INVARIANT SUBSPACES OF VECTOR-VALUED FUNCTION SPACES ON BOHR GROUPS

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INTRODUCTION

The theory of invariant subspaces of various function-spaces of complex-valued and vector-valued functions on the circle group is well known through the 'Lectures on Invariant Subspaces' by Helson ([4]). Replacing the circle group by a Bohr group B (that is, a compact abelian group whose dual is a subgroup of the real line R, dense in the topology of R), Helson and Lowdenslager initiated the study of invariant subspaces of $L_2(B)$ in [6]. They discovered that after suitable normalisation, the simply invariant subspaces of $L_2(B)$ are in one-to-one correspondence with a certain class of functions on $R \times B$, which are called cocycles. Further contributions came from many authors : Helson ([5] ; I, II), Gamelin ([2], ch. VII), Yale ([16]) and Helson and Kahane ([8]). Also the work of Helson and Lowdenslager on multivariate prediction ([7] ; I, II) has points of contact with this theory.

In this dissertation we extend certain parts of the above theory to invariant subspaces of L_2-spaces of Hilbert space-valued functions under the simplifying assumption that B has a countable dual. The Helson-Lowdenslager correspondence extends with the difference that the cocycles, now, are operator-valued. This is accomplished by resorting to the theory of systems of imprimitivity which is extensively employed in representation theory of locally compact groups.

The material is divided into three chapters. In the first chapter we go into the relationship of cocycles with systems of imprimitivity for the special case where a second countable locally compact abelian group acts densely into another. This leads to dual systems of imprimitivity which are not available in the general case. Next, our notion of cocycles is wider than usual ; we allow them to have partial isometries as values. This permits us to describe systems of imprimitivity which do not act in the whole Hilbert space. We also obtain a generalisation of the structure of cocycles given by Gamelin ([2], ch. VII, sec. 11). We use this to discuss the irreducibility of certain systems of imprimitivity.

In chapter II we come to invariant subspaces. Here most of the work consists in recognising some structures, to which, then, the results of chapter I readily apply. The principal correspondence theorem between simply invariant subspaces and cocycles appears in section 6. In the same section we show that there are simply invariant subspaces of the vectorial L_2, which are not equivalent to a direct sum of simply invariant subspaces of the scalar L_2. In section 8 we obtain an analytic description of simply invariant subspaces in terms of cocycles similar to one given by Helson ([5], I) for the scalar case.

Chapter III deals with an application of the results of chapter II, to prediction theory. We consider a multivariate stationary stochastic process with time Γ (a countable dense subgroup of R). We find necessary and sufficient conditions for such a process to be purely non-deterministic. This is similar to the result of Helson and Lowdenslager ([7], II) for processes with integer time.

A sectionwise summary appears at the end of each chapter.

CHAPTER I

SYSTEMS OF IMPRIMITIVITY AND COCYCLES

1. The Basic Function Spaces

1.1. Let $(X, \underline{X})$ be a measurable space where $\underline{X}$ is a countably generated σ-algebra of subsets of X. Let $\mathbb{H}$ be a complex separable Hilbert space. A function F on X with values in $\mathbb{H}$ is said to be (weakly) measurable if for all $\eta \in \mathbb{H}$, the function $x \rightarrow (F(x), \eta)$ is measurable. Similarly, if $\mathbb{H}_1$ and $\mathbb{H}_2$ are complex separable Hilbert spaces, a function A on X whose values are bounded operators from $\mathbb{H}_1$ to $\mathbb{H}_2$ is said to be measurable if for arbitrary $\xi \in \mathbb{H}_1$ and $\eta \in \mathbb{H}_2$, the numerical function $x \rightarrow (A(x)\xi, \eta)$ is measurable.

Let $(\xi_1, \xi_2, \xi_3, \ldots)$ be a complete orthonormal basis of $\mathbb{H}$ (which is finite if $\mathbb{H}$ is finite-dimensional). With any function F on X taking values in $\mathbb{H}$ we associate a sequence of numerical functions $(f_1, f_2, f_3, \ldots)$ by setting $f_i(x) = (F(x), \xi_i)$, $x \in X$. We call f_i the i^{th} co-ordinate function of F with respect to the basis $(\xi_1, \xi_2, \xi_3, \ldots)$. Then $\sum_{i \geq 1} |f_i(x)|^2 = \| F(x) \|^2$ and if F is measurable, so is every f_i.

1.2. Let α be a non-negative σ-finite measure on $\underset{\sim}{X}$. By $L_2(X ; \mathbb{H}, \alpha)$ we shall mean the linear space of measurable functions $F : X \to \mathbb{H}$ for which

$$\int_X \| F(x) \|^2 \, d\,\alpha(x) < \infty ,$$

after identification of functions differing on α-null sets. $F \varepsilon L_2(X ; \mathbb{H}, \alpha)$ if and only if each f_i is measurable and $\| F(x) \|^2 = \sum_i |f_i(x)|^2 \varepsilon L_2(X, \alpha)$ where $L_2(X, \alpha)$ denotes the L_2 of complex-valued functions. (It follows that each $f_i \varepsilon L_2(X, \alpha)$.) Equipped with the inner product

$$(1.2.1) \qquad (F_1, F_2) = \int_X (F_1(x), F_2(x)) \, d\,\alpha(x),$$

$F_i \varepsilon L_2 (X ; \mathbb{H}, \alpha)$, $i = 1, 2$, $L_2(X ; \mathbb{H}, \alpha)$ is a Hilbert space which is separable as $\underset{\sim}{X}$ is countably generated. Analogous to $L_2(X, \alpha)$, $L_2(X ; \mathbb{H}, \alpha)$ has the property that from any norm-convergent sequence we can extract a subsequence which converges almost everywhere.

REMARK. For any Hilbert space, the inner product will be denoted by $(\cdot , \cdot)$ and the norm by $\| \cdot \|$. This should cause no confusion since it will be clear from the context which Hilbert space is referred to. For instance, in (1.2.1),

(F_1, F_2) refers to the inner product in $L_2(X ; \mathbb{H}, \alpha)$ and $(F_1(x), F_2(x))$ to the inner product in $\mathbb{H}$.

2. Spectral Measures

2.1. By a spectral measure on $(X, \underline{X})$ (or on X, when there is no confusion about the σ-algebra $\underline{X}$), we mean a countably additive set function P on $\underline{X}$, whose values are orthogonal projections in a separable Hilbert space $\mathbb{K}$, $P(X)$ being the identity projection.

We refer to Halmos ([3]) for general properties of a spectral measure and merely state here a few results of interest to us.

2.2 Let α be a non-negative, σ-finite measure on $\underline{X}$. For a subset $D \varepsilon \underline{X}$, define the operator $P(D)$ on $L_2(X ; \mathbb{H}, \alpha)$ by setting

$$P(D)F = 1_D . F , \quad F \varepsilon L_2(X ; \mathbb{H}, \alpha),$$

where 1_D is the characteristic function of D. Then $P(D)$ is a projection on $L_2(X ; \mathbb{H}, \alpha)$. $D \rightarrow P(D)$ defines a spectral measure P on $(X, \underline{X})$. We call P the canonical spectral measure on $(X, \underline{X})$ acting in $L_2(X ; \mathbb{H}, \alpha)$. Let $\mathbb{H}_n$ stand for a fixed Hilbert space of dimension n, $1 \leq n \leq \infty$. We denote by $P^{(n,\alpha)}$ (or by $P^{(n)}$ when α is known from the context) the canonical spectral measure on $(X, \mathfrak{X})$ acting in $L_2(X, \mathbb{H}_n, \alpha)$, $1 \leq n \leq \infty$.

2.3. The canonical spectral measures derive their importance from the following fundamental theorem about spectral measures.

HELLINGER-HAHN THEOREM. Let Q be a spectral measure on $(X, \mathfrak{X})$, the values of Q being orthogonal projections in a separable Hilbert space $\mathbb{K}$. Then there exist mutually singular numerical measures α_∞, α_1, α_2, .. on X (each non-negative σ-finite and some of them may possibly be the zero-measure) and an invertible isometry S of $\mathbb{K}$ onto the direct sum $\bigoplus_{1\leq n\leq\infty} L_2(X; \mathbb{H}_n, \alpha_n)$ such that for each $D \in \mathfrak{X}$,

$$S\,Q(D)S^{-1} = \bigoplus_{1\leq n\leq\infty} P^{(n,\alpha_n)}(D).$$

Further, if α'_∞, α'_1, α'_2, ... be another sequence of mutually singular measures on X such that for some invertible isometry S' of $\mathbb{K}$ onto $\bigoplus_{1\leq n\leq\infty} L_2(X; \mathbb{H}_n, \alpha'_n)$,

$$S'\,Q(D)\,S'^{-1} = \bigoplus_{1\leq n\leq\infty} P^{(n,\alpha'_n)}(D), \quad D \in \mathfrak{X},$$

then for each n, α_n and α'_n are mutually absolutely continuous.

For any measure α on $(X, \underline{X})$ we shall write $\vec{\alpha}$ to denote the class of all measures on $(X, \underline{X})$ mutually absolutely continuous with respect to α and call $\vec{\alpha}$ the measure class of α. The Hellinger-Hahn theorem assigns to each spectral measure Q a uniquely determined sequence of mutually singular measure classes $(\bar{\alpha}_{\infty}, \bar{\alpha}_1, \bar{\alpha}_2, \ldots)$ in a particular order so that $\alpha_{\infty}, \alpha_1, \alpha_2, \ldots$ satisfy the conditions of the theorem. In case for a spectral measure Q there exists n, $1 \leq n \leq \infty$ such that $\bar{\alpha}_k = \bar{0}$ for all $k \neq n$, then Q is said to be <u>homogeneous of multiplicity n</u>.

We say that a spectral measure Q is absolutely continuous with respect to a non-negative measure α if $Q(D) = 0$, whenever $\alpha(D) = 0$, $D \in \underline{X}$. In this case α_k is absolutely continuous with respect to α for each k.

2.4. Let Q_1 and Q_2 be spectral measures on $(X, \underline{X})$, their values being orthogonal projections in Hilbert spaces $\mathbb{K}_1$ and $\mathbb{K}_2$ respectively. Q_1 and Q_2 are said to be <u>unitarily equivalent</u> if there exists an invertible isometry S of $\mathbb{K}_1$ onto $\mathbb{K}_2$ such that $S\, Q_1(D)\, S^{-1} = Q_2(D)$ for all $D \in \underline{X}$. Now it can be shown that Q_1 and Q_2 are unitarily equivalent if and only if they determine the same sequence of measure classes (in the same order) in Hellinger-Hahn theorem.

2.5. DEFINITION. By a range function J on X we mean a function on X whose values are closed subspaces of $\mathbb{H}$. J determines an operator-function R on X, $R(x)$ being the projection, defined in $\mathbb{H}$, onto $J(x)$. J is said to be measurable if R is measurable.

The notion of range functions is going to be important for us. Most of our considerations will involve them in some way or other. The notion was introduced by Helson and Lowdenslager ([7], II) in their work on multivariate prediction. Investigations on special types of range functions were made by Cambern, a report of which appears in Helson's book ([4]). In the same book, range functions are used, among other things, to describe doubly invariant subspaces, objects we shall consider in Chapter II of this dissertation. Here we shall employ them to describe projections in $L_2(X ; \mathbb{H}, \alpha)$ which commute with the canonical spectral measure P, that is, to describe subspaces which reduce every $P(D)$, $D \in \mathfrak{X}$.

We give a method of obtaining measurable range functions which will be frequently used. For any subset $\mathfrak{L}$ of a Hilbert space, $\mathfrak{S}(\mathfrak{L})$ will denote the closed linear span of $\mathfrak{L}$.

LEMMA. Let $\{F_n : n \geq 1\}$ be a finite or countable set of measurable functions on X with values in $\mathbb{H}$. Then the range function J, defined by

$$J(x) = \underline{C}\ (\{F_n(x) : n \geq 1\}),\quad x \in X,$$

is measurable.

PROOF. Let $(G_1, G_2, G_3, \ldots)$ be obtained from $(F_1, F_2, F_3, \ldots)$ by pointwise Gram-Schmidt orthogonalisation. That is, for each fixed $x \in X$, set

$$G_1'(x) = F_1(x)$$

$$G_1(x) = G_1'(x)/\|G_1'(x)\| \quad \text{if} \quad G_1'(x) \neq 0,$$

$$= \quad 0 \quad \text{if} \quad G_1'(x) = 0,$$

and after defining $G_1(x), \ldots, G_{n-1}(x)$,

$$G_n'(x) = F_n(x) - \sum_{k=1}^{n-1} (F_n(x),\ G_k(x))\ G_k(x)$$

$$G_n(x) = G_n'(x)/\|G_n'(x)\| \quad \text{if} \quad G_n'(x) \neq 0$$

$$= \quad 0 \quad \text{if} \quad G_n'(x) = 0.$$

It is clear that each G_n is measurable and $J(x) = \underline{C}\ (\{G_n(x): n \geq 1\}$. Let R be the projecti operator-function associated to J. Now for each $\xi, \xi' \in \mathbb{H}$,

$$(R(x)\,\xi,\ \xi') = \sum_{n\geq 1} (\xi,\ G_n(x))\,(G_n(x),\ \xi'),\ x \in X.$$

From the measurability of the functions G_n, the right hand side is measurable. Therefore, R is weakly measurable and hence, by definition, J is measurable. QED

2.6. Consider a measurable range function J on X having R as the associated projection function. If $(\xi_1, \xi_2, \ldots)$ is dense in $\mathbb{H}$, then the functions $(R(\cdot)\,\xi_1,\ R(\cdot)\,\xi_2, \ldots)$ will define J as in the lemma above. Let now $F_1, F_2, F_3, \ldots$ be obtained from $(R(\cdot)\,\xi_1,\ R(\cdot)\,\xi_2, \ldots)$ by point-wise Gram-Schmidt orthogonalisation. It follows that the dimension function $n(x) = \dim J(x) = \sum_{i\geq 1} \|F_i(x)\|$, $x \in X$, is measurable on X. Further, if α is a measure on $\mathfrak{X}$, and $J(x) \neq \{0\}$ on a set of positive measure, then at least one of the F_i's will not vanish almost everywhere and, if necessary, by changing this function outside a set of finite measure, we can claim that there exists $F \in L_2(X\,;\ \mathbb{H},\ \alpha)$, $F \neq 0$, such that $F(x) \in J(x)$ a.e.

We shall now examine a special case where J has constant dimension (finite or denumerable), $n(x) = n$, for all $x \in X$.

For each $x \in X$, the nonzero elements in $(F_1(x), F_2(x), \ldots)$, say, $(F^{(x)}_{n_1(x)}, F^{(x)}_{n_2(x)}, \ldots)$, $n_1(x) < n_2(x) < \ldots$, form an an orthonormal basis of $J(x)$. Since each F_i $i = 1,2,\ldots$ is measurable, the functions n_k, $k = 1,2,\ldots$ are also measurable and they all terminate at $k = n$ or go an indefinitely according as n is finite or infinite. Set

$$F_k'(x) = F_{n_k(x)}(x), \quad x \in X.$$

Then F_k' is a measurable function for each $k \geq 1$, and

$$\text{(i)} \quad \|F_k'(x)\| = 1, \quad k \geq 1,\ x \in X$$

$$(2.6.1)\text{(ii)} \quad (F_k'(x), F_{k'}'(x)) = 0, \quad k \neq k',\ x \in X$$

$$\text{(iii)} \quad \mathcal{G}(\{F_k'(x) : k \geq 1\}) = J(x), \quad x \in X.$$

A sequence of measurable functions $\{F_k' : k = 1,2,\ldots\}$ satisfying (i), (ii), (iii) will be called a basis of J. Note that J has a basis if and only if J has constant dimension. Next, if α is a measure on X and J has constant dimension on a set $X' \subsetneq X$ of full measure we can define a basis $\{F_k' : k \geq 1\}$ of J on X' and extending them arbitrarily on $X - X'$, (2.6.1) will be satisfied a.e. In this case we say that the family of extended functions defines a basis of J almost everywhere.

2.7. Consider the canonical spectral measure P on $(X, \mathfrak{X})$ acting in $L_2(X ; \mathbb{H}, \alpha)$. We ask the question : What are the subspaces of $L_2(X ; \mathbb{H}, \alpha)$ that reduce $P(D)$ for every $D \in \mathfrak{X}$? To answer this we need the following :

LEMMA. Let M be a subspace of $L_2(X ; \mathbb{H}, \alpha)$ which is invariant under every $P(D)$ for $D \in \mathfrak{X}$. Let $F \in L_2(X ; \mathbb{H}, \alpha)$ be orthogonal to M. Then for any $G \in M$, $(F(x), G(x)) = 0$ almost everywhere.

PROOF. Let $G \in M$ be fixed. For every $D \in \mathfrak{X}$, $P(D)\, G \in M$ and hence $(F, P(D)\, G) = 0$. This means

$$\begin{aligned}(F, P(D)\, G) &= \int_X (F(x), 1_D(x)\, G(x))\, d\alpha(x) \\ &= \int_D (F(x), G(x))\, d\alpha(x) \\ &= 0\end{aligned}$$

for each $D \in \mathfrak{X}$. Therefore, $(F(x), G(x)) = 0$ a.e.

QED

Consider a measurable range function J on X whose values are closed subspaces of $\mathbb{H}$. Write

(2.7.1) $\quad J\,L_2(X\,;\,\mathbb{H},\,\alpha) = \{F \in L_2(X\,;\,\mathbb{H},\alpha) : F(x) \in J(x) \text{ a.e.}\}$

Clearly $J\,L_2(X\,;\,\mathbb{H},\,\alpha)$ is linear. Further $J\,L_2(X\,;\,\mathbb{H},\,\alpha)$ is a closed subspace, because if

$$F_n \in J\,L_2(X\,;\,\mathbb{H},\alpha),\ n \geq 1, \quad \text{and} \quad F_n \xrightarrow[n\to\infty]{} F,$$

then there exists a subsequence F_{n_k}, $k \geq 1$ such that $F_{n_k}(x) \xrightarrow[k\to\infty]{} F(x)$ for a.e. x and hence $F \in J\,L_2(X;\,\mathbb{H},\alpha)$. We note that $J\,L_2(X\,;\,\mathbb{H},\,\alpha)$ is invariant under $P(D)$ for every $D \in \mathfrak{X}$. Conversely we have

2.8. THEOREM. Let M be a subspace of $L_2(X\,;\,\mathbb{H},\,\alpha)$ which is invariant under every $P(D)$, $D \in \mathfrak{X}$. Then there exists a measurable range function J such that $M = J\,L_2(X\,;\,\mathbb{H},\,\alpha)$. Further, J is determined uniquely almost everywhere.

PROOF. Since $L_2(X\,;\,\mathbb{H},\,\alpha)$ is separable, M has a countable dense set $\{F_1, F_2, \ldots\}$. Let J be the measurable range function obtained from F_i, $i \geq 1$, as in Lemma, 2.5. Consider the subspace $J\,L_2(X\,;\,\mathbb{H},\,\alpha)$. Obviously $F_i \in J\,L_2(X\,;\,\mathbb{H},\,\alpha)$ for each i and hence $M \subseteq J\,L_2(X\,;\,\mathbb{H},\,\alpha)$. Let now $F \in J\,L_2(X\,;\,\mathbb{H},\,\alpha)$ be orthogonal to M. By the lemma above for all $i \geq 1$, $(F(x), F_i(x)) = 0$ a.e. By definition of J this means that $F(x)$ is orthogonal to $J(x)$ for a.e. x. But $F \in J\, L_2(X ; \mathbb{H}, \alpha)$. Therefore, $F = 0$ and $M = J\, L_2(X ; \mathbb{H}, \alpha)$.

Next, if J' is any other measurable range function such that $M = J'L_2(X ; \mathbb{H}, \alpha)$, then since $F_i \in J'L_2(X ; \mathbb{H}, \alpha), i=1,2,\ldots,$ we have $J(x) \subseteq J'(x)$ for a.e. x. Defining

$$(J'-J)(x) = J'(x) \ominus J(x), \quad x \in X,$$

if $F \in (J'-J)\, L_2(X ; \mathbb{H}, \alpha)$, then $F \in J'L_2(X ; \mathbb{H}, \alpha)$ and $F \perp J\, L_2(X ; \mathbb{H}, \alpha)$. Therefore, $F = 0$ and hence $(J'-J)(x) = 0$ a.e.x. This proves the uniqueness (upto a null set) of J.

QED

REMARK. The definition (2.7.1) is meaningful even if J is not measurable. The same proof shows that $J\, L_2(X ; \mathbb{H}, \alpha)$ is closed while it is obvious that $J\, L_2(X ; \mathbb{H}, \alpha)$ reduces every $P(D)$, $D \in \mathfrak{X}$. Then by the theorem above there exists a measurable range function J' such that $J\, L_2(X ; \mathbb{H}, \alpha) = J'L_2(X ; \mathbb{H}, \alpha)$. We further have

(a) $J'(x) \subseteq J(x)$ a.e.x.,

(b) if J'' is a measurable range function such that $J''(x) \subseteq J(x)$ a.e., then $J''(x) \subseteq J'(x)$ a.e., and

(c) if $F \varepsilon L_2(X ; \mathbb{H}, \alpha)$ is such that $F(x) \varepsilon J(x)$ a.e., then $F(x) \varepsilon J'(x)$ a.e.

2.9 Let J be a measurable range function on X whose values are closed subspaces of $\mathbb{H}$. Write

$$P^J(D)\ F = 1_D . F$$

for every $D \varepsilon \mathfrak{X}$ and $F \varepsilon J\, L_2(X ; \mathbb{H}, \alpha)$. Then P^J is a spectral measure on X whose values are orthogonal projections in $J\, L_2(X ; \mathbb{H}, \alpha)$. P^J is, in fact, the restriction of the canonical spectral measure P on the subspace $J\, L_2(X ; \mathbb{H}, \alpha)$.

It is clear that P^J is absolutely continuous with respect to α. The question when P^J is homogeneous with $\bar{\alpha}$ as the associated measure class is answered by

THEOREM. The spectral measure P^J is homogeneous with $\bar{\alpha}$ as the associated measure class if and only if J has constant dimension almost everywhere (α). In that case the multiplicity n of P^J is the dimension of J.

PROOF. Assume that P^J is homogeneous with multiplicity n and measure class $\bar{\alpha}$. We take α to be a finite measure in the measure class $\bar{\alpha}$. There exists an isometry S of $L_2(X ; \mathbb{H}_n, \alpha)$ onto $J\, L_2(X ; \mathbb{H}, \alpha)$ such that

(2.9.1) $$S\, P^{(n)}\, S^{-1} = P^{J}$$

where $P^{(n)}$ is the canonical spectral measure acting in $L_2(X\,;\,\mathbb{H}_n,\,\alpha)$. Let $(\xi_1,\,\xi_2,\,\ldots)$ be an orthonormal basis of $\mathbb{H}_n$. We write ξ_i for the constant function equal to ξ_i and set

$$F_i = S(\xi_i),\ i = 1, 2,\ \ldots$$

Each $F_i \in J\,L_2(X\,;\,\mathbb{H},\,\alpha)$ and by (2.9.1) we have for $D \in \underline{X}$,

$$S(1_D.\xi_i) = 1_D.S(\xi_i) = 1_D.F_i,\ i = 1, 2,\ \ldots$$

Since S is an isometry we have for each $D \in \underline{X}$

$$\int_D (F_i(x),\ F_j(x))\, d\alpha(x) = \int_X (1_D.F_i(x), F_j(x))\, d\alpha(x)$$

$$= \int_X (1_D(x)\ \xi_i, \xi_j)\, d\alpha(x)$$

$$= \begin{cases} \alpha(D) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

That is, $(F_i(x),\ F_j(x)) = \begin{cases} 0 & \text{a.e. if } i \neq j \\ 1 & \text{a.e. if } i = j \end{cases}$

Therefore, if J' is the measurable range function defined by the functions F_i, $i = 1, 2, \ldots$, then $(F_1, F_2, F_3,\ \ldots)$ is a basis of J' almost everywhere. We shall now show that $J'(x) = J(x)$ a.e. This will prove that J has a basis a.e. and hence is of constant dimension a.e. To that end observe that the subspace $J' L_2(X ; \mathbb{H}, \alpha) \subseteq J\, L_2(X ; \mathbb{H}, \alpha)$ is invariant under every $P^J(D)$, $D \in \mathfrak{X}$. Hence $S^{-1}(J' L_2(X ; \mathbb{H}, \alpha))$ is invariant under every $P^{(n)}(D)$, $D \in \mathfrak{X}$ and further, contains each constant function. Thus $1_D \cdot \xi_i \in S^{-1}(J' L_2(X ; \mathbb{H}, \alpha))$ for each $D \in \mathfrak{X}$ and $i = 1,2,\ldots$ Therefore, $S^{-1}(J' L_2(X ; \mathbb{H}, \alpha)) = L_2(X ; \mathbb{H}_n, \alpha)$ and hence $J' L_2(X ; \mathbb{H}, \alpha) = J\, L_2(X ; \mathbb{H}, \alpha)$. It follows from the uniqueness of J, that $J'(x) = J(x)$ a.e.

Conversely, Suppose that J has constant dimension a.e. and $(F_1, F_2, \ldots)$ is a basis of J a.e. We define a measurable operator function s on X as follows. Let $(\xi_1, \xi_2, \ldots)$ be a basis of $\mathbb{H}_n$ where n is the dimension of J. For $x \in X$ where $(F_1, F_2, \ldots)$ satisfies conditions (i), (ii) and (iii) of (2.6.1) set

$$s(x)\, \xi_i = F_i(x) , \quad i = 1,2, \ldots$$

$s(x)$ extends to an isometry of $\mathbb{H}_n$ onto $J(x)$. At other points x (on a set of measure zero) define $s(x)$ to be equal to a fixed isometry of $\mathbb{H}_n$ into $\mathbb{H}$. Now s induces an isometry S of $L_2(X ; \mathbb{H}_n, \alpha)$ onto $J\, L_2(X ; \mathbb{H}, \alpha)$ by the formula

$$(S\ F)\ (x) = s(x)\ F(x),\ F \in L_2(X\ ;\ \mathbb{H}_n,\ \alpha),$$

which satisfies

$$S(1_D . F) = 1_D\ S(F)$$

for each $D \in \mathfrak{X}$ and $F \in L_2(X\ ;\ \mathbb{H}_n,\ \alpha)$. That is,

$$S\ P^{(n)}\ S^{-1} = P^J\ .$$

QED

2.10. COROLLARY. Let P^J be homogeneous with multiplicity n and having the measure class $\bar{\alpha}$. Consider an invertible isometry S of $L_2(X\ ;\ \mathbb{H}_n\ \alpha)$ onto $J\ L_2(X\ ;\ \mathbb{H},\ \alpha)$ such that

(2.10.1) $$S\ P^{(n)}\ S^{-1} = P^J.$$

Then there exists a measurable operator-function s on X, where $s(x)$ is an isometry of $\mathbb{H}_n$ onto $J(x)$ for a.e. x, such that

$$(S\ F)\ (x) = s(x)\ F(x)$$

for each $F \in L_2\ (X\ ;\ \mathbb{H}_n,\ \alpha)$, for a.e. x.

PROOF. Assume for the moment that α is a finite measure. As in the proof of the preceding theorem $F_i = S\ \xi_i$, $i = 1,2,\ldots$ define a basis of J a.e., where $(\xi_1, \xi_2, \ldots)$ is an orthonormal basis of $\mathbb{H}_n$. Then from the second part of the proof of the theorem we get the measurable operator function s on X where

$$s(x)\,\xi_i = F_i(x), \text{ a.e.,}$$

for $i = 1, 2, \ldots$. By (2.10.1) we have for any $D \in \mathfrak{X}$ and $i \geq 1$

$$\begin{aligned}[S(1_D \cdot \xi_i)](x) &= 1_D(x)\ (S\,\xi_i)\ (x)\\ &= 1_D(x)\ s(x)\ \xi_i\\ &= s(x)\ (1_D(x)\ \xi_i)\ , \quad \text{a.e.x.}\end{aligned}$$

If S' stands for the operator

$$(S'F)\ (x) = s(x)\ F(x)$$

for each $F \in L_2(X\ ;\ \mathbb{H}_n\ ,\ \alpha)$, for a.e.x , we find that S and S' agree on the subset $\{1_D \cdot \xi_i : D \in \mathfrak{X},\ i = 1, 2, \ldots\}$ which is dense in $L_2(X\ ;\ \mathbb{H}_n,\ \alpha)$. Hence $S = S'$.

Next, if α is any σ-finite measure, choose a finite measure α_o from the measure class of α . Let ω denote the Radon-Nikodym derivative of α_o with respect to α . Then multiplication by $\omega^{1/2}$ is an invertible isometry of $L_2(X\ ;\ \mathbb{H}_n\ ,\ \alpha)$ onto $L_2(X\ ;\ \mathbb{H}_n,\ \alpha_o)$ as well as of $J\ L_2(X\ ;\ \mathbb{H}_n,\ \alpha)$ onto $J\ L_2(X\ ;\ \mathbb{H}_n\ ,\ \alpha_o)$. This reduces the problem to the previous case.

QED

3. Representations of Locally Compact Abelian Groups.

3.1 Let G be a second countable locally compact abelian group and let $\mathbb{K}$ be a separable Hilbert space. By a representation of G acting in $\mathbb{K}$ we mean a continuous homomorphism U of G into $\underline{U}(\mathbb{K})$ with the weak topology on the latter. It is well known that for a homomorphism $U : G \to \underline{U}(\mathbb{K})$ to be continuous it is enough to have U weakly measurable.

For any locally compact abelian group G, we shall write $\hat{G}$ to denote the dual group. For $x \in G$, $\hat{x} \in \hat{G}$, $< x, \hat{x} >$ will have the dual meaning : the character $\hat{x}$ evaluated at x or the character x evaluated at $\hat{x}$.

3.2 STONE'S THEOREM. If $U : g \to U_g$ be a representation of G acting in $\mathbb{K}$, then there exists a unique spectral measure P on the Borel subsets of $\hat{G}$ acting in $\mathbb{K}$ such that for all $g \in G$,

$$U_g = \int_{\hat{G}} < -x, g > dP(x) \tag{3.2.1}$$

in the sense that for any $\xi, \eta \in \mathbb{K}$,

$$(U_g \xi, \eta) = \int_{\hat{G}} < -x, g > d\,\alpha_{\xi,\eta}(x)$$

where $\alpha_{\xi,\eta}$ is the numerical measure on $\hat{G}$ defined by $\alpha_{\xi,\eta}(D) = (P(D))\xi, \eta)$ for every Borel subset $D \subseteq \hat{G}$.

We refer to Loomis ([11], Art. 36) for a proof of Stone's theorem.

Two representations of G, U and V, acting respectively in $\mathbb{K}_1$ and $\mathbb{K}_2$ are said to be equivalent if there exists an invertible isometry $S : \mathbb{K}_1 \to \mathbb{K}_2$ such that for all $g \in G$, $S\, U_g\, S^{-1} = V_g$. If the spectral measures P and Q correspond respectively to U and V in (3.2.1), then it is known that U and V are equivalent if and only if P and Q are equivalent and that a subspace $\mathbb{K}_1 \subseteq \mathbb{K}$ is invariant under U_g for each $g \in G$ if and only if $\mathbb{K}_1$ is invariant under $P(D)$ for each Borel subset D of $\hat{G}$.

3.3 EXAMPLE. Let σ be the Haar measure on G. Take $\mathbb{K} = L_2(G ; \mathbb{H}, \sigma)$. For $g \in G$, define U_g by

$$(U_g F)(x) = F(x+g), \quad x \in G, \quad F \in L_2(G ; \mathbb{H}, \sigma).$$

Then U_g is a unitary operator on $L_2(G ; \mathbb{H}, \sigma)$ and $g \to U_g$ is a representation of G, known as the regular representation of G acting in $L_2(G ; \mathbb{H}, \sigma)$.

4. The Pair (G_o, G)

4.1 A pair (G_o, G) will mean that

(i) G_o and G are locally compact second countable abelian groups and

(ii) there exists a one-to-one continuous homomorphism ϕ of G_o into G such that $\phi(G_o)$ is dense in G.

4.2 Given a pair (G_o, G) there arises another pair in a natural way. Consider the dual groups $\hat{G}_o$ and $\hat{G}$ and the map $\hat{\phi} : \hat{G} \to \hat{G}_o$ defined by

$$< x, \hat{\phi}(\hat{y}) > = < \phi(x), \hat{y} >, \ \hat{y} \in \hat{G}, \ x \in G_o.$$

$\hat{\phi}$ is a continuous homomorphism of $\hat{G}$ into $\hat{G}_o$. Further, if $\hat{y}_1$ and $\hat{y}_2$ are distinct elements of $\hat{G}$, then from the fact that $\phi(G_o)$ is dense in G, there exists $x \in G_o$ such that $< \phi(x), \hat{y}_1 > \neq < \phi(x), \hat{y}_2 >$, whence $< x, \hat{\phi}(\hat{y}_1) > \neq < x, \hat{\phi}(\hat{y}_2) >$. Thus $\hat{\phi}(\hat{y}_1) \neq \hat{\phi}(\hat{y}_2)$, that is, $\hat{\phi}$ is one-to-one. Next, $\hat{\phi}(\hat{G})$ is dense in $\hat{G}_o$. Because, if not, there exists $x \in G_o$, $x \neq 0$ such that $< x, \hat{\phi}(\hat{y}) > = 1$ for all $\hat{y} \in \hat{G}$. But this means that $< \phi(x), \hat{y} > = 1$ for all $\hat{y} \in \hat{G}$ and hence $\phi(x) = 0$. ϕ being one-to-one this is a contradiction.

The pair $(\hat{G}, \hat{G}_o)$ will be called the dual pair of (G_o, G).

4.3 **EXAMPLES.** (i) For any locally compact second countable abelian group H, (H, H) is an obvious pair, ϕ being the identity map.

(ii) Let T be the circle group

$$T = \{ z : z \text{ is a complex no. with } |z| = 1\}$$

and N the group of integers. Choose $\theta \in T$ such that θ has an irrational argument. Define $\phi : N \to T$ by

$$\phi(n) = \theta^n, \ n \in N.$$

It is well-known that $\{\theta^n, n \in N\}$ is dense in T. Thus we have a pair (N, T). Notice that the dual pair of (N, T) is (N, T) itself.

4.4. The more important pairs for us come from the Bohr groups. A Bohr group B is a compact abelian group whose discrete dual $\hat{B} = \Gamma$ is a subgroup of the additive group of real numbers R, dense in the usual topology of R. We shall consider only such B as has a countable dual group Γ. Then both B and Γ are second countable. Now the identity map of Γ into R is a one-to-one (continuous) homomorphism having a dense range. This gives us a pair (Γ, R). Again since $\hat{R} = R$, the dual pair of (Γ, R) is (R, B). The continuous homomorphism of R into B will be denoted by $t \to e_t$. The subgroup

$\{ e_t, t \in R \}$ is dense in B and the elements e_t are identified by the relation $\langle e_t, \delta \rangle = \exp(it\delta)$, $t \in R$, $\delta \in \Gamma$.

4.5. The Bohr groups give yet another class of pairs of considerable interest to us. This comes out of a direct product decomposition of a Bohr group B. This decomposition was introduced by Hoffman ([10]) and also considered by Gamelin ([2], ch. VIII). We shall follow the latter author.

Let B be a Bohr group with Γ countable. Assume, for simplicity, that $2\pi \in \Gamma$. Let K be the subgroup of B defined by

$$K = \{x \in B : \langle x, 2\pi \rangle = 1\}.$$

K is a compact subgroup of B. Note that an element $e_t \in K$ if and only if t is an integer. Consider now the Borel subset $\{e_t, t \in [0, 1)\}$ of B. It contains exactly one point from each coset of K in B. Hence it is easy to see that each element $x \in B$ has a unique representation $x = y + e_t$, $y \in K$, $t \in [0, 1)$. This gives a one-to-one bimeasurable mapping $(y, t) \longrightarrow x = y + e_t$ of $K \times [0,1)$ onto B. Furthermore, if σ_0 is the normalised Haar measure on K, the measure σ on B induced by the measure $\sigma_0 \times dt$ on $K \times [0,1)$, is easily seen to be translation invariant. Therefore, σ is the normalised Haar measure on B. Thus the measure space (B, σ) is identified with the product space $(K \times [0,1), \sigma_0 \times dt)$, the σ-algebras being the Borel σ-algebras.

Let $\Gamma_o = \{n\,2\pi : n \in N\}$. In standard terminology, K is the annihilator of Γ_o and, therefore, the dual of K is the quotient group Γ / Γ_o (see, for instance, Rudin [14], chapter II). Observe that as Γ is a dense subgroup of R, from the standard identification of the circle T with R/Γ_o, Γ/Γ_o is a dense subgroup of T. This gives us the pair $(\Gamma/\Gamma_o, T)$. Its dual is the pair (N, K) where the homomorphism is seen to be $n \longrightarrow e_n$ of N into K.

4.6. EXAMPLE. Let Γ be the subgroup of R

$$\Gamma = \{2\pi m + n : m,n \text{ integers}\}$$

Γ can be identified with $N \times N$ and its dual T^2 is conveniently written as $T^2 = [0,1) \times [0,2\pi)$ where elements (x_1,y_2) and (x_2,y_2) are added according to
$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2 (\text{mod } 1),\ y_1 + y_2 \ (\text{mod } 2\pi))$. We have

$$< 2\pi m + n,\ (x,y) > = \exp(i\ 2\pi\ mx).\ \exp(iny)$$

for $2\pi m + n \in \Gamma$ and $(x,y) \in T^2$. For $t \in R$,

$$< 2\pi m + n,\ e_t > = \exp(it\ (2\pi n + m)),\ m,n \in N,$$

gives $e_t = (t\ (\text{mod } 1),\ t (\text{mod } 2\pi))$. This describes the pair (R, T^2).

Next, the annihilator of the subgroup $\Gamma_0 = \{2\pi m : m \in N\}$ is the subgroup $K = \{0\} \times [0, 2\pi)$ and the pair (N, K) is described by $n \to (0, n(\text{mod } 2\pi))$. Thus in this case (N, K) can be identified with (N, T). Hence the dual pair $(\Gamma / \Gamma_0, T)$ can also be identified with (N, T).

4.7. Consider a pair (G_0, G). A non-negative σ-finite measure α on the Borel subsets of G is said to be quasi-invariant if for every $g \in G_0$, $\alpha(D) = 0$ if and only if $\alpha(D + \phi(g)) = 0$. A quasi-invariant measure α is said to be ergodic if for each Borel subset $D \subseteq G$ such that $\alpha(D \triangle D + \phi(g)) = 0$ for each $g \in G$, one has either $\alpha(D) = 0$ or $\alpha(D^c) = 0$. Equivalently, α is ergodic if for any Borel function f on G such that $f(x + \phi(g)) = f(x)$ almost everywhere for each g, one has f = constant a.e. It is well-known that if σ is the Haar measure on G, then σ is ergodic. We adapt below a lemma and a proposition from Gamelin's book ([2], ch. VII, sec.7) to prove a more general fact which will be useful later.

LEMMA. Let ν be a non-negative finite measure on G whose closed support is G. If E is a Borel subset of G of positive Haar measure, then $\nu(E + x) > 0$ for σ-almost every x.

PROOF. Enough to consider E such that $\sigma(E) < \infty$. The assertion of the lemma is equivalent to saying that

$h * \nu(x) > 0$ for σ-a.e. x where h is the indicator function of $-E$. Let D be a subset of finite σ-measure where $h * \nu$ vanishes. Denote by f the characteristic function of $-D$. Then $h * f$ is a continuous non-negative function on G. Since

$$\begin{aligned} \int_G h * f(-x)\, d\nu(x) &= (h * f * \nu)(0) \\ &= \int_G f(-x)\, h * \nu(x)\, d\sigma(x) \\ &= 0 \end{aligned}$$

and since the closed support of ν is G, $h * f = 0$. Therefore,

$$0 = \int h * f(x)\, d\sigma(x) = \sigma(E).\sigma(D).$$

This shows that $\sigma(D) = 0$, proving the lemma. QED

PROPOSITION. If f is a Borel function on G such that for σ-a.e. x, $f(x + \phi(g)) = f(x)$ for λ-a.e. g, then f is constant σ-almost everywhere. In particular, σ is ergodic.

PROOF. Let D be an arbitrary Borel subset of the complex plane. It suffices to show that $f^{-1}(D)$ has either zero σ-measure or full σ-measure.

By the condition on $f^{-1}(D)$ it follows that for a.e. x, the set $\{g \in G_o : x + \phi(g) \in f^{-1}(D)\}$ has either zero λ-measure of full λ-measure. Now consider a probability measure ν on G having support $\phi(G_o) \subseteq G$ such that ν is mutually absolutely continuous with respect to λ (regarded as a measure on $\phi(G_o)$). Then the closed support of ν is G. Suppose now $\sigma(f^{-1}(D)) > 0$. Then by the lemma, for σ-almost every $x \in G$, $\nu(f^{-1}(D) + x) > 0$. This implies that for σ-almost every x, $\nu(f^{-1}(D) + x) = 1$, that is, $f^{-1}(D)$ contains $x + \phi(g)$ for σ-a.e. x, for a.e. g. By Fubini's theorem there exists $g \in G_o$ such that $f^{-1}(D)$ contains $x + \phi(g)$ for a.e. x. Thus $f^{-1}(D)$ has full measure.

QED

5. Unitary Operator-Valued Cocycles

5.1. Let $\underline{U}(\mathbb{H})$ stand for the class of unitary operators on a complex separable Hilbert space $\mathbb{H}$. Consider the pair (G_o, G) with the Haar measures λ and σ on G_o and G respectively.

DEFINITION. By a $(G_o, G, \underline{U}(\mathbb{H}))$-cocycle A we mean a measurable function $A : G_o \times G \to \underline{U}(\mathbb{H})$ such that

$$(5.1.1) \quad A(g_1 + g_2, x) = A(g_1, x) \; A(g_2, x + \phi(g_1))$$

for a.e. (g_1, g_2, x) (with respect to the product measure $\lambda \times \lambda \times \sigma$ on $G_o \times G_o \times G$).

When there is no scope for confusion over the pair (G_o, G), we shall refer to a $(G_o, G, \underline{U}(\mathbb{H}))$-cocycle as a $\underline{U}(\mathbb{H})$-cocycle. Two cocycles are identified if they agree outside a $\lambda \times \sigma$-null set.

(5.2.2) $$A(g, x) = \rho(x)\, \rho^*(x + \phi(g)) \quad \text{a.e.}$$

for some measurable function $\rho : G \longrightarrow \underline{U}(\mathbb{H})$. The cocycle (5.2.2) is referred to as the coboundary of the function ρ. Similarly (5.2.1) is described by saying that A_1 is cohomologous (ρ) to A_2. In the lemma below we come across a situation where every cocycle is a coboundary.

LEMMA. If $\phi(G_o)$ has full measure in G, then every $\underline{U}(\mathbb{H})$-cocycle is a coboundary.

PROOF. Let A be a $(G_o, G, \underline{U}(\mathbb{H}))$ cocycle satisfying the cocycle identity (5.1.1) on a subset $D \subseteq G_o \times G_o \times G$ of full $\lambda \times \lambda \times \sigma$-measure. Choose an $x_o \in G$ such that for a.e. $(g_1, g_2) \in G_o \times G_o$, $(g_1, g_2, x_o) \in D$. Define the function ρ on G by setting

$$\rho(y) = \begin{cases} \rho(x_o + \phi(g_1)) = A^*(g_1, x_o), g_1 \in G_o, \text{if } y \in \phi(G_o) + x_o, \\ I \quad , \qquad \text{if } y \notin \phi(G_o) + x_o. \end{cases}$$

From (5.1.1),

(5.2.3) $$A(g_2, x_o + \phi(g_1)) = A^*(g_1, x_o)\, A(g_1 + g_2, x_o)$$

for a.e. $(g_1, g_2) \in G_o \times G_o$. Since $\phi(G_o)$ has full measure in G if in $y = x_o + \phi(g_1)$, g_1 runs over a set of full

(5.2.2) $$A(g, x) = \rho(x)\, \rho^*(x + \phi(g)) \quad \text{a.e.}$$

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$$\rho(y) = \begin{cases} \rho(x_o + \phi(g_1)) = A^*(g_1, x_o),\, g_1 \in G_o, & \text{if } y \in \phi(G_o) + x_o, \\ I, & \text{if } y \notin \phi(G_o) + x_o. \end{cases}$$

From (5.1.1),

(5.2.3) $$A(g_2, x_o + \phi(g_1)) = A^*(g_1, x_o)\, A(g_1 + g_2, x_o)$$

for a.e. $(g_1, g_2) \in G_o \times G_o$. Since $\phi(G_o)$ has full measure in G if in $y = x_o + \phi(g_1)$, g_1 runs over a set of full measure in G_0, then y also runs over a set of full measure in G. This follows from the properties of Haar measure. Therefore, (5.2.3) means

$$A(g_2, y) = \rho(y)\ \rho^*(y + \phi(g_2))$$

for a.e. $(g_2, y) \in G_0 \times G$. This proves the lemma.

.3 Strict Cocycles. A $(G_0, G, \underline{U}(\mathbb{H}))$-cocycle is called a strict cocycle if the identity (5.1.1) is satisfied for all $(g_1, g_2, x) \in G_0 \times G_0 \times G$.

Observe that the right hand side of (5.2.2) always defines a strict cocycle, so that a coboundary has a strict version.

LEMMA. If G_0 is a countable group, then every $\underline{U}(\mathbb{H})$ cocycle A has a strict version.

PROOF. Suppose the cocycle identity is satisfied for A on a subset $D \subseteq G_0 \times G_0 \times G$ of full $\lambda \times \lambda \times \sigma$-measure. Since $G_0 \times G_0$ is countable and hence every element $(g_1, g_2) \in G_0 \times G_0$ has positive $\lambda \times \lambda$ measure, D contains a rectangle $G_0 \times G_0 \times E$, where $E \subsetneq G$ has full σ-measure. Replacing E by $\bigcap E + \phi(g)$, we can assume that $E + \phi(g) = E$ for all $g \in G_0$

$g \in G_0$. Define A' by

$$A'(g, x) = \begin{cases} A(g, x) & \text{if } x \in E, \\ I & \text{if } x \notin E. \end{cases}$$

It is easy to check that A' is a strict cocycle which agrees with A almost everywhere. QED

5.4 We shall now consider $(R, B, \underline{U}(\mathbb{H}))$-cocycles. In ([2], ch. VII, sec. 11) Gamelin has given the structure of numerical valued cocycles defined on $R \rtimes B$. The rest of this section is devoted to extending this structure theorem to $\underline{U}(\mathbb{H})$-valued cocycles on $R \rtimes B$. For simplicity we shall assume that $2\pi \in \Gamma = \hat{B}$. K is the annihilator of the subgroup $\{2\pi n, n \in N\} \subset \Gamma$. We get the pair (N, K). It is necessary that we start with a discussion of $(N, K, \underline{U}(\mathbb{H}))$-cocycles.

We shall call an $(N, K, \underline{U}(\mathbb{H}))$-cocycle a <u>Gamelin cocycle</u> if it is strict. By the lemma last proved every $(N, K, \underline{U}(\mathbb{H}))$-cocycle is equal to some Gamelin cocycle a.e. Let C be a Gamelin cocycle. The identity (5.1.1) gives

$$C(n, y) = C(1, y)\, C(n-1, y) \tag{5.4.1}$$

for each $y \in K$ and $n \in N$. If β is the measurable function on K, $\beta(y) = C(1,y), y \in K$, then repeated application of (5.4.1) yields

$$(5.4.2)\qquad C(n,y) = \begin{cases} \beta(y)\,\beta(y+e_1)\,\ldots\,\beta(y+e_{n-1}), & n > 0 \\ I\,, & n = 0 \\ \beta^*(y+e_{-1})\,\ldots\,\beta^*(y+e_n), & n < 0\,. \end{cases}$$

Conversely, if β is any measurable function $\beta : K \to \underline{U}(H)$, (5.4.2) defines a Gamelin cocycle which we shall denote by C_β. Further, if β_1 and β_2 agree except on a σ_o-null (σ_o is the Haar measure on K) set, then C_{β_1} and C_{β_2} agree except on a $N \rtimes K$-null set. Next, suppose C_{β_1} and C_{β_2} are cohomologous. Then there exists a measurable function $\rho : K \to \underline{U}(H)$ such that

$$C_{\beta_1}(n,\, y) = \rho(y)\;\; C_{\beta_2}(n,\, y)\; \rho^*(y + e_n)$$

for a.e. $(n,y) \in N \rtimes K$. In particular,

$$(5.4.3)\qquad \beta_1(y) = \rho(y)\, \beta_2(y)\, \rho^*(y + e_1)$$

for a.e. $y \in K$. Conversely, if β_1 and β_2 are related as in (5.4.3) for some measurable function $\rho : K \to \underline{U}(H)$, then C_{β_1} and C_{β_2} are cohomologous. Finally, C_β is the coboundary of the function ρ if and only if

$$\beta(y) = \rho(y)\, \rho^*(y + e_1), \quad \text{a.e. } y \in K.$$

Summing up, we have

5.5 THEOREM. Every Gamelin cocycle is of the *form* C_β for a measurable function $\beta : K \to \underline{U}(\mathbb{H})$. C_{β_1} is cohomologous to C_{β_2} if and only if $\beta_1(y) = \rho(y)\,\beta_2(y)\,\rho^*(y + e_1)$, a.e. $y \in K$, for some measurable function $\rho : K \to \underline{U}(\mathbb{H})$.

5.6 Let C be a Gamelin cocycle. Then C can be extended to a strict $(R, B, \underline{U}(\mathbb{H}))$-cocycle by setting

$$(5.6.1) \qquad \overline{C}\,(t,\ y + e_s) = C([t+s],\ y)$$

for all $t \in R$, $y \in K$ and $s \in [0,1)$, where $[t + s]$ stands for the largest integer not exceeding $t + s$. Since every $x \in B$ has a unique representation $x = y + e_s$, $y \in K$ and $s \in [0, 1)$, $\overline{C}$ is well defined on $R \times B$. That $\overline{C}$ is a strict cocycle will follow from the observation

$$[a+b+c] = [a+b] + [c + \langle a+b \rangle]$$

for all real numbers a, b, c where $\langle a + b \rangle$ stands for the fractional part of $a+b$, $\langle a + b \rangle = a+b - [a+b]$.

We shall call the cocycle $\overline{C}$ of (5.6.1) an extended Gamelin cocycle. Since any Gamelin cocycle is of the form C_β for some measurable function $\beta : K \to \underline{U}(\mathbb{H})$, the extended Gamelin cocycles have their structure determined by (5.4.2) and (5.6.1). We have the following extension of a theorem of Gamelin's ([2], chap. VII, theorem 11.1).

5.7 THEOREM. Every strict $(R, B, \underline{\underline{U}}(H))$-cocycle is cohomologous to an extended Gamelin cocycle. Further, if C_{β_1} and C_{β_2} are *cohomologous, then* $\overline{C}_{\beta_1}$ and $\overline{C}_{\beta_2}$ are also cohomologous.

PROOF. Let A be a strict $(R, B, \underline{\underline{U}}(H))$-cocycle. We define functions ρ and β on B and K respectively by

$$\rho(y + e_s) = A^*(s,y), \ y \in K, \ \ s \in [0,1), \quad \text{and}$$

$$\beta(y) = A(1, y), \ \ y \in K.$$

ρ and β are clearly measurable. Now

$$\begin{aligned} &\rho(y + e_s)\ \overline{C}_\beta(t, y + e_s)\ \rho^*(y + e_s + e_t) \\ = \ & A^*(s,y)\ C_\beta([t+s],y)\ \ A(\langle s+t\rangle, y + e_{[s+t]}) \\ = \ & A^*(s,y)\ A([t+s],y)\ A(\langle s+t\rangle, y + e_{[s+t]}) \\ = \ & A^*(s,y)\ A(t+s,y) \\ = \ & A(t, y + e_s), \end{aligned}$$

for each $t \in R$, $y \in K$ and $s \in [0,1)$. Thus A is cohomologous to $\overline{C}_\beta$.

For the second assertion, suppose that for the Gamelin cocycles C_{β_1} and C_{β_2}, C_{β_1} is cohomologous (ρ_0) to C_{β_2}. Define a function ρ on B by setting $\rho(y + e_s) = \rho_0(y)$, $y \in K$, $s \in [0,1)$ which is clearly measurable. Then for any $t \in R$, $s \in [0,1)$

$$\rho(y + e_s)\, \overline{C}_{\beta_2}(t, y + e_s)\, \rho^*(y + e_s + e_t)$$

$$= \rho_o(y)\ C_{\beta_2}([t+s], y)\, \rho_o^*(y + e_{[s+t]})$$

$$= C_{\beta_1}([t+s], y)$$

$$= \overline{C}_{\beta_1}(t, y + e_s)$$

for a.e. $y \in K$. That is, $\overline{C}_{\beta_1}$ is cohomologous (ρ) to $\overline{C}_{\beta_2}$ QED

REMARK. We do not know if $\overline{C}_{\beta_1}$ and $\overline{C}_{\beta_2}$ can be cohomologous without C_{β_1} and C_{β_2} being cohomologous. In case the cocycles are numerical valued, Gamelin has shown ([2], chap VII, p.184) that $\overline{C}_{\beta_1}$ and $\overline{C}_{\beta_2}$ are cohomologous if and only if C_{β_1} and C_{β_2} are cohomologous.

Next we shall show that every (R, B, $\underline{U}$ ($\mathbb{H}$))-cocycle has a strict version, completing the study of structure of these cocycles. This was proved in the numerical case by Gamelin ([2], ch. VII, lemma 12.3). We shall adapt his proof.

5.8. THEOREM. To every (R, B, $\underline{U}(\mathbb{H})$)-cocycle A, there corresponds a strict cocycle A' such that

$$A(t, x) = A'(t, x) \quad \text{for a.e. } (t, x) \in R \times B.$$

PROOF. Since A is a cocycle, we have

$$(5.8.1) \qquad A(t+s,\ x) = A(t,\ x)\ A(s,\ x + e_t)$$

for a.e. $(t,s,x) \in R \times R \times B$. Using the representation $x = y + e_r$ for $y \in K$, $0 \leq r < 1$, (5.8.1) can be rewritten as

$$A(t+s,\ y+e_r) = A(t,\ y+e_r)\ A(s,\ y+e_r+e_t)$$

for a.e. $(y,r,t,s) \in K \times [0,1) \times R \times R$. By Fubini's theorem, there exists an $r_0 \in [0,1)$ such that for r_0 and a.e. $(y,s,t) \in K \times R \times R$, the identity holds. We define a unitary operator-valued function A_1 on $R \times B$ by setting

$$(5.8.2) \qquad A_1(t,\ x) = A(t,\ x + e_{r_0}).$$

It is immediate that A_1 is a cocycle; further, A_1 satisfies

$$(5.8.3) \qquad A_1(t+s,\ y) = A_1(t,\ y)\ A_1(s,\ y + e_t)$$

for a.e. $(y,\ s,\ t) \in K \times R \times R$.

Let N be a Borel subset of K of σ_0-measure zero such that for each $y \notin N$, (5.8.3) holds for a.e. $(s,t) \in R \times R$ (the null set may depend on $y \notin N$). Taking $\bigcup_{n=-\infty}^{\infty} (N + e_n)$, if necessary, we can assume that $N \pm e_1 = N$. We now define a cocycle A_2 on $R \times B$ by setting

$$A_2(t, y + e_s) = \begin{cases} I \quad , & y \in N \\ A_1(t, y+e_s), & y \notin N. \end{cases}$$

Then $A_2(t, x) = A_1(t, x)$ for a.e. (t,x). Note that A_2 satisfies (5.8.3) for each $y \in K$, for a.e. $(t,s) \in R \times R$ (the null set depending on y).

Now let ρ be the measurable operator-function on B defined by

$$\rho(y + e_s) = A_2(s, y), \; y \in K, \; s \in [0,1).$$

Let A_3 be the cocycle

$$A_3(t, x) = \rho(x) \, A_2(t,x) \, \rho^*(x + e_t), \; (t,x) \in R \times B.$$

For each $y \in K$, A_3 continues to satisfy (5.8.3) for a.e. $(s, t) \in R \times R$. Moreover,

(5.8.4) $$A_3(s, y) = I, \; y \in K, \; s \in [0,1).$$

Next, we <u>fix $y \in K$</u>. For a.e. s, $1 \leq s < 2$, we have

$$A_3(s-1+t, y+e_1) = A_3(s-1, y+e_1) \, A_3(t, y+e_s)$$
$$= A_3(t, y+e_s), \quad a.e. \; t \in R,$$

and $A_3(s+t, y) = A_3(s,y) \, A_3(t, y+e_s)$, a.e. $t \in R$.

Regarding s as fixed and setting $u = s+t-1$ we obtain

$$A_3(u, y + e_1) = A_3^*(s, y) \quad A_3(u+1, y)$$

for every fixed s in a set of full measure in $[1,2)$, for a.e. $u \in R$. This implies that $A_3(s,y)$ must be a constant for a.e. $s \in [1,2)$. We define this constant value to be $\beta(y)$.

This procedure, carried out for every $y \in K$ gives a function $\beta : K \longrightarrow \underline{U}(\mathbb{H})$. β is measurable, because, for every $\xi, \xi' \in \mathbb{H}$, the right hand side in

$$(\beta(y)\,\xi,\ \xi') = \int_1^2 (A_3(s,y)\,\xi,\ \xi')\,ds$$

is measurable in y. Also for each $y \in K$, we have

(5.8.5) $\qquad A_3(s, y) = \beta(y)$

for a.e. $s \in [1,2)$.

Consider now the extended Gamelin cocycle $\overline{C}_\beta$ obtained from β. We <u>claim</u> that $\overline{C}_\beta(t, x) = A_3(t,x)$ for a.e. $(t,x) \in R \times B$. Granting this for a moment, observe that the strict cocycle $\rho^*(x)\ \overline{C}_\beta(t,x)\ \rho(x + e_t)$ and the cocycle A_2 agree except on a null set. But A_2 is equal almost everywhere to A_1, so that then A_1 has a strict version. Since A_1 is a translate of A, this proves the theorem.

Proof of the claim. From the definition of $\overline{C}_\beta$ and from (5.8.4)

$$\overline{C}_\beta(s,y) = I = A_3(s,y) \ , \quad y \in K, \ 0 \le s < 1.$$

Similarly, from (5.8.5),

$$\overline{C}_\beta(s, y) = \beta(y) = A_3(s, y)$$

for each $y \in K$, for a.e. s, $1 \le s < 2$. Recall from (5.8.3) that

$$A_3(t+s, y) = A_3(s, y) \, A_3(t, y + e_s)$$

for each $y \in K$, for a.e. $(t,s) \in [0,1) \times [0,1)$. We also have

$$\overline{C}_\beta(t+s, y) = \overline{C}_\beta(s,y) \, \overline{C}_\beta(t, y + e_s)$$

for all $t,s,y \in [0,1) \times [0,1) \times K$. The first two identities applied to the last two, yield

$$A_3(t, y+e_s) = \overline{C}_\beta(t, y+e_s)$$

for each $y \in K$, for a.e. $(s,t) \in [0,1) \times [0,1)$. By Fubini's theorem (both the sides being measurable) and the identification of $K \times [0,1)$ with B, this gives

$$A_3(t,x) = \overline{C}_\beta(t, x)$$

for a.e. $(t,x) \in [0,1) \times B$. Since A_3 and $\overline{C}_\beta$ are cocycles, the cocycle-identity now implies that the equality persists for a.e. $(t,x) \in R \times B$. QED

6. Partial Isometry-valued Cocycles.

6.1 In this section we consider cocycles whose values are partial isometries.. This extension will be necessary where the unitary operator-valued cocycles apply only through the intervention of an auxiliary Hilbert space. The new class of cocycles is going to contain the $\underline{U}(\mathbb{H})$-cocycles and will have a cohomology structure compatible with that of $\underline{U}(\mathbb{H})$-cocycles.

By $\underline{P}(\mathbb{H})$ we shall denote the class of non-zero partial isometries of $\mathbb{H}$. We recall that a partial isometry is an operator T on $\mathbb{H}$ into itself such that $\mathbb{H}$ breaks up into a direct sum $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$, where $\|T\xi\| = \|\xi\|$ for $\xi \in \mathbb{H}_1$ and $T\xi = 0$ for $\xi \in \mathbb{H}_2$. $\mathbb{H}_1$ and $\mathbb{H}_2$ are called the initial space and the kernel of T respectively. We allow $\mathbb{H}_2 = \{0\}$.

For any operator T on $\mathbb{H}$, ker T stands for the stands for the kernel of T and $\underline{R}(T)$ for the range of T. For $T \in \underline{P}(\mathbb{H})$, In.$(T)$ will denote the initial space of T.

6.2. DEFINITION. By a $(G_o, G, \mathcal{P}(\mathbb{H}))$-cocycle A we mean a measurable operator-function $A : G_o \times G \to \mathcal{P}(\mathbb{H})$ such that

$$A(g_1+g_2, x) = A(g_1, x)\ A(g_2, x+\phi(g_1))$$

for a.e. $(g_1, g_2, x) \in G_o \times G_o \times G$ (with respect to product measure $\lambda \times \lambda \times \sigma$).

A $(G_o, G, \mathcal{P}(\mathbb{H}))$-cocycle is also called a $\mathcal{P}(\mathbb{H})$-cocycle when there is no scope of confusion. As before, two cocycles are identified if they agree except on a $\lambda \times \sigma$-null set. Now for a $(G_o, G, \mathcal{P}(\mathbb{H}))$-cocycle A, there arises two range functions on $G_o \times G$, one considering $\mathcal{R}\ A(g, x)$, $(g, x) \in G_o \times G$ and the other considering $\text{In}.\ A(g, x)$, $(g, x) \in G_o \times G$. It is necessary to examine them carefully before we go on to define the cohomology classes of $\mathcal{P}(\mathbb{H})$-cocycles.

6.3. We shall call a $(G_o, G, \mathcal{P}(\mathbb{H}))$-cocycle A <u>strict</u> if it satisfies (6.2.1) for all $(g_1, g_2, x) \in G_o \times G_o \times G$. Suppose A is a strict $(G_o, G, \mathcal{P}(\mathbb{H}))$-cocycle. Then we have

$$A(g_1+g_2, x) = A(g_1, x)\ A(g_2, x + \phi(g_1))$$

(6.3.1)

$$A(g_1, x) = A(g_1+g_2, x)\ A(-g_2, x + \phi(g_1+g_2))$$

for all $(g_1, g_2, x) \in G_o \times G_o \times G$, *where the second equation* is obtained from the first. These equations give

(6.3.2) $$\underset{=}{B}\ A(g_1, x) = \underset{=}{B}\ A(g_1+g_2, x)$$

for all (g_1, g_2, x). Fixing g_1 and varying g_2, it follows that $\underset{=}{B}\ A(g, x)$ is independent of g and depends only on x. Write $\underset{=}{B}\ A(g, x) = J(x)$. Then J is a measurable range function. Again we have

$$A(g_1+g_2, x) = A(g_1, x)\ A(g_2, x + \phi(g_1))$$

$$A(g_2, x + \phi(g_1)) = A(-g_1, x+\phi(g_1))\ A(g_1+g_2, x)$$

which give

(6.3.3) $$\text{In. } A(g_1+g_2, x) = \text{In. } A(g_2, x + \phi(g_1)).$$

But now (6.3.2) applied to the first equation in (6.3.1) gives

$$\text{In. } A(g_1, x) \subseteq \underset{=}{B}\ A(g_2, x + \phi(g_1)),$$

while (6.3.3) implies

$$\text{In. } A(g_1, x) \supseteq \underset{=}{B}\ A(g_2, x + \phi(g_1)).$$

Hence $$\text{In. } A(g, x) = J(x + \phi(g)), \quad (g, x) \in G_0 \rtimes G.$$

Thus, we have proved that given a strict cocycle A, there exists a range function J on G such that

$$\underset{=}{B}\ A(g, x) = J(x)$$

$$\text{In. } A(g, x) = J(x + \phi(g))$$

for all $(g, x) \in G_0 \rtimes G$. Our next result will show that much the same situation obtains when A is not necessarily strict.

6.4. LEMMA. Let A be a $(G_o, G, \mathcal{P}(\mathcal{H}))$-cocycle. Then there exists a measurable range function J on G such that

(i) $\mathcal{R}\, A(g, x) = J(x)$

(ii) $\mathrm{In}.\, A(g,x) = J(x + \phi(g))$

for a.e. (g,x).

PROOF. By definition, A satisfies the identity

(6.4.1) $$A(g_1+g_2, x) = A(g_1, x)\, A(g_2, x + \phi(g_1))$$

for all (g_1,g_2,x) belonging to a subset C of $G_o \times G_o \times G$ having full measure. Consider the transformation $T : (g_1,g_2,x) \longrightarrow (g_1+g_2, -g_2, x)$ which is a measure-preserving homeomorphism of $G_o \times G_o \times G$ onto itself. Let $C_o = \bigcap_{-\infty<n<\infty} T^n C$. Then C_o is of full measure and $T C_o = C_o$. If $(g_1,g_2,x) \in C_o$, then $(g_1+g_2, -g_2, x) \in C_o$ and (6.4.1) written out for this triple gives

(6.4.2) $$A(g_1,x) = A(g_1+g_2, x)\, A(-g_2, x + \phi(g_1) + \phi(g_2)).$$

From (6.4.1) and (6.4.2),

$$\mathcal{R}\, A(g_1+g_2, x) = \mathcal{R}\, A(g_1, x)$$

for all $(g_1,g_2,x) \in C_o$. Since C_o is of full measure, by ubini's theorem, there exists a null set $N \subseteq G$ such that for $x \in N^c$, $(g_1, g_2, x) \in C_0$ for a.e. $(g_1, g_2) \in G_0 \times G$, that is, the x-section C_0^x is of full measure in $G_0 \times G_0$. For such an x, consider a $g_1 = g_1^x$ such that $(C_0^x)^{g_1}$ is of full measure in G_0. We have for each $x \in N^c$

$$\mathcal{B}\, A(g_1^x + g_2,\ x) = \mathcal{B}\, A(g_1^x,\ x)$$

for a.e. $g_2 \in G_0$. This means for $x \in N^c$

(6.4.4) $$\mathcal{B}\, A\,(g,\ x) = \mathcal{B}\, A(g_1^x,\ x)$$

for a.e. $g \in G_0$. Define a range function J on G by setting,

(6.4.5) $$J(x) = \begin{cases} \mathcal{B}\, A(g_1^x,\ x)\ , & x \in N^c, \\ \{0\} & ,\ x \in N\ . \end{cases}$$

We claim that J is measurable. Consider the range function Z on $G_0 \times G$ defined by $Z(g,\ x) = \mathcal{B}\, A(g,\ x)$. By lemma, 2.5, Z is seen to be measurable by looking at the functions $A(g,\ x)\, \xi_i$, $i \geq 1$, where $(\xi_1,\ \xi_2,\ \ldots\ )$ is a basis of $\mathbb{H}$. Now let Q_Z stand for the projection operator-function on $G_0 \times G$ such that $Q_Z(g,\ x)$ is the projection onto $Z(g,\ x)$, defined in $\mathbb{H}$, $(g,\ x) \in G_0 \times G$. Similarly, let Q_J be the projection operator-function on G defined by J. Take $\xi,\ \eta$ arbitrarily in $\mathbb{H}$. The relations (6.4.4) and (6.4.5) imply that for each $x \in N^c$,

$$(Q_Z(g, x)\,\xi, \eta) = (Q_J(x)\,\xi, \eta)$$

for a.e. $g \in G_0$. If ν is a probability measure on G_0 which is absolutely continuous with respect to the Haar measure λ on G_0, then we have

$$(Q_J(x)\,\xi, \eta) = \int_{G_0} (Q_Z(g, x)\,\xi, \eta)\, d\ \ (g)$$

for each $x \in N^c$. As a function of x the right hand side is measurable. ξ, η being arbitrary, this proves that J is measurable. We already have $J(x) = R\, A(g, x)$ for a.e. x, for a.e. g. Both the sides being measurable, assertion (i) follows from Fubini's theorem.

To prove (ii) consider a second transformation $(g_1, g_2, x) \rightarrow (-g_1, g_1+g_2, x + \phi(g_1))$ which is again a measure-preserving homeomorphism of $G_0 \times G_0 \times G$ onto itself. Let $C_1 \subsetneq C_0$ be a set of full measure which is invariant under this transformation. Fix $(g_1, g_2, x) \in C_1$. Then $(-g_1, g_1+g_2, x+\phi(g_1)) \in C_1$. Since $C_1 \subseteq C$, we have

$$A(g_1+g_2, x) = A(g_1, x)\, A(g_2, x + \phi(g_1)),$$

(6.4.6)

$$A(g_2, x + \phi(g_1)) = A(-g_1, x+\phi(g_1))\, A(g_1+g_2, x).$$

(6.4.6) implies

$$\text{(6.4.7)} \qquad \text{In. } A(g_1+g_2,\ x) = \text{In. } A(g_2,\ x + \phi(g_1))$$

for $(g_1, g_2, x) \in C_1$. But $C_1 \subseteq C_0$ and so one has from (6.4.3)

$$\text{(6.4.8)} \qquad \mathcal{R}\ A(g_1+g_2,\ x) = \mathcal{R}\ A(g_1,\ x).$$

Applying (6.4.7) and (6.4.8) to the first relation in (6.4.6) one has

$$\text{(6.4.9)} \qquad \mathcal{R}\ A(g_2,\ x + \phi(g_1)) = \text{In. } A(g_1,\ x)$$

for all $(g_1, g_2, x) \in C_1$. Now by Fubinis' theorem choose a subset $D_1 \subseteq G_0 \rtimes G$ having full measure such that for each $(g_1, x) \in D_1$, $(g_1, g_2, x) \in C_1$ for a.e. $g_2 \in G_0$ (the null set depending on (g_1, x)). Again let D_2 be a subset of $G_0 \rtimes G$ of full measure such that if $(g_1, x) \in D_2$, then $x + \phi(g_1) \in N^c$. (This is possible as N is null subset of G). Writing $D = D_1 \cap D_2$, it follows from (6.4.5) and (6.4.9) that

$$\text{In. } A(g_1,\ x) = J(x + \phi(g_1))$$

for all $(g_1, x) \in D$. Hence (ii). QED

In the lemma above the range function J is uniquely (upto null sets) determined by A and will be called the range of A, in notations, $\mathcal{R}(A) = J$.

COROLLARY. If J is the range of a $\mathfrak{P}(\mathbb{H})$-cocycle, then J has constant dimension a.e.

PROOF. Let A be a $\mathfrak{P}(\mathbb{H})$-cocycle having range J. Define $n(x) = \dim.(J(x))$, $x \in G$. From the lemma above for a.e. (g,x), $A(g,x)$ is a partial isometry with initial space $J(x + \phi(g))$ and range space $J(x)$. Therefore for a.e.$(g,x) \in G_0 \times G$, $n(x + \phi(g)) = n(x)$. Since J is measurable, n is a measurable function. Hence from the proposition, 4.7, $n(x)$ is a constant a.e. QED

6.5 Cohomology classes of $\mathfrak{P}(\mathbb{H})$-cocycles.

DEFINITION. Let A be a $(G_0, G, \mathfrak{P}(\mathbb{H}))$-cocycle having range J. A $\mathfrak{P}(\mathbb{H})$-cocycle A will be said to be cohomologous to A' if there exists a measurable function $\rho : G \to \mathfrak{P}(\mathbb{H})$ such that

(i) for a.e. $x \in G$, In. $\rho(x) = J(x)$, and

(6.5.1)

(ii) for a.e.$(g,x) \in G_0 \times G$, $A'(g,x) = \rho(x)A(g,x)\rho^*(x+\phi(g))$.

If (6.5.1) holds, it is clear that the range J' of A' is given by $J'(x) = \rho(x)\ [J(x)]$ a.e. Now the function $\rho^* : \rho^*(x) = [\rho(x)]^*$, satisfies (i) with respect to J' and further

$$A(g,x) = \rho^*(x)\ A'(g,x)\ \rho(x + \phi(g))$$

for a.e. (g, x). Thus A is also cohomologous to A'. In fact, one can verify that ' A' is cohomologous to A ' defines an equivalence relation. The resulting equivalence classes will again be called cohomology classes and as before, we shall describe (6.5.1) by saying ' A' is cohomologous (ρ) to A '.

6.6. From the lemma, 6.4, it will follow that a $\underline{P}(\mathbb{H})$-cocycle A is a $\underline{U}(\mathbb{H})$-cocycle if and only if A has full range, that is, if $J = \underline{R}(A)$, then $J(x) = \mathbb{H}$ a.e. Now if the $\underline{P}(\mathbb{H})$-cocycles A and A' have full range and if A' is cohomologous (ρ) to A, then it follows from (6.5.1) that $\rho(x) \in \underline{U}(\mathbb{H})$, a.e. x. Therefore, A and A' are cohomologous as $\underline{P}(\mathbb{H})$-cocycles if and only if they are cohomologous as $\underline{U}(\mathbb{H})$-cocycles. However, we should point out that if $\mathbb{H}$ is infinite-dimensional, then a $\underline{U}(\mathbb{H})$-cocycle A can be cohomologous to a $\underline{P}(\mathbb{H})$-cocycle without the latter having full range. To see this, consider in (6.5.1) a $\underline{U}(\mathbb{H})$-cocycle A and a measurable function ρ whose values are isometries of $\mathbb{H}$ onto proper subspaces. The cocycle A' will have range J ; $J(x) = \underline{R}(\rho(x))$, a.e. x. In view of Corollary, 6.4 we should also note that the range functions J, J' of cohomologous cocycles A and A' have necessarily the same dimension.

6.7 Consider the cocycle

$$A(g,x) = I_{\mathbb{H}_1}, \quad (g,x) \in G_0 \times G,$$

where $I_{\mathbb{H}_1}$ is the projection on a closed subspace $\mathbb{H}_1 \subsetneq \mathbb{H}$, $\mathbb{H}_1 \neq \{0\}$. $\underline{P}(\mathbb{H})$-cocycles which are cohomologous to such a cocycle A for some $\mathbb{H}_1$ are called <u>coboundaries in the extended sense.</u> A coboundary in the extended sense is, therefore, of the form

$$A(g, x) = \rho(x)\ \rho^*(x + \phi(g)),\ \text{a.e. } (g, x)$$

where $\rho : G \to \underline{P}(\mathbb{H})$ is a measurable function such that In. $\rho(x)$ = constant subspace of $\mathbb{H}$ a.e. It is called the coboundary of the function ρ.

Finally, the fact that any two coboundaries (not in the extended sense) are cohomologous, has the following counterpart here.

LEMMA. Two coboundaries in the extended sense are cohomologous if and only if they have range functions of the same dimension.

PROOF. The 'only if' part has already been observed. For the 'if part', let A_1 and A_2 be the coboundaries of the functions ρ_1 and ρ_2 respectively. If In. $\rho_i(x) = \mathbb{H}_i$ a.e., $i = 1,2$, then the condition of the lemma implies that $\mathbb{H}_1$ and $\mathbb{H}_2$ are of the same dimension. Choose $q \in \underline{P}(\mathbb{H})$ such that

$$\text{In. } q = \mathbb{H}_1, \quad \text{and}$$

$$\underline{R}\,(q) = \mathbb{H}_2$$

Consider the measurable function $q_0 = r_2\, q\, r_1^*$. Then $q_0(x) \in \underline{P}(\mathbb{H})$ a.e. and

$$r_2(x)\, r_2^*(x + \phi(g)) = q_0(x)\, r_1(x)\, r_1^*(x+\phi(g))\, q_0^*(x + \phi(g))$$

for a.e. $(g, x) \in G_0 \rtimes G$. QED

6.8. In 6.6 we have observed that the range of a $\underline{P}(\mathbb{H})$-cocycle has constant dimension almost everywhere. Let J be a measurable range function on G having constant dimension almost everywhere. Then the next theorem describes the class of all $\underline{P}(\mathbb{H})$-cocycles having range J (this will turn out to be a nonempty class !).

THEOREM. Let J be a measurable range function on G having constant dimension n. Consider a measurable isometry-function s on G where each $s(x)$ is an isometry of $\mathbb{H}_n$ into $\mathbb{H}$, range of $s(x)$ being $J(x)$ almost everywhere. Then the formula

$$(6.8.1) \qquad A'(g,x) = s(x)\, A(g,x)\, s^*(x+ \phi(g)), \quad (g,x) \in G_0 \rtimes G$$

gives a one-to-one correspondence between the class of $(G_0, G, \underline{U}(\mathbb{H}_n))$-cocycles A and the class of $(G_0, G, \underline{P}(\mathbb{H}))$-cocycles A' having range J. Under this correspondence the $\underline{U}(\mathbb{H}_n)$-cocycles A_1 and A_2 are cohomologous if and only if their images, the $\underline{P}(\mathbb{H})$-cocycles A_1' and A_2', are cohomologous. In particular A is a coboundary if and only if A' is a coboundary in the extended sense. Moreover, A is strict if and only if A' is strict.

PROOF. The theorem follows on noting that the map $A \longrightarrow A'$ given by (6.8.1) has the inverse

$$(6.8.2) \qquad A(g,x) = s^*(x)\, A'(g,x)\, s(x + \phi(g)), \; (g,x) \in G_o \times G.$$

COROLLARY. If in the pair (G_o, G), $\phi(G_o)$ has full σ-measure in G, then every $(G_o, G, \underline{P}(\mathbb{H}))$-cocycle is a coboundary in the extended sense. Again if in (G_o, G), G_o is countable, or if $(G_o, G) = (R, B)$, then every $\underline{P}(\mathbb{H})$-cocycle has a strict version.

PROOF. The first assertion follows from lemma, 5.2, and the theorem above. The second assertion is a consequence of lemma, 5.3, theorem, 5.7, and theorem above.

7. Systems of Imprimitivity.

7.1 DEFINITION. Let $\mathbb{K}$ be a separable Hilbert space. By a system of imprimitivity based on (G_o, G) and acting in $\mathbb{K}$ we mean a pair (U, P) where (i) U is a representation of G_o acting in $\mathbb{K}$ and (ii) P is a spectral measure on the Borel subsets of G, acting in the same space $\mathbb{K}$ and such that for each Borel subset $D \subseteq G$, and for each $g \in G_o$,

$$(7.1.1) \qquad U_g^{-1}\, P(D)\, U_g = P(D + \phi(g)).$$

Two systems of imprimiti ity (U, P) and (U', P') based on (G_o, G) and acting in $\mathbb{K}$ and $\mathbb{K}'$ respectively are said to be equivalent if there exists an isometric isomorphism S of $\mathbb{K}$

onto $\mathbb{K}'$ such that

$$S\,P(D)\,S^{-1} = P'(D)$$

$$S\,U_g\,S^{-1} = U'_g$$

for each Borel subset $D \subseteq G$ and for each $g \in G_0$.

If (U^i, P^i) is a system of imprimitivity based on (G_0, G) and acting in $\mathbb{K}_i$ for each i in a finite or countable index set, then $\underset{i}{\oplus}(U^i, P^i)$ stands for the system of imprimitivity (U, P) acting in $\underset{i}{\oplus}\mathbb{K}_i$, where

$$U = \underset{i}{\oplus} U^i$$

$$P = \underset{i}{\oplus} P^i .$$

7.2. **EXAMPLES:**

(i) Define the representation U of G_0 acting in $L_2(G ; \mathbb{H}, \sigma)$ by

$$(U_g F)(x) = F(x + \phi(g)),\ F \in L_2(G ; \mathbb{H}, \sigma),\ g \in G_0.$$

Let P stand for the canonical spectral measure on G acting in $L_2(G ; \mathbb{H}, \sigma)$. Then (U,P) is a system of imprimitivity which we call the regular system of imprimitivity.

(ii) Let A be a strict $(G_0, G, \underline{P}(\mathbb{H}))$-cocycle with range J. Let P^J be the canonical spectral measure acting in

$J L_2(G ; \mathbb{H}, \sigma)$. Define the representation U^A by

(7.2.1) $(U_g^A F)(x) = A(g, x) F(x+\phi(g))$, $F \in JL_2(G; \mathbb{H}, \sigma)$, $g \in G_o$.

(That U^A is indeed a representation can be verified by using the cocycle identity.) Then (U^A, P^J) is a system of imprimitivity based on (G_o, G) and acting in $JL_2(G ; \mathbb{H}, \sigma)$.

(iii) Let A be a $(G_o, G, \mathfrak{U}(\mathbb{H}))$-cocycle, not necessarily strict, with range J. The right hand side of (7.2.1) defines a unitary operator L_g on $J L_2(G ; \mathbb{H}, \sigma)$ for a.e. g (when for the particular g, for a.e. x, $\mathfrak{R}\, A(g, x) = J(x)$ and $\text{In}_r\, A(g, x) = J(x + \phi(g))$). Also from the cocycle identity (6.2.1), $L_{g_1} \cdot L_{g_2} = L_{g_1+g_2}$ for a.e. $(g_1, g_2) \in G_o \times G_o$. Hence by a standard result (see Varadarajan [15], lemma 9.6) there exists a unitary representation U^A of G_o such that

$$U_g^A = L_g \text{ , a.e. } g.$$

Therefore, for a.e. $g \in G_o$

(7.2.2) $$(U_g^A F)(x) = A(g, x) F(x + \phi(g))$$

for each $F \in L_2(G ; \mathbb{H}, \sigma)$, for a.e. x. Now for each Borel set $E \subsetneq G$, we still have

$$(U_g^A)^{-1} P^J(E) U_g^A = P^J(E + \phi(g))$$

for a.e. g and so, by continuity of $g \longrightarrow U_g^A$, for all $g \in G_0$. Thus (U^A, P^J) is a system of imprimitivity acting in $JL_2(G ; \mathbb{H}, \sigma)$.

Next, suppose the $\mathfrak{P}(\mathbb{H})$-cocycle A' is cohomologous (ρ) to A. The operator T_ρ :

$$(T_\rho F)(x) = \rho(x) F(x) \quad \text{a.e.}, \; F \in J L_2(G ; \mathbb{H}, \sigma),$$

is an invertible isometry of $J L_2(G ; \mathbb{H}, \sigma)$ onto $J' L_2(G; \mathbb{H}, \sigma)$, J' being the range of A'. Further,

$$T_\rho P^J T_\rho^{-1} = P^{J'}$$

whereas from (7.2.2) it follows that

$$T_\rho U_g^A T_\rho^{-1} = U_g^{A'}$$

for a.e. g and hence by continuity, for all $g \in G_0$. Therefore, (U^A, P^J) and $(U^{A'}, P^{J'})$ are equivalent.

(iv) Let (U, P) be a system of imprimitivity based on (G_0, G) and acting in $\mathbb{K}$. Apply Stone's theorem to U to yield a spectral measure Q on $\hat{G}_0$ and to P to yield a representation V of $\hat{G}$:

$$(7.2.3) - \begin{cases} U_g = \int_{\hat{G}_0} \langle -y, g \rangle \, d\, Q(y), & g \in G_0, \\ V_h = \int_G \langle x, h \rangle \, d\, P(x), & h \in \hat{G}. \end{cases}$$

Recall that if ϕ is the map associated to the pair (G_0, G) then the map $\hat{\phi}$ associated to $(\hat{G}, \hat{G}_0)$ is given by

$$\langle \hat{\phi}(h), g \rangle = \langle h, \phi(g) \rangle, \quad h \in \hat{G}, \ g \in G_0.$$

Since (U, P) is a system of imprimitivity, we have

$$\begin{aligned} U_g^{-1} V_h U_g &= \int_G \langle x, h \rangle \ d(U_g^{-1} P(x) U_g) \\ &= \int_G \langle x, h \rangle dP(x + \phi(g)) \\ &= \langle -\phi(g), h \rangle V_h , \end{aligned}$$

whence, $V_h^{-1} U_g^{-1} V_h = \langle -\phi(g), h \rangle U_g^{-1} = \langle -g, \hat{\phi}(h) \rangle U_g^{-1}$.

From the first integral in (7.2.3)

$$V_h^{-1} U_g^{-1} V_h = \int_{\hat{G}_0} \langle y, g \rangle d(V_h^{-1} Q(y) V_h).$$

Also, $\langle -g, \hat{\phi}(h) \rangle U_g^{-1} = \int_{\hat{G}_0} \langle y - \hat{\phi}(h), g \rangle d\, Q(y)$

$$= \int_{\hat{G}_0} \langle y, g \rangle d\, Q(y + \hat{\phi}(h)).$$

Therefore, $V_h^{-1} \ Q(D) V_h = Q(D + \hat{\phi}(h))$ for each Borel subset $D \subseteq \hat{G}_0$ and each $h \in \hat{G}$. Hence (V, Q) is a system of imprimitivity based on $(\hat{G}, \hat{G}_0)$ and acting in $\mathbb{K}$. We shall call (V, Q) the dual system of (U, P).

7.3. Hereafter we shall use the notations U^A and P^J in the sense of example (iii), 7.2. Our problem is to look for representations U of G_o such that (U, P^J) is a system of imprimitivity. We shall show that if J has constant dimension such a representation U exists and has the form U^A for some $\mathfrak{Y}(\mathbb{H})$-cocycle A having range J. The lemma below is crucial for our development.

LEMMA. Let (U, P^J) be a system of imprimitivity based on (G_o, G) and acting in $J\, L_2(G ; \mathbb{H}, \sigma)$. Then P^J is homogeneous having $\bar{\sigma}$ as the associated measure class.

PROOF. Note that P^J is absolutely continuous with respect to σ. Now let $\bar{\alpha}_n$ be a measure class that comes in the Hellinger-Hahn decomposition of P^J. It is known that by virtue of the system of imprimitivity, α_n is quasi-invariant (see Varadarajan [15], lemma 9.9.). Since α_n is absolutely continuous with respect to σ and since σ is ergodic, α_n must be either zero or equivalent to σ. But the α_n's are supposed to be mutually singular. Hence only one among them can be nontrivial.

QED

Next, we need the following theorem which is proved in (Varadarajan [15], theorem 9.11). We omit the proof.

7.4. THEOREM. Let (U, P) be a system of imprimitivity based on (G_o, G) and acting in $\mathbb{K}$ such that P is homogeneous of multiplicity n having the measure class $\tilde{\sigma}$ Then there exists a $\underline{U}(\mathbb{H}_n)$-cocycle A such that the system of imprimitivity $(U^A, P^{(n)})$ acting in $L_2(G ; \mathbb{H}_n, \sigma)$ is equivalent to (U, P).

We come to the main result of this section.

7.5. THEOREM. To any system of imprimitivity (U, P^J) based on (G_0, G) and acting in $J\, L_2(G ; \mathbb{H}, \sigma)$ there exists a unique $(G_0, G, \underline{P}(\mathbb{H}))$-cocycle A such that $U = U^A$. A must have range J.

PROOF. Let (U, P^J) be as in the hypothesis. From lemma, 7.3, P^J is homogeneous, say, with multiplicity n. Therefore, by theorem, 7.4, there exists a $\underline{U}(\mathbb{H}_n)$-cocycle A' such that $(U^{A'}, P^{(n)})$ is equivalent to (U, P^J). That is, there exists an invertible isometry S of $L_2(G ; \mathbb{H}, \sigma)$ onto $J\, L_2(G; \mathbb{H}, \sigma)$ such that

$$S\, P^{(n)}\, S^{-1} = P^J$$

(7.5.1)

$$A\, U^{A'}\, S^{-1} = U$$

By Corollary, 2.10, S is induced by a meas urable isometry-valued function s, where each $s(x)$ is an isometry of $\mathbb{H}_n$ into $\mathbb{H}$, range of $s(x)$ being $J(x)$ a.e. Explicitly

$$(S\, F)\, (x) = s(x)\, F(x)$$

for each $F \in L_2(G ; \mathbb{H}_n, \sigma)$ for a.e. x. Consider now the $\underline{P}(\mathbb{H})$-cocycle A defined by (see 6.8)

$$A(g,x) = s(x)\ A'(g,x)\ s^*(x + \phi(g)),\ \text{a.e. } (g,x).$$

Compare the definitions of U^A and $U^{A'}$ from 7.2(ii) :

$$\begin{aligned} \text{for a.e. } g,\ (U^A_g F)(x) &= A(g, x)\ F(x + \phi(g)) \\ &= s(x)\ A'(g,x)\ s^*(x + \phi(g))\ F(x + \phi(g)) \\ &= [S\ U^{A'}_g (S^{-1}\ F)]\ (x) \end{aligned}$$

for each $F \in J\ L_2(G\ ;\ \mathbb{H},\ \sigma)$, for a.e. x. That is,

$$U^A = S\ U^{A'}\ S^{-1}.$$

Hence from (7.5.1) $U^A = U$. Range of A is, of course, J.

As for uniqueness, let A_1 and A_2 be $\mathcal{P}(\mathbb{H})$-cocycles such that $U^{A_1} = U = U^{A_2}$. In the first place A_1 and A_2 must have the same range J. Next, for each $F \in J\,L_2(G\ ;\ \mathbb{H},\sigma)$,

$$A_1(g,x)\ F(x + \phi(g)) = A_2(g,x)\ F(x + \phi(g))$$

for a.e. g, for a.e. x. By considering a countable family of functions $\{F_i : i \geq 1\} \subseteq J\ L_2(G\ ;\ \mathbb{H},\ \sigma)$ such that for a.e. x, $\mathcal{S}\{\ F_i(x)\ :\ i \geq 1\} = J(x)$, it follows that

$$A_1(g,\ x) = A_2(g,\ x)$$

for a.e. g, for a.e. x, and hence by the measurability of both the sides, for a.e. $(g,\ x)$. QED

.6. In 7.2 (iii) we have observed that if A and A' are cohomologous $\mathcal{P}(\mathbb{H})$-cocycles having range functions J and J' respectively, then (U^A, P^J) and $(U^{A'}, P^{J'})$ are equivalent. We have the converse as well.

THEOREM. (U^A, P^J) and $(U^{A'}, P^{J'})$ are equivalent if and only if A and A' are cohomologous.

PROOF. The only if part remains to be proved. Suppose (U^A, P^J) and $(U^{A'}, P^{J'})$ are equivalent. Then there exists an invertible isometry T of $L_2(G; \mathbb{H}, \sigma)$ onto $J'L_2(G; \mathbb{H}, \sigma)$ such that

$$T\, U^A\, T^{-1} = U^{A'}$$

$$T\, P^J\, T^{-1} = P^{J'} .$$

Consider (U^A, P^J). As in the proof of theorem, 7.5, there exists an invertible isometry S of $L_2(G; \mathbb{H}_n, \sigma)$ onto $J\,L_2(G; \mathbb{H}, \sigma)$, where n is the multiplicity of P^J, such that $S\,P^{(n)}\,S^{-1} = P^J$. Further, S is induced by a measurable isometry function s where each $s(x)$ is an isometry of $\mathbb{H}_n$ into $\mathbb{H}$, range of $s(x)$ being $J(x)$ a.e. Now $Q = RS$ is an invertible isometry of $L_2(G; \mathbb{H}_n, \sigma)$ onto $J'L_2(G; \mathbb{H}, \sigma)$ such that

$$Q\, P^{(n)}\, Q^{-1} = P^{J'} .$$

Therefore, by Corollary, 2.10, Q is induced by a measurable isometry-function q where $q(x)$ is an isometry of $\mathbb{H}_n$ onto

$J'(x)$ a.e. Set $\rho(x) = q(x)\, s^*(x)$, $x \in G$. Then each $\rho(x)$ is a partial isometry having $\text{In.}\ \rho(x) = J(x)$ and $\mathcal{R}(\rho(x)) = J'(x)$ almost everywhere. It is also clear that ρ is a measurable function. Observing that for each $F' \in L_2(G ; \mathbb{H}_n, \sigma)$,

$$S\, F'(x) = s(x)\, F'(x)$$

$$Q\, F'(x) = q(x)\, F'(x)$$

for a.e. x and that $T = QS^*$, it follows that for each $F \in J\, L_2(G ; \mathbb{H}, \sigma)$

$$(T\, F)\ (x) = \rho(x)\ F(x) \text{ a.e.}$$

Recall from (7.6.1) that

$$T^{-1}\ U^{A'}\ T = U^{A},$$

so that for a.e. g, for each $F \in J\, L_2(G ; \mathbb{H}, \sigma)$

$$[\rho^*(x)\ A'(g,x)\ \rho(x + \phi(g)]\ F(x + \phi(g)) = A(g,x)\ F(x + \phi(g))$$

for a.e. x. From the uniqueness assertion of theorem, 7.5, this implies

$$\rho^*(x)\ A'(g,x)\ \rho(x + \phi(g)) = A(g,x), \text{ a.e.} (g,x).$$

QED

COROLLARY. (U^A, P^J) is equivalent to the regular system acting in $L_2(G, \mathbb{H}_1, \sigma)$ for some subspace $\mathbb{H}_1 \subseteq \mathbb{H}$ if and only if A is a coboundary in the extended sense.

PROOF. The Corollary follows on noting that the regular system (Example (1), 7.2) acting in $L_2(G ; H_1, \sigma)$ is the system (U^{A_0}, P^J) where

$$A_0(g, x) = I_{H_1}, \text{ a.e. } (g, x).$$

8. Irreducible Systems

8.1. Let (U, P) be system of imprimitivity based on (G_0, G) acting in K. Let R be an orthogonal projection defined in K, $R \neq 0$, such that

$$(8.1.1) \qquad \begin{aligned} U_g R &= R U_g \\ P(D)R &= R P(D) \end{aligned}$$

for every $g \in G_0$ and every Borel set $D \subseteq G$. Then the restriction of U_g's to the range of R defines a representation U^0 of G_0. Similarly, the restrictions of $P(D)$ define a spectral measure P^0 on G and obviously, (U^0, P^0) is a system of imprimitivity. A system (U^0, P^0) obtained in the manner from (U, P) through a projection R satisfying (8.1.1) is called a subsystem of (U, P). Note that in that case the complementary projection $R^{\perp}$ also defines a subsystem (U^1, P^1) of (U, P) and moreover,

$$(U, P) = (U^0, P^0) \oplus (U^1, P^1).$$

DEFINITION. A system of imprimitivity (U, P) based on (G_0, G) and acting in $\mathbb{K}$ is said to be irreducible if (U, P) does not have any nontrivial subsystem, that is, if for any orthogonal projection R in $\mathbb{K}$, $R \neq 0$, satisfying (8.1.1), $R = I$.

In this section we shall study the subsystems of the systems of imprimitivity (U^{A}, P^{J}) and find out when (U^{A}, P^{J}) is irreducible.

8.2. THEOREM. A system of imprimitivity (U, P) based on (G_0, G) is a sub-system of (U^{A}, P^{J}) if and only if $(U, P) = (U^{A'}, P^{J'})$, where J' is a measurable range function on G such that

$$(8.2.1)\quad \begin{array}{ll} \text{(i)} & J'(x) \subseteq J(x) \quad \text{a.e., and} \\ \text{(ii)} & A(g, x)\, J'(x+\phi(g)) = J'(x) \quad \text{a.e. } (g, x) \end{array}$$

and A' is obtained from A by the formula

$$(8.2.2)\quad A'(g, x)\xi = \begin{cases} A(g, x)\,\xi & \text{if } \xi \in J'(x + \phi(g)) \\ 0 & \text{if } \xi \in [J'(x+\phi(g))]^{\perp}. \end{cases}$$

PROOF. Let the sub-system (U, P) of (U^{A}, P^{J}) be defined by the orthogonal projection in $JL_2(G ; \mathbb{H}, \sigma)$ onto the subspace M. Then (8.1.1) gives

$$(8.2.3)\quad \begin{array}{l} P^{J}(D)\, M \subseteq M \\ U^{A}_{g}\, M = M \end{array}$$

for all Borel subsets $D \subseteq G$ and $g \in G_0$ and U is the restriction of U^A to M and P that of P^J to M. The first relation of (8.2.3) implies, by theorem, 2.7, that $M = J'L_2(G ; \mathbb{H}, \sigma)$ for a measurable range functions J' and since $M \subseteq J\, L_2(G ; \mathbb{H}, \sigma)$, (i) of (8.2.1) is satisfied. Again

$$(U_g^A F)(x) = A(g,x)\, F(x + \phi(g))$$

for each $F \in J\, L_2(G ; \mathbb{H}, \sigma)$, for a.e.x. Therefore, the second relation in (8.2.3) implies condition (ii) of (8.2.1). Next, it is easy to see that if A' is defined by (8.2.2), then $U^{A'}$ is indeed the restriction of U^A to M.

Conversely, if J' satisfies (8.2.1) it is easy that $M = J'L_2(G ; \mathbb{H}, \sigma)$ will satisfy (8.2.3) and that with A' defined by (8.2.2), $U^{A'}$ is the restriction of U^A to $J'L_2(G ; \mathbb{H}, \sigma)$. Hence $(U^{A'}, P^{J'})$ is a sub-system of (U^A, P^J).

QED

The following important corollary is immediate from the theorem.

COROLLARY. (U^A, P^J) is irreducible if and only if for any measurable range function J', $J'(x) \subseteq J(x)$ a.e. x, which satisfies

(8.2.4) $$A(g,x)\, J'(x + \phi(g)) = J'(x) \quad \text{a.e. } (g,x),$$

one has

(8.2.5) either $J'(x) = \{0\}$ a.e. or $J'(x) = J(x)$ a.e.

8.3 Consider now the regular system of imprimitivity (U,P) based on (G_o, G) and acting in $L_2(G ; \mathbb{H}, \sigma)$. We claim that (U,P) is irreducible if and only if $\mathbb{H}$ is one-dimensional. This is because if $\mathbb{H}$ can be written as $\mathbb{H}_1 \oplus \mathbb{H}_2$, $\mathbb{H}_1$ being non-trivial, then $J'(x) = \mathbb{H}_1$ a.e. would satisfy (8.2.4). On the other hand, more generally, if J has dimension 1, then (U^A, P^J) is irreducible because any J' satisfying (8.2.4) would have constant dimension, and hence satisfy (8.2.5). Again if (U^{A_1}, P^{J_1}) and (U^{A_2}, P^{J_2}) are equivalent then there exists a one-one correspondence between their subsystems and one is irreducible if and only if the other is. Thus, if A is a coboundary in the extended sense, then (U^A, P^J) is irreducible if and only if J is one-dimensional. In the case when $\phi(G_o)$ has full measure in G, we conclude from corollary, 6.8, that (U^A, P^J) is irreducible if and only if J is one-dimensional. This, however, is not true in general and we give an example of an irreducible system (U^A, P^J) where J has dimension greater than one. This example, due to M.G.Nadkarni, is indeed an observation on an example in Varadarajan ([15]; Ch.I, Sec.5, Ex.1).

8.4. Let (U, P) be a system of imprimitivity based on (G_o, G) and acting in $\mathbb{K}$ and (V, Q) the dual system based on $(\hat{G}, \hat{G}_o)$. If an orthogonal projection R satisfies (8.1.1) with (U,P), then from the remarks following Stone's theorem, 3.2,

$$V_h\, R = R\, V_h.$$

$$Q(E)R = R\, Q(E)$$

for all $h \in \hat{G}$ and Borel subsets $E \subseteq \hat{G}_o$. Therefore, (U, P) is irreducible if and only if the duel system (V, Q) is irreducible.

EXAMPLE. Consider the pair (N, T) of (ii), 4.3. Let p be a fixed integer $p \geq 2$. On T let β be the function

$$\beta(z) = z^p, \quad z \in T.$$

Let now C_β be the (N, T) cocycle obtained from β :

$$C_\beta(n, z) = \begin{cases} z^p(\theta z)^p \ldots (\theta^{n-1} z)^p, & n > 0 \\ 1, & n = 0 \\ (\theta^{-1} z)^p \ldots (\theta^n z)^p, & n < 0, \end{cases}$$

where $\theta = \phi(1)$, $\theta^n = \phi(n)$, $n \in N$. Then

$$C_\beta(n, z) = c_n\, z^{np}, \; n \in N, z \in T$$

where c_n is a constant of absolute value one.

Consider the system of imprimitivity (U^{C_β}, P) based on (N, T) acting in $L_2(T, \sigma)$, which is clearly irreducible.

Next, observe that the dual pair of (N, T) is (N, T) itself. If (V, Q) is the dual system of (U, P), then (V, Q) is an irreducible system. We shall now show that (V, Q) is equivalent to a system

(U^A, P), where A is an $(N, T, \mathcal{U}(H_p))$-cocycle. By theorem, 7.4, it is enough to show that Q is homogeneous of multiplicity p having the measure class $\bar{\sigma}$.

For each λ, $0 \leq \lambda \leq p-1$, let f_λ be the function in $L_2(T, \sigma)$, $f_\lambda(z) = z^\lambda$, $z \in T$. Now for each $n \in N$ and each λ, $0 \leq \lambda \leq p-1$,

$$U_n f_\lambda(z) = c_n z^{pn} z^\lambda = c_n z^{pn+\lambda}, \text{ a.e. } z.$$

Thus, for each λ, $0 \leq \lambda \leq p-1$, $\mathcal{C}(\{U_n f_\lambda, n \in N\}) = \mathcal{C}(\{z^{pn+\lambda}, n \in N\})$ is a cyclic subspace for U and hence, a cyclic subspace for Q, generated by f_λ. Further $L_2(T, \sigma) = \bigoplus_{0 \leq \lambda \leq p-1} \mathcal{C}(\{z^{pn+\lambda}, n \in N\})$. Now for any λ, $0 \leq \lambda \leq p-1$, the numerical measure on T, $E \to (Q(E) f_\lambda, f_\lambda)$ has the Fourier coefficients

$$\begin{aligned} &\int_T z^{-n} (Q(dz) f_\lambda, f_\lambda) \\ &= (U_n f_\lambda, f_\lambda) \\ &= \int_T c_n z^{pn+\lambda} \bar{z}^\lambda \, d\sigma(z) \\ &= \int_T c_n z^{pn} \, d\sigma(z) \\ &\neq 0, \text{ if and only if } n = 0. \end{aligned}$$

Therefore, the measure $E \to (Q(E) f_\chi, f_\chi)$ is a constant multiple of σ. Hence, Q is homogeneous of multiplicity p having the measure class $\bar{\sigma}$.

8.5. We shall restrict our attention to the pair (R, B) for the rest of this section. Let us call a $(G_0, G, \underline{P}(\mathbb{H}))$-cocycle A irreducible if the system of imprimitivity (U^A, P^J) is irreducible. From corollary, 8.2, A is irreducible if and only if for any measurable range function J' satisfying (8.2.4), one has (8.2.5). Now if A has range J of dimension n and A' is the $\underline{U}(\mathbb{H}_n)$-cocycle corresponding to A in theorem, 6.8, then A and A' are simultaneously irreducible. We can, therefore, restrict our search for irreducible cocycles to the class of $\underline{U}(\mathbb{H})$-cocycles. Further, since every $(R, B, \underline{U}(\mathbb{H}))$-cocycle is cohomologous to an extended Gamelin cocycle, it is enough to consider only the latter. Our next theorem gives necessary and sufficient conditions for an extended Gamelin cocycle to be irreducible.

THEOREM. Let β be a measurable function on K taking values in $\underline{U}(\mathbb{H})$. Then the Gamelin cocycle C_β and the extended Gamelin cocycle $\overline{C}_\beta$ are both irreducible if one of them is. Moreover, this happens if and only if for any measurable range function J on K such that

(8.5.1) $$J(y) = \beta(y)\, J(y + e_1) \text{ a.e. } y,$$

one has

(8.5.2) $\quad$ either $J(y) = \{0\}$ a.e. y or $J(y) = H$ a.e. y.

PROOF. First, we shall show that C_β is irreducible if and only if $\overline{C}_\beta$ is irreducible. Suppose J_o is a measurable range function on K such that

(8.5.3) $\quad C_\beta(n,y)\, J_o(y + e_n) = J_o(y), \quad$ a.e. $(n,y) \in N \times K$.

Define a range function J on B, using the direct product representation $x = y + e_s$, $x \in B$, $y \in K$, $0 \leq s < 1$,

$$J(y + e_s) = J_o(y)\ , \quad y \in K,\ 0 \leq s < 1.$$

J is a measurable range function and J is trivial if and only if J_o is trivial. Recall that

$$\overline{C}_\beta(t, y+e_s) = C_\beta([t+s], y), \quad t \in R,\ y \in K,\ 0 \leq s < 1.$$

Now $\quad \overline{C}_\beta(t, y+e_s) \ J(y+e_s+e_t)$

$$= C_\beta([t+s], y) \ J_o(y + e_{[s+t]})$$

$$= J_o(y) \qquad [\text{ by } (8.5.3)]$$

$$= J(y + e_s)$$

for a.e. $(t,s,y) \in R \times [0,1) \times K$. That is,

(8.5.4) $\quad \overline{C}_\beta(t,x)\, J(x+e_t) = J(x)$ a.e. $(t,x) \in R \times B$.

Now if C_β is not irreducible, there exists a nontrivial range function J_o on K satisfying (8.5.3). Then $\bar{C}_\beta$ satisfies (8.5.4) for a nontrivial measurable range function J and hence is not irreducible.

Conversely, suppose that $\bar{C}_\beta$ is not irreducible. This means that there exists a non-trivial measurable range function J on B such that

(8.5.5) $\quad \bar{C}_\beta(t,x)\, J(x+e_t) = J(x)$, a.e. (t,x) .

Using the direct product representation we get

(8.5.6) $\quad \bar{C}_\beta(t,\ y+e_s)\, J(y+e_s+e_t) = J(y+e_s)$,

for a.e. $(t,s,y) \varepsilon R \times [0,1) \times K$. By Fubini's theorem, for a.e. $s \varepsilon [0,1)$, (t,s,y) satisfies (8.5.6) for a.e. $(t,y) \varepsilon R \times K$. Again since J is non-trivial and has constant dimension, for a.e. $s \varepsilon [0,1)$, $y \to J(y + e_s)$ is non-trivial on K. Now choose $s_o \varepsilon [0,1)$ such that (i) for a.e. $(t,y) \varepsilon R \times K$, $(t,\ s_o,y)$ satisfies (8.5.6) and (ii) if $J_o(x) = J(x + e_{s_o})$, $x \varepsilon B$, then the restriction of J_o to K is non-trivial. We recall

$$\bar{C}_\beta\ (t,\ y + e_{s_o}) = C_\beta\ ([t + s_o],\ y)$$
$$= C_\beta(n,\ y) \quad \text{for} \quad t \varepsilon [n-s_o,\ n+1-s_o).$$

From (8.5.6)

(8.5.7) $\quad C_\beta(n,y)\, J_o(y + e_t) = J_o(y)$,

a.e. $(t,y) \varepsilon [n-s_o,\ n+1-s_o) \times K$. Noting that $C_\beta(0,\ y) = I, y \varepsilon K$, (8.5.7) gives

(8.5.8) $J_o(y+e_t) = J_o(y)$, a.e. $(t,y) \in [-s_o, 1-s_o) \times K$.

Writing $y + e_n$ for y in (8.5.8) we conclude

(8.5.9) $J_o(y + e_t) = J_o(y + e_n)$,

a.e. $(t,y) \in [n-s_o, n+1-s_o) \times K$, for each n. Now (8.5.9) and (8.5.7) together imply, for each n,

$$C_\beta(n,y)\, J_o(y + e_n) = J_o(y), \quad \text{a.e. } y$$

Since J_o restricted to K is assumed to be nontrivial this shows that C_β is not irreducible.

The second assertion will be proved if we show that for any measurable range function J on K

$$\beta(y)\, J(y + e_1) = J(y), \quad \text{a.e. } y \in K$$

if and only if

$$C_\beta(n,y)\, J(y + e_n) = J(y), \quad \text{a.e. } (n,y) \in N \times K.$$

But this follows readily on recalling the definition of C_β, (5.4.1).

QED

8.6 **EXAMPLE.** Consider the special case of (R, B), when Γ is the free group generated by 1 and 2π and consequently $B = T^2$ (Example, 4.6). The pair (N, K) here is the pair (N, T). Now consider the example, 8.4. For every $p \geq 2$, there exists an irreducible system of imprimitivity (U, P) acting in $L_2(T; \mathbb{H}_p, \sigma)$, so that there exists an irreducible Gamelin cocycle C_β where $\beta(x) \in \underline{U}(\mathbb{H}_p)$, $x \in K$. By theorem, 8.1, we have for every $p \geq 2$ an irreducible extended Gamelin cocycle $\overline{C}_\beta$ where $\beta(x) \in \underline{U}(\mathbb{H}_p)$. This shows the existence, for each $p \geq 2$, of an irreducible system of imprimitivity (U, P^J) based on (R, T^2) where J has dimension p.

9. SUMMARY. The spaces $L_2(X ; \mathbb{H}, \alpha)$ are introduced in section 1. $L_2(X ; \mathbb{H}, \alpha)$ is the Hilbert space of square integrable functions on the measure space $(X, \underline{X}, \alpha)$ having values in a separable Hilbert space $\mathbb{H}$.

Section 2 is devoted to spectral measures. The emphasis is on canonical spectral measures ; we find the projections commuting with the canonical spectral measure on $(X, \underline{X})$ acting in $L_2(X ; \mathbb{H}, \alpha)$. We also determine conditions under which the restriction of the canonical spectral measure becomes equivalent to a canonical spectral measure itself. In section 3 we state a few useful facts from the theory of representations of locally compact abelian groups. In section 4 we introduce our setting for cocycles and systems of imprimitivity : a pair of locally compact second countable abelian groups (G_0, G). We demand that G_0 be mapped onto a dense subgroup of G by a one-to-one continuous homomorphism ϕ. Such pairs were considered by de Leeuw and Glicksberg ([1]).

In section 5 we discuss cocycles on $G_0 \rtimes G$ whose values are unitary operators on a Hilbert space. This is the usual kind of cocycles studied in group representations. In the special case when $G_0 = R$ and $G = B$, a Bohr group with a countable dual, we determine the structure of cocycle by unitary operator-valued functions on a subgroup of B, generalising Gamelin's work on scalar-valued cocycles ([2], Ch. VII). Next, in section 6, we extend the notion of cocycles by allowing them to have partial isometry-values. Through certain measure-theoretic observations, we set forth the cohomology theory of these cocycles. We also show that theoretically the extended class of cocycles can be obtained from unitary operator-valued cocycles.

In section 7 we give our main theorems. We show that the class of systems of imprimitivity based on (G_0, G) and acting in subspaces of $L_2(G ; H, \sigma)$ (σ is the Haar measure on G) arises from cocycles on $G_0 \rtimes G$ whose values are partial isometries on H. We show that this correspondence between cocycles and systems of imprimitivity is one-to-one. Section 8 deals with irreducible systems of imprimitivity. Using the structure theorem of section 5 we find a necessary and sufficient condition on a cocycle on $R \rtimes B$ such that the corresponding

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system of imprimitivity is irreducible. It turns out that a system of imprimitivity is irreducible if it acts in $L_2(G, \sigma)$. We give an example to show that there are irreducible systems which are not equivalent to systems acting in $L_2(G, \sigma)$.

CHAPTER II

INVARIANT SUBSPACES

1. Definitions.

B is a Bohr group with $\hat{B} = \Gamma$ countable. For $\delta \in \Gamma$, χ_δ will stand for the character corresponding to δ, that is, $\chi_\delta(x) = \langle x, \delta \rangle$, $x \in B$.

A subspace $M \subseteq L_2(B ; \mathbb{H}, \sigma)$ is said to be invariant if for each $\delta \in \Gamma$, $\delta > 0$, one has $\chi_\delta . M \subseteq M$.

If M is an invariant subspace, then $\chi_\delta . M \subseteq \chi_{\delta'} . M$, whenever $\delta, \delta' \in \Gamma$, $\delta > \delta'$. If M is such that $\chi_{\delta_0} . M = M$ for some $\delta_0 \in \Gamma$, $\delta_0 > 0$, then $\chi_{n\delta_0} . M = M$ for each integer n and if δ is an arbitrary element in Γ, we have, for $n\delta_0 \leq \delta \leq (n+1)\delta_0$,

$$M = \chi_{(n+1)\delta_0} . M \subseteq \chi_\delta . M \subseteq \chi_{n\delta_0} . M = M.$$

Thus in this case $\chi_\delta . M = M$ for all $\delta \in \Gamma$.

An invariant subspace M of $L_2(B ; \mathbb{H}, \sigma)$ is said to be doubly invariant if $\chi_\delta . M = M$ for all $\delta \in \Gamma$. M is called simply invariant if $\chi_\delta . M \subset M$ for each $\delta \in \Gamma$, $\delta > 0$.

2. Fourier Series and the Plancherel Theorem.

2.1 The concept of Fourier series has a natural extension to $L_2(B ; \mathbb{H}, \sigma)$ (see Helson [4], lecture VI). Let $F \in L_2(B ; \mathbb{H}, \sigma)$ and set, for $\delta \in \Gamma$,

$$\omega(\xi) = \int_B (F(x), \xi)\, \chi_\sigma(-x)\, d\sigma(x), \quad \xi \in H.$$

Then ω is a conjugate linear functional on $\mathbb{H}$ which is bounded because

$$|\omega(\xi)| \leq \|\xi\| \int_B \|F(x)\|\, d\sigma(x).$$

Hence, there exists a unique element $\hat{F}(\sigma) \in \mathbb{H}$ such that $\omega(\xi) = (\hat{F}(\sigma), \xi)$, $\xi \in \mathbb{H}$. This defines a function $\hat{F}$ on Γ which is called the Fourier transform of F. Note that the function $(F(\cdot), \xi) \in L_2(B, \sigma)$ has as its Fourier transform the function $(\hat{F}(\cdot), \xi)$ on Γ.

2.2. **THE PLANCHEREL THEOREM.** Let $(\xi_1, \xi_2, \ldots)$ be an orthonormal basis of $\mathbb{H}$. Let $F, F' \in L_2(B; \mathbb{H}, \sigma)$ and for each i, let f_i and f_i' be the i^{th} co-ordinate functions of F and F' respectively (with respect to $(\xi_1, \xi_2, \ldots)$),

$$\left.\begin{aligned} f_i(x) &= (F(x), \xi_i) \\ f_i'(x) &= (F'(x), \xi_i) \end{aligned}\right\} \quad x \in X,\ i = 1, 2, \ldots.$$

Then $f_i, f_i' \in L_2(B, \sigma)$, $i = 1, 2, \ldots$ and

$$(F, F') = \sum_{i \geq 1} (f_i, f_i').$$

By the ordinary Plancherel theorem

$$(f_i, f_i') = \sum_{\sigma \in \Gamma} \hat{f}_i(\sigma) \; \bar{\hat{f}}_i'(\sigma), \quad i = 1, 2, \ldots .$$

Therefore,

$$(F, F') = \sum_{i \geq 1} \sum_{\sigma \in \Gamma} \hat{f}_i(\sigma) \; \bar{\hat{f}}_i'(\sigma)$$

$$= \sum_{i \geq 1} \sum_{\sigma \in \Gamma} (\hat{F}(\sigma), \xi_i) \; \overline{(\hat{F}'(\sigma), \xi_i)}$$

$$= \sum_{\sigma \in \Gamma} \sum_{i \geq 1} (\hat{F}(\sigma), \xi_i) \; \overline{(\hat{F}'(\sigma), \xi_i)}$$

$$= \sum_{\sigma \in \Gamma} (\hat{F}(\sigma), \hat{F}'(\sigma)).$$

This is the **Plancherel** Theorem for $L_2(B ; \mathbb{H}, \sigma)$. By virtue of this theorem we can express any function $F \in L_2(B ; \mathbb{H}, \sigma)$ as

$$F = \sum_{\sigma \in \Gamma} \hat{F}(\sigma) \chi_\sigma ,$$

where the sum converges in $L_2(B ; \mathbb{H}, \sigma)$.

2.3. By a trigonometric polynomial we shall mean functions of the form

$$P(x) = \sum_{\sigma \in \Gamma} \xi_\sigma \chi_\sigma(x), \quad x \in B$$

where $\xi_\sigma \in \mathbb{H}$ and $\xi_\sigma \neq 0$ for only a finite no. of σ's in Γ. Note that

$$\hat{P}(\sigma) = \xi_\sigma , \quad \sigma \in \Gamma$$

and by Plancherel theorem the trigonometric polynomials are dense in $L_2(B\,;\,\mathbb{H},\sigma)$.

2.4. With the help of the Fourier transform we define a few subspaces of $L_2(B\,;\,\mathbb{H},\sigma)$ which are important for us.

Let $\mathbb{H}_1 \subseteq \mathbb{H}$ be a closed subspace. We shall always look upon $L_2(B\,;\,\mathbb{H}_1,\sigma)$ as a subspace of $L_2(B\,;\,\mathbb{H},\sigma)$. Now let

$$H_2(B\,;\,\mathbb{H}_1,\sigma) = \{F \in L_2(B;\mathbb{H}_1,\sigma) : \hat{F}(\delta) = 0 \text{ for all } \delta < 0\}$$

$$H_0(B\,;\,\mathbb{H}_1,\sigma) = \{F \in L_2(B;\mathbb{H}_1,\sigma) : \hat{F}(\delta) = 0 \text{ for all } \delta \leq 0\}.$$

Taking $\mathbb{H}_1 = \mathbb{H}$ we get the spaces $H_2(B\,;\,\mathbb{H},\sigma)$ and $H_0(B\,;\,\mathbb{H},\sigma)$ which are of special interest to us. The functions in the former subspace will be referred to as analytic functions.

3. Doubly Invariant Subspaces.

3.1. Consider the operators of multiplication by characters χ_δ = $\delta \in \Gamma$, on the space $L_2(B\,;\,\mathbb{H},\sigma)$. They define a representation of the group Γ acting in $L_2(B\,;\,\mathbb{H},\sigma)$. Now for F, F' in $L_2(B\,;\,\mathbb{H},\sigma)$,

$$(\chi_\delta F, F') = \int_B \chi_\delta(x)\,(F(x), F'(x))\,d\sigma(x)$$

$$= \int_B \chi_\delta(x)\,d\,\alpha_{F,F'}(x)$$

where the measure $\alpha_{F,F'}$ is given by

$$\alpha_{F,F'}(D) = \int_D (F(x), F'(x))\, d\sigma(x)$$
$$= (1_D.F, F')$$

for each Borel subset $D \subseteq B$. Thus the spectral measure P corresponding to the representation, $\theta \to$ multiplication by χ_θ, of Γ (appearing in Stone's theorem, I 3.2)is the canonical spectral measure. Therefore, a subspace $M \subseteq L_2(B; \mathbb{H}, \sigma)$ is invariant under multiplication by each χ_θ, $\theta \in \Gamma$ if and only if M is invariant under $P(D)$ for each Borel subset $D \subseteq B$. Hence theorem, I 2.7, yields the following characterisation of doubly invariant subspaces.

THEOREM. For every measurable range function J on B whose values are closed subspaces of $\mathbb{H}$, $J\, L_2(B ; \mathbb{H}, \sigma)$ is a doubly invariant subspace of $L_2(B ; \mathbb{H}, \sigma)$. Conversely, any doubly invariant subspace M of $L_2 (B ; \mathbb{H}, \sigma)$ is of the form $J\, L_2(B ; \mathbb{H}, \sigma)$ for a measurable range function J which is determined uniquely almost everywhere.

3.2. For any collection $\mathcal{L}$ of functions in $L_2(B ; \mathbb{H}, \sigma)$ we shall define the range of $\mathcal{L}$ to be the range function J associated to the smallest doubly invariant subspace containing $\mathcal{L}$. We shall say that $\mathcal{L}$ has full range if its range J is such that $J(x) = \mathbb{H}$ a.e. In particular, we shall be interested in the range of a simply invariant subspace. We record here an observation which will be useful later.

PROPOSITION. If M_1 and M_2 are mutually orthogonal invariant subspaces of $L_2(B\,;\,\mathbb{H},\sigma)$ with range functions J_1 and J_2 respectively, then for a.e. x, $J_1(x) \perp J_2(x)$.

PROOF. From the definition of range functions of M_1 and M_2, $J_i\, L_2(B\,;\,\mathbb{H},\sigma)$ is the smallest doubly invariant subspace containing M_i, $i = 1,2$. Let $F_1 \in M_1$ and $F_2 \in M_2$; since M_1 and M_2 are invariant

$$F_1 \perp \chi_\delta F_2 \quad \text{for all } \delta \geq 0, \quad \text{and}$$

$$\chi_\delta F_1 \perp F_2 \quad \text{for all } \delta \geq 0.$$

Combining, $\chi_\delta F_1 \perp \chi_{\delta'} F_2$ for all $\delta, \delta' \in \Gamma$. Therefore, $J_1\, L_2(B\,;\,\mathbb{H},\sigma)$ and $J_2\, L_2(B\,;\,\mathbb{H},\sigma)$ are mutually orthogonal. Hence the proposition.

4. <u>Simply Invariant Subspaces.</u>

4.1. Let M be a simply invariant subspace of $L_2(B\,;\,\mathbb{H},\sigma)$. If $\delta, \delta' \in \Gamma$ and $\delta > \delta'$, then $\chi_\delta . M$ is a proper subset of $\chi_{\delta'} . M$. Now for each $s \in R$, define

$$M_s = \bigcap_{\substack{\delta < s \\ \delta \in \Gamma}} \chi_\delta . M.$$

family $\{M_s, s \in R\}$ obtained from M will be called the <u>characteristic family of M</u>. It has the following properties :

(i) $\chi_\delta . M_s = M_{s+\delta}$ for each $\delta \in \Gamma$, $s \in R$.

(ii) Each M_s, $s \in R$ is a simply invariant subspace of $L_2(B ; \mathbb{H}, \sigma)$.

(iii) $\{M_s, s \in R\}$ is a strictly decreasing left-continuous family, that is, for $s' < s$, $M_s \subset M_{s'}$ and for each $s \in R$ $\bigcap_{s'<s} M_{s'} = M_s$.

(iv) If $M_\infty = \bigcap_{s \in R} M_s$ and $M_{-\infty} = \bigvee_{s \in R} M_s$, then M_∞ and $M_{-\infty}$ are doubly invariant. M_∞ is the largest doubly invariant subspace contained in M and $M_{-\infty}$ is the smallest doubly invariant subspace containing M.

(v) If $M_{0+} = \bigvee_{s>0} M_s$, then $M_{0+} \subsetneqq M \subsetneqq M_0$; M_{0+} and M_0 are respectively called the right hand and the left hand versions of M.

(vi) If M' is a simply invariant subspace, such that $M_{0+} \subsetneqq M' \subsetneqq M_0$, then for each $s \in R$, $M_s = M'_s$, where $\{M'_s ; s \in R\}$ is the characteristic family of M'.

4.2, DEFINITION. A simply invariant subspace $M \subsetneqq L_2(B ; \mathbb{H}, \sigma)$ is said to be exact if $M_\infty = \{0\}$. M is called left - (right -)

continuous if $M = M_0$ ($M = M_{0+}$, respectively) and continuous if it is both left and right-continuous.

Note that according to this definition M can be continuous without $\{M_s, s \in R\}$ being continuous at all points.

EXAMPLES :

(i) Consider the case when $\mathbb{H}$ is one-dimensional, that is, when we work with the *ordinary* $L_2(B, \sigma)$. The two prominent simply invariant subspaces are

$$H_2(B,\sigma) = \{f \in L_2(B,\sigma) : \hat{f}(\delta) = 0 \text{ for all } \delta < 0\}$$

$$H_0(B,\sigma) = \{f \in L_2(B,\sigma) : \hat{f}(\delta) = 0 \text{ for all } \delta \leq 0\}$$

If M denotes either of the spaces, then $M_\infty = \{0\}$ and $M_{-\infty} = L_2(B, \sigma)$. Therefore, both the spaces are exact. Further $M_0 = H_2(B, \sigma)$ and $M_{0+} = H_0(B, \sigma)$ so that the first space is left-continuous but not right-continuous and the second space is only right-continuous.

(ii) Let $\mathbb{H}_1 \subseteq \mathbb{H}$ be a closed subspace. From 2.4, recall the spaces $H_2(B ; \mathbb{H}_1, \sigma)$ and $H_0(B ; \mathbb{H}_1, \sigma)$. By Plancherel theorem, 2.2, they are seen to be exact simply invariant subspaces of $L_2(B ; \mathbb{H}_1, \sigma)$. If $\mathbb{H}_1 = \mathbb{H}$, they have full range. Finally $H_2(B ; \mathbb{H}_1, \sigma)$ is left-continuous without being right-continuous and the opposite is true for $H_0(B ; \mathbb{H}_1, \sigma)$.

(iii) Let $s_0 \in R$, $s_0 \notin \Gamma$. Consider the subspace

$$M = \bigcap_{\delta < s_0} \chi_\delta H_2(B ; \mathbb{H}, \sigma).$$

From the Plancherel theorem, M is an exact simply invariant subspace having $M = M_{0+} = M_0$. That is, M is continuous.

Observe that the family $\{M_s, s \in R\}$ is not continuous at $s = s_0$, and M may be regarded as a trivial example of a continuous simply invariant subspace. However, there are nontrivial continuous subspaces and indeed, they are responsible for most of the developments in this chapter. The existence of nontrivial continuous simply invariant subspaces of $L_2(B, \sigma)$ was shown by Helson and Lowdenslager ([6]), Gamelin ([2], Ch. VII), Helson and Kahane ([8]) and Yale ([16]).

(iv) If $\mathbb{H}_1$ and $\mathbb{H}_2$ are complex separable Hilbert spaces, $H_2(B ; \mathbb{H}_1, \sigma) \oplus H_0(B ; \mathbb{H}_2, \sigma)$ is a simply invariant subspace of $L_2(B ; \mathbb{H}, \sigma)$ (where $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$) which is neither left-continuous nor right-continuous.

(v) Let J be a measurable range function such that there exists a finite or countable family of functions $F_i \in H_2(B ; \mathbb{H}, \sigma)$ for which $J(x) = \underline{C}(\{F_i(x) : i \geq 1\})$, a.e. Then the subspace $J\,H_2(B ; \mathbb{H}, \sigma) = \{F \in H_2(B ; \mathbb{H}, \sigma) : F(x) \in J(x) \text{ a.e.}\}$ is a simply invariant subspace of $L_2(B ; \mathbb{H}, \sigma)$. Since $H_2(B ; \mathbb{H}, \sigma)$ is exact, this subspace is also exact.

othing can be said, in general, about the continuity properties f $J H_2(B ; \mathbb{H}, \sigma)$. Note also that $J H_2(B ; \mathbb{H}, \sigma)$ has range

3. PROPOSITION. If M is a simply invariant subspace of $_2(B ; \mathbb{H}, \sigma)$, $M_0 \ominus M_{0+}$ has at most the dimension of $\mathbb{H}$. oreover, if $(F_1, F_2, \ldots)$ is an orthonormal basis of $_0 \ominus M_{0+}$, then for a.e. x, $(F_1(x), F_2(x), \ldots)$ is an rthonormal set in $\mathbb{H}$.

ROOF. The first assertion follows from the second. Since $_i \in M_0 \ominus M_{0+}$, $\chi_\delta \cdot F_i \in M_{0+}$ whenever $\delta > 0$, for each $i = 1, 2, \ldots$. For $i \neq j$

$$(\chi_\delta \cdot F_i, F_j) = 0$$

$$(F_i, \chi_\delta F_j) = 0$$

for $\delta \in \Gamma$, $\delta \geq 0$. This gives

$$\int \chi_\delta(x)\ (F_i(x), F_j(x))\ d\sigma(x) = 0$$

for all $\delta \in \Gamma$ (as $\bar{\chi}_\delta = \chi_{-\delta}$). Therefore, $(F_i(x), F_j(x)) = 0$ a.e. Again for each $i = 1, 2, \ldots$, $(F_i, \chi_\delta F_i) = 0$ for all $\delta > 0$, whence one has

$$\int \chi_\delta(x)\ (F_i(x), F_i(x))\ d\sigma(x) = 0$$

for all $\delta > 0$, and taking conjugates, for all $\delta < 0$ as well. Thus $(F_i(x), F_i(x))$ is a constant function. Since, $\| F_i \| = 1$, we conclude $(F_1(x), F_1(x)) = 1$ a.e. x. Hence a.e., $(F_1(x), F_2(x), \dots)$ is an orthonormal set, proving the proposition.

It is easy to see that the conclusions of the proposition are valid for the spaces $M_0 \ominus M$ and $M \ominus M_{0+}$ as well. In fact, let I stand for any of the spaces $M_0 \ominus M_{0+}$, $M_0 \ominus M$ and $M \ominus M_{0+}$ and assume that $I \neq \{0\}$. Then the crucial property of I is that $I \perp \chi_\delta \cdot I$ whenever $\delta > 0$ and hence $\chi_\delta \cdot I \perp \chi_{\delta'} \cdot I$ whenever $\delta \neq \delta'$. Now set $N = \bigoplus_{\delta \geq 0} \chi_\delta . I$. Clearly, N is a simply invariant subspace whose characteristic family $\{N_s, s \in R\}$ is given by $N_s = \bigoplus_{\delta \geq s} \chi_\delta . I$, $s \in R$. Therefore, N is exact and $N_0 = N$, $N_0 \ominus N_{0+} = I$. Prompted by this we make the following

DEFINITION. We call a **simply** invariant subspace M of $L_2(B ; \mathbb{H}, \sigma)$ free, or more explicitly, freely generated over a subspace I if $\chi_\delta . I \perp \chi_{\delta'} . I$ whenever $\delta \neq \delta'$ and $M = \bigoplus_{\delta \geq 0} \chi_\delta . I$.

If M is freely generated over I, M is exact and left-continuous, but not right-continuous, with $M_0 \ominus M_{0+} = I$. $M_{0+} = \bigoplus_{\delta > 0} \chi_\delta . I$ is the right-hand version of M. As an example, $H_2(B ; \mathbb{H}, \sigma)$ is a free simply invariant subspace, freely generated over the subspace I of constant functions, having the right hand version $H_0(B ; \mathbb{H}, \sigma)$. In 4.5 we shall determine all free simply invariant subspaces of $L_2(B ; \mathbb{H}, \sigma)$ in terms of $H_2(B ; \mathbb{H}, \sigma)$.

4.4. (i) Let M be a simply invariant subspace of $L_2(B; \mathbb{H}, \sigma)$ which is not exact. Consider the subspace $M \ominus M_\infty$, which is easily seen to be simply invariant. Further, for each $s \in R$, $M'_s = M_s \ominus M_\infty$, where $M' = M \ominus M_\infty$. Thus $M'_\infty = \{ 0 \}$, and that is, $M \ominus M_\infty$ is exact.

(ii) Suppose M is an exact simply invariant subspace which is not right-continuous. Consider the free simply invariant subspace N generated over $I = M \ominus M_{0+}$. Then $N \subseteq M$ and if $N \neq M$, we claim that $M \ominus N = M'$ is an exact simply invariant subspace which is right-continuous. To see this look at $N_{-\infty} = \underset{\delta \in \Gamma}{\oplus} \chi_\delta . I$. Since for $\delta < 0$, $\chi_\delta . I \perp M$, one has $M \ominus N = M \cap N_{-\infty}^{\perp}$. If $F \in M$ and $F \perp N_{-\infty}$ and if $\delta \geq 0$, then $\chi_\delta . F \in M$ as M is simply invariant and $\chi_\delta . F \perp N_{-\infty}$ as $N_{-\infty}$ is doubly invariant. Therefore, M' is invariant. But M' cannot be doubly invariant, because if it were, $\{ 0 \} \neq M' \subseteq M_\infty$ contradicts the fact that M is exact. Again $M'_\infty \subsetneq M_\infty$ and hence M' is exact as well. What is more important, M' is right-continuous as $\bigvee_{\delta > 0} \chi_\delta . M' = \bigvee_{\delta > 0} (M_\delta \ominus N_\delta) = M_{0+} \ominus N_{0+}$ $M \ominus N$. Thus M is the direct sum of a right-continuous simply invariant subspace and a free simply invariant space.

(iii) Similarly, if M is an exact simply invariant subspace which is not left-continuous M is the direct sum of a simply invariant subspace M' which is left-continuous and the right hand version N of a free simply invariant subspace. This is obtained by setting $N = \bigoplus_{\delta > 0} \chi_\delta \cdot (M_0 \ominus M)$ and $M' = M \ominus N$.

(iv) Observe that if N and M are simply invariant subspaces of $L_2(B ; \mathbb{H}, \sigma)$ and $N \subseteq M$, then $M \ominus N$ need not be simply invariant, even if it is non-empty.

4.5. In the theorem below we shall describe all free simply invariant subspaces. Recall that $\mathbb{P}(\mathbb{H})$ is the class of all non-zero partial isometries defined on $\mathbb{H}$. A measurable function $\rho : B \to \mathbb{P}(\mathbb{H})$ will be said to have constant initial space if for a.e. $x \in B$

$$\text{In. } \rho(x) = \mathbb{H}_1$$

for some fixed subspace $\mathbb{H}_1 \subseteq \mathbb{H}$. For any measurable function ρ, T_ρ will stand for the operators on $L_2(B ; \mathbb{H}, \sigma)$ induced by ρ through

$$(4.5.1) \qquad (T_\rho F)(x) = \rho(x) F(x)$$

for each $F \in L_2(B ; \mathbb{H}, \sigma)$, for a e. x.

THEOREM. If ρ is a measurable function, $\rho : B \to \mathbb{P}(\mathbb{H})$ having constant initial space, then $T_\rho H_2(B ; \mathbb{H}, \sigma)$ is a free simply invariant subspace of $L_2(B ; \mathbb{H}, \sigma)$. Conversely, given any free simply invariant subspace $M \subseteq L_2(B ; \mathbb{H}, \sigma)$ there exists a measurable function $\rho : B \to \mathbb{P}(\mathbb{H})$ having constant initial space such that $M = T_\rho H_2(B ; \mathbb{H}, \sigma)$. Further, if ρ_1 and ρ_2 are measurable $\mathbb{P}(\mathbb{H})$-valued functions with constant initial spaces such that $T_{\rho_1} H_2(B ; \mathbb{H}, \sigma) = T_{\rho_2} H_2(B ; \mathbb{H}, \sigma)$, then there exists $q \in \mathbb{P}(\mathbb{H})$ such that $\rho_1(x) = \rho_2(x)\, q$ almost everywhere.

PROOF. Let ρ be a measurable function $: B \to \mathbb{P}(\mathbb{H})$ having constant initial space $\mathbb{H}_1$. Then T_ρ defined by (4.5.1) is a partial isometry commuting with multiplication by χ_δ, $\delta \in \Gamma$. We have

$$T_\rho H_2(B ; \mathbb{H}, \sigma) = T_\rho \Big(\bigoplus_{\delta \geq 0} \chi_\delta \cdot \mathbb{H} \Big) = \bigoplus_{\delta \geq 0} T_\rho (\chi_\delta \cdot \mathbb{H}) = \bigoplus_{\delta \geq 0} \chi_\delta \cdot T_\rho (\mathbb{H})$$

showing that $T_\rho H_2(B ; \mathbb{H}, \sigma)$ is freely generated over $T_\rho(\mathbb{H})$.

Conversely, suppose that M is a simply invariant subspace freely generated over a subspace I. Let $(F_1, F_2, F_3, \dots)$ be an orthonormal basis of I. Since $I = M_0 \ominus M_{0+}$, from the proposition, 4.3, $(F_1(x), F_2(x), \dots)$ is an orthonormal set for a.e. x, say, for all $x \in B - N$, N being a null set. Now choose an orthonormal basis $(\xi_1, \xi_2, \xi_3, \dots)$ of $\mathbb{H}$. For $x \in B - N$, define

$$\rho(x)\,\xi_i = \begin{cases} F_i(x), & \text{if } i \leq \text{card.} \{F_j : j \geq 1\} \\ 0, & \text{if } i > \text{card.} \{F_j : j \geq 1\} \end{cases}$$

(notice that by proposition, 4.3, card. $\{F_j : j \geq 1\} \leq$ card. $\{\xi_i, i \geq 1\}$.). Then $\rho(x)$ extends to a partial isometry of $\mathbb{H}$ and In. $\rho(x) = \mathcal{G}\{\xi_i, i \leq \text{card.} \{F_j : j \geq 1\}\}$ for all $x \in B - N$. Next, define $\rho(x)$ for $x \in N$ to be a fixed partial isometry. The function ρ is measurable because $F_i(x) = \rho(x)\,\xi_i$ for all $x \in B-N$, $i = 1, 2, \dots$. Observe, now, that

$$I = \mathcal{G}\{F_i : i \geq 1\} = T_\rho(\mathbb{H}).$$

Since $M = \bigoplus_{\varepsilon \geq 0} X_\varepsilon . I$, it follows that

$$\begin{aligned} M &= \bigoplus_{\varepsilon \geq 0} X_\varepsilon . T_\rho(\mathbb{H}) \\ &= \bigoplus_{\varepsilon \geq 0} T_\rho(X_\varepsilon . \mathbb{H}) \\ &= T_\rho H_2(B ; \mathbb{H}, \sigma). \end{aligned}$$

Finally, suppose that ρ_1 and ρ_2 are measurable functions : $B \to \mathcal{P}(\mathbb{H})$ such that In. $\rho_i(x) = \mathbb{H}_i$ a.e., $i = 1,2$, and

$$T_{\rho_1} H_2(B ; \mathbb{H}, \sigma) = T_{\rho_2} H_2(B ; \mathbb{H}, \sigma)$$
$$= M \text{ (say)}.$$

Then from the first part of the theorem, M is freely generated over $I = M_0 - M_{0+}$ where $I = T_{\rho_1}(\mathbb{H}) = T_{\rho_1}(\mathbb{H}_1)$, and at the same time $I = T_{\rho_2}(\mathbb{H}) = T_{\rho_2}(\mathbb{H}_2)$. It follows immediately that $\mathcal{R}(\rho_1(x)) = \mathcal{R}(\rho_2(x))$ a.e., so that $\rho_2^*(x)\, \rho_1(x)$ has, almost everywhere, the constant initial space $\mathbb{H}_1$. Now if $\xi \in \mathbb{H}_1$, there exists $\eta \in \mathbb{H}_2$ such that

$$T_{\rho_1} \xi = T_{\rho_2} \eta ,$$

that is, $$\rho_1(x)\, \xi = \rho_2(x)\, \eta ,$$

or $$\rho_2^*(x)\, \rho_1(x)\, \xi = \eta , \quad \text{a.e. } x.$$

That is $\rho_2^*(x)\, \rho_1(x)$ sends constant functions into constant functions. Hence, there exists $q \in \mathcal{P}(\mathbb{H})$, such that

$$q = \rho_2^*(x)\ \rho_1(x) \quad \text{a.e.},$$

whence, $$\rho_2(x)\, q = \rho_1(x) \quad \text{a.e.}$$

QED

COROLLARY. A simply invariant subspace M of $L_2(B ; \mathbb{H}, \sigma)$ is the right hand version of a free simply invariant subspace if and only if M is of the form $T_\rho \; H_0(B ; \mathbb{H}, \sigma)$ where ρ is a measurable function $: B \to \mathfrak{P}(\mathbb{H})$ with constant initial space.

PROOF. The corollary follows on noting that $T_\rho \; H_0(B ; \mathbb{H}, \sigma)$ is the right hand version of $T_\rho \; H_2 (B ; \mathbb{H}, \sigma)$:

$$\underset{\delta>0}{\oplus} \chi_\delta \; T_\rho(\mathbb{H}) = T_\rho(\underset{\delta>0}{\oplus} \chi_\delta . \mathbb{H}) = T_\rho \; H_0(B ; \mathbb{H}, \sigma).$$

5. Simply invariant subspaces and systems of imprimitivity.

5.1. From this point onwards we shall consider exact simply invariant subspaces which are left-continuous and call them ESI subspaces. That is, a simply invariant subspace $M \subseteq L_2(B, \mathbb{H}, \sigma)$ is an ESI subspace if the characteristic family $\{ M_s, s \in R\}$ satisfies $\bigcap_{s \in R} M_s = M_\infty = \{ 0 \}$ and $M_0 = M$.

The characteristic family $\{M_s, s \in R\}$ of an ESI subspace M defines a spectral measure Q^M on R, where for any interval $[s_1, s_2) \subseteq R$, $Q^M([s_1, s_2))$ is the orthogonal projection defined in $M_{-\infty}$ onto the subspace $M_{s_1} \ominus M_{s_2}$. Let U be the representation of R defined by

(5.1.1) $U_t = \int_R \exp(its)\, d\, Q^M(s), \quad t \in R.$

U will be referred to as the representation given by M.

THEOREM. Let M be an ESI subspace of $L_2(B ; \mathbb{H}, \sigma)$ having range J. If U is the representation of R, given by M and acting in $M_{-\infty} = J\, L_2(B ; \mathbb{H}, \sigma)$, then (U, P^J) is a system of imprimitivity based on (R, B). Conversely, if for some measurable range function J, U is a representation of R acting in $J\, L_2(B ; \mathbb{H}, \sigma)$ such that (U, P^J) is a system of imprimitivity based on (R, B), then there exists a unique ESI subspace M having range J such that U is the representation given by M.

PROOF. Let M be an ESI subspace and Q^M the spectral measure on R obtained from the characteristic family $\{M_s, s \in R\}$. Consider the representation V of $\Gamma = \hat{B}$, where each V_σ consists in multiplication by $\chi_{-\sigma}$ on $JL_2(B ; \mathbb{H}, \sigma)$, $\sigma \in \Gamma$. Since

$$\chi_\sigma \cdot M_s = M_{s+\sigma}, \quad \sigma \in \Gamma, \ s \in R,$$

one has

(5.1.2) $V_\sigma^{-1}\, Q^M([s_1, s_2))\, V_\sigma = Q^M([s_1 + \sigma, s_2 + \sigma))$

for all $\sigma \in \Gamma$ and all intervals $[s_1, s_2)$. Hence for all Borel subsets $D \subseteq R$

$$V_\delta^{-1}\, Q^M(D)\, V_\delta = Q^M(D+\delta)\ ,\quad \delta \in \Gamma.$$

That is (V, Q^M) is a system of imprimitivity based on (Γ, R). Recall now the relationship of V and P^J from 3.1 and that of U and Q^M from (5.1.1). It is then clear that (U, P^J) is the dual system of (V, Q^M) and a fortiori, a system of imprimitivity.

Conversely, if U is such that (U, P^J) is a system of imprimitivity based on (R, B) and acting in $J\, L_2(B\, ;\, \mathbb{H}, \sigma)$, consider the dual system (V, Q) based on (Γ, R). For each $\delta \in \Gamma$, then, V_δ consists of multiplication by $\chi_{-\delta}$ and for each $s \in R$, let M_s stand for the range of the projection $Q([s, \infty))$. Since (V, Q) is a system of imprimitivity,

$$V_\delta^{-1}\, Q([s, \infty))\, V_\delta = Q([s+\delta, \infty)),$$

so that $\chi_\delta \cdot M_s = M_{s+\delta}$, $s \in R$, $\delta \in \Gamma$. Since $\{M_s,\ s \in R\}$ is a decreasing left-continuous family with $\bigcap_{s\in R} M_s = \{0\}$ (all this by virtue of Q^M being a spectral measure), M_0 is an SSI subspace and further $\{M_s,\ s \in R\}$ is the characteristic family of $M = M_0$. It follows that $Q = Q^M$ and since (V, Q) is the dual system of (U, P^J),

$$U_t = \int \exp(its)\, d\, Q(s),\quad t \in R.$$

that is, U is the representation given by M. Next, observe that if M_1 and M_2 are distinct ESI subspaces of $L_2(B\,;\mathbb{H},\sigma)$, then $Q^{M_1} \neq Q^{M_2}$ and hence the representations given by M_1 and are distinct. This proves the uniqueness assertion of the theorem. QED

COROLLARY 1. If J is the range of an exact simply invariant subspace M, then J has constant dimension.

PROOF. Consider M_0; it is an ESI subspace and has range J. By the theorem above, M_0 gives rise to a system of imprimitivity (U, P^J). Hence by lemma, I 7.3, P^J is homogeneous and therefore, from theorem, I 2.9, J has constant dimension.

COROLLARY 2. If M is a simply invariant subspace having one-dimensional range J, then M is exact.

PROOF. There is no loss of generality in assuming M to be left-continuous. Now $M = (M \ominus M_\infty) \oplus M_\infty$. If $M \ominus M_\infty$ has range J_1 and M_∞ has J_2, then $J = J_1 \oplus J_2$. Since $M \ominus M_\infty \neq \{0\}$, by Corollary 1, J_1 must have constant dimension 1, and hence $J_1 = J$. Hence $J_2 = \{0\}$ a.e. and $M_\infty = \{0\}$.

The theorem above establishes a one-to-one correspondence between ESI subspaces $M \subsetneq L_2(B, \mathbb{H}, \sigma)$ and systems of imprimitivity of the type (U, P^J) based on (R, B). The next result will show how two ESI subspaces M and M' relate to each other when the corresponding systems of imprimitivity (U, P^J) and $(U', P^{J'})$ are equivalent, U and U' being the representations given by M and M' respectively.

5.2 THEOREM. Let U and U' be the representations given by ESI subspaces M and M' respectively. Then (U, P^J) and $(U', P^{J'})$ are equivalent if and only if there exists a measurable function $\rho : B \to \underset{\sim}{P}(\mathbb{H})$ such that

(i) $\operatorname{In.} \rho(x) = J(x)$, $\underset{\sim}{R}\, \rho(x) = J'(x)$ for a.e. $x \in B$ and

(ii) $T_\rho M = M'$ where T_ρ is the operator on $JL_2(B ; \mathbb{H}, \sigma)$ induced by ρ by the relation

$$(T_\rho F)(x) = \rho(x) F(x) \quad \text{a.e.}, \; F \in J L_2(B ; \mathbb{H}, \alpha).$$

PROOF. From 7.6, Chapter I, (U, P^J) and $(U', P^{J'})$ are equivalent if and only if there exists a measurable function $\rho : B \to \underset{\sim}{P}(\mathbb{H})$ satisfying (i) such that for each $t \in R$,

$$U'_t = T_\rho U_t T_\rho^{-1} .$$

Hence,

$$Q^{M'} = T_\rho Q^M T_\rho^{-1} .$$

$$
\begin{aligned}
M' &= Q^{M'}([0, \infty))\,(J' L_2(B\,;\,\mathbb{H}, \sigma)] \\
&= T_\rho\, Q^M([0, \infty))\, T_\rho^{-1}[J' L_2(B\,;\,\mathbb{H}, \sigma)] \\
&= T_\rho\, Q^M([0, \infty))\, [J\, L_2(B\,;\,\mathbb{H}, \sigma)] \\
&= T_\rho(M)).
\end{aligned}
$$

QED

3. The next result shows that orthogonal sums of ESI subspaces rrespond to orthogonal sums of the corresponding systems of primitivity.

EOREM. Let M^i be an ESI subspace of $L_2(B\,;\,\mathbb{H}, \sigma)$ having nge J_i for each i in a finite or denumerable index set ch that for $i \neq j$, M^i and M^j are mutually orthogonal. t M be the ESI subspace $M = \underset{i}{\oplus} M^i$ having range $= \underset{i}{\oplus} J_i$. Then (U, P^J) $\underset{i}{\oplus} (U^i, P^{J_i})$, where U is the presentation given by M and for each i, U^i is the presentation given by M^i. Conversely, if for an ESI subspace with range J, the corresponding system of imprimitivity (U, P^J) is the direct sum of systems of imprimitivity (U^i, P^{J_i}), r i in a finite or denumerable index set, then the ESI bspaces M^i corresponding to (U^i, P^{J_i}) are mutually rthogonal and $M = \underset{i}{\oplus} M^i$.

ROOF. Let M be the orthogonal sum of ESI subspaces M^i aving range. J_i, $i \geq 1$. By proposition, 3.2, for $i \neq j$,

$J_i L_2(B\,;\,\mathbb{H},\sigma)$ and $J_j L_2(B\,;\,\mathbb{H},\sigma)$ are orthogonal. If now $\{M^i_s,\ s \in R\}$ be the characteristic family of M^i, $i \geq 1$, and $\{M_s,\ s \in R\}$ that of M, then for each $s \in R$,

$$M_s = \underset{i}{\oplus}\ M^i_s\ .$$

By the manner in which Q^M and the Q^{M_i}'s are defined we see that

$$Q^M = \underset{i}{\oplus}\ Q^{M_i}$$

whence

$$U = \underset{i}{\oplus}\, U^i\ .$$

Also $J = \underset{i}{\oplus}\ J_i$. Therefore,

$$(U,\ P^J) = \underset{i}{\oplus}\ (U^i,\ P^{J_i}).$$

To prove the converse, observe that since J_i, $i \geq 1$ are mutually orthogonal we have M^i and M^j orthogonal whenever $i \neq j$. Consider the ESI space $\underset{i}{\oplus}\ M^i$; by the previous part, the corresponding system of imprimitivity is $\underset{i}{\oplus}\ (U^i,\ P^{J_i})$, where $(U^i,\ P^{J_i})$ correspond to M^i, $i \geq 1$. Since $(U,\ P^J) = \underset{i}{\oplus}\ (U^i,\ P^{J_i})$ it follows that $M = \underset{i}{\oplus}\ M^i$. QED

We recall from 4.4 (ii) that, in general, if M' and M are ESI subspaces, $M' \subsetneq M$, then $M \ominus M'$ need not be simply invariant. If however $M \ominus M'$ is simply invariant, must be left-continuous and exact.

COROLLARY. Let M be an ESI subspaces having range J and the associated system of imprimitivity (U, P^J). Then the sub-systems (U^1, P^{J_1}) of (U, P^J) are precisely the systems of imprimitivity corresponding to ESI subspaces $M^1 \subsetneq M$ such that $M \ominus M^1$ is ESI.

PROOF. Let $M^1 \subseteq M$ and $M^2 = M \ominus M^1$ be both ESI subspaces. By the preceding theorem

$$(U, P^J) = (U^1, P^{J_1}) \oplus (U^2, P^{J_2})$$

where (U^i, P^{J_i}) is the system of imprimitivity corresponding to M^i, $i = 1,2$. Therefore, (U^1, P^{J_1}) is a sub-system of (U, P^J).

Conversely, if (U^1, P^{J_1}) is a sub-system of (U, P^J), there is the complementary sub-system (U^2, P^{J_2}) such that

$$(U, P^J) = (U^1, P^{J_1}) \oplus (U^2, P^{J_2}).$$

Suppose now that the ESI subspaces corresponding to (U^1, P^{J_1}) and (U^2, P^{J_2}) are M_1 and M_2 respectively. Then by the theorem above $M = M_1 \oplus M_2$. This shows that $M_2 = M \ominus M_1$ is simply invariant.

QED

4. THEOREM. Let $\mathbb{H}_1 \subseteq \mathbb{H}$ be a subspace. Then $H_2(B; \mathbb{H}_1, \sigma)$ [c]orresponds to the regular system of imprimitivity acting in $(B; \mathbb{H}_1, \sigma)$.

[PR]OOF. By Plancherel theorem, the characteristic family of [M =] $H_2(B; \mathbb{H}_1, \sigma)$ is

$$M_s = \bigoplus_{\substack{\delta \geq s \\ \delta \in \Gamma}} \chi_\delta \cdot \mathbb{H}_1 , \qquad s \in R.$$

[H]ence the spectral measure Q^M is concentrated on Γ, for [a]ny $\delta \in \Gamma$, $Q^M(\{\delta\})$ being the projection onto the subspace [χ_δ] . $\mathbb{H}_1$. Then the representation U given by M is

$$(U_t F)(x) = F(x + e_t)$$

[f]or each $F \in L_2(B; \mathbb{H}_1, \sigma)$, for a.e. x, as can be seen from

$$\begin{aligned} (U_t F, G) &= (\widehat{U_t F}, \hat{G}) \text{ [by Plancherel theorem]} \\ &= \Sigma\ e^{it\delta} (\hat{F}(\delta), \hat{G}(\delta)) \\ &= \Sigma\ e^{it\delta} (Q^M(\{\delta\}) F, G). \end{aligned}$$

Hence the theorem.

6. Simply Invariant Subspaces and Cocycles.

6.1. We shall now have the results of sec. 7, chapter I to bear upon the correspondence established in the last section. This will give the vectorial generalisation of the Helson-Lowdenslager ory of correspondence of simply invariant subspaces with cycles. Our first result is the principal correspondence theorem which is an immediate consequence of theorem, I 7.5, d theorem, 5.1, above.

THEOREM. There is a one-one correspondence between ESI subspaces $M \subseteq L_2(B ; \mathbb{H}, \sigma)$ and $(R, B, \mathfrak{P}(\mathbb{H}))$-cocycles under which A corresponds to M if and only if A and have the same range J and (U^A, P^J) is the system of primitivity corresponding to M.

If the cocycle A corresponds to M in the theorem, shall say that <u>A and M are in correspondence.</u>

COROLLARY. Let the ESI subspace M and the $(R, B, \mathfrak{P}(\mathbb{H}))$-cocycle A be in correspondence. Then M has full range if and only if A is a $\mathfrak{U}(\mathbb{H})$-cocycle.

PROOF. The corollary follows from the fact that a $\mathfrak{P}(\mathbb{H})$-cocycle having full range is a $\mathfrak{U}(\mathbb{H})$-cocycle.

The above theorem explains our insistence on the use of partial isometry-valued cocycles. They allow us to give an intrinsic description of ESI subspaces and there is a gain in definiteness.

6.2. THEOREM. Let the $\mathfrak{P}(\mathbb{H})$-cocycle A and the ESI subspace M be in correspondence. Then a $\mathfrak{P}(\mathbb{H})$-cocycle A', cohomologous (p) to A, is in correspondence with $M' = T_p M$.

OOF. From theorem, I 7.3, if J and J' stand for the ges of A and A' respectively, then (U^{A}, P^{J}) and $(U^{A'}, P^{J'})$ are equivalent through T_ρ, that is,

$$U^{A'} = T_\rho U^{A} T_\rho^{-1}$$

$$P^{J'} = T_\rho P^{J} T_\rho^{-1}$$

e result now follows from theorem, 5.2.

COROLLARY. An ESI subspace is free if and only if it corresponds a coboundary in the extended sense. If M and the coboundary f the operator function ρ are in correspondence, then $= T_\rho H_2(B ; \mathbb{H}, \sigma)$.

PROOF. For any subspace $\mathbb{H}_1 \subseteq \mathbb{H}$, it follows from theorem, 4.4, that $H_2(B ; \mathbb{H}_1, \sigma)$ is in correspondence with the cocycle given by

$$A(t, x) = I_{\mathbb{H}_1} \text{ a.e. } (t, x).$$

Next, if ρ has constant initial space $\mathbb{H}_1$, then

$$T_\rho H_2(B ; \mathbb{H}, \sigma) = T_\rho H_2 (B ; \mathbb{H}_1, \sigma).$$

The corollary follows from the theorem above and theorem, 4.5.

QED

6.3. In 4.4(ii) we have observed that an ESI subspace M is, in general, the orthogonal sum of two ESI subspaces M_1 and M_2, M_1 being free and M_2 being continuous, where one of the components may be missing. The Corollary, 6.2 shows that M_2 is missing if and only if M is in correspondence with a coboundary in the extended sense. The following necessary and sufficient condition for M_1 to be absent, that is, for M to be continuous, is obtained from theorem, I 8.2 and theorem, 5.3 above.

THEOREM. An ESI subspace M having range J is continuous if and only if for the corresponding cocycle A, there does not exist a measurable range function J' such that

(i) $J'(x) \subsetneq J(x)$ a.e., $J' \neq \{0\}$ and for a.e. (t, x), $A(t, x)\, J'(x + e_t) = J'(x)$,

(ii) the restriction of A to J' is a coboundary in the extended sense.

6.4. The subspaces M_s, $s \in R$ in the characteristic family of an ESI subspace M are ESI subspaces themselves. Here we shall examine the system of imprimitivity given by M_s in relation to the same given by M. Fix $s_0 \in R$. It is clear that

$$Q^{M_{s_0}}(D) = Q^M(D + s_0)$$

for any Borel subset $D \subseteq R$. If (U, P) is the system of imprimitivity given by M, then

$$U_t = \int_R \exp(its)\, d\, Q^M(s), \quad t \in R.$$

Hence $$\int_R \exp(its)\, d\, Q^{M_{s_0}}(s) = \exp(its_0)\, U_t, \quad t \in R.$$

Thus the system of imprimitivity given by M_{s_0} is $(\exp(its_0)U, P^J)$. Hence if A and M are in correspondence, then M_{s_0} is in correspondence with A' given by

$$A'(t, x) = \exp(its_0)\, A(t, x) \quad \text{a.e.}(t,x).$$

If $s_0 = \delta \in \Gamma$, then

$$A'(t, x) = X_{-\delta}(x)\, A(t, x)\, \overline{X_{-\delta}(x + e_t)}, \quad \text{a.e.}(t,x)$$

and hence A' and A are cohomologous. But in case $s_0 \notin \Gamma$, it can be shown that A and A' are not cohomologous (see Helson [5], I, p.149).

6.5. DEFINITION. A simply invariant subspace M is said to be irreducible if there does not exist a nontrivial simply invariant subspace $M' \subsetneq M$ such that $M \ominus M'$ is also simply invariant.

Since the range of an ESI subspace has constant dimension, it follows that any ESI subspace M having a one-dimensional

...ge function is irreducible. Next, from the Corollary, 5.3, ...get that an ESI subspace M is irreducible if and only if ...e corresponding system of imprimitivity is irreducible. From ...eorem, I 8.5, we immediately obtain

THEOREM. An ESI subspace M is irreducible if and only if the corresponding $\underline{U}(\mathbb{H})$-cocycle A is irreducible.

We have considered irreducible cocycles in section 8 of Chapter I and have characterised a class of measurable operator-functions $\beta : B \to \underline{U}(\mathbb{H})$ which give rise to irreducible $(\mathbb{H})$-cocycles and hence, irreducible ESI subspaces of $L_2(B ; \mathbb{H}, \sigma)$. Our examples, I 8.6, now yield irreducible ESI subspaces of $L_2(B ; \mathbb{H}_p, \sigma)$ in the special case $B = T^2$, one for each p having a range of dimension p, $2 \leq p < \infty$.

7. Analytic Functions on the Upper Half-plane.

7.1. Analytic functions on the upper half-plane $\Omega = \{ \omega : \text{Im } \omega > 0 \}$ are studied with the help of the linear fractional transformation

$$\tau : z \longrightarrow \omega = -i \frac{z+1}{z-1}$$

giving a conformal equivalence of the disc $D = \{ z : |z| < 1 \}$ with the upper half-plane. A function f on Ω is analytic if and only if $f \circ \tau$ is analytic on D. The map τ extends to the boundary $T = \{ z : |z| = 1\}$ of D taking T onto $R \cup \{\infty\}$ through

$$\tau : \quad z = e^{i\theta} \rightarrow t = \tan(\theta - \pi)/2, \; 0 \leq \theta < 2\pi .$$

An easy calculation on the latter map gives

$$\frac{1}{2\pi} \frac{d\theta}{dt} = \frac{1}{\pi} (1 + t^2)^{-1}$$

that is the normalised Lebesgue measure σ on the circle induces, through τ, the Cauchy measure μ on the line having the density $d\mu(t) = [\pi (1 + t^2)]^{-1} dt$. This makes the map : $f \rightarrow f \circ \tau$ an isometric isomorphism of $L_2(R, \mu)$ onto $L_2(T, \sigma)$.

7.2. $H_2(D)$ is the Hardy space consisting of analytic functions g on the open unit disc for which

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |g(r e^{i\theta})|^2 d\theta < \infty .$$

For the general theory of Hardy spaces we refer to Hoffman's book [9]. It is well-known that a Taylor series $\sum_{n=0}^{\infty} a_n z^n$ defines a function $g \in H_2(D)$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, that is, if and only if the one-sided Fourier series $\sum_{n=0}^{\infty} a_n \; e^{in\theta}$ defines a function g_0 in $L_2(T, \sigma)$ and, moreover,

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 = \| g_0 \|^2 .$$

This correspondence $g \to g_0$ allows us to identify $H_2(D)$ with the subspace $H_2(T) = \{h \in L_2(T,\sigma) : \hat{h}(n) = 0, \ n < 0\}$. Finally, by the celebrated Fatou's lemma, given any $g \in H_2(D)$, one has for a.e. $\theta \in [0, 2\pi)$, $g_0(e^{i\theta}) = \lim g(z)$ as $z \to e^{i\theta}$ non-tangentially which describes the identification. In particular,

$$g_0(e^{i\theta}) = \lim_{r \to 1} g(re^{i\theta}) \quad \text{a.e. } \theta.$$

7.3 Let λ stand for the Lebesgue measure on R. $H_2(\Omega)$ is the space of analytic functions f in the upper half-plane Ω such that the L_2-norms

(7.3.1) $$\| f_y \| = \left(\int_R |f(x + iy)|^2 \, d\lambda(x) \right)^{\frac{1}{2}}$$

are bounded for $y > 0$. It is a Banach space with the norm $\| f \| = \sup_{y>0} \| f_y \|$. We list a few properties of functions in $H_2(\Omega)$.

(7.3.2) For each $f \in H_2(\Omega)$, the boundary function $f_0(x) = \lim_{\omega \to x \text{ nontangentially}} f(\omega)$ is defined for a.e. $x \in R$ and $f_0 \in L_2(R, \lambda)$. Further,

$$\|f\|_{H_2(\Omega)} = \|f_o\|_{L_2(R,\lambda)}.$$

This makes $H_2(\Omega)$ into a Hilbert space with the inner product

$$(f,g)_{H_2(\Omega)} = (f_o, g_o)_{L_2(R,\lambda)}.$$

(7.3.3) PALEY-WIENER THEOREM. A function f on Ω belongs to $H_2(\Omega)$ if and only if

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(t)\, e^{i\omega t}\, d\lambda(t), \quad \omega \in \Omega$$

for some function $\hat{f} \in L_2([0, \infty), \lambda)$. Moreover, the boundary function f_o is the (inverse) Fourier-Plancherel transform of $\hat{f}$,

$$f_o(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(t)\, e^{ixt}\, d\lambda(t), \quad \text{a.e.} \quad x \in R.$$

(7.3.4) From the Paley-Wiener theorem the boundary functions f_o for functions $f \in H_2(\Omega)$ are seen to be precisely the class of (inverse) Fourier-Plancherel transforms of elements in $L_2([0, \infty), \lambda)$. Using the Plancherel theorem, therefore, they form the subspace $H_2(R,\lambda)$ of $L_2(R,\lambda)$ consisting of functions $f_o \in L_2(R,\lambda)$ for which the Fourier-Plancherel transform $\hat{f}$,

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f_o(x)\, e^{-itx}\, d\lambda(x), \quad \text{a.e.}\, t,$$

nishes for a.e. $t < 0$. Further, the mapping $f \to f_0$ is n isometric isomorphism of $H_2(\Omega)$ onto $H_2(R, \lambda)$.

(7.3.5) Let f and g be analytic functions on Ω and related by the map τ of 7.1

$$f(\omega) = g(z), \quad \omega = \tau(z).$$

Then $g \in H_2(D)$ if and only if $(1 - i\omega)^{-1} f \in H_2(\Omega)$.

For proofs of (7.3.2), (7.3.4) and (7.3.5) we refer to Hoffman's book ([9], chapter 8) where these results have been proved for the right half-plane Ω'. Observing that $\omega' \to i\omega' = \omega$ gives a conformal equivalence of Ω' and Ω the present results follow.

Next, we derive an easy characterisation from (7.3.4).

LEMMA. A function $f \in L_2(R, \lambda)$ belongs to $H_2(R, \lambda)$ if and only if

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, h(t)\, d\lambda(t) = 0$$

for every $h \in H_2(R, \lambda)$.

PROOF. Let $h \in L_2(R, \lambda)$, $\bar{h}$ the complex conjugate of h. The Fourier-Plancherel transforms are related by

$$\hat{\bar{h}}(x) = \hat{h}(-x), \quad \text{a.e. } x.$$

Applying the Plancherel theorem

$$(f, \bar{h}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, h(t)\, d\lambda(t)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x)\, \overline{\hat{h}(-x)}\, d\lambda(x) .$$

As h runs over $H_2(R, \lambda)$, by (7.3.4) $\hat{h}(-x)$ runs over all $L_2(R, \lambda)$-functions which vanish for a.e. $x > 0$. Therefore, $(f, \bar{h}) = 0$ for all $h \in H_2(R, \lambda)$ if and only if $\hat{f}$ vanishes for a.e. $x < 0$, i.e., if and only if $f \in H_2(R, \lambda)$.

7.4. The map τ sets up an isomorphism of $H_2(T)$ onto a subspace $H_2(R, \mu) \subseteq L_2(R, \mu)$. That is

$$H_2(R, \mu) = \{f \in L_2(R, \mu) : f \circ \tau \in H_2(T)\} .$$

Now the analytic extensions of functions in $H_2(T)$ to the disc D constitute the space $H_2(D)$. From (7.3.5), τ transforms $H_2(D)$ into the space $(1 - i\omega) H_2(\Omega)$. It is, therefore, immediate that the functions in $H_2(R, \mu)$ have analytic extensions to the upper half-plane which constitute the space $(1 - i\omega) H_2(\Omega)$. Since the boundary functions of $H_2(\Omega)$ form the space $H_2(R, \lambda)$, we conclude that $H_2(R,\mu) = (1-it) H_2(R,\lambda)$.

Next, if we note that $f \in L_2(R, \mu)$ if and only if $(1 - it)^{-1} f \in L_2(R, \lambda)$, we have the following characterisation of $H_2(R, \mu)$ from (7.3.4).

(7.4.1) Let a function $f \in L_2(R, \mu)$. Then f is in $H_2(R, \mu)$ if and only if

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_R f(t)\,(1 - it)^{-1}\, e^{-itx}\, d\lambda(t)$$

vanishes for a.e. $x < 0$.

A consequence of (7.4.1) is

(7.4.2) If $f \in H_2(R, \mu)$, then $e^{ist} f \in H_2(R, \mu)$ for each $s \geq 0$.

A second characterisation of $H_2(R, \mu)$ results from the lemma, 7.3 :

(7.4.3) A function $f \in L_2(R, \mu)$ is in $H_2(R, \mu)$ if and only if

$$\int f(t)\, g(t)\, (1-it)^{-2}\, dt = 0 \quad \text{for all } g \in H_2(R, \mu).$$

Finally, we prove an analogue of Paley-Wiener theorem for the space $H_2(R, \mu)$. It was used by Helson in ([5],I,p. 152).

THEOREM. If ν is a bounded measure on $[0, \infty)$ and

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{[0,\infty)} e^{itx}\, d\nu(x), \quad \text{then } f \in H_2(R, \mu).$$

Conversely, if $f \in H_2(R, \mu)$ is the Fourier-Stieltjes transform of a bounded measure on R, then ν is concentrated on $[0, \infty)$.

PROOF. Let $f(t) = \frac{1}{\sqrt{2\pi}} \int_{[0,\infty)} e^{itx}\, d\nu(x),\ t \in R.$

Then f is bounded and hence belongs to $L_2(R, \mu)$. Take $g \in H_2(R, \mu)$. Then

$$\int_R f(t)\, g(t)\, (1-it)^{-2}\, dt = \int_R g(t)\, (1-it)^{-2} \frac{1}{\sqrt{2\pi}} \int_{[0,\infty)} e^{itx}\, d\nu(x) dt$$

$$= \int_{[0,\infty)} \left(\int_R e^{itx} g(t)(1-it)^{-2}\, dt\right) d\nu(x).$$

Since for $x \geq 0$, $e^{itx} g \in H_2(R,\mu)$ and $1 \in H_2(R,\mu)$, by (7.4.3), the inside integral vanishes. Therefore, $f \in H_2(R,\mu)$ by (7.4.3).

Conversely, let $f \in H_2(R, \mu)$ and

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_R e^{itx}\, d\nu(x),\quad t \in R.$$

By the first part of the theorem, if f_1 is defined by

$$f_1(t) = \frac{1}{\sqrt{2\pi}} \int_{[0,\infty)} e^{itx}\, d\nu(x),\quad t \in R$$

then $f_1 \in H_2(R, \mu)$. Therefore, $f_2 = f - f_1 \in H_2(R, \mu)$. But

$$f_2(t) = \frac{1}{\sqrt{2\pi}} \int_{(-\infty,0)} e^{itx}\, d\nu(x)$$

so that $\overline{f_2(t)} = \frac{1}{\sqrt{2\pi}} \int_{(0,\infty)} e^{itx}\, d\bar{\nu}(-x)$

and again by the first part of the theorem, $\overline{f}_2 \in H_2(R, \mu)$. Therefore, the function f_2 must reduce to a constant and hence can be the Fourier-Stieltjes transform of a nontrivial measure only if the measure is concentrated on $\{0\}$. Hence, restricted to $(-\infty, 0)$, ν is the zero measure. This proves the result.

8. Analytical Description of Simply Invariant Subspaces in Terms of Cocycles.

8.1. DEFINTION. Let $\mathbb{H}$ be a separable Hilbert space. By $H_2(R ; \mathbb{H}, \mu)$ we shall mean the subspace of $L_2(R; \mathbb{H}, \mu)$ consisting of functions F such that for each $\xi \in \mathbb{H}$, $(F(\cdot), \xi) \in H_2(R, \mu)$.

We shall now prove the vectorial extension of a theorem of Helson's ([5], I, p.152).

THEOREM. Let M be an ESI subspace of $L_2(B ; \mathbb{H}, \sigma)$ with range J and let A be the $(R, B, \mathcal{P}(\mathbb{H}))$-cocycle corresponding to M. Let $F \in J\, L_2(B ; \mathbb{H}, \sigma)$. Then F belongs to M if and only if for a.e. x, the function on R : $t \to A(t,x)\, F(x + e_t)$ belongs to $H_2(R ; \mathbb{H}, \mu)$.

PROOF. Let Q^M be the spectral measure given by M.

Claim. F belongs to M if and only if the measure $(Q^M F, \chi_\delta\, \xi_n)$ is concentrated on $[0, \infty)$ for each $\delta \in \Gamma$

and for each $n \geq 1$, where $\{\xi_n, n \geq 1\}$ is an orthonormal basis of $\mathbb{H}$.

Proof of claim. From the definition of Q^M, $F \varepsilon M$ if and only if $Q^M((-\infty, 0))\, F = 0$. This is equivalent to demanding that the numerical measures $(Q^M F, F')$ are concentrated on $[0, \infty)$ for F' varying over a subset generating $J\, L_2(B; \mathbb{H}, \sigma)$. If $\{\xi_n, n \geq 1\}$ is an orthonormal basis of $\mathbb{H}$, then $\{\chi_\delta . \xi_n ; \delta \varepsilon \Gamma, n \geq 1\}$ generates $L_2(B ; \mathbb{H}, \sigma)$ and hence the projections of $\chi_\delta . \xi_n$ generate $J\, L_2(B ; \mathbb{H}, \sigma)$. Noting that $(Q^M F, F') = (Q^M F, F'_o)$ for any $F' \varepsilon L_2(B ; \mathbb{H}, \sigma)$ and the projection F'_o of F' on $J\, L_2(B ; \mathbb{H}, \sigma)$, the claim is settled.

Next, since for each $t \varepsilon R$

$$(U_t\, F, F') = \int_R \exp(its)\, d(Q^M(s)\, F, F') ; \quad F' \varepsilon L_2(B ; \mathbb{H}, \sigma)$$

where U is the representation given by M, from the claim and from theorem, 7.4, we conclude that

$$F \varepsilon M <==> [t \to (U_t\, F, \chi_\delta . \xi_n)] \varepsilon H_2(R, \mu)$$

for each $\delta \varepsilon \Gamma$ and $n \geq 1$. Note that $(U_t\, F, F')$ is a bounded function of t and hence belongs to $L_2(R, \mu)$ for any $F' \varepsilon L_2(B ; \mathbb{H}, \sigma)$. Therefore, by (7.4.3), $F \varepsilon M$ if and only if for each $\delta \varepsilon \Gamma$ and $n \geq 1$,

$$\int_R (U_t F, \chi_\delta \xi_n)\, g(t)\, (1 - it)^{-2}\, dt = 0$$

for every $g \in H_2(R, \mu)$ and hence, equivalently for each g in a countable dense set $\{ g_m : m \geq 1\}$ of $H_2(R, \mu)$. Now recall that for a.e. t,

$$(U_t F)(x) = A(t, x)\, F(x + e_t)$$

for a.e. x. Thus $F \in M$ if and only if

$$(8.1.1) \quad \int_R \Big[\int_B (A(t,x)F(x+e_t), \chi_\delta(x)\, \xi_n)\, d\sigma(x)\Big] g_m(t)(1-it)^{-2}\, dt = 0$$

for all $\delta \in \Gamma$, $n, m \geq 1$. Interchanging the integrals, (8.1.1) is equivalent to

$$(8.1.2) \quad \int_B \chi_\delta(x)\Big[\int_R (A(t,x)F(x+e_t), \xi_n)\, g_m(t)(1-it)^{-2} dt\Big]\, d\sigma(x) = 0$$

for each $n, m \geq 1$, for every $\delta \in \Gamma$. This demands that for each $n, m \geq 1$, the inside integral vanishes for a.e. x. Taking a countable union of null sets, (8.1.2) is equivalent to

$$(8.1.3) \quad \int (A(t,x)\, F(x+e_t), \xi_n)\, g_m(t)\, (1-it)^{-2}\, dt = 0$$

for a.e. x, for all $n, m \geq 1$. Again from (7.4.3) and the facts that $\{g_m, m \geq 1\}$ is dense in $H_2(R, \mu)$ and $\{\xi_n, n \geq 1\}$ generates $\mathbb{H}$, (8.1.3) is equivalent to

$$[t \longrightarrow (A(t,x)\, F(x + e_t), \xi)] \in H_2(R, \mu)$$

for a.e. x, for all $\xi \in \mathbb{H}$. This proves the theorem.

9. The Spectral Measures Q^M.

9.1. If M is an ESI subspace of $L_2(B ; \mathbb{H}, \sigma)$, we have seen that the characteristic family $\{M_s, s \in R\}$ gives rise to a spectral measure Q^M on R. In this section we shall consider the Hellinger-Hahn decomposition of Q^M.

For each $\delta \in \Gamma$, V_δ is the operator of multiplication by $\chi_{-\delta}$. Then (V, Q^M) is a system of imprimitivity based on (Γ, R) (see the proof of theorem 4.1). This leads to the following proposition.

PROPOSITION. Let the spectral measure Q^M have the Hellinger-Hahn decomposition $Q^M = \bigoplus_{1 \le n \le \infty} P^{(n, \alpha_n)}$ where for each n, $P^{(n, \alpha_n)}$ is the canonical spectral measure on R acting in $L_2(R ; \mathbb{H}_n, \alpha_n)$. Then every α_n is quasi-invariant with respect to translations by elements of Γ. If M is irreducible, in particular, if M is a simply invariant subspace with one-dimensional range, then Q^M is homogeneous with the nonzero α_n ergodic.

PROOF. The proposition will follow from lemmas 9.9 and 9.10 of Varadarajan's book [15].

9.2. Let M be an ESI subspace of $L_2(B\ ;\ \mathbb{H}, \sigma)$. First, we shall account for the discrete measures that may come in the decomposition of Q^M.

Suppose that M is free. Then Q^M is concentrated on Γ, range of $Q^M(\{\delta\})$ having the same dimension n for every $\delta \in \Gamma$. Therefore, Q^M is homogeneous of multiplicity n, α_n being supported on Γ. Next, for $s \in R$, Q^{M_s} is Q^M translated by s and hence is homogeneous of multiplicity n with the measure class supported on $\Gamma - s$. Conversely, if Q^M is homogeneous of multiplicity n whose measure class rests on $\Gamma - s$, then it is easy to see that M_{-s} is a free simply invariant subspace.

Observe now that a discrete quasi-invariant measure must sit on a countable no. of cosets of Γ. Therefore, by taking orthogonal sums of simply invariant subspaces, we see that any discrete measure with any given multiplicity can occur in the decomposition of Q^M, M being suitably chosen. Finally, the examples of Helson and Lowdenslager [6], Gamelin ([2], ch. VII), Yale [16] and Helson and Kahane [8] show that non-discrete measures are also expected in the decomposition of Q^M. They have given examples of ESI subspaces of $L_2(B, \sigma)$ which do not belong to the characteristic family of free simply invariant subspaces.

9.3. Beyond the proposition, 8.1, and the discussion in 8.2 not much is known about the measures α_n and their relation to the cocycle A corresponding to M. In this connection we raise the following specific questions.

QUESTION 1. Given an ESI subspace M of $L_2(B, \sigma)$ with associated (R, B, T)-cocycle A, to describe the system of imprimitivity (V, Q^M) in terms of A, that is to find the ergodic measure class $\tilde{\alpha}$ associated with Q^M and the multiplicity n with which it occurs and then the $(\Gamma, R, \mathcal{U}(\mathbb{H}_n))$-cocycle corresponding to (V, Q^M) (see theorem, I 7.4).

QUESTION 2. Given a measure class $\tilde{\alpha}$ ergodic with respect to Γ and a cardinal number $n \leq \infty$, does there exist an ESI subspace M of $L_2(B, \sigma)$ such that $\tilde{\alpha}$ is the measure class associated to E^M with multiplicity n?

QUESTION 3. Suppose M is an ESI subspace of $L_2(B\,;\,\mathbb{H}, \sigma)$, having a range of dimension greater than one. Is it possible that Q^M is homogeneous of multiplicity one, with an ergodic class? In that case M would be irreducible.

9.4. We give here an example where the measure class of Q^M is continuous but singular with respect to λ. On the other hand we do not know of a simply invariant subspace M, such that Q^M has the measure class of λ.

EXAMPLE. Assume that $\pi \notin \Gamma$ but $2\pi \in \Gamma$. Let K be the annihilator of the subgroup $\Gamma_0 = \{2\pi n ; n \text{ integer}\}$ of Γ. Let β be the function $\beta = -1_K$. Gamelin ([2], chap. VII ; theorem, 11.4) has observed that the cocycle $\overline{C}_\beta$ (see I 5.4) is neither a coboundary nor a constant cocycle times a coboundary. If $M \subsetneq L_2(B, \sigma)$ is the ESI subspace corresponding to $\overline{C}_\beta$, then M does not occur in the characteristic family of a free simply invariant subspace. Therefore, Q^M has a continuous measure class.

Now consider the measure $(Q^M(\cdot)\, 1, 1)$. Its Fourier-Stieltjes coefficients are given by

$$\int e^{its}\, d(Q^M(s)1, 1) = (U_t^{\overline{C}_\beta} 1, 1).$$

Recall the definition of C_β. Observe that

$$C_\beta(n, y) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

for all $y \in K$. Then we have

$$\overline{C}_\beta(t, y + e_s) = C_\beta([t + s], y)$$

for all $y \in K$, $t \in R$ and $s \in [0,1)$. In particular,

$$\overline{C}_\beta(2n,\ y + e_s) = C_\beta(2n,\ y) = 1, \quad n \in N$$

and hence, $(U_{2n}^{\overline{C}_\beta} 1,1) = \int_B \overline{C}_\beta(2n,\ x)\, d\sigma(x) = 1$ for all $n \in N$.

Therefore, the measure $(Q^M(\cdot)\ 1,1)$ is not absolutely continuous with respect to λ. Let α represent the measure class of Q^M. Since $\alpha(D) = 0$ implies $(Q^M(D)\ 1,1) = 0$, α is also not absolutely continuous with respect to λ. But α is ergodic and the part of α singular with respect to λ is also quasi-invariant. This forces α to be singular to λ.

9.5. We note that the example of I 8.4 has some relevance to the questions we raised. There, for the pair (N, T) whose dual is itself, a system of imprimitivity (U, P) based on (N, T) where P is the canonical spectral measure of dimension 1, has as its dual a system (V, Q), Q being the canonical spectral measure of higher dimension.

10. SUMMARY. In section 1 we define invariant subspaces of $L_2 (B ; \mathbb{H}, \sigma)$ where B is a Bohr group with a countable dual, $\mathbb{H}$ is a separable Hilbert space and σ is the Haar measure on B. We also define doubly invariant subspaces and simply invariant subspaces.

Section 2 deals with the vectorial extension of the Plancherel theorem to $L_2(B ; \mathbb{H}, \sigma)$. In section 3 we dispose of doubly invariant subspaces. We observe that they are precisely the subspaces reducing the canonical spectral measure and then they are determined by results in section 2, chapter I. In section 4 we study elementary properties of simply invariant subspaces and classify them. Picking out what we call free simply invariant subspaces we determine their form.

From section 5 onwards we work with a convenient class of simply invariant subspaces, called ESI subspaces. In section 5 we observe that the ESI subspaces are in one-one correspondence with the systems of imprimitivity based on (R, B) and acting in subspaces of $L_2(B ; \mathbb{H}, \sigma)$. In section 6, we apply the results of section 7, chapter I to get the principal correspondence theorem : between ESI subspaces on one hand and partial isometry-valued cocycles on the other. We examine various relationships between ESI subspaces to see how they are reflected on the corresponding cocycles. Using our results on irreducible systems of imprimitivity here we show the existence of simply invariant subspaces of $L_2(B ; \mathbb{H}, \alpha)$ which are not equivalent to direct sums of simply invariant subspaces of the ordinary L_2.

In section 7, we record some facts from the theory of analytic functions on the upper half-plane. In section 8, we obtain an analytic description of an ESI subspace in terms of the corresponding cocycle. This generalises a result of Helson's ([5], I). In section 9, we discuss the Hellinger-Hahn decomposition of a spectral measure on R given by an ESI subspace.

CHPATER III

APPLICATIONS TO MULTIVARIATE STATIONARY STOCHASTIC PROCESSES

1. The Problem.

1.1. DEFINITION. Let Γ be a dense countable subgroup of R and $\mathbb{H}$ a Hilbert space. Let q be a positive integer. A q-variate stationary stochastic process Y in $\mathbb{H}$ with time Γ is a sequence $Y = \{Y_\delta, \delta \in \Gamma\}$ where for each δ, Y_δ is q-ple of elements of $\mathbb{H}$, $Y_\delta = (Y_\delta^1, \ldots, Y_\delta^q)$ such that the inner product $(Y_\delta^i, Y_{\delta'}^j)$ depends on δ and δ' through the difference $\delta - \delta'$, once i and j, $1 \leq i, j \leq q$, are fixed.

Associated with a q-variate stationary stochastic process Y with time Γ are functions ρ_{ij} on Γ, called correlation functions, such that

$$(Y_\delta^i , Y_{\delta'}^j) = \rho_{ij} (\delta - \delta')$$

for all $\delta, \delta' \in \Gamma$ and all i, j. Two q-variate stationary stochastic processes X and Y in $\mathbb{H}$ and $\mathbb{H}'$ with time Γ are said to be isomorphic if they have the same set of correlation functions, that is, if

$$(X_\delta^i , X_{\delta'}^j) = (Y_\delta^i , Y_{\delta'}^j)$$

for all $\ell, \ell' \in \Gamma$ and $1 \leq i,\ j \leq q$.

1.2. Let Y be a q-variate stationary stochastic process in $\mathbb{H}$ with time Γ. For each $t \in R$, we define the space of the process upto time t as the subspace $\mathcal{L}(\{Y^j_\ell : \ell \leq t, 1 \leq j \leq q\}) = \mathbb{H}_t$. The space $\mathbb{H}_{-\infty} = \bigcap_{t > -\infty} \mathbb{H}_t$ is called the remote past and the space $\mathbb{H}_\infty = \bigvee_{t > -\infty} \mathbb{H}_t$ is called the space of the process Y. A fundamental result in the theory of stationary stochastic processes is the

SPECTRAL REPRESENTATION THEOREM. Given a q-variate stationary stochastic process Y with time Γ and having correlation functions ρ_{ij}, $1 \leq i,\ j \leq q$, there exists a matrix-valued measure $N = [\nu_{ij}]_{1 \leq i, j \leq q}$ on B (which is the same thing as a matrix whose entries are measures) such that

(i) for each Borel set E, $[\nu_{ij}(E)]$ is positive-semi-definite and

(ii) for each i and j and for every $\ell \in \Gamma$,

$$\rho_{ij}(\ell) = \int_R \chi_\ell(-x)\, d\,\nu_{ij}(x).$$

PROOF. - The map $U_\gamma : Y^j_\ell \rightarrow Y^j_{\ell+\gamma}$, $\ell \in \Gamma$, $1 \leq j \leq q$, extends to a unitary operator on $\mathbb{H}_\infty$ such that $\{U_\gamma, \gamma \in \Gamma\}$ is a representation of Γ acting in $\mathbb{H}_\infty$. Hence there exists a spectral measure P on B such that

$$U_\gamma = \int_B \chi_\gamma(-x)\, dP(x).$$

Therefore,

(1.2.1) $(U_\gamma Y_0^i, Y_0^j) = \int_B \chi_\gamma(-x)\, d(P(x)Y_0^i, Y_0^j) = \rho_{ij}(\gamma)$

for each $\gamma \in \Gamma$ and $1 \leq i, j \leq q$. Therefore, defining ν_{ij} by

$$\nu_{ij}(E) = (P(E)\, Y_0^i, Y_0^j),$$

(i) follows because $[\nu_{ij}(E)]$ is the gram-matrix of $(P(E)\, Y_0^1, \ldots, P(E)\, Y_0^q)$ with itself. (ii) is immediate from (1.2.1). QED

We shall be interested only in the case when each ν_{ij} is absolutely continuous with respect to σ. Let $\nu_{ij} = w_{ij}\, d\sigma$ and W be the matrix-valued function $W(x) = [w_{ij}(x)]$, $x \in B$. Then W is called the spectral density matrix and conditions (i) and (ii) imply

(i)' for σ - a.e. x, $W(x)$ is positive semi-definite

(ii)' for each i and j, $w_{ij} \in L_1(B, \sigma)$.

Also we shall have $\rho_{ij}(\gamma) = \int_B \chi_\gamma(-x)\, w_{ij}(x)\, d\sigma(x)$ for each i and j.

1.3. DEFINITION. The process Y is called a purely deterministic process if $\mathbb{H}_t = \mathbb{H}_{-\infty}$ for each t. It is said to be purely non-deterministic in the other extreme case, $\mathbb{H}_{-\infty} = \{0\}$.

Our problem in this chapter is to decide when a stationary stochastic process Y with spectral density matrix W is purely non-deterministic. Observe in this connection that if X and Y are isomorphic processes, then X and Y are purely deterministic or purely nondeterministic together.

2. <u>The Matrix-valued Weight Function and Canonical Processes.</u>

2.1. Let $(X, \mathcal{A}, \alpha)$ be a measure-space and $W(x) = [w_{ij}(x)]_{1 \le i, j \le q}$ be a matrix-valued weight function on X, that is,

(i) $w_{ij} \in L_1(X, \alpha)$ for each i and j, and

(ii) W(x) is positive semi-definite for a.e. x.

By $L_2(X, W)$ we shall mean the space of all $1 \times q$ matrix-valued measurable functions on X such that

$$\| F \|_W^2 = \int_X F(x)\, W(x)\, F^*(x)\, d\alpha(x) < \infty$$

after identification of functions F and F' for which $\| F - F' \|_W = 0$. $L_2(X, W)$ is a Hilbert space with the inner product

$$(F, F')_W = \int_X F(x)\, W(x) F'^*(x)\, d\alpha(x)$$

(see Rosenberg ([13]), for a proof). $(\cdot,\cdot)_W$ gives rise to the norm $\|\cdot\|_W$.

In the sequel we shall consider spaces $L_2(X, W)$ where $X = B$, T or R, the measures α being the normalised Haar measures in the case of B and T, and the Cauchy measure μ in the case of R.

2.2. Let $W = [w_{ij}]$ be a $q \times q$ matrix valued weight function on B. In $L_2(B, W)$ we define a q-variate process by setting, for $\delta \in \Gamma$, $1 \leq j \leq q$,

$$X_\delta^j(x) = \chi_\delta(-x)\, I_j$$

where I_j is the j^{th} row of the $q \times q$ identity matrix. It is easy to see that X is a stationary stochastic process with time Γ having the correlation functions

$$(2.2.1) \quad \rho_{ij}(\delta-\delta') = (X_\delta^i, X_{\delta'}^j)_W = \int_B \chi_{\delta-\delta'}(-x)\, w_{ij}(x)\, d\sigma(x)$$

for all $\delta, \delta' \in \Gamma$, $1 \leq i, j \leq q$. X is called the canonical process in $L_2(B, W)$. (2.2.1) shows, moreover, that any process Y with the spectral density matrix W is isomorphic to X. From the remarks in 1.3 our problem reduces to finding conditions on W so that the canonical process in $L_2(B, W)$ is purely non-deterministic.

3. Stationary Stochastic Processes and Simply Invariant Subspaces.

3.1. Let **W** be a matrix-valued weight function on B and **Q** be its positive-semi-definite square root, $Q(x) = W^{1/2}(x), x \in B$.

For $F \in L_2(B, W)$, the function $Q[F]$ defined by

(3.1.1) $$Q[F](x) = F(x) \cdot Q(x), \quad x \in B$$

belongs to $L_2(B ; \mathbb{C}^q, \sigma)$, elements of $\mathbb{C}^q$ being written as row vectors. Further,

$$\| Q[F] \| = \int_B (F(x) Q(x), F(x) Q(x)) \, d\sigma(x)$$

$$= \int_B (F(x) W(x), F(x)) \, d\sigma(x)$$

$$= \| F \|_W .$$

Therefore, the map $F \to Q[F]$ is an isometry of $L_2(B, W)$ into $L_2(B ; \mathbb{C}^q, \sigma)$. The range of this map is easily seen to be a doubly invariant subspace $J L_2(B ; \mathbb{C}^q, \sigma)$, where,

(3.1.2) $$J(x) = \underline{\underline{R}}\, Q(x) = \underline{\underline{R}}\, W(x), \text{ a.e. } x.$$

3.2. Let X be the canonical process in $L_2(B, W)$. Consider the subspace $\mathbb{H}_0 = \underline{\underline{C}} \{ X_s(-x) I_j, s \leq 0, 1 \leq j \leq q \}$, the space of the process upto time 0. Let $M_0 = Q[\mathbb{H}_0]$. Then M_0

is clearly an invariant subspace.

PROPOSITION. With the above notation, one has

(i) M_0 is doubly invariant if and only if X is purely deterministic.

(ii) M_0 is exact simply invariant if and only if X is purely non-deterministic.

PROOF. By definition of $\mathbb{H}_t$, $t \in R$,

$$\mathbb{H}_\delta = \chi_\delta \cdot \mathbb{H}_0 .$$

Applying Q on both sides, $Q[\mathbb{H}_\delta] = \chi_\delta . Q[\mathbb{H}_0]$. Therefore, $\mathbb{H}_0 = \mathbb{H}_t$ for all t if and only if $\chi_\delta M_0 = M_0$ for all $\delta \in \Gamma$. Hence (i). Again

$$\mathbb{H}_{-\infty} = \bigcap_{t \in R} \mathbb{H}_t = \bigcap_{\delta \in \Gamma} \chi_\delta \mathbb{H}_0$$

whence, $$Q[\mathbb{H}_{-\infty}] = \bigcap_{\delta \in \Gamma} \chi_\delta M_0$$

Now since Q is an isometry $\mathbb{H}_{-\infty} = \{0\}$ if and only if $Q[\mathbb{H}_{-\infty}] = \{0\}$. This proves (ii) completing the proof.

4. Facts about Continuous Time-parameter Stationary Stochastic Processes.

4.1. Let $W_1(t) = [f_{ij}(t)]_{1 \le i,\, j \le q}$ be a μ-almost everywhere positive semi-definite matrix-valued function on R whose entries $f_{ij} \in L_1(R, \mu)$. As on B, we define the space $L_2(R, W_1)$. The canonical process in $L_2(R, W_1)$ is the q-variate stationary stochastic process

$$Z = \{Z_s = (Z_s^1, \ldots, Z_s^q), s \in R\},$$

where $Z_s^j(t) = e^{-ist} I_j$

for each $s \in R$ and $1 \leq j \leq q$, for μ-a.e. t. As before we define the spaces

$$\mathbb{H}_r(Z) = \mathcal{G}\{Z_s^j, s \leq r \text{ and } 1 \leq j \leq q\}, r \in R.$$

The remote past of Z is the space $\mathbb{H}_{-\infty}(Z) = \bigcap_{r > -\infty} \mathbb{H}_r(Z)$. The theorem below gives two equivalent conditions for Z to be purely non-deterministic, i.e. for $H_{-\infty}(Z) = \{0\}$. We need a definition.

DEFINITION. Let $\mathbb{H}$ be a separable Hilbert space. A measurable range function J on R with closed subspaces of $\mathbb{H}$ as values is said to be an analytic range function if there exists a finite or countable sequence of functions $F_i \in H_2(R; \mathbb{H}, \mu)$, $i \geq 1$, such that for a.e. $s \in R$, $J(s) = \mathcal{G}(\{F_i(s); i \geq 1\})$.

4.2. For any positive semi-definite matrix W det W will stand for the determinant of W regarded as a transformation on its range, where it is nonsingular.

THEOREM. The following are equivalent

(a) The canonical process Z in $L_2(R, W_1)$ is purely non-deterministic, that is, $\mathbb{H}_{-\infty}(Z) = \{0\}$.

(b) $W_1 = A_1 A_1^*$ where A_1 is a $q \times q$ matrix valued function with entries belonging to $H_2(R, \mu)$.

(c) $\int_R \log \det W_1(t)\, d\mu(t) > -\infty$ and W has analytic range.

PROOF. We shall use the map τ of II 7.1,

$$\tau : e^{i\theta} \longrightarrow t = \tan(\theta - \pi)/2, \quad 0 \leq \theta < 2\pi$$

of the circle T (parametrised by $[0, 2\pi)$) onto $R \cup \{\infty\}$. This map transforms the normalised Lebesgue measure σ on the circle into the Cauchy measure μ on R. Let now $\bar{W}_1(\theta) = W_1(\tau(\theta))$. Then $\bar{W}_1$ is a positive semi-definite matrix function on T with the entries $\bar{f}_{ij} \in L_1(T, \sigma)$. Now consider the space $L_2(T, \bar{W}_1)$ (the map $F \to F \circ \tau$ is an isometric isomorphism of $L_2(T, \bar{W}_1)$ onto $L_2(R, W_1)$) and the canonical process $\bar{Z}$ in $L_2(T, \bar{W}_1)$ with time N,

$$\bar{Z} = \{\bar{Z}_n = (\bar{Z}_n^1, \ldots, \bar{Z}_n^q), \ n \in N\}$$

where $\bar{Z}_n^j(\theta) = e^{-in\theta} I_j$ a.e. θ.

For this process we have the spaces

$$\mathbb{H}_n(\bar{\bar{Z}}) = \mathfrak{S}\{\bar{\bar{Z}}_m^j, \quad m \le n, \ 1 \le j \le q\}$$

and $\quad \mathbb{H}_{-\infty}(\bar{\bar{Z}}) = \bigcap_{n \in N} \mathbb{H}_n(\bar{\bar{Z}}).$

It is known that the remote past $\mathbb{H}_{-\infty}(\bar{\bar{Z}}) = \{0\}$ if and only if $\mathbb{H}_{-\infty}(\bar{\bar{Z}}) = \{0\}$ (Rozanov, [12], p.112). Next, for the process $\bar{\bar{Z}}$, Helson and Lowdenslager [[7] II, theorem 13] showed that the following three conditions are equivalent

(a') $\quad \mathbb{H}_{-\infty}(\bar{\bar{Z}}) = \{0\}$

(b') $\quad \bar{\bar{W}}_1 = \bar{\bar{A}}_1 \bar{\bar{A}}_1^*$ where $\bar{\bar{A}}_1$ is a matrix function on T with the entries belonging to $H_2(T)$.

(c') (i) $\bar{\bar{W}}_1$ has analytic range function (that is, there exists a finite or countable sequence of functions $\bar{F}_i \in H_2(T ; \mathbb{C}^q, \sigma)$ such that for a.e. θ, $\mathfrak{S}\{\bar{F}_i(\theta), i \ge 1\} = \mathcal{R}\,\bar{\bar{W}}_1(\theta)$).

(ii) $\displaystyle\int_0^{2\pi} \log \det \bar{\bar{W}}_1(\theta) \, d\sigma(\theta) > -\infty$.

Now from the definition of $H_2(R, \mu)$ and $H_2(R ; \mathbb{C}^q, \mu)$, the map τ (4.1.1) gives an isomorphism of $H_2(T)$ onto $H_2(R, \mu)$ and an isomorphism of $H_2(T ; \mathbb{C}^q, \sigma)$ onto $H_2(R ; \mathbb{C}^q, \mu)$. Therefore, it is clear that (b) and (b') as well as (c) and (c') are equivalent. We have already mentioned that (a) and (a') are equivalent. Hence our theorem follows from the result of Helson and Lowdenslager.

4.3. We note here a simple observation. Suppose the canonical process Z in $L_2(R, W_1)$ has remote past $\mathbb{H}_{-\infty}(Z) = \{0\}$. Let $F \neq 0$ be fixed in $L_2(R, W_1)$ and let $d(F, \mathbb{H}_r(Z))$ stand for the metric distance of F from $\mathbb{H}_r(Z)$ in $L_2(R, W_1)$. Then from the way $\mathbb{H}_{-\infty}$ is defined, $\lim_{r \to -\infty} d(F, \mathbb{H}_r(Z)) = \|F\|_{W_1}$. Suppose next, for each $r \in R$, $F_r \in \mathbb{H}_r(Z)$ is arbitrarily chosen. Since $\|F - F_r\|_{W_1} \geq d(F, \mathbb{H}_r(Z))$, it follows that

$$\liminf_{r \to -\infty} \|F - F_r\|_{W_1} \geq \|F\|_{W_1} > 0.$$

5. <u>The Main Theorem.</u>

5.1. Using the analytic description of ESI subspaces (sec. 8, chap. II) we now prove

THEOREM. Let W be a $q \times q$ matrix-valued weight function on B. The canonical process X in $L_2(B, W)$ is purely non-deterministic if and only if

(i) for a.e. x, $W(x + e_t)$ has an analytic range function on R, and

(ii) for a.e. x, $\int_B \log|\det W(x + e_t)| \, d\mu(t) > -\infty$.

PROOF. Let Q, $Q(x) = W^{\frac{1}{2}}(x)$, be the pointwise positive semi-definite square root of W. Assume that X is purely non-deterministic. Then by the proposition, 3.2, $M_0 = Q[\mathbb{H}_0]$ is an exact simply invariant subspace of $L_2(B;\mathbb{C}^q, \sigma)$. M_0, however, need not be left-continuous and let $(M_0)_0$ be its left-continuous version. Let A be the $(R, B, \underset{\sim}{P}(\mathbb{H}))$-cocycle in correspondence with $(M_0)_0$. Now $M_0 \subsetneq (M_0)_0$ and so, by the theorem, II 8.1, for each $F \in M_0$

$$[t \longrightarrow F(x + e_t)\ A(t,x)] \in H_2(R, \mathbb{C}^q, \mu)$$

for a.e. $x \in B$ (the elements $A(t,x) \in \underset{\sim}{P}(\mathbb{H})$ act by matrix multiplication from the right). Since the constant functions $I_j \in \mathbb{H}_0$, one has $I_j\ Q \in M_0$, for each j, $1 \leq j \leq q$, and hence

$$I_j\ Q(x + e_t)\ A(t, x) \in H_2(R, \mathbb{C}^q, \mu)$$

for each j, for a.e. x. This means that for a.e. x, the entries of the matrix-function $Q(x + e_t)\ A(t, x)$ belong to $H_2(R, \mu)$. Next, for $(t, x) \in R \times B$

$$Q(x + e_t)\ A(t, x)\ [\ Q(x + e_t)\ A(t, x)]^*$$
$$= Q(x + e_t)\ A(t, x)\ A^*(t,x)\ Q(x + e_t).$$

But $\underset{\sim}{R}(A) = \underset{\sim}{R}\ (M_0)_0 = \underset{\sim}{R}\ (Q[\mathbb{H}_0]) = \underset{\sim}{R}\ (Q)$, where the first equality follows from the correspondence of A and $(M_0)_0$, and the last was observed in (3.1.2). Let J stand for this common range function. Then $A(t, x)\ A^*(t, x)$ is the identity transformation on $J(x + e_t) = \mathbb{R}\ Q(x + e_t)$ for a.e. (t, x). Therefore, from the above, for a.e. x, the matrix function $W(x + e_t)$ has the factorisation

$$W(x + e_t) = Q(x + e_t)\ Q(x + e_t)$$
$$= Q(x + e_t)\ A(t, x)\ [Q(x + e_t)\ A(t, x)]^*$$

for a.e. t. By theorem, 4.2, the necessity part follows.

Conversely suppose W satisfies conditions (i) and (ii) of the theorem. We shall show that $\mathbb{H}_{-\infty}(X) = \{0\}$. Let $F \in \mathbb{H}_{-\infty}(X)$, that is $F \in \mathbb{H}_t$ for each $t \in R$. By definition, $\mathbb{H}_t$ is the closure in $L_2(B, W)$ of all trigonometric polynomials having the spectrum contained in $[-t, \infty)$. For each integer n, choose a trigonometric polynomial P_n having the spectrum in $[n, \infty)$ such that

$$\| F - P_n \|_W < \frac{1}{n} \quad \text{and} \quad P_n \in \mathbb{H}_{-n}.$$

Now, $\| F - P_n \|_W$

$$= \int_B (F(x) - P_n(x))\ W(x)\ (F(x) - P_n(x))^*\ d\sigma(x)$$

$$= \int_B d\sigma(x) \int_R (F(x+e_t) - P_n(x+e_t))\ W(x+e_t)(F(x+e_t)-P_n(x+e_t))^*\ d\mu(t)$$

$$= \int_B H_n(x)\ d\sigma(x),$$

where $H_n(x)$ is the inside integral in the previous step.

Since $F \in L_2(B, W)$, for a.e. x, $F(x + e_t)$ as a function of t belongs to $L_2(R, W(x + e_{(\cdot)}))$. This is because

$$\int_B F(x)\ W(x)\ F^*(x)\ d\sigma(x)$$

$$= \int_R \left[\int_B F(x+e_t)\ W(x+e_t)\ F^*(x+e_t)\ d\sigma(x) \right] d\mu(t)$$

$$= \int_B d\sigma(x) \int_R F(x+e_t)\ W(x+e_t)\ F^*(x+e_t)\ d\mu(t) < \infty$$

and hence the inside integral must be finite for a.e. x. Consider an $x \in B$ for which this happens and for which $W(x + e_{(\cdot)})$ satisfies conditions (i) and (ii) of the theorem. Then by the theorem, 4.2, the canonical process Z in $L_2(R, W(x + e_{(\cdot)}))$ has no remote past. Further $t \to F_n(x + e_t)$ is a trigonometric polynomial on R having the spectrum contained in $[n, \infty)$ and so belonging to $\mathcal{H}_{-n}(Z)$. Therefore, from 4.3 if the function $F(x + e_{(\cdot)}) \neq 0$ then $\liminf_{n \to \infty} H_n(x) > 0$.

But then we have, using Fatou-Lebesgue theorem

$$0 = \lim_{n \to \infty} \int_B H_n(x)\ d\sigma(x) \geq \int_B \liminf_{n \to \infty} H_n(x)\ d\sigma(x)$$

which shows that for a.e. x, $F(x + e_{(\cdot)}) = 0$.

An application of Fubini's theorem gives $F = 0$. Thus $H_{-\infty}(X) = \{ 0 \}$, as F was arbitrary. This proves the sufficiency part and the proof is complete.

Our theorem generalises the following result of Helson and Lowdenslager ([7] II, theorem 5).

COROLLARY. If w is a non-negative integrable function on B, then the canonical process in $L_2(B, w)$ is purely non-deterministic if and only if $\int_R \log w(x + e_t)\, d\mu(t) > -\infty$ for a.e. x.

6. SUMMARY. We consider a q-variate stationary stochastic process X with time Γ, where Γ is a countable dense subgroup of the real group R. We assume that X has a spectral density W (a $q \times q$-matrix function) on the Bohr group B whose dual is Γ. We obtain a necessary and sufficient condition on W in order that X may be purely non-deterministic.

The requisite definitions appear in section 1. In section 3 we state our problem in terms of simply invariant subspaces of $L_2(B\,;\,\mathbb{C}^q, \sigma)$. In section 4 we deduce a result for some continuous-time multivariate processes from the work of Helson and Lowdenslager on integer-time processes ([7], II). Finally, in section 5 the main result is proved.

APPENDIX

INVARIANT SUBSPACES OF $L_2(R; \mathbb{H}, \lambda)$

The methods of chapter II apply to invariant subspaces of $L_2(R; \mathbb{H}, \lambda)$ as well, where λ is the Lebesgue measure on R. One calls a subspace M of $L_2(R; \mathbb{H}, \lambda)$ <u>doubly invariant</u> if $e^{it(\cdot)} M = M$ for all $t \in R$, and <u>simply invariant</u> if for all $t > 0$, $e^{it(\cdot)} M$ is a proper subspace of M. Most of the concepts in chapter II have their obvious analogues here.

The doubly invariant subspaces M are of the form $J L_2(R; \mathbb{H}, \lambda)$; the arguments of section 3, chapter II go through without any essential difference. For simply invariant subspaces one notes at the beginning that there cannot exist any free simply invariant subspace. This is because a subspace of the form

$$\underset{t \geq 0}{\oplus} e^{it(\cdot)} I ,$$

if nonempty, cannot be separable. Every simply invariant space is, therefore, continuous. In fact, the characteristic family is necessarily continuous at all points. Thus, in the place of ESI subspaces in sections 5 and 6 of chapter II one considers exact simply invariant subspaces. This leads to a one-to-one correspondence of exact simply invariant subspaces with $(R, R, \underline{P}(\mathbb{H}))$-cocycles. But from corollary, I 6.8 every $(R, R, \underline{P}(\mathbb{H}))$-cocycle is a coboundary in the extended sense.

Now it is easy to see that the Plancherel theorem generalises to $L_2(R ; \mathbb{H}, \lambda)$ giving a unitary operator on $L_2(R ; \mathbb{H}, \lambda)$. Therefore, if A is the coboundary of the operator-functions ρ (on R having constant initial space) one can proceed as in II. 6.2 to find the corresponding simply invariant subspace. It turns out to be the subspace

$$T_\rho \, H_2(R ; \mathbb{H}, \lambda)$$

where $H_2(R ; \mathbb{H}, \lambda) = \{F \in L_2(R ; \mathbb{H}, \lambda) : \text{ for each } \xi \in \mathbb{H}, \; (f(\cdot), \xi) \in H_2(R, \lambda)\}$.

Thus every exact simply invariant subspace of $L_2(R ; \mathbb{H}, \lambda)$ is of the form

$$T_\rho \, H_2(R ; \mathbb{H}, \lambda)$$

where $\rho : R \longrightarrow \underline{P}(\mathbb{H})$ is a measurable function with constant initial space.

In the special case of $L_2(R, \lambda)$ this result appears in Helson's book ([4], Lecture V).

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LIST OF SPECIAL SYMBOLS

Additional Notes to My Thesis "Invariant Subspaces of Vector-valued Function Spaces on Bohr Groups"

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Regarding some questions treated in the above-mentioned thesis a few examples have been recently studied by Nadkarni and the author. We present them below pointing out their relevance.

1^{o} The first example is of an infinite-dimensional irreducible system of imprimitivity. We recall the situation in Example, I 8.4. We have the pair (N, T). Writing $T = [0, 2\pi)$, the action of N is given by, say,

$$n \rightarrow e_n = n \pmod{2\pi}$$

For any positive integer p we consider the function β ; $\beta(y) = e^{ipy}$, $y \in T$. Then the Gamelin cocycle C_β is given by

(1) $$C_\beta(n,y) = e^{i\frac{p}{2}n(n-1)} e^{ipy}$$

for all $n \in N$ and $y \in [0, 2\pi)$. Let (U^{C_β}, P) the system of imprimitivity given by C_β and (V, Q) its dual. We have shown that Q is homogeneous of multiplicity p having the measure class of Lebesgue measure on $[0, 2\pi)$. Thus (V, Q) is a p-dimensional irreducible system of imprimitivity based on (N, T).

Now instead of β let us consider an inner function q on the circle having infinitely many distinct zeros in the disc. We consider the Gamelin cocycle C_q obtained from q (see I5.4). Observe that

$$U_1^{C_q} H_2(T) = \{q(\cdot)\, f(\cdot + e_1) : f \in H_2(T)\} = q.\, H_2(T) \subsetneq H_2(T)$$

and the condition on q ensures that $H_2(T) \ominus q.\, H_2(T)$ is infinite-dimensional having a complete orthonormal system $(f_1, f_2, \ldots)$. Then the cyclic subspaces $\{U_n^{C_q} f_i\}_{n \in N}$ for $i = 1, 2, \ldots$ are mutually orthogonal and together span $L_2(T)$. Also the function $(U_n^{C_q} f_i, f_i) = 1_{\{0\}}$ for each i. Thus if Q is the spectral measure corresponding to $\{U_n^{C_q} : n \in N\}$ then Q is homogeneous of multiplicity ∞ having the measure class $\bar{\sigma}$. Then (V, Q), the dual system of (U^{C_q}, P), is an irreducible system of imprimitivity based on (N, T) and acting in $L_2(T ; \mathbb{H}_\infty, \sigma)$.

The arguments in I 8.6 now yield irreducible systems of imprimitivity based on (R, T^2) which are infinite-dimensional.

2^0 The following article has since come to our notice.

A. A. Kirillov, ''Dynamical systems, factors and representation of groups'' - Uspekhi Mat. Nauk 22 : 5(1967) 67-80 ; Russ. Math. Surveys 22 : 5 (1967) 63-75.

[3]

This paper contains a construction, due to A.M. Gleason, of (N,T) cocycles having values in 2×2 unitary matrices giving rise to irreducible systems of imprimitivity of dimension 2. The construction was communicated to A.A.Kirillov by G.W.Mackey. The paper also quotes results of O.P. Chopenko modifying this construction to exhibit irreducible systems of dimension p for any positive integer p. In Gleason's example the proof of irreducibility is direct whereas we prove irreducibility by referring to the dual system.

3° We refer to Question 2, II 9.3. We have a Bohr group B whose dual is Γ, an ergodic measure class $\bar{\alpha}$ on R and a cardinal number $1 \leq n \leq \infty$. We raised the question if there exists a simply invariant subspace M of $L_2(B)$ for which Q^M is homogeneous of multiplicity n having the measure class $\bar{\alpha}$. In the special case when $B = T^2$ we shall show that such subspaces M exist when n is finite and α is the Lebesgue measure. This also answers a question we raised in II 9.4 whether Q^M can have the measure class of the Lebesgue measure.

Consider the pair (R, T^2) where the dual of T^2 is the subgroup $\Gamma = \{2\pi n + m : n,m \text{ integers}\}$ of R. Writing T^2 as $[0,1] \times [0, 2\pi)$ with co-ordinate-wise addition modulo 1 and 2π respectively, one has the characters

$$< 2\pi n + m, (x,y) > = e^{2\pi i n x} e^{i m y}$$

for all $2\pi n + m \in \Gamma$ and $(x,y) \in T^2$. Hence the action of R is given by

$$t \to e_t = (t \pmod 1,\ t \pmod{2\pi}),\quad t \in R.$$

The annihilator of $\{2\pi n\}$ is the subgroup $K = \{0\} \times [0, 2\pi)$. Hence by the direct product representation (see I 4.5) every $(x,y) \in T^2$ is the sum of the uniquely determined elements

$$(x,y) = (x,x) + (0,\ y-x \pmod{2\pi})$$

$$= e_x + (0,\ y-x \pmod{2\pi}).$$

Finally the pair (N, K) here works out to be $(N, \{0\} \times [0, 2\pi))$ with the map $n \to e_n = (0, n \pmod{2\pi})$—essentially the pair in 1°. Now if C_β be given by (1), the extended Gamelin cocycle $\bar{C}_\beta$ on $R \ltimes T^2$ is

$$\bar{C}_\beta (t, (x,y)) = c_{[t+x]}\ e^{ip[t+x]\,(y-x)}$$

for $t \in R$, $(x,y) \in T^2$, where $c_n = e^{i \frac{p}{2} n(n-1)}$ for all $n \in N$.

Consider the representation $\{V_t, t \in R\}$ corresponding to $\bar{C}_\beta$:

$$V_t f = \bar{C}_\beta (t, \cdot\,)\ f(\,\cdot + e_t)$$

for $f \in L_2(T^2)$. Let, for $\lambda = 0, 1, \ldots, p-1$,

$$f_\lambda(x,y) = e^{i\lambda y}, \quad (x,y) \in [0,1) \times [0, 2\pi).$$

Then elementary calculations using Fubini's theorem yield :

$$\text{(i)} \quad (V_t f_\lambda, f_\lambda) = \begin{cases} e^{it\lambda} \quad (1 - |t|), & |t| \leq 1 \\ 0 \;, & |t| > 1, \end{cases}$$

which is the familiar Fourier transform of a translate of the measure $\frac{1}{2\pi} \left(\frac{\sin x/2}{x/2} \right)^2 dx$ on R.

$$\text{(ii)} \quad (V_t f_\lambda, f_{\lambda'}) = 0 \quad \text{if} \quad \lambda \neq \lambda' ,$$

which shows that the multiplicity of the spectral measure of $\{V_t, t \in R\}$ is at least p with the class of Lebesgue measure as the associated measure class. Now we shall show that the multiplicity is exactly p. Let $g \in L_2(T^2)$ and $g \perp V_t f_\lambda$ for all t and λ. We show that g is the zero function. Let $n \leq t < s < n+1$. $g \perp (e^{it\lambda} V_t f_\lambda - e^{is\lambda} V_s f_\lambda)$ means

$$((e^{it\lambda} V_t - e^{is\lambda} V_s) f, g) = \int_\alpha^\beta dx \left(\int^{2\pi} (c_n e^{-ipnx} e^{i(pn+\lambda)} - c_{n+1} e^{-ip(n+1)x} e^{i(pn+p+\lambda)}) \bar{g}(x,y) \, dy \right.$$

$$= 0$$

where $\alpha = n + 1 - s$ and $\beta = n + 1 - t$. Varying t and s, this is true for all α, β with $0 \leq \alpha < \beta < 1$. Hence the inside integral vanishes for a.e. x. That is

$$c_n \; e^{-ipnx} \; \hat{\bar{g}}_x(-pn-\lambda) - c_{n+1} \; e^{-ip(n+1)x} \; \hat{\bar{g}}_x(-pn-p-\lambda)$$

$$= 0 \quad \text{a.e. } x,$$

where $\bar{g}_x$ is the x-section of the function $\bar{g}$. Thus

$$|\hat{\bar{g}}_x(-pn - \lambda)| = |\hat{\bar{g}}_x(-pn-p-\lambda)| \quad \text{a.e. } x$$

for all n which is impossible unless $\hat{\bar{g}}_x(-pn-\lambda) = 0$ for all n. Taking $\lambda = 0, 1, \ldots, p-1$, we have $\bar{g}_x = 0$ for a.e. x. That is, $g = 0$ a.e.

Now if M is the simply invariant subspace of $L_2(T^2)$ corresponding to $\bar{C}_\beta$, then M answers our purpose.