TH121

Zero-cycles and K-theory on Normal Surfaces

INDIAN STATISTICAL INSTITUTE
LIBRARY

The Book / Periodical is to be returned on or before the date last stamped.

Subr
fo

INDIAN STATISTICAL INSTITUTE LIBRARY
T 121
18 DEC 2003
CALCUTTA

Tata Institute of Fundamental Research
Mumbai
2001

Declaration

The work presented in this thesis has been carried out by me jointly with my thesis supervisor Professor V. Srinivas at School of Mathematics, Tata Institute of Fundamental Research, Mumbai.

The work reported in this thesis is original and has not been submitted in part or in full for a Degree, a Diploma or a Fellowship in any other university or institution.

Amalendu Krishna
(Amalendu Krishna)

Counter-signed by the supervisor:

V. Srinivas
(V. Srinivas)

Date : 08/03/2001

STATISTICAL INSTITUTE
18 DEC 2003

Statement of Joint Work

This thesis is based on my work titled " Zero-cycles and K-theory on Normal Surfaces", done jointly with my thesis supervisor Prof. V. Srinivas at Tata Institute of Fundamental Research, Mumbai.

Signature of the candidate : Amalendu Krishna

Date : 08/03/2001

Signature of the supervisor : V. Srinivas

Date : 08/03/2001

Acknowledgements

The results of this thesis are joint work with my thesis supervisor Prof. V Srinivas. It gives me great pleasure to thank him for the collaboration, this thesis would not have been written without his guidance and encouragement. I am grateful to him for teaching me a lot of mathematics and showing me patience, mathematically and otherwise.

I thank Prof. S. M. Bhatwadekar for teaching me Commutative Algebra and helping me all the time.

I thank Rev. Gurujee for teaching me music. I would like to thank Amitava, Debu, Ilangovan, Kavitha, Krishnan, Maneesh, Manoj, Pablo, Pralay, Pranab, Preeti, Rahul, Siddhartha and Venkatesh for friendship and for helping me in many ways.

Finally I thank my parents for the support they have given me all these years.

Contents

Chapter 0

Introduction

The main theme of this thesis is to study the theory of algebraic cycles on singular varieties over a field. This has been studied before extensively by Collino, Barbieri-Viale, Levine, Srinivas among several others. Our interest in this thesis is to address some well known problems in the theory of zero-cycles over normal varieties. The use of K- theoretic techniques in our proofs illustrate the interplay between the study of algebraic cycles and algebraic K-theory.

For a quasi-projective surface X over a field k, we define $F^2K_0(X)$ to be the subgroup of the Grothendieck group $K_0(X)$ of vector bundles generated by the classes of smooth codimension 2 points of X. By a result of Levine [L3] (see also [BS]), one knows that if k is algebraically closed, then $F^2K_0(X)$ is naturally isomorphic to $CH^2(X)$, the (cohomological) *Chow group of zero cycles* on X modulo rational equivalence. We recall here that zero cycles are elements of the free abelian group on smooth, codimension 2 points of X. Following Levine and Weibel [LW], the cycles rationally equivalent to 0 are defined to be sums of divisors of suitable rational functions on "Cartier curves" in X. It is known by the work of Bloch and Levine [L2] that $CH^2(X) \cong H^2(X, \mathcal{K}_{2,X})$ for any quasi-projective surface X over an algebraically closed field.

For any closed subscheme Z of X, we denote by $K_0(X, Z)$, the relative K-

group as defined, for example, in [CS], and let $F^2K_0(X, Z)$ be the subgroup of $K_0(X, Z)$ defined by the classes of smooth points of $X \setminus Z$ as in [CS].

These definitions are particular instances of the more general theory of [TT], where for example the negative relative K-groups are defined as well.

In Chapter 1, we set up various definitions, notations etc. and discuss some background material. We also prove some lemmas and other results needed for the rest of this thesis. In Chapter 2, we consider the problem of understanding the relation between the Chow group of 0-cycles on a normal quasi-projective surface X over a field, and that on its desingularization. Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities of X, with reduced exceptional divisor E. We prove a formula, conjectured by Bloch and Srinivas. This formula describes the Chow group of 0-cycles $CH^2(X)$ of X, as an inverse limit of the relative Chow groups $F^2K_0(\widetilde{X}, nE)$ of $\widetilde{X}$ relative to multiples of E. Later in Chapter 4, we prove a result, which will imply that, if the ground field k is algebraically closed of characteristic $p > 0$, then the relative Chow groups of $\widetilde{X}$ relative to all non-zero multiples of E are same. Hence in this case, $CH^2(X)$ is isomorphic to the relative Chow group $F^2K_0(\widetilde{X}, E)$ of $\widetilde{X}$ relative to E. In the end of Chapter 2, we demonstrate the necessity of taking higher multiples of the exceptional divisor in characteristic 0 (see [S5]).

In Chapter 3, we study some problems which are analogues in the singular case of conjectures of Bloch-Beilinson and of Bloch. The first result of this chapter is motivated by a conjecture of Bloch and Beilinson, that the Chow group of 0-cycles of a smooth projective surface over the field $\overline{\mathbb{Q}}$ of algebraic numbers is always "finite dimensional". This is in contrast to Mumford's infinite dimensionality theorem for complex projective surfaces with positive geometric genus. Equivalent formulations of the conjecture are:
(a) for smooth projective surfaces over $\overline{\mathbb{Q}}$, the Albanese map is always injective on cycle classes of degree 0, or
(b) the Chow group of 0-cycles of a smooth *affine* surface over $\overline{\mathbb{Q}}$ is always 0. We will prove that for a 2-dimensional graded normal affine domain over

$\overline{\mathbb{Q}}$, the Chow group of 0-cycles vanishes, and all projective modules are free. We remark here that this result is not true in general when $R_0 = \mathbb{C}$, by an analogue of the Mumford theorem; examples can be found in [S5].

This result suggests that the Bloch-Beilinson conjecture, as formulated in (a) or (b) above, should be valid even for singular surfaces. Here the Chow group is to be taken in the sense of [LW]; the Albanese variety in (a) should be that defined in [ESV].

As an explicit example, if $A = \overline{\mathbb{Q}}[x, y, z]/(x^n + y^n + z^n)$, where $n \geq 4$, then $F^2K_0(A) = 0$ (in fact all projective A-modules are free), while $F^2K_0(A \otimes \mathbb{C})$ has uncountable rank (and is in fact infinite dimensional, in an appropriate sense). As far as we are aware, our result yields the first known examples of this phenomenon for *normal* (but singular) affine surfaces. In contrast, as far as we are aware, there is no similar example known at present of this phenomenon for a *smooth* affine surface, though the Bloch-Beilinson conjecture predicts the abundance of such examples. For example, the ring $B = \overline{\mathbb{Q}}[x, y, z]/(x^n + y^n + z^n - 1)$ is expected to have the same property: conjecturally $F^2K_0(B) = 0$, while Mumford's theorem implies $F^2K_0(B \otimes \mathbb{C})$ has uncountable rank.

The characteristic 0 part of our other main result of Chapter 3 is a relative version of the famous Bloch Conjecture, which says that if X is a smooth projective surface over $\mathbb{C}$ with $p_g(X) = 0$ (or equivalently if $H^2(X, \mathcal{O}_X) = 0$), then the Albanese map

$$\alpha_X : A_0(X) \to \mathrm{Alb}\,(X)$$

is an isomorphism (here $A_0(X) = F^2K_0(X)_{\deg 0}$). This conjecture is known in certain cases, e.g., for X which is not of general type. We show that if X is a normal projective surface over an algebraically closed field k of characteristic 0, and if $\widetilde{X}$ and E are as before, then $CH^2(X) \cong CH^2(\widetilde{X})$, when $H^2(X, \mathcal{O}_X) \cong H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. If k is uncountable, the converse also holds. In the case of characteristic $p > 0$, the isomorphism of the cohomology groups above is replaced by the surjectivity of the natural map $\mathrm{Pic}^0(\widetilde{X}) \longrightarrow \mathrm{Pic}^0(E)$.

We remark that one can restate the condition for isomorphism of Chow groups in positive characteristics in the following equivalent ways, using ℓ-adic cohomology:

$$H^1_{\text{ét}}(\tilde{X}, \mathbb{Q}_\ell(1)) \twoheadrightarrow H^1_{\text{ét}}(E, \mathbb{Q}_\ell(1)), \text{ or}$$

$$H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))/NS(X) \otimes \mathbb{Q}_\ell \cong H^2_{\text{ét}}(\tilde{X}, \mathbb{Q}_\ell(1))/NS(\tilde{X}) \otimes \mathbb{Q}_\ell.$$

We also prove some interesting corollaries of this, about normal rational singularities.

The final chapter of this thesis is devoted to proving Roitman's torsion theorem for normal varieties over algebraically closed fields. We show that the torsion in the Chow group of 0-cycles on a normal projective variety coincides with the torsion subgroup of its Albanese variety. This is a culmination of work of Roitman, Milne, Collino, Levine, and Srinivas, the last progress being in the mid 1980's. The outstanding open problem remaining, after the above earlier work, was the case of p-primary torsion for singular varieties in positive characteristic $p > 0$. This technically difficult case has been resolved in the present work, completing the proof of the theorem in the correct generality.

As a consequence, we prove that the Chow group of 0-cycles on a normal affine domain over an algebraically closed field is torsion free. This was known before in characteristic 0, and also in positive characteristics for normal varieties (see [S6]) of dimension three and higher. So the new result of this kind in this thesis is for affine normal surfaces.

Chapter 1

Preliminaries

In this thesis, a *variety* X will mean a reduced, connected and separated scheme of finite type over a field. We will usually restrict to *quasi-projective* varieties. X will be called a *surface*, if it is 2-dimensional.

1.1 The Chow group of 0-cycles on a singular variety

Let X be a quasi-projective variety of dimension n over a field k. Let X_{reg} denote the smallest Zariski open set containing all non-singular closed points of X which have codimension n, i.e., which lie on n-dimensional irreducible components of X. Let $X_{\text{sing}} = X \setminus X_{\text{reg}}$. A zero-cycle on X is an element of the free abelian group $Z^n(X)$ on the set of closed points of X_{reg}. For a 0-cycle $\delta = \sum_{i=1}^{r} n_i x_i, n_i \neq 0 \ \forall\ i$, let supp $\delta = \{x_1, \cdots, x_r\}$, denote the support of δ.

A *Cartier curve* C on X is a purely 1-dimensional closed subscheme of X such that no component of C is contained in X_{sing}, and C is a local complete intersection in X at all points of $C \cap X_{\text{sing}}$. From the definition, it is clear that if C is a Cartier curve on X, then any $\mathcal{O}_C$-module has finite homological dimension as an $\mathcal{O}_X$-module and hence there is a well-defined homomorphism $K_0(C) \longrightarrow K_0(X)$. Here $K_0(X)$ is the Grothendieck group of vector bundles on X (i.e., of locally free coherent $\mathcal{O}_X$-modules).

Let C be a reduced Cartier curve on X and let $C_1, \cdots, C_r$ be its irreducible components with generic points $\eta_1, \cdots, \eta_r$ respectively. For a closed point $x \in X_{\text{reg}} \cap C_i$, denote by $\mathcal{O}_{x,C_i}$, the local ring of C_i at x. Let $f \in k(C)^* = \prod_{i=1}^r k(C_i)^*$, which is a unit on $C \cap X_{\text{sing}}$, and let $f_i = a_x/b_x$ be its component in $k(C_i)^*$. We set

$$(f_i)_{C_i} = \sum_{x \in C_i} (l(\mathcal{O}_{x,C_i}/a_x\mathcal{O}_{x,C_i}) - l(\mathcal{O}_{x,C_i}/b_x\mathcal{O}_{x,C_i})).[x].$$

Here $l(M)$ denotes the length of an Artinian module M. Finally, we set

$$(f)_C = \sum_i (f_i)_{C_i}.$$

It is easy to see that this definition of $(f)_C$ is independent of the choice of chosen local expressions for the f_i's. Also $(f)_C$ is a 0-cycle on X supported on C. Let $Z^n(X)$ denote the free abelian group on the set of closed points of X_{reg} and let $R^n(X)$ denote the subgroup of $Z^n(X)$ generated by the cycles of the form $(f)_C$ for all reduced Cartier curves C on X, and all $f \in k(C)^*$ which are units on $C \cap X_{\text{sing}}$.

Definition 1.1.1 *The Chow group of 0-cycles $CH^n(X) = CH^n(X, X_{\text{sing}})$ is defined to be the quotient*

$$CH^n(X) = \frac{Z^n(X)}{R^n(X)}$$

If X is irreducible, and *normal*, then it follows from the results of [L3] (see also [BS1], section 2) that

$$CH^n(X) = \frac{Z^n(X)}{R^n(X)'}$$

where $R^n(X)'$ is the subgroup of $Z^n(X)$ generated by the set

$$\{(f)_C |\ C \text{ is irreducible, } C \cap X_{\text{sing}} = \varnothing, \text{ and } f \in k(C)^*\}.$$

The reader is referred to [LW] and [BS1] for more details regarding the Chow group nof 0-cycles on singular varieties.

Let $\mathcal{K}_{2,X}$ be the Zariski sheaf on X associated to the presheaf $U \mapsto K_2(\Gamma(U, \mathcal{O}_U))$, where K_2 denotes Milnor's (or equivalently Quillen's) K_2-group of a ring (see [Mi], [S4]). The following theorem relates 0-cycles on a surface with Zariski cohomology.

Theorem 1.1.2 *(***Bloch, Levine,** *[L2]) Let X be quasi-projective surface over an infinite field k. Then, there is a natural isomorphism*

$$CH^2(X) \cong H^2(X, \mathcal{K}_{2,X}).$$

Next we give another kind of identification of the Chow group of 0-cycles on a surface X. This relates $CH^2(X)$ with a subgroup of $K_0(X)$. Let $F^1K_0(X)$ be the kernel of the rank map $K_0(X) \longrightarrow \mathbb{Z}$, and let $F^2K_0(X)$ be the kernel of the determinant map $F^1K_0(X) \longrightarrow \text{Pic}(X)$. Then one has $F^1/F^2 \cong \text{Pic}(X)$.

Theorem 1.1.3 *(cf. [L3], [BS]) There is a natural isomorphism*

$$CH^2(X) \cong F^2K_0(X).$$

We see that the (cohomological) Chow group as defined above, is closely related to the Grothendieck group $K_0(X)$ of locally free sheaves, just as the (homological) Chow group defined by Fulton (see [S4]) is related to the Grothendieck group $G_0(X)$ of coherent sheaves.

1.2 Relative K-groups

In this section, we recall the definitions and some properties of relative K-groups. Let T be a Noetherian scheme. Let $\mathcal{P}(T)$ denote the exact category of coherent locally free $\mathcal{O}_T$-modules, and let $\mathbf{K}(T) = BQ\mathcal{P}(T)$ be the simplicial complex defined by Quillen whose homotopy groups $\pi_{i+1}(\mathbf{K}(T))$ are the Quillen K-groups $K_i(T)$ (see [S4] for example). As usual, the base point is the class of a 0-object. Let $S \subset T$ be a closed subscheme. Let F denote the homotopy fiber of the natural map $\mathbf{K}(T) \to \mathbf{K}(S)$ of topological spaces, induced by the exact functor $\mathcal{P}(T) \to \mathcal{P}(S)$ given by restriction of locally free sheaves; then we define

$$K_i(T,S) = \pi_{i+1}(F)$$

for all $i \geq 0$. Using the long exact sequence of homotopy groups associated to a fibration, the relative K-groups fit into a long exact sequence, which we refer to as the relative K-theory *exact sequence of the pair* (T,S),

$$\cdots \to K_i(T,S) \longrightarrow K_i(T) \longrightarrow K_i(S) \xrightarrow{\partial} K_{i-1}(T,S) \to \cdots \to K_0(T) \to K_0(S).$$

The relative K-groups satisfy some functorial properties (see [CS]). For example, a morphism of pairs $f : (T', S') \to (T, S)$ induces maps $f^* : K_i(T,S) \to K_i(T', S')$ giving a commutative diagram of the associated long exact sequences. Another property is that if $W \subset T$ is a closed subscheme of finite homological dimension, such that $\mathcal{O}_W$ and $\mathcal{O}_S$ are Tor-independent over $\mathcal{O}_T$ (for instance, if W is disjoint from S), then the scheme theoretic intersection $W \cap S$ has finite $\mathcal{O}_S$-homological dimension, and there are natural maps $K_i(W, S \cap W) \to K_i(T,S)$ compatible with the natural ("Gysin") maps $K_i(W) \to K_i(T)$ and $K_i(W \cap S) \to K_i(S)$. Here "natural" means that if $f : (T', S') \to (T, S)$ is a morphism of pairs which is flat over the support of W, and if $W' = W \times_T T'$, then we have commutative diagrams

$$\begin{array}{ccc} K_i(W', W' \cap S') & \longrightarrow & K_i(T', S') \\ f^* \uparrow & & \uparrow f^* \\ K_i(W, W \cap S) & \longrightarrow & K_i(T, S) \end{array}$$

In particular, for a quasi-projective surface X and a closed subscheme Z, taking W to be a smooth codimension 2 point of $X \setminus Z$, and $i = 0$, we can define the *relative cycle class* $[W] \in K_0(X, Z)$ as the image of the unit element of $K_0(W) = \mathbb{Z}$ under $K_0(W) = K_0(W, W \cap Z) \to K_0(X, Z)$. Then the *relative Chow group* $F^2K_0(X, Z)$ is defined to be the subgroup generated by such classes $[W]$. With the relative K-groups and relative Chow groups defined, we prove the first lemma in this thesis, which will be used in the sequel.

Lemma 1.2.1 *(a) For any quasi-projective surface X and closed subschemes $S \subset Z \subset X$ of dimension ≤ 1, the natural map $F^2K_0(X, Z) \to F^2K_0(X, S)$ is surjective. If $U \subset X$ is an open dense subscheme, then $F^2K_0(X, Z)$ is generated by the classes of smooth codimension 2 points of $U \setminus Z$.*

(b) If X, X' are quasi-projective surfaces, $Z \subset X$, $Z' \subset X'$ closed subschemes, and $f : (X', Z') \to (X, Z)$ is a morphism of pairs such that $X' \setminus Z' \to X \setminus Z$ is an isomorphism, then $f^* : K_0(X, Z) \to K_0(X', Z')$ induces a surjection $f^* : F^2K_0(X, Z) \to F^2K_0(X', Z')$.

Proof. The assertion (b) is obvious from the functoriality properties of relative K-groups listed earlier, since $F^2K_0(X, Z)$ and $F^2K_0(X', Z')$ have the same sets of generators, namely the relative cycle classes of smooth codimension 2 points of $X \setminus Z = X' \setminus Z'$.

The first assertion in (a) is an immediate consequence of the second, which we now prove. Let $U \subset X$ be dense open, and $x \in X \setminus Z$ a smooth codimension 2 point. By Bertini's theorem, choose a reduced hypersurface section $C \subset X$ such that $x \in C$ is a smooth point, $C \cap U$ is non-empty, $C \cap Z$ is finite, and C does not contain any associated point of Z. Then $\mathcal{O}_C$ and $\mathcal{O}_Z$ are Tor-independent over $\mathcal{O}_X$, so that there is a natural homomorphism $K_0(C, C \cap Z) \to K_0(X, Z)$. This induces a homomorphism from the relative Picard group $\mathrm{Pic}\,(C, C \cap Z) \to F^2K_0(X, Z)$ such that the class $[x] \in F^2K_0(X, Z)$ is in the image. It is standard that $\mathrm{Pic}\,(C, C \cap Z)$ is generated by the classes of smooth points contained in any dense open subset of $C \setminus Z$, for instance $(C \cap U) \setminus Z$. □

1.3 The group $SK_1(T)$ as Zariski cohomology

In this section, we define another K-theoretic notion and its cohomological interpretation in some situations. Let T be a Noetherian scheme. Let S be the finite set of associated points of T, and let $\mathcal{O}_{S,T}$ denote the semi-local ring of S on T. We define

$$SK_1(T) = \mathrm{Ker}(K_1(T) \longrightarrow K_1(\mathcal{O}_{S,T})).$$

It is easy to see that $SK_1(T)$ can also be expressed as the kernel of the natural map $K_1(T) \longrightarrow \Gamma(T, \mathcal{O}_T^*)$. We recall that this map is naturally split and surjective. The splitting is given as follows.

Fix an element $\alpha \in \Gamma(T, \mathcal{O}_T^*)$. Then α determines a unique morphism

$$\tau_\alpha : T \longrightarrow \mathrm{Spec}(\mathbb{Z}[t, t^{-1}]),$$

such that $\tau_\alpha{}^*(t) = \alpha$. Here, $t \in (\mathbb{Z}[t, t^{-1}])^* = K_1(\mathbb{Z}[t, t^{-1}])$ is the standard class. Now we map α to the image of t in $K_1(T)$ under the induced map

$$\tau_\alpha{}^* : K_1(\mathbb{Z}[t, t^{-1}]) \longrightarrow K_1(T).$$

This gives the required splitting. Thus, one has the following short exact sequence, which is naturally split.

$$0 \longrightarrow SK_1(T) \longrightarrow K_1(T) \longrightarrow \Gamma(T, \mathcal{O}_T^*) \longrightarrow 0.$$

We prove the following lemma to give a cohomological understanding of the group $SK_1(T)$ in some situations.

Lemma 1.3.1 *Let T be a quasi-projective scheme of dimension ≤ 1 over a field k.*

(a) There is a natural isomorphism

$$SK_1(T) \longrightarrow H^1(T, \mathcal{K}_{2,T}).$$

In particular, if dim $T = 0$, then $SK_1(T) = 0$.

(b) If $T' \subset T$, the restriction map $SK_1(T) \to SK_1(T')$ is surjective.

(c) If T is a closed subscheme of a smooth surface Y over k, then there is a surjection $SK_1(Y) \longrightarrow H^1(Y, \mathcal{K}_{2,Y})$ making the following diagram commute

$$\begin{array}{ccc} SK_1(Y) & \longrightarrow & H^1(Y, \mathcal{K}_{2,Y}) \\ \downarrow & & \downarrow \\ SK_1(T) & \longrightarrow & H^1(T, \mathcal{K}_{2,T}) \end{array}$$

Proof. The proof of (a) is based on the Brown-Gersten spectral sequence constructed in [TT]

$$E_2^{p,q} = H^p_{\text{Zar}}(T, \mathcal{K}_{-q,T}) \Rightarrow K_{-q-p}(T).$$

Since $\dim T \leq 1$, $E_2^{p,q} = 0$ for $p \neq 0, 1$, so we compute easily that

$$F^0K_1(T)/F^1K_1(T) = E_\infty^{0,-1} = E_2^{0,-1} = H^0(T, \mathcal{K}_{1,T}),$$

$$F^1K_1(T)/F^2K_1(T) = E_\infty^{1,-2} = E_2^{1,-2} = H^1(T, \mathcal{K}_{2,T}),$$

and

$$F^iK_1(T)/F^{i+1}K_1(T) = 0 \;\; \forall \;\; i \geq 2$$
$$\Rightarrow F^2K_1(T) = 0.$$

Hence we get a short exact sequence

$$0 \longrightarrow H^1(T, \mathcal{K}_{2,T}) \longrightarrow K_1(T) \longrightarrow H^0(T, \mathcal{K}_{1,T}) \longrightarrow 0$$

which gives the desired isomorphism of (a). The assertion (b) is a consequence of (a) and the following

Sublemma 1.3.2 *For any scheme T and a closed subscheme T', the canonical map of sheaves $\mathcal{K}_{2,T} \to \mathcal{K}_{2,T'}$ is surjective.*

Proof. It is enough to prove the surjectivity at stalks. But we know that for a local ring A, $K_2(A)$ is generated by Steinberg symbols (see [S4]), and hence the map on K_2 is surjective for surjective map of local rings. □

To prove (c), we use the BGQ-spectral sequence for smooth varieties (which is a special case of that of [TT])

$$E_2^{p,q} = H^p_{\text{Zar}}(Y, \mathcal{K}_{-q,Y}) \Rightarrow K_{-q-p}(Y)$$

to get $F^1K_1(Y) = SK_1(Y)$ and $F^1K_1(Y)/F^2K_1(Y) = E_\infty^{1,-2} = H^1(Y, \mathcal{K}_{2,Y})$. Hence, there is a surjection

$$SK_1(Y) \twoheadrightarrow H^1(Y, \mathcal{K}_{2,Y}).$$

INDIAN STATISTICAL INSTITUTE LIBRARY
T 121
18 DEC 2003
CALCUTTA

The commutativity of the diagram follows from the functoriality of the spectral sequence of [TT]. □

Remark We remark here that the more sophisticated methods of [TT] give generalizations of the relative K-groups also for negative indices i, and yield a Brown-Gersten spectral sequence in the relative case (see [PW]): for a pair (T, S) there is a spectral sequence

$$E_2^{p,q} = H^p(T, \mathcal{K}_{-q,(T,S)}) \implies K_{-p-q}(T, S).$$

Here $\mathcal{K}_{j,(T,S)}$ is the sheaf associated to the presheaf $U \mapsto K_j(U \cap T, U \cap S)$, and in this spectral sequence, the indices of the K-groups and sheaves are also allowed to be negative integers (this is exploited in [We]). For an irreducible surface $\widetilde{X}$ and a closed subscheme Z of dimension ≤ 1, we obtain from the spectral sequence an exact sequence

$$0 \to H^2(\widetilde{X}, \mathcal{K}_{2,(\widetilde{X},Z)}) \to K_0(\widetilde{X}, Z) \to H^1(\widetilde{X}, \mathcal{K}_{1,(\widetilde{X},Z)}) \to 0 \tag{1.1}$$

where

$$H^1(\widetilde{X}, \mathcal{K}_{1,(\widetilde{X},Z)}) = H^1(\widetilde{X}, \mathcal{O}^*_{(\widetilde{X},Z)}) \cong \mathrm{Pic}\,(\widetilde{X}, Z)$$

is the relative Picard group of isomorphism classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf on $\widetilde{X}$, and s is trivialization (*i.e.*, a nowhere vanishing section) on Z of $\mathcal{L}\mid_Z = \mathcal{L} \otimes \mathcal{O}_Z$. One can show using Proposition 1.4.4 below that the subgroup $H^2(\widetilde{X}, \mathcal{K}_{2,(\widetilde{X},Z)}) \subset K_0(\widetilde{X}, Z)$ coincides with our relative Chow group $F^2K_0(\widetilde{X}, Z)$, at least when Z is contained in the exceptional divisor of a resolution of a normal surface. We will not explicitly use any of these remarks in the sequel; however, a result of [PW], used in the proof of Theorem 2.0.5 in a latter chapter, is a consequence of (1.1) combined with [GW]. If, however, we define the relative Chow groups to be the appropriate K-cohomology groups, then one gets a version of Theorem 2.0.5 expressed in terms of these groups, which can be proved "by elementary methods", *i.e.*, without the use of [TT]. A similar comment applies to the remaining results of this thesis, which are phrased in terms of relative Chow groups.

Next we prove a simple lemma, which we will refine later in a special situation. This will give us a bound on the kernel of the map from the relative Chow group of the pair (X, Z) to the absolute Chow group of X in terms of the group $SK_1(Z)$, if the underlying subscheme Z is 'small'. This often reduces the problem of understanding the Chow group of 0-cycles on surfaces to the understanding of relative Chow groups and groups SK_1 of subschemes.

Lemma 1.3.3 *Let X be a quasi-projective surface over a field k and let Z be a closed subscheme of X. Then*

$$\ker\left(F^2K_0(X,Z) \longrightarrow F^2K_0(X)\right) \subset \text{image}\,(\partial : SK_1(Z) \to K_0(X,Z)).$$

where $\partial : K_1(Z) \longrightarrow K_0(X,Z)$ is the boundary map in the exact sequence for the pair (X, Z), obtained in Section 1.2. In particular, if $\dim Z = 0$, then $F^2K_0(X,Z) \longrightarrow F^2K_0(X)$ is an isomorphism.

Proof. Let $\alpha \in \ker(F^2K_0(X,Z) \to F^2K_0(X))$, so that $\alpha = \partial(\beta)$ for some $\beta \in K_1(Z)$. Write $\beta = \beta_0 + \beta_1$ with $\beta_0 \in \Gamma(Z, \mathcal{O}_Z^*)$ and $\beta_1 \in SK_1(Z)$. We claim $\partial(\beta_0) = 0$, so that $\alpha = \partial(\beta_1)$.

Indeed, let δ be a 0-cycle representing α, and let $D = \operatorname{supp}\delta$. Then $D \cap Z = \emptyset$ and D is a finite set of points. Clearly $\alpha \mapsto 0$ in $K_0(X \setminus D, Z)$, so that there exists $\gamma \in K_1(X \setminus D)$ with $\gamma \mapsto \beta$ under the restriction map $K_1(X \setminus D) \longrightarrow K_1(Z)$. Write $\gamma = \gamma_0 + \gamma_1$, with $\gamma_0 \in \Gamma(X \setminus D, \mathcal{O}_X^*)$, $\gamma_1 \in SK_1(X \setminus D)$. Then $\gamma_i \mapsto \beta_i$. But D consists of smooth, codimension 2 points of X, so that γ_0 extend to a global section $\tilde{\gamma}_0 \in \Gamma(X, \mathcal{O}_X^*) \subset K_1(X)$, which restricts to β_0, and so $\partial(\beta_0) = 0$.

In the case when $\dim Z = 0$, we have $SK_1(Z) = 0$, so lemma 1.2.1 implies $F^2K_0(X,Z) \cong F^2K_0(X)$. $\square$

1.4 K-groups of blow-ups

This section will be devoted to proving some results about the K-groups of blow-ups of Noetherian schemes. We will be particularly interested in the case of blow-up of normal quasi-projective varieties. Recall that a morphism $f : X \longrightarrow Y$ of Noetherian schemes is said to be of *finite Tor dimension*, if there exists a positive integer N such that the $\mathcal{O}_X$-modules $Tor_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}) = 0$ for all coherent $\mathcal{O}_Y$-modules $\mathcal{F}$ for all $i \geq N$.

Proposition 1.4.1 *Let $f : X \longrightarrow Y$ be a proper morphism of finite Tor-dimension between schemes supporting ample line bundles. Then there is a push forward map $f_* : K_i(X) \longrightarrow K_i(Y)$, such that the projection formula holds, i.e., for any $x \in K_0(X)$, $y \in K_i(Y)$, we have the formula*

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y.$$

Proof. See [S4], Proposition 5.12. □

Proposition 1.4.2 *Let $\pi : X' \longrightarrow X$ be a proper morphism of finite Tor-dimension between Noetherian schemes supporting ample line bundles. Assume that for a dense open subscheme $V \subset X$, the restriction $\pi^{-1}(V) \to V$ is an isomorphism. Then the natural maps $\pi^* : K_*(X) \longrightarrow K_*(X')$ are split injective for all $i \geq 0$.*

Proof. From Proposition 1.4.1, there are homomorphisms $\pi_* : K_i(X') \to K_i(X)$, such that $\pi_* \circ \pi^*(x) = \pi_*(1) \cdot x$ for $x \in K_i(X)$. Here $\pi_*(1) \in K_0(X)$ is the image of the unit element $1 = [\mathcal{O}_{X'}] \in K_0(X')$. Since π is an isomorphism over a dense open subscheme V of X, $\pi_*(1) \in K_0(X)$ restricts to the unit element in $K_0(V)$, and hence $\pi_*(1) \in K_0(X)$ is a unit in the ring $K_0(X)$. We see that π^* clearly splits the push forward map. □

Corollary 1.4.3 *If X is integral, and $\pi : X' \to X$ is the blow-up along a local complete intersection proper subscheme, then $\pi^* : K_i(X) \to K_i(X')$ is split injective for all i.*

Proof. π is an isomorphism outside the local complete intersection closed subscheme. □

In fact $\pi_*(1) = 1 \in K_0(X)$ in the situation of the corollary, but we don't need to use this.

To conclude this section, we prove the following refinement of lemma 1.3.3. When k is algebraically closed of characteristic 0, and X, $\widetilde{X}$ are defined over a number field, this was proved in [S8] by a different method. Though one can get by without using this result in the proofs of the theorems given in this thesis, it has a certain conceptual significance, since

(i) it gives a formula for the kernel of the map on Chow groups under a resolution of singularities, instead of just an upper bound, in terms of SK_1 of exceptional divisors

(ii) using [TT], it implies that $K_0(X, Z)$, which was defined in a formal way using homotopy theory, is in fact generated by the classes of relative 0-cycles and by elements of Pic (X, Z), and that there is a "Bloch formula" $F^2K_0(X, Z) = H^2(X, \mathcal{K}_{2,(X,Z)})$

(iii) it suggests that the sheaf $R^1\pi_*\mathcal{K}_{2,\widetilde{X}}$ satisfies a *Formal Function Theorem*, if $\pi : \widetilde{X} \to X$ is a resolution of singularities of a normal quasi-projective surface X. For technical reasons (basically, in order to use results of Levine as formulated in [S4], Chapter 9), we assume the ground field k is infinite, though presumably this hypothesis can be eliminated with some more work.

Proposition 1.4.4 *Let X be an integral normal quasi-projective surface over an infinite field k, and $\pi : \widetilde{X} \to X$ a resolution of singularities, with reduced exceptional divisor E. Let Z be a closed subscheme of $\widetilde{X}$ supported within E. Then*

$$\ker\left(F^2K_0(\widetilde{X}, Z) \longrightarrow F^2K_0(\widetilde{X})\right) = \text{image}\left(\partial : SK_1(Z) \to K_0(\widetilde{X}, Z)\right).$$

In particular, there is an exact sequence

$$SK_1(\widetilde{X}) \to SK_1(Z) \to F^2K_0(\widetilde{X}, Z) \to F^2K_0(\widetilde{X}) \to 0$$

Proof. By lemma 1.3.3 it suffices to prove that

$$\text{image}\left(\partial : SK_1(Z) \to K_0(\widetilde{X}, Z)\right) \subset F^2K_0(\widetilde{X}, Z).$$

Clearly Z is a subscheme of nE for some positive integer n. From lemma 1.2.1 and lemma 1.3.1, we are reduced to considering the special case when $Z = nE$.

Let $W = \widetilde{X} \times_X \operatorname{Spec} \mathcal{O}_{S,X}$, where $S = X_{sing}$ is the singular locus. Then W is regular, and (lemma 1.3.1) there is a commutative diagram

$$\begin{array}{ccc} SK_1(W) & \longrightarrow & SK_1(Z) \\ \downarrow & & \downarrow\cong \\ H^1(W, \mathcal{K}_{2,W}) & \longrightarrow & H^1(Z, \mathcal{K}_{2,Z}) \end{array}$$

and hence by composition with $\partial : SK_1(Z) \to K_0(\widetilde{X}, Z)$, an induced map

$$\varphi : H^1(W, \mathcal{K}_{2,W}) \to K_0(\widetilde{X}, Z).$$

We claim that the image of this map lies in $F^2K_0(\widetilde{X}, Z)$.

In fact, we claim this map has the following "more concrete" description. Comparing the Gersten resolution (see [S4], Chapter 5) for $\mathcal{K}_{2,\widetilde{X}}$,

$$0 \to \mathcal{K}_{2,\widetilde{X}} \to i_*K_2(k(\widetilde{X})) \to \oplus_{x \in \widetilde{X}^1} i_{x*}k(x)^* \to \bigoplus_{x \in \widetilde{X}^2} i_{x*}\mathbb{Z} \to 0$$

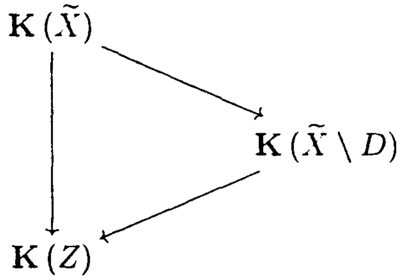
and its restriction to W, a 1-cocycle for $\alpha \in H^1(W, \mathcal{K}_{2,W})$ is an element

$$\widetilde{\alpha} \in \bigoplus_{x \in W^1} k(x)^* \subset \bigoplus_{x \in \widetilde{X}^1} k(x)^*,$$

and hence by taking the coboundary on $\widetilde{X}$, yields a 0-cycle δ supported on $\widetilde{X} \setminus E$; the class of this 0-cycle in $F^2K_0(\widetilde{X}, Z)$ depends only on the element in $H^1(W, \mathcal{K}_{2,W})$, and is the element $\varphi(\alpha)$ (up to sign). To see this, note that if $D = \operatorname{supp} \delta$, there is a commutative diagram with exact rows

$$\begin{array}{ccccc} SK_1(\widetilde{X}) & \to SK_1(\widetilde{X} \setminus D) \to & K_0(D) \to \\ =\downarrow & \downarrow & \downarrow \vartheta \\ SK_1(\widetilde{X}) & \to SK_1(Z) \to & K_0(\widetilde{X}, Z) \to \end{array}$$

where the top row is a localization sequence for the inclusion of the open subset $\tilde{X} \setminus D \subset \tilde{X}$, and the bottom row is the exact sequence of the pair $(\tilde{X}, Z)$. This is induced from the commutative diagram of spaces



and standard properties of homotopy fibers. The homotopy fiber of $\mathbf{K}(\tilde{X}) \to \mathbf{K}(\tilde{X} \setminus D)$ is identified with $\mathbf{K}(D)$ by Quillen's localization theorem (see [S4], Chapter 5). One verifies from the definitions that if F is the homotopy fiber of $\mathbf{K}(\tilde{X}) \to \mathbf{K}(Z)$, then the induced map $\mathbf{K}(D) \to F$ from the above commutative triangle coincides with the "Gysin" map, inducing $K_i(D) \to K_i(\tilde{X}, Z)$ on homotopy groups. In particular, the induced map $\vartheta : K_0(D) \to K_0(\tilde{X}, Z)$ is just the relative cycle class map restricted to points of D. Now $\tilde{\alpha}$ determines an element of $H^1(\tilde{X} \setminus D, \mathcal{K}_{2,\tilde{X}\setminus D})$, which lifts to an element $\alpha' \in SK_1(\tilde{X} \setminus D)$. So it suffices to observe that the localization boundary map $SK_1(\tilde{X} \setminus D) \to K_0(D)$ maps α' to the class of δ.

So we are reduced to proving that the natural map $H^1(W, \mathcal{K}_{2,W}) \to H^1(Z, \mathcal{K}_{2,Z})$ is *surjective*. Now from results of Levine (see [S4], Chapter 9, pp. 221-224) one knows that

$$H^1(W, \mathcal{K}_{2,W})/\text{image} H^1_E(W, \mathcal{K}_{2,W}) \cong SK_0(\mathcal{C}),$$

where $\mathcal{C}$ is the category of $\mathcal{O}_{S,X}$-modules of finite length and finite projective dimension ([S4] deals with the case when S is a single point, *i.e.*, $\mathcal{O}_{S,X}$ is a local ring, but the arguments there are easily adapted to the semi-local case), and $SK_0(\mathcal{C})$ is the kernel of a natural map $K_0(\mathcal{C}) \to G_0(E)$ (here $G_0(E)$ is the Grothendieck group of coherent $\mathcal{O}_E$-modules). Now $SK_0(\mathcal{C})$ remains invariant under étale base change of Spec $\mathcal{O}_{S,X}$-schemes (preserving residue fields of $\mathcal{O}_{S,X}$); so does the subscheme $Z = nE$, and also $H^1_E(W, \mathcal{K}_{2,W})$. So, after passing to a limit, we reduce to proving the surjectivity of

$$H^1(W^h, \mathcal{K}_{2,W^h}) \to H^1(nE, \mathcal{K}_{2,nE}),$$

where $W^h = W \times_{\operatorname{Spec} \mathcal{O}_{S,X}} \operatorname{Spec} \mathcal{O}^h_{S,X}$ is the base change to the Henselization of $\mathcal{O}_{S,X}$ with respect to its Jacobson radical.

Since nE is Cohen-Macaulay, for each closed point $P \in E$, the total quotient ring of $\mathcal{O}_{P,nE}$ is the product of the local rings $\mathcal{O}_{\eta_j,nE_j}$ of the generic points η_j of the irreducible components E_j of E which contain P. If $\mathcal{H}_P$ is the category of $\mathcal{O}_{P,nE}$-modules of finite length and finite projective dimension, we have a localization sequence

$$K_2(\mathcal{O}_{P,nE}) \to \bigoplus_{E_j \ni P} K_2(\mathcal{O}_{\eta_j,nE_j}) \to K_1(\mathcal{H}_P) \to 0 \tag{1.2}$$

(the surjectivity of the right hand arrow is because the Cohen-Macaulay property implies that $K_1(\mathcal{O}_{P,nE}) = \mathcal{O}^*_{P,nE}$ injects into $\oplus_j K_1(\mathcal{O}_{\eta_j,nE_j}) = \oplus_j \mathcal{O}^*_{\eta_j,nE_j}$). If P is a smooth point of E, then in fact $\mathcal{O}_{P,nE} \cong \mathcal{O}_{P,E}[t]/(t^n)$ is a truncated polynomial algebra, and so $K_2(\mathcal{O}_{P,nE})$ injects into K_2 of its total quotient ring (see [Gr], or [B], [BL1]). Hence we obtain an exact sheaf sequence (compare [S2], lemma 2.1)

$$0 \to \mathcal{S} \to \mathcal{K}_{2,nE} \to \oplus_j i_{j*} K_2(\mathcal{O}_{\eta_j,nE_j}) \to \bigoplus_{P \in E^1} i_{P*} K_1(\mathcal{H}_P) \to 0$$

analogous to a Gersten resolution, where $\eta_j \in E_j$ are the generic points of irreducible components of E, $i_{j*}K_2(\mathcal{O}_{\eta_j,E_j})$ is a constant sheaf on E_j, and $i_{P*}K_1(\mathcal{H}_P)$ is a constant sheaf on the closed point $P \in E$. The sheaf $\mathcal{S}$ is supported on the finite set E_{sing}. This yields a presentation for $H^1(nE, \mathcal{K}_{2,nE})$ as

$$H^1(nE, \mathcal{K}_{2,nE}) = \operatorname{coker}\left(\bigoplus_j K_2(\mathcal{O}_{\eta_j,nE_j}) \to \bigoplus_P K_1(\mathcal{H}_P)\right). \tag{1.3}$$

Similarly there is an exact sequence

$$0 \to \mathcal{K}_{2,W^h} \to i_* \mathcal{K}_{2,\operatorname{Spec} \mathcal{O}_{E,W^h}} \to \bigoplus_{x \in (W^h \setminus E)^1} i_{x*} k(x)^* \to \bigoplus_{P \in E} i_{P*}\mathbb{Z} \to 0$$

where $\mathcal{O}_{E,W^h}$ is the 1-dimensional semi local ring of the generic points of E on W^h, and $i : \operatorname{Spec} \mathcal{O}_{E,W^h} \to W^h$ the natural map. This is a consequence of the Gersten resolutions for $\mathcal{K}_2$ on W^h and on $\operatorname{Spec} \mathcal{O}_{E,W^h}$. Since W^h is birational to $\operatorname{Spec} \mathcal{O}^h_{S,X}$, each $x \in (W^h \setminus E)^1$ is such that its closure in W^h is of the form $\{x, P\}$ for a *unique* $P \in E$ (since $\mathcal{O}^h_{S,X}$ is Henselian!). Hence if we write

$$\mathcal{A} = \ker \left(\bigoplus_{x \in (W^h \setminus E)^1} i_{x*} k(x)^* \to \bigoplus_{P \in E} i_{P*} \mathbb{Z} \right),$$

then in fact

$$\mathcal{A} = \bigoplus_{P \in E} \mathcal{A}(P)$$

where

$$\mathcal{A}(P) = \ker \left(\bigoplus_{x \in (W^h \setminus E)^1,\, x \rightsquigarrow P} i_{x*} k(x)^* \to i_{P*} \mathbb{Z} \right).$$

The map $H^1(W^h, \mathcal{K}_{2,W^h}) \to H^1(nE, \mathcal{K}_{2,nE})$ is clearly that induced by the sheaf map $\mathcal{K}_{2,W^h} \to i_{E*}\mathcal{K}_{2,nE}$, where $i_E : E \to W^h$ is the inclusion. From the commutative diagram with exact rows (we omit the symbol i_{E*} for simplicity)

$$\begin{array}{ccccc}
0 \to \mathcal{K}_{2,W^h} \to & i_* \mathcal{K}_{2,\operatorname{Spec} \mathcal{O}_{E,W^h}} & \to \oplus_{P \in E} \mathcal{A}(P) \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 \to \mathcal{K}_{2,nE}/S \to & \oplus_j i_{j*} K_2(\mathcal{O}_{\eta_j, E_j}) & \to i_{P*} K_1(\mathcal{H}_P) \to 0
\end{array}$$

whose vertical arrows are surjective, and since $H^1(\operatorname{Spec} \mathcal{O}_{E,W^h}, \mathcal{K}_{2,\operatorname{Spec} \mathcal{O}_{E,W^h}}) = 0$, we obtain a commutative square

$$\begin{array}{ccc}
\oplus_{P \in E} H^0(W^h, \mathcal{A}(P)) & \twoheadrightarrow & H^1(W^h, \mathcal{K}_{2,W^h}) \\
\downarrow & & \downarrow \\
\oplus_{P \in E} K_1(\mathcal{H}_P) & \twoheadrightarrow & H^1(nE, \mathcal{K}_{2,nE})
\end{array}$$

Hence we are reduced to proving that $H^0(W^h, \mathcal{A}(P)) \twoheadrightarrow K_1(\mathcal{H}_P)$ for each P. Now by the Gersten resolution for $K_2(\mathcal{O}_{P,W^h})$, we see that

$$H^0(W^h, \mathcal{A}(P)) = \ker \left(\bigoplus_{x \in (W^h \setminus E)^1,\, x \rightsquigarrow P} k(x)^* \to \mathbb{Z} \right)$$

$$= \text{coker}\ \left(K_2(\mathcal{O}_{P,W^h}) \to K_2(\mathcal{O}_{\cup_j E_j, W^h})\right)$$

where E_j ranges over the components of E containing P. Now it is clear from comparison with the presentation (1.2) for $K_1(\mathcal{H}_P)$ that $H^0(W^h, \mathcal{A}(P)) \twoheadrightarrow K_1(\mathcal{H}_P)$ for all P. □

1.5 Reduction ideals

In this section, we review the notion of reduction ideals and different equivalent conditions for reduction ideals. Reduction ideals will play an important role in the proof of Theorem 2.0.5 in the next chapter and also in Chapter 3. Throughout in this section, R will denote a commutative Noetherian ring.

Definition 1.5.1 *An ideal I is called a reduction ideal of an ideal J, if $I \subset J$ and there exists a positive integer n so that $IJ^n = J^{n+1}$.*

We give an alternate interpretation of reduction ideals. First, recall that an element $x \in R$ is said to *integral* over I, if there are $a_i \in I^i$ so that

$$x^m + a_1 x^{m-1} + \cdots + a_{m-i} x^i + \cdots + a_m = 0.$$

Lemma 1.5.2 *Let $I \subset J$ be two ideals of R. Then I is a reduction ideal of J if and only if J is integral over I in the sense that every $x \in J$ is integral over I.*

Proof. See [We], Lemma 1.2. □

Before stating the next result related to reduction ideals, we recall some basic facts about the blowing-up of Spec$R$ along I, i.e. the scheme $\mathbf{Proj}(R[It])$. If $x \in I$ and T denotes xt, then $R[It][T^{-1}] \cong R'[T, T^{-1}]$, where $R' = R[I/x]$. Thus the affine open subsets $D_+(xt)$ of $\mathbf{Proj}(R[It])$ are isomorphic to SpecR'.

Theorem 1.5.3 *Let $I \subset J$ be ideals in a Noetherian domain R. Then I is a reduction ideal of J if and only if $\mathbf{Proj}(R[Jt]) \longrightarrow \mathbf{Proj}(R[It])$ is a finite morphism.*

Proof. See [We], Theorem 1.5. □

The following result will show that there is an abundance of reduction ideals generated by systems of parameters.

Proposition 1.5.4 *Suppose that $(R, \mathfrak{m})$ is a d-dimensional Noetherian local domain, with infinite residue field. If $J = (x_1, \cdots, x_r)R$ is an $\mathfrak{m}$-primary ideal. Then for any sufficiently general $d \times r$ matrix (r_{ij}) over R, the elements $y_i = \sum_j r_{ij}x_j$ satisfy:*

(1) the ideal $I = (y_1, \cdots, y_d)$ is a reduction ideal of J;

(2) the elements $y_1, \cdots, y_d$ form a system of parameters of R;

(3) If R is Cohen-Macaulay, the y_i's form a regular sequence on R.

Proof. See [Ma], 14.14 and 17.4. □

Remark. When $R/\mathfrak{m}$ is finite, the conclusions still hold for some power J^e of J, except that the term " sufficiently general" should be replaced by terms like "most". This was proved by Northcott and Rees ([NR], theorem 3.4).

Chapter 2

Inverse limit formula

Let X be a connected (equivalently, integral), normal quasi-projective surface over a field k and let $S \subset X$ be the singular locus of X. Let $\pi : \widetilde{X} \rightarrow X$ be a resolution of singularities of X. Let $E = \pi^{-1}(S)$ denote the reduced exceptional divisor on $\widetilde{X}$, and let nE denote the n-th infinitesimal neighbourhood of E in $\widetilde{X}$. Then (as explained in [S2]) there is a commutative diagram as follows, for each $n > 1$, with surjective arrows.

$$\begin{array}{ccccc} & & F^2K_0(X) & & \\ & \swarrow & \downarrow & \searrow^{\pi^*} & \\ F^2K_0(\widetilde{X}, nE) & \twoheadrightarrow & F^2K_0(\widetilde{X}, (n-1)E) & \twoheadrightarrow & F^2K_0(\widetilde{X}) \end{array}$$

The surjectivity of each arrow is a consequence of Lemma 1.2.1. It was conjectured by Bloch and Srinivas (see [S2] and [S8]) that the maps $F^2K_0(\widetilde{X}, nE) \longrightarrow F^2K_0(\widetilde{X}, (n-1)E)$ and $F^2K_0(X) \longrightarrow F^2K_0(\widetilde{X}, nE)$ should be isomorphisms for all sufficiently large n. This chapter is devoted to the proof of this conjecture. In precise terms, we prove:

Theorem 2.0.5 *Let X be a connected normal quasi-projective surface over a field k, $\pi : \widetilde{X} \rightarrow X$ a resolution of singularities, with reduced exceptional divisor E. Then for all sufficiently large n, the maps*

(i) $F^2K_0(\widetilde{X}, nE) \longrightarrow F^2K_0(\widetilde{X}, (n-1)E)$ and

(ii) $F^2K_0(X) \longrightarrow F^2K_0(\widetilde{X}, nE)$

are isomorphisms. In particular,

$$F^2K_0(X) \;\cong\; \varprojlim_n F^2K_0(\widetilde{X}, nE).$$

We will also discuss special cases of this theorem when the characteristic of the ground field is 0 or it is $p > 0$. In particular, we will see that it is in general really necessary to take higher thickenings of the reduced exceptional curve.

2.1 Proof of Bloch-Srinivas Conjecture

Our proof of Theorem 2.0.5 is inspired by a technique in [We], of factoring the morphism $\pi : \widetilde{X} \to X$ into a composition of morphisms $\widetilde{X} \to X'$ and $X' \to X$, where $X' \to X$ is the blow-up of a local complete intersection subscheme supported at the singular locus, and $\widetilde{X} \to X'$ is the normalization map. The morphism $X' \to X$ is a proper morphism of finite Tor dimension, and hence the maps $K_i(X) \to K_i(X')$ are split inclusions, inducing in particular an isomorphism $F^2K_0(X) \cong F^2K_0(X')$. On the other hand, the relation between $K_0(X')$ and $K_0(\widetilde{X})$ is governed by a suitable Mayer-Vietoris sequence.

We proceed with the proof of Theorem 2.0.5 in detail. We first note that if E is the reduced exceptional divisor on $\widetilde{X}$, then for all $n \geq 1$, nE is an effective Cartier divisor on $\widetilde{X}$ with ideal sheaf $\mathcal{O}_{\widetilde{X}}(-nE)$. We give the set S, the reduced subscheme structure and denote by nS, the n-th infinitesimal neighbourhood of S in X. By [Li], π is the blow-up of X along a sheaf of ideals $\mathcal{I}$ of the structure sheaf $\mathcal{O}_X$ such that the subscheme Y defined by $\mathcal{I}$ has support S. Thus one has

$$\widetilde{X} = \mathrm{Proj}_X(\oplus_{n\geq 0}\mathcal{I}^n), \quad E = \mathrm{Proj}_Y((\oplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1})_{\mathrm{red}}).$$

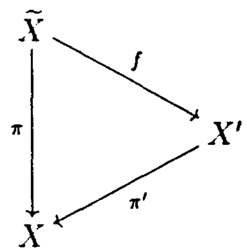
Putting $\widetilde{Y} = \text{Proj}_Y(\oplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1})$, we get $S \subset Y \subset nS$ and $E \subset \widetilde{Y} \subset nE$ for all sufficiently large n.

Next, as in [We], we observe that by the Northcott-Rees theory of reductions of ideals (see Section 1.5), there exists $m \geq 1$ for which there exists a minimal reduction ideal sheaf $\mathcal{J}$ of $\mathcal{I}^m$; here one may take $m = 1$ if the ground field k is infinite (see for example [Ma], Theorem 14.14). Replacing $\mathcal{I}$ by $\mathcal{I}^m$ if needed, which doesn't change the blow-up, we may assume $m = 1$. Then $\mathcal{J}$ will have the following properties:

(a) $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{J}\mathcal{I}^n = \mathcal{I}^{n+1}$ for all sufficiently large n (this is a direct consequence of the definition of a reduction ideal given in Chapter 1)

(b) the stalks of $\mathcal{J}$ at points of S are generated by 2 elements (minimality of $\mathcal{J}$); since X is 2-dimensional and Cohen-Macaulay, $\mathcal{J}$ is a local complete intersection ideal sheaf in $\mathcal{O}_X$.

From (a), the Rees algebra of $\mathcal{I}$ is, locally on X, a finite module over that of $\mathcal{J}$; further, the subscheme of $\widetilde{X} = \text{Proj}\,(\oplus\mathcal{I}^n)$ defined by the ideal sheaf generated by $\mathcal{J}$ (in degree 1) is empty. Hence, if $\pi' : X' \to X$ denotes the blow up of $\mathcal{J}$ on X, then the graded inclusion $\oplus\mathcal{J}^n \hookrightarrow \oplus\mathcal{I}^n$ induces a finite morphism $f : \widetilde{X} \to X'$ of X-schemes, *i.e.*, the map $\pi : \widetilde{X} \longrightarrow X$ factories as



Here f is the normalization morphism, *i.e.*, $\widetilde{X}$ is the normalization of X'. Note that since $\mathcal{J}$ is a reduction ideal sheaf of $\mathcal{I}$, one has $\sqrt{\mathcal{J}} = \mathcal{I}_S$, the ideal sheaf of S in $\mathcal{O}_X$, and so we have inclusions

$$\mathcal{I}^n \subset \mathcal{J} \subset \mathcal{I}$$

for all large n. Hence if Y_1 denotes the local complete intersection closed subscheme of X defined by the sheaf of ideals $\mathcal{J}$ then the support of Y_1 is S.

Let

$$Y' = Y \times_X X', \qquad Y_1' = Y_1 \times_X X', \qquad \widetilde{Y} = Y \times_X \widetilde{X},$$

$$\widetilde{Y}_1 = Y_1 \times_X \widetilde{X}, \qquad S' = (S \times_X X')_{\mathrm{red}}.$$

Also let $Z \subset X'$ be the conducting subscheme for the normalization morphism f, *i.e.*, Z is the subscheme of X' defined by the $\mathcal{O}_{X'}$-annihilator of the coherent sheaf $f_*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_{X'}$ in $\mathcal{O}_{X'}$. Let $\widetilde{Z} = Z \times_{X'} \widetilde{X}$. The support of Z is contained in $f^{-1}(S)$, and hence $\widetilde{Z}_{\mathrm{red}} \subset E$. We have $\widetilde{Z}_{\mathrm{red}} \subset \widetilde{Z} \subset n\widetilde{Z}_{\mathrm{red}}$ for all sufficiently large n. This implies $\widetilde{Z} \subset n\widetilde{Z}_{\mathrm{red}} \subset nE$ which in turn implies that for every given $m > 0$, we have $m\widetilde{Z} \subset nE$ for all sufficiently large n.

Hence for a given $m > 0$ we have, for all sufficiently large n, the following commutative diagram with surjective arrows.

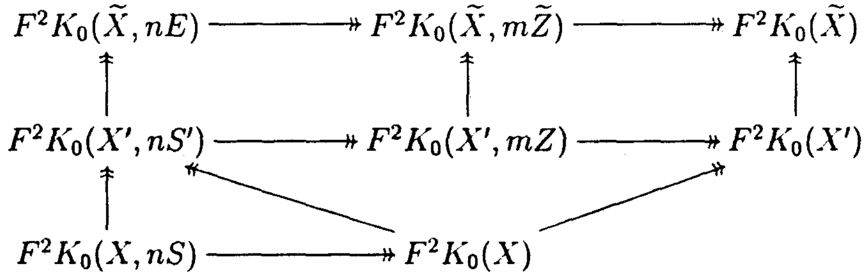


Diagram A

The surjectivity of all the maps in Diagram A follows from lemma 1.2.1. Furthermore the map $F^2K_0(X, nS) \longrightarrow F^2K_0(X)$ is an isomorphism by lemma 1.3.3.

Now by corollary 1.4.3, the map $F^2K_0(X) \longrightarrow F^2K_0(X')$ is also *injective*, and hence in fact an isomorphism. This, combined with the surjectivity of all arrows in Diagram A, gives the following diagram, all of whose arrows are omorphisms.

$$\begin{array}{ccccc}
F^2K_0(X', nS) & \longrightarrow & F^2K_0(X', mZ) & \longrightarrow & F^2K_0(X') \\
\uparrow & \nwarrow & & \nearrow & \\
F^2K_0(X, nS) & \longrightarrow & F^2K_0(X) & &
\end{array}$$

Diagram B

Next, we study the relation between $F^2K_0(X', mZ)$ and $F^2K_0(\tilde{X}, m\tilde{Z})$ for $m > 0$. By [PW], Corollary A.6, one has for each $m > 0$ a functorial exact sequence

$$H^1(mZ, \mathcal{I}_{mZ}/\mathcal{I}_{mZ}^2 \otimes \Omega_{m\tilde{Z}/mZ}) \longrightarrow K_0(X', mZ) \longrightarrow K_0(\tilde{X}, \tilde{Z}) \qquad (2.1)$$

(as remarked earlier, this follows from (1.1) combined with [GW]). Comparing the exact sequences for $m = 1$ and 2, we have the following commutative diagram

$$\begin{array}{ccccc}
H^1(2Z, \mathcal{I}_{2Z}/\mathcal{I}_{2Z}^2 \otimes \Omega_{2\tilde{Z}/2Z}) & \longrightarrow & K_0(X', 2Z) & \longrightarrow & K_0(\tilde{X}, 2\tilde{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(Z, \mathcal{I}_Z/\mathcal{I}_Z^2 \otimes \Omega_{\tilde{Z}/Z}) & \longrightarrow & K_0(X', Z) & \longrightarrow & K_0(\tilde{X}, \tilde{Z})
\end{array}$$

But

$$\mathcal{I}_{2Z}/\mathcal{I}_{2Z}^2 \otimes \Omega_{2\tilde{Z}/2Z} \longrightarrow \mathcal{I}_Z/\mathcal{I}_Z^2 \otimes \Omega_{\tilde{Z}/Z}$$

is the zero map, and hence it is zero on cohomology (compare [BPW], Theorem 3.3). Note that the isomorphisms in Diagram B for each m imply that $F^2K_0(X', 2Z) \longrightarrow F^2K_0(X', Z)$ is also an isomorphism.

Now let $A_m = \ker(F^2K_0(X', mZ) \longrightarrow F^2K_0(\tilde{X}, m\tilde{Z}))$. Then we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_2 & \longrightarrow & F^2K_0(X', 2Z) & \longrightarrow & F^2K_0(\tilde{X}, 2\tilde{Z}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & F^2K_0(X', Z) & \longrightarrow & F^2K_0(\tilde{X}, \tilde{Z}) & \longrightarrow & 0
\end{array}$$

From the discussion above, we see that the left vertical map is zero, and the middle vertical map is an isomorphism. Hence a simple diagram chase shows that $F^2K_0(X', 2Z) \longrightarrow F^2K_0(\tilde{X}, 2\tilde{Z})$ is an isomorphism. This, combined with the isomorphism $F^2K_0(X', nS') \longrightarrow F^2K_0(X', 2Z)$ obtained in Diagram B, and a simple diagram chase in Diagram A, gives us the isomorphism $F^2K_0(X, nS) \longrightarrow F^2K_0(\tilde{X}, nE)$ for all sufficiently large n. This implies the desired isomorphisms

$$F^2K_0(\tilde{X}, nE) \longrightarrow F^2K_0(\tilde{X}, (n-1)E), \quad F^2K_0(X) \longrightarrow F^2K_0(\tilde{X}, nE)$$

for all sufficiently large n, as well as the formula

$$F^2K_0(X) \;\cong\; \varprojlim_n F^2K_0(\tilde{X}, nE).$$

This proves the theorem. □

2.2 Special cases in different characteristics

The above theorem says that the Chow group of 0-cycles on X is isomorphic to the relative Chow groups of the desingularization X relative to "sufficiently large" thickenings of the reduced exceptional divisor E on the desingularization $\tilde{X}$ of X. We discuss here the special cases of the theorem, if one considers ground field of different characteristics.

Positive characteristic. Let the ground field k be algebraically closed of characteristic $p > 0$. In this case, it will follow from proposition 4.2.9, proved later in the last chapter, that the natural maps

$$F^2K_0(X) \longrightarrow F^2K_0(\tilde{X}, nE)$$

$$F^2K_0(\tilde{X}, nE) \longrightarrow F^2K_0(\tilde{X}, E)$$

are isomorphisms for all $n > 0$, and hence

$$F^2K_0(X) \cong F^2K_0(\tilde{X}, E).$$

This is a stronger statement than that in Theorem 2.0.5.

Characteristic 0. *If the ground field k has characteristic 0, we consider* two different possibilities.

In the first case, when $k = \overline{\mathbb{Q}}$, the situation turns out to be same as in the positive characteristic case. That is, the maps

$$F^2K_0(X) \longrightarrow F^2K_0(\widetilde{X}, nE)$$

$$F^2K_0(\widetilde{X}, nE) \longrightarrow F^2K_0(\widetilde{X}, E)$$

are isomorphisms for all $n > 0$, and hence

$$F^2K_0(X) \cong F^2K_0(\widetilde{X}, E).$$

This is corollary 3.1.8 in the next chapter of this thesis.

In the second case, when k contains transcendental elements, we give examples to demonstrate that it may, in general, be necessary to take non-trivial thickenings of the reduced exceptional divisor. That is, if $\mathrm{char}(k) = 0$, and k is not algebraic over $\mathbb{Q}$, then the map

$$F^2K_0(X) \longrightarrow F^2K_0(\widetilde{X}, E)$$

is not an isomorphism in general.

To give examples, let $C \subset \mathbb{P}^r$ be a projectively normal curve over $\mathbb{C}$ such that $H^1(C, \mathcal{O}_C(1)) \neq 0$, e.g. a smooth curve of degree at least 4 in $\mathbb{P}^2$. Let Y be the affine cone over C, and let P be the isolated singular point of Y. Let $\pi : \widetilde{Y} \longrightarrow Y$ be the blow-up of Y along the point P, with the exceptional divisor E. Then we know that there is a morphism $\phi : \widetilde{Y} \longrightarrow C$, such that ϕ is an $\mathbb{A}^1$-bundle, with a section given by E.

Now we have a commutative diagram of K-theory long exact sequences of pairs

$$\begin{array}{ccccccc}
K_1(Y) & \xrightarrow{\alpha} & K_1(P) & \longrightarrow & K_0(Y,P) & \longrightarrow & K_0(Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(\widetilde{Y}) & \xrightarrow{\alpha} & K_1(E) & \longrightarrow & K_0(\widetilde{Y},E) & \longrightarrow & K_0(\widetilde{Y})
\end{array}$$

But since ϕ is an $\mathbb{A}^1$-bundle, we have isomorphisms

$$\begin{array}{ccc} K_1(\widetilde{Y}) & \xrightarrow{\tilde{\alpha}} & K_1(E) \\ & \nwarrow^{\cong} & \uparrow{\scriptstyle\cong} \\ & & K_1(C), \end{array}$$

which implies that $\tilde{\alpha}$ is an isomorphism. Now we see from the above commutative diagram of exact sequences that the map

$$F^2K_0(\widetilde{Y}, E) \longrightarrow F^2K_0(\widetilde{Y})$$

is an isomorphism. But $\widetilde{Y}$ is a non complete birationally ruled surface and hence $F^2K_0(\widetilde{Y}, E) = F^2K_0(\widetilde{Y}) = 0$. However, it is follows from Corollary 3.3.5 (see also [S5]) that $F^2K_0(Y)$ is not trivial.

Chapter 3

Analogues of Bloch-Beilinson and Bloch Conjectures

Our aim in this chapter will be to prove the following two of the main results of this thesis. The first result is motivated by a conjecture of Bloch and Beilinson, that the Chow group of 0-cycles of a smooth projective surface over the field $\overline{\mathbb{Q}}$ of algebraic numbers is always "finite dimensional". This is in contrast to Mumford's infinite dimensionality theorem for complex projective surfaces with positive geometric genus. Equivalent formulations of the conjecture are:

(a) for smooth projective surfaces over $\overline{\mathbb{Q}}$, the Albanese map is always injective on cycle classes of degree 0, or

(b) the Chow group of 0-cycles of a smooth *affine* surface over $\overline{\mathbb{Q}}$ is always 0. Our first main result of this chapter is

Theorem 3.0.1 *Let $R = \oplus_{n\geq 0} R_n$ be a 2-dimensional graded normal affine domain over the field $R_0 = \overline{\mathbb{Q}}$. Then $K_0(R) = \mathbb{Z}$, and in fact every projective module over R is free.*

We remark here that the above result is not true in general when $R_0 = \mathbb{C}$, by an analogue of the Mumford theorem; examples can be found in [S5]. Theorem 3.0.1 suggests that the Bloch-Beilinson conjecture, as formulated in (a) or (b) above, should be valid even for singular surfaces. Here the Chow group is to be taken in the sense of [LW]; the Albanese variety in (a) should be that defined in [ESV].

As an explicit example, if $A = \overline{\mathbb{Q}}[x, y, z]/(x^n + y^n + z^n)$, where $n \geq 4$, then $F^2K_0(A) = 0$ from Theorem 3.0.1 (in fact all projective A-modules are free), while $F^2K_0(A \otimes \mathbb{C})$ has uncountable rank, as shown in [S5] (and it is in fact infinite dimensional, in an appropriate sense). As far as we are aware, Theorem 3.0.1 yields the first known examples of this phenomenon for *normal* (but singular) surfaces. In contrast, as far as we are aware, there is no similar example known at present of this phenomenon for a *smooth* surface, though the Bloch-Beilinson conjecture predicts the abundance of such examples. For example, the ring $B = \overline{\mathbb{Q}}[x, y, z]/(x^n + y^n + z^n - 1)$ is expected to have the same property: conjecturally $F^2K_0(B) = 0$, while Mumford's theorem implies $F^2K_0(B \otimes \mathbb{C})$ has uncountable rank.

Before stating our next result of this chapter, we recall *Bloch's Conjecture: if X is a smooth projective surface over $\mathbb{C}$ with $p_g(X) = 0$ (or equivalently if $H^2(X, \mathcal{O}_X) = 0$), then the Albanese map*

$$\alpha_X : A_0(X) \to \mathrm{Alb}\,(X)$$

is an isomorphism (here $A_0(X) = F^2K_0(X)_{\deg 0}$). This conjecture is known in certain cases, e.g., for X which is not of general type. The characteristic 0 case of our result below is then a relative version of this conjecture, for normal projective surfaces. We now state the second main result of this chapter.

Theorem 3.0.2 *Let X be a connected normal projective surface over an algebraically closed field k. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of X, with reduced, normal crossing exceptional divisor E.*
(a) If char $k = 0$, then $CH^2(X) \cong CH^2(\widetilde{X})$ if $H^2(X, \mathcal{O}_X) \cong H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$; if k is uncountable, the converse also holds.
(b) If k is of characteristic $p > 0$, then $CH^2(X) \cong CH^2(\widetilde{X})$ if $\mathrm{Pic}^0(\widetilde{X}) \twoheadrightarrow \mathrm{Pic}^0(E)$; if k is uncountable, the converse also holds.

We remark that in (b) above, one can restate the condition for isomorphism of Chow groups in positive characteristics in the following equivalent ways, using ℓ-adic cohomology:

$$H^1_{\text{ét}}(\tilde{X}, \mathbb{Q}_\ell(1)) \to H^1_{\text{ét}}(E, \mathbb{Q}_\ell(1)), \text{ or}$$

$$H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))/NS(X) \otimes \mathbb{Q}_\ell \cong H^2_{\text{ét}}(\tilde{X}, \mathbb{Q}_\ell(1))/NS(\tilde{X}) \otimes \mathbb{Q}_\ell.$$

We will obtain some interesting corollaries of this theorem towards the end of this chapter.

To obtain the main results of this chapter, we refine techniques used earlier by Srinivas in several papers (see for example [S1], [S2], [S3]). Apart from the fundamental inverse limit formula given by Theorem 2.0.5, the main new ingredient in the characteristic 0 proofs (Theorems 3.0.1 and 3.0.2(a)) is an exact sequence

$$H^1(\tilde{X}, \mathcal{I}_E/\mathcal{I}_E^n) \otimes_k \Omega^1_{k/\mathbb{Q}} \to SK_1(nE) \to SK_1(E) \to 0 \qquad (3.1)$$

where $\tilde{X} \to X$ is a resolution of singularities with a reduced normal crossing exceptional divisor E. This is proved by a detailed analysis of the K_2-sheaves of the non-reduced Cartier divisors $nE \subset \tilde{X}$, as in the earlier works cited, and finally reducing the result to the Grauert-Riemenschneider vanishing theorem. This allows one to get the detailed results stated above.

Before going into the proofs of above results, we will prove some technical lemmas, which will be used crucially in the proofs of main results.

3.1 Analysis of $SK_1(nE)$ in characteristic 0

In this section, we obtain a general result, lemma 3.1.7, which is useful in computing $SK_1(nE)$, where E is a normal crossing divisor which is the reduced exceptional divisor of a resolution of singularities of a normal surface over an algebraically closed field of characteristic 0. More generally, we will analyze the structure of $\ker(SK_1(nE) \to SK_1(E))$ where E is any reduced normal crossing divisor on a smooth surface $\tilde{X}$ in characteristic 0 (see lemma 3.1.6); this will be done by first considering the case when E is a smooth curve, and then in general via a Mayer-Vietoris argument.

We first recall a theorem of Bloch [B] which we use below.

Theorem 3.1.1 *Let B be a local $\mathbb{Q}$-algebra, A an augmented B-algebra and assume the kernel J of the augmentation homomorphism satisfies $J^n = 0$. Write Ω^1_A and Ω^1_B for the groups of absolute Kähler differentials and let*

$$\Omega^1_{A,J} = \ker(\Omega^1_A \longrightarrow \Omega^1_B)$$

$$K_2(A, J) = \ker(K_2(A) \longrightarrow K_2(B))$$

Then there is a canonical isomorphism

$$\tau : \ \Omega^1_{A,J}/d(J) \ \cong \ K_2(A, J)$$

As an application of Bloch's result we obtain the following corollary. The reader is referred to [S3] for a proof.

Corollary 3.1.2 *Let X be a $\mathbb{Q}$-scheme, Y an infinitesimal extension of X. Suppose that $\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is locally split (i.e., for each $x \in Y$, $\mathcal{O}_{x,Y}$ is an augmented $\mathcal{O}_{x,X}$ algebra). Define*

$$\Omega^1_{(Y,X)/\mathbb{Z}} = \ker(\Omega^1_{Y/\mathbb{Z}} \rightarrow \Omega^1_{X/\mathbb{Z}}).$$

Let $\mathcal{I}$ be the ideal sheaf of X on Y. Then there is a natural isomorphism of sheaves

$$\frac{\Omega^1_{(Y,X)/\mathbb{Z}}}{d(\mathcal{I})} \cong \ker(\mathcal{K}_{2,Y} \rightarrow \mathcal{K}_{2,X}).$$

Let $\widetilde{X}$ be a smooth quasi-projective surface over k, algebraically closed of characteristic 0, and let V be a nonsingular closed irreducible subvariety of $\widetilde{X}$ of dimension at most one with ideal sheaf $\mathcal{I}$. We fix the following notations which will be used in the rest of this section.

For V as above, we define for $n > 0$,

$$\begin{aligned}
\mathcal{I} &= \text{the ideal sheaf of } V \\
\mathcal{I}^n &= \text{the ideal sheaf defining the subscheme } nV \subset \widetilde{X} \\
\mathcal{I}_n &= \begin{cases} \mathcal{I}^n, \text{ if } V \text{ is a curve,} \\ \text{or} \\ \text{the ideal sheaf locally defined by } (x^n, y^n), \text{ if } V \text{ is a closed point} \\ \text{locally defined by the maximal ideal } (x, y) \end{cases} \\
V_n &= \text{the subscheme of } \widetilde{X} \text{ defined by } \mathcal{I}_n.
\end{aligned}$$

Note that if V is a curve then V_n and nV coincide, but this is not the case if V is a closed point, and this distinction will be crucial in what follows.

With the above notations, we have for every $n > 0$, the following commutative diagram of exact sequences of Kähler differentials (the terms in the top row of the middle and right hand columns are defined by the exactness of the respective columns).

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}_n & \longrightarrow & \Omega^1_{(V_n,V)/\mathbb{Z}} & \longrightarrow & \Omega^1_{(V_n,V)/k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{V_n} & \longrightarrow & \Omega^1_{V_n/\mathbb{Z}} & \longrightarrow & \Omega^1_{V_n/k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{V} & \longrightarrow & \Omega^1_{V/\mathbb{Z}} & \longrightarrow & \Omega^1_{V/k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Diagram C

Let

$$\mathcal{K}_{2,(V_n,V)} = \operatorname{Ker}(\mathcal{K}_{2,V_n} \longrightarrow \mathcal{K}_{2,V}).$$

Then by corollary 3.1.2, there is an isomorphism of sheaves

$$\frac{\Omega^1_{(V_n,V)/\mathbb{Z}}}{d(\mathcal{I}/\mathcal{I}_n)} \longrightarrow \mathcal{K}_{2,(V_n,V)}. \tag{3.2}$$

Note that by lemma 1.2.1, there is an exact sequence

$$H^1(V, \mathcal{K}_{2,(V_{n+1},V)}) \to SK_1(V_{n+1}) \to SK_1(V) \to 0. \tag{3.3}$$

Our next step is to unravel the isomorphism (3.2).

Lemma 3.1.3 *Let V be a nonsingular irreducible closed subvariety of $\tilde{X}$ of dimension ≤ 1. Then, with the above notation, there is a short exact sequence*

$$0 \longrightarrow \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}_{n+1} \longrightarrow \mathcal{K}_{2,(V_{(n+1)},V)} \longrightarrow \frac{\Omega^1_{(V_{(n+1)},V)/k}}{d(\mathcal{I}/\mathcal{I}_{n+1})} \longrightarrow 0$$

Proof. We claim that, with the above notation, the natural map

$$\frac{\mathcal{I}}{\mathcal{I}_{n+1}} \xrightarrow{d} \Omega^1_{(V_{(n+1)},V)/k} \tag{3.4}$$

is injective, where we recall that

$$\Omega^1_{(V_{n+1},V)/k} = \ker(\Omega^1_{V_{n+1}/k} \longrightarrow \Omega^1_{V/k})$$

Proof of the claim. We can check this locally.

Case 1. V is a nonsingular irreducible curve.
Let $P \in V$ and let $\mathfrak{m}_P = (x, y)$ be the maximal ideal of the local ring $\mathcal{O}_{P,\tilde{X}}$ with x the defining ideal of V. Let

$$S = \mathcal{O}_{P,\tilde{X}}/(x^{n+1}) \text{ and } R = \mathcal{O}_{P,V} = \mathcal{O}_{P,\tilde{X}}/(x)$$

Since R is a regular local ring essentially of finite type over k and hence smooth over K as $\text{char}(k) = 0$, this implies that $S \cong R[x]/(x^{n+1})$ and R is a discrete valuation ring with uniformising parameter y. Thus we have

$$\Omega^1_{S/k} = \Omega^1_{R/k} \otimes_R S \ \oplus \ \frac{S}{x^n S} dx$$

$$\begin{aligned} \Omega^1_{(S,R)/k} &= \text{Ker}(\Omega^1_{S/k} \longrightarrow \Omega^1_{R/k}) \\ &\cong \Omega^1_{R/k} \otimes_R xS \ \oplus \ \tfrac{S}{x^n S} dx \end{aligned}$$

Any $a \in (x) \subset S$ can be written as $f_n x^n + \cdots + f_1 x$, where $f_i \in R \ \forall i$ and so $f_i = u_i y^{n_i}$ $(1 \leq i \leq n)$ with $u_i \in R^*$. This implies

$$\begin{aligned} d(a) &= \sum_{i=1}^{n} (x^i df_i + i x^{i-1} f_i dx) \\ &= (\sum_{i=1}^{n} x^i df_i) \ + \ (\sum\nolimits_{i=1}^{n} i x^{i-1} f_i dx) \end{aligned}$$

Clearly $a \neq 0 \Rightarrow \sum_{i=1}^{n} ix^{i-1} f_i dx \neq 0$ using the expression for $\Omega^1_{(S,R)/k}$ and for f_i's.
Hence

$$\frac{(x)}{(x^{n+1})} \xrightarrow{d} \Omega^1_{(S,R)/k} \text{ is injective.}$$

Case 2. When $V = \{P\}$ is a closed point of $\tilde{X}$.
Put

$$\mathfrak{m}_P = (x, y)\ , \quad S = \mathcal{O}_{P,\tilde{X}}/(x^{n+1}, y^{n+1})\ , \quad R = \mathcal{O}_{P,\tilde{X}}/(x, y) \cong k.$$

We need to show

$$\frac{(x, y)}{(x^{n+1}, y^{n+1})} \xrightarrow{d} \Omega^1_{(S,R)/k} \quad \text{is injective.}$$

Identifying R with k, we can write $S = k[x, y]/(x^{n+1}, y^{n+1})$. This gives

$$\Omega^1_{S/k} = \frac{Sdx \oplus Sdy}{x^n Sdx + y^n Sdy} = \Omega^1_{(S,R)/k}$$

which implies that $\Omega^1_{S/k}$ is a finite dimensional k-vector space with a basis given by the the union of the following sets

$$\{x^i dx\}_{0 \leq i \leq n-1}\ , \quad \{y^j dy\}_{0 \leq j \leq n-1}$$

$$\{(i+1)x^i y^j dx\}_{0 \leq i \leq i \leq n-1,\ 1 \leq j \leq n}\ , \quad \{(j+1)x^i y^j dy\}_{1 \leq i \leq n,\ 0 \leq j \leq n-1}.$$

Applying a linear isomorphism to the span of the third and fourth sets above, we get another k-basis of $\Omega^1_{S/k}$ as

$$\{x^i dx\}_{0 < i < n-1}\ , \quad \{y^j dy\}_{0 \leq j \leq n-1}$$

$$\{ix^{i-1}y^j dx + jx^i y^{j-1} dy\}_{1 \leq i,j \leq n}\ , \quad \{ix^{i-1}y^j dx - jx^i y^{j-1} dy\}_{1 \leq i,j \leq n}.$$

Further from the definition of the map d ,we see that

$$\text{Image}(d) = \oplus_{1 \leq i,j \leq n} k\alpha_{ij}\ , \text{where } \ \alpha_{ij} = (ix^{i-1}y^j dx + jx^i y^{j-1} dy).$$

But

$$\dim_k\left(\frac{(x, y)}{(x^{n+1}, y^{n+1})}\right) = n^2 = \dim_k(\oplus_{1 \leq i,j \leq n} k(\alpha_{ij}))$$

This implies

$$\frac{(x,y)}{(x^{n+1},y^{n+1})} \xrightarrow{d} \oplus_{1\le i,j\le n} k(\alpha_{ij})$$

is an isomorphism and hence d is injective. This proves the claim.

Now the factorization

$$\frac{\mathcal{I}}{\mathcal{I}_{n+1}} \xrightarrow{d} \Omega^1_{(V_{(n+1)},V)/\mathbb{Z}} \longrightarrow \Omega^1_{(V_{(n+1)},V)/k}$$

implies that the map

$$\frac{\mathcal{I}}{\mathcal{I}_{n+1}} \xrightarrow{d} \Omega^1_{(V_{(n+1)},V)/\mathbb{Z}}$$

is also injective.

Using the isomorphism (3.2), the injectivity in (3.4) and Diagram C, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{I}/\mathcal{I}_{n+1} & = & \mathcal{I}/\mathcal{I}_{n+1} & & \\
 & & & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}_{n+1} & \longrightarrow & \Omega^1_{(V_{(n+1)},V)/\mathbb{Z}} & \longrightarrow & \Omega^1_{(V_{(n+1)},V)/k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}_{n+1} & \longrightarrow & \mathcal{K}_{2,(V_{(n+1)},V)} & \longrightarrow & \frac{\Omega^1_{(V_{(n+1)},V)/k}}{d(\mathcal{I}/\mathcal{I}_{n+1})} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the vertical sequences on the middle and right, as well as the two horizontal sequences, are exact, and the left hand vertical map is the identity. The bottom row is the desired exact sequence. This finishes the proof of the lemma. □

The next lemma will further clarify the structure of the sheaf $\mathcal{K}_{2,(Y_{(n+1)},X)}$. Let $\Omega^2_{Y_{(n+1)}/k} = \bigwedge^2(\Omega^1_{Y_{(n+1)}/k})$. Then one has a natural (exterior derivative) morphism of sheaves of abelian groups

$$\Omega^1_{(Y_{(n+1)},X)/k} \xrightarrow{d} \Omega^2_{Y_{(n+1)}/k}$$

Since $d^2 = 0$, we get an induced map

$$\frac{\Omega^1_{(Y_{(n+1)},X)/k}}{d(\mathcal{I}/\mathcal{I}_{n+1})} \xrightarrow{d} \Omega^2_{Y_{(n+1)}/k}. \tag{3.5}$$

Lemma 3.1.4 *The map d in 3.5 is an isomorphism of sheaves of abelian groups.*

Proof. It is enough to check the isomorphism of d locally. We prove this separately for curves and closed points on $\widetilde{X}$.

Case 1. Y is a nonsingular irreducible curve.

As before, putting $S = \mathcal{O}_{P,\widetilde{X}}/(x^{n+1})$ and $R = \mathcal{O}_{P,X}$ for any closed point $P \in Y$, we have $\Omega^1_{(S,R)/k} \cong \Omega^1_{R/k} \otimes_R xS \oplus \frac{S}{x^n S}dx$ and we have also seen that $\mathcal{I}/\mathcal{I}_{n+1} \hookrightarrow \Omega^1_{(S,R)/k}$ under the map d.

We claim that

$$\frac{\Omega^1_{(S,R)/k}}{d((x)/(x^{n+1}))} \cong \Omega^1_{R/k} \otimes_k xS \tag{3.6}$$

We have the surjection

$$\Omega^1_{R/k} \otimes_R xS \oplus \frac{S}{x^n S}dx \cong \Omega^1_{(S,R)/k} \to \frac{\Omega^1_{(S,R)/k}}{d((x)/(x^{n+1}))}.$$

Any element $a \in (x)/(x^{n+1})$ can be written as

$$a = \sum_{i=1}^{n} f_i x^i \Rightarrow da = \sum_{i=1}^{n} (df_i \otimes x^i) \oplus \sum_{i=1}^{n} (i f_i x^{i-1} dx).$$

clearly

$$a \neq 0 \Rightarrow \sum_{i=1}^{n} (i f_i x^{i-1} dx) \neq 0 \Rightarrow da \notin \Omega^1_{R/k} \otimes_R xS.$$

This implies

$$\Omega^1_{R/k} \otimes_R xS \longrightarrow \frac{\Omega^1_{(S,R)/k}}{d((x)/(x^{n+1}))} \quad \text{is injective.} \tag{3.7}$$

Next for any element $fx^i dx$, $(0 \leq i \leq n-1) \in S/(x^n S)dx$, with $f \in R$. We have

$$d(\frac{f}{i+1}x^{i+1}) = fx^i dx + \frac{x^{i+1}}{i+1}df$$

$$\Rightarrow \quad fx^i dx - \frac{df}{-(i+1)} \otimes x^{i+1} = d(\frac{fx^{i+1}}{i+1})$$

$$\Rightarrow \quad \text{image}(fx^i dx) = \text{image}(\frac{df}{-(i+1)})$$

under the surjection

$$\Omega^1_{R/k} \otimes_R xS \oplus \frac{S}{(x^n S)}dx \longrightarrow \frac{\Omega^1_{(S,R)/k}}{d((x)/(x^{n+1}))}$$

This implies that

$$\Omega^1_{R/k} \otimes_R xS \longrightarrow \frac{\Omega^1_{(S,R)/k}}{d((x)/(x^{n+1}))}$$

is surjective and hence by (3.7), it is an isomorphism which proves the claim. Now once we have established (3.6), to prove the lemma, it is enough to show that

$$\begin{aligned} \Omega^1_{R/k} \otimes_R xS \xrightarrow{d} \Omega^2_{S/k} \quad &\cong \quad \frac{\mathcal{O}_{P,\tilde{X}}}{x^n \mathcal{O}_{P,\tilde{X}}} dx \wedge dy \\ &\cong \quad \frac{R[x]}{(x^n)} dx \wedge dy \end{aligned}$$

$$d(\sum_{i=1}^{n} f_i dy \otimes x^i) \longmapsto \sum\nolimits_{i=1}^{n} i f_i x^{i-1} dx \wedge dy$$

is an isomorphism.

We define a map

$$\tilde{d} : \Omega^2_{S/k} \longrightarrow \Omega^1_{R/k} \otimes_R xS$$

by "integrating" the expression on the right with respect to x to define the inverse map to d. Every element in $\Omega^2_{S/k}$ can be written uniquely as $(\sum_{i=0}^{n-1} f_i x^i)dx \wedge dy$ with $f_i \in R$, and so it is enough to define $\tilde{d}$ for elements of the form $(fx^i)dx \wedge dy$ where $f \in R$ and $0 \leq i \leq n-1$. For such elements, define $\tilde{d}$ as

$$\tilde{d}((fx^i)dx \wedge dy) = \frac{f}{(i+1)}x^{i+1}dy = \frac{f}{(i+1)}dy \otimes x^{i+1}.$$

Note that for $i \geq n$, this is zero, and $\tilde{d}$ is a well defined homomorphism. Now from the definition of d and $\tilde{d}$, it is clear that they are inverse to each other. This proves the lemma when V is a curve.

Case 2. $V = \{P\}$ is a closed point of $\tilde{X}$. As before, let $\mathfrak{m}_P = (x, y)$ be the maximal ideal of $\mathcal{O}_{P,\tilde{X}}$ and let

$$S = \frac{\mathcal{O}_{P,\tilde{X}}}{(x^{n+1}, y^{n+1})} \text{ and } R = \frac{\mathcal{O}_{P,\tilde{X}}}{(x, y)} \cong k$$

By Theorem 3.1.1, this implies

$$\frac{\Omega^1_{(S,R)/k}}{d((x,y)/(x^{n+1}, y^{n+1}))} \cong \oplus_{1 \leq i,j \leq n} k\beta_{ij}$$

where $\beta_{ij} = ix^{i-1}y^j dx - jx^i y^{j-1}dy$. Also we have

$$\begin{aligned}\Omega^1_{S/k} \cong \Omega^1_{(S,R)/k} &\cong \frac{Sdx \oplus Sdy}{x^n Sdx + y^n Sdy} \\ &\cong \frac{S}{x^n S}dx \oplus \frac{S}{y^n S}dy\end{aligned}$$

and hence taking the second exterior power, we get

$$\Omega^2_{S/k} \cong \frac{S}{(x^n, y^n)S}dx \wedge dy = \frac{\mathcal{O}_{P,\tilde{X}}}{(x^n, y^n)}dx \wedge dy \tag{3.8}$$

We have the differential map

$$d : \frac{\Omega^1_{S/k}}{d((x,y)/(x^{n+1}, y^{n+1}))} \longrightarrow \Omega^2_{S/k}$$

and hence using the above isomorphism in (3.7), we get a map

$$d : \frac{\Omega^1_{S/k}}{d((x,y)/(x^{n+1}, y^{n+1}))} \longrightarrow \frac{\mathcal{O}_{P,\tilde{X}}}{(x^n, y^n)} dx \wedge dy$$

given by

$$\begin{aligned} d(ix^{i-1}y^j dx \; - \; jx^i y^{j-1}) &= (ijx^{i-1}y^{j-1})dx \wedge dy - \; (ijx^{i-1}y^{j-1})dy \wedge dx \\ &= ijx^{i-1}y^{j-1}(dx \wedge dy \; + \; dx \wedge dy \\ &= (2ijx^{i-1}y^{j-1})dx \wedge dy \end{aligned}$$

A basis for $\frac{\mathcal{O}_{P,\tilde{X}}}{(x^n,y^n)} dx \wedge dy$ as a k-vector space is given by the set $\{x^i y^j dx \wedge dy\}_{0 \leq i,j \leq n-1}$ and

$$d(\frac{1}{2(i+1)(j+1)}((i+1)x^i y^{j+1} dx \; - \; (j+1)x^{i+1}y^j dy)) \; = \; x^i y^j dx \wedge dy.$$

This shows that the elements of the basis of $\Omega^2_{S/k}$ are in the image of the map d, and since d is k-linear we see that d is surjective, and hence an isomorphism, on comparing dimensions of the two k-vector spaces. This proves the lemma in all cases. □

Next we prove

Lemma 3.1.5 *Let V be as above. Then there is a natural isomorphism*

$$\frac{\omega_{\tilde{X}}}{\mathcal{I}_n \omega_{\tilde{X}}} \longrightarrow \Omega^2_{V_{(n+1)}/k}.$$

Proof. We have a natural surjection

$$\Omega^1_{\tilde{X}/k} \twoheadrightarrow \Omega^1_{V_{(n+1)}/k}$$

and taking the second exterior power on both sides we get a surjection

$$\omega_{\tilde{X}} \twoheadrightarrow \Omega^2_{V_{(n+1)}/k}$$

To show that this induces an isomorphism as defined in the lemma, we can check locally. If V is a curve, then at a point P of V, we have stalks

$$\omega_{\tilde{X},P} \cong \mathcal{O}_{P,\tilde{X}} dx \wedge dy \quad \text{and} \quad \Omega^2_{V_{(n+1)}/k,P} \cong \frac{\mathcal{O}_{P,\tilde{X}}}{(x^n)} dx \wedge dy$$

This immediately gives a short exact sequence

$$0 \longrightarrow \mathcal{I}_n \omega_{\tilde{X}} \longrightarrow \omega_{\tilde{X}} \longrightarrow \Omega^2_{V_{(n+1)}/k} \longrightarrow 0$$

which proves the lemma in this case.
Similarly for $V = \{P\}$,

$$\frac{\omega_{\tilde{X},P}}{(x^n, y^n)\omega_{\tilde{X}}} \cong \frac{\mathcal{O}_{P,\tilde{X}}}{(x^n, y^n)} dx \wedge dy$$

and from (3.8)

$$\Omega^2_{P_{(n+1)}/k} \cong \frac{\mathcal{O}_{P,\tilde{X}}}{(x^n, y^n)} dx \wedge dy.$$

This proves the lemma in the second case. □

Now let $\tilde{X} \to X$ be a resolution of singularities of a normal, quasi-projective surface X over an algebraically closed field k of arbitrary characteristic, with a reduced, normal crossing divisor E. Let $\mathcal{I}_E$ be the ideal sheaf of E on $\tilde{X}$. Let $S = \{E_1, \ldots, E_r\}$ be the set of irreducible components of E. Set $S_{ij} = E_i \cap E_j$, $1 \le i < j \le r$. By [Mi], 6.4, we have an exact (Mayer-Vietoris) sequence of sheaves

$$\mathcal{K}_{2,(n+1)E} \longrightarrow \oplus_{1 \le i \le r} \mathcal{K}_{2,(n+1)E_i} \longrightarrow \oplus_{1 \le i < j \le r} \oplus_{P \in S_{ij}} \mathcal{K}_{2,P_{(n+1)}} \longrightarrow 0 \qquad (3.9)$$

The surjectivity of the last map is easily seen from the surjectivity of the restriction maps $\mathcal{K}_{2,E_i} \to \mathcal{K}_{2,S_{ij}}$, which follows from sublemma 1.3.2.

For $n \ge 0$, let

$$\overline{\mathcal{K}_{2,(n+1)E}} = \text{image} \left(\mathcal{K}_{2,(n+1)E} \longrightarrow \bigoplus_{1 \le i \le r} \mathcal{K}_{2,(n+1)E_i} \right). \qquad (3.10)$$

This gives a short exact sequence

$$0 \longrightarrow \overline{\mathcal{K}_{2,(n+1)E}} \longrightarrow \oplus_{1\le i\le r}\mathcal{K}_{2,(n+1)E_i} \longrightarrow \oplus_{1\le i<j\le r}\oplus_{P\in S_{ij}}\mathcal{K}_{2,P_{(n+1)}} \longrightarrow 0 \tag{3.11}$$

Let

$$\overline{\mathcal{K}_{2,(n+1)}} = \ker(\overline{\mathcal{K}_{2,(n+1)E}} \longrightarrow \overline{\mathcal{K}_{2,E}}). \tag{3.12}$$

We then have the following commutative diagram of short exact sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathcal{K}_{2,(n+1)}} & \longrightarrow & \oplus_{1\le i\le r}\mathcal{K}_{2,((n+1)E_i,E_i)} & \longrightarrow & \oplus_{1\le i<j\le r}\oplus_{P\in S_{ij}}\mathcal{K}_{2,(P_{(n+1)},P)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathcal{K}_{2,(n+1)E}} & \longrightarrow & \oplus_{1\le i\le r}\mathcal{K}_{2,(n+1)E_i} & \longrightarrow & \oplus_{1\le i<j\le r}\oplus_{P\in S_{ij}}\mathcal{K}_{2,P_{(n+1)}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathcal{K}_{2,E}} & \longrightarrow & \oplus_{1\le i\le r}\mathcal{K}_{2,E_i} & \longrightarrow & \oplus_{1\le i<j\le r}\oplus_{P\in S_{ij}}\mathcal{K}_{2,P} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here $\mathcal{K}_{2,((n+1)E_i,E_i)}$ and $\mathcal{K}_{2,(P_{(n+1)},P)}$ are defined to make the corresponding columns exact. In the above diagram, the middle and bottom rows are exact from the previous exact sequence (3.11), and from the snake lemma, we see that the top row is also exact. We write below the exact sequence of the top row separately as this is the part of the above diagram which will be used later in this section.

$$0 \longrightarrow \overline{\mathcal{K}_{2,(n+1)}} \longrightarrow \oplus_{1\le i\le r}\mathcal{K}_{2,((n+1)E_i,E_i)} \longrightarrow \oplus_{1\le i<j\le r}\oplus_{P\in S_{ij}}\mathcal{K}_{2,(P_{(n+1)},P)} \longrightarrow 0 \tag{3.13}$$

Remark. We make an important remark here that the above diagram, and (3.13), are valid for an algebraically closed ground field of arbitrary characteristic. This observation will be used to prove the positive characteristic version of Theorem 3.0.2. in the next chapter.

The following lemma will be our main tool to compute the cohomology groups of the sheaf $\overline{\mathcal{K}_{2,(n+1)}}$ in characteristic 0. We will use these cohomology groups to compare $SK_1(E)$ and $SK_1(nE)$ for any thickening nE of E.

Lemma 3.1.6 *For each $n \geq 1$ there is a short exact sequence*

$$0 \longrightarrow \Omega^1_{k/\mathbb{Z}} \otimes \mathcal{I}_E/\mathcal{I}_E^{n+1} \longrightarrow \overline{\mathcal{K}_{2,(n+1)}} \longrightarrow \omega_{\tilde{X}} \otimes \mathcal{O}_{nE} \longrightarrow 0$$

Proof. Note that for all n, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{nE} \longrightarrow \oplus_{1\leq i\leq r}\mathcal{O}_{nE_i} \longrightarrow \oplus_{1\leq i<j\leq r}\oplus_{P\in S_{ij}}\mathcal{O}_{P_n} \longrightarrow 0$$

This follows because E has smooth components with normal crossings and so locally at any closed point P of some S_{ij}, the sequence

$$0 \longrightarrow \frac{\mathcal{O}_{P,\tilde{X}}}{(xy)^n} \longrightarrow \frac{\mathcal{O}_{P,\tilde{X}}}{(x^n)} \oplus \frac{\mathcal{O}_{P,\tilde{X}}}{(y^n)} \longrightarrow \frac{\mathcal{O}_{P,\tilde{X}}}{(x^n, y^n)} \longrightarrow 0$$

is exact, where $\mathfrak{m}_P = (x, y)$ is the maximal ideal of the local ring $\mathcal{O}_{P,\tilde{X}}$. Tensoring the above exact sequence of coherent sheaves on $\tilde{X}$ with the canonical bundle $\omega_{\tilde{X}}$ of $\tilde{X}$, we get the following exact sequence.

$$0 \longrightarrow \omega_{\tilde{X}} \otimes \mathcal{O}_{nE} \longrightarrow \oplus_{1\leq i\leq r}\omega_{\tilde{X}} \otimes \mathcal{O}_{nE_i} \longrightarrow \oplus_{1\leq i<j\leq r} \oplus_{P\in S_{ij}}\omega_{\tilde{X}} \otimes \mathcal{O}_{P_n} \longrightarrow 0 \quad (3.14)$$

Combining lemmas 3.1.3, 3.1.4 and lemma 3.1.5, we get for any nonsingular irreducible closed subvariety $V \subset \tilde{X}$ of dimension ≤ 1, the following short exact sequence

$$0 \longrightarrow \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}_{n+1} \longrightarrow \mathcal{K}_{2,(V_{(n+1)},V)} \longrightarrow \omega_{\tilde{X}} \otimes \mathcal{O}_{V_n} \longrightarrow 0 \qquad (3.15)$$

Letting $\mathcal{I}_{E_i}$ = (ideal sheaf of E_i), $(1 \leq i \leq r)$, and $\mathcal{I}_P$ = (ideal sheaf of the closed point P on $\tilde{X}$), and putting $V = E_i$, $(1 \leq i \leq r)$, and $V = P$, In the exact sequence (3.15), we get the following commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}_E/\mathcal{I}_E^{n+1} & \longrightarrow & \overline{\mathcal{K}_{2,n+1}} & \longrightarrow & \omega_{\widetilde{X}} \otimes \mathcal{O}_{nE} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \oplus_i \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}_{E_i}/\mathcal{I}_{E_i}^{n+1} & \longrightarrow & \oplus_i \mathcal{K}_{2,((n+1)E_i,E_i)} & \longrightarrow & \oplus_i \omega_{\widetilde{X}} \otimes \mathcal{O}_{nE_i} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \oplus_{i<j} \oplus_{P \in S_{ij}} \Omega^1_{k/\mathbb{Z}} \otimes_k \frac{\mathcal{I}_{E_i,P}+\mathcal{I}_{E_j,P}}{\mathcal{I}^{n+1}_{E_i,P}+\mathcal{I}^{n+1}_{E_j,P}} & \longrightarrow & \oplus_{i<j} \oplus_{P \in S_{ij}} \mathcal{K}_{2,(P_{(n+1)},P)} & \longrightarrow & \oplus_{i<j} \oplus_{P \in S_{ij}} \omega_{\widetilde{X}} \otimes \mathcal{O}_{P_n} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

In the above diagram, the middle and bottom rows are exact from the exact sequence (3.15). The exactness of the middle and right hand columns follow from exact sequences (3.13)and (3.14) respectively, and it is easily seen that the sequence in the left hand column is exact. Now by a diagram chase we see that the maps in the top row are defined, and the top row is exact. This proves the lemma. □

Finally, we specialize further to the situation where E is an exceptional divisor and characteristic of the ground field is 0, which leads to the exact sequence (3.1) stated in the beginning of this chapter.

Lemma 3.1.7 *Let $\widetilde{X}$ be a connected smooth quasi-projective surface over an algebraically closed field k of characteristic 0. Let $E \subset \widetilde{X}$ be a normal crossing divisor which is contained in the reduced exceptional divisor of a proper birational morphism $\pi : \widetilde{X} \to X$ to a normal surface X. Then for any $n \geq 2$, there is an exact sequence*

$$H^1(nE, \mathcal{I}_E/\mathcal{I}_E^n) \otimes_k \Omega^1_{k/\mathbb{Q}} \to SK_1(nE) \to SK_1(E) \to 0.$$

Proof. First from (3.9), we see that $\mathrm{Ker}(\mathcal{K}_{2,nE} \longrightarrow \overline{\mathcal{K}}_{2,nE})$ is supported on $\bigcup_{i,j} S_{ij}$, so that $H^1(nE, \mathcal{K}_{2,nE}) \cong H^1(nE, \overline{\mathcal{K}}_{2,nE})$. Now from lemmas 1.3.1 and 3.1.6, we are reduced to proving that the group $H^1(nE, \omega_{\widetilde{X}} \otimes \mathcal{O}_{(n-1)E}) = 0$. From the right exactness of H^1 on 1-dimensional schemes, it suffices to prove this when E is the full exceptional divisor of $\pi : \widetilde{X} \to X$ (E may not have normal crossings); further, it suffices to prove that

$$\varprojlim_n H^1(nE, \omega_{\widetilde{X}} \otimes \mathcal{O}_{nE}) = 0.$$

From the Formal Function Theorem, this is equivalent to $R^1\pi_*\omega_{\widetilde{X}} = 0$, which holds by the Grauert-Riemenschneider Vanishing Theorem (see [EV], or [Li] for the surface case). This proves the lemma. □

Corollary 3.1.8 *Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities of a connected normal quasi-projective surface X over $\overline{\mathbb{Q}}$. Then the maps*

$$F^2K_0(X) \longrightarrow F^2K_0(\widetilde{X}, E)$$

and

$$F^2K_0(\widetilde{X}, nE) \longrightarrow F^2K_0(\widetilde{X}, E), \ \forall\ n > 1$$

are isomorphisms.

Proof. We immediately reduce to the case when X and $\widetilde{X}$ are projective. From Theorem 2.0.5 and lemma 1.2.1, we see that both the conclusions are equivalent and are independent of the resolution $\widetilde{X} \longrightarrow X$ chosen, since any two resolutions are dominated by a third. So we may assume that E is a normal crossing divisor. From proposition 1.4.4, we get the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
SK_1(\widetilde{X}) & \longrightarrow & SK_1(nE) & \longrightarrow & F^2K_0(\widetilde{X}, nE) & \longrightarrow & F^2K_0(\widetilde{X}) \\
\| & & \downarrow & & \downarrow & & \| \\
SK_1(\widetilde{X}) & \longrightarrow & SK_1(E) & \longrightarrow & F^2K_0(\widetilde{X}, E) & \longrightarrow & F^2K_0(\widetilde{X}).
\end{array}$$

Now the corollary follows from lemma 3.1.7 and a diagram chase, since $\Omega^1_{\overline{\mathbb{Q}}/\mathbb{Q}} = 0$. □

3.2 0-cycles on normal affine surfaces over $\overline{\mathbb{Q}}$

This section contains the proof of Theorem 3.0.1. But we begin with an analysis of a more general situation. Let

$$R = k \oplus R_1 \oplus R_2 \cdots$$

be a finitely generated 2-dimensional normal graded algebra over a field k (in particular, R is an integral domain). We recall a construction of Grothendieck, generalizing the blow-up of the vertex singularity of a cone (see [W] for an exposition).

Let $\mathfrak{m} \subset R$ be the irrelevant graded maximal ideal. Put $I_n = \oplus_{m \geq n} R_m$ (thus $I_1 = \mathfrak{m}$, but $I_n \neq I_1^n$ in general). Define

$$R^\natural = R \oplus I_1 \oplus I_2 \cdots \qquad \text{and } R' = R^\natural/\mathfrak{m}R^\natural = k \oplus I_1/\mathfrak{m}I_1 \oplus I_2/\mathfrak{m}I_2 \oplus \cdots .$$

Then we see that there is an isomorphism of graded k-algebras

$$R'_{\mathrm{red}} = k \oplus I_1/I_2 \oplus I_2/I_3 \oplus \cdots \cong R.$$

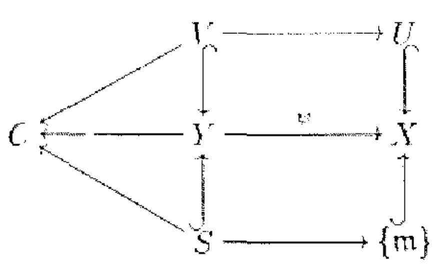
Set

$$X = \operatorname{Spec} R, \quad U = X - \{\mathfrak{m}\}, \quad Y = \operatorname{Proj} R^\natural,$$

$$C = \operatorname{Proj} R, \quad S = \operatorname{Proj} R'_{\mathrm{red}} \hookrightarrow Y, \quad V = Y - S.$$

Note that C is a normal projective curve over k since the ring R is normal of dimension 2.

The inclusion of graded rings $R \hookrightarrow R^\natural$ given by $R_i \hookrightarrow I_i$ induces a surjective morphism $\pi : Y \longrightarrow C$; also, the inclusion of R as the subring of degree zero elements of $R^\natural$ gives the restriction map $\psi : Y \longrightarrow X$. Thus there is a commutative diagram



Note that since $I_i\mathcal{O}_U = \mathcal{O}_U\ \forall\ i \geq 0$, the morphism ψ induces an isomorphism $Y - S = V \cong U$ and hence it is birational (though it is not exactly e blow-up morphism at the closed point $\mathfrak{m}$, in general), with S as the ceptional set.

Now one easily verifies the following properties (compare [W]).

. The induced morphism between S and C is an isomorphism, giving a section of π (since $R'_{\mathrm{red}} \cong R$).

2. π is an affine morphism, whose fiber over any $x \in C$ is isomorphic to a monomial curve $\mathrm{Spec}\,(k(x)[t^{a_1}, \ldots, t^{a_s}])$, such that the intersection $\pi^{-1}(x)\cap S$ is the origin (determined by the maximal ideal $(t^{a_1}, \ldots, t^{a_s})$).

3. π is Zariski locally trivial over a non-empty open subset of C. In fact, let $\{f_1, \ldots, f_r\}$ be homogeneous generators of R as a k-algebra with $d_i = \deg f_i$, and let $g = \prod_i f_i$; then for the open subset $D_+(g) = \mathrm{Spec}\,(R_g)_0 \subset C$ (notation as in [H]), one has that $\pi^{-1}(D_+(g)) \cong D_+(g) \times \mathrm{Spec}\,k[t^{d_1}, \ldots, t^{d_r}]$.

Let $\varphi : Z \longrightarrow Y$ be the normalization morphism. Then we have the following properties:

1. φ is bijective, and $\varphi^{-1}(V) \cong V$ (since V is smooth)

2. $\varphi^{-1}(S)_{\mathrm{red}} \to S \cong C$ is birational, hence an isomorphism (since C is smooth)

3. $(\pi \circ \varphi)^{-1}(D_+(g)) \cong D_+(g) \times_k \mathbb{A}^1_k$, such that the intersection with $\varphi^{-1}(S)_{\mathrm{red}}$ is $D_+(g) \times \{0\}$ where $0 \in \mathbb{A}^1_k$ is the origin

4. all fibers of $\pi \circ \varphi$ are monomial curves.

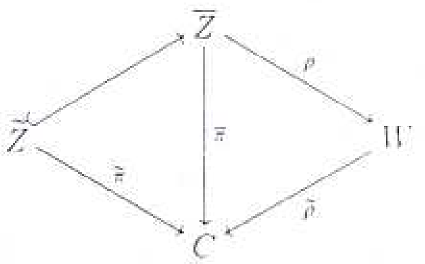
Let $\widetilde{p} : \widetilde{Z} \longrightarrow Z$ be a resolution of singularities of Z (and hence of Y and X). Put

$$p = \psi \circ \varphi \circ \widetilde{p} : \widetilde{Z} \to X, \quad \widetilde{\pi} = \pi \circ \varphi \circ \widetilde{p} : \widetilde{Z} \to C.$$

Then $\widetilde{\pi}^{-1}(D_+(g)) = D_+(g) \times_k \mathbb{A}^1_k$ is also generically an $\mathbb{A}^1$-bundle. Note that $Y_{\mathrm{sing}} \subset S$, and $Z_{\mathrm{sing}} \subset \varphi^{-1}(S)_{\mathrm{red}}$. Let $E_1 = \sum_{1\leq i\leq m} E^i$ be the reduced exceptional divisor for $\widetilde{p}$, with irreducible components E^i, and let $\widetilde{S}$ be the strict transform of $\varphi^{-1}(S)_{\mathrm{red}}$ under the map $\widetilde{p}$. Then the reduced exceptional divisor for the resolution of singularities map $p : \widetilde{Z} \to X$ is $E = E_1 + \widetilde{S}$.

Now since $\varphi^{-1}(S)_{\mathrm{red}} \to S$ is an isomorphism, so is $\widetilde{S} \to S$. Hence $\widetilde{S}$ is a section of $\widetilde{\pi} : \widetilde{Z} \to C$, and the image of E_1 is a finite set of points in C. Let $\overline{Z}$ be a smooth compactification of $\widetilde{Z}$, such that there is an induced morphism $\overline{\pi} : \overline{Z} \to C$ extending $\widetilde{\pi}$. Then $\overline{Z} \to C$ is a possibly non-minimal ruling, such that $\widetilde{S}$ is a section, and E_1 consists of curves contained in fibers.

From the classification of surfaces (see [H]) we may choose a relative minimal model W over C, which is a $\mathbb{P}^1$-bundle, such that there is a birational morphism $\rho : \overline{Z} \longrightarrow W$ making the following diagram commute.



Here ρ is a composition of point blow-ups, and thus all fibers of $\overline{\pi}$ are trees of smooth rational curves.

It is now easy to see that $E = \widetilde{S} + E_1$ has normal crossings, its dual graph has no loops and all irreducible components (except possibly $\widetilde{S}$) are smooth rational curves. This implies that

$$\begin{aligned} \mathrm{Pic}\,(E) &= \mathrm{Pic}\,(\widetilde{S}) \oplus (\oplus_i \mathrm{Pic}\,(E^i)) \\ &= \mathrm{Pic}\,(C) \oplus \mathbb{Z}^t \end{aligned}$$

where E_1 has t irreducible components.

Now since $E = E_1 + \widetilde{S}$ is exceptional for the resolution of singularities $\pi : \widetilde{X} \to X = \mathrm{Spec}\, R$, the intersection pairing matrix of E is negative definite. This implies that under the homomorphism on Néron-Severi groups $NS(\overline{Z}) \to NS(E) = \mathbb{Z}^{t+1}$, the images of the classes of components of E generate a subgroup of finite index. Combined with the isomorphisms $\mathrm{Pic}^0(C) \cong \mathrm{Pic}^0(\overline{Z}) \cong \mathrm{Pic}^0(\widetilde{S})$, we deduce easily that the natural homomorphism $\mathrm{Pic}\,\overline{Z} \to \mathrm{Pic}\,E$ has a finite cokernel. Since this factors through $\mathrm{Pic}(\widetilde{Z})$, we deduce:

$$\mathrm{Pic}(\widetilde{Z}) \otimes k^* \longrightarrow \mathrm{Pic}(E) \otimes k^* \quad \text{is surjective} \tag{3.16}$$

since k^* is a divisible group.

The product maps in K-theory induce a natural cup product map

$$H^1(X, \mathcal{K}_{1,X}) \otimes H^0(X, \mathcal{K}_{1,X}) \longrightarrow H^1(X, \mathcal{K}_{2,X}),$$

and we may identify $H^1(X, \mathcal{K}_{1,X})$ with $\mathrm{Pic}(X)$ for any quasi- projective variety X. We thus have the following diagram

$$\begin{array}{ccc}
\mathrm{Pic}(\widetilde{Z}) \otimes k^* & \longrightarrow & \mathrm{Pic}(E) \otimes k^* \\
\downarrow & & \downarrow \\
H^1(\widetilde{Z}, \mathcal{K}_{2,\widetilde{Z}}) & \longrightarrow & H^1(E, \mathcal{K}_{2,E})
\end{array}$$

We see that the top arrow is surjective by (3.16) above. Also, since E is reduced, the right vertical arrow is surjective (see [G,1.27] for example, or use the presentation (1.3)). This implies that the bottom arrow is also surjective. Now from the commutative diagram (by lemma 1.3.1(c))

$$\begin{array}{ccc}
SK_1(\widetilde{Z}) & \twoheadrightarrow & H^1(\widetilde{Z}, \mathcal{K}_{2,\widetilde{Z}}) \\
\downarrow & & \downarrow \\
SK_1(E) & \xrightarrow{\cong} & H^1(E, \mathcal{K}_{2,E})
\end{array}$$

we conclude that

$$SK_1(\widetilde{Z}) \twoheadrightarrow SK_1(E) \tag{3.17}$$

Now assume we are in the situation of Theorem 3.0.1, so that $k = \overline{\mathbb{Q}}$. Then $\Omega_{k/\mathbb{Q}} = 0$, so that from lemma 3.1.7 we have

$$SK_1(nE) \cong SK_1(E) \ \ \forall \ \ n.$$

Combining this with (3.17) above, we get

$$SK_1(\tilde{Z}) \twoheadrightarrow SK_1(nE) \cong SK_1(E) \ \ \forall \ \ n. \tag{3.18}$$

Now by lemma 1.3.3, it follows that $F^2K_0(\tilde{Z}, nE) \cong F^2K_0(\tilde{Z})$. But $F^2K_0(\tilde{Z}) = 0$ since $\tilde{Z}$ is a non-complete smooth birationally ruled surface over $\overline{\mathbb{Q}}$. Hence $F^2K_0(X) = 0$ from Theorem 2.0.5.

Since R is a normal graded ring, $\operatorname{Pic} X = 0$ as observed by Murthy. Hence, using the filtration on $K_0(R)$ as discussed in Section 1.1, we see that $K_0(R) = \mathbb{Z}$. This implies that if P is any finitely generated projective R-module, then P is stably free. Now the cancellation theorem of Murthy and Swan [MS] implies that P is free. □

Remarks. We remark here that if, instead of working over $k = \overline{\mathbb{Q}}$, we work over an algebraically closed field k of characteristic $p > 0$, then (3.17) is still valid, so that $F^2K_0(\tilde{Z}, E) = 0$. From Proposition 4.2.9 in the next chapter, it will follow that when k is algebraically closed of characteristic $p > 0$, then

$$F^2K_0(\tilde{Z}, nE) \cong F^2K_0(\tilde{Z}, E) \ \ \forall \ \ n \geq 1$$

and hence we conclude as above that $F^2K_0(X) = 0$, *i.e.*, $K_0(X) = \mathbb{Z}$, and projective R-modules are free, in this situation as well. This recovers the characteristic p result in [S5], from a new perspective.

We further remark here that, for any normal quasi-projective surface X over $\overline{\mathbb{Q}}$, and a resolution of singularities $\pi : \tilde{X} \longrightarrow X$, it will follow from the proof of Theorem 3.0.1 above and from Theorem 2.0.5 that $F^2K_0(X) \cong F^2K_0(\tilde{X})$, provided another case of the Bloch-Beilinson Conjecture holds: for any smooth projective curve C over $\overline{\mathbb{Q}}$, the natural map $SK_1(C) \longrightarrow \overline{\mathbb{Q}}^*$ is an isomorphism.

3.3 Relative Bloch Conjecture for normal surfaces

Let X be a connected normal projective surface over an algebraically closed field of characteristic 0, and let $\pi : \tilde{X} \longrightarrow X$ be a resolution of singularities of X as in Theorem 3.0.2. In this section, we will give the proof of part (a) of the theorem. The other part will be considered in the next chapter.

First note that the groups $F^2K_0(\tilde{X}) = CH^2(\tilde{X})$ and $H^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$ are birational invariants for the smooth projective surface $\tilde{X}$. Thus we can assume (by further blow-up of $\tilde{X}$ if necessary) that the reduced exceptional divisor E has smooth components with normal crossings. Let $\{E_1, \cdots, E_r\}$ be the irreducible components of E. As before let nE be the n-th infinitesimal thickening of E.

Lemma 3.3.1 *If the natural map $H^2(X, \mathcal{O}_X) \longrightarrow H^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$, is an isomorphism, then the map*

$$SK_1(nE) \longrightarrow SK_1(E) \quad \textit{is also an isomorphism} \; \forall \; n \geq 1.$$

Proof. By lemma 3.1.7, it suffices to prove

$$H^1(\tilde{X}, \mathcal{I}_E/\mathcal{I}_E^{n+1}) = 0 \; \forall \; n > 0.$$

By [FGA], Expose V, we know that for all n, $\mathrm{Pic}^0(nE)$ is a connected algebraic group with $\mathrm{Lie}\,\mathrm{Pic}^0(nE) \cong H^1(nE, \mathcal{O}_{nE})$.

By the Leray spectral sequence for π, we have an exact sequence

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow H^0(X, R^1\pi_*\mathcal{O}_{\tilde{X}}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow 0$$

Now since E is 1-dimensional, $\{H^1(nE, \mathcal{O}_{nE})\}_{n>0}$ is an inverse system of surjective maps of finite dimensional vector spaces and so it stabilizes. Hence by the Formal Function Theorem, one gets

$$H^0(X, R^1\pi_*\mathcal{O}_{\tilde{X}}) \cong \varprojlim_n H^1(nE, \mathcal{O}_{nE}) \cong H^1(nE, \mathcal{O}_{nE}) \text{ for all large } n$$

Hence we see that

$$H^2(X, \mathcal{O}_X) \cong H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \text{ implies}$$

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \twoheadrightarrow H^1(nE, \mathcal{O}_{nE}) \ \forall n$$

This implies

$$\mathrm{Lie\,Pic}^0(\tilde{X}) \twoheadrightarrow \mathrm{Lie\,Pic}^0(nE) \ \forall n\,.$$

We deduce that the natural map

$$\mathrm{Pic}^0(\tilde{X}) \longrightarrow \mathrm{Pic}^0(nE)$$

is surjective for all $n > 0$.
Thus we get the following commutative diagram with surjective arrows

$$\begin{array}{ccc} \mathrm{Pic}^0(\tilde{X}) & \xrightarrow{\ \alpha_n\ } & \mathrm{Pic}^0(nE) \\ & {\scriptstyle \alpha_1}\searrow \quad \swarrow{\scriptstyle \beta_n} & \\ & \mathrm{Pic}^0(E) & \end{array}$$

Now the map $\mathrm{Pic}^0(nE) \longrightarrow \mathrm{Pic}^0(E)$ is affine by [FGA] Expose V. This implies that $\ker\beta_n$ is affine. On the other hand the surjectivity of α_n implies that $\mathrm{Pic}^0(nE)$ is an abelian variety for all n, since $\mathrm{Pic}^0(\tilde{X})$ is an abelian variety (as $\tilde{X}$ is a smooth projective variety). Hence $\ker\beta_n$ must be a finite subgroup scheme of $\mathrm{Pic}^0(nE)$. Since the characteristic is 0, one has a sheaf exact sequence

$$0 \to \mathcal{I}_E^{n-1}/\mathcal{I}_E^n \overset{\exp}{\to} \mathcal{O}_{nE}^* \to \mathcal{O}_{(n-1)E}^* \to 0$$

(with $\exp(a) = 1 + a$ since $a^2 = 0$). From the resulting cohomology sequence

$$H^1(E, \mathcal{I}_E^{n-1}/\mathcal{I}_E^n) \to \mathrm{Pic}\, nE \to \mathrm{Pic}\,(n-1)E \to 0,$$

we see that $\ker(\mathrm{Pic}\,(nE) \to \mathrm{Pic}\,(n-1)E)$ is divisible for all $n > 1$, since the group on the left in the above exact sequence is a finite dimensional vector space over a characteristic 0 field and is hence divisible. But the short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(nE) \longrightarrow \mathrm{Pic}(nE) \longrightarrow NS(E) \longrightarrow 0$$

for all n implies that $\ker(\mathrm{Pic}^0(nE) \to \mathrm{Pic}^0(n-1)E) = \ker(\mathrm{Pic}(nE) \to \mathrm{Pic}(n-1)E)$, and hence also $\ker \beta_n$ is divisible. Hence $\ker \beta_n = 0$, and β_n is an isomorphism for all n.

This implies that

$$\mathrm{Lie}\,\mathrm{Pic}^0(nE) \cong \mathrm{Lie}\mathrm{Pic}^0(E),$$

that is,

$$H^1(nE,\ \mathcal{O}_{nE}) \cong H^1(E,\ \mathcal{O}_E) \text{ for } n \geq 1. \tag{3.19}$$

The sheaf exact sequence

$$0 \longrightarrow \mathcal{I}_E/\mathcal{I}_E^{n+1} \longrightarrow \mathcal{O}_{(n+1)E} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

gives an exact cohomology sequence

$$H^0((n+1)E,\ \mathcal{O}_{(n+1)E}) \longrightarrow H^0(E,\ \mathcal{O}_E) \longrightarrow H^1((n+1)E,\ \mathcal{I}_E/\mathcal{I}_E^{n+1}) \longrightarrow$$

$$H^1((n+1)E,\ \mathcal{O}_{(n+1)E}) \longrightarrow H^1(E,\ \mathcal{O}_E) \longrightarrow 0.$$

Now the surjectivity of $H^0((n+1)E,\ \mathcal{O}_{(n+1)E}) \longrightarrow H^0(E,\ \mathcal{O}_E)$ and (3.19) give

$$H^1((n+1)E,\ \mathcal{I}_E/\mathcal{I}_E^{n+1}) = 0 \ \forall\ n > 0.$$

□

Lemma 3.3.2 *Let $\tilde{X}$ and E be as in the previous lemma. Then the map*

$$SK_1(\tilde{X}) \longrightarrow SK_1(nE)$$

is surjective for all $n \geq 1$.

Proof. From lemma 3.3.1, it suffices to take $n = 1$. We have seen in the previous lemma that

$$\mathrm{Pic}^0(\widetilde{X}) \twoheadrightarrow \mathrm{Pic}^0(E).$$

If E has r irreducible components, the natural map

$$NS(\widetilde{X}) \to NS(E) = \mathbb{Z}^r,$$

given by taking the intersection numbers with the components of E, has finite cokernel, since the subgroup of $NS(\widetilde{X})$ spanned by the components of E has an image of finite index, by the negative definiteness of the intersection matrix of these components. Hence

$$\mathrm{coker}\,(\mathrm{Pic}\,(\widetilde{X}) \to \mathrm{Pic}\,(E))$$

is finite, and hence

$$\mathrm{Pic}\,(\widetilde{X}) \otimes_{\mathbb{Z}} k^* \twoheadrightarrow \mathrm{Pic}\,(E) \otimes_{\mathbb{Z}} k^*, \tag{3.20}$$

since k^* is divisible as k is algebraically closed. Now consider the following two commutative diagrams

$$\begin{array}{ccc}
\mathrm{Pic}\,(\widetilde{X}) \otimes_{\mathbb{Z}} k^* & \longrightarrow & \mathrm{Pic}\,(E) \otimes_{\mathbb{Z}} k^* \\
\downarrow & & \downarrow \\
H^1(\widetilde{X}, \mathcal{K}_{2,\widetilde{X}}) & \longrightarrow & H^1(E, \mathcal{K}_{2,E})
\end{array}$$

$$\begin{array}{ccc}
SK_1(\widetilde{X}) & \twoheadrightarrow & H^1(\widetilde{X}, \mathcal{K}_{2,\widetilde{X}}) \\
\downarrow & & \downarrow \\
SK_1(E) & \longrightarrow & H^1(E, \mathcal{K}_{2,E})
\end{array}$$

In the first diagram, $\mathrm{Pic}\,(E) \otimes k^* \twoheadrightarrow H^1(E, \mathcal{K}_{2,E})$ (for example, see [G], 1.27, or use the presentation (1.3), where $K_1(\mathcal{O}_P) = k^*$ for all smooth points P). Hence using (3.20), it follows that $H^1(\widetilde{X}, \mathcal{K}_{2,\widetilde{X}}) \twoheadrightarrow H^1(E, \mathcal{K}_{2,E})$. Now in the second diagram (obtained from lemma 1.3.1(c)), the bottom arrow is an isomorphism by lemma 1.3.1(a) and hence we have

$$SK_1(\widetilde{X}) \twoheadrightarrow SK_1(E).$$

$\square$

Now we complete the proof of Theorem 3.0.2 in characteristic 0. Suppose first that

$$H^2(X,\ \mathcal{O}_X)\ \cong\ H^2(\tilde{X},\ \mathcal{O}_{\tilde{X}}).$$

From lemma 1.3.3,

$$\ker(F^2K_0(\tilde{X},\ nE) \longrightarrow F^2K_0(\tilde{X})) \subset \operatorname{coker}(SK_1(\tilde{X}) \to SK_1(nE)).$$

By lemma 3.3.2 we conclude that

$$F^2K_0(\tilde{X},\ nE) \hookrightarrow F^2K_0(\tilde{X}).$$

and hence by lemma 1.2.1, is an isomorphism for all n. Now by Theorem 2.0.5, it follows that

$$CH^2(X) = F^2K_0(X)\ \cong\ F^2K_0(\tilde{X}) = CH^2(\tilde{X}).$$

Conversely, suppose k is uncountable, and

$$CH^2(X)\ \cong\ CH^2(\tilde{X}),$$

or equivalently

$$F^2K_0(X)\ \cong\ F^2K_0(\tilde{X}).$$

This implies that all arrows in the diagram

$$\begin{array}{ccc} & & F^2K_0(X) \\ & \swarrow & \downarrow \\ F^2K_0(\tilde{X}, nE) & \longrightarrow & F^2K_0(\tilde{X}) \end{array}$$

are isomorphisms. Now suppose

$$H^2(X,\mathcal{O}_X)\ \ncong\ H^2(\tilde{X},\mathcal{O}_{\tilde{X}}).$$

Then by the Leray spectral sequence, as in the proof of lemma 3.3.1, we see that $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \longrightarrow H^1(nE, \mathcal{O}_{nE})$ is not surjective for some $n > 0$. Choose the smallest such value of n. It is proved in [S2], Section 3, that for this n, the map $F^2K_0(\widetilde{X}, nE) \longrightarrow F^2K_0(\widetilde{X})$ has a non-trivial kernel. This contradicts the isomorphisms in the above diagram.

This completes the proof of the theorem 3.0.2 in characteristic 0. □

Assuming the Bloch Conjecture for smooth surfaces, Theorem 3.0.2 immediately gives

Corollary 3.3.3 *Let X be a connected normal projective surface over $\mathbb{C}$. Then the Chow group of 0-cycles on X is "finite dimensional" if and only if $H^2(X, \mathcal{O}_X)$ is zero.*

Proof. Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities of X. Then

$$H^2(X, \mathcal{O}_X) \to H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}).$$

Now the corollary follows from the characteristic 0 version of Theorem 3.0.2. □

Corollary 3.3.4 *Let X be a connected normal quasi-projective surface over an algebraically closed field k of characteristic 0 with only rational singularities, and let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities of X. Then* $CH^2(X) \cong CH^2(\widetilde{X})$.

Proof. We immediately reduce to the case when X is projective. That the surface X has only rational singularities means that the higher direct images of the structure sheaf of $\widetilde{X}$ vanish, *i.e.*, $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0 \ \forall \ i > 0$. Hence by the Leray spectral sequence, as in the proof of lemma 3.3.1, we see that $H^2(X, \mathcal{O}_X) \cong H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Now the result follows from Theorem 3.0.2 in characteristic 0. □

This result was known earlier in a special case of quotient singularities (see

[S4], Chapter 9).

Another immediate consequence of Theorem 3.0.2 is the following converse to the main result of [S5] on vector bundles on cones.

Corollary 3.3.5 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a projectively normal curve, and $C(X) \subset \mathbb{A}^{n+1}_{\mathbb{C}}$ the affine cone over X. Then $K_0(C(X)) = \mathbb{Z}$ if and only if $n = \deg(X) - \text{genus}\,(X)$, or equivalently, the curve is embedded by a "non-special linear system", (i.e., $H^1(X, \mathcal{O}_X(1)) = 0$).*

Remark. We remark here that without the projective normality of X, the same conclusion holds with $C(X)$ replaced by its normalization.

Chapter 4

Roitman's torsion theorem for normal varieties

This chapter will be devoted to the last main result of this thesis. Here we prove the analogue of Roitman's torsion theorem for 0-cycles on normal projective varieties.

Let X be a connected normal projective variety over an algebraically closed field k. Then we define the Albanese variety $\mathrm{Alb}(X)$ as given in [L]. If X is a surface, then $\mathrm{Alb}(X)$ coincides with the Albanese variety of its desingularizations. Let $f : X_{reg} \longrightarrow \mathrm{Alb}(X)$ be the Albanese mapping, obtained by fixing a base point on X_{reg}. One obtains a group homomorphism

$$\alpha_X : CH_0(X) \longrightarrow \mathrm{Alb}(X),$$

and its restriction to the subgroup $A_0(X)$ of degree 0 zero cycles is independent of the base point. Hence one obtains a well defined group homomorphism

$$\alpha_X : A_0(X) \longrightarrow \mathrm{Alb}(X).$$

It was known by the works of Roitman and Milne that this map is isomorphism on torsion subgroups, when X is smooth. For X possibly singular, this was known *in characteristic 0, and for torsion of order prime to p*, if characteristic is $p > 0$, by works of Collino and Levine. So the new result of this thesis will be the proof of torsion theorem for p-primary torsion cycles in characteristic p. However, we will also give a new proof for the theorem in the other cases as well. As a consequence, we will obtain an important result about the Chow group of 0-cycles on normal affine varieties. We will also give the proof of part (b) of Theorem 3.0.2 mentioned in Chapter 3.

Our main result of this chapter is

Theorem 4.0.6 *Let X be a connected normal projective variety over an algebraically closed field k of arbitrary characteristic. Then the Albanese map*

$$\alpha_X : A_0(X) \longrightarrow \mathrm{Alb(X)}$$

is an isomorphism on torsion subgroups.

We proceed with the proof of this theorem. We first remark that α_X is surjective on torsion subgroups. Let $Y \subset X$ be a smooth complete intersection curve missing the singular locus of X. Then one knows that the composite map $\mathrm{Pic}^0(Y) \longrightarrow A_0(X) \longrightarrow \mathrm{Alb}(X)$ is a surjective map of abelian varieties, and hence is surjective on torsion subgroups (see [L1]).

Now we prove injectivity of α_X on torsion subgroups. As in Bloch's proof of Roitman's theorem, this is done by first reducing to the case of surfaces, by induction on the dimension of X.

4.1 Reduction to the case of surfaces

In this section, we discuss injectivity of the Albanese map on torsion subgroups for higher dimensional normal varieties, assuming that this holds for surfaces and using induction on dimension. This reduction has been shown by S. Bloch in [BL] in case of smooth varieties. For normal varieties, this was proved by M. Levine (see [L1]), following the ideas of Bloch in the smooth case. We will give below an outline of the argument in [L1]. The proof of injectivity for surfaces will be given in the next section.

Let X be a normal projective variety over an algebraically closed field k of arbitrary characteristic with $\dim X \geq 3$. Let δ be a 0-cycle such that $[\delta] \neq 0$ in $A_0(X)$, while $n[\delta] = 0$ for some $n > 1$. We will show that $\alpha_X([\delta]) \neq 0$. Since $n[\delta] = 0$ in $A_0(X)$, this implies that there exists a finite collection of irreducible projective curves C_i contained in $U = X \setminus X_{\text{sing}}$ and rational functions $f_i \in k(C_i)^*$ such that $n\delta = \sum (f_i)_{C_i}$ as 0-cycles. Let $\pi : \widetilde{X} \longrightarrow X$ be the successive blow-up at smooth points such that if D_i is the strict transform of C_i on $\widetilde{X}$, then

(i) D_i's are all smooth,

(ii) D_i's are pairwise disjoint and

(iii) D_i's meet the reduced exceptional divisor of π transversely at smooth points.

Let δ' be a 0-cycle on $\cup D_i$ such that $\pi_*(\delta') = \delta$. Now f_i can be considered as a rational function on D_i and we get $\pi_*(\sum (f_i)_{D_i} - n\delta') = 0$. Put $\eta = n\delta' - \sum (f_i)_{D_i}$. Then we have $\pi_*(\eta) = 0$. Thus, as explained in [BL], we can find smooth complete rational curves L_j, with $\pi(L_j) =$ point., and rational functions g_j on L_j such that $\eta = \sum (g_j)_{L_j}$. This gives $n\delta' = \sum (f_i)_{D_i} + \sum (g_j)_{L_j}$.

Put $D = (\cup D_i) \cup (\cup L_j)$. One can choose L_j's in such a way that the curve D is reduced with smooth components and only ordinary double point singularities. In particular, the curve D has local embedding dimension two (see [BL]). Suppose that $\widetilde{X}$ is a closed subvariety of $\mathbb{P}^r$ with the embedding given by a line bundle $\mathcal{L}$. Let $\mathcal{I}$ be the ideal sheaf of D on $\widetilde{X}$, and let $p : Z \longrightarrow \widetilde{X}$ denote the blow-up along $\mathcal{I}$. Put $Y = p^{-1}(D)$. Then $\mathcal{I}_Y \otimes p^*\mathcal{L}^m$ is a very ample line bundle on Z for large m. By [S], we can find an element Z_1 in the linear system $|\mathcal{I}_Y \otimes p^*\mathcal{L}^m|$, which is irreducible and geometrically normal outside of Y. On the other hand, since D has embedding dimension two, one can choose Z_1 so that $X_1 = p(Z_1)$ is smooth along D and hence X_1 is irreducible and normal.

Now by ([L], Chap.8, Sec.2, Theorem 5), the natural map $\mathrm{Alb}(Z_1) \longrightarrow \mathrm{Alb}(Z)$ is an isomorphism of groups. Further, since p is birational, we have following commutative diagram with all arrows isomorphisms.

$$\begin{array}{ccc} \mathrm{Alb}(Z_1) & \longrightarrow & \mathrm{Alb}(Z) \\ \downarrow & & \downarrow \\ \mathrm{Alb}(X_1) & \longrightarrow & \mathrm{Alb}(\widetilde{X}) \end{array}$$

By induction and from the proof of the theorem for surfaces below, the map $\alpha_{X_1} : A_0(X_1) \longrightarrow \mathrm{Alb}(X_1)$ is isomorphism on torsion subgroups. This implies that $\alpha_{\widetilde{X}}([\delta']) \neq 0$, as δ' is a torsion cycle on X_1 by choice (since $D \subset X_1$). Now it follows from the commutative diagram

$$\begin{array}{ccc} A_0(\widetilde{X}) & \xrightarrow{\alpha_{\widetilde{X}}} & \mathrm{Alb}(\widetilde{X}) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow \\ A_0(X) & \xrightarrow{\alpha_X} & \mathrm{Alb}(X) \end{array}$$

that $\alpha_X([\delta]) = \alpha_X \circ \pi_*([\delta']) \neq 0$, since the right vertical arrow is an isomorphism. This completes the reduction to the surface case. □

4.2 The case of surfaces

In this section, we will be concerned with the proof of injectivity of α_X and hence the proof of Theorem 4.0.6 for normal projective surfaces. The main focus will be on the case of characteristic $p > 0$, which is the new result.

Since the Albanese variety of the normal projective surface X remains unchanged if one takes the resolution of singularities of X, it suffices to prove that if $\pi : \widetilde{X} \longrightarrow X$ is a resolution of singularities of X, then the induced composite map

$$CH^2(X)_{\text{torsion}} \xrightarrow{\pi^*} CH^2(\widetilde{X})_{\text{torsion}} \xrightarrow{\alpha_{\widetilde{X}}} \mathrm{Alb}(\widetilde{X})_{\text{torsion}}$$

is an inclusion.

By the results of Roitman [R] and Milne [M], we know that $\alpha_{\widetilde{X}}$ is isomorphism on torsion subgroups. Hence it suffices to prove that π^* is injective on torsion subgroups, *i.e.*, it is enough to show that $\pi^* : F^2K_0(X)_{\text{torsion}} \hookrightarrow F^2K_0(\widetilde{X})_{\text{torsion}}$.

Let E be the reduced exceptional curve on $\widetilde{X}$ as before and let nE be the n-th infinitesimal thickening of E. Then by Theorem 2.0.5, we see that it is enough to prove that $\theta_n : F^2K_0(\widetilde{X},\ nE)_{\text{torsion}} \longrightarrow F^2K_0(\widetilde{X})_{\text{torsion}}$ is injective for all $n > 0$. We may assume, choosing $\widetilde{X}$ suitably, that E has normal crossings, i.e. all the irreducible components of E are smooth and they intersect transversely.

Let $F_n = \ker(F^2K_0(\widetilde{X}, nE) \to F^2K_0(\widetilde{X}))$ for any $n \geq 1$. Our goal is thus to prove that F_n is torsion-free for each n.

We recall below a structure theorem, a corollary of which will be used later in this chapter.

Theorem 4.2.1 *Let R be a smooth local ring which is essentially of finite type over a perfect field k of characteristic $p > 0$, and let $n \geq 1$. Then*

$$\ker\left(K_2(R[t]/(t^{n+1})) \longrightarrow K_2(R[t]/(t^n))\right)$$

is isomorphic with one of the following (unless $n = 1$, $p = 2$):

$$\begin{array}{cl} \Omega^1_{R/\mathbb{Z}} & \text{if } n \neq 0,\ -1 \bmod p \\ \Omega^1_{R/\mathbb{Z}} \oplus R/R^{p^r} & \text{if } n = mp^r - 1,\ (m,p) = 1,\ r \geq 1,\ n \geq 2 \\ \Omega^1_{R/\mathbb{Z}}/D_{r,R} & \text{if } n = mp^r,\ (m,p) = 1,\ r \geq 1 \end{array}$$

Here $D_{r,R}$ is the subgroup of $\Omega^1_{R/\mathbb{Z}}$ generated by the forms $a^{p^j-1}da$ with $0 \leq j < r$.

If $n = 1$ and $p = 2$, then there is an exact sequence of $\mathbb{F}_2$-vector spaces

$$0 \longrightarrow R/R^2 \longrightarrow k_2(R[t]/(t^2),(t)) \longrightarrow \Omega^1_{R/\mathbb{Z}} \longrightarrow 0,$$

which splits, but not naturally.

The reader is referred to [KS] (see also [BL1]) for a proof of this theorem. Note that $\Omega^1_{R/\mathbb{Z}} = \Omega^1_{R/k}$ since k is perfect of char $p > 0$. This result has the following immediate corollary, which is what is used in this thesis.

Corollary 4.2.2 *Let D be a smooth curve on the smooth surface $\widetilde{X}$ over k, where k is algebraically closed of characteristic p. Then* $\ker(\mathcal{K}_{2,(n+1)D} \longrightarrow \mathcal{K}_{2,D})$ *is a sheaf of p^N-torsion abelian groups, for some $N > 0$ depending only on n and p.*

Fix an integer $n > 1$, and consider the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_n & \longrightarrow & F^2K_0(\widetilde{X},\ nE) & \xrightarrow{\theta_n} & F^2K_0(\widetilde{X}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{\theta}_{n,1}} & & \downarrow{\scriptstyle \theta_{n,1}} & & \| & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F^2K_0(\widetilde{X},\ E) & \xrightarrow{\theta_1} & F^2K_0(\widetilde{X}) & \longrightarrow & 0
\end{array}$$

Diagram D

Lemma 4.2.3 *If k has characteristic $p > 0$, then* $\ker(\widetilde{\theta}_{n,1} : F_n \to F_1)$ *is a torsion group of finite exponent p^N.*

Proof. By lemma 1.3.1 and lemma 1.3.3, it is enough to prove that the kernel $L_{n,1}$ of the map

$$\frac{H^1(nE,\ \mathcal{K}_{2,nE})}{\text{image}(H^1(\widetilde{X},\ \mathcal{K}_{2,\widetilde{X}}))} \longrightarrow \frac{H^1(E,\ \mathcal{K}_{2,E})}{\text{image}(H^1(\widetilde{X},\ \mathcal{K}_{2,\widetilde{X}}))}$$

is torsion of finite exponent p^N.

Let $\{E_1, \cdots, E_r\}$ be the set of irreducible components of E and let $\overline{\mathcal{K}_{2,nE}}$ and $\overline{\mathcal{K}_{2,n}}$ be the sheaves on nE defined in Chapter 3 (Section 3.1) (see (3.10), (3.12) and (3.13)). Since $H^1(nE, \mathcal{K}_{2,nE}) \cong H^1(nE, \overline{\mathcal{K}_{2,nE}})$ for all $n \geq 1$, the exact sequence

$$0 \longrightarrow \overline{\mathcal{K}_{2,n}} \longrightarrow \overline{\mathcal{K}_{2,nE}} \longrightarrow \overline{\mathcal{K}_{2,E}} \longrightarrow 0 \tag{4.1}$$

yields a diagram with exact rows

$$\begin{array}{ccccccccc}
 & & H^1(nE, \overline{\mathcal{K}_{2,n}}) & \longrightarrow & H^1(nE,\ \mathcal{K}_{2,nE}) & \longrightarrow & H^1(E,\ \mathcal{K}_{2,E}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{n,1} & \longrightarrow & \frac{H^1(nE,\ \mathcal{K}_{2,nE})}{\text{image}(H^1(\widetilde{X},\ \mathcal{K}_{2,\widetilde{X}}))} & \longrightarrow & \frac{H^1(E,\ \mathcal{K}_{2,E})}{\text{image}(H^1(\widetilde{X},\ \mathcal{K}_{2,\widetilde{X}}))} & \longrightarrow & 0
\end{array}$$

From the snake lemma, we deduce that

$$H^1(nE, \overline{\mathcal{K}_{2,n}}) \longrightarrow\!\!\!\!\rightarrow L_{n,1}.$$

The exact sequence (3.13) and corollary 4.2.2 imply that $\overline{\mathcal{K}_{2,n}}$ is itself a sheaf of p^N-torsion abelian groups, for some $N \geq 0$, and hence so is any cohomology group of this sheaf. This proves the lemma. □

We recall the following result, which summarizes in convenient form some well-known properties about the sheaf $\mathcal{K}_2$, and K-cohomology of smooth curves and surfaces.

Lemma 4.2.4 *(a) Let D be an irreducible non-singular curve over an algebraically closed field k. Then we have the following.*

(i) $\mathcal{K}_{2,D}$ is a divisible sheaf of abelian groups, and if characteristic $k = p > 0$, it has no p-torsion.

(ii) If k has characteristic 0, or if $\operatorname{char} k = p \nmid m$, its m-torsion subsheaf is isomorphic to the image of the symbol map $\mathcal{O}_D^* \otimes \mu_m \to \mathcal{K}_{2,D}$, where μ_m is the group of m-th roots of unity in k.

(iii) If D is projective, then the natural maps

$$\operatorname{Pic}(D) \otimes_{\mathbb{Z}} k^* \to H^1(D, \mathcal{K}_{2,D}) \to K_1(k) = k^*$$

both induce isomorphisms on torsion subgroups.

(b) Let Y be a non-singular projective surface over an algebraically closed field. Then

$$H^1(Y, \mathcal{K}_{2,Y}) = (\textit{maximal divisible subgroup}) \oplus (\textit{finite group})$$

Proof. The proof of (a)(i) and (ii) is by a standard argument, using
(1) Suslin's description of the torsion subgroup of K_2 of any field (see [S4], Theorem 8.24),
(2) the divisibility of K_2 of the function field of a curve over an algebraically

closed field (or more generally of any C_1 field K, in the sense of Lang, for which the norm map $N_{L/K} : L^* \to K^*$ is surjective for any finite algebraic extension field L; see [S4], Lemma 8.4), and
(3) the Gersten resolution for $\mathcal{K}_2$ and the Bloch-Ogus resolution for the étale cohomology sheaves (see [BL], Chapter 4 and [S4], Section (8.4)).
From the resulting exact sheaf sequence

$$0 \to \mathcal{O}_D^* \otimes_{\mathbb{Z}} \mu_m \to \mathcal{K}_{2,D} \xrightarrow{m} \mathcal{K}_{2,D} \to 0$$

for any m with $1/m \in k$, the induced map

$$(\operatorname{Pic} D) \otimes_{\mathbb{Z}} \mu_m \xrightarrow{\cong} H^1(D, \mathcal{O}_D^* \otimes_{\mathbb{Z}} \mu_m) \to {}_mH^1(D, \mathcal{K}_{2,D})$$

is a surjection. Now $(\operatorname{Pic} D) \otimes_{\mathbb{Z}} \mu_m \cong \mu_m$ via the degree homomorphism $\operatorname{Pic} D \to \mathbb{Z}$, since $\operatorname{Pic}^0(D)$ is divisible. On the other hand, the composite $(\operatorname{Pic} D) \otimes_{\mathbb{Z}} k^* \to H^1(D, \mathcal{K}_{2,D}) \to K_1(k) = k^*$ is the map $(\operatorname{Pic} D) \otimes_{\mathbb{Z}} k^* \to k^*$ induced by the degree homomorphism $\operatorname{Pic} D \to \mathbb{Z}$; hence this composition is an isomorphism on torsion subgroups. This proves (a)(iii).

The proof of (b) is by using the exact sequence (see [S4], Section (8.4))

$$0 \to H^1(Y, \mathcal{K}_{2,Y}) \otimes \mathbb{Z}/m\mathbb{Z} \to H^1(Y, \mathcal{K}_{2,Y} \otimes \mathbb{Z}/m\mathbb{Z}) \to {}_mCH^2(Y) \to 0$$

valid for any $m \geq 1$, combined with
(1) the Roitman theorem on torsion 0-cycles for Y
(2) the Mercurjev-Suslin and Bloch-Ogus theorems, if $1/m \in k$, and
(3) analogues involving logarithmic de Rham-Witt sheaves of 2-forms, if $m = p^s$, where k has characteristic $p > 0$.
First consider the case when $1/m \in k$. Then (see [S4], Chapter 8) one has isomorphisms

$$H^1(Y, \mathcal{K}_{2,Y} \otimes \mathbb{Z}/m\mathbb{Z}) \cong H^3_{\text{ét}}(Y, \mu_m^{\otimes 2}) \cong \operatorname{Hom}({}_m\operatorname{Pic} Y, \mu_m)$$

while the Roitman theorem gives

$${}_mCH^2(Y) \cong {}_m\operatorname{Alb} Y \cong \operatorname{Hom}({}_m\operatorname{Pic}^0(Y), \mu_m).$$

Since $NS(Y) = \text{Pic}(Y)/\text{Pic}^0(Y)$ is a finitely generated abelian group, we deduce that $H^1(Y, \mathcal{K}_{2,D}) \otimes \mathbb{Z}/m\mathbb{Z} \cong {}_mNS(Y)$, which implies (b) in characteristic 0. Milne, in his proof in [M] of the Roitman theorem for p-primary torsion 0-cycles in characteristic $p > 0$, gives an analogous description of $H^1(Y, \mathcal{K}_{2,Y}) \otimes \mathbb{Z}/p^\nu\mathbb{Z}$, and deduces that in arbitrary characteristic, $H^1(Y, \mathcal{K}_{2,Y})$ modulo its divisible subgroup is a finite abelian group. □

Next, returning to our situation, we prove:

Lemma 4.2.5 *Let $\overline{\mathcal{K}_{2,nE}}$ be the sheaf as defined by the exact (Mayer-Vietoris) sequence*

$$0 \to \overline{\mathcal{K}_{2,nE}} \to \bigoplus_{1\le i\le r} \mathcal{K}_{2,nE_i} \to \bigoplus_{1\le i<j\le r} \bigoplus_{P\in S_{ij}} \mathcal{K}_{2,P_n} \to 0$$

where E_i, $1 \le i \le r$ are the irreducible components of E. Then

(a) ${}_m\overline{\mathcal{K}_{2,E}} \cong \oplus_{1\le i\le r}\mathcal{O}^*_{E_i} \otimes_{\mathbb{Z}} \mu_m$ *if* $p \nmid m$

(b) if k has characteristic $p > 0$, then ${}_p\overline{\mathcal{K}_{2,E}} = 0$

(c) $\overline{\mathcal{K}_{2,E}}$ *is a divisible sheaf of abelian groups*

(d) if k has characteristic 0, then $\overline{\mathcal{K}_{2,nE}}$ is divisible for any $n \ge 1$, and the restriction map

$$\overline{\mathcal{K}_{2,nE}} \to \overline{\mathcal{K}_{2,E}}$$

induces an isomorphism on torsion subsheaves.

Proof. First consider the case $n = 1$, *i.e.*, of the reduced divisor E. We have the following diagram of exact sequences, where the vertical arrows are multiplication by m.

$$\begin{array}{ccccccc}
0 \longrightarrow & \overline{\mathcal{K}_{2,E}} & \longrightarrow & \oplus_{1\le i\le r}\mathcal{K}_{2,E_i} & \longrightarrow & \oplus_{1\le i,j\le r}\oplus_{P\in S_{ij}}(i_P)_*K_2(k) & \longrightarrow 0 \\
& \downarrow m & & \downarrow m & & \downarrow m & \\
0 \longrightarrow & \overline{\mathcal{K}_{2,E}} & \longrightarrow & \oplus_{1\le i\le r}\mathcal{K}_{2,E_i} & \longrightarrow & \oplus_{1\le i,j\le r}\oplus_{P\in S_{ij}}(i_P)_*K_2(k) & \longrightarrow 0
\end{array}$$

Diagram E

Since the ground field k is algebraically closed, $K_2(k)$ is uniquely divisible by a lemma of Bass and Tate (see [S4],Theorem 8.24). This implies the last vertical arrow in the above diagram is an isomorphism. Hence by the snake lemma, combined with lemma 4.2.4(a), we get that $\overline{\mathcal{K}_{2,E}}$ is m-divisible, and

$$_m\overline{\mathcal{K}_{2,E}} \cong \oplus_{1 \leq i \leq r}(_m\mathcal{K}_{2,E_i}) \cong \oplus_{1 \leq i \leq r}\mathcal{O}^*_{E_i} \otimes_{\mathbb{Z}} \mu_m.$$

For $n > 1$, the conclusions about $\overline{\mathcal{K}_{2,nE}}$ in characteristic 0 follow from the case $n = 1$, using lemma 3.1.6 and the exact sequence (3.13), which imply that

$$\ker(\overline{\mathcal{K}_{2,nE}} \twoheadrightarrow \overline{\mathcal{K}_{2,E}})$$

is a sheaf of $\mathbb{Q}$-vector spaces. □

Corollary 4.2.6 *(a) Let E be the reduced exceptional curve in $\tilde{X}$. Then $SK_1(E)$ is divisible, and the natural maps*

$$SK_1(E) \to \bigoplus_{i=1}^{r} SK_1(E_i) \to (k^*)^{\oplus r}$$

induce isomorphisms on torsion subgroups.

(b) If k has characteristic 0, then $SK_1(nE)$ is divisible for any $n \geq 1$, and $SK_1(nE) \to SK_1(E)$ is an isomorphism on torsion subgroups.

Proof. This is an immediate corollary of the lemma 4.2.5, combined with lemma 4.2.4(a). □

Lemma 4.2.7 *Let E and $\tilde{X}$ be as above. Then*

$$SK_1(E)_{\text{torsion}} \subset \text{image}\,(SK_1(\tilde{X}))$$

under the natural map $SK_1(\tilde{X}) \longrightarrow SK_1(E)$. If k has characteristic 0, then a similar statement holds with $SK_1(nE)$ (in place of $SK_1(E)$).

Proof. After corollary 4.2.6(a), it suffices in the case $n = 1$ to show that the composite

$$\operatorname{Pic}\widetilde{X} \otimes_{\mathbb{Z}} k^* \to SK_1(\widetilde{X}) \to SK_1(E) \to \oplus_i SK_1(E_i) \to (k^*)^{\oplus r}$$

is surjective on torsion subgroups. We consider the following diagram

$$\begin{array}{ccc}
\operatorname{Pic}(E) \otimes k^* & \twoheadrightarrow & \oplus_{1\leq i\leq r}\operatorname{Pic}(E_i) \otimes k^* \\
\downarrow\!\!\downarrow & \searrow^{d_E} & \downarrow \\
SK_1(E) & \xrightarrow{p_*} & k^{*\oplus r}
\end{array}$$

where the map d_E is given by

$$d_E(\mathcal{L} \otimes a) \;=\; (a^{d_1}, \cdots, a^{d_r}) \text{ with } d_i = \deg(\mathcal{L}|_{E_i}).$$

Now the composite map

$$M : \oplus_{1\leq i\leq r}\mathbb{Z}E_i \longrightarrow \operatorname{Pic}(\widetilde{X}) \longrightarrow \operatorname{Pic}(E) \longrightarrow \mathbb{Z}^{\oplus r}$$

is given by the intersection matrix M of E. Since E is the exceptional curve on $\widetilde{X}$, the intersection matrix is negative definite, i.e., the map M is injective, and hence has finite cokernel because of the rank comparison. This implies that this map is surjective when tensored with the group μ_∞ of roots of unity, since the latter group is divisible. This implies that the composite map

$$\operatorname{Pic}(\widetilde{X}) \otimes \mu_\infty \longrightarrow \operatorname{Pic}(E) \otimes \mu_\infty \longrightarrow SK_1(E)_{\text{torsion}} \cong \mu_\infty{}^{\oplus r}$$

is surjective..

The stronger conclusion for arbitrary $n \geq 1$ in characteristic 0 now follows immediately by applying corollary 4.2.6(b). □

Let E and $\widetilde{X}$ be as above. Recall that we have defined the group

$$F_n = \ker(F^2K_0(\widetilde{X}, nE) \longrightarrow F^2K_0(\widetilde{X}))$$

in the beginning of this section.

Lemma 4.2.8 *(a) With the above notation, the group F_1 is torsion free, and*

$$(\ker \widetilde{\theta_{n,1}})_{\text{torsion}} = (F_n)_{\text{torsion}}.$$

(b) If k has characteristic 0, then F_n is torsion-free for all $n \geq 1$.

Proof. We first prove (a). Using the relative K-theory long exact sequence of the pair $(\widetilde{X}, E)$ and lemma 1.3.3, we see that $F_1 \hookrightarrow \frac{SK_1(E)}{\text{image} SK_1(\widetilde{X})}$, and so by the commutative diagram of lemma 1.3.1(c), we have

$$F_1 \hookrightarrow \frac{H^1(E, \mathcal{K}_{2,E})}{\text{image} H^1(\widetilde{X}, \mathcal{K}_{2,\widetilde{X}})}.$$

But we have seen (corollary 4.2.6) that $SK_1(E) = H^1(E, \mathcal{K}_{2,E})$ is divisible, and its torsion subgroup is contained in image $(SK_1(\widetilde{X})) = $ image $(H^1(\widetilde{X}, \mathcal{K}_{2,\widetilde{X}}))$. But the latter group itself is the direct sum of a divisible group and a finite group by lemma 4.2.4. Hence we see easily that the quotient group

$$\frac{H^1(E, \mathcal{K}_{2,E})}{\text{image} H^1(\widetilde{X}, \mathcal{K}_{2,\widetilde{X}})}$$

must be torsion-free and divisible and so F_1 is torsion free. Since $\widetilde{\theta}_{n,1}$ is the natural map $F_n \to F_1$, the other conclusion is obvious.

In characteristic 0, the statement (b) follows from the corresponding stronger conclusion in lemma 4.2.7. □

We now prove a general result in characteristic p, which has independent interest, apart from its use in the proof of Theorem 4.0.6.

Proposition 4.2.9 *Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities of a normal quasi-projective surface X over an algebraically closed field k of characteristic $p > 0$, and let E be the reduced exceptional curve on $\widetilde{X}$. Then the maps*

$$F^2K_0(X) \to F^2K_0(\widetilde{X}, E)$$

and

$$F^2K_0(\widetilde{X}, nE) \longrightarrow F^2K_0(\widetilde{X}, E), \ \forall\, n > 1$$

are isomorphisms.

Proof. As in corollary 3.1.8,we reduce to the case when X and $\widetilde{X}$ are projective, and E is a normal crossing divisor. Also, from Theorem 2.0.5, combined with lemma 1.2.1, we see that the two conclusions are equivalent.

We know (see [S1]), that the group

$$\ker(F^2K_0(X) \longrightarrow F^2K_0(\widetilde{X}))$$

is divisible. Hence by lemma 1.2.1,

$$F_n = \ker(F^2K_0(\widetilde{X}, nE) \longrightarrow F^2K_0(\widetilde{X}))$$

is also divisible for all n, and hence also $(F_n)_{\text{torsion}}$ is divisible for all n. Now by lemma 4.2.8(a), F_1 is torsion free, and

$$(F_n)_{\text{torsion}} = (\ker \widetilde{\theta}_{n,1})_{\text{torsion}}.$$

By lemma 4.2.3, $\ker(\widetilde{\theta}_{n,1})$ is torsion of a fixed exponent. Hence F_n is torsion-free, and

$$\ker(\widetilde{\theta_{n,1}}) = 0.$$

But from Diagram D above, we see that

$$\ker(\widetilde{\theta}_{n,1}) = \ker(\theta_{n,1} : F^2K_0(\widetilde{X}, nE) \twoheadrightarrow F^2K_0(\widetilde{X}, E)).$$

Hence $\theta_{n,1}$ is an isomorphism, which proves the proposition. □

Proof of Theorem 4.0.6. Now we finish proof of Theorem 4.0.6, which was the main goal of this chapter. If k has characteristic $p > 0$, then lemma 4.2.8(a) combined with proposition 4.2.9 implies that $F^2K_0(X) \rightarrow$

$F^2K_0(\tilde{X})$ is injective on torsion, and hence the map α_X is injective on torsion subgroups, as desired.

In characteristic 0, we use lemma 4.2.8(b), which implies that $F^2K_0(\tilde{X}, nE) \to F^2K_0(\tilde{X})$ is injective on torsion subgroups. Now we use Theorem 2.0.5 to finish the proof. □

Corollary 4.2.10 *Let X be a normal projective variety over algebraic closure of a finite field. Then the Albanese map*

$$A_0(X) \longrightarrow \mathrm{Alb}(X)$$

is an isomorphism.

Proof. In this case, both groups are torsion. Now the corollary follows from Theorem 4.0.6. □

We have another important corollary to Theorem 4.0.6. This result is new only in characteristic $p > 0$ for p-torsion cycles; even for this, [S6] contains a proof in the case $\dim A \geq 3$, using a simple trick (based on a suggestion of M. P. Murthy). The remaining case, that of surfaces, follows from Theorem 4.0.6, as is explained in [MS].

Corollary 4.2.11 *Let A be a normal affine domain of dimension $n \geq 2$ over an algebraically closed field k. Then $F^nK_0(A)$ is torsion-free.*

By removing a point from a smooth projective curve of positive genus, it is easily seen that the above result is not true in general for a smooth affine curve of positive genus.

4.3 Proof of Theorem 3.0.2 in characteristic $p > 0$

We now prove Theorem 3.0.2 in positive characteristic, using proposition 4.2.9. First suppose that $\mathrm{Pic}^0(X) = \mathrm{Pic}^0(E)$. As in characteristic 0, we remark that, because the intersection matrix of the components of E is negative definite, $NS(\tilde{X}) \to NS(E)$ has finite cokernel, so that $\mathrm{Pic}\,\tilde{X} \to \mathrm{Pic}\,E$ has finite cokernel as well. This implies as before that $SK_1(\tilde{X}) \longrightarrow SK_1(E)$ is surjective, which in turn implies that $F^2K_0(\tilde{X}, E) \longrightarrow F^2K_0(\tilde{X})$ is an isomorphism as before, since these implications are independent of the characteristic of the ground field.

But by the proposition 4.2.9, the map

$$F^2K_0(\tilde{X},\ nE) \longrightarrow F^2K_0(X,\ E)$$

is an isomorphism for all $n > 0$. From this, we get that

$$F^2K_0(\tilde{X},\ nE) \longrightarrow F^2K_0(\tilde{X})$$

is isomorphism for all $n > 0$. Now by Theorem 2.0.5, we conclude that

$$CH^2(X) = F^2K_0(X) \longrightarrow F^2K_0(\tilde{X}) = CH^2(\tilde{X})$$

is an isomorphism.

Conversely, suppose k is uncountable, and

$$\mathrm{Pic}^0(\tilde{X}) \longrightarrow \mathrm{Pic}^0(E)$$

is not surjective. It is then shown in [S2], Section 3, that the map $F^2K_0(\tilde{X}, E) \longrightarrow F^2K_0(\tilde{X})$ has a non-trivial kernel, and hence so does

$$CH^2(X) = F^2K_0(X) \to F^2K_0(\tilde{X}) = CH^2(\tilde{X}).$$

This completes the proof of theorem 3.0.2 in characteristic $p > 0$. □

Corollary 4.3.1 *Let X be a connected normal quasi-projective surface over an algebraically closed field k of characteristic $p > 0$, and let $\pi : \tilde{X} \longrightarrow X$ be a resolution of singularities of X, with reduced, normal crossing exceptional divisor E. Suppose all components of E are rational, and the dual graph of E has no loops. Then $CH^2(X) \cong CH^2(\tilde{X})$.*

Proof. The hypotheses imply that $\mathrm{Pic}^0(E) = 0$, so the result follows from the positive characteristic version of Theorem 3.0.2 proved above. □

Bibliography

[BPW] L. Barbieri-Viale, C. Pedrini, C. A. Weibel, *Roitman's theorem for singular complex projective surfaces*, Duke Math. J. **84** (1996) 155-190.

[BS] J. G. Biswas, V. Srinivas, *The Chow ring of a singular surface*, Proc. Indian Acad. Sci. (Math. Sci.) **108** (1998) 227-249.

[BS1] J. G. Biswas, V. Srinivas, *Roitman's theorem for singular projective varieties*, Compositio Math. **119** (1999) 213-237.

[B] S. Bloch, *K_2 of Artinian $\mathbb{Q}$-algebras with applications to algebraic cycles*, Comm. Algebra **3** (1975) 405-426.

[BL] S. Bloch, *Lectures on Algebraic Cycles*, Duke Univ. Math. Ser. IV, Duke Univ. Press (1976).

[BL1] S. Bloch, *Algebraic K-theory and crystalline cohomology*, Publ. Math. IHES **47** (1977) 187-268.

[CS] K. R. Coombes, V. Srinivas, *Relative K-Theory and Vector Bundles on Certain Singular Varieties*, appendix *Relative Algebraic K-Theory* by K. R. Coombes, Invent. Math. **70** (1982) 1-25.

[ESV] H. Esnault, V. Srinivas, E. Viehweg, *The universal regular quotient of the Chow group of points on projective varieties*, Invent. Math. **135** (1999) 595-664.

[EV] H. Esnault, E. Viehweg, *Lectures On Vanishing Theorems*, DMV Seminar No. 20, Birkhauser (1992).

[GW] S. Geller, C. A. Weibel, $K_1(A, B, I)$, J. reine. angew. Math. **342** (1983) 12-34.

[G] H. Gillet, *On the K-theory of surfaces with multiple curves and a conjecture of Bloch*, Duke Math. J. **51** (1984) 195-233.

[Gr] D. Grayson, *Localization for flat modules in algebraic K-theory*, J. Algebra **61** (1979) 463-496.

[FGA] A. Grothendieck, *Fondements de Géométrie Algébrique*, Extraits du Séminaire Bourbaki, 1957-62, Secrétariat mathématique, Paris, 1962 (see also Séminaire Bourbaki, Vol. 7, Année 1961/62, Exposs 223–240, reprinted edition, Société Mathématique de France, Paris, 1995.)

[H] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. No. 52, Springer-Verlag (1977).

[KS] W. van der Kallen, J. Stienstra, *The relative K_2 of truncated polynomial rings*, Mathematical Institute, University of Leiden, The Netherlands, **16** (1983) 1-17.

[KrS] A. Krishna, V. Srinivas *Zero-cycles and K-theory on normal surfaces*, Preprint, submitted for publication.

[L] S. Lang, *Abelian Varieties*, Interscience Tracts in Pure and Applied Mathematics No. 7, Interscience Publishers (1959).

[L1] M. Levine, *Torsion zero-cycles on singular varieties*, Amer. J. Math. **107** (1985) 737-757.

[L2] M. Levine, *Bloch's formula for singular surfaces*, Topology **24(2)** (1985) 165-174.

[L3] M. Levine, *A geometric theory of the Chow ring for singular varieties*, Preprint (circa 1983).

[LW] M. Levine, C. A. Weibel, *Zero cycles and complete intersections on singular varieties*, J. reine ang. Math. **359** (1985) 106-120.

[Li] J. Lipman, *Desingularisation of two-dimensional schemes*, Ann. Math.(2) **107** (1978) No.1 151-207.

[Ma] H. Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced math. No. 8, Cambridge University Press (1992).

[M] J. Milne, *Zero-cycles on algebraic varieties in non-zero characteristic: Roitman's Theorem*, Compositio Math. **47** (1982) 271-287.

[Mi] J. Milnor, *Introduction to Algebraic K-Theory*, Annals of Math Studies No. 72, Princeton (1971).

[MS] M. P. Murthy, R. G. Swan, *Vector bundles over affine surfaces*, Invent. Math. **36** (1976) 125-165.

[NR] D. Northcott, D. Rees, *A note on reductions of ideals with an application to the generalized Hilbert function*, Proc. Camb. Phil. soc. **50** (1954) 353-359.

[PW] C. Pedrini, C. A. Weibel, *Divisibility in the Chow group of zero-cycles on a singular surface*, Asterisque **226** (1994) 371-409.

[R] A. A. Roitman, *The torsion of the group of zero- cycles modulo rational equivalence*, Ann. Math. **111** (1980) 553-569.

[S] A. Seidenberg, *The Hyperplane Sections Of Normal Varieties* , Trans. of A.M.S. **69** (1950) 357-383.

[S1] V. Srinivas, *Zero cycles on a singular surface I*, J. reine angew. Math. **359** (1985) 90-105.

[S2] V. Srinivas, *Zero cycles on a singular surface II*, J. reine angew. Math. **362** (1985) 4-27.

[S3] V. Srinivas, *Grothendieck groups of polynomial and Laurent polynomial rings*, Duke Math J. **53** (1986) 595-633.

[S4] V. Srinivas, *Algebraic K-Theory*, Second Ed., Progress in Math. No. 90, Birkhauser (1996).

[S5] V. Srinivas, *Vector Bundles on the Cone over a Curve*, Compositio Math. **47** (1982) 249-269.

[S6] V. Srinivas, *Torsion zero-cycles on affine varieties in characteristic p*, Journal of Algebra **120** (1989) 428-432.

[S7] V. Srinivas, *Zero cycles on singular varieties*, in *The Arithmetic and Geometry of Algebraic Cycles*, NATO Science Series C, No. 548, Kluwer (2000) 347-382.

[S8] V. Srinivas, Ph.D. Thesis (1982), University of Chicago, USA.

[TT] R. W. Thomason, T. Trobaugh, *Higher Algebraic K-Theory Of Schemes And Of Derived Categories*, in *The Grothendieck Festschrift III*, Progress in Math. No. 88, Birkhauser (1990).

[W] K.-I. Watanabe, *Rational singularities with k^*-action*, Proceedings of the Trento Conference, edited by S. Greco and G. Valla, Lecture notes in pure and applied Mathematics, No. 84, Marcel Decker, Vol. 84, 339-351.

[We] C. A. Weibel, *The negative K-theory of normal surfaces*, K-Theory Preprint Archive - Univ. Illinois, Preprint No. 367, 22nd Sept. 1999.

INDIAN STATISTICAL INSTITUTE LIBRARY
No. T 121
Dt. 18 DEC 2003
CALCUTTA