condition (4.17) is equivalent to the bi-invariance of the Gaussian generator l on $\mathcal{Q}$. However, let us show that it is indeed so for the class of quantum spaces which are quotient (hence in particular for the classical ones).

We recall (see definition 1.3.12) that $\mathcal{A}$ will be called a quotient of the CQG $(\mathcal{Q}, \Delta)$ by a quantum subgroup H if $\mathcal{A}$ is C^*-algebra isomorphic to the algebra $\{x \in \mathcal{Q} \ : (\pi \otimes id)\Delta(x) = 1 \otimes x\}$, where $\pi : \mathcal{Q} \to H$ is the CQG morphism.

Theorem 4.3.12. *Let $l : \mathcal{Q}_0 \to \mathbb{C}$ be the generator of a quantum Gaussian process on $\mathcal{Q}$. Suppose that $(\mathcal{Q}, \Delta)$ acts on a quantum space $\mathcal{A}$ such that $\mathcal{A}$ is a quotient space. Denote the action by α and define $\mathcal{L}_{\mathcal{A}} := (id_{\mathcal{A}} \otimes l)\alpha$. Then the following conditions are equivalent:*

1. $(l \otimes id)\Delta = (id \otimes l)\Delta$.
2. $(l \otimes id_{\mathcal{Q}})\alpha = (id_{\mathcal{A}} \otimes l)\alpha$.
3. $(\mathcal{L}_{\mathcal{A}} \otimes id_{\mathcal{Q}})\alpha = \alpha \circ \mathcal{L}_{\mathcal{A}}$.

Proof. It can be shown (see [58]) that $\alpha = \Delta|_{\mathcal{A}}$ in case of quotien spaces, where $\mathcal{A}$ has been identified with the algebra $\{x \in \mathcal{Q} \ : (\pi \otimes id)\Delta(x) = 1 \otimes x\}$. Thus (1) $\Rightarrow$ (2) is trivial. Let us prove (2) $\Rightarrow$ (1). It can be shown (see [58, page 5]) that if $\mathcal{A}$ is a quotient space, then the subspaces $W_{\gamma i}$ for $\gamma \in Irr_{\mathcal{Q}}$, as described in Proposition 1.4.11 are spanned by $\{u_{ij}^{\gamma}\}_{j=1}^{d_\gamma}$ and cardinality of the set I_γ is n_γ. So for a fixed $i, j, \ i = 1, 2, ...n_\gamma; \ j = 1, 2, ...d_\gamma$, we have

$$\begin{aligned}
&(l \otimes id_{\mathcal{Q}})\alpha(u_{ij}^{\gamma}) = (id_{\mathcal{A}} \otimes l)\alpha(u_{ij}^{\gamma}) \\
&i.e. \\
&\sum_{k=1}^{d_\gamma} l(u_{ik}^{\gamma})u_{kj}^{\gamma} = \sum_{k=1}^{d_\gamma} u_{ik}^{\gamma} l(u_{kj}^{\gamma}) \ ;
\end{aligned}$$

comparing the coefficients, we get $l(u_{ii}^{\gamma}) = l(u_{jj}^{\gamma}), \ l(u_{ij}^{\gamma}) = 0$ for $i \neq j$, where $1 \leq i \leq n_\gamma$ and $1 \leq j \leq d_\gamma$. As a vector space, $\mathcal{Q}_0 = \oplus_{\gamma \in Irr_{\mathcal{Q}}} \oplus_{i=1}^{d_\gamma} W_{\gamma i}$. From the preceding discussions, it follows that $(l \otimes id)\Delta(u_{ij}^{\gamma}) = (id \otimes l)\Delta(u_{ij}^{\gamma})$ which implies that $(l \otimes id)\Delta = (id \otimes l)\Delta$ i.e. (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (3) was already observed right after defining covariant quantum Gaussian process. The proof of the theorem will be completed if we show (3) $\Rightarrow$ (2). This can be argued as follows:

Since $\mathcal{A}$ is a quotient, we have $\alpha = \Delta|_{\mathcal{A}}$. Consider the functional $\epsilon|_{\mathcal{A}_0}$, where $\mathcal{A}_0 := \mathcal{A} \cap \mathcal{Q}_0$. Note that $\epsilon|_{\mathcal{A}_0} \circ \mathcal{L}_{\mathcal{A}} = l$. So applying $\epsilon|_{\mathcal{A}_0} \otimes id_{\mathcal{Q}}$ on both sides of (3), we get $(l \otimes id_{\mathcal{Q}})\alpha = \mathcal{L}_{\mathcal{A}} := (id_{\mathcal{A}} \otimes l)\alpha$. Thus (3) $\Rightarrow$ (2). $\square$

4.3.3 Deformation of Quantum Brownian motion

Recall the set-up and notations of subsection 1.3.3 of chapter 1, where the Rieffel deformation of $(\mathcal{Q},\Delta)$, denoted by $\mathcal{Q}_{\theta,-\theta}$, for some skew symmetric matrix θ, of a CQG was described. As a C^*-algebra, it is the fixed point subalgebra $(\mathcal{Q}\otimes C^*(\mathbb{T}^{2n}_{\theta}))^{\sigma\times\tau^{-1}}$, and has the same coalgebra structure as that of $\mathcal{Q}$.

Theorem 4.3.13. *Let l be the generator of a quantum Gaussian process and $\mathcal{L} := \widetilde{l}$. Suppose that $\mathcal{L}\circ\sigma_{\overline{z}} = \sigma_{\overline{z}}\circ\mathcal{L}$, for $\overline{z}\in\mathbb{T}^{2n}$. Let $\mathcal{W} := \langle\{U_i\}_{i=1}^n\rangle_{\mathbb{C}}$ Then we have the following:*

(i) $(\mathcal{L}\otimes id)((\mathcal{Q}_0\otimes_{alg}\mathcal{W})^{\sigma\times\tau^{-1}})\subseteq(\mathcal{Q}_0\otimes_{alg}\mathcal{W})^{\sigma\times\tau^{-1}}$;

(ii) $\mathcal{L}_\theta := (\mathcal{L}\otimes id)|_{(\mathcal{Q}_0\otimes_{alg}\mathcal{W})^{\sigma\times\tau^{-1}}}$ *is a generator of a quantum Gaussian process;*

(iii) with respect to the natural identification of $(\mathcal{Q}_{\theta,-\theta})_{-\theta,\theta}$ with $\mathcal{Q}$, we have $(\mathcal{L}_\theta)_{-\theta} = \mathcal{L}$.

Proof. Notice that the counit ϵ and the coproduct Δ remains the same in the deformed algebra, as the coalgebra $\mathcal{Q}_0$ is vector space isomorphic to $(\mathcal{Q}_0\otimes_{alg}\mathcal{W})^{\sigma\times\tau^{-1}}$. By our hypothesis, $\sigma_{\overline{z}}\circ\mathcal{L} = \mathcal{L}\circ\sigma_{\overline{z}}$, which implies (i).

Since $\mathcal{L}$ is a CCP map, it follows that $\mathcal{L}_\theta$ is a CCP map. Moreover, since we have the identity $l(abc) = l(ab)\epsilon(c)-\epsilon(ab)l(c)+l(bc)\epsilon(a)-\epsilon(bc)l(a)+l(ac)\epsilon(b)-\epsilon(ac)l(b)$ for $a,b,c\in\mathcal{Q}_0$, it follows that $l_\theta := \epsilon\circ\mathcal{L}_\theta$ also satisfies the same identity on the coalgebra $(\mathcal{Q}_0\otimes_{alg}\mathcal{W})^{\sigma\times\tau^{-1}}$. Thus l_θ, or equivalently $\mathcal{L}_\theta$, generates a quantum Gaussian process on $\mathcal{Q}_{\theta,-\theta}$, which proves (ii).

(iii) follows from the natural identification of $(\mathcal{Q}_{\theta,-\theta})_{-\theta,\theta}$ with $\mathcal{Q}$ and an application of the result in (ii). □

We have the following obvious corollary:

Corollary 4.3.14. *For a bi-invariant quantum Gaussian process, the conclusions of Theorem 4.3.13 hold.*

Thus we have a $1-1$ correspondence given by $\mathcal{L}\to\mathcal{L}_\theta$, between the set of generators of quantum Gaussian processes on $\mathcal{Q}$ satisfying the hypothesis of Theorem 4.3.13 and those of $\mathcal{Q}_{\theta,-\theta}$. In case $\mathcal{Q}$ is co-commutative, i.e. $\Sigma\circ\Delta=\Delta$, where Σ is the flip operation, it is easily seen that any quantum Gaussian process on $\mathcal{Q}$ will be bi-invariant and so the $1-1$ correspondence $\mathcal{L}\leftrightarrow\mathcal{L}_\theta$ holds for arbitrary quantum Gaussian processes in such a case. It is not clear, however, whether we can get $1-1$ correspondence between bi-invariant QBM on the deformed and undeformed CQGs.

Theorem 4.3.15. *If in the setup of Theorem 4.3.13, we have $\mathcal{Q} = C(G)$ for a compact Lie-group G with abelian Lie-algebra $\mathfrak{g}$, then the hypothesis of Theorem 4.3.13 holds for generators of quantum Gaussian processes on $\mathcal{Q}$ as well as on $\mathcal{Q}_{\theta,-\theta}$ and hence we have a $1-1$ correspondence between quantum Gaussian processes on $\mathcal{Q}$ and its Rieffel deformation.*

Proof. Let $G = G_e \bigsqcup_{i\in\Lambda} G_i$, where $e \in G$ is the identity element and G_e, G_i are the connected components of G, G_e being the identity component. Let the coproduct of the Rieffel-deformed algebra $\mathcal{Q}_{\theta,-\theta}$ be denoted by Δ_θ (note that it is the same coproduct as the original one). Observe that since the action σ is strongly continuous, $\overline{z} \cdot G_e \subseteq G_e$ $\forall \overline{z} \in \mathbb{T}^{2n}$, or equivalently, we have $\sigma_{\overline{z}}(C(G_e)) \subseteq C(G_e)$. Thus one has the following decomposition:

$$(C(G))_{\theta,-\theta} := (C(G_e))_{\theta,-\theta} \oplus (\mathcal{B})_{\theta,-\theta},$$

where $\mathcal{B} := \oplus_{i\in\Lambda} C(G_i)$ and $C(G_e)_{\theta,-\theta}$ itself is a quantum group satisfying $\Delta_\theta\left(C(G_e)_{\theta,-\theta}\right) \subseteq C(G_e)_{\theta,-\theta} \otimes C(G_e)_{\theta,-\theta}$. Note that since G_e is an abelian Lie-group, $C(G_e)_{\theta,-\theta}$ is a co-commutative quantum group. Let l be the generator of a quantum Gaussian process on $\mathcal{Q}_{\theta,-\theta}$. We claim that l is supported on $C(G_e)_{\theta,-\theta}$. Observe that χ_{G_e} (the indicator function of G_e) $\in C(G_e)$. Moreover, we have $\sigma_{\overline{z}}(\chi_{G_e}) = \chi_{G_e}$. Thus χ_{G_e} is identified with $\chi^\theta_{G_e} := \chi_{G_e} \otimes 1 \in (C(G) \otimes C^*(\mathbb{T}^{2n}_\theta))^{\sigma\times\tau^{-1}}$. In particular, $\chi^\theta_{G_e}$ is a self-adjoint idempotent in $C(G)_{\theta,-\theta}$. It now suffices to show that $l((1-\chi^\theta_{G_e})a) = 0$ for all $a \in (C(G)_0 \otimes_{alg} \langle U_i | i = 1, 2, ...2n\rangle_{\mathbb{C}})^{\sigma\times\tau^{-1}}$. Let (l, η, ϵ) be a Schürmann triple for l. Now

$$l((1-\chi^\theta_{G_e})a) = l(1-\chi^\theta_{G_e})\epsilon(a) + \epsilon(1-\chi^\theta_{G_e})l(a) + \langle \eta(1-\chi^\theta_{G_e}), \eta(a)\rangle.$$

Now as $(1-\chi^\theta_{G_e})^2 = (1-\chi^\theta_{G_e})$, and $\epsilon(1-\chi^\theta_{G_e}) = 0$, we have $1-\chi^\theta_{G_e} \in ker(\epsilon)^2$. By conditions 2 and 6 of Proposition 4.2.13, we have $l(1-\chi^\theta_{G_e}) = \eta(1-\chi^\theta_{G_e}) = 0$. This implies that $l((1-\chi^\theta_{G_e})a) = 0$ for all $a \in (C(G)_0 \otimes_{alg} \langle U_i | i = 1, 2, ...2n\rangle_{\mathbb{C}})^{\sigma\times\tau^{-1}}$. Now as $(C(G_e))_{\theta,-\theta}$ is a co-commutative quantum group, we have

$$(l \otimes id)\Delta_\theta = (id \otimes l)\Delta_\theta \text{ on } C(G_e)_{\theta,-\theta}. \tag{4.18}$$

Let $\overline{z} = (u, v)$ for $u, v \in \mathbb{T}^n$. Let us recall that $\sigma_{\overline{z}} = (\Omega(u) \otimes id)\Delta_\theta(id \otimes \Omega(-v))\Delta_\theta$, where we have $\Omega(u) := ev_u \circ \pi$, $\pi : C(G) \to C(\mathbb{T}^n)$ being the surjective CQG morphism. Let $R(x) := \sigma_{(0,x)}$ and $L(x) := \sigma_{(x,0)}$ for $x \in \mathbb{T}^n$. By equation (4.18), we have $l(R(u)a) = l(L(u)a)$ for all $a \in C(G_e)_{\theta,-\theta}$. Now $L(u)(C(G_i)_{\theta,-\theta}) \subseteq C(G_i)_{\theta,-\theta}$ and

$R(u)(C(G_i)_{\theta,-\theta}) \subseteq C(G_i)_{\theta,-\theta}$ for all i and $l(C(G_i)_{\theta,-\theta}) = 0$, which, in combination with equation (4.18), gives $l(R(u)a) = l(L(u)a)$ for all $a \in C(G)_{\theta,-\theta}$. From this, it easily follows that $\mathcal{L} \circ \sigma_{\overline{z}} = \sigma_{\overline{z}} \circ \mathcal{L}$ for all $\overline{z} \in \mathbb{T}^{2n}$. □

Moreover, in subsection 4.3.4, we shall see that condition of Theorem 4.3.13 is indeed necessary, i.e. there may not be a 'deformation' of a general quantum Gaussian generator.

4.3.4 Computation of Quantum Brownian motion

In this subsection, we compute the generators of QBM on the QISO of various non-commutative manifolds. We refer the reader to section 1.4 for a recollection of the description of QISO of the non-commutative manifolds which we will consider here.

a. **non-commutative 2-tori:** Recall from subsection 1.3.3 that $C^*(\mathbb{T}^2_\theta)$ is the universal C^*-algebra generated by a pair of unitaries U, V satisfying the relation $UV = e^{2\pi i\theta}VU$. The QISO of $C^*(\mathbb{T}^2_\theta)$ is a Rieffel deformation of the compact quantum group $C\left(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2)\right)$ (see [9]). Moreover, $\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2)$ is a Lie-group with abelian Lie-algebra. Hence an application of Theorem 4.3.15 and Theorem 4.3.13 leads to the conclusion that the generators of quantum Gaussian processes on the QISO of $C^*(\mathbb{T}^2_\theta)$ are precisely those coming from QISO($C(\mathbb{T}^2)$)=ISO($\mathbb{T}^2$)$\cong$ $C(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2))$ i.e. they are of the form l_θ, where l is a generator of classical Gaussian process on $\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2)$, which is supported on its identity component namely $\mathbb{T}^2$. Using the formulae for coproduct, it easily follows that if η be an ϵ-derivation, then we it is given on the generators by:

$$\begin{aligned}
&\eta(U_{11}) = c,\ \eta(U_{12}) = d\\
&\eta(U_{11}^*) = -c,\ \eta(U_{12}^*) = -d,\\
&\eta(U_{kj}) = 0\ \forall\ k > 1,\ j = 1, 2.
\end{aligned}$$

These are precisely the ϵ-derivation on the undeformed algebra $C(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2))$. Thus it follows that the space of ϵ-derivations on QISO($C^*(\mathbb{T}^2_\theta)$) is same as the space of ϵ-derivations on $C(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2))$. Moreover, all the ϵ-derivations are supported on the identity component namely $C(\mathbb{T}^2)$, which remains undeformed as a quantum subgroup of QISO($C^*(\mathbb{T}^2_\theta)$). Thus it follows that in this case, a QBM on the undeformed CQG remains a QBM on the deformed CQG.

Here $\mathcal{V}_{QISO(C^*(\mathbb{T}^2_\theta))}$ is same as $\mathcal{V}_{C(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2))}$. $\mathcal{V}_{C(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2))}$ is 2 dimensional. A basis for $\mathcal{V}_{C(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2))}$ is given by $\eta_{(1)}, \eta_{(2)}$, where $\eta_{(1)}(U_{11}) = 1, \eta_{(1)}(U_{11}^*) = -1, \eta_{(1)}(U_{12}) = 0$, $\eta_{(2)}(U_{11}) = 0, \eta_{(2)}(U_{12}) = 1, \eta_{(2)}(U_{12}^*) = -1$ and $\eta_{(l)}(U_{kj}) =$

0 for all $l = 1, 2; k > 1; j = 1, 2$. Let $\eta_1(\cdot) := c_1^{(1)}\eta_{(1)}(\cdot) + c_2^{(1)}\eta_{(2)}(\cdot)$ and $\eta_2(\cdot) := c_1^{(2)}\eta_{(1)}(\cdot) + c_2^{(2)}\eta_{(2)}(\cdot)$. Let (e_1, e_2) be the standard basis for $\mathbb{C}^2$. Note that by Lemma 4.2.12, the noise space of a quantum Gaussian process on $QISO(C^*(\mathbb{T}^2_\theta))$ is atmost 2 dimensional. Let (e_1, e_2) be a basis for $\mathbb{C}^2$. Let $\eta(\cdot) := \eta_1(\cdot)e_1 + \eta_2(\cdot)e_2$. The generator l of the quantum Gaussian process which has (l, η, ϵ) as the surjective Schürmann triple is given by:
$l(U_{11}) := -\frac{1}{2}\sum_{i=1}^{2}|c_i^{(1)}|^2 + ik_1$, $l(U_{12}) := -\frac{1}{2}\sum_{i=1}^{2}|c_i^{(2)}|^2 + ik_2$, $l(U_{kj}) = 0$ for all $k > 1, j = 1, 2$, and extending the map to other elements by using the formula $l(ab) = l(a)\epsilon(b) + \epsilon(a)l(b) + \langle \eta(a^*), \eta(b)\rangle$.

Using the action α of $\mathbb{T}^2$ as described in subsection 1.3.3, we can construct a QBM on $C^*(\mathbb{T}^2_\theta)$ as described in section 4.3.2, and conclude that

Theorem 4.3.16. *Any QBM k_t on $C^*(\mathbb{T}^2_\theta)$ is essentially driven by a classical Brownian motion on $\mathbb{T}^2$, in the sense that $k_t : C^*(\mathbb{T}^2_\theta) \to C^*(\mathbb{T}^2_\theta)'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbb{C}^2))) \cong \mathcal{B}(L^2(\omega_1, \omega_2))$, where (ω_1, ω_2) is the 2-dimensional standard Wiener measure, is given by $k_t(a)(\omega_1, \omega_2) = \alpha_{(e^{2\pi i\omega_1}, e^{2\pi i\omega_2})}(a)$.*

We now give an intrinsic characterization of a qunatum Gaussian (QBM) generator on $C^*(\mathbb{T}^2_\theta)$:

Let $\mathcal{A}_0$ denote the $*$-subalgebra spanned by the unitaries U, V.

Theorem 4.3.17. *A linear CCP map $\mathcal{L} : \mathcal{A}_0 \to \mathcal{A}_0$ is a generator of a quantum Gaussian process (QBM) on $C^*(\mathbb{T}^2_\theta)$ if and only if $\mathcal{L}$ satisfies:*

1. *$\mathcal{L}(abc) = \mathcal{L}(ab)c - ab\mathcal{L}(c) + \mathcal{L}(bc)a - bc\mathcal{L}(a) + \mathcal{L}(ac)b - ac\mathcal{L}(b)$, for all $a, b, c \in \mathcal{A}_0$.*
2. *$(\mathcal{L} \otimes id) \circ \alpha = \alpha \circ \mathcal{L}$, where α is the action of $\mathbb{T}^2$ on $C^*(\mathbb{T}^2_\theta)$.*
   Moreover, $\mathcal{L}$ will generate a QBM if and only if
   $$l_{(1,1)} - l_{(1,0)} - l_{(0,1)} < 2\sqrt{Re(l_{(1,0)})Re(l_{(0,1)})},$$
   where $l(U) = l_{(1,0)}U,\ l(V) := l_{(0,1)},\ l(UV) := l_{(1,1)}UV$.

Proof. Suppose that $\mathcal{L}$ is the generator of a quantum Gaussian process (QBM) on $C^*(\mathbb{T}^2_\theta)$. Notice that condition (2.) implies that U, V, UV are the eigenvectors of $\mathcal{L}$, which follows from the fact that since $\mathcal{L}$ commutes with the action α, it preserves the spectral subspaces of the action α.

Let the eigenvalues be denoted by $l_{(1,0)}, l_{(0,1)}, l_{(1,1)}$ respectively. Then suppose that there exists a Gaussian (Brownian) functional l on QISO$(C^*(\mathbb{T}^2_\theta))(= \mathcal{Q})$

with surjective Schürmann triple (l, η, ϵ), such that $\mathcal{L} = (id \otimes l)\alpha$, Let $(\eta_i)_{i=1,2}$ be the coordinates of η. Then since $l(abc) = l(ab)\epsilon(c) - \epsilon(ab)l(c) + l(bc)\epsilon(a) - \epsilon(bc)l(a) + l(ac)\epsilon(b) - \epsilon(ac)l(b)$ for $a, b, c \in \mathcal{Q}_0$, we have condition 1. of the present theorem.

Let us prove condition (2):

l generates a QBM. As a result η_1 and η_2 spans $\mathcal{V}_{C^*(\mathbb{T}^2_\theta)}$. Now let $\eta_{(1)}$ and $\eta_{(2)}$ be the canonical basis for $\mathcal{V}_{C^*(\mathbb{T}^2_\theta)}$. Let $\eta_1(\cdot) = a_1\eta_{(1)}(\cdot) + b_1\eta_{(2)}(\cdot)$ and $\eta_2(\cdot) = a_2\eta_{(1)}(\cdot) + b_2\eta_{(2)}(\cdot)$, $a_i, b_i \in \mathbb{C}$ for $i = 1, 2$. Note that $|a_1|^2 + |a_2|^2 = -2Re(l_{(1,0)})$ and $|b_1|^2 + |b_2|^2 = -2Re(l_{(0,1)})$. Observe that since by hypothesis η_1 and η_2 spans $\mathcal{V}_{C^*(\mathbb{T}^2_\theta)}$, we must have $a_1\overline{b_1} + a_2\overline{b_2} < 2\sqrt{Re(l_{(1,0)})Re(l_{(0,1)})}$, i.e. the matrix
$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ is non-singular.
Now we have

$$l(U_{11}U_{12}) - l(U_{11}) - l(U_{12}) = \langle \eta(U_{11}^*), \eta(U_{12})) \rangle.$$

The L.H.S equals $l_{(1,1)} - l_{(1,0)} - l_{(0,1)}$ and the R.H.S equals $a_1\overline{a_2} + b_1\overline{b_2}$. Thus we have the required inequality.

Conversely, suppose that we are given a CCP functional $\mathcal{L}$, satisfying conditions (1.) and (2.). Choose two vectors $(c_1, c_2), (d_1, d_2) \in \mathbb{R}^2$ such that $c_1^2 + c_2^2 = -2Re(l_{(1,0)})$, $d_1^2 + d_2^2 = -2Re(l_{(0,1)})$, and $c_1d_1 + c_2d_2 = l_{(1,1)} - l_{(1,0)} - l_{(0,1)}$. Consider the two ϵ-derivations $\eta_1 := c_1\eta_{(1)} + d_1\eta_{(2)}$ and $\eta_2 := c_2\eta_{(1)} + d_2\eta_{(2)}$. Define a CCP finctional l_{new} on $\mathcal{Q}$ as : $l_{new}(U_{11}) = l_{(1,0)}$ and $l_{new}(U_{12}) = l_{(0,1)}$, $l_{new}(U_{kj}) = 0$ for $k > 1, j = 1, 2$, and extend the definition to $(\mathcal{Q})_0$ by the rule $l(a^*b) = l(a^*)\epsilon(b) + l(b)\epsilon(a^*) + \sum_{p=1}^{2} \overline{\eta_p(a^*)}\eta_p(b)$. Note that we have $l_{new}(abc) = l_{new}(ab)\epsilon(c) - \epsilon(ab)l_{new}(c) + l_{new}(bc)\epsilon(a) - \epsilon(bc)l_{new}(a) + l_{new}(ac)\epsilon(b) - \epsilon(ac)l_{new}(b)$ for $a, b, c \in \mathcal{Q}_0$. It follows that $\mathcal{L}_{new} := (id \otimes l_{new})\alpha$ satisfies conditions (1.) and (2.) Thus $\mathcal{L} = \mathcal{L}_{new}$ on $\mathcal{A}_0$ and since $\mathcal{L}_{new}$ generates a quantum Gaussian process (QBM) on $C^*(\mathbb{T}^2_\theta)$, so does $\mathcal{L}$. □

Remark 4.3.18. *For general rotational irrational algebra $C^*(\mathbb{T}^n_\theta)$, which is the universal C^* algebra generated by n unitaries $\{U_i\}_{i=1}^n$ satisfying $U_iU_j = e^{2\pi i\theta_{ij}}U_jU_i$, for some skew-symmetric matrix $\theta := (\theta_{ij})_{ij}$, we have:*

Theorem 4.3.19. *A linear CCP map $\mathcal{L} : \mathcal{A}_0 \to \mathcal{A}_0$ is a generator of a quantum Gaussian process (QBM) on $C^*(\mathbb{T}^2_\theta)$ if and only if $\mathcal{L}$ satisfies:*

1. $\mathcal{L}(abc) = \mathcal{L}(ab)c - ab\mathcal{L}(c) + \mathcal{L}(bc)a - bc\mathcal{L}(a) + \mathcal{L}(ac)b - ac\mathcal{L}(b)$, *for all* $a, b, c \in \mathcal{A}_0$.
2. $(\mathcal{L} \otimes id) \circ \alpha = \alpha \circ \mathcal{L}$, *where* α *is the action of* $\mathbb{T}^n$ *on* $C^*(\mathbb{T}^2_\theta)$.
   Moreover, $\mathcal{L}$ *will generate a QBM if and only if the matrix given* $\mathbf{L} := (l_{ij} - l_i - l_j)_{ij}$ *is non-singular, where* $l_m, l_{pq} \in \mathbb{C}$ *such that* $\mathcal{L}(U_m) = l_m U_m$ *and* $\mathcal{L}(U_p U_q) = l_{pq} U_p U_q$.

It follows from this that in a similar way, we can also characterize generators of quantum Gaussian processes on quantum spaces on which $\mathbb{T}^n$ *acts ergodically.*

b. **The θ deformed sphere S^n_θ:**

Theorem 4.3.20. *(i) Suppose that* l *is the generator of a quantum Gaussian process on* $O_\theta(2n)$*. Then it satisfies the following:*

There exists $2n$ *complex numbers* $\{z_1, z_2, ...z_{2n}\}$ *with* $Re(z_i) \leq 0$ *for all* i *and* $\mathbf{A} \in M_{2n}(\mathbb{C})$ *with* $A_{ii} = 0 \ \forall \ i$ *and* $[A_{ij} - \overline{z_i} - z_j]_{ij} \geq 0$, *such that*

$$l(a^i_i) = z_i, \ \ l(a^{i*}_i a^j_j) = A_{ij} \ \ i, j = 1, 2, ...2n. \tag{4.19}$$

Conversely, given $2n$ *complex numbers* $\{z_1, z_2, ...z_{2n}\}$ *and* $\mathbf{A} \in M_{2n}(\mathbb{C})$, *such that* $Re(z_i) \leq 0$, $A_{ii} = 0 \ \forall \ i$ *and* $[A_{ij} - \overline{z_i} - z_j]_{ij} \geq 0$, *there exists a unique map* l, *such that* l *generates a quantum Gaussian process and satisfies equation (4.19).*

(ii) The generator of a quantum Gaussian process say l *generates a QBM if and only if the matrix*

$$\left[l(a^{\mu*}_\mu a^\nu_\nu) - l(a^{\mu*}_\mu) - l(a^\nu_\nu)\right]_{\mu,\nu} \in M_{2n}(\mathbb{C}) \tag{4.20}$$

is invertible.

(iii) l *generates a bi-invariant quantum Gaussian process if and only if* $z_\beta = z$ *for all* $\beta = 1, 2, ..2n$, *where* $z \in \mathbb{R}$ *such that* $z \leq 0$.

Proof. Let us first calculate all possible ϵ-derivations. Let η be an ϵ-derivation on this CQG. Put $\eta(a^\mu_\nu) = c^\mu_\nu$, $\eta(\overline{a^\mu_\nu}) = \widehat{c^\mu_\nu}$, $\eta(b^\mu_\nu) = d^\mu_\nu$, $\eta(\overline{b^\mu_\nu}) = \widehat{d^\mu_\nu}$, $\mu, \nu = 1, 2, ...2n$. Using condition (a), we get

$$c^\mu_\nu \delta^\tau_\rho + c^\tau_\rho \delta^\mu_\nu = \lambda_{\mu\tau}\lambda_{\rho\nu}(c^\mu_\nu \delta^\tau_\rho + c^\tau_\rho \delta^\mu_\nu);$$

putting $\tau = \rho$, we get $c^{\mu}_{\nu} = 0$ for $\mu \neq \nu$. Likewise using conditions (b) and (c), we get $\widehat{c}^{\mu}_{\nu} = d^{\mu}_{\nu} = \widehat{d}^{\mu}_{\nu} = 0$ for $\mu \neq \nu$. Using condition (d) with $\alpha = \beta$, we arrive at the following relations:

$$\begin{aligned}
&\widehat{c}^{\alpha}_{\alpha} + c^{\alpha}_{\alpha} = 0 \ (since\ \eta(1) = 0),\\
&d^{\alpha}_{\alpha} + d^{\alpha}_{\alpha} = 0,\\
&\widehat{d}^{\alpha}_{\alpha} + \widehat{d}^{\alpha}_{\alpha} = 0;
\end{aligned}$$

this implies that $c^{\alpha}_{\beta} = c_{\beta}\delta_{\alpha\beta}, \widehat{c}^{\alpha}_{\beta} = -c_{\beta}\delta_{\alpha\beta}$ for $2n$ complex numbers $\{c_1, c_2, ...c_{2n}\}$. Thus as a matrix, $(\eta(a^{\alpha}_{\beta}))_{\alpha,\beta} = (c_{\beta}\delta_{\alpha\beta})_{\alpha,\beta}$ and $(\eta(a^{\alpha *}_{\beta}))_{\alpha,\beta} = (-c_{\beta}\delta_{\alpha\beta})_{\alpha,\beta}$ and $(\eta(b^{\alpha}_{\beta}))_{\alpha,\beta} = (\eta(b^{\alpha *}_{\beta}))_{\alpha,\beta} = \mathbf{0}$. It may be noted that all the above steps are reversible, and hence this also characterizes ϵ-derivations on $O_{\theta}(2n)$. It is clear that the space of ϵ-derivations, $\mathcal{V}_{O_{\theta}(2n)}$ is $2n$-dimensional and is spanned by $2n$ ϵ-derivations $\{\eta_{(1)}, \eta_{(2)}, ...\eta_{(2n)}\}$, where $(\eta_{(k)}(a^{\alpha}_{\beta}))_{\alpha,\beta} = E_{kk}$, $(\eta_{(k)}(a^{\alpha *}_{\beta}))_{\alpha,\beta} = -E_{kk}$, $(\eta_{(k)}(b^{\alpha}_{\beta}))_{\alpha,\beta} = (\eta_{(k)}(b^{\alpha *}_{\beta}))_{\alpha,\beta} = \mathbf{0}$, where E_{ij} denote an elementary matrix.

Now we prove (i) as follows:

Let l be the generator of a quantum Gaussian process. Let the surjective Schürmann triple of l be (l, η, ϵ). Let $(\eta_i)_i$ be the coordinates of η, which are ϵ-derivations. By Lemma 4.2.12, there can be atmost $2n$ such coordinates. Let $\eta_i(a^{\alpha}_{\beta}) = c^{(i)}_{\alpha\beta}$ and $\eta_i(a^{\alpha *}_{\beta}) = \widehat{c}^{(i)}_{\alpha\beta}$ such that $c^{(i)}_{\alpha\beta} = c^{(i)}_{\beta}\delta_{\alpha\beta}$ and $\widehat{c}^{(i)}_{\alpha\beta} = -c^{(i)}_{\beta}\delta_{\alpha\beta}$. Suppose that $l(a^{\alpha}_{\beta}) = l_{\alpha\beta}$ and $l(b^{\alpha}_{\beta}) = l'_{\alpha\beta}$. Then using the relations among the generators of $O_{\theta}(2n)$, as given in section 1.4, we arrive at the following results:

$$\begin{aligned}
&l_{\alpha\beta} = 0 \text{ for all } \alpha \neq \beta,\\
&\bar{l}_{\alpha\alpha} + l_{\alpha\alpha} = -\sum_i |c^{(i)}_{\alpha}|^2 \text{ for all } \alpha = 1, 2, ...2n,\\
&l'_{\alpha\beta} = 0 \text{ for all } \alpha, \beta.
\end{aligned}$$

Moreover, we have $l(a^*b) - l(a^*)\epsilon(b) - \epsilon(a^*)l(b) = \langle \eta(a), \eta(b) \rangle$, so that by taking $z_i := l(a^i_i)$, $\mathbf{A} := [l(a^{i*}_i a^j_j)]_{ij}$ we have the result.

Conversely, suppose that we are given $2n$ complex numbers $\{z_1, z_2, ...z_{2n}\}$ such that $Re(z_i) \leq 0$ for all i and $\mathbf{A} \in M_{2n}(\mathbb{C})$, satisfying the hypothesis. Let $\mathbf{B} := [A_{ij} - \overline{z_i} - z_j]_{ij}$. Suppose that $\mathbf{P} := \mathbf{B}^{\frac{1}{2}}$. Let us define $2n$ ϵ-derivations $(\eta_i)_{i=1}^{2n}$ by $\eta_k := \sum_{i=1}^{2n} \overline{P_{ik}}\eta_{(i)}$, $k = 1, 2, ...2n$. Let $\eta := \sum_{i=1}^{2n} \eta_i \otimes e_i$, where

$\{e_i\}_i$ is the standard basis of $\mathbb{C}^{2n}$. Define a CCP map l on $(O_\theta(2n))_0$ by the prescription $l(a^i_i) = z_i, l(a^i_j) = 0$ for $i \neq j$, $l(b^i_j) = 0 \;\forall\; i, j$ and extending the map to $(O_\theta(2n))_0$ by the rule $l(a^*b) = l(a^*)\epsilon(b) + \epsilon(a^*)l(b) + \langle\eta(a), \eta(b)\rangle$. Such a map is clearly the generator of a quantum Gaussian process on $O_\theta(2n)$ and it satisfies $l(a^{i*}_i a^j_j) = A_{ij}$. The uniqueness follows from the fact that a generator of a quantum Gaussian process on $O_\theta(2n)$ must satisfy the identity:

$$l(abc) = l(ab)\epsilon(c) - \epsilon(ab)l(c) + l(bc)\epsilon(a) - \epsilon(bc)l(a) + l(ac)\epsilon(b) - \epsilon(ac)l(b),$$

for all $a, b, c \in (O_\theta(2n))_0$.

For proving (ii), let us proceed as follows:

Let l be the generator of a QBM and let (l, η, ϵ) be the surjective Schürmann triple associated with l. Suppose that $(\eta_i)_i$ are the cordinates of η. Then by hypothesis, $\{\eta_1, \eta_2, ...\eta_{2n}\}$ forms a basis for $\mathcal{V}$. Let $\eta_k = \sum_i c^{(k)}_i \eta_{(i)}$. Consider the $2n \times 2n$ matrix $\mathbf{P}$ such that $P_{ij} := \overline{c^{(j)}_i}$. Then $\mathbf{P}^*\mathbf{P}$ is an invertible matrix. Moreover, we have $[l(a^{i*}_i a^j_j) - l(a^{i*}_i) - l(a^j_j)]_{ij} = \mathbf{P}^*\mathbf{P}$, which proves our claim.

Conversely, suppose that l is the generator of a quantum Gaussian process, such that $\mathbf{B} := [l(a^{i*}_i a^j_j) - l(a^{i*}_i) - l(a^j_j)]_{ij}$ is an invertible matrix. Let (l, η, ϵ) be the surjective Schürmann triple associated to l. Let $(\eta_i)_i$ be the coordinates of η. Let $\eta_k(\cdot) = \sum_i c^{(k)}_i \eta_{(i)}(\cdot)$, for all k. Let $\mathbf{P} := [c^{(j)}_i]_{ij}$. Then we have $\mathbf{P}^*\mathbf{P} = \mathbf{B}$, which implies that the matrix $\mathbf{P}$ is invertible, and hence $\{\eta_i\}_{i=1}^{2n}$ forms a basis for $\mathcal{V}_{O_\theta(2n)}$, which proves the claim.

(iii) This can be proved as follows:
$(l \otimes id)\Delta(a^\mu_\nu) = (id \otimes l)\Delta(a^\mu_\nu)$. Using the formula for co-product we have $\sum_{\lambda=1}^{2n}[l(a^\mu_\lambda)a^\nu_\lambda] = \sum_{\lambda=1}^{2n}[a^\mu_\lambda l(a^\nu_\lambda)]$. From (i) of the present theorem, this implies that $z_\mu a^\mu_\nu = z_\nu a^\mu_\nu$ i.e. $z_\mu = z_\nu$ for all $\mu, \nu = 1, 2, ...2n$. On the other hand, $(l \otimes id)\Delta(b^\mu_\nu) = (id \otimes l)\Delta(b^\mu_\nu)$ implies $z_\mu = \overline{z}_\nu$ which implies that $z_\mu \in \mathbb{R}$ for all μ. □

We have the following obvious corollary, which follows from (iii) of the theorem above and the definition of quantum Gaussian process on quantum homogeneous space.

Corollary 4.3.21. *A map $\mathcal{L}_{S^{2n-1}_\theta}$, which generates a qunatum Gaussian process on S^{2n-1}_θ, satisfy: $\mathcal{L}_{S^{2n-1}_\theta}(z^\mu) = cz^\mu$, for some real number $c \leq 0$.*

Remark 4.3.22. *Notice that the space of ϵ-derivations on the undeformed algebra $O(2n)$ has dimension more than $2n$, since there are ϵ-derivations, which takes non-zero values on $(b^{\mu}_{\nu})_{\mu\nu}$ and hence there are quantum Gaussian processes on $O(2n)$ such that their generators take non-zero values on b^{α}_{β}, and so there is no 1-1 correspondence between quantum Gaussian processes on the deformed and undeformed algebra in this case.*

c. **The free orthogonal group** $O_+(2n)$: We refer the reader to section 1.4 again, for the definition and formulae for the free orthogonal group. Before stating the main theorem, we introduce some notations for convenience. Let $\mathbf{A} \in M_{n(2n-1)}(\mathbb{C})$. We will index the elements of $\mathbf{A}$ by the set $I\!N^4$ instead of $I\!N^2$ as follows:

Let $\mathbf{A} = \begin{pmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ . \\ . \\ . \\ \mathbf{A_{2n-1}} \end{pmatrix}$, where $\mathbf{A_i}$ is a $n(2n-1) \times (2n-i+1)$ matrix, such that

$$
\begin{aligned}
(\mathbf{A_i})_{\mathbf{kl}} =& a_{(i,i+k,1,1+l)} \chi_{\{1,2,...2n-1\}}(l) \\
&+ a_{(i,i+k,2,3+(l-2n))} \chi_{\{2n,2n+1,...4n-3\}}(l) \\
&+ a_{(i,i+k,3,4+(l-(4n-2)))} \chi_{\{4n-2,...6n-2\}}(l) \\
&+ \\
&. \\
&. \\
&. \\
&+ a_{(i,i+k,2n-1,2n)} \chi_{\{n(2n-1)\}}(l),
\end{aligned}
$$

for $k = 1, 2, ...2n - i + 1$, where χ_B denotes the indicator function of the set B. We now state the main theorem:

Theorem 4.3.23. *(i) There exists a 1-1 correspondence between generators of quantum Gaussian processes on $O_+(2n)$ and matrices $\mathbf{L} := [L_{ij}]_{ij} \in M_{2n}(\mathbb{C})$ and $\mathbf{A} := [A_{ij}]_{ij} \in M_{n(2n-1)}(\mathbb{C})$, satisfying*

a. *$\mathbf{B} \in M_{n(2n-1)}(\mathbb{C})$, defined by $\mathbf{B} := [a_{(i,j,k,l)} - \overline{L_{ij}} - L_{kl}]$, $i < j$, $k < l$ is positive definite,*

b. *$L_{ij} + L_{ji} := -\sum_{k=1}^{i-1} a_{(k,i,k,j)} + \sum_{k=i+1}^{j-1} a_{(i,k,k,j)} - \sum_{k=j+1}^{2n} a_{(i,k,j,k)}$, $i < j$.*

(ii) l will generate a QBM if and only if the matrix $\mathbf{B}$, defined above, is invertible.

(iii) There exists no bi-invariant quantum Gaussian process on $O_+(2n)$.

Proof. Using the relations among the generators, as given in section 1.4, it is seen that the epsilon-derivations on this algebra are given by

$$\eta(x_{ij}) = A_{ij};$$

such that $A_{ij} = -A_{ji}$. Clearly this characterizes the ϵ-derivations on the CQG. Observe that the space of ϵ-derivations, $\mathcal{V}_{O_+(2n)}$ has dimension $n(2n-1)$. A basis for the space is given by $\{\eta_{(ij)}\}_{i<j}$, such that $\eta_{(ij)}(x_{ij}) = 1,\ \eta_{(ij)}(x_{ji}) = -1$ and $\eta_{(ij)}(x_{kl}) = 0$ for $k \neq i, j$ or $l \neq i, j$. So after a suitable re-indexing, let us denote the basis by $\{\eta_{_{(p)}}\}_{p=1}^{n(2n-1)}$.

We prove (i):

Let l be the generator of a quantum Gaussian process on $O_+(2n)$, with the surjective Schürmann triple (l, η, ϵ). Let $(\eta_i)_i$ be the coordinates of η. By Lemma 4.2.12, there can be atmost $n(2n-1)$ coordinates. Let $\mathbf{A^{(i)}} := ((\eta_i(x_{kl})))_{kl}$. Now using the relations among the generators, as described in section 1.4, we see that

$$((l(x_{ij})))_{i,j} = \mathbf{L} := [L_{ij}]_{ij}$$

such that

$$L_{ij} + L_{ji} = -\sum_{s=1}^{n(2n-1)} \sum_{k=1}^{2n} \overline{A_{ik}^{(s)}} A_{jk}^{(s)},\ i < j.$$

Thus by taking $a_{_{(i,j,k,l)}} := l(x_{ij}x_{kl})$ and $L_{ij} := l(x_{ij})$, the conclusion follows.

Conversely, suppose that we are given matrices $\mathbf{L} \in M_{2n}(\mathbb{C}),\ \mathbf{A} \in M_{n(2n-1)}(\mathbb{C})$ satisfying the hypothesis in (i). Let $\mathbf{P} := [P_{ij}]_{ij} := \mathbf{B}^{\frac{1}{2}}$. Define $n(2n-1)$ ϵ-derivations by $\eta_{_p}(\cdot) := \sum_{k=1}^{n(2n-1)} P_{pk}\eta_{_{(k)}}(\cdot)$,
$p = 1, 2, ...n(2n-1)$. Define a CCP map by the prescription $l(x_{ij}) := L_{ij}$, and extending the definition to $(O_+(2n))_0$ by the rule $l(a^*b) = l(a^*)\epsilon(b) + \epsilon(a^*)l(b) + \sum_{p=1}^{n(2n-1)} \overline{\eta_{_p}(a)}\eta_{_p}(b)$. Clearly such a functional satisfies $l(x_{ij}x_{kl}) = a_{_{(i,j,k,l)}},\ i < j,\ k < l$. The uniqueness follows from the fact that l satisfies $l(abc) = l(ab)\epsilon(c) - \epsilon(ab)l(c) + l(bc)\epsilon(a) - \epsilon(bc)l(a) + l(ac)\epsilon(b) - \epsilon(ac)l(b),\ a, b, c \in (O_+(2n))_0$.

(ii) follows from the fact that the invertibility of the matrix $\mathbf{B}$ implies the invertibility of the matrix $\mathbf{P} := \mathbf{B}^{\frac{1}{2}}$, so that $\{\eta_i\}_{i=1}^{n(2n-1)}$, as defined in (i), forms a basis for $\mathcal{V}_{O_+(2n)}$.

(iii) can be proven as follows:

Theorem 4.3.24. *Suppose $\mathcal{L}$ is the generator of a bi-invariant QBM on the free orthogonal group. Then $\mathcal{L} \equiv 0$.*

Proof. Since $\mathcal{L}$ is bi-invariant, we have

$$(id \otimes \mathcal{L})\Delta(x_{ij}) = (\mathcal{L} \otimes id)\Delta(x_{ij}) \tag{4.21}$$

and

$$(id \otimes \mathcal{L})\Delta(x_{ij}x_{kl}) = (\mathcal{L} \otimes id)\Delta(x_{ij}x_{kl}) \text{ where } i \neq j \text{ and } k \neq l; \tag{4.22}$$

comparing the coefficients in (4.21) and (4.22), we get

$$\mathcal{L}(x_{ij}) = 0 \text{ for } i \neq j;$$
$$\mathcal{L}(x_{ij}x_{kl}) = 0 \text{ for } i \neq j \text{ and } k \neq l;$$

substituting $k = i,\ l = j, (i \neq j)$ in the second equation, we get

$$0 = \mathcal{L}(x_{ij}x_{kl}) = \sum_{p \geq 1} \overline{\eta_{(p)}(x_{ij})}\eta_{(p)}(x_{ij}) = \sum_{p \geq 1} |\eta_{(p)}(x_{ij})|^2,\ i \neq j;$$

where η_p is an ϵ-derivation for each p. This implies $\eta_{(p)} \equiv 0$, since $\eta_{(p)}(x_{ii}) = 0$. Thus $\mathcal{L}$ becomes an ϵ-derivation. But $\mathcal{L}(x_{ij}) = 0$ for $i \neq j$. Thus we have $\mathcal{L} \equiv 0$. □

Remark 4.3.25. *Theorem 4.3.24 implies that there does not exist any quantum Brownian motion on the quantum space S^+_{2n-1} (i.e. the free sphere) in the sense described in subsection 4.3.2.So, in some sense, free orthogonal group behaves like a totally disconnected group, such as the group of $n \times n$ orthogonal matrices over the p-adic field, in admitting no Brownian motion.*

□

4.4 Exit time of Quantum Brownian motion on non-commutative torus.

4.4.1 Motivation and formulation

We shall first recast the classical results about the asymptotics of exit time of Brownian motion in a form which will be easily generalized to the quantum set-up.

Let M be a Riemannian manifold of dimension d which is also a homogeneous space. Therefore M can be realized as K/G, where G is the isometry group of M and K is a compact subgroup of G. For $m \in M$, let $\mathcal{B}_t^m$ denote the standard Brownian motion on M starting at m, as described in section 4.3.2. Let $\widetilde{\mathcal{A}}$ denote the universal enveloping von-Neumann algebra of $C(M)$. Let us define a map j_t : $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} \otimes \mathcal{B}(L^2(\mathbb{P}))$ by: $j_t(f)(x,\omega) := f(\mathcal{B}_t^x(\omega))$, for $f \in C(M)$ and extending the map to $\widetilde{\mathcal{A}}$, where $\mathbb{P}$ denote the d-dimensional Wiener measure.

Let B_r^x denote a ball of radius r around $x \in M$. Let $\tau_{B_r^x}$ be the exit time of the Brownian motion from the ball B_r^x. Then $\{\tau_{B_r^x} > t\} = \{\mathcal{B}_s^x \in B_r^x \forall\ 0 \leq s \leq t\}$, so that we have $\chi_{\{\tau_{B_r^x} > t\}} = \bigwedge_{s \leq t} \left(\chi_{\{\mathcal{B}_s^x \in B_r^x\}} \right)$, where $\bigwedge$ denotes infimum and for a set A, χ_A denotes the indicator function on the set A. In terms of the map j_t, we have

$$\chi_{\{\tau_{B_r^x} > t\}}(\cdot) = \bigwedge_{s \leq t} j_s(\chi_{B_r^x})(x, \cdot) = \bigwedge_{s \leq t} ((ev_x \otimes id) \circ j_s(\chi_{B_r^x}))(\cdot).$$

Now by the Wiener-Itô isomorphism (see [52]), $L^2(\mathbb{P}) \cong \Gamma(L^2(\mathbb{R}_+, \mathbb{C}^d))$. Thus we may view $\tau_{B_r^x}$ as a family of projections in $\widetilde{\mathcal{A}} \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbb{C}^d)))$ defined by

$$\tau_{B_r^x}([0,t)) = \mathbf{1} - \wedge_{s \leq t}(j_s(\chi_{B_r^x}))\ .$$

We recall from subsection 4.2.1, the asymptotic behaviour of $\mathbb{E}(\tau_{B_r^x})$ as $r \to 0$. Now one has
$\mathbb{E}(\tau_{B_r^x}) = \int_0^\infty \mathbb{P}(\tau_{B_r^x} > t)dt = \int_0^\infty \langle e(0), \{(ev_x \otimes 1) \left(\wedge_{s \leq t} j_s(\chi_{B_r^x}) \right) \} e(0) \rangle dt$, since $\tau_{B_r^x}$ is a positive random variable. Note that the points of M are in $1-1$ correspondence with the pure states and $\{P_r = \chi_{B_r^x}\}_{r \geq 0}$ is a family of projections on $\widetilde{\mathcal{A}}$ satisfying $vol(P_r) \to 0$ as $r \to 0$ and $ev_x(P_r) = 1\ \forall r$. One can slightly generalize this as follows:

Choose a sequence $(x_n)_n \in M$ and positive numbers ϵ_n such that $x_n \to x$ and $\epsilon_n \to 0$. Now for large n_0 the random variable $\chi_{\{\mathcal{B}_s^{x_n} \in B_{\epsilon_n}^{x_n}\}}(\cdot)$ has the same distribution as the random variable $\chi_{\{\mathcal{B}_s^x \in B_{\epsilon_n}^x\}}$ for each $s \geq 0$. Thus,

$$\mathbb{E}(\tau_{B_{\epsilon_n}^{x_n}}) = \mathbb{E}(\tau_{B_{\epsilon_n}^x}) = \int_0^\infty \langle e(0), \{(ev_{x_n} \otimes id) \left(\wedge_{s \leq t} j_s(\chi_{B_{\epsilon_n}^{x_n}}) \right) \} e(0) \rangle dt$$

which implies that the asymptotic behaviour of $\mathbb{E}(\tau_{B_{\epsilon_n}^{x_n}})$ and $\mathbb{E}(\tau_{B_{\epsilon_n}^x})$ will be the same.

For a non-commutative generalization of the above, we need the notion of quantum stop time. There are several formulations of this concept as discussed in chapter 1, section 1.6.7. We adopt definition 1.7.13 for our purpose.

Observe that by our definition, $\tau_{B_r}([0,t))$ is adapted to the filtration $(\mathfrak{A}_t)_{t\geq 0}$, where
$\mathfrak{A}_t := \widetilde{\mathcal{A}} \otimes \mathcal{B}(\Gamma_{t]})$ $\left(\Gamma_{t]} := \Gamma\left(L^2([0,t],\mathbb{C}^n)\right)\right)$, for $\tau_{B_r}([0,t]) \in \mathfrak{A}_t \otimes 1_{\Gamma_{[t}}$.

Suppose that we are given an E-H flow $j_t : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+,k_0)))$, where $\mathcal{A}$ is a C^* or von-Neumann algebra. For a projection $P \in \mathcal{A}$, the family $\{\mathbf{1} - \wedge_{s\leq t}(j_s(P))\}_{t\geq 0}$ defines a quantum random time adapted to the filtration $\left(\mathcal{A}'' \otimes \mathcal{B}(\Gamma_{t]})\right)_{t\geq 0}$. Let us assume, furthermore, that $\mathcal{A}$ is the C^* or von-Neumann closure of the 'smooth algebra' $\mathcal{A}^\infty$ of a Θ-summable, admissible spectral triple (see section 1.4 in chapter 1) and j_t is a QBM on it.

Definition 4.4.1. *We refer to the quantum random time* $\{1 - \bigwedge_{s\leq t} j_s(P)\}_{t\geq 0}$ *as the 'exit time from the projection P'.*

Motivated by the Propostion 4.2.3 and the discussion after it, we would like to formulate a quantum analogue of the exit time asymptotics and study it in concrete examples.

Let τ be the non-commutative volume form corresponding to the spectral triple, and assume that we are given a family $\{P_n\}_{n\geq 1}$ of projections in $\mathcal{A}$, and a family $\{\omega_n\}_{n\geq 1}$ of pure states of $\mathcal{A}$ such that

- ω_n is weak* convergent to a pure state ω,
- $\omega_n(P_n) = 1$ for all n,
- $v_n \equiv \tau(P_n) \to 0$ as $n \to \infty$.

Definition 4.4.2. *Let* $\gamma_n := \int_0^\infty dt\langle e(0), (\omega_n \otimes id) \circ \bigwedge_{s\leq t} j_s(P_n)e(0)\rangle$. *We say that there is an exit time asymptotic for the family* $\{P_n;\omega_n\}$ *of intrinsic dimension* n_0 *if*

$$\lim_{n\to\infty} \frac{\gamma_n}{v_n^{\frac{2}{m}}} = \begin{cases} \infty \ \textit{if}\ m < n_0 - \epsilon \ \textit{for}\ 0 < \epsilon < 1, \\ \neq 0 \ \textit{if}\ m \neq n_0, \\ = 0 \ \textit{if}\ m > n_0; \end{cases}$$

and

$$\gamma_n = c_1 v_n^{\frac{2}{n_0}} + c_2 v_n^{\frac{4}{n_0}} + \cdots c_k v_n^{\frac{2^k}{n_0}} + O(v_n^{\frac{2^{k+1}}{n_0}}) \ \textit{as}\ n \to \infty. \tag{4.23}$$

It is not at all clear whether such an asymptotic exists in general, and even if it exists, whether it is independent of the choice of the family $\{P_n;\omega_n\}$. If it is the case, one may legitimately think of c_1, c_2 as geometric invariants and imitating the classical formulae (4.4) and (4.5), the extrinsic dimension d and the mean curvature H of the non-commutative manifold may be defined to be

$$d := \frac{1}{2c_1}(\frac{n_0}{\alpha_{n_0}})^{\frac{2}{n_0}} + 1, \tag{4.24}$$

$$H^2 := 8(d+1)c_2(\frac{\alpha_{n_0}}{n_0})^{\frac{4}{n_0}}. \tag{4.25}$$

4.4.2 A case-study: non-commutative 2-Torus.

Fix an irrational number $\theta \in [0,1]$. We refer the reader to [[18],page 173] for a natural class of projections in $C^*(\mathbb{T}^2_\theta)$, which we will be using in this section.

Let tr be the canonical trace in $C^*(\mathbb{T}^2_\theta)$, given by $tr(\sum_{m,n} a_{mn}U^mV^n) = a_{00}$. This trace will be taken as an analogue of the volume form in $C^*(\mathbb{T}^2_\theta)$. Throughout the section, we will consider $C^*(\mathbb{T}^2_\theta)$ as a concrete C^*-subalgebra of $\mathcal{B}(\widetilde{H})$, where $\widetilde{H}$ denote the so-called universal enveloping Hilbert space for $C^*(\mathbb{T}^2_\theta)$, and let $W^*(\mathbb{T}^2_\theta)$ be the universal enveloping von-Neumann algebra of it. i.e. the weak closure of $C^*(\mathbb{T}^2_\theta)$ in $\mathcal{B}(\widetilde{H})$. For $(x,y) \in \mathbb{T}^2$, let $\alpha_{_{(x,y)}}$ denote the canonical action of $\mathbb{T}^2$ on $C^*(\mathbb{T}^2_\theta)$ given by $\alpha_{_{(x,y)}}(\sum_{m,n} a_{mn}U^mV^n) = \sum_{m,n} x^my^na_{mn}U^mV^n$. For a projection P, let $A_{(s,t)}(P) := \alpha_{_{e^{2\pi is},e^{2\pi it}}}(P)$. Note that each $\alpha_{(x,y)}$ extends as a normal automorpihsm of $W^*(\mathbb{T}^2_\theta)$. On $C^*(\mathbb{T}^2_\theta)$, there are two conditional expectations denoted by ϕ_1, ϕ_2, which are defined as:

$$\phi_1(A) := \int_0^1 \alpha_{_{(1,e^{2\pi it})}}(A)dt, \quad \phi_2(A) := \int_0^1 \alpha_{_{(e^{2\pi it},1)}}(A)dt.$$

By universality of $W^*(\mathbb{T}^2_\theta)$, ϕ_1, ϕ_2 extend on $W^*(\mathbb{T}^2_\theta)$ as well.

Let $\mathfrak{X} = \{A \in W^*(\mathbb{T}^2_\theta) | \ A = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V, \ f_1, f_0 \in L^\infty(\mathbb{T})\}$.

Lemma 4.4.3. *The subspace $\mathfrak{X}$ is closed in the ultraweak topology.*

Proof. Let $A_\beta := f^{(\beta)}_{_{-1}}(U)V^{-1} + f^{(\beta)}_{_0}(U) + f^{(\beta)}_{_1}(U)V$ be a convergent net in the ultraweak topology. Now $\phi_1(A_\beta) = f^{(\beta)}_{_0}(U)$, $\phi_1(A_\beta V) = f^{(\beta)}_{_{-1}}(U)$ and $\phi_1(A_\beta V^{-1}) = f^{(\beta)}_{_1}(U)$ Since ϕ_1 is a normal map, which implies that $f^{(\beta)}_{_0}(U)$, $f^{(\beta)}_{_1}(U)$ *and* $f^{(\beta)}_{_{-1}}(U)$ (all of which are elements of $L^\infty(\mathbb{T})$) are ultraweakly convergent, to $f_0(U), f_1(U), f_{-1}(U)$ (say) and this proves the lemma. □

Lemma 4.4.4. *Suppose that $A \in \mathfrak{X}$, such that $f_{-1}(l) = \overline{f_1(l+\theta)}$. Define*

$$A_{s,t} := f_{-1}(e^{2\pi is}U)V^{-1}e^{-2\pi it} + f_0(e^{2\pi is}U) + f_1(e^{2\pi is}U)Ve^{2\pi it}.$$

Suppose $s, s' \in [0,1)$ *be such that* $|s - s'| \leq \frac{\epsilon}{4}$ *where* $0 < \epsilon < \frac{4\theta}{5}$, *and* $|supp(f_1)| < \epsilon$, *where* $|C|$ *denotes the Lebesgue measure of a Borel subset* $C \subseteq \mathbb{R}$. *Then* $A_{s,t} \cdot A_{s',t'} \in \mathfrak{X}$.

Proof. It suffices to show that the coefficient of V^2 and that of V^{-2} in $A_{s,t} \cdot A_{s',t'}$ is zero. By a direct computation, the coefficient of V^2 is $g(l) := f_1(s+l)f_1(s'+l-\theta)e^{2\pi i(t+t')}$ and that of V^{-2} is $g'(l) := \overline{f_1(s+l+\theta)f_1(s'+l+2\theta)}e^{2\pi i(-t-t')}$, $l \in [0,1)(mod 1)$. But $|(s+l)-(s'+l-\theta)| = |\theta+s-s'| > \epsilon$ and $|(s+l+\theta)-(s'+l+2\theta)| = |\theta+s-s'| > \epsilon$. Now by hypothesis, we have $|supp(f_1)| < \epsilon$, so that both $g(l)$ and $g'(l)$ will vanish, and hence the lemma is proved. □

Lemma 4.4.5. *Suppose that* $A = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$, *such that* $f_1(l)f_1(l+\theta) = 0$, $f_{-1}(l)f_{-1}(l+\theta) = 0$ *for* $l \in [0,1)(mod 1)$. *Then* $A^{2n} \in \mathfrak{X}$, *for* $n \in I\!N$.

Proof. The coefficient of V^2 in A^2 is $f_1(l)f_1(l+\theta)$ and that of V^{-2} is $f_{-1}(l)f_{-1}(l+\theta)$ for $l \in [0,1)(mod 1)$ and this is zero by the hypothesis. Hence $A^2 \in \mathfrak{X}$. The coefficient of V in A^2 is $f_1^{(2)}(\cdot) := f_1(\cdot)\left(f_0(\cdot) + \tau_{-\theta}(f_0)(\cdot)\right)$, and that of V^{-1} is $f_{-1}^{(2)}(\cdot) := f_{-1}(\cdot)\{f_0(\cdot) + \tau_\theta(f_0)(\cdot)\}$, where τ_x is translation by $x \in \mathbb{R}$. We have $f_1^{(2)}(l)f_1^{(2)}(l+\theta) = 0$, $f_{-1}^{(2)}(l)f_{-1}^{(2)}(l+\theta)$ so that, applying the same argument as before, we conclude that $A^4 \in \mathfrak{X}$. Proceeding like this we get the required result. □

Lemma 4.4.6. *Suppose that* $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$, *such that* P *is a self-adjoint projection. Furthermore, assume that* $|supp(f_1)| < \epsilon$. *Then* $\left(A_{s,t}(P)\right) \bigwedge \left(A_{s',t'}(P)\right) \in \mathfrak{X}$ *for* $|s - s'| < \frac{\epsilon}{4}$, *where for two self-adjoint projections* P *and* Q, $P \bigwedge Q$ *denotes the projection onto the subspace* $Ran(P) \cap Ran(Q)$.

Proof. We start with the following well-known formula due to von-Neumann:

$$(P \cdot Q)^n \stackrel{SOT}{\longrightarrow} P \bigwedge Q.$$

Thus in particular:

$$A_{s,t}(P) \bigwedge A_{s',t'}(P) = \lim_{n\to\infty} \{A_{s,t}(P) \cdot A_{s',t'}(P)\}^n,$$

where the convergence is in SOT. Now by the hypothesis, $|s-s'| < \frac{\epsilon}{4}$ and $|supp(f_1)| < \epsilon$, and note that since P is a self-adjoint projection, we must have $f_{-1}(l) = \overline{f_1(l+\theta)}$, $f_1(l)f_1(l+$

$\theta) = 0$ (see [18]). It follows by Lemma 4.4.4, that $A_{s,t}(P) \cdot A_{s',t'}(P) \in \mathfrak{X}$. The coefficient of V in $A_{s,t}(P) \cdot A_{s',t'}(P)$ is

$$f_1^{(2)}(l) := \{f_1(s+l)f_0(s'+l-\theta)e^{2\pi it} + f_0(s+l)f_1(s'+l)e^{2\pi it'}\}$$

and that of V^{-1} is

$$f_{-1}^{(2)}(l) := \{f_{-1}(s+l)f_0(s'+t+\theta)e^{-2\pi it} + f_0(s+l)f_{-1}(s'+l)e^{-2\pi it'}\}.$$

Now we have

$$\begin{aligned} f_1^{(2)}(l)f_1^{(2)}(l+\theta) = &\{f_1(s+l)f_0(s'+l-\theta)e^{2\pi it} + f_0(s+l)f_1(s'+l)e^{2\pi it'}\} \\ &\times \{f_1(s+l+\theta)f_0(s'+l)e^{2\pi it} + f_0(s+l+\theta)f_1(s'+l+\theta)e^{2\pi it'}\}. \end{aligned} \tag{4.26}$$

Observe that by the hypothesis $f_1(s+l)f_1(s+l+\theta) = 0$. Since $|supp(f_1)| < \frac{\epsilon}{4}$, we have $f_1(s+l)f_1(s'+l+\theta) = 0$. Thus we have $f_1^{(2)}(l)f_1^{(2)}(l+\theta) = 0$.

Likewise, observe that

$$\begin{aligned} f_{-1}^{(2)}(l)f_{-1}^{(2)}(l+\theta) = &\{f_{-1}(s+l)f_0(s'+l+\theta)e^{-2\pi it} + f_0(s+l)f_{-1}(s'+l)e^{-2\pi it'}\} \\ &\times \{f_{-1}(s+l+\theta)f_0(s'+l+2\theta)e^{-2\pi it} + f_0(s+l+\theta)f_{-1}(s'+l+\theta)e^{-2\pi it'}\}. \end{aligned} \tag{4.27}$$

Using the fact that $f_{-1}(l) = \overline{f_1(l+\theta)}$, $|supp(f_1)| < \frac{\epsilon}{4}$ and arguing as before, we have $f_{-1}^{(2)}(l)f_{-1}^{(2)}(l+\theta) = 0$.

Thus by Lemma 4.4.5, $\{A_{s,t}(P) \cdot A_{s',t'}(P)\}^{2n} \in \mathfrak{X}$ for $n \geq 1$. Now by Lemma 4.4.3, the subspace $\mathfrak{X}$ is closed in SOT. Thus

$$\lim_{n\to\infty} \{A_{s,t}(P) \cdot A_{s',t'}(P)\}^{2n} \in \mathfrak{X};$$

i.e. $\left(A_{s,t}(P)\right) \bigwedge \left(A_{s',t'}(P)\right) \in \mathfrak{X}$. $\square$

Lemma 4.4.7. *Let* $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *and* $A = f_{-1}^{(A)}(U)V^{-1} + f_0^{(A)}(U) + f_1^{(A)}(U)V$ *be projections. Then* $A \leq A_{s,t}(P)$ *and* $A \leq A_{s',t'}(P)$ *if and only if the following hold:*

1. $f_1(s+l)f_1^{(A)}(l-\theta) = 0$;
2. $f_{-1}(s+l)f_{-1}^{(A)}(l+\theta) = 0$;
3. $f_0(s+l)f_0^{(A)}(l) + f_1(s+l)f_{-1}^{(A)}(l-\theta)e^{2\pi it} + f_{-1}(s+l)f_1^{(A)}(l+\theta)e^{-2\pi it} = f_0^{(A)}(l)$;

4. $f_1(s+l)f_0^{(A)}(l-\theta)e^{2\pi it} + f_0(s+l)f_1^{(A)}(l) = f_1^{(A)}(l)$;

5. $f_{-1}(s+l)f_0^{(A)}(l+\theta)e^{-2\pi it} + f_0(s+l)f_{-1}^{(A)}(l) = f_{-1}^{(A)}(l)$;

6. $f_1(s'+l)f_1^{(A)}(l-\theta) = 0$;

7. $f_{-1}(s'+l)f_{-1}^{(A)}(l+\theta) = 0$;

8. $f_0(s'+l)f_0^{(A)}(l)+f_1(s'+l)f_{-1}^{(A)}(l-\theta)e^{2\pi it'}+f_{-1}(s'+l)f_1^{(A)}(l+\theta)e^{-2\pi it'} = f_0^{(A)}(l)$;

9. $f_1(s'+l)f_0^{(A)}(l-\theta)e^{2\pi it'} + f_0(s'+l)f_1^{(A)}(l) = f_1^{(A)}(l)$;

10. $f_{-1}(s'+l)f_0^{(A)}(l+\theta)e^{-2\pi it'} + f_0(s'+l)f_{-1}^{(A)}(l) = f_{-1}^{(A)}(l)$;

for $l \in [0,1)$.

Proof. By comparing the coefficients of V^{-1}, V and 1 from the equations

$$A_{s,t}(P)A = A; \quad A_{s',t'}(P)A = A,$$

we have the following:
Relations (1),(2),(6),(7) follow from the fact that the coefficients of V^2 and V^{-2} are zero, since $A_{s,t}(P)A = A$ and $A_{s',t'}(P)A = A$ and A has no term involving V^2 and V^{-2}. By comparing the coefficients of 1 from the equations $A_{s,t}(P)A = A$ and $A_{s',t'}(P)A = A$, we have relations (3) and (8). Comparing the coefficients of V in $A_{s,t}(P)A = A$ and $A_{s',t'}(P)A = A$, we have relations (4) and (9). A comparison of the coefficients of V^{-1} in $A_{s,t}(P)A = A$ and $A_{s',t'}(P)A = A$ yields relations (5) and (10). □

Lemma 4.4.8. *For two projections A and B such that*

$$A = f_{-1}^{(A)}(U)V^{-1} + f_0^{(A)}(U) + f_1^{(A)}(U)V,$$

$$B = f_{-1}^{(B)}(U)V^{-1} + f_0^{(B)}(U) + f_1^{(B)}(U)V;$$

we have $A \leq B$ if and only if

1. $f_1^{(B)}(l)f_1^{(A)}(l-\theta) = 0$;

2. $f_1^{(B)}(l+\theta)f_1^{(A)}(l+2\theta) = 0$;

3. $f_0^{(B)}(l)f_0^{A}(l) + f_1^{(B)}(l)f_1^{(A)}(l) + f_1^{(B)}(l+\theta)f_1^{(A)}(l+\theta) = f_0^{(A)}(l)$;

4. $f_1^{(B)}(l)f_0^{(A)}(l-\theta) + f_0^{(B)}(l)f_1^{(A)}(l) = f_1^{(A)}(l)$;

5. $f_1^{(B)}(l+\theta)f_0^{(A)}(l+\theta)+f_0^{(B)}(l)f_1^{(A)}(l+\theta)=f_1^{(A)}(l+\theta)$;

for $l\in[0,1)$.

Proof. Consider the equation $BA=A$. Relations (1) and (2) follows from the fact that the coefficients of V^2 and V^{-2} are zero in A. A comparison of the coefficient of 1 yields relation (3). Relations (4) and (5) follow from a comparison between the coefficients of V and V^{-1} respectively. $\square$

Lemma 4.4.9. *Let* $P=f_{-1}(U)V^{-1}+f_0(U)+f_1(U)V$ *such that* P *is a projection and suppose* $f_0(t)=0$ *for some* t. *Then* $f_1(t)=f_1(t+\theta)=0$.

Proof. The fact that $P^2=P$ implies that

$$\begin{aligned} f_0(t)-(f_0(t))^2&=|f_1(t-\theta)|^2+|f_1(t)|^2 \text{ (see [18],page 173)},\\ f_0(t+\theta)-(f_0(t+\theta))^2&=|f_1(t)|^2+|f_1(t+\theta)|^2. \end{aligned} \tag{4.28}$$

The first expression in (4.28) implies that $f_1(t)=0$. Moreover we have

$$f_1(t+\theta)\left(1-f_0(t)-f_0(t+\theta)\right)=0\ [18,page173];$$

so that if $f_0(t+\theta)=0$ implies $f_1(t+\theta)=0$; else if $f_0(t+\theta)=1$, the second expression in (4.28) gives $f_1(t+\theta)=0$. $\square$

For a set $A\subseteq\mathbb{R}$ and real numbers $a\in\mathbb{R}$, $\tau_a(A):=A+a$.

Define functions f_0 and f_1 by: $f_0(t)=\begin{cases}\epsilon^{-1}t\ \ if\ 0\le t\le\epsilon\\ 1\ \ if\ \epsilon\le t\le\theta\\ \epsilon^{-1}(\theta+\epsilon-t)\ if\ \theta\le t\le\theta+\epsilon\\ 0\ if\ \theta+\epsilon\le t\le 1\end{cases}$

$f_1(t)=\begin{cases}\sqrt{f_0(t)-f_0(t)^2}\ if\ \theta\le t\le\theta+\epsilon\\ 0\ if\ otherwise.\end{cases}$

It is known (see [18]) that $P:=f_{-1}(U)V^{-1}+f_0(U)+f_1(U)V$ is a projection in $C^*(\mathbb{T}_\theta^2)$.

Theorem 4.4.10. *Let* $P=f_{-1}(U)V^{-1}+f_0(U)+f_1(U)V$ *be a projection with* f_0,f_1 *as described above. Consider the projections* $A_{s,t}(P)$, $A_{s',t'}(P)$ *such that* $|s-s'|<\frac{\epsilon}{4}$. *Then*

$$\left(A_{s,t}(P)\right)\bigwedge\left(A_{s',t'}(P)\right)=\chi_S(U),$$

for the set $S=X_1\cap X_2\cap X_3\cap X_4$, *where* $X_1=\tau_{-s}(\{x|f_1(x)=0\}),X_2:=\tau_{-s'}(\{x|f_1(x)=0\})$, $X_3:=\tau_{-s}(\{x|f_0(x)=1\})$ *and* $X_4:=\tau_{-s'}(\{x|f_0(x)=1\})$.

Proof. The hypothesis of the theorem and Lemma 4.4.6 together implies that

$$\left(A_{s,t}(P)\right) \bigwedge \left(A_{s',t'}(P)\right) \in \mathfrak{X}.$$

Let $B = \chi_S(U)$. Now in the notation of Lemma 4.4.7, let $f_1^{(A)} \equiv 0$ and $f_0^{(A)}(U) = \chi_S(U)$. Then relations (1),(2),(6),(7) of Lemma 4.4.7 trivially hold. Notice that $f_0(s+l) = f_0(s'+l) = 1$ for $l \in S$ which implies that relations (3) and (8) of Lemma 4.4.7 hold. Observe that Relations (4) and (9) follow by re-writing them with l replaced by $l+\theta$ and the fact that for $l \in S$, we have $f_1(s+l) = f_1(s'+l) = 0$ which implies by Lemma 4.4.9, that $f_1(s+l+\theta) = f_1(s'+l+\theta) = 0$. Re-writing relations (5) and (10) with $l+\theta$ replaced by l, we see that they hold because of the fact that for $l \in S$, $f_1(s+l) = f_1(s'+l) = 0$. Thus $B \leq A_{s,t}(P)$, $B \leq A_{s',t'}(P)$.

Again let $A = f_{-1}^{(A)}(U)V^{-1} + f_0^{(A)}(U) + f_1^{(A)}(U)V$ be a projection, such that $A \leq A_{s,t}(P)$ and $A \leq A_{s',t'}(P)$. Let $l \in X_1^c \cup X_2^c \cup X_3^c \cup X_4^c$. Then we have the following four mutually exclusive and exhaustive cases:

(a) $f_1(s+l) = 0, f_1(s'+l) = 0$;

(b) $f_1(s+l) \neq 0, f_1(s'+l) = 0$;

(c) $f_1(s+l) = 0, f_1(s'+l) \neq 0$;

(d) $f_1(s+l) \neq 0, f_1(s'+l) \neq 0$.

Relation (1) of Lemma 4.4.7 and cases (b) or (d) imply $f_1^{(A)}(l-\theta) = 0$ which, by Lemma 4.4.9 implies that $f_1^{(A)}(l) = 0$. Likewise, Relation (6) of Lemma 4.4.7 and cases (c) or (d) imply that $f_1^{(A)}(l) = 0$. Consider case (a). From the exact formulae of $f_1(\cdot)$, it follows that either $0 \leq s+l, s'+l \leq \theta$ or $\theta+\epsilon \leq s+l, s'+l \leq 1$. If $\theta+\epsilon \leq s+l, s'+l \leq 1$, we have $f_0(s+l) = 0 = f_0(s'+l)$. Thus relation (4) of Lemma 4.4.7 implies that $f_1^{(A)}(l) = 0$. If $0 \leq s+l, s'+l \leq \theta$, then we cannot have $\epsilon \leq s+l, s'++, \theta$. This is because, then $f_0(s+l) = 1 = f_0(s'+l)$ which implies that $l \in X_1 \cap X_2 \cap X_3 \cap X_4$, which is a contradiction. Thus we must have $0 \leq s+l, s'+l < \epsilon$. Relations (4) and (9) of Lemma 4.4.7 then yields $f_0(s+l)f_1^{(A)}(l) = f_0(s'+l)f_1^{(A)}(l)$. Using the exact formula for f_0 in the region $[0,\epsilon)$, we have $\frac{s+l}{\epsilon} f_1^{(A)}(l) = \frac{s'+l}{\epsilon} f_1^{(A)}(l)$. Since $s \neq s'$, this implies that we must have $f_1^{(A)}(l) = 0$.

Thus we have proved that $f_1^{(A)}(\cdot)$ vanishes outside S.

Note that relations (5) and (10) of Lemma 4.4.7 imply that for $l \in S^c$, we have $f_1(s+l)f_0^{(A)}(l) = 0$, $f_1(s'+l)f_0^{(A)}(l) = 0$. So cases (b),(c) and (d) implies that we must have $f_0^{(A)}(l) = 0$. Relations (3) and (8) of Lemma 4.4.7 imply that $f_0(s+l)f_0^{(A)}(l) = f_0(s'+l)f_0^{(A)}(l)$. Thus case (a) implies that if $f_0^{(A)}(l) \neq 0$, we must have $f_0(s+l) = f_0(s'+l)$ which implies that $f_0(s+l) = 0 = f_0(s'+l)$ since $l \in S^c$. Relation (3) of Lemma 4.4.7 implies that $f_0(s+l)f_0^{(A)}(l) = f_0^{(A)}(l)$ and $f_0(s+l) = 0$ implies that $f_0^{(A)}(l) = 0$, contrary to our assumption $f_0^{(A)}(l) \neq 0$. Thus we must have $f_0^{(A)}(l) = 0$. Thus we have proven that $f_0^{(A)}(\cdot)$ also vanishes outside S.

Now let A be as above and $B := \chi_S(U)$. An application of Lemma 4.4.9 implies trivially that $A \leq B$. Thus B is the largest among all projections in $\mathfrak{X}$ which are smaller than both $A_{s,t}(P)$ and $A_{s',t'}(P)$. Thus $B = A_{s,t}(P) \bigwedge A_{s',t'}(P)$. □

Observe that $S = ([\epsilon,\theta] - s) \cap ([\epsilon,\theta] - s')$.

It is worthwhile to note that the conclusion of the above theorem holds if we replace U by U^k, V by V^k, and θ by $\{k\theta\}$ ($\{\cdot\}$ denoting the fractional part).

Let $P_n = f_{-1}^{(k_n)}(U^{k_n}) + f_0^{(k_n)}(U^{k_n}) + f_1^{(k_n)}(U^{k_n})U^{k_n}$, be projections as described in [18, page 173] such that $\{k_n\theta\} \to 0$. Put $\epsilon_n := \frac{\{k_n\theta\}}{2}$. Consider a standard Brownian motion in $\mathbb{R}^2$, given by $(W_t^{(1)}, W_t^{(2)})$. Define $j_t : W^*(\mathbb{T}_\theta^2) \to W^*(\mathbb{T}_\theta^2) \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbb{C}^2)))$ by $j_t(\cdot) := \alpha_{(e^{2\pi i W_t^{(1)}}, e^{2\pi i W_t^{(2)}})}(\cdot)$.

Theorem 4.4.11. *Almost surely, $\bigwedge_{s\leq t}(j_s(P_n)(\omega)) \in W^*(U)$, for all n, i.e.*

$$\bigwedge_{s\leq t}(j_s(P_n)) \in W^*(U) \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbb{C}^2))),$$

for each n, where $W^*(U)$ is the universal enveloping von-Neumann algebra of $C^*(U)$, $C^*(U)$ being the universal C^* algebra generated by a single unitary.

Proof. In the strong operator topology,

$$\bigwedge_{0\leq s\leq t}(j_s(P_n)) = \lim_{m\to\infty} \bigwedge_i \{j_{\frac{it}{2^m}}(P_n) \wedge j_{\frac{(i+1)t}{2^m}}(P_n)\}. \tag{4.29}$$

Now almost surely a Brownian path restricted to $[0,t]$ is uniformly continuous, so that the for sufficiently large m, and for almost all ω, $|W^{(1)}_{\frac{it}{2^m}} - W^{(1)}_{\frac{(i+1)t}{2^m}}|$ can be made small, uniformly for all i such that $i = 0,1,..2^m$. So $\bigwedge_i\{j_{\frac{it}{2^m}}(P_n) \wedge j_{\frac{(i+1)t}{2^m}}(P_n)\} = \bigwedge_i \chi_{[([\frac{\{k_n\theta\}}{2},\{k_n\theta\}]-W^{(1)}_{\frac{it}{2^m}}(\omega))\cap([\frac{\{k_n\theta\}}{2},\{k_n\theta\}]-W(\omega)^{(1)}_{\frac{(i+1)t}{2^m}})]}(U) \in W^*(U) \cap \mathfrak{X}$ by Theorem 4.4.10, for almost all ω. Now Lemma 4.4.3 implies that $W^*(U) \cap \mathfrak{X}$ is closed in the WOT-topology. Thus

$$\lim_{m\to\infty} \bigwedge_i \{j_{\frac{it}{2^m}}(P_n) \wedge j_{\frac{(i+1)t}{2^m}}(P_n)\} \in W^*(U) \cap \mathfrak{X}.$$

□

So, $\bigwedge_{0\leq s\leq t} j_s(P_n) \in W^*(U)\otimes L^\infty(\Omega, I\!\!P)(\cong L^\infty(\Omega, W^*(U)))$, where $(\Omega, I\!\!P)$ is the 2-dimensional Wiener probability space.

Thus for almost all $\omega \in \Omega$, we have:

$$\{\bigwedge_{0\leq s\leq t}(j_s(P_n))\}(U,\omega) = \chi_{[\frac{\{k_n\theta\}}{2},\{k_n\theta\}]-W_s^{(1)}(\omega):\ 0\leq s\leq t}\}(U).$$

Let $z_n = e^{2\pi i \frac{3\{k_n\theta\}}{4}}$. Consider the sequence of states $\phi_{z_n} := ev_{z_n}\circ E_1$. By [29], this is a sequence of pure states on $C^*(\mathbb{T}^2_\theta)$ converging in the weak-* topology to $\phi_1 := ev_1 \circ E_1$. Following the discussion in the beginning, consider

$$\langle e(0), (\phi_{z_n}\otimes 1)\circ \bigwedge_{0\leq s\leq t}(j_s(P_n))e(0)\rangle.$$

Using the fact that under the isomorphism $\Gamma(L^2(\mathbb{R}_+,\mathbb{C}^2)) \cong L^2(I\!\!P)$, $e(0)$ gets mapped to the constant function $\mathbf{1} \in L^2(I\!\!P)$ (see [52]), it follows from the discussion above that this is equal to

$$I\!\!P\{e^{2\pi i W_s^{(1)}} \in \mathcal{B},\ 0\leq s\leq t\} = I\!\!P\{\tau_{[\frac{-\{k_n\theta\}}{4},\frac{\{k_n\theta\}}{4}]} > t\},$$

where $\mathcal{B} := \{e^{2\pi i x} :\ x \in [\frac{-\{k_n\theta\}}{4}, \frac{\{k_n\theta\}}{4}]\}$. So we have a family of $(\tau_n)_n$ random times defined by

$$\tau_n([t,+\infty)) = \bigwedge_{0\leq s\leq t}(j_s(P_n));$$

so that $\int_0^t \langle e(0), (\phi_{z_n}\otimes 1)\circ \bigwedge_{0\leq s\leq t}(j_s(P_n))e(0)\rangle dt$ can be taken as the expectation of the random time τ_n. Note that here the analogue for balls of decreasing volume is $(P_n)_n$, such that
$tr(P_n) = \{k_n\theta\} \to 0$, tr being the canonical trace in $W^*(\mathbb{T}^2_\theta)$, and $\{k_n\theta\}$ being the volume of the projection P_n.

Note that viewing $\mathbb{T} \subseteq \mathbb{R}^2$, the radius (with respect to the metric of $\mathbb{R}^2$) of a ball intercepted between the points $e^{\frac{2\pi i a}{4}}$ and $e^{-\frac{2\pi i a}{4}}$ is given by $2sin\frac{a}{8}\sqrt{1+cos^2\frac{a}{8}} = 2x$ (say).

Thus by Proposition 4.2.3, we have

$$\begin{aligned}
&\int_0^t \langle e(0), (\phi_{z_n}\otimes 1)\circ \bigwedge_{0\leq s\leq t}(j_s(P_n))e(0)\rangle dt \\
&= I\!\!E(\tau_{[\frac{-\{k_n\theta\}}{4},\frac{\{k_n\theta\}}{4}]}) \\
&= 2x^2 + \frac{2}{3}x^4 + O\left(x^5\right)\ \text{ where } x := 2sin\frac{\{k_n\theta\}}{8}\sqrt{1+cos^2\frac{\{k_n\theta\}}{8}}, \text{ since the mean curvature of } \mathbb{T}\subseteq\mathbb{R}^2 \text{ is } 1.
\end{aligned} \tag{4.30}$$

In terms of the volume $\{k_n\theta\}$, applying the alternative expansion of the series as discussed after proposition 4.2.3, we have

$$\int_0^t \langle e(0), (\phi_{z_n} \otimes 1) \circ \bigwedge_{0 \leq s \leq t} (j_s(P_n))e(0)\rangle dt$$
$$= \frac{\{k_n\theta\}^2}{2^5} + \frac{\{k_n\theta\}^4}{2^{11}.3} + O(\{k_n\theta\}^5), \text{ since the mean curvature of the circle viewed inside } \mathbb{R}^2 \text{ is } 1. \tag{4.31}$$

Let us compare the final expansion with that given before the formulae 4.24 and 4.25. In the notation of the formulae 4.24 and 4.25, we have $n_0 = 1$. Moreover, since $\alpha_1 := 2$, we have $c_1 = \frac{1}{2^5}$, so that $d = 5$. Furthermore, we have $c_2 = \frac{1}{2^{11}.3}$, so that we have $H = \frac{1}{2\sqrt{2}}$.

Remark 4.4.12. *In view of equations (4.4),(4.24) and (4.25), we see that the 'intrinsic dimension' $n_0 = 1$, the 'extrinsic diimension' $d = 5$, and the 'mean curvature' is $\frac{1}{2\sqrt{2}}$. As we have already remarked in the introduction, the instrinsic one-dimensionality may be interpreted as a manifestation of the local one-dimensionality of the 'leaf space' of the Kronecker foliation (see [16] for details). It is worth pointing out that the spectral behaviour of the standard Dirac operator or the Laplacian coming from it for this noncommutative manifold is identical with that of the commutative two-torus, and thus it does not recognize the one-dimensionality of the leaf space of Kronecker foliation. Thus, it is a remarkable success of our (quantum) stochastic analysis using exit time to reveal the association of the noncommutative geometry of $\mathcal{A}_\theta$ with the leaf space of Kronecker foliation, and also to distinguish it from the commutative two-torus. All these give a good justification for developing a general theory of quantum stochastic geometry.*

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