Some Contributions To Reliability Analysis of A Consecutive-k-out-of-n:F System

Ph.D. Thesis
Submitted to the Indian Statistical Institute

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Contents

Synopsis

Present day technology has been characterized by development of complex systems or equipments containing a large number of subsystems and components. *Reliability*, as a human attribute, has been praised for a very long time. For technical systems, however, the reliability concept has not been applied for more than about 50 years. Reliability is the concern of all scientists and engineers engaged in developing a system, from design, through the manufacturing, to its ultimate use. Reliability technology has a potentially wide range of application areas like safety or risk analysis, environmental protection, quality, optimization, maintenance, engineering design, etc.

For a highly complex system, formal optimisation of system reliability may not be possible. In such cases, in attempting to achieve high reliability, a basic problem facing the system analyst and reliability engineers is that of evaluating the relative importance of the various components comprising the system. Measuring the relative importance of components may permit the analyst to determine which components merit the most additional research and development efforts to improve overall system effectiveness. A number of different importance measures have been defined to quantify the relative importance of components of a system and provide component ranking in order of importance. These measures can be classified as *structural importance measures* and *reliability importance measures.* Structural importance measures require only the knowledge of the *structure function* of the system. Whereas reliability importance measures require the additional information about component reliabilities. Structural measures of importance are more suitable during system design and development phases.

It is an undesirable fact that the reliability of a series system is low and, on the other hand, the parallel system has high reliability but tends to be very expensive. In last two decades a new system, a *consecutive-k-out-of-n:F system*, has caught the attention of many engineers and researchers because of its high reliability and low cost.

The present work deals with problems of reliability analysis and component importance measures of a consecutive-k-out-of-n:F system. In reliability theory, a consecutive-k-out-of-n:F system has been studied since 1980. It was first introduced by Kontoleon [37]. It consists of n linearly ordered and interconnected components. The system fails if and only if it has at least k consecutive failed components. If components be arranged on a circle we then have a *circular* consecutive-k-out-of-n:F system. Such systems find applications in telecommunication systems and pipeline networks [21], design of integrated circuits [11], vacuum systems in accelerators [35], computer networks [32], spacecraft relay stations [20], etc. A consecutive-k-out-of-n:F system is always more reliable than a conventional k-out-of-n:F system, in which the system fails if and only if at least k components fail. Since the family of minimal cut sets of the former is a subset of the family of minimal cut sets of the latter. However for $k = 1$ both the systems reduce to a series system and for $k = n$ reduce to a parallel system.

In this dissertation, the entire work has been divided into five chapters.

Chapter 1 covers the preliminaries needed for understanding the work done. A brief history of the development of reliability is traced and concepts and notations used in the subsequent chapters are defined and described. It also describes different measures of component importance.

Chapters 2, 3, 4 and 5 mainly present the work of the author.

Chapter 2, starts with a review of the literature on a consecutive-k-out-of-n:F system. We then study the *path sets* of a consecutive-k-out-of-n:F system and its applications.

In Section 2, we present a new and direct formula for determining the number of path sets with known size in a consecutive-k-out-of-n:F system.

Section 3, considers reliability function of the system with i.i.d. components and examines other results in the literature.

In Section 4, we study the *structural matrix* and its applications. Seth and Ramamurthy [61] introduced the concept of structural matrix and presented a unified approach for determining different structural importance measures. We give a combinatorial expression for the elements of the structural matrix and apply this to a consecutive-k-out-of-n:F system.

Birnbaum reliability measure of importance and Vesely-Fussell reliability measure of importance are defined using the concepts of *critical vectors* and *minimal cut sets.*

In **Chapter 3,** we study these concepts and also *minimal path sets* of a consecutive-k-out-of-n:F system and provide algorithms for *lexicographically* generating all minimal path sets and critical vectors.

Section 2, gives the necessary and sufficient conditions for a subset of components to be a minimal path set of the system. The problem of generating all minimal path sets of the system was considered by Chan et al. [14]. They proposed a recursive procedure to find all minimal path sets of a consecutive-k-out-of-n:F system. Their method starts with the generation of all minimal path sets of a consecutive-k-out-of-$2k$:F system and uses them to generate all minimal path sets of a consecutive-k-out-of-$2k+1$:F system and so on. This method recursively generates all minimal path sets of $n-2k+1$ different systems resulting in a large number of repetitions.

In Section 3, using a linear ordering on the subsets of components (*lexicographical ordering*), we present a nonrecursive algorithm for lexicographically generating and listing the collection of all minimal path sets of a linear consecutive-k-out-of-n:F system. Our algorithm generates minimal path sets only for the consecutive-k-out-of-n:F system. Hence unnecessary generations of the minimal path sets is avoided as in the case of Chan et al. [14] procedure. We also study the minimal path sets of a circular consecutive-k-out-of-n:F system and show that all minimal path sets of a circular system can be generated using minimal path sets of a linear system.

In Section 4, we present a nonrecursive algorithm for generating all minimal path sets containing a given component of the system. This algorithm is used for evaluating the Vesely-Fussell reliability measure of component importance in a consecutive-k-out-of-n:G system, in Chapter 4. A consecutive-k-out-of-n:G system is a *dual* of a consecutive-k-out-of-n:F system and this system functions if and only if at least k consecutive components function.

Section 5, gives a nonrecursive algorithm for generating all critical vectors for a given component in a consecutive-k-out-of-n:F system. This is applied for computing the Birnbaum reliability measure of component importance, in Chapter 4.

Chapter 4 considers the system reliability and component importance of consecutive-k-out-of-n systems. If the components of a consecutive-k-out-of-n:F system are independent but not identical, the problem of computing system reliability leads to a recurrence relationship [63]. Using the minimal path(cut) sets of a consecutive-k-out-of-n:F(G) system and the algorithm presented in Chapter 3, we introduce a nonrecursive algorithm for determining the system reliability with different component reliabilities, in Section 2. This is an efficient alternative to the *inclusion-exclusion* principle for evaluating system reliability in consecutive-k-out-of-n systems. It has no cancelling terms and number of terms equals the number of minimal path(cut) sets. We show that this algorithm can also be used for determining system reliability of a k-out-of-n system with different component reliabilities. Furthermore, we show that this approach is applicable for a general case when the components of the system are not independent.
Recent related results in the literature are considered and examined.

Section 3, considers the determination of the Vesely-Fussell reliability and structural measures of component importance in consecutive-k-out-of-n systems. We show that in case of a consecutive-k-out-of-n:F system, these measures can be computed easily. Using the approach presented in Chapter 3, we present an algorithm to compute Vesely-Fussell reliability and structural measures of component importance in a consecutive-k-out-of-n:G system.

In Section 4, we consider the problem of determining Birnbaum reliability measure of component importance of a consecutive-k-out-of-n:F system with different component reliabilities. We present an algorithm for this using the approach described in Section 5 of Chapter 3. This algorithm does not require computation of the reliability function.

In Section 5, we provide a different recursive relation for determining the reliability function of a consecutive-k-out-of-n:F system with non iid components. Using this we give new upper and lower bounds for reliability function of the system in iid case.

Although the minimal path sets of a consecutive-k-out-of-n:F system can be generated using the algorithms presented in Chapter 3, but the combinatorial problem of determining the number of minimal path sets of the system still remains a difficult task. On the other hand we note that the number of terms in the nonrecursive formula for determining the reliability function of the system with non iid components, as given in Algorithm 1 of Chapter 4, equals the number of minimal path sets of the system. Hence in **Chapter 5**, we consider the enumeration of the minimal path sets of a consecutive-k-out-of-n:F system.

Section 2 of Chapter 5, gives explicit formulae for determining the number of minimal path sets and also the number of minimal path sets with known size, for a linear and a circular consecutive-2-out-of-n:F systems.

In Section 3, we consider direct computation of the number of minimal path sets in consecutive-3-out-of-n:F systems for both linear as well as circular.

In Section 4, we present a recurrence relationship for determining the number of minimal path sets of a general consecutive-k-out-of-n:F system.

We may add that, direct computation of the number of minimal path sets of a consecutive-k-out-of-n:F system, still remains a problem.

Chapter 1

Preliminaries

1.1 Introduction

At the time when the electronic computers appeared as a main tool in the service of scientists and researchers, advancement in technology became fast. Technology has been characterized by development of complex systems or equipments containing large number of subsystems, components and parts. The trend to ever larger and more complex systems is continuing with the development of space vehicles, electronic computers, communication systems, weapons system, etc. Reliability, as a human attribute, has been praised for a very long time. For technical systems, however, the reliability concept has not been applied for more than some 50 years. Reliability is the concern of all scientists and engineers engaged in developing a system, from design, through the manufacturing, to its ultimate use. Reliability is: "The ability of an item to perform a required function, under given environmental and operational conditions and for a stated period of time." If, under the appropriate conditions, the item achieves the required performance or continue to achieve it, then the item may, in a qualitative sense, be termed reliable, otherwise it may be described as unreliable. Today complex systems must be developed to not only work, but to work reliably. The degree of acceptable reliability depends on the consequences of failure. Reliability application started just after World War I; it was then used in connection with comparing operational safety of one, two, and four engine airplanes. Initially, the reliability was measured as the number of accidents per hour of flight time.

During World War II, the product probability law of series components was derived. This theorem concerns with systems functioning only if all the components are functioning and is valid under special assumptions. If the system comprises a large number of components, the system reliability may be rather low, even though the individual components have high reliabilities. The failure of a single inexpensive component causes the failure of the entire system.

To compensate a low system reliability, it was improved by improving the quality of individual components. Better raw materials and better designs for the products were demanded. A higher system reliability was obtained, but extensive systematic analysis of the problem was probably not carried out at that time.

For a highly complex system, formal optimisation of system reliability may not be possible. In such cases, in attempting to achieve high reliability, a basic problem facing the system analyst and reliability engineers is that of evaluating the relative importance of the various components comprising the system. Measuring the relative importance of components may permit the analyst to deterimine which components merit the most additional research and development efforts to improve overall system reliability at minimum cost or effort.

Reliability technology has a potentially wide range of application areas like safety or risk analysis, environmental protection, quality, optimization, maintenance, engineering design etc.

It is an undesirable fact that the reliability of a series system is low and, on the other hand, the parallel system has high reliability but tends to be very expensive. A new system, consecutive-k-out-of-n:F, and its related systems, have caught the attention of many engineers and researchers because of their high reliability and low cost.

1.2 System Representation

Assessment of the reliability of a system from its basic elements is one of the most important aspects of reliability analysis. A system is a collection of components (subsystems, units, items, blocks, etc.) whose proper, coordinated function leads to the proper functioning of the system. In reliability analysis, it is therefore important to model the relationship between various components as well as the reliability of the individual components to determine the reliability of the system as a whole. Whether we are dealing with a single component or a complex system, we will content ourselves to classifying a system as being in one of two possible states, either in a functioning state or in a failed state.
We will assume binary state and apply this to each component as well as to the system itself. The state of the system is completely determined by the state of its components, and the dependence of the system state on the component states is expressed through a structure function.

Structure Function

A system composed of n components will be denoted as a system or structure of order n. The components are assumed to be numbered consecutively from 1 to n from the set $N = \{1, 2, \ldots, n\}$. The state of component i, $i = 1, 2, \ldots, n$ can be described by a binary variable x_i, where

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is in a failed state.} \end{cases}$$

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called the state vector. Furthermore we assume that by knowing the states of all the n components, we also know whether the system is functioning or not. Similarly the state of the system can then be described by a binary function

$$\phi(\mathbf{x}) = \phi(x_1, x_2, \ldots, x_n), \quad \mathbf{x} \in B^n$$

where $B = \{0, 1\}$ and

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is in a failed state.} \end{cases}$$

$\phi(\mathbf{x})$ is called the *structure* function of the system.

Example 1. A system that is functioning if and only if all of its n components are functioning is called a *series* system. Its structure function is given by

$$\phi(\mathbf{x}) = x_1 x_2 \cdots x_n = \prod_{i=1}^{n} x_i.$$

A series structure of order n can be illustrated as in Figure 1.1.

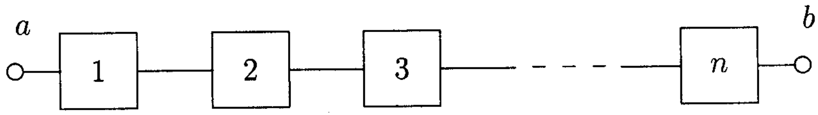

Figure 1.1: series structure

Example 2. A system that is functioning if and only if at least one of its components is functioning is called a *parallel* system. A parallel structure of order n is illustrated in Figure 1.2. In this case the structure function can be written as :

$$\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2) \cdots (1 - x_n) = 1 - \prod_{i=1}^{n} (1 - x_i).$$

The expression on the right hand side is often written as $\coprod_{i=1}^{n} x_i$. Since x_i's are binary variables, we have $\coprod_{i=1}^{n} x_i = \max_{1 \le i \le n} x_i$.

Similarly in series system we have $\phi(\mathbf{x}) = \prod_{i=1}^{n} x_i = \min_{1 \le i \le n} x_i$.

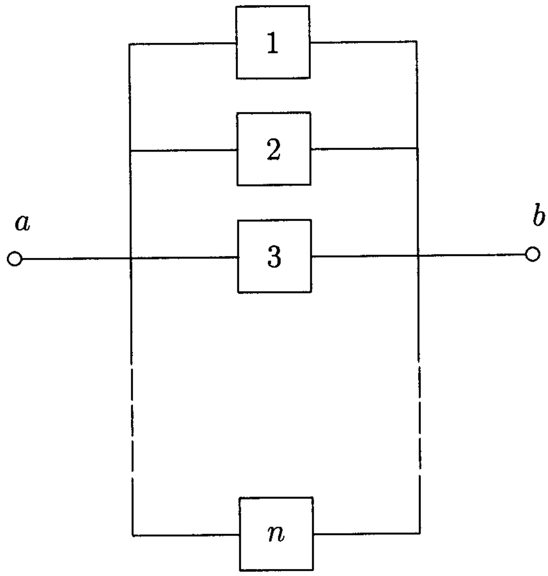

Figure 1.2: parallel structure

Example 3. A system that is functioning if and only if at least k of the n components are functioning is called a k-out-of-n system. A series system is therefore an n-out-of-n system, and a parallel system is a 1-out-of-n system. The structure function of a k-out-of-n system is given by

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{if } \sum_{i=1}^{n} x_i < k \end{cases}$$

Definition 1. Let ϕ be a structure of order n. Its *dual* ϕ^D is another structure of order n defined by

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}) \text{ for all } \mathbf{x}$$

where $(\mathbf{1} - \mathbf{x}) = (1 - x_1, 1 - x_2, \ldots, 1 - x_n)$.

Example 4. The dual of a series (parallel) structure is a parallel (series) structure. More generally, the dual of a k-out-of-n structure is $(n-k+1)$-out-of-n structure.

1.2.1 Coherent Structure

When establishing the structure of a system, it seems reasonable first to leave out all components that do not play any direct role in the functioning of the system. The components we are left with are called *relevant.* The components that are not relevant are called *irrelevant.* If component i is irrelevant, then

$$\phi(1_i, \mathbf{x}) = \phi(0_i, \mathbf{x}) \text{ for all } (\cdot_i, \mathbf{x})$$

$(1_i, \mathbf{x})$ represents a state vector where the state of the ith component is 1, $(0_i, \mathbf{x})$ represents a state vector where the state of the ith component is 0, and $(\cdot_i, \mathbf{x})$ represents a state vector where the state of the ith component is 0 or 1. That is

$$(1_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 1_i, x_{i+1}, \ldots, x_n)$$

and

$$(0_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 0_i, x_{i+1}, \ldots, x_n).$$

We assume that the system will not run worse than before if we replace a component in a failed state with the one that is functioning. This is obviously the same as requiring that the structure function shall be nondecreasing in each of its arguments.

Definition 2. A system of components is said to be *semi-coherent* if at least one of its components is relevant and the structure function is nondecreasing in each argument. A system is also said to be *monotone system* if its structure function is nondecreasing in each of its arguments.

Let ϕ be a semi-coherent structure of order n, we then have $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$ where

$$\mathbf{0} = (0, 0, \ldots, 0) \quad \text{and} \quad \mathbf{1} = (1, 1, \ldots, 1)$$

Definition 3. A semi-coherent system is called *coherent* if all its components are relevant.

For detailed exposition on coherent structures see Barlow and Proschan [6], Kaufmann [36], Ramamurthy [55], Høyland and Rausand [30] and Gertsbakh [26].

Structure Representation by Minimal Path Sets and Cut Sets

Definition 4. A state vector $\mathbf{x}$ is called a *path vector* if $\phi(\mathbf{x}) = 1$. Then $C_1(\mathbf{x}) = \{i : i \in N \text{ and } x_i = 1\}$ is called a *path set*. If $\phi(\mathbf{y}) = 0$ for all $\mathbf{y} < \mathbf{x}$, then $\mathbf{x}$ is called a *minimal path vector* and the corresponding set $C_1(\mathbf{x})$ is a *minimal path set*. In other words, a minimal path set is a minimal set of components whose functioning causes the functioning of the system.
Similarly a state vector $\mathbf{x}$ is called a *cut vector* if $\phi(\mathbf{x}) = 0$ and $C_0(\mathbf{x}) = \{i : i \in N \text{ and } x_i = 0\}$ is called a *cut set*. If $\phi(\mathbf{y}) = 1$ for all $\mathbf{y} > \mathbf{x}$, then $\mathbf{x}$ is called a *minimal cut vector* and the corresponding set $C_0(\mathbf{x})$ is a *minimal cut set*. In other words, a minimal cut set is a minimal set of components whose failure causes the failure of the system. $\mathbf{x} > \mathbf{y}$ means coordinate-wise inequality $x_i \geq y_i$, $i = 1, 2, \ldots, n$ with at least one strict inequlity.
We denote by $\alpha(\phi)$ and $\beta(\phi)$, respectively the collections of all minimal path and cut sets of ϕ.

Example 5. consider a bridge structure given in Figure 1.3. The minimal path sets are
$P_1 = \{1, 4\}$, $P_2 = \{2, 5\}$, $P_3 = \{1, 3, 5\}$ and $P_4 = \{2, 3, 4\}$.

The minimal cut sets are

$C_1 = \{1,2\},\ C_2 = \{4,5\},\ C_3 = \{1,3,5\}$ and $C_4 = \{2,3,4\}$.

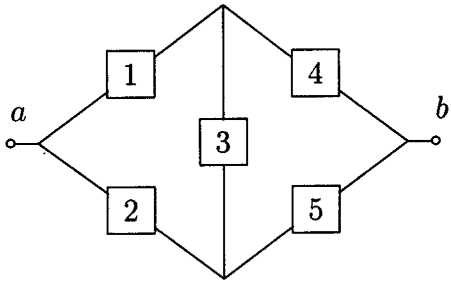

Figure 1.3 Bridge Structure

The concepts of minimal path sets and minimal cut sets play a crucial role in the study of reliability analysis of semi-coherent structures.

Consider a structure ϕ on N with the minimal path sets $P_1, P_2, \ldots, P_p$ and the minimal cut sets $C_1, C_2, \ldots, C_c$. We now state, without proofs, some relevant results of coherent and semi-coherent structures. Proofs are given in [55], [6] and [36].

Theorem 1. Let ϕ be a semi-coherent structure on N. A subset P of N is a path set of ϕ if and only if $P \cap C_j \neq \emptyset\ \forall j = 1, 2, \ldots, c$. And a subset Q of N is a cut set of ϕ if and only if $Q \cap P_i \neq \emptyset\ \forall i = 1, 2, \ldots, p$.
That is P (Q) is a path (cut) set of the system if and only if it has nonempty intersection with every minimal cut (path) set of the system.

Theorem 2. Let ϕ be a semi-coherent structure on N. We have:

$$\phi(\mathbf{x}) = 1 - \prod_{i=1}^{p}\left(1 - \prod_{j\in P_i} x_j\right) = \prod_{j=1}^{c}\left(1 - \prod_{i\in C_j}(1 - x_i)\right) \quad \text{for all } \mathbf{x} \in B^n.$$

That is, structure ϕ may be interpreted as parallel (series) structure of the minimal path (cut) series (parallel) structures.

Definition 5. A structure function ϕ, can be expressed as

$$\phi(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j \quad \text{for} \quad \mathbf{x} \in B^n$$

which is called the *simple form* of $\phi(\mathbf{x})$. For $S = \emptyset$, we take $\prod_{j \in S} x_j = 1$ and the a_S's are some integers.

Ramamurthy [55], showed that this representation for structure function always exists and is unique.

Theorem 3. Minimal path (cut) sets of a coherent system completely determine its structure function and vice versa.

Theorem 4. Suppose $\alpha(\phi)$ and $\beta(\phi)$ be the collections of all minimal path and cut sets of ϕ, respectively. We then have

$$\alpha(\phi^D) = \beta(\phi) \text{ and } \beta(\phi^D) = \alpha(\phi).$$

Theorem 5. Any structure ϕ of order n is a linear composition of two structures of order at most $n-1$, that is

$$\phi(\mathbf{x}) = x_i \phi(1_i, \mathbf{x}) + (1 - x_i)\phi(0_i, \mathbf{x})$$

for all $\mathbf{x} \in B^n$ and $i = 1, 2, \ldots, n$. This relation is called the *pivotal decomposition* of the structure function.

1.3 Reliability Function

In Section 2, the structural relationship between a system and its components was established by using a deterministic model of the structure. However, the failures of components can not usually be predicted with certainty. For studying the occurrence of such failures, one looks for statistical regularity. Hence it seems reasonable to interpret the state variables of the n components X_i's and consequently the state variable of the system $\phi(\mathbf{X})$ as random variables. The following probabilities are of interest:

$$P(X_i = 1) = p_i \text{ for } i = 1, 2, \ldots, n.$$

$$P(\phi(\mathbf{X}) = 1) = h(\mathbf{p}) \text{ where } \mathbf{p} = (p_1, p_2, \ldots, p_n).$$

p_i's are called *component reliabilities* and $h(\mathbf{p})$ is called *system reliability* or *reliability function.* Since the state variables X_i's and $\phi(\mathbf{X})$ are binary, we note that $p_i = E(X_i)$ and $h(\mathbf{p}) = E(\phi(\mathbf{X}))$.

Components and systems that are replaced or repaired after failure are called *repairable.* We consider *nonrepairable* components and systems. In this case p_i and $h(\mathbf{p})$ correspond to the survivor function of component i and of the system, respectively. We also assume that the components are stochastically independent, that is $X_1, X_2, \ldots, X_n$ are independently distributed. Furthermore if $p_1 = p_2 = \cdots = p_n = p$, that is components are i.i.d., we shall denote the system reliability function by $h(p)$.

Example 6. For a series structure, reliability function is $h(\mathbf{p}) = p_1 p_2 \ldots p_n$.

Example 7. For a parallel structure $h(\mathbf{p}) = 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n)$.

Example 8. For a k-out-of-n structure with i.i.d. components, reliability function is given by :

$$h(p) = E(\phi(\mathbf{X})) = P(\phi(\mathbf{X}) = 1) = P\left(\sum_{i=1}^{n} X_i \geq k\right) = \sum_{x=k}^{n} \binom{n}{x} p^x (1-p)^{n-x}.$$

Example 9. Using Theorem 2, structure function of a bridge structure as illustrated in Example 5 is :

$\phi(\mathbf{X}) = 1 - (1 - X_1X_3X_5)(1 - X_2X_3X_4)(1 - X_2X_5)(1 - X_1X_4).$

Using the fact that $X_i^k = X_i$, we have

$\phi(\mathbf{X}) = X_1X_3X_5 + X_2X_3X_4 + X_2X_5 + X_1X_4 - X_1X_2X_3X_5 - X_1X_2X_4X_5 -$

$X_1X_3X_4X_5 - X_1X_2X_3X_4 - X_2X_3X_4X_5 + 2X_1X_2X_3X_4X_5$

and reliability function is

$h(\mathbf{p}) = E(\phi(\mathbf{X})) = p_1p_3p_5 + p_2p_3p_4 + p_2p_5 + p_1p_4 - p_1p_2p_3p_5 - p_1p_2p_4p_5 - p_1p_3p_4p_5 - p_1p_2p_3p_4 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5$.

We see that in spite of the fact that a compact formula (Theorem 2) has been obtained for $\phi(\mathbf{X})$, the computation of the system reliability $h(\mathbf{p}) = E(\phi(\mathbf{X}))$ still remains a serious task for complex systems.

The most straightforward method for computation of the system reliability is based on the structure function. We need to find out all minimal path (or cut) sets and represent $\phi(\mathbf{X})$ in its simple form and apply the E[.]-operator. This method for system reliability evaluation is called *sum of disjoint products.*

Another method used for calculating system reliability is called, *inclusion-exclusion principle.* We note that $\bar{h}(\mathbf{p}) = 1 - h(\mathbf{p}) = P\left(\bigcup_{i=1}^{c} E_i\right)$ where E_i is the event that all components of the minimal cut set C_i are in a fail state, that is ith minimal cut parallel structure is failed and c is the number of minimal cut sets.

In general, the individual events E_i, $i = 1, 2, \ldots, c$ are not disjoint. Hence probability $P\left(\bigcup_{i=1}^{c} E_i\right)$ is determined by using the general addition theorem in probability theory (e.g., see Dudewicz and Mishra [23] or Feller [24], p. 99)

$$\bar{h}(\mathbf{p}) = \sum_{j=1}^{c}(-1)^{j-1} \sum_{i_1<i_2<\cdots<i_j} P\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_j}\right).$$

According to this method we have to calculate the probability of a large number of terms that later cancel each other. A number of alternatives to the inclusion-exclusion principle have been proposed. For example see Satyanarayana and Prabhakar [59] and Aven [4].

Theorem 6. Reliability function satisfies

$$h(\mathbf{p}) = p_i h(1_i, \mathbf{p}) + (1 - p_i) h(0_i, \mathbf{p})$$

where

$$h(1_i, \mathbf{p}) = E\left(\phi(1_i, \mathbf{X})\right) = E\left(\phi(X_1, X_2, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)\right)$$

and

$$h(0_i, \mathbf{p}) = E\left(\phi(0_i, \mathbf{X})\right) = E\left(\phi(X_1, X_2, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n)\right).$$

This representation for $h(\mathbf{p})$ is called the *pivotal decomposition of the reliability function.*

Remark. By repeated pivotal decomposition of the structure function, $\phi(\mathbf{X})$ can always be written as

$$\phi(\mathbf{X}) = \sum_{\blacksquare \in B^n} \prod_{j=1}^{n} X_j^{y_j} (1 - X_j)^{1-y_j} \phi(\mathbf{Y})$$

where $0^0 \equiv 1$. Therefore the reliability function may be written

$$h(\mathbf{p}) = E\left(\phi(\mathbf{X})\right) = \sum_{\blacksquare} \prod_{j=1}^{n} p_j^{y_j} (1 - p_j)^{1-y_j} \phi(\mathbf{Y}).$$

Theorem 7. Reliability function $h(\mathbf{p})$ of a monotone system is a monotone function of $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

The first work on system reliability, was given by Moore and Shannon [45]. It dealt with two-terminal relay network reliability. Research on coherent structures was initiated by Birnbaum, Esary and Saunders [9]. They introduced the notions of minimal cut and minimal path sets and derived the representation of structure function by using its minimal cut or path sets.

Computation of system reliability is relatively simple, if the system is of the series-parallel type. As we have seen, there is a very simple structure (e.g., a bridge structure) which is not a series-parallel system. An enumeration of all cut or path sets of an arbitrary network is a problem of outstanding algorithmical difficulty, and its solution for a large system may take a long time. Many algorithms for computing system reliability are based on the pivotal decomposition formula (Theorem 6). The goal of a repeated application of pivoting is to obtain series-parallel subsystems for which further computations are relatively easy. However, the efficiency of the computational algorithms will depend on the selection strategy. This topic interfaces closely with the computational complexity of the algorithms for reliability computation. For details see, Barlow [5], Satanarayana and Chang [58] and Agrawal and Barlow [1].

1.4 Component Importance

During the design of a system, the choice of components and their arrangement may render some to be more important than others in determining whether the system is functioning or not. For example, placing a component in series within a system causes it to have a higher importance for system reliability than any other component in the system. It would be of a great practical significance to the designer, the reliability analyst, as well as the repairperson to have a quantitative measure of the importance of each component. Such a measure would, for example, identify the components that, by being improved, would increase the reliability of the system the most or, by means of a list, tell the repairperson in which order to check the components that may have caused the system failure.

The importance of components should be used during design or evaluation of systems to determine which components or subsystems are important to the reliability of the system. Those with high importance could prove to be candidates for further improvements. In an operational context, items with high importance should be watched by the operators, since they are critical for the continuous operation of the system.

A number of different measures of component importance have been defined. We describe three methods of measuring the importance of components: Birnbaum, Barlow-Proschan, and Vesely-Fussell measures of importance.

These measures can be classified as *structural importance measures* and *reliability importance measures.* Structural measures require only the knowledge of the structure function of the system. Whereas reliability importance measures require the additional information about component reliabilities. Structural measures of importance are most suitable during system design and development phases.

The following definitions are required.

Definition 6. Let $\phi(\mathbf{x})$ be a structure function of order n. If $\phi(1_i, \mathbf{x}) = 1$ while $\phi(0_i, \mathbf{x}) = 0$ then the state vector $(\cdot_i, \mathbf{x})$ is called a *critical vector* for component i. This is equivalent to requiring that

$$\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 1.$$

In other words, given the states of the other components $(\cdot_i, \mathbf{x})$, the system is functioning if and only if component i is functioning. It is therefore natural to call $(1_i, \mathbf{x})$ and $(0_i, \mathbf{x})$ as a *critical path vector* and *critical cut vector* for component i, respectively. When $(\cdot_i, \mathbf{x})$ is a critical vector for component i, we say that component i is critical for the system. We note that component i is critical for the system tells nothing about the state of component i.

Definition 7. A *critical path set,* $P(1_i, \mathbf{x})$ corresponding to the critical path vector $(1_i, \mathbf{x})$ for component i is defined by

$$P(1_i, \mathbf{x}) = \{i\} \cup \{j; x_j = 1, j \neq i\}\,.$$

Similarly corresponding to the critical cut vector, $(0_i, \mathbf{x})$ for component i, a *critical cut set,* $C(0_i, \mathbf{x})$ is defined by

$$C(0_i, \mathbf{x}) = \{i\} \cup \{j; x_j = 0, j \neq i\}.$$

We note that the total number of critical path (cut) vectors, as well as critical path (cut) sets, for component i is

$$n_\phi(i) = \sum_{(\cdot_i, \mathbf{x})} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})].$$

By binary assumption of variables x_j's, the total number of state vectors $(\cdot_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ is 2^{n-1}.

Birnbaum Measure of Importance

Birnbaum [8], introduced the concept of *component importance.*

When component reliabilities are not known, we use structural measure of component importance that can be calculated using only the structure function of the system.

Birnbaum Structural Measure

Birnbaum proposed the following measure for the structural importance of component i

$$I_B^\phi(i) = \frac{n_\phi(i)}{2^{n-1}}$$

where $n_\phi(i)$, the number of critical vectors for component i. $I_B^\phi(i)$ expresses the relative proportion of the 2^{n-1} possible state vectors $(\cdot_i, \mathbf{x})$ which are critical vectors for component i.

Birnbaum Reliability importance Measure

Reliability measure of component importance makes use of probabilistic information about the components of the system.

Birnbaum reliability measure of importance of component i is defined by

$$I_B^h(i, \mathbf{p}) = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}) \text{ for } i = 1, 2, \ldots, n.$$

We note that $I_B^h(i, \mathbf{p})$ does not depend on p_i.

We have $I_B^h(i, \mathbf{p}) = E\left(\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X})\right) = P\left(\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X}) = 1\right)$.

From Theorem 6, we can write $I_B^h(i, \mathbf{p}) = \dfrac{\partial h(\mathbf{p})}{\partial p_i}$.

We note that $I_B^\phi(i) = I_B^h(i;\ 1/2, 1/2, \ldots, 1/2)$.

Barlow-Proschan Measure of Importance

Barlow and Proschan [7], introduced a new measure of component importance. For details see [7].

Barlow-Proschan Structural Measure

We define *size* of vector $(\cdot_i, \mathbf{x})$ as well as size of vector $(0_i, \mathbf{x})$ by $\sum_{j=1, j\neq i}^{n} x_j$ and size of vector $(1_i, \mathbf{x})$ by $1 + \sum_{j=1, j\neq i}^{n} x_j$.

Definition 8. We define by $n_\phi(i, r)$, the number of critical path vectors of ϕ of size of r for component i. Then the number of critical vectors for component i, is $n_\phi(i) = \sum_{r=1}^{n} n_\phi(i, r)$.

Barlow-Proschan measure of structural importance for component i, $\Psi(i, \phi)$ is defined as the probability that failure of component i causes system failure under the assumption that all the $n!$ orderings (or permutations) of components are equiprobable.

Remarks.

(i) We note that $\sum_{i=1}^{n} \Psi(i, \phi) = 1$. But this is not true for $I_B^\phi(i)$.

(ii) In fact $\Psi(i, \phi)$, is the probability that component i causes system failure under the assumption that component life distributions of X_j's, $j = 1, 2, \cdots, n$ are i.i.d. random variables.

Barlow-Proschan measure of structural importance for component i, can be obtained as follows :

Theorem 8.

$$\Psi(i, \phi) = \frac{1}{n} \sum_{r=1}^{n} \frac{n_\phi(i, r)}{\binom{n-1}{r-1}}.$$

Theorem 9.

$$\Psi(i, \phi) = \int_0^1 (h(1_i, p) - h(0_i, p))\, dp = \int_0^1 I_B^h(i, p) dp.$$

Barlow-Proschan Reliability Importance Measure

This requires the life time distribution of each component. Let the distribution function $F_i(t)$ of the life of component i be absolutely continuous. Then Barlow-Proschan reliability importance of component i, $\Psi(i, h)$ is defined as the probability that component i causes system failure.

Theorem 10.

$$\Psi(i, h) = \int_0^\infty \left[h\left(1_i, \bar{\mathbf{F}}(\mathbf{t})\right) - h\left(0_i, \bar{\mathbf{F}}(\mathbf{t})\right)\right] dF_i(t)$$

where

$\left(\cdot_i, \bar{\mathbf{F}}(\mathbf{t})\right) = \left(\bar{F}_1(t), \ldots, \bar{F}_{i-1}(t), ., \bar{F}_{i+1}(t), \ldots, \bar{F}_n(t)\right)$ and $\bar{F}_j(t) = 1 - F_j(t)$.

Vesely-Fussell Measure of Importance

Vesely-Fussell measure of importance takes into account the fact that a component may contribute to system failure or system reliability without being critical. The component contributes to system failure when a minimal cut, containing the component, is failed. We say that a minimal cut set is failed when all the components in the minimal cut set are failed.

This measure was introduced by Vesely [66] and later applied by Fussell [25].

Vesely-Fussell measure of reliability importance of component i, $I_{VF}^h(i, \mathbf{p})$ is defined as the probability that at least one minimal cut set that contains component i is failed, given that the system is failed.

Consider a system with c minimal cut sets $C_1, C_2, \ldots, C_c$. According to Theorem 2, the system can then be represented logically by a series structure of c minimal cut parallel structures. The system is failed if and only if at least one of the c minimal cut sets is failed. We use the following notations.

D_i : The event that at least one minimal cut set that contains component i is failed.

m_i : The number of minimal cut sets that contain component i.

E_j^i : The event that minimal cut set j among those containing component i is failed for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m_i$.

We note that D_i implies that system is failed ($\phi(\mathbf{X}) = 0$) and, $D_i = \bigcup_{j=1}^{m_i} E_j^i$ and $P(\phi(\mathbf{X}) = 0) = 1 - h(\mathbf{p})$. Then

$$I_{VF}^h(i, \mathbf{p}) = P(D_i | \phi(\mathbf{X}) = 0) = \frac{P\left(\bigcup_{j=1}^{m_i} E_j^i\right)}{1 - h(\mathbf{p})}.$$

We also note that same component may be a member of different minimal cut sets. Hence the events E_j^i's, $j = 1, 2, \ldots, m_i$ are usually not disjoint. For the same reason these events will not, in general be independent, even if all the components are independent.

Remark. We note that $1 - h(\mathbf{p})$, the denominator of $I_{VF}^h(i, \mathbf{p})$ is a constant for a given $\mathbf{p}$. Hence in order to get the ranking of components by using Vesely-Fussell reliability measure $I_{VF}^h(i, \mathbf{p})$, it is not necesssary to compute the denominator.

Vesely-Fussell structural measure of importance of component i, $I_{VF}^\phi(i)$ is also given by :

$$I_{VF}^\phi(i) = I_{VF}^h(i;\ 1/2, 1/2, \ldots, 1/2).$$

It should be noted that the ranking of components, using different measures may lead to different rankings. It is expected, since the measures are defined differently. When analyzing a specific system, one must choose the measure that is relevant to the situation at hand.

To identify the component that should be improved to increase system reliability, Birnbaum measure is normally the most appropriate. On the other hand, to identify the component that has the largest probability of being the cause of system failure, the Barlow-Proschan measure or Vesely-Fussell measure is the most appropriate. These two measures may also be used to set up a repairperson's checklist. Because of simplicity, Vesely-Fussell measure is widely used.

Chapter 2

A Consecutive-k-out-of-n:F System

2.1 Introduction

In this chapter, we first describe a *consecutive-k-out-of-n:F system* then we study the path sets of this system and its applications. We also consider the problem of finding component structural importance in a consecutive-k-out-of-n:F system.

A consecutive-k-out-of-n:F system consists of n linearly ordered and interconnected components. The system fails if and only if it has at least k consecutive failed components. Because of high reliability and low cost, this system has caught the attention of many engineers and researchers in last two decades. The reliability of a consecutive-k-out-of-n:F system was first studied by Kontoleon [37], but the name, consecutive-k-out-of-n:F, was first coined by Chiang and Niu [21]. Such a system finds applications in telecommunication systems and oil pipeline networks [21], design of integrated circuits [11], vacuum systems in accelerators [35], computer networks [32], spacecraft relay [20], etc.

This system is always more reliable than a conventional k-out-of-n:F system in which the system fails if and only if at least k components fail, since the family of minimal cut sets of the former is a subset of the family of minimal cut sets of the latter. However for $k = 1$ both the systems reduce to the usual series system and for $k = n$ reduce to the parallel system.

Since 1980 many papers have been published on the reliability of a consecutive-k-out-of-n:F and related systems, under various assumptions.

Since 1990, this area has expanded very fast and connected to many other promising areas. Thus, recent results associated with this field appears not only in reliability journals, but also in many other applied probability, operational research, and statistics journals. In a survey article by Chao et al [18] on the reliability aspect of this system alone, more than hundred papers have been cited.

The reliability analysis of a consecutive-k-out-of-n:F system is closely related to some discrete distributions of order k which have been studied extensively, recently. Relationships between the reliability of this system to discrete distributions of order k and distributions of longest success run, have been studied by Aki [2], Aki and Hirano [3], Hirano and Aki [29], Philippou et al [51], Philippou and Makri [52] and [53], Philippou and Muwafi [54], Philippou [50], and Johnson, Kotz and Kemp [34].

The simplest variation of a consecutive-k-out-of-n:F system is the *circular consecutive-k-out-of-n:F system* in which the n components are placed on a circle so that component number 1 and component number n become adjacent. Such a system has been considered by Derman *et al* [22].

A dual of a consecutive-k-out-of-n:F system is called *consecutive-k-out-of-n:G system.* This system works if and only if at least k consecutive components work. Kuo, Zhang and Zuo [40], studied the reliability of a consecutive-k-out-of-n:G system.

Section 2 of this chapter provides a direct formula for determining the number of path sets with known size in a consecutive-k-out-of-n:F system. In view of computational efforts, some efficiency properties of this formula with those of already known, are compared.

In Section 3, we derive an expression for determining the reliability function of a consecutive-k-out-of-n:F system with i.i.d. components. Other related results in the literature are also considered and reviewed.

In Section 4, we study the *structural matrix*. Seth and Ramamurthy [61], introduced the concept of a structural matrix and presented a unified approach for determining structural importance of components. Different measures of structural importance can then be obtained by using the structural matrix. We give a combinatorial expression for the elements of the structural matrix. Using the results of Section 2, we apply this approach to a consecutive-k-out-of-n:F system.

We shall use the following notations.

n: number of components in the system.

k: minimum number of consecutive failed components which cause the system to fail.

p, q: component reliability, unreliability; if all components have the same reliability p and $q = 1 - p$.

$h_k(p, n)$: system reliability of a con$|k|n$:F system with component reliability p.

$\lfloor x \rfloor$: largest integer less than or equal to x.

$\binom{n}{r}, \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k}$: usual binomial and multinomial coefficients.

$S(n, k)$: the set of all k-tuples $(n_1, n_2, \ldots, n_k)$ whose elements are nonnegative integers such that $\sum_{i=1}^{k} i n_i = n$.

$P_k(n)$: number of path sets of a con$|k|n$:F system.

$g_k(n, r)$: number of path sets of a con$|k|n$:F system with r failed components or equivalently $(n - r)$ components functioning, i.e., path sets of size $n - r$.

We also use con$|k|n$:F system as an abbreviation of consecutive-k-out-of-n:F system.

2.2 Path Sets With Known Size In A Con$|k|n$:F System

In this Section we give a closed formula for determining the number of path sets with known size of a con$|k|n$:F system.

Chiang and Niu [21], have shown that

$$g_2(n,r) = \binom{n-r+1}{r}. \tag{1}$$

In this section we now derive a generalized expression of (1).

Definition 1. A sequence of integers $\{x_n\}$ determined by the difference equation :

$$x_n = x_{n-1} + x_{n-2} + \cdots + x_{n-k}; \quad n > k, \tag{2}$$

with initial conditions $x_1 = a_1,\ x_2 = a_2,\ \ldots\ ,\ x_k = a_k$, is called a *Fibonacci sequence of order* k.

Seth [60], showed that the number of path sets, $P_k(n)$ of a con$|k|n$:F system follows the Fibonacci sequence of order k with the following initial conditions:

$a_1 = 2,\ a_2 = 2^2,\ a_3 = 2^3,\ \ldots,\ a_{k-1} = 2^{k-1},\ a_k = 2^k - 1.$

We define :

$$x_n = \begin{cases} 1 & \text{if} \quad n = -1, 0 \\ 0 & \text{if} \quad n \leq -2. \end{cases}$$

Now (2) holds true for all $n \geq 0$.

Miles [44], derived a closed form for the Fibonacci sequence $\{f_n\}$ of order k, defined as follows :

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k}, \qquad n \geq k$$

with initial conditions

$$f_0 = 0,\ f_1 = 0,\ \ldots\ ,\ f_{k-2} = 0,\ f_{k-1} = 1.$$

He showed that

$$f_n = \sum_{S(n-k+1,k)} \binom{\sum_{i=1}^{k} n_i}{n_1, n_2, \ldots, n_k}$$

using the following relation

$$\binom{\sum_{i=1}^{k} n_i}{n_1, n_2, \ldots, n_k} = \binom{\sum_{i=1}^{k} n_i - 1}{n_1 - 1, n_2, \ldots, n_k} + \binom{\sum_{i=1}^{k} n_i - 1}{n_1, n_2 - 1, n_3, \ldots, n_k} +$$

$$\cdots + \binom{\sum_{i=1}^{k} n_i - 1}{n_1, n_2, \ldots, n_{k-1}, n_k - 1} \qquad (3)$$

We assume that $\binom{\sum_{i=1}^{k} n_i - 1}{n_1, n_2, \ldots, n_{j-1}, n_j - 1, n_{j+1}, \ldots, n_k} = 0 \quad \text{if } n_j = 0.$

Lemma 1. In a con$|k|n$:F system, the number of path sets $P_k(n)$ is given by:

$$P_k(n) = \sum_{S(n+1,k)} \binom{\sum_{i=1}^{k} n_i}{n_1, n_2, \ldots, n_k}, \qquad n \geq 0. \qquad (4)$$

Proof. The proof is an immediate consequence of the fact that $P_k(n) = f_{n+k}$.

Bollinger [10], showed that $g_k(n, r)$ satisfies following recurrence relation.

$$g_k(n,r) = g_k(n-1,r) + g_k(n-2,r-1) + \cdots + g_k(n-k,r-k+1), \quad \text{for } k \leq r < n \qquad (5)$$

We have

$$g_k(n,r) = \begin{cases} \binom{n}{r} & \text{if } \ 0 \leq r \leq k-1, n \geq 1, n \geq r \\ 0 & \text{if } \ n = r \geq k \text{ or } n < r \text{ or } r < 0 \text{ or } n = 0 \end{cases} \qquad (6)$$

From (5) we note that $g_k(n, r)$ does not follow a Fibonacci sequence of order k.

We now consider a modification of (4) for computing $g_k(n, r)$ directly.

Lemma 2. Suppose r denotes the number of failed components in a path set of a con$|k|n$:F system. Then $0 \leq r \leq n - \lfloor n/k \rfloor$.

Proof. Suppose $n = sk+t$ where $0 \leq s, 0 \leq t \leq k-1$ and s, t are integers. We note that $s = \lfloor n/k \rfloor$ and $t = n - \lfloor n/k \rfloor k$. Consider the following path vector:

$$\underbrace{0\,0\ldots0}_{k-1}\;1\;\underbrace{0\,0\ldots0}_{k-1}\;1\;\ldots\ldots\ldots\;1\;\underbrace{0\,0\ldots0}_{k-1}\;1\;\underbrace{0\,0\ldots}_{t}$$

We know that the number of failed components r, in this path vector is maximum and we have $n = sk+t = s(k-1)+s+t$. Therefore the maximum value of $r = s(k-1)+t = n - \lfloor n/k \rfloor$

Remark 1. For $k = 2$, we know that $\lfloor (n+1)/2 \rfloor = n - \lfloor n/2 \rfloor$.

Remark 2. It is easy to see that

(i) minimum value of $\sum_{i=1}^{k} n_i$ subject to $\sum_{i=1}^{k} in_i = n+1$ is greater than or equal to n_k+(minimum value of $\sum_{i=1}^{k-1} n_i$ subject to $\sum_{i=1}^{k-1} in_i = n+1-kn_k$) and

(ii) for nonnegative integers x and k, $\lfloor x/(k-1) \rfloor \geq \lfloor (x+1)/k \rfloor$.

Lemma 3. Suppose $(n_1, n_2, \ldots, n_k) \in S(n+1, k)$. Then distinct values of $\sum_{i=1}^{k} n_i$ are $\lfloor n/k \rfloor + 1, \lfloor n/k \rfloor + 2, \ldots, n+1$.

Proof. It is obvious that the maximum value of $\sum_{i=1}^{k} n_i$ subject to $\sum_{i=1}^{k} in_i = n+1$ occurs for the case $n_1 = n+1$ and $n_i = 0$ for $i = 2, 3, \ldots, k$.
In this case, we have $\sum_{i=1}^{k} n_i = \sum_{i=1}^{k} in_i = n+1$.

We now show that the minimum value of $\sum_{i=1}^{k} n_i$ subject to $\sum_{i=1}^{k} in_i = n+1$ is $\lfloor n/k \rfloor + 1$. We prove this using induction. Suppose $k = 2$. We note that the minimum value of $n_1 + n_2$ subject to $n_1 + 2n_2 = n+1$ occurs when n_2 takes its maximum value. We have $n_2 \leq \lfloor (n+1)/2 \rfloor$. Hence the minimum value of $n_1 + n_2$ is :

$$\lfloor (n+1)/2 \rfloor + (n+1-2\lfloor (n+1)/2 \rfloor) = \lfloor n/2 \rfloor + 1.$$

Assume that the result is true for $k-1$, that is $\min \sum_{i=1}^{k-1} n_i$ subject to $\sum_{i=1}^{k-1} in_i = n+1$ is $\lfloor n/(k-1) \rfloor + 1$.

We show that $\min \sum_{i=1}^{k} n_i$ subject to $\sum_{i=1}^{k} in_i = n+1$ is $\lfloor n/k \rfloor + 1$. Suppose $s = \lfloor n/k \rfloor$. We consider two cases.

(I) Let $n+1 = (s+1)k$.

We show that $n_k > s$, suppose not, that is $n_k \leq s$, hence $n_k = s - y$ for some $0 \leq y \leq s$. Therefore $\sum_{i=1}^{k-1} in_i = n + 1 - kn_k = k(y+1)$. By induction hypothesis, we note that $\min \sum_{i=1}^{k-1} n_i$ subject to $\sum_{i=1}^{k-1} in_i = k(y+1)$ is $\left\lfloor \frac{k(y+1)-1}{k-1} \right\rfloor + 1$ and by part (ii) of Remark 2, it is greater than or equal to $y+2$. And using part (i) of Remark 2, it follows that $\min \sum_{i=1}^{k} n_i$ subject to $\sum_{i=1}^{k} in_i = n+1$ is greater than or equal to $s+2$. Whereas if we take $n_i = 0$ for $i = 1, 2, \ldots, k-1$ and $n_k = s+1$ then in this case we have $\sum_{i=1}^{k} n_i = s+1$ and also $\sum_{i=1}^{k} in_i = n+1$. Hence we have a contradiction as $\sum_{i=1}^{k} n_i \geq s+2$ subject to $\sum_{i=1}^{k} in_i = n+1$. Therefore we get $n_k > s$. Hence $\sum_{1}^{k} n_i \geq s+1$ and equality holds only in above case.

(II) Let $n = ks + t$ for $0 \leq t \leq k-2$ where $t = n - k\lfloor n/k \rfloor$.

If $n_k > s$, resulting in a contradiction (since $\sum_{i=1}^{k} in_i \geq k(s+1) > n+1$).

Hence, $n_k = s - y$, $0 \leq y \leq s$. We have $\sum_{i=1}^{k-1} in_i = n + 1 - kn_k = n - k\lfloor n/k \rfloor + ky + 1$. By induction hypothesis, we get

$$\min \sum_{i=1}^{k-1} n_i = \left\lfloor \frac{n - k\lfloor n/k \rfloor + ky}{k-1} \right\rfloor + 1 \geq \left\lfloor \frac{n - k\lfloor n/k \rfloor + ky + 1}{k} \right\rfloor + 1$$

which is equal to

$$\left\lfloor \frac{n+1}{k} - \lfloor n/k \rfloor + y \right\rfloor + 1 = y + 1.$$

Hence $\min \sum_{i=1}^{k} n_i \geq n_k + \min \sum_{i=1}^{k-1} n_i \geq s - y + y + 1 = s + 1.$
Equality holds if we assume

$$n_i = 0 \text{ for } i \neq k, n-k\lfloor n/k \rfloor+1, \ n_i = 1 \text{ for } i = n-k\lfloor n/k \rfloor+1 \text{ and } n_k = \lfloor n/k \rfloor.$$

This completes the proof.

We now consider the following problem.
$\sum_{i=1}^{k} n_i = s$ subject to $\sum_{i=1}^{k} i n_i = n+1$ for $s = \lfloor n/k \rfloor + 2, \lfloor n/k \rfloor + 3, \ldots, n.$
This problem has the following solution.

$$n_k = \left\lfloor \frac{n+1-s}{k-1} \right\rfloor, \quad n_{k-1} = \left\lfloor \frac{n+1-s-(k-1)n_k}{k-2} \right\rfloor,$$

$$n_{k-2} = \left\lfloor \frac{n+1-s-(k-1)n_k-(k-2)n_{k-1}}{k-3} \right\rfloor$$

$$n_2 = n+1-s-(k-1)n_k-(k-2)n_{k-1}-\cdots-2n_3 \text{ and } n_1 = s - \sum_{i=2}^{k} n_i.$$

We note that $\sum_{i=1}^{k} n_i = s$ and $\sum_{i=1}^{k} i n_i = n+1$. This completes the proof of the lemma.

Remark 3. From Lemma 2 and Lemma 3, we note that the number of distinct values of failed components (r) in a path set of a con$|k|n|$:F system and the number of distinct values of $\sum_{i=1}^{k} n_i$ where $(n_1, n_2, \ldots, n_k) \in S(n+1, k)$, are the same $(n - \lfloor n/k \rfloor + 1)$. Using this property, we can find the number of path sets with r failed components, as shown in Theorem 1.

Theorem 1. Suppose

$$g_k^*(n,r) = \begin{cases} \displaystyle\sum_{\begin{cases} S(n+1,k) \\ \sum_{i=1}^k n_i = n-r+1 \end{cases}} \binom{\sum_{i=1}^k n_i}{n_1, n_2, \ldots, n_k} & \text{if } 0 \le r < n \text{ or } 1 \le r = n < k \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where n, k and r are nonnegative integers. Then $g_k^*(n,r)$ satisfies (5).

Proof. Using (3) we can write

$$\sum_{\begin{cases} S(n+1,k) \\ \sum_{i=1}^k n_i = n-r+1 \end{cases}} \binom{n_1+n_2+\cdots+n_k}{n_1, n_2, \ldots, n_k} = \sum_{\begin{cases} S(n+1,k) \\ \sum_{i=1}^k n_i = n-r+1 \end{cases}} \sum_{j=1}^{k} \binom{n_1+n_2+\cdots+n_k-1}{n_1, n_2, \ldots, n_{j-1}, n_j-1, n_{j+1}, \ldots, n_k}$$

Note that for a given j, $1 \le j \le k$ such that $n_j > 0$ we have

$$\sum_{\begin{cases} S(n+1,k) \\ \sum_{i=1}^k n_i = n-r+1 \end{cases}} \binom{n_1+n_2+\cdots+n_k-1}{n_1, n_2, \ldots, n_{j-1}, n_j-1, n_{j+1}, \ldots, n_k} = \sum_{\begin{cases} S(n+1-j,k) \\ \sum_{i=1}^k n_i = n-r \end{cases}} \binom{n_1+n_2+\cdots+n_k}{n_1, n_2, \ldots, n_k}$$

(if we define $n_i' = \begin{cases} n_i & \text{if } i \neq j \\ n_j - 1 & \text{if } i = j \end{cases}$).

Therefore $g_k^*(n,r) = \sum_{j=1}^{k} g_k^*(n-j, r-j+1)$. This completes the proof of the theorem. ■

Remark 4. We give a direct combinatorial argument for $g_k^*(n,r)$ as stated in Theorem 1. We note that $g_k^*(n,r)$ can be interpreted as the number of ways of placing r failed components in among $n-r+1$ distinct spaces between $n-r$ working components (including the two ends) subject to at most $k-1$ failed components are being placed in any space. Now suppose n_i is the number of spaces which contain $i-1$ failed components, $i = 1, 2, \ldots, k$. Therefore we have

$\sum_{i=1}^{k} n_i =$total number of spaces$= n-r+1$ and

$\sum_{i=1}^{k} (i-1)n_i =$total number of failed components$= r$.

It implies that $\sum_{i=1}^{k} in_i = r + \sum_{i=1}^{k} n_i = r + (n-r+1) = n+1$.

With these two conditions on $(n_1, n_2, \ldots, n_k)$, the multinomial coefficient $\binom{\sum_{i=1}^{k} n_i}{n_1, n_2, \ldots, n_k}$, in fact gives the number of ways such that we have n_i different spaces contain $i-1$ failed components for $i = 1, 2, \ldots, k$.

Remarks. $g_k^*(n,r)$ also satisfies (6). For this we show that it reduces to $\binom{n}{r}$, when $0 \le r < k$. But before that we consider two special cases, that are required in the sequel.

(i) Suppose $1 \le n = r \le k-1$, then $\sum_{i=1}^{k} n_i = 1$ and since $\sum_{i=1}^{k} in_i = n+1$ it follows that, there exists j, $1 \le j \le k$ such that $n_j = 1, n_i = 0$, for $i \ne j$ and $j = n+1$. We have $r = n \le k-1$ or $j = n+1 \le k$. Hence

$$g_k^*(n,r) = 1!/1! = 1 = \binom{n}{r}.$$

(ii) If $r = 0 < n$, then $\sum_{i=1}^{k} n_i = n+1$ and $\sum_{i=1}^{k} in_i = n+1$, it follows that $n_1 = n+1, n_i = 0$, for $i = 2, 3, \ldots, k$.

$$\text{Hence } g_k^*(n,r) = \binom{\sum_1^k n_i}{n_1, n_2, \ldots, n_k} = (n+1)!/(n+1)! = 1 = \binom{n}{r}.$$

In these two special cases, $g_k^*(n,r)$ equals $g_k(n,r)$ as stated in (6). We now show that if $0 \le r \le k-1$ and $2 \le n$, then $g_k^*(n,r)$ satisfies the following relationship

$$g_k^*(n,r) = g_k^*(n-1,r) + g_k^*(n-1,r-1) \tag{8}$$

If $r = 0$, using special case (ii), (8) is trivially true as $n \ge 2$. If $2 \le r = n < k$, in view of special case (i), (8) is again trivially true. Now suppose $0 < r < n$, that is, $0 \le r-1 < n-1$. Therefore we can apply the recursive relation (5) for $g_k^*(n-1,r-1)$. We have

$$g_k^*(n-1,r) + g_k^*(n-1,r-1) =$$

$$g_k^*(n-1,r) + g_k^*(n-2,r-1) + g_k^*(n-3,r-2) + \cdots + g_k^*(n-k-1,r-k).$$

But as $r \le k-1$, that is $r-k \le -1$, it follows that $g_k^*(n-k-1,r-k) = 0$ (by definition of $g_k^*(n,r)$ as stated in (7)). Hence using (5) we have

$$g_k^*(n-1,r) + g_k^*(n-1,r-1) = g_k^*(n,r).$$

We now use the principle of induction to show that

$$g_k^*(n,r) = \binom{n}{r}, 0 \le r \le k-1 \tag{9}$$

For $r = 0$, we have shown that $g_k^*(n,r) = 1$ (special case (ii)).
For $r = 1$, we have $\sum_{i=1}^{k} n_i = n$, $\sum_{i=1}^{k} in_i = n+1$. This gives
$n_1 = n-1$, $n_2 = 1$, $n_i = 0$ for $i = 3,4,\ldots,k$.

Hence $g_k^*(n,r) = n!/(n-1)! = n = \binom{n}{r}$.

For $r = 2$, we have $\sum_{i=1}^{k} n_i = n-1$, $\sum_{i=1}^{k} in_i = n+1$.

Hence $n_2 + 2n_3 + 3n_4 + \cdots + (k-1)n_k = 2$ and $\sum_{i=1}^{k} n_i = n-1$.

We have only two solutions for $(n_1, n_2, \ldots, n_k)$ as follows. The first solution is

$n_1 = n-2$, $n_2 = 0$, $n_3 = 1$ and $n_i = 0$ for $i = 4, 5, \ldots, k$

and the second solution is

$n_1 = n-3$, $n_2 = 2$ and $n_i = 0$ for $i = 3, 4, \ldots, k$.

It follows that

$$g_k^*(n,r) = (n-1)!/(n-2)! + (n-1)!/[(n-3)!2!] = \binom{n}{2} = \binom{n}{r}$$

For $r = 3$, equations $n_2 + 2n_3 + 3n_4 + \cdots + (k-1)n_k = 3$ and $\sum_{i=1}^{k} n_i = n-2$ have only three feasible solutions as follows. The first solution is

$n_1 = n-4$ $(n \geq 4)$, $n_2 = 1$, $n_3 = 1$ and $n_i = 0$ for $i = 4, 5, \ldots, k$

the second solution is

$n_1 = n-5$ $(n \geq 5)$, $n_2 = 3$ and $n_i = 0$ for $i = 3, 4, \ldots, k$

and the third solution is

$n_1 = n-3$, $n_4 = 1$ and $n_i = 0$ for $i = 2, 3, 5, \ldots, k$.

Hence we have

$$g_k^*(n,r) = (n-2)!/n-4)! + (n-2)!/[3!(n-5)!] + (n-2)!/(n-3)! = \binom{n}{3} = \binom{n}{r}$$

after simplification.

Now suppose (9) is true for $3 \leq r < k-1$ (as induction hypothesis).
If $r+1 = n$, we have shown (special case (i)) $g_k^*(n, r+1) = g_k^*(n,n) = 1$ (since $4 \leq n = r+1 \leq k-1$) and validity of (9) is trivial.
Suppose $r+1 < n$, that is $x = n-(r+2) \geq 0$. We note that $4 \leq r+1 \leq k-1$ and $n - x = r+2 \geq 5$. Hence using (8) repeatedly we have

$$g_k^*(n, r+1) = g_k^*(n-1, r+1) + g_k^*(n-1, r)$$

$$g_k^*(n-1, r+1) = g_k^*(n-2, r+1) + g_k^*(n-2, r)$$

$$g_k^*(n-2, r+1) = g_k^*(n-3, r+1) + g_k^*(n-3, r)$$

$$g_k^*(n-x, r+1) = g_k^*(n-x-1, r+1) + g_k^*(n-x-1, r)$$

But $n-x-1 = r+1$. This gives us $g_k^*(n-x-1, r+1) = 1$ (since $4 \leq n-x-1 = r+1 \leq k-1$ and using special case (i)) and by induction hypothesis we get

$$g_k^*(n, r+1) = \binom{n-1}{r} + \binom{n-2}{r} + \binom{n-3}{r} + \cdots + \binom{r+1}{r} + 1$$

We know that

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n-1}{r} = \binom{n}{r+1}, \; n \geq r+1$$

Hence (9) is true for all $0 \leq r \leq k-1$. That is $g_k^*(n, r)$ satisfies (6).

In the remainder of this section we consider and compare the available results in the literature.

Remark 5. Lambiris and Papastavridis [41], using the generating function of $g_k(n, r)$, gave an expression for $g_k(n, r)$ as follows

$$g_k(n, r) = \sum_{j=0}^{n-r+1} (-1)^j \binom{n-r+1}{j} \binom{n-kj}{n-r} \tag{10}$$

In their formula, we note that $n - r \leq n - kj$, hence $j \leq r/k$. Therefore $0 \leq j \leq \min\{n-r+1, \lfloor r/k \rfloor\} = s_{r,k}$.

Remark 6. The above formula for $g_k(n, r)$, was also obtained by Hwang [32] independently. In [32], it is stated that the number of terms in $g_k(n, r)$ is n/k. Here we note that this number is $s_{r,k}$. In Lemma 5, it is shown that $s_{r,k} = \lfloor r/k \rfloor$.

Lemma 4. If r is the number of failed components in a path set of a con$|k|n$:F system, then the minimum value of the system size is given by $n = r + \lfloor (r-1)/(k-1) \rfloor$.

Proof. Suppose $r < k$, hence the minimum value of the system size is $n = r$. Now suppose $r \geq k$. We consider two cases as follows.

(i) Let $r = s(k-1)$ for some integer $s > 1$. We know that the following path vector has the minimum value of the system size.

$$\underbrace{000\cdots0}_{k-1}\ 1\ \underbrace{000\cdots0}_{k-1}\ 1\ \ldots\ldots\ 1\ \underbrace{000\cdots0}_{k-1}$$

We note that

$n = s(k-1)+s-1 = r+s-1 = r+\lfloor r/(k-1)\rfloor -1 = r+\lfloor (r-1)/(k-1)\rfloor$.

(ii) Let $r = s(k-1)+t$ for some integers $0 < t \leq k-2$ and $s \geq 1$. In this case the minimum value of system size occurs in following path vector

$$\underbrace{000\cdots0}_{k-1}\ 1\ \underbrace{000\cdots0}_{k-1}\ 1\ \ldots\ldots\ \underbrace{000\cdots0}_{k-1}\ 1\ \underbrace{00\cdots0}_{t}$$

We note that

$n = s(k-1)+s+t = r+\lfloor r/(k-1)\rfloor = r+\lfloor (r-1)/(k-1)\rfloor$.

This completes the proof of the lemma. ∎

Therefore whenever we refer to a path sets of size $n-r$ it is assumed that $n \geq r + \lfloor (r-1)/(k-1)\rfloor$ (otherwise, we have no path set of size $n-r$).

Lemma 5. $s_{r,k} = \lfloor r/k \rfloor$.

Proof. If $r < k$, then $\lfloor r/k \rfloor = 0$ and also $s_{r,k} = 0$ (since $n - r + 1 > 1$).

If $r \geq k$, then it is easy to verify that $(r-1)/(k-1) \geq r/k$. Hence $\lfloor (r-1)/(k-1)\rfloor \geq \lfloor r/k \rfloor$.

On the other hand we have $n - r + 1 > n - r \geq \lfloor (r-1)/(k-1)\rfloor$. Therefore $n - r + 1 > \lfloor r/k \rfloor$ and hence $s_{r,k} = \lfloor r/k \rfloor$. ∎

Lemma 6. For a given nonnegative integer r, let C_r^2 denote the number of solutions of $n_1 + 2n_2 = r$, where n_1, n_2 are nonnegative integers.
Then $C_r^2 = \lfloor r/2 \rfloor + 1$.

Proof. This can be verified easily.

We now consider some special cases of Theorem 1. For $k = 2, 3$, $g_k^*(n,r)$ has a simple form as follows.
If $k = 2$ we have

$$g_2^*(n,r) = \sum_{\begin{cases} n_1 + 2n_2 = n+1 \\ n_1 + n_2 = n - r + 1 \end{cases}} \binom{n-r+1}{n_1, n_2} = \binom{n-r+1}{r}.$$

This is same as Chiang and Niu's [21] formula.
If $k = 3$ we have

$$g_3^*(n,r) = \sum_{\begin{cases} n_1 + 2n_2 + 3n_3 = n+1 \\ n_1 + n_2 + n_3 = n - r + 1 \end{cases}} \binom{n-r+1}{n_1, n_2, n_3} = \sum_{\begin{cases} n_2 + 2n_3 = r \\ n_1 + n_2 + n_3 = n - r + 1 \end{cases}} \binom{n-r+1}{n_1, n_2, n_3}.$$

(I) Suppose $r \leq n - r + 1$ (or equivalently $r \leq \lfloor (n+1)/2 \rfloor$) from Lemma 6, we have

$$g_3^*(n,r) = \sum_{j=0}^{\lfloor r/2 \rfloor} \binom{n-r+1}{n - 2r + 1 + j, r - 2j, j}$$

(II) And if $r > \lfloor (n+1)/2 \rfloor$ we then have

$$g_3^*(n,r) = \sum_{j=0}^{\lfloor (2n-3r)/2 \rfloor + 1} \binom{n-r+1}{j, 2n - 3r - 2j + 2, 2r - n - 1 + j}$$

Hence we have

$$g_3^*(n,r) = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor r/2 \rfloor} \binom{n-r+1}{n-2r+1+j, r-2j, j} & \text{if } r \leq \lfloor (n+1)/2 \rfloor \\ \displaystyle\sum_{j=0}^{\lfloor (2n-3r)/2 \rfloor + 1} \binom{n-r+1}{j, 2n-3r-2j+2, 2r-n-1+j} & \text{if } r > \lfloor (n+1)/2 \rfloor \end{cases} \tag{11}$$

Lemma 7. For a given nonnegative integer r, suppose C_r^3 denotes the number of solutions of equation $n_1 + 2n_2 + 3n_3 = r$ and let $r = 6s + t$, $s = \lfloor r/6 \rfloor$ and $0 \leq t \leq 5$. We then have

$C_r^3 = 3s^2 + (t+3)s + a_t$, where $a_t = \begin{cases} t & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$ and n_i's, r, s and t are nonnegative integers.

Remark 7. From Lemma 7, we can verify that for the cases where $k \geq 4$, (10) is more efficient than (7) in view of computational efforts.

We now consider and compare available results in the literature.

Remarks. We note that the result of Theorem 1 of this section (formula (7)) can be used not only to derive the reliability of a consecutive-k-out-of-n:F system (that is given in the next section), but also for obtaining the distribution of longest success run in n independent trials with success probability p $(0 < p < 1)$.

Philippou and Makri [52], have given following formula:

$$P(L_n \leq k, S_n = r) = p^r q^{n-r} \sum_{j=0}^{k} \sum_{\substack{S(n-j, k+1) \\ \sum_1^{k+1} n_i = n-r}} \binom{\sum_1^{k+1} n_i}{n_1, n_2, \ldots, n_{k+1}}$$

where S_n and L_n are random variables denoting respectively the number of successes and the length of the longest success run in n independent trials with success probability p.

In their formula, if we replace k by $k-1$ and ignore p and $q = 1-p$, we then obtain $g_k(n,r)$ as follow:

$$g_k(n,r) = \sum_{j=0}^{k-1} \sum_{\substack{S(n-j,k) \\ \sum_1^k n_i = n-r}} \binom{\sum_1^k n_i}{n_1, n_2, \ldots, n_k}.$$

For example using this formula for $n=5$, $k=3$ and $r=3$ we have:

$$g_3(5,3) = \binom{2}{0,1,1} + \binom{2}{0,2,0} + \binom{2}{1,0,1} + \binom{2}{1,1,0} = 2+1+2+2 = 7$$

and using top formula in equation (11), we have:

$$g_3^*(5,3) = \binom{3}{0,3,0} + \binom{3}{1,1,1} = 1+6 = 7.$$

Although their formula for $g_k(n,r)$ is similar to formula (7) but it contains double summations, whereas formula (7) contains a single summation.

Remark 8. K.G.Ramamurthy [56], introduced a simple formula for $h_k(p,n)$. Using this he obtained a closed formula for $L_n(x) = \sum_{r=0}^{n-\lfloor n/k \rfloor} g_k(n,r)x^r$, the generating function of $g_k(n,r)$. He also proved that

$$P_k(n) = 2^n - \sum_{j=1}^{\lfloor (n+1)/(k+1) \rfloor} (-1)^{j-1} \binom{n-jk}{j-1} \frac{n-j(k-1)+1}{j} 2^{n-j(k+1)}$$

(Corollary p. 218) and

$$g_k(n,r) = \binom{n}{r} - (n-r+1) \sum_{j=1}^{s_{r,k}} \frac{(-1)^{j-1}}{n-jk+1} \binom{n-jk+1}{j, n-r-j+1, r-jk}$$

(Theorem 3 p. 219), where $s_{r,k} = min\{n-r+1, \lfloor r/k \rfloor\}$.
He claimed that two above formulae for $P_k(n)$ and $g_k(n,r)$ are new.

In fact we have proved that $s_{r,k} = \lfloor r/k \rfloor$ in Lemma 5.

Remark 9. Note that the expression

$$g_k(n,r) = \binom{n}{r} - (n-r+1)\sum_{j=1}^{\lfloor r/k \rfloor} \frac{(-1)^{j-1}}{n-jk+1} \binom{n-jk+1}{j, n-r-j+1, r-jk}$$

as given by Ramamurthy [56], can be also obtained by using Lambiris and Papastavridis [41] formula. In [41], it is proved that

$$g_k(n,r) = \sum_{j=0}^{\lfloor r/k \rfloor} (-1)^j \binom{n-r+1}{j} \binom{n-kj}{n-r}.$$

We can write

$$g_k(n,r) = \sum_{j=0}^{\lfloor r/k \rfloor} (-1)^j \frac{(n-r+1)(n-kj)!}{j!(n-r-j+1)!(r-kj)!}$$

$$= \binom{n}{r} + (n-r+1)\sum_{j=1}^{\lfloor r/k \rfloor} (-1)^j \frac{(n-kj)!}{j!(n-r-j+1)!(r-kj)!}$$

$$= \binom{n}{r} - (n-r+1)\sum_{j=1}^{\lfloor r/k \rfloor} (-1)^{j-1} \frac{1}{n-kj+1} \binom{n-kj+1}{j, n-r-j+1, r-kj}$$

We note that in this formula the number of trinomial coefficients is $\lfloor r/k \rfloor$. In the next lemma, we compare the number of trinomial coefficients of $g_3^*(n,r)$ with that of the Ramamurthy formula.

Lemma 8. Suppose r is the number of failed components in a path set of a con|3|n:F system. If $r \geq \left\lfloor \frac{6(n+1)}{11} + 0.5 \right\rfloor$ then the number of trinomial coefficients of $g_3^*(n,r)$ as given in formula (11), is less than $\lfloor r/3 \rfloor$.

Proof. It is obvious that if $r \geq \left\lfloor \frac{6(n+1)}{11} + 0.5 \right\rfloor$ then $r > \left\lfloor \frac{n+1}{2} \right\rfloor$. We note that the number of trinomial coefficients in $g_3^*(n,r)$ is $\left\lfloor \frac{2n-3r}{2} \right\rfloor + 1$. Hence for a fixed n, by increasing values of r, the number of trinomial coefficients in $g_3^*(n,r)$ decreases, whereas $\lfloor r/k \rfloor$ increases. We have

$$\frac{2n-3r}{2} + 1 - \frac{r}{3} = n + 1 - \frac{11r}{6} \leq n + 1 - \frac{11}{6}\left\lfloor \frac{6(n+1)}{11} + 0.5 \right\rfloor$$

which is less than or equal to $n + 1 - \frac{11}{6}\left(\frac{6(n+1)}{11}\right) = 0$. On the other hand we know that

$$\left\lfloor \frac{2n-3r}{2} \right\rfloor + 1 - \left\lfloor \frac{r}{3} \right\rfloor \leq \frac{2n-3r}{2} + 1 - \frac{r}{3} + 0.5$$

Therefore we get the result

$$\left\lfloor \frac{2n+1}{2} \right\rfloor + 1 - \left\lfloor \frac{r}{3} \right\rfloor \leq 0.5$$

and since $\left\lfloor \frac{2n+1}{2} \right\rfloor + 1 - \left\lfloor \frac{r}{3} \right\rfloor$ is an integer number hence we have $\left\lfloor \frac{2n+1}{2} \right\rfloor + 1 - \left\lfloor \frac{r}{3} \right\rfloor \leq 0$. This completes the proof of the lemma. ■

Remarks. The number of path sets with known size in a circular consecutive-k-out-of-n:F system is given by Hwang and Yao [33].

2.3 Reliability Function of A Con$|k|n$:F System With I.I.D. Components

Most of the literature on con$|k|n$:F system is devoted to derive recursive relationships for computing system reliability. Bollinger [10], gave a direct combinatorial method for determining the system failure probability. In his method the coefficients in failure probability function are obtained recursively. Bollinger and Salvia [11], introduced a counting scheme for determining the reliability function. Their method requires the construction of triangular tables. Hwang [31], gave a simple formula for computing system reliability for the case $n \leq 2k$. For the case $n > 2k$, he introduced a recursive formula. Chen and Hwang [19], derived a direct formula for computing system reliability.

Using Theorem 1 and Lemma 2, we get the following direct formula for computing system reliability of a con$|k|n$:F system for i.i.d. case.

$$h_k(p,n) = \sum_{r=0}^{n-\lfloor n/k \rfloor} g_k(n,r) q^r p^{n-r} = p^n \sum_{r=0}^{n-\lfloor n/k \rfloor} (q/p)^r \sum_{\begin{cases} S(n+1,k) \\ \sum_{i=1}^{k} n_i = n-r+1 \end{cases}} \binom{n_1+n_2+\cdots+n_k}{n_1,n_2,\ldots,n_k}. \tag{12}$$

Chen and Hwang [19], gave the following direct formula

$$h_k(p,n) = 1 - q^k \sum_{r=0}^{n-k} q^r \sum_{S(r,k)} \binom{n_1+n_2+\cdots+n_k}{n_1,n_2,\ldots,n_k} (p/q)^{\sum_{i=1}^{k} n_i}. \tag{13}$$

One of the advantages of (12) over (13) is that in (12), the second sum has no power of (p/q).
We note that if $n - \lfloor n/k \rfloor \leq n - k$ (or equivalently $n \geq k^2$) then number of terms for computing system reliability using formula (12) is less than (13). We also note that for some values of n ($n < k^2$), formula (12) is more efficient, in view of computational efforts. Here we give an example.

Example 1. Table 1 gives the number of multinomial coefficients, that are computed in (12) and (13), for some values of k and n.

Table 1

k	2		3					4					5			
n	3	4	6	7	8	9	10	11	12	13	14	15	13	14	15	16
formula (12)	3	3	8	10	12	15	16	34	39	47	54	63	70	82	98	114
formula (13)	2	4	7	11	16	23	31	33	42	69	91	117	60	83	113	150

From Table 1, we see that for $k = 2$ and $n \geq 4$, $k = 3$ and $n \geq 7$, $k = 4$ and $n \geq 12$ and $k = 5$ and $n \geq 14$, formula (12) requires fewer computations.

We now consider other results in the literature.

Remarks. Philippou et al [51], introduced the geometric distribution of order k ($G_k(x, p)$) as follows:

$$P(X = x) = \sum_{S(x-k,k)} \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k} p^x (q/p)^{\sum_1^k n_i},\ x \geq k$$

and then showed that N_k is distributed as $G_k(x, p)$, where N_k is a random variable denoting the number of trials until the occurrence of the k-th consecutive success in independent trials with success probability p.

We note that, in $P(N_k = x)$, if we replace p by $q = 1 - p$ we get the result:

$$1 - h_k(p, n) = P(N_k \leq n) = \sum_{x=k}^{n} \sum_{S(x-k,k)} \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k} q^x (p/q)^{n_1 + \cdots + n_k}$$

which matches with formula (13), given by Chen and Hwang [19].

Philippou and Makri [52], also showed that $N_n^{(k)}$ is distributed as follows:

$$P(N_n^{(k)} = x) = \sum_{i=0}^{k-1} \sum_{S(n-i-kx,k)} \binom{n_1 + \cdots + n_k + x}{n_1, \ldots, n_k, x} p^n (q/p)^{\sum_1^k n_i}$$

for $x = 0, \ldots, \lfloor n/k \rfloor$. Where $N_n^{(k)}$ is a random variable denoting the number of success runs of length k in n independent trials with success probability p. They denoted the distribution on $N_n^{(k)}$ by $B_k(n, p)$ as the binomial distribution of order k. Here we also note that, in $P(N_n^{(k)} = x)$, if we replace p by $q = 1-p$ we then have:

$$h_k(p,n) = P(N_n^{(k)} = 0) = \sum_{i=0}^{k-1} \sum_{S(n-i,k)} \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k} q^n (p/q)^{n_1+\cdots+n_k}$$

Therefore in view of the above formulae, we note that formula (12) presents another expression for $1 - P(N_k \leq n)$ and $P(N_n^{(k)} = 0)$.

Remark 10. Lambiris and Papastavridis [41], using their formula for determining $g_k(n,r)$ (as stated in formula (10)), gave the following formula for determinig reliability function

$$h_k(p,n) = \sum_{i=0}^{n} \binom{n-ik}{i} (-1)^i (pq^k)^i - q^k \sum_{i=0}^{n} \binom{n-ik-k}{i} (-1)^i (pq^k)^i.$$

This formula consists of two single sums.

Remark 11. K.G. Ramamurthy [56], introduced a simpler formula for determining $h_k(p,n)$ as follows.

$$h_k(p,n) = 1 - \sum_{j=1}^{\lfloor (n+1)/(k+1) \rfloor} (-p)^{j-1} (1-p)^{jk} \binom{n-jk}{j-1} \left\{ \frac{n-jk+1}{j} p + 1 - p \right\}.$$

(For details see Corollary 3, p. 217 in [56]).

Remark 12. M. Muselli [46], showed that

$$Pr\,(L_n \leq k-1) = \sum_{j=0}^{\lfloor (n+1)/(k+1) \rfloor} (-1)^j p^{jk} (1-p)^{j-1} \left\{ \binom{n-jk}{j-1} + (1-p) \binom{n-jk}{j} \right\}$$

(Corollary 1, p.127) where L_n is the length of the longest success run in n *Bernoulli* trials.

Remarks. We note that if we replace p by $1-p$ in Muselli formula we then get $h_k(p,n)$ as given by Ramamurthy formula.

Remark 13. Using a simple combinatorial argument, E.A. Peköz and S.M. Ross [48], derived a formula for the failure probability of a con$|k|n$:F system as follows

$$1 - h_k(p,n) = \sum_{m=1}^{\lfloor (n+1)/(k+1) \rfloor} (-1)^{m+1} \left[\binom{n-mk}{m} + \frac{1}{p} \binom{n-mk}{m-1} \right] (q^k p)^m.$$

Remark. We note that all formulae given by Ramamurthy [56], Muselli [46] and Peköz and Ross [48], using different approaches are equivalent.

2.4 Combinatorial Approach for Structural Matrix

A number of structural importance measures like Birnbaum importance [8], Barlow-Proschan importance [7] and Butler cut importance ranking [12] have been considered in the literature using varying approaches. A structural importance measure requires only the knowledge of the structure function of the system. These measures are more suitable during system design and development phases when the component reliabilities are generally not known. On the other hand reliability importance measures require additional information about component reliabilities apart from the structure function.

A unified approach for determining structural importance was introduced by Seth and Ramamurthy [61]. This approach uses the concept of structural matrix which needs to be determined from the simple form of the structure function. Different measures of structural importance of components can then be obtained using the structural matrix. In this section, we first give a combinatorial expression for the elements of a structural matrix in terms of the number of path sets. This approach provides a new look for better understanding of the structural importance problem. We then apply this approach to a con$|k|n$:F system. The concepts of path set, cut set, structure function, semi-coherent structure and coherent structure will be used in this section, as defined in Chapter 1.

A structure function ϕ, can be expressed as

$$\phi(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j \quad \text{for} \quad \mathbf{x} \in B^n.$$

This expression is called the *simple form* of $\phi(\mathbf{x})$ (see Ramamurthy [55]).
$N = \{1, 2, \ldots, n\}$ is the set of components of the system and $B = \{0, 1\}$.
For $S = \emptyset$, we take $\prod_{j \in S} x_j = 1$ and the a_S's are some integers. The structural matrix $M(\phi) = (m(\phi))_{ij}$ is a square matrix of order n with elements given by

$$m(\phi)_{ij} = \sum_{S \in A_{ij}} a_S$$

where

$$A_{ij} = \{S : S \subseteq N, \;\; i \in S, \;\; \text{and} \;\; |S| = j\}$$

for all $i, j \in N$.

Different measures of importance of component have been proposed in the literature using different approaches like critical path vectors, minimal path (cut) sets, etc. The concept of structural matrix was introduced by Seth and Ramamurthy [61], to develop a unified approach for calculating the structural importance of components. They showed that $M(\phi)\mu$ gives the vector of Birnbaum structural importance of components where $\mu \in R^n$ is a column vector with $\mu_j = (1/2)^{j-1}$ $(j = 1, 2, ..., n)$, similarly $M(\phi)\mu$ gives the vector of Birnbaum measure of reliability importance of components where $\mu_j = p^{j-1}$. They also showed that $M(\phi)\mu$ gives the vector of Barlow-Proschan structural importance if we take $\mu_j = 1/j$, $(j = 1, 2, ..., n)$. Furthermore they have proved that Butler's cut importance ranking of components (due to Butler [12]) is equivalent to the *lexicographic ordering* of the rows of the structural matrix of the dual of a structure ϕ.

The determination of $M(\phi)$ as per their methods requires the availability of $\phi(\mathbf{x})$ in its simple form. However, a problem arises if we know only the path sets or cut sets of the system and $\phi(\mathbf{x})$ is not known in the required form.

We consider a different route for calculating the structural matrix. We basically develop a combinatorial approach using path sets instead of the simple form of a structure for determining $M(\phi)$.

Combinatorial Approach

For any $S \subseteq N$, we associate a binary vector $e^S \in \{0,1\}^n$ where $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. Let p_{ij} denote the number of path sets of ϕ that contain component i and are of cardinality j. Further let q_{ij} represent the number of path sets of ϕ that do not contain component i but are of cardinality j. Obviously, we have

$$p_{ij} = \sum_{S \in A_{ij}} \phi(e^S) \quad \text{and} \quad q_{ij} = \sum_{S \in \bar{A}_{ij}} \phi(e^S)$$

where

$$\begin{aligned} A_{ij} &= \{S : S \subseteq N,\ i \in S \text{ and } |S| = j\} \text{ and} \\ \bar{A}_{ij} &= \{S : S \subseteq N,\ i \notin S \text{ and } |S| = j\} \text{ for } i, j \in N. \end{aligned}$$

Also we have $p(j) = p_{ij} + q_{ij}$, where $p(j)$ is the number of path sets of ϕ which are of size j.

Lemma 9. Let ϕ be a structure function defined on N. For any $S \subseteq N$, we have

$$a_S = \sum_{T \subseteq S} (-1)^{|S-T|} \phi(e^T).$$

Proof. We have $\phi(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j$ for $\mathbf{x} \in B^n$. This gives us for any $S \subseteq N$

$$\phi(e^S) = \sum_{T \subseteq S} a_T$$

and the required result follows from the well known Mobius Inversion Theorem (see Ramamurthy [55], p.31).

Lemma 10. For a semi-coherent structure ϕ, the element $m(\phi)_{ij}$ of the structural matrix $M(\phi)$ is given by

$$m(\phi)_{ij} = p_{ij} + \sum_{r=1}^{j-1}(-1)^{j-r}\left\{\binom{n-r}{j-r}p_{ir} + \binom{n-r-1}{j-r-1}q_{ir}\right\}.$$

Proof. For $T \subseteq N, 1 \leq i,j \leq n$, define $h_{ij}(T) = |\{S : S \in A_{ij}, S \supseteq T\}|$, we then have

$$\begin{aligned} h_{ij}(T) &= 0 && \text{if } |T| > j \\ & && \text{or } |T| = j \text{ and } i \notin T \\ &= 1 && \text{if } |T| = j \text{ and } i \in T \\ &= \tbinom{n-r}{j-r} && \text{for } i \in T, \text{ and } |T| = r < j \ (1 \leq r < j) \\ &= \tbinom{n-r-1}{j-r-1} && \text{for } i \notin T \text{ and } |T| = r < j \ (0 \leq r < j). \end{aligned}$$

This gives us

$$\begin{aligned} m(\phi)_{ij} &= \sum_{S\in A_{ij}} a_S \\ &= \sum_{S\in A_{ij}} \sum_{T\subseteq S}(-1)^{|S-T|}\phi(e^T) \\ &= \sum_{T\subseteq N} h_{ij}(T)\ \phi(e^T)(-1)^{j-|T|} \\ &= \sum_{i\in T\subseteq N}(-1)^{j-|T|}\ h_{ij}(T)\phi(e^T) + \sum_{i\notin T\subseteq N}(-1)^{j-|T|}\ h_{ij}(T)\phi(e^T) \\ &= p_{ij} + \sum_{r=1}^{j-1}(-1)^{j-r}\left[\binom{n-r}{j-r}p_{ir} + \binom{n-r-1}{j-r-1}q_{ir}\right] \end{aligned}$$

Remark 14. $m(\phi)_{ij} = 0$ for $j < r_0$ and $i = 1, 2, ..., n$ where r_0 is the smallest integer for which $p(r_0) > 0$.

Remark 15. If the cut sets of the system are known we can use the same approach to get $M(\phi^D)$ where ϕ^D is a dual structure of ϕ given by $\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x})$ for $\mathbf{x} \in B^n$.

Example 2. Consider the system of components given in Figure 1.

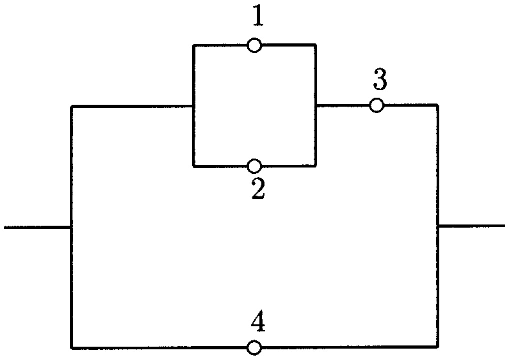

Figure 1.

The minimal path sets of this system are : $\{1,3\}, \{2,3\}$ and $\{4\}$. Also we have

$$((p_{ij})) = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix}, \quad ((q_{ij})) = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$p(1) = 1,\ p(2) = 5,\ p(3) = 4,\ p(4) = 1$. This gives us

$$\begin{aligned} m(\phi)_{i1} &= p_{i1} \\ m(\phi)_{i2} &= p_{i2} - 3p_{i1} - q_{i1} \\ m(\phi)_{i3} &= p_{i3} + 3p_{i1} + 2q_{i1} - 2p_{i2} - q_{i2} \\ m(\phi)_{i4} &= p_{i4} - p(1) + p(2) - p(3). \end{aligned}$$

for $i = 1, 2, 3, 4$. Hence we obtain

$$M(\phi) = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -3 & 1 \\ 1 & 0 & -2 & 1 \end{bmatrix}$$

We now consider the structural matrix of a con$|k|n$:F system.

Using Lemma 10, we note that for determining $m(\phi)_{ij}$ we need to compute p_{ir} and q_{ir}, for $r = 1, 2, \ldots, j$. We also note that:

$$p_n(r) = p_{ir} + q_{ir} = g_k(n, n-r)$$

where $p_n(r)$ is the number of path sets of a con$|k|n$:F system which are of size r (or equivalently the path sets containing $n-r$ failed components).
Using Lambiris and Papastavridis' formula [41], (formula (10)) and in view of Lemma 5, we then have

$$p_n(r) = \sum_{m=0}^{\lfloor\frac{n-r}{k}\rfloor} (-1)^m \begin{pmatrix} r+1 \\ m \end{pmatrix} \begin{pmatrix} n-mk \\ r \end{pmatrix}, \quad 0 < r \leq n \text{ or } r = 0,\ n < k \tag{14}$$

Using formula (6), we also have

$$p_n(r) = \begin{cases} \begin{pmatrix} n \\ r \end{pmatrix} & \text{if } n-k+1 \leq r \leq n,\ n \geq 1,\ r \geq 0 \\ 0 & \text{if } r = 0,\ n \geq k \text{ or } r < 0 \text{ or } r > n \text{ or } n = 0 \end{cases} \tag{15}$$

Therefore for determining $m(\phi)_{ij}$ we need only to compute p_{ir}.

Lemma 11. Let p_{ir} denote the number of path sets of a con$|k|n$:F system that contain component i, which are of size r. If $1 < i < n$, we then have

$$p_{ir} = \sum_{\ell=o}^{r-1} p_{i-1}(\ell) p_{n-i}(r-1-\ell)$$

where $p_n(r)$ is given by formulae (14) and (15).

Proof. Consider two subsystems a con$|k|i-1$:F system and a con$|k|n-i$:F system of the original system, where a con$|k|i-1$:F system is made of first $i-1$ components of original system and a con$|k|n-i$:F system is made of last $n-i$ components of original system. Suppose P is a path set of a con$|k|n$:F system that contains component i, and of size r. We note that P is of the form

$$P = P_1 \cup \{i\} \cup P_2$$

where P_1 is a path set of a con$|k|i-1$:F system and P_2 is a path set of a con$|k|n-i$:F system such that $|P_1 \cup P_2| = r-1$. This completes the proof of the lemma. ∎

Remark 16.

(i) For special cases $i = 1$ and $i = n$ we note that $p_{ir} = p_{n-1}(r-1)$.

(ii) In view of Lemma 2, if r denotes the size of a path set of a con$|k|n$:F system, then $\left\lfloor \frac{n}{k} \right\rfloor \leq r \leq n$.

Remark 17. Seth [60], showed that if $M(\phi)$ represents the structural matrix of a con$|k|n$:F system, then $m(\phi)_{ij} = m(\phi)_{n+1-i,j}$ for $i = 1, 2, \ldots, n$ and all $j = 1, 2, \ldots, n$.

In view of Remark 17 and part (ii) of Remark 16, We note that it is not necessary to compute all the n^2 elements of structural matrix of a con$|k|n$:F system.

Chapter 3

Generation of Minimal Path Sets and Critical Vectors in a Consecutive-k-out-of-n:F System

3.1 Introduction

In this chapter we study the concepts of *minimal path sets*, *critical vectors* and also *minimal cut sets* of a consecutive-k-out-of-n:F (con$|k|n$:F) system. These concepts play a crucial role in the study of *measures of component importance* and *system reliability* of a con$|k|n$:F system. The problem of generating all minimal path sets of the system was considered by Chan et al. [14]. They proposed a recursive procedure to find all minimal path sets of a con$|k|n$:F system. Their method starts with the generation of all minimal path sets of a con$|k|2k$:F system and uses them to generate all minimal path sets of a con$|k|2k+1$:F system and so on. This method recursively generates all minimal path sets of $n-2k+1$ different systems resulting in a large number of repetitions.

Section 2 gives necessary and sufficient conditions for a subset of components to be a minimal path set of the system.

In Section 3 using a linear ordering on the subsets of components (*lexicographical ordering*) we present a nonrecursive algorithm for lexicographically generating and listing the collection of all minimal path sets of a linear con$k|n$:F system. Our algorithm generates minimal path sets only for the con$|k|n$:F system. Hence unnecessary generations of the minimal path sets is avoided as in the case of Chan et al. [14] procedure. We also study the minimal path sets of a circular con$|k|n$:F system and show that all minimal path sets of a circular system can be generated using the minimal path sets of a linear system.

In Section 4 we present a nonrecursive algorithm for generating all minimal path sets containing a given component of the system. This algorithm is used for evaluating the Vesely-Fussell reliability measure of component importance in a con$|k|n$:G system, in Chapter 4.

Section 5 gives a nonrecursive algorithm for generating all critical vectors for a given component in a con$|k|n$:F system. This is applied for computing the Birnbaum reliability measure of component importance, in Chapter 4.

3.2 Characterization of Minimal Path Sets

Recall that a con$|k|n$:F system fails if and only if it has at least k consecutive failed components. Hence a minimal cut set of a con$|k|n$:F system is of the form $\{i, i+1, \ldots, i+k-1\}$, $i = 1, 2, \ldots, n-k+1$. However there is no such simple representation for a minimal path set of a con$|k|n$:F system.

We shall denote by $\alpha_k(n)$, the collection of all minimal path sets of a con$|k|n$:F system.

It is easy to verify that a con$|k|n$:F system is coherent for $n \geq k$. We extend the definition of a con$|k|n$:F system for the cases where $n = 0, 1, \ldots, k-1$. In these cases we adapt the convention that a con$|k|n$:F system is always in an operating state irrespective of the states of the components. With this convention, we note that for $n = 0, 1, \ldots, k-1$ a con$|k|n$:F system is noncoherent, it has no cut sets and $\alpha_k(n) = \{\emptyset\}$ that is, the empty set is the only minimal path set. The following results are required in the sequel.

Theorem 1. For $m \geq k \geq 2$ and $S \subseteq \{1, 2, \ldots, m\}$, we have $S \in \alpha_k(m)$ if and only if

(i) $|S \cap \{j, j+1, \ldots, j+k-1\}| \geq 1$ for $1 \leq j \leq m-k+1$

(ii) $|(S \cup \{0, m+1\}) \cap \{j-1, j, j+1, \ldots, j+k-1\}| \leq 2$ for $1 \leq j \leq m-k+2$

Proof. Suppose $S = \{a_1, a_2, \ldots, a_r\}$ be a subset of $\{1, 2, \ldots, m\}$ such that $a_1 < a_2 < \cdots < a_r$. We note that $a_i \in \{1, 2, \ldots, m\}, i = 1, 2, \ldots, r$. It is easy to verify that part (i) and part (ii) are equivalent to :

(I) $a_i - a_{i-1} \leq k$, for $i = 1, 2, \ldots, r+1$.

(II) $a_{i+1} - a_{i-1} \geq k+1$, for $i = 1, 2, \ldots, r$ where $a_{r+1} = m+1$ and $a_0 = 0$.

From Theorem 1, in Chapter 1, we note that, S is a path set of the system if and only if it has nonempty intersection with each minimal cut set. Therefore part (I) means S is a path set of a con$|k|m$:F system. And part (II) means $S - \{a_i\}$ is not a path set. That is, S is a minimal path set of the system. ∎

Remark 1. Chan *et al.* [14], have given the following necessary and sufficient conditions for a state vector, $(x_1, x_2, \ldots, x_n)$, to be a minimal path vector of the system.

$$x_i + x_{i+1} + \cdots + x_{i+k-1} \geq 1, \quad i = 1, 2, \ldots, n-k+1,$$

and

$$x_{i-1}\left(\sum_{i \leq j_1 < j_2 \leq i+k-1} x_{j_1} x_{j_2}\right) = 0, \quad i = 1, 2, \ldots, n,$$

where $x_0 = 0,\ x_{n+1} = 1,$ and $x_{n+i} = 0$ for $i = 2, 3, \ldots, k-1$.

We see that these conditions are difficult to apply and less tractable than the conditions of Theorem 1.

Corollary. For $n \geq k$ and $S \in \alpha_k(n)$ we have

[(*i*)] $i \in S \Longleftrightarrow S \cap (\{1, 2, \ldots, k\}/\{i\}) = \emptyset$ for $i = 1, 2, \ldots, k$.

[(*ii*)] $i \in S \Longleftrightarrow S \cap (\{n-k+1, n-k+2, \ldots, n-2, n-1\}/\{i\}) = \emptyset$.

for $i = n-k+1, n-k+2, \ldots, n$.

[(*iii*)] $\{j, j+1, \ldots, j+k-2\} \cap S = \emptyset \Longleftrightarrow \{j-1, j+k-1\} \subseteq S$

for $2 \le j \le n-k+1$

[(iv)] $\{j, j{+}1\} \subseteq S \Longleftrightarrow \{j{-}k{+}1, j{-}k{+}2, \ldots, j{-}2, j{-}1, j{+}2, j{+}3, \ldots, j{+}k\} \cap S = \emptyset.$

for $k \le j \le n-k$

Theorem 2. For $n \ge k+1$ we have

$$\{S : S \in \alpha_k(n), n \in S\} = \{S : S = T \cup \{n-k, n\}, T \in \alpha_k(n-k-1)\}.$$

Further for $n \ge k$ we have

$$\{S : S \in \alpha_k(n), i \notin S, \;\; i = n{-}k{+}2, n{-}k{+}1, \ldots, n\} = \{S : S = T \cup \{n-k+1\}\}$$

where $T \in \alpha_k(n-k)$.
The proof is easy and omitted.

The next theorem provides a recursive algorithm for computing the minimal path sets of a con$|k|n+1$:F system using the minimal path sets of a con$|k|n$:F system.

Theorem 3. Suppose $n \ge k$ and $R = \{x_1, x_2, \ldots, x_r\}$ are given, where $1 \le x_1 < x_2 \cdots < x_{r-1} < x_r \le n$ and $R \in \alpha_k(n)$. We then have

(i) If $x_r \in \{n-k+2, n-k+3, \ldots, n-1, n\}$, then R is also a minimal path set for a con$|k|n+1$:F system.

(ii) If $x_r = n-k+1$ and $x_{r-1} = n-k-j$, for some $j \in \{0, 1, \ldots, k-1\}$, we then have $j+1$ more minimal path sets for a con$|k|n+1$:F system as follows

$R_i = R \cup \{n+1-i\}$, $i = 0, 1, 2, \ldots, j$. (for $r = 1$ we assume $x_{r-1} = 0$)

We note that $x_r \in \{n-k+1, n-k+2, \ldots, n-1, n\}$ and also $x_{r-1} \in \{n-2k+1, n-2k+2, \ldots, n-k\}$.

Proof. Using Theorem 1, the proof is easy and omitted.

3.3 Lexicographical Ordering of Minimal Path Sets

In this Section, we propose a nonrecursive algorithm for generating minimal path sets of a linear con$|k|n$:F system. This is done by the defining a linear order on the subsets of components. To start with we write down the first element of the ordered collection of the minimal path sets of a con$|k|n$:F system and then generate its immediate successor and so on. In the remainder of this Section, we study the minimal path sets of a circular con$|k|n$:F system. We show that all minimal path sets of a circular system can be generated using all minimal path sets of a linear system.

Definition 1. Let S be any subset of N and $\ell \in N$. The translate of S through ℓ, denoted by $S + (\ell)$ is defined as:

$$S + (\ell) = \{j : j = i + \ell, i \in S\}.$$

We use lex ordering of the subsets of N, a concept used by Butler [12], for the purpose of importance ranking for components of coherent systems.
For any subset S of N, we associate a binary vector $\mathbf{x}^S \in \{0,1\}^n$ as follows:

$$x_j^S = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise} \end{cases}$$

We now make use of the binary vector associated with each subset of N to order them using lex ordering.

Definition 2. Let S and T be two subsets of N. We say S is lexicographically less than T if and only if the binary vector $\mathbf{x}^S$ is lexicographically less than the vector $\mathbf{x}^T$ and we denote this by writing $S \prec T$.

For example, if $N = \{1, 2, \ldots, 10\}$, $S = \{1, 3, 5, 8\}$ and $T = \{1, 3, 5, 7\}$, we have $\mathbf{x}^S = (1, 0, 1, 0, 1, 0, 0, 1, 0, 0)$, $\mathbf{x}^T = (1, 0, 1, 0, 1, 0, 1, 0, 0, 0)$. We observe that $\mathbf{x}^S$ is lexicographically less than $\mathbf{x}^T$, hence we say S is lex less than T. The following lemma is required in the sequel.

Lemma 1. For any two subsets S and T of N, we have $S \prec T$ if and only if there exists $r \in T/S$ such that $\{1, 2, \ldots, r-1\} \cap S = \{1, 2, \ldots, r-1\} \cap T$. We define $\{1, 2, \ldots, r-1\} = \emptyset$ if $r = 1$.

Proof. Let $\mathbf{x}^S$ and $\mathbf{x}^T$ be the binary vectors associated with S and T respectively. Suppose $S \prec T$ and recall that by definition $S \prec T$ if and only if $\mathbf{x}^S$ is lexicographically less than $\mathbf{x}^T$. Let r be the smallest j for which $x_j^S \neq x_j^T$. We note that $x_r^S = 0$ and $x_r^T = 1$. If $r = 1$, the result is trivial. If $r > 1$ then it is easy to see that

$$\{1, 2, \ldots, r-1\} \cap S = \{1, 2, \ldots, r-1\} \cap T \text{ and } r \in T/S.$$

Now suppose that there exists $r \in T/S$ such that $\{1, 2, \ldots, r-1\} \cap S = \{1, 2, \ldots, r-1\} \cap T$.

If $r = 1$ then obviously $\mathbf{x}^S$ is lex less than $\mathbf{x}^T$, since $r \in T/S$, and hence $S \prec T$.

If $r > 1$ we then have $x_r^T = 1, x_r^S = 0$ and $x_j^T = x_j^S$ for $j < r$.

It follows that $\mathbf{x}^S$ is lexicographically less than $\mathbf{x}^T$ and hence $S \prec T$.

Remarks. Whenever we talk of the first element or last element or immediate predecessor or immediate successor, it is always with respect to the lex ordering. Also whenever we talk of a con$|k|n$:F system it always refers to a linear con$|k|n$:F system. A circular con$|k|n$:F system will be specifically mentioned.

3.3.1 Algorithm For a Linear System

The following theorem introduces the first and the last elements of $\alpha_k(n)$.

Theorem 4. Let $n \geq k$ and the subsets A and B of N be as defined by

$A = \{k, 2k, 3k, \ldots, k\lfloor n/k \rfloor\}$ and

$$B = \begin{cases} \{1, k+1, k+2, 2k+2, 2k+3, \ldots, n-t-k-1, n-t-k, n-t\} \\ \text{if } 0 \leq t \leq k-1 \\ \\ \{1, k+1, k+2, 2k+2, 2k+3, \ldots, n-t-k-1, n-t-k, n-t, n-t+1\} \\ \text{if } t = k \end{cases}$$

where $t = n - (k+1)\lfloor n/(k+1) \rfloor$ and $\lfloor x \rfloor$ denotes the integer part of x (the largest integer less than or equal to x).

Then, A is the first element of $\alpha_k(n)$ and B is the last element of $\alpha_k(n)$.

Proof. Using Theorem 1, we can verify that $A, B \in \alpha_k(n)$. Suppose $S \in \alpha_k(n)$ is a subset of N such that $S \prec A$. By Lemma 1, there exists an $r \in A/S$ such that $\{1, 2, \ldots, r-1\} \cap S = \{1, 2, \ldots, r-1\} \cap A$. We have $r \geq k$ and r is a positive integer multiple of k, since $r \in A$. We note that $r-1, r-2, r-3, \ldots, r-k+1$ do not belong to A. Hence $r, r-1, r-2, \ldots, r-k+1$ do not belong to S. It follows that S is not a path set of a con$|k|n$:F system and this contradicts the assumption that $S \in \alpha_k(n)$.

Suppose now that $T \in \alpha_k(n)$ is a subset of N such that $T \succ B$. By Lemma 1, there exists an $r \in T/B$ such that $\{1, 2, \ldots, r-1\} \cap B = \{1, 2, \ldots, r-1\} \cap T$. We have $r \notin B$ therefore we can write $r = (k+1)s + x$ for some nonnegative integer s and $x \in \{2, 3, \ldots, k\}$. We have $\{1, r\} \subseteq T$. If $s = 0$ then in this case, $2 \leq r \leq k$, hence T is not minimal. If $s \neq 0$, we have $r \in T, r-x \in T, r-x+1 \in T$ since $r - x \in B$ and $r - x + 1 \in B$. Hence $\{r-x, r-x+1, r\} \subseteq T$. That is T is not minimal, since $x \leq k$. This completes the proof of the theorem. ∎

Definition 3. Let $n \geq k$ and $S \in \alpha_k(n)$ be given. Define M_S a subset of $N \cup \{0\}$ by:

$$M_S = \{j : j \in S \cup \{0\}, \ j \leq n-k \text{ and } \{j-1, j+1\} \cap (S \cup \{0\}) = \emptyset\}$$

Lemma 2. Let $n \geq k$ and $S \in \alpha_k(n)$ be given. Then $M_S = \emptyset$ if and only if

$S = B$ where M_S is as defined earlier and B is the last element of $\alpha_k(n)$.

Proof. Suppose $S = B = \{1, k+1, k+2, 2k+2, 2k+3, \ \ldots\}$. We note that if $x \in B$ then $x = s(k+1)+1$ or $x = (s+1)(k+1)$ for some nonnegative integer s. If $x = 1$, we note that $1 \notin M_S$. If $x = (s+1)(k+1)$ such that $x \leq n-k$ then $x+1 \in S$, hence $x \notin M_S$. Now suppose $x = s(k+1)+1$ for some integer $s > 0$, then we note that $x-1 \in S$, hence $x \notin M_S$. Therefore there is no element of S that belongs to M_S and also we have $0 \notin M_S$, since $1 \in S$. Hence, we get the result $M_S = \emptyset$.

Conversely suppose M_S is empty set. We show that S is the last element of $\alpha_k(n)$. Suppose not, we then have $S \prec B$, that is S is lex less than B. By Lemma 3, there exists $r \in B/S$ such that $\{1, 2, \ldots, r-1\} \cap S = \{1, 2, \ldots, r-1\} \cap B$. We note that $r > 1$. Because if $r = 1$, we then have $1 \notin S$ and hence $0 \in M_S$. This contradicts the assumption that $M_S = \emptyset$. Therefore $r > 1$. We note that $r = jk+j$ or $r = jk+j+1$ for some positive integer j. If $r = jk+j$, we have

$$B \cap \{1, 2, \ldots, r-1\} = \{1, 2, \ldots, jk+j-1\} \cap S.$$

which is equal to

$$\{1, k+1, k+2, 2k+2, 2k+3, \ldots, (j-1)k+(j-1), (j-1)k+j\}.$$

We get the result that $jk+j-1, jk+j-2, \ldots, jk+j-(k-1)$ do not belong to S. On the other hand we know $r = jk+j$ does not belong to S. Hence S is not a path set, since $S \cap \{jk+j, jk+j-1, jk+j-2, \ldots, jk+j-(k-1)\} = \emptyset$ resulting in a contradiction.

If $r = jk+j+1$ we have

$B \cap \{1, 2, \ldots, r-1\} = \{1, k+1, k+2, 2k+2, 2k+3, \ldots, (j-1)k+j, jk+j\} = \{1, 2, \ldots, jk+j\} \cap S$.

It implies that $jk+j \in S$, $jk+j-1 \notin S$ and also we have $r = jk+j+1 \notin S$.

We also note that $jk + j \leq n - k$, since $r = jk + j + 1 \in B$ and $jk + j \in B$. Hence $jk + j \in M_S$ and this contradicts the hypothesis that $M_S = \emptyset$. This completes the proof of the lemma. ■

Definition 4. Let $n \geq k$ and $S \in \alpha_k(n)$ are given.
For each $\ell \in M_S$ define the collection β_ℓ by

$$\beta_\ell = \{T : T = (\{1, 2, \ldots, \ell\} \cap S) \cup (H + (\ell)),\ H \in \alpha_k(n - \ell), k \notin H\}$$

where $H + (\ell)$ and M_S are as defined in Definitions 1 and 3, respectively. In addition to $k \notin H$, choice of H depends on the following conditions given below. We define

$$s_1 = \begin{cases} \max\{s | s \in S,\ s < \ell\} & \text{if } \{s | s \in S,\ s < \ell\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$s_2 = \min\{s \in S,\ s \geq \ell + 1\}$.

(condition 1.) $1 \in H$ if $\ell - s_1 = k$ for $\ell \neq 0$. For $\ell = 0$, we also assume $1 \in H$.

(condition 2.) $i \in H$ if $\ell + i < s_2$, for all ℓ and $\ell + i - s_1 \geq k + 1$ for $\ell \neq 0$.
$i = 2, 3, \ldots, k - 2$

(condition 3.) $k - 1 \in H$ if $\ell + k - 1 < s_2$.
For a given ℓ, we take $\beta_\ell = \emptyset$ if these conditions do not hold.

Theorem 5. Let $n \geq k$ and $S \in \alpha_k(n)$ be given and M_S, β_ℓ are as defined earlier. We then have

$$\{T : T \in \alpha_k(n) \text{ and } T \succ S\} = \bigcup_{\ell \in M_S} \beta_\ell$$

Furthermore $p, q \in M_S$, $p > q$, $P \in \beta_p$ and $Q \in \beta_q$ implies that $Q \succ P$.

Proof. We note that if $S = B$, then $\{T : T \in \alpha_k(n) \text{ and } T \succ S\} = \emptyset$. On the other hand, from Lemma 2, we have $M_S = \emptyset$, therefore $\bigcup_{\ell \in M_S} \beta_\ell = \emptyset$ and hence theorem is trivially true. Now suppose $S \prec B$. Let $T \in \alpha_k(n)$ and $T \succ S$. By Lemma 1, there exists an $r \in T/S$ such that

$$\{1,2,\ldots,r-1\} \cap S = \{1,2,\ldots,r-1\} \cap T.$$

We consider three following cases for r.

case 1. If $1 \leq r \leq k-1$, we note that $r \notin S$. Take $\ell = 0$, $H = T$ and note that $H \in \alpha_k(n)$ and $k \notin H$. We note that $1 \notin S$, since if $1 \in S$ then $k+1 \in S$ therefore $\{1, k+1\} \subseteq T$, since $T \succ S$. It implies that $r \geq k+2$ resulting in a contradiction. Therefore $1 \notin S$ and hence $\ell = 0 \in M_S$. If $r = 1 \in T$, obviously $1 \in H$, since $\ell = 0$. Now suppose $r > 1$ we then have $r \in T \Rightarrow \{1,2,\ldots,r-1\} \cap T = \emptyset \Rightarrow \{1,2,\ldots,r-1\} \cap S = \emptyset$
It implies that $s_2 \in \{r+1, r+2, \ldots, k\}$, since $r \notin S$. We have $\ell + r = r < s_2$, hence $r \in H$, that is β_0 is not empty and H is the correct selection. Therefore $T \in \beta_0$.

case 2. We also note that $r \neq k$, because if $r = k$ then $k \in T$, $k \notin S$ It implies that

$$\{1,2,\ldots,k-1\} \cap T = \{1,2,\ldots,k-1\} \cap S = \emptyset.$$

This contradicts the fact that S is a path set.

case 3. Now suppose $r > k$, we have
$\{1,2,\ldots,r-1\} \cap S = \{1,2,\ldots,r-1\} \cap T$, $r \in T/S$. Let
$S_r = \{x : \; x < r, \; x \in S, \; \{x-1, x+1\} \cap (S \cup \{0\}) = \emptyset\}$
We take $\ell = \max\{x : \; x \in S_r\}$.
We note that S_r is a subset of M_S. We now show that $S_r \neq \emptyset$. Suppose $S = \{x_1, x_2, \ldots, x_{r_s}\}$ and $T = \{y_1, y_2, \ldots, y_{r_t}\}$. We have $r > k$ and $r \in T/S$. Therefore there exists an integer j, $2 \leq j$ such that $y_j = r$. We show that $x_{j-1} \in S_r$. We know that $x_i = y_i$ for $i = 1, 2, \ldots, j-1$. It is

enough to show that $x_{j-1}+1 \notin S$ and $x_{j-1}-1 \notin S \cup \{0\}$.

We know that $x_{j-1}+1 = y_{j-1}+1 \leq y_j = r < x_j$. Therefore $x_{j-1} < x_{j-1}+1 < x_j$. Hence $x_{j-1}+1 \notin S$. We now show that $x_{j-1}-1 \notin S \cup \{0\}$.

Suppose $j = 2$. If $x_1 = 1$ we then have $x_2 = k+1$ and therefore $\{1, k+1\} \subset T$, since $T \succ S$. It implies that $j \geq 3$ and this contradicts the fact that $j = 2$. Hence we have $x_1 > 1$ or $x_1 - 1 \neq 0$. On the other hand we have $r < x_2$ hence $x_1 < r < x_2$. Therefore $x_1 + 1 < x_2$ that is $x_1 + 1 \notin S$.

Now suppose $j \geq 3$. We have $x_{j-1} \geq k+1$, hence $x_{j-1} - 1 \neq 0$.

If $x_{j-1} - 1 = x_{j-2}$ we then have $x_j = x_{j-1} + k$. On the other hand $r < x_j$ hence we can write: $r - y_{j-2} = r - x_{j-2} = r - x_{j-1} + 1 = r - x_j + k + 1 < k + 1$ and this contradicts the second part of Theorem 1. Therefore $x_{j-1} - 1 > x_{j-2}$ that is $x_{j-1} - 1 \notin S$ and hence $S_r \neq \emptyset$.

Therefore, $\ell = \max\{x : x \in S_r\}$ is well defined. It is trivial to show that $2 \leq \ell \leq r-1$. Let $K = \{i \in T,\ i \geq \ell + 1\}$ and $H = K - (\ell)$.

We show that $k \notin H$ and $\beta_\ell \neq \emptyset$. We also show that $H \in \alpha_k(n-\ell)$, for all ℓ, $2 \leq \ell \leq r-1$. It is obvious that $\ell \in M_S$ since $\ell \in S_r \subseteq M_S$.

Since $T \in \alpha_k(n)$ and $\ell \in T$, we have

(i) $|H \cap \{j, j+1, \ldots, j+k-1\}| \geq 1$ for $1 \leq j \leq n - \ell - k + 1$

(ii) $|(H \cup \{0, n-\ell+1\}) \cap \{j-1, j, \ldots, j+k-1\}| \leq 2$

for $1 \leq j \leq n - \ell - k + 2$.

It follows that H satisfies the conditions of Theorem 1 for $m = n - \ell$, and hence $H \in \alpha_k(n-\ell)$.

Now we show that $k \notin H$ and also H satisfies at least one of the conditions of Definition 4. We consider following cases for ℓ.

(a) Suppose $\ell = r - t$, $1 \leq t \leq k-1$. We have
$\ell = r - t \in T$, $r \in T \Rightarrow \ell + k = r - t + k \notin T \Rightarrow k \notin H$ and $r = t + \ell \in T$.

It implies that $t \in H$. We also have

$\ell + 1 \notin T, \ell + 2 \notin T, \ldots, \ell + t - 1 \notin T$ which implies that

$\ell + 1 \notin S, \ell + 2 \notin S, \ldots, \ell + t - 1 \notin S, \ell + t = r \notin S$.

By definition of s_2 we then have $\ell + t = r < s_2$, that is, first part of condition 2, for $t \in H$ is satisfied. On the other hand we have

$$\ell - 1 = r - t - 1 \notin T, r - t - 2 \notin T, \ldots, r - k \notin T.$$

Hence $r - t - 1 \notin S, r - t - 2 \notin S, \ldots, r - k \notin S$. By definition of s_1 we then have $s_1 \leq r - k - 1$, therefore $\ell + t - s_1 \geq \ell + t - r + k + 1 = k + 1$, that is second part of condition 2, for $t \in H$ is also satisfied. We note that for the special case $t = 1$, we have $\{r - 2, r - 3, \ldots, r - k\} \cap T = \emptyset$, therefore $r - k - 1 \in T$ and hence $s_1 = r - k - 1$. It implies that $\ell - s_1 = r - 1 - (r - k - 1) = k$ that is $1 \in H$.

Therefore $\beta_\ell \neq \emptyset$ and H is the correct selection and so $T \in \beta_\ell$.

Now we show that $\ell \not\leq r - k$.

(b) We note that $\ell \neq r - k$, because if $\ell = r - k \in T$, we then have $r \in T/S$, hence

$r - k + 1 \notin T, r - k + 2 \notin T, \ldots, r - 1 \notin T$ which implies that

$r - k + 1 \notin S, r - k + 2 \notin S, \ldots, r - 1 \notin S$.

We also have $r \notin S$. It implies that k consecutive components do not belong to S and this contradicts the fact that $S \in \alpha_k(n)$.

(c) Now suppose $\ell < r - k$. We note that $\ell \in S$ and $\ell < r$, $\ell \in T$ and $r \in T/S$. If $k < r - \ell$ then there exists $t \in T$ such that $\ell < t < r$. Let $t^* \in T$ be the first element after ℓ. We can write

$$S = \{x_1, x_2, \ldots, \ell, t^*, \ldots, x_{j-1}, x_j, \ldots, x_{r_s}\}.$$

$$T = \{y_1, y_2, \ldots, \ell, t^*, \ldots, y_{j-1}, r, \ldots, y_{r_t}\}.$$

From definition ℓ, we have $\ell + 1 < t^*$ and hence $t^* - 1 \notin S$. Once again from definition ℓ, we have $t^* + 1 \in S$ and hence $t^* + k + 1 \in S$. Therefore we can write

$$\{t^*, t^*+1, t^*+k+1, t^*+k+2, t^*+2k+2, t^*+2k+3, \ldots, t^*+sk+s, t^*+sk+s+1\} \subset S$$

where s is the largest integer such that $t^* + sk + s + 1 < r$.

We note that $x_{j-1} = y_{j-1} < y_j = r$ and also $r < x_j$. Therefore $x_{j-1} = t^* + sk + s + 1$ and hence $x_j = t^* + sk + s + k + 1 = t^* + (s+1)(k+1)$.

Now we have

$r - y_{j-2} = r - x_{j-2} < x_j - x_{j-2} = t^* + (s+1)(k+1) - (t^* + sk + s) = k+1$.

This means $|T \cap \{y_{j-2}, y_{j-1}, r\}| = 3$, and this contradicts the second part of Theorem 1, since $T \in \alpha_k(n)$.

From above argument, we get the result $r - k < \ell \leq r - 1$.

Therefore if $r > k$ then $H = K - (\ell)$, where $K = \{i \in T,\ i \geq \ell + 1\}$ and $\ell = \max\{x | x \in S_r\}$, is the correct selection and we have $T \in \beta_\ell$. This completes the proof of necessary condition of the theorem.

Conversely, now suppose $\ell \in M_S$, $T \in \beta_\ell$. We then have

$T = (\{1, 2, \ldots, \ell\} \cap S) \cup (H + (\ell))$ where $H \in \alpha_k(n - \ell)$, $k \notin H$ and also H satisfies the conditions of Definition 4. We have

$T \in \beta_\ell \Rightarrow \beta_\ell \neq \emptyset \Rightarrow t \in H$ for some $1 \leq t \leq k - 1$.

If $t = 1$, we have $1 \in H$ and hence $1 + \ell \in T$. On the other hand since $\ell \in M_S$ it follows that $\ell + 1 \notin S$. Therefore we have $\ell + 1 \in T/S$ and by definition of T we have $T \cap \{1, 2, \ldots, \ell\} = S \cap \{1, 2, \ldots, \ell\}$. Hence using Lemma 1, we get the result $T \succ S$.

Now suppose $1 < t \leq k - 1$.

We have $t + \ell \in T$ and $\{1, 2, \ldots, t-1, t+1, \ldots, k\} \cap H = \emptyset$. From Definition 4, we have $\ell + t < s_2$, since $t \in H$.

Hence $\{\ell+1, \ell+2, \ldots, \ell+t\} \cap S = \emptyset$. That is, $\ell+t \notin S$ and therefore $\ell+t \in T/S$. On the other hand $\{\ell+1, \ell+2, \ldots, \ell+t-1\} \cap T = \emptyset$. By definition of T we get the result $\{1, 2, \ldots, \ell+t-1\} \cap T = \{1, 2, \ldots, \ell+t-1\} \cap S$. Hence by Lemma 1, it implies that $T \succ S$. It can be seen that T satisfies the conditions of Theorem 1, and hence $T \in \alpha_k(n)$. It completes the proof of sufficient condition of the theorem.

Finally, let $p, q \in M_S$, $p > q$, $P \in \beta_p$ and $Q \in \beta_q$. We show that $Q \succ P$. The definition of M_S implies that $p \geq q + 2$. We have

$$P = (\{1, 2, \ldots, p\} \cap S) \cup (H_p + (p))$$

where $H_p \in \alpha_k(n-p)$ and $k \notin H_p$ and H_p satisfies the conditions of Definition 4, for $\ell = p$. *Similarly we also have*

$$Q = (\{1, 2, \ldots, q\} \cap S) \cup (H_q + (q))$$

where $H_q \in \alpha_k(n-q)$, $k \notin H_q$ and H_q satisfies the conditions of Definition 4 for $\ell = q$. We have $Q \in \beta_q$ hence $\beta_q \neq \emptyset$ that is, $s \in H_q$, for some $1 \leq s \leq k-1$. If $s = 1$ then $q+1 \in H_q + (q)$ and hence $q+1 \in Q$. On the other hand, we have $q \in M_S$ and hence $q + 1 \notin S$ and from the expression for P we get $q + 1 \notin P$. Therefore we have $q + 1 \in Q/P$ and from the expressions for P and Q we have $\{1, 2, \ldots, q\} \cap P = \{1, 2, \ldots, q\} \cap Q$. From Lemma 1, it implies that $Q \succ P$. Now suppose, $1 < s \leq k - 1$. From Definition 4, we note that $s + q < s_2$, since $s \in H_q$. Hence using the definition of s_2, we have $\{1+q, 2+q, \ldots, s+q\} \cap S = \emptyset$. On the other hand, we have $p \geq q+2$, $p \in M_S \Rightarrow p \in S$. Therefore $p \geq s+q+1$. We note that $\{1, 2, \ldots, s-1\} \cap H_q = \emptyset$ and using the expression for Q we have $\{q+1, q+2, \ldots, q+s-1\} \cap Q = \emptyset$. Now in view of the expressions of P and Q we can write $\{1, 2, \ldots, q\} \cap P = \{1, 2, \ldots, q\} \cap Q = \{1, 2, \ldots, q\} \cap S$. On the other hand, we have $s + q \notin S$ and using the expression for P we also have $s + q \notin P$ hence $s + q \in Q/P$.

$\{1 + q, 2 + q, \ldots, s + q\} \cap S = \emptyset$, $p \geq s + q + 1$. It implies that

$\{1+q, 2+q, \ldots, s+q\} \cap P = \emptyset$.

Therefore we have $\{1, 2, \ldots, q+s-1\} \cap P = \{1, 2, \ldots, q+s-1\} \cap Q$ and $s+q \in Q/P$. By Lemma 1, it implies that $Q \succ P$. This completes the proof of the theorem. ■

Remark 2. For the case $k = 2$, we always have $1 \in H$, where $H \in \alpha_2(n-\ell)$ for all $\ell \in M_S$. Since

$\ell \in M_S \Rightarrow \ell+1 \notin S, \ell \in S, \ell-1 \notin S \Rightarrow \ell+2 \in S, \ell-2 \in S$. Therefore $s_1 = \ell - 2$. Hence $\ell - s_1 = \ell - \ell + 2 = 2 = k$. It implies that $1 \in H$. Therefore we don't need to put any further condition for $1 \in H$.

Now using Theorem 5, we can obtain the immediate successor of S in $\alpha_k(n)$ with respect to lex ordering.

Theorem 6. Let $n \geq k$ and $S \in \alpha_k(n)$ be given. Further let M_S be as stated in Definition 3. Suppose $M_S \neq \emptyset$ and let $t = \max\{\ell | \ \ell \in M_S, \text{ and } \beta_\ell \neq \emptyset\}$. Suppose $L = \{t+i^*, t+i^*+k, t+i^*+2k, \ldots, t+i^*+j^*k\}$ where

$i^* = \max\{i: \ 1 \leq i \leq k-1, \ \ i \in H, \ \ H \in \alpha_k(n-t)$ and $k \notin H\}$ and also H satisfies the conditions of Definition 4, and j^* is the largest integer such that $t + i^* + j^*k \leq n$.

Let $J = (\{1, 2, \ldots, t\} \cap S) \cup L$, then J is the immediate successor of S in $\alpha_k(n)$.

Proof. We note that $M_S \neq \emptyset$ that is $S \neq B$ ($S \prec B$) where B is the last element of $\alpha_k(n)$. Using Theorem 5, we have $\{\ell: \ \ell \in M_S \text{ and } \beta_\ell \neq \emptyset\} \neq \emptyset$, therefore t is well defined.

Now from Theorem 5, we have $\{T: \ T \in \alpha_k(n), \ T \succ S\} = \bigcup_{\ell \in M_S} \beta_\ell$ where M_S and β_ℓ are defined in Definitions 3 and 4 respectively. If $T \succ S$ then there exists $\ell \in M_S$ such that $T \in \beta_\ell$. From definition of β_ℓ, as given in Definition 4, we have

$T = (\{1, 2, \ldots, \ell\} \cap S) \cup (H + (\ell))$ where $H \in \alpha_k(n-\ell), k \notin H$ and H satisfies the conditions of Definition 4. It is obvious that in order to get the immediate successor of S, first we have to select maximum value of $\ell \in M_S$ (by definition of $T \in \beta_\ell$ and noting that $T \succ S$). Hence we should take $t = \max\{\ell : \ell \in M_S, \beta_\ell \neq \emptyset\}$. Therefore the immediate successor of S belongs to β_t. Secondly we have to find first element of β_t. It is just immediate successor of S. Let $\bar{H} = L - (t)$. In view of conditions of Theorem 1, it is easy to see that $\bar{H} \in \alpha_k(n-t)$, and by definition of L we have $k \notin \bar{H}$ and $\{1, 2, \ldots, k-1\} \cap \bar{H} \neq \emptyset$. Hence we have $J = (\{1, 2, \ldots, t\} \cap S) \cup \left(\bar{H} + (t)\right)$ where $\bar{H} \in \alpha_k(n-t)$ and $k \notin \bar{H}$ and also $\bar{H}$ satisfies the conditions of Definition 4. Therefore $J \in \beta_t$. We need to show that $\bar{H}$ is the first element of $\alpha_k(n-t)$. We have

$$\bar{H} = \{i^*, k + i^*, 2k + i^*, 3k + i^*, \ldots, i^* + j^* k\}.$$

Using similar argument as given in the proof of Theorem 4, it can be shown that $\bar{H}$ is the first element of $\alpha_k(n-t)$. Hence $J = (\{1, 2, \ldots, t\} \cap S) \cup L$ is the immediate successor of S in $\alpha_k(n)$. This completes the proof of the theorem. ■

Remark 3. We are now in a position to state the algorithm to generate the minimal path sets of a con$|k|n$:F system. It is based on the results of Theorems 5 and 6.

ALGORITHM 1.

Input. Positive integers n, k. $(n \geq k)$

Output. The collection of minimal path sets of a con$|k|n$:F system arranged in ascending lex ordering.

Step 0. Put $P_{(1)} = A$ and $r = 1$. Go to step 1. (A is defined in Theorem 4)

Step 1. Compute the subset M defined by

$$M = \left\{j : j \in \left(P_{(r)} \cup \{0\}\right), \; j \leq n-k \;, \; \{j-1, j+1\} \cap \left(P_{(r)} \cup \{0\}\right) = \emptyset\right\}.$$

If $M = \emptyset$ put $s = r$ and go to step 3.

Otherwise let $t = \max\{j \;:\; j \in M \;,\; \beta_j \neq \emptyset\}$ and go to step 2.

Step 2. Put $P_{(r+1)} = \Big(\{1, 2, \ldots, t\} \cap P_{(r)}\Big) \cup L$. ($L$ is defined in Theorem 6) Replace $r + 1$ by r and go to step 1.

Step 3. $P_{(1)}, P_{(2)}, \ldots, P_{(s)}$ are the minimal path sets of a con$|k|n$:F system arranged in ascending lex ordering.

Example 1. We shall use the above algorithm to obtain the minimal path sets of a con$|k|n$:F system for the case where $n = 10$ and $k = 4$.

r	$P_{(r)}$	t
1	$\{4, 8\}$	4
2	$\{4, 7\}$	4
3	$\{4, 6, 10\}$	6
4	$\{4, 6, 9\}$	4
5	$\{4, 5, 9\}$	0
6	$\{3, 7\}$	3
7	$\{3, 6, 10\}$	6
8	$\{3, 6, 9\}$	6
9	$\{3, 6, 8\}$	3
10	$\{3, 5, 9\}$	5
11	$\{3, 5, 8\}$	0
12	$\{2, 6, 10\}$	6
13	$\{2, 6, 9\}$	6
14	$\{2, 6, 8\}$	6
15	$\{2, 6, 7\}$	2
16	$\{2, 5, 9\}$	5
17	$\{2, 5, 8\}$	5
18	$\{2, 5, 7\}$	0
19	$\{1, 5, 9\}$	5
20	$\{1, 5, 8\}$	5
21	$\{1, 5, 7\}$	5
22	$\{1, 5, 6, 10\}$	-

3.3.2 Algorithm For a Circular System

In this subsection, we consider minimal path sets of a circular con$|k|n$:F system. We show that all minimal path sets in a circular system can be generated using all minimal path sets in a linear system. We present a nonrecursive algorithm for generating all minimal path sets of a circular system on the basis of the algorithm given in the previous subsection. We assume that the minimal path sets of a linear system are arranged in lex ordering.

In a circular con$|k|n$:F system components are arranged on a circle and the minimal cut sets are as given below

$C_1 = \{1, 2, \ldots, k\}$

$C_2 = \{2, 3, \ldots, k+1\}$

$C_{n-k+1} = \{n-k+1, n-k+2, \ldots, n\}$

$C_{n-k+2} = \{n-k+2, \ldots, n, 1\}$

$C_{n-k+3} = \{n-k+3, \ldots, n, 1, 2\}$

$C_n = \{n, 1, 2, \ldots, k-1\}$

Lemma 3. Let $R = \{a_1, a_2, \ldots, a_r\}$ be a minimal path set of a circular con$|k|n$:F system such that $a_1 < a_2 < \cdots < a_r$. We then have $a_1 + n - a_r \leq k$ and $a_1 + n - a_{r-1} \geq k+1$.

Proof. We note that R has nonempty intersection with every minimal cut set of a circular con$|k|n$:F system. On the other hand we have

$R \cap \{a_r+1, a_r+2, \ldots, n, 1, 2, \ldots, a_1-1\} = \emptyset$. Therefore we have

$|\{a_r+1, a_r+2, \ldots, n, 1, 2, \ldots, a_1-1\}| \leq k-1$, that is $n - a_r + a_1 \leq k$.

We also note that

$|\{a_{r-1}+1, a_{r-1}+2, \ldots, a_r, a_r+1, \ldots, n, 1, 2, \ldots, a_1\}| \geq k+1$ otherwise we can delete a_r from R and still be path set, that is R is not a minimal path set and resulting in a contradiction. Therefore we have $n - a_{r-1} + a_1 \geq k + 1$. ■

Next lemma gives the necessary and sufficient conditions for a subset of components to be a minimal path set of a circular con$|k|n$:F system.

Lemma 4. Let $R = \{a_1, a_2, \ldots, a_r\}$ be a subset of $N = \{1, 2, \ldots, n\}$ such that $a_1 < a_2 < \cdots < a_r$. Then R is a minimal path set of a circular con$|k|n$:F system if and only if we have

(i) $a_i - a_{i-1} \leq k$ for $i = 1, 2, \ldots, r$ where $a_0 = a_r - n$. And

(ii) $a_{i+1} - a_{i-1} \geq k + 1$ for $i = 1, 2, \ldots, r$ where $a_{r+1} = a_1 + n$.

Proof. The proof of part (i) for $i = 2, 3, \ldots, r$ and the proof of part (ii) for $i = 1, 2, \ldots, r - 1$ are same as that of Theorem 1, and in view of Lemma 3, the proof of part (i) for $i = 1$ and the proof of part (ii) for $i = 1, 2, \ldots, r - 1$ are same as that of Theorem 1, and in view of Lemma 3, the proof of part (i) for $i = 1$ and the proof of part (ii) for $i = r$ are trivial. ■

Let P_C and P_L be a minimal path sets of a circular con$|k|n$:F system and of a linear con$|k|n$:F system, respectively. We note that difference between P_C and P_L is

(i) $|C_i \cap P_C| \leq 2$ for $i = 1, 2, \ldots, n$.

(ii) $|C_i \cap P_L| = 1$ for $i = 1, n - k + 1$.

(iii) $|C_i \cap P_L| \leq 2$ for $i = 2, 3, \ldots, n - k$.

The next lemma, gives the necessary and sufficient conditions for a minimal path set of a linear system to be a minimal path set of a circular system.

Lemma 5. Suppose $P = \{a_1, a_2, \ldots, a_r\}$, $r \geq 1$, $a_1 < a_2 < \cdots < a_r$ is a minimal path set of a linear con$|k|n$:F system. Then P is a minimal path set of a circular con$|k|n$:F if and only if $n - a_r + a_1 \leq k$.

Proof. Suppose $n - a_r + a_1 \leq k$, then it is easy to show that P satisfies the conditions of Lemma 4, and hence P is a minimal path set of a circular system. In view of Lemma 3, converse part is obvious. This completes the proof of the lemma. ■

We note that if P be a minimal path set of size one in a linear system, that is $r = 1$, then P is a minimal path set of a circular system only if $n = k$. In this case the minimal path sets of a linear and a circular con$|k|n$:F systems are same and given by $\{1\}, \{2\}, \{3\}, \ldots, \{k\}$. So we assume that $n > k$.

We know that every minimal path set of a circular con$|k|n$:F system is a path set of a linear con$|k|n$:F system. Therefore using the minimal path sets of a linear system we can generate all minimal path sets of a circular system.

Let P be a minimal path set of a linear system that satisfies the condition of Lemma 5, then P is also a minimal path set of a circular system. We now consider the problem of generation of minimal path sets of a circular system using minimal path sets of a linear system that do not satisfy the condition of Lemma 5.

Suppose $P = \{a_1, a_2, \ldots, a_r\}$, $1 \leq a_1 < a_2 < \cdots < a_r \leq n$, is a minimal path set of a linear system which is of size r.

If P satisfies the condition of Lemma 5, then P is also a minimal path set of a circular system. Otherwise, we add one component namely i_1, $i_1 \in \{1, 2, \ldots, a_1 - 1\}$ and or one component namely i_2, $i_2 \in \{a_r + 1, a_r + 2, \ldots, n\}$ to P, such that

$$n - \bar{a}_r + \bar{a}_1 - 1 \leq k - 1 \tag{1}$$

where $\bar{a}_r$ and $\bar{a}_1$ are the last element and the first element in the new set after adding new components and arranging the components in increasing order. We note that the new set obtained is a path set of a circular system as it satisfies condition (i) of Lemma 4.

While we add new components to P, in order to get a minimal path set for a circular system we must ensure that the second condition of Lemma 4 holds. It may be noted that this condition should be considered only for $i = 2$ and $i = r - 1$, as P is a minimal path set of a linear system.

Remark 4. As P is a minimal path set of a linear system, we need to add at most two component to P.

We now consider all four possible and disjoint cases of P, when it does not satisfy the condition of Lemma 5.

Case 1 : When $r \geq 2$, $a_2 = k + 1$ and $a_{r-1} = n - k$.
In this case we can not modify P to get a minimal path set of a circular system. Because if $a_2 = k + 1$ then $a_2 - i_1 \leq k + 1 - 1 = k$, as $i_1 \in \{1, 2, \ldots, a_1 - 1\}$ and this contradicts the second condition of Lemma 4. Hence we can not add i_1 to P.
Furthemore if $a_{r-1} = n - k$ then $i_2 - a_{r-1} \leq n - (n - k) = k$, since $i_2 \in \{a_r + 1, \ldots, n\}$ resulting in a contradiction. Hence we can not add i_2 to P. Therefore in this case we can not obtain any minimal path set for the circular system using P. It is easy to show that $a_2 = k+1$ and $a_{r-1} = n-k$ if and only if immediate predecessor minimal path set of P, namely $\underline{P} = \{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_{\underline{r}}\}$ satisfies the following conditions $n = (\underline{r} - 1)k + 1$, $\underline{a}_{i+1} - \underline{a}_{i-1} = k + 1$, for $i = 2, 3, \ldots, r - 1$ and $\underline{a}_2 = k + 2$.
Therefore if $\underline{P}$ satisfies these conditions then we don't need to generate P, its immediate successor minimal path set.

Case 2 : When $a_2 > k + 1$ and $a_{r-1} = n - k$.
In this case we need to add only $i_1 \in \{1, 2, \ldots, a_1 - 1\}$ to P and get

$P_c = \{i_1, a_1, a_2, \ldots, a_r\}$. We choose i_1 such that $a_2 - i_1 \geq k+1$ and $n - a_r + i_1 \leq k$. It should be noted that we may get more than one minimal path set of a circular system using P.

Case 3 : When $a_2 = k+1$ and $a_{r-1} < n-k$.
In this case we need to add only $i_2 \in \{a_r+1, a_r+2, \ldots, n\}$ to P and get $P_c = \{a_1, a_2, \ldots, a_r, i_2\}$. The choice of i_2 depends on the following conditions $i_2 - a_{r-1} \geq k+1$ and $n - i_2 + a_1 \leq k$. As in case 2, choice of i_2 may not be unique.

Case 4 : When $a_2 > k+1$ and $a_{r-1} < n-k$.
Consider the following conditions

$$a_2 - i_1 \geq k+1,\ n - a_r + i_1 > k,\ i_2 - a_{r-1} \geq k+1 \text{ and } n - i_2 + a_1 > k \quad (*)$$

We note that under conditions $(*)$, we can not add i_1 or i_2 to P alone. We need to add i_1 and i_2 to P if we also have $n - i_2 + i_1 \leq k$. We get $P_c = \{i_1, a_1, \ldots, a_r, i_2\}$ as a minimal path set of a circular system. We also note that if the conditions of $(*)$ do not hold then we can add only i_1 or i_2 to P but not both. In other words, if we have $a_2 - i_1 \geq k+1$, $n - a_r + i_1 \leq k$ we then can add only i_1 to P and if we have $i_2 - a_{r-1} \geq k+1$, $n - i_2 + a_1 \leq k$ we then can add only i_2 to P.

Example 2. Consider a con|3|14:F system. We note that $P_1 = \{3, 4, 7, 8, 11, 12\}$, $P_2 = \{3, 6, 8, 11, 13\}$, $P_3 = \{2, 4, 7, 9, 12\}$ and $P_4 = \{3, 6, 9, 12\}$ are minimal path sets.of a linear con|3|14:F system that satisfy the conditions described in case 1 to case 4, respectively.
We see that using P_1 we can not obtain any minimal path set for a circular con|3|14:F system.
Using P_2 we have the following minimal path sets for a circular system

$3, 6, 8, 11, 13\}$ and $\{2, 3, 6, 8, 11, 13\}$.

P_3 we have the following minimal path sets for a circular system $4, 7, 9, 12, 13\}$ and $\{2, 4, 7, 9, 12, 14\}$.

consider P_4 and suppose $i_1 = 2$ and $i_2 = 13$. These values of i_1 and i_2 *fy the conditions of* $(*)$, *(stated in case 4)* and also we have $n - i_2 + i_1 \leq k$. we need to add i_1 and i_2 to P_4 and get $\{2, 3, 6, 9, 12, 13\}$ as a minimal set of a circular system. We note that, $i_1 = 1$ or $i_2 = 14$, can be added P_4 *alone, as they don't satisfy the conditions of* $(*)$.

now state a nonrecursive algorithm to generate all minimal path sets of a lar con$|k|n$:F system. It is based on the Algorithm 1, for generating the al path sets of a linear con$|k|n$:F system, stated in previous subsection.

GORITHM 2.

ut. Positive integers $n, k (n > k)$.

put. All minimal path sets of a circular con$|k|n$:F system.

0. Generate P, the first minimal path set of a linear con$|k|n$:F system then go to step 1.

1. Put $P = \{a_1, a_2, \ldots, a_r\}$. If $n - a_r + a_1 - 1 \leq k - 1$ then P is also a mal path set of a circular system and then go to step 2.

$= k + 1$ and $a_{r-1} = n - k$ go to step 2. If $a_2 > k + 1$, then

$$\left[\begin{array}{l} \text{Do for } i_1 = 1, \min\{k + a_r - n + 1, a_2 - k - 1\} \\ \text{Put } P_c = \{i_1\} \cup P \end{array}\right.$$

$a_{r-1} < n - k$, then $\left[\begin{array}{l} \text{Do for } i_2 = \max\{a_{r-1} + k + 1, n + a_1 - k\}, n \\ \text{Put } P_c = P \cup \{i_2\} \end{array}\right.$

$a_2 > k+1$, $a_{r-1} < n-k$, $a_{r-1}+k+1 < n+a_1-k$ and $a_r+k-n < a_2-k-1$

then

$$\left[\begin{array}{l}\text{Do for } i_1 = k + a_r - n + 1, a_2 - k - 1 \\ \left[\begin{array}{l}\text{Do for } i_2 = \max\{n + i_1 - k, a_{r-1} + k + 1\}, n + a_1 - k - 1 \\ \text{Put } P_c = \{i_1\} \cup P \cup \{i_2\}\end{array}\right.\end{array}\right.$$

Step 2. If all minimal path sets of a linear system are generated then stop. Otherwise generate next minimal path set, namely P and go to step 1.

Remark 5. Although the minimal path sets of a linear system are generated lexicographically, but this is not the case for a circular system.

3.4 Minimal Path Sets Containing a Given Component

In order to compute Vesely-Fussell reliability measure of component importance in a con$|k|n$:G system, we need to obtain all minimal cut sets of the system, that contain a given component. We know that a con$|k|n$:G system is a dual of a con$|k|n$:F system. Hence in view of Theorem 4 of Chapter 1, we note that the collection of all minimal path sets of a con$|k|n$:F system and the collection of all minimal cut sets of a con$|k|n$:G system are the same.
This section gives an algorithm to generate all minimal path sets of a con$|k|n$:F system which contain component i and arranged in lex ordering.

Let P_1 be a minimal path sets of a con$|k|n_1$:F subsystem that consists of first $n_1 = i - 1$ components of the original system and P_2 be a minimal path set of a con$|k|n_2$:F subsystem that consists of last $n_2 = n - i$ components of the original system.
We consider two cases for generating P_n^i where P_n^i is a minimal path set of a con$|k|n$:F system that contains component i.

(a) Suppose $n < 2k$.

(a1) Let $i \leq n-k < k$. Then we note that P_n^i is of the form $P_n^i = \{i\} \cup P_2$, such that $P_2 \cap \{i+1, i+2, \ldots, k\} = \emptyset$. We know that in this case P_2 is given by $\{i+k\}, \{i+k-1\}, \{i+k-2\}, \ldots, \{k+1\}$.

(a2) Let $n-k < i \leq k$. In this case we note that the only minimal path set is $P_n^i = \{i\}$.

(a3) Let $n-k < k < i$. In this case P_n^i, the minimal path set that contains i is given by $P_n^i = \{n-k, i\}, \{n-k-1, i\}, \ldots, \{i-k, i\}$.

(b) Suppose $n \geq 2k$.

(b1) Let $i < k \leq n-k$. In this case, similar to case (a1) we have $P_n^i = \{i\} \cup P_2$ where $P_2 \cap \{i+1, i+2, \ldots, k\} = \emptyset$.

(b2) Let $i = k \leq n-k$. In this case we also have $P_n^i = \{i\} \cup P_2$.

(b3) Let $k < i \leq n-k$. In this case P_n^i is of the form $P_n^i = P_1 \cup \{i\} \cup P_2$. Suppose $P_1 = \{a_1, a_2, \ldots, a_r\}$ we note that if $a_r + k > i$ then we should have $P_2 \cap \{i+1, i+2, \ldots, a_r + k\} = \emptyset$.

(b4) Let $k < i = n-k+1$. In this case we have $P_n^i = P_1 \cup \{i\}$.

(b5) Let $k \leq n-k < n-k+1 < i$. In this case we have $P_n^i = P_1 \cup \{i\}$ where $P_1 \cap \{n-k+1, n-k+2, \ldots, i-1\} = \emptyset$.

Algorithm developed for lexicographically generating of the minimal path sets of a con$|k|n$:F system, first generates all minimal path sets starting with component k, and then generates all minimal path sets starting with component $k-1$ and so on. Therefore we see that, in all cases, except case (b5), P_n^i can be generated using Algorithm 1. In other words, using lex ordering defined on the minimal path sets of the system we can generate only those minimal path sets that contain component i and no more. This is not true for the recursive algorithm presented by Chan *et al.* [14].

We note that in case (b5), P_1 has empty intersection with last $i+k-n-1$ components of a con$|k|n_1$:F subsystem. Here we examine case (b5) in details. Suppose $P_1 = \{a_1, a_2, \ldots, a_r\}$ is the first minimal path set of a con$|k|n_1$:F subsystem, where $n_1 = i-1$ and $k \leq n-k < n-k+1 < i$. If $a_r > n-k$ then we should replace a_r by $n-k$, because in case (b5), we should have $P_1 \cap \{n-k+1, n-k+2, \ldots, i-1\} = \emptyset$. Suppose $\bar{P}_1 = \{b_1, b_2, \ldots, b_s\}$ denotes the immediate successor of P_1 in a con$|k|n_1$:F subsystem. We should replace b_s by $n-k$ if $b_s > n-k$. We consider $P_1 = \{a_1, a_2, \ldots, a_r\}$, $r \geq 2$ and confine our attention to the next minimal path set after P_1, in a con$|k|n_1$:F subsystem, such that it has empty intersection with $\{n-k+1, n-k+2, \ldots, i-1\}$ and we call it as an acceptable minimal path set. Next lemma is useful in the sequel.

Lemma 6. Suppose $n_1 = i-1$, $k \leq n-k < n-k+1 < i$ and $P = \{a_1, a_2, \ldots, a_r\}$, $r \geq 2$ is a minimal path set of a con$|k|n_1$:F subsystem such that $P \cap \{n-k+1, n-k+2, \ldots, i-1\} = \emptyset$ (that is P is an acceptable minimal path set) and suppose $\bar{P}$ is its immediate successor minimal path set. Then $\bar{P}$ is not an acceptable minimal path set if and only if

$$n-k-a_{r-1} \leq k \text{ and } a_r = i-k > a_{r-2}+k+1 \tag{2}$$

Proof. We know that if (2) holds then the next minimal path set is $\bar{P} = \{a_1, a_2, \ldots, a_{r-2}, a_{r-1}, i-k-1, i-1\}$ which is obviously not acceptable. We also note that if we replace $i-1$ by $n-k$ then $\bar{\bar{P}} = \{a_1, a_2, \ldots, a_{r-1}, i-k-1, n-k\}$ is not a minimal path set, since we also have $n-k-a_{r-1} \leq k$.

Conversely, suppose (2) does not hold, we then show that $\bar{P}$ is an acceptable minimal path set. We note that $a_r \geq a_{r-2}+k+1$. Suppose $a_r > a_{r-2}+k+1$

and $n-k-a_{r-1} \geq k+1$ we then have

$$\bar{P} = \begin{cases} \{a_1, a_2, \ldots, a_{r-2}, a_{r-1}, a_r - 1\} & \text{if } a_r - 1 \geq i-k \\ \{a_1, a_2, \ldots, a_{r-2}, a_{r-1}, i-k-1, n-k\} & \text{if } a_r = i-k \end{cases}$$

which is an acceptable minimal path set.

Now suppose $a_r = a_{r-2}+k+1$. In this case we note that $\bar{P}$ can be generated by reducing a_{r-1} or a_{r-2} and or a_{r-3} and so on but not by reducing a_r. Suppose $\bar{P}$ is generated by reducing a_{r-1} we then have

$a_{r-1}-1 > a_{r-2} \Rightarrow a_{r-1}-1+k > a_{r-2}+k \Rightarrow a_{r-1}-1+k \geq a_{r-2}+k+1 = a_r$

and

$$\bar{P} = \begin{cases} \{a_1, a_2, \ldots, a_{r-2}, a_{r-1}-1, a_{r-1}-1+k\} & \text{if } a_{r-1}-1+k \leq n-k \\ \{a_1, a_2, \ldots, a_{r-2}, a_{r-1}-1, n-k\} & \text{if } a_{r-1}-1+k > n-k \end{cases}$$

which is an acceptable minimal path set. Other cases can be dealt similarly and this completes the proof of the lemma. ∎

From Lemma 6, we note that if P satisfies the conditions of (2), we then should not generate $\bar{P}$, since $\bar{P}$ is not acceptable.

On the other hand if P satisfies (2) then consider $\bar{\bar{P}}$ given by

$\bar{\bar{P}} = \{a_1, a_2, \ldots, a_{r-2}, a_{r-1}, a_{r-2}+k+1, a_{r-1}+k+1\}$.

We note that $\bar{\bar{P}}$ is a minimal path set of a con$|k|n_1$:F subsystem which is lex greater than P. We also note that $\bar{\bar{P}}$ is not acceptable, since $a_{r-1}+k+1 > n-k$. It may be noted that if we replace $a_{r-1}+k+1$ by $n-k$ then $\bar{\bar{P}}$ is not a minimal path set.

Now in view of Lemma 6, we note that immediate successor of $\bar{\bar{P}}$ is acceptable.

Using above arguments and Lemma 6, we can generate all minimal path sets of a con$|k|n_1$:F subsystem such that they have empty intersection with $\{n-k+1, n-k+2, \ldots, i-1\}$.

We also note that for the cases (a1),(b1) and (b2) we have $P_n^i = \{i\} \cup P_2$ such that $P_2 \cap \{i+1, i+2, \ldots, k\} = \emptyset$ (this condition for case (b2) is trivial). Therefore we can reduce these three cases to a single case with
$i \le \min\{n-k, k\}$.
Similarly all three cases (a3),(b4) and (b5) can be reduced to a single case with
$i > \max\{n-k, k\}$.

We now state an algorithm to generate all minimal path sets containing component i of a con$|k|n$:F system, arranged in ascending lex ordering.

ALGORITHM 3.

Input. n, k and i ($n \ge k$ and $1 \le i \le n$).

Output. All minimal path sets containing component i and arranged in lex ordering.

Step 0. If $i \le \min\{n-k, k\}$, go to case 1. If $i > \max\{n-k, k\}$, go to case 2. If $k < i \le n-k$, go to case 3. And if $n-k < i \le k$, then go to case 4.

Case 1. Generate P_2, the first minimal path set of a con$|k|n_2 = n-i$:F subsystem and put $P_2 = \{a_1, a_2, \ldots, a_r\}$.

Step 1.1. If P_2 is the last minimal path set of a con$|k|n_2$:F subsystem, or if $a_1 = k$ then stop. Otherwise put $P_n^i = \{i\} \cup P_2$ and then go to step 1.2.

Step 1.2. Generate next minimal path set of a con$|k|n_2$:F subsystem as $P_2 = \{a_1, a_2, \ldots, a_r\}$ and then go to step 1.1.

Case 2. Generate P_1, the first minimal path set of a con$|k|n_1 = i-1$:F subsystem and put $P_1 = \{a_1, a_2, \ldots, a_r\}$. Replace a_r by $n-k$, if $a_r > n-k$. If $r = 1$ then all minimal path sets containing component i, which are of size 2 are given by $\{a_r, i\}, \{a_r - 1, i\}, \ldots, \{i-k, i\}$. If $r = 1$ and $n \le 2k$ then stop.

If $r = 1$ and $n > 2k$ put $P_1 = \{i - k\}$ and then go to step 2.1. If $r > 1$ put $P_n^i = P_1 \cup \{i\}$ and then go to step 2.2.

Step 2.1. If P_1 is the last minimal path set of a con$|k|n_1$:F subsystem then stop, otherwise generate the next minimal path set of a con$|k|n_1$:F subsystem as $P_1 = \{a_1, a_2, \ldots, a_r\}$. Replace $n - k$ by a_r if $a_r > n - k$. Put $P_n^i = P_1 \cup \{i\}$ and then go to step 2.2.

Step 2.2. If $n - k - a_{r-1} \leq k$ and $a_r = i - k > a_{r-2} + k + 1$ (we assume that $a_0 = 0$) then put $a_r = a_{r-2} + k + 1$ and $a_{r+1} = a_{r-1} + k + 1$. Then go to step 2.1.

Case 3. Generate P_1, the first minimal path set of a con$|k|n_1$:F subsystem and put $P_1 = \{a_1, a_2, \ldots, a_r\}$. Then go to step 3.1.

Step 3.1. Generate P_2, the first minimal path set of a con$|k|n_2$:F subsystem and put $P_2 = \{b_1, b_2, \ldots, b_s\}$ and go to step 3.2.

Step 3.2. Put $P_n^i = P_1 \cup \{i\} \cup P_2$.

Step 3.3. If P_2 is the last minimal path set of a con$|k|n_2$:F subsystem or if $b_1 = a_r + k$ then go to step 3.4, otherwise generate next minimal path set of a con$|k|n_2$:F subsystem as P_2 and then go to step 3.2.

Step 3.4. If P_1 is the last minimal path set of a con$|k|n_1$:F subsystem then stop. Otherwise generate next minimal path set of a con$|k|n_1$:F subsystem as P_1 and then go to step 3.1.

Case 4. Put $P = \{i\}$ and stop.

3.5 Critical Vectors For a Given Component

In this Section we study critical vectors for a given component in a con$|k|n$:F system. It is needed to evaluate the Birnbaum reliability measure of component importance in a con$|k|n$:F system.

From Chapter 1, recall that a vector $(\cdot_i, \mathbf{x})$ is a critical vector for component i if and only if

$$\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 1.$$

This section gives a nonrecursive algorithm for generating all critical vectors of component i in a con$|k|n$:F system, which are arranged in ascending lex ordering. In order to compute Birnbaum reliability measure of component importance, this arrangement is required, as shown in the subsequent Chapter.

The next theorem, given by F.C Meng [43], introduces a useful method for generating all critical vectors for component i, for a general coherent system.

Theorem 7. If $(\cdot_i, \mathbf{x})$ is a critical vector for component i, then there exists a minimal path set P and a minimal cut set Q such that

$$P \cap Q = \{i\} \text{ and } x_s = \begin{cases} 1 & \text{if } \ s \in P - \{i\} \\ 0 & \text{if } \ s \in Q - \{i\} \end{cases}$$

Proof. For the sake of completeness, we present the proof given by Meng [43]. Let $\phi(1_i, \mathbf{x}) = 1$ and $\phi(0_i, \mathbf{x}) = 0$. Let $\phi(1_i, \mathbf{1}_A, \mathbf{0}_B) = 1$ and $\phi(0_i, \mathbf{1}_A, \mathbf{0}_B) = 0$ We then have

$\phi(1_i, \mathbf{1}_{A1}, \mathbf{0}_{A2}, \mathbf{0}_B) = 1$ for some minimal path set $A1 \cup \{i\}$, and

$\phi(0_i, \mathbf{1}_A, \mathbf{1}_{B2}, \mathbf{0}_{B1}) = 0$ for some minimal cut set $B1 \cup \{i\}$, where $A1 \subseteq A$ and $B1 \subseteq B$. Let $P = A1 \cup \{i\}$ and $Q = B1 \cup \{i\}$, then the proof follows. ■

Remark 6. In Theorem 7, we note that, it is not necessary that $P \cup Q = N$. Let P be a minimal path set and Q be a minimal cut set such that $P \cap Q = \{i\}$. Then $(\cdot_i, \cdot_{N-\{P \cup Q\}}, \mathbf{x})$ is a critical vector for component i where

$$x_s = \begin{cases} 1 & \text{if } \ s \in P - \{i\} \\ 0 & \text{if } \ s \in Q - \{i\} \end{cases} \quad \text{and } N = \{1, 2, \ldots, n\}.$$

Hence using all minimal path sets and minimal cut sets which have intersection $\{i\}$, we can generate all critical vectors for component i.

We note that the states of components that belong to $N - (P \cup Q)$, do not play any role in determining Birnbaum reliability measure of importance for component i. Therefore we call $\mathbf{x} = (\cdot_i, \cdot_{N-\{P\cup Q\}}, \mathbf{1}_{P-\{i\}}, \mathbf{0}_{Q-\{i\}})$ as a *minimal critical vector* for component i.

In view of the above argument, we can now generate minimal critical vectors for component i, in a con$|k|n$:F system, which are arranged in lex ordering as follows. We consider three cases for i.

(i) Let $i \leq k$.

Suppose P^i be a minimal path set of a con$|k|n$:F system that contains component i. In this case, we note that P^i is of the form $P^i = \{i, a_2, a_3, \ldots, a_r\}$, where $r \geq 2$ is the cardinality of P^i. From Theorem 1, we note that $k+1 \leq a_2 \leq k+i$. All minimal cut sets containing component i and satisfying Theorem 7, are as follows:

$Q_1 = \{a_2 - k, a_2 - k + 1, \ldots, a_2 - 1\}$

$Q_2 = \{a_2 - k - 1, a_2 - k, \ldots, a_2 - 2\}$

$Q_{a_2-k} = \{1, 2, \ldots, k\}$.

Remark 7. In $n < 2k$, we may have $n - k < i \leq k$. In this case, we have $P^i = \{i\}$, that is, $r = 1$. We note that in this case the corresponding minimal cut sets are as follows.

$Q_1 = \{n - k + 1, n - k + 2, \ldots, n\}$

$Q_2 = \{n - k, n - k + 1, \ldots, n - 1\}$

$Q_{n-k+1} = \{1, 2, \ldots, k\}$.

Assuming $a_{r+1} = a_2 = n + 1$ as in Theorem 1, we note that this special case can be obtained from case (i).

(ii) Let $k + 1 \leq i \leq n - k$.

In this case we have $P^i = \{a_1, a_2, \ldots, a_{r_0-1}, i, a_{r_0+1}, \ldots, a_r\}$, for some integer r_0, $1 < r_0 < r$. All minimal cut sets containing component i that satisfy Theorem 7, are given below.

$Q_1 = \{a_{r_0+1} - k, a_{r_0+1} - k + 1, \ldots, a_{r_0+1} - 1\}$

$Q_2 = \{a_{r_0+1} - k - 1, a_{r_0-1} - k, \ldots, a_{r_0+1} - 2\}$

$Q_x = \{a_{r_0-1} + 1, a_{r_0-1} + 2, \ldots, a_{r_0-1} + k\}$ where $x = a_{r_0+1} - a_{r_0-1} - k$.

(iii) Let $k + 1 \leq n - k < i$. In this case we note that P^i is of the form $\{a_1, a_2, \ldots, a_{r_0-1}, i\}$. The corresponding minimal cut sets containing component i that satisfy Theorem 7, are given as follows.

$Q_1 = \{n - k + 1, n - k + 2, \ldots, n\}$

$Q_2 = \{n - k, n - k + 1, \ldots, n - 1\}$

$Q_x = \{a_{r_0-1} + 1, a_{r_0-1} + 2, \ldots, a_{r_0-1} + k\}$ where $x = n + 1 - k - a_{r_0-1}$.

Remark 8. We note that case (iii), can be obtained from case (ii), if we assume that $a_{r_0+1} = a_{r+1} = n + 1$. Similarly case (i), can also be obtained from case (ii), if we assume that $a_{r_0-1} = a_0 = 0$, as in Theorem 1. Therefore case (i) and case (iii) are special cases of case (ii).

We now can state an algorithm to generate all minimal critical vectors for component i, in a con$|k|n$:F system which are arranged in ascending lex ordering. It is based on the Algorithm 3, Theorem 7 and Remark 6.

ALGORITHM 4.

Input. Integers n and k ($n \geq k$) and i ($1 \leq i \leq n$).

Output. All minimal critical vectors for component i, in a con$|k|n$:F system which are arranged in ascending lex ordering.

Step 1. Generate P_1^i, the first minimal path set that contains component i and put $P^i = P_1^i$.

Step 2. Put $P^i = \{a_1, a_2, \ldots, a_{r_0-1}, i, a_{r_0+1}, \ldots, a_r\}$ and

$Q_1 = \{a_{r_0+1} - k, a_{r_0+1} - k + 1, \ldots, a_{r_0+1} - 1\}$

$Q_2 = \{a_{r_0+1} - k - 1, a_{r_0+1} - k, \ldots, a_{r_0+1} - 2\}$

$Q_j = \{a_{r_0-1} + 1, a_{r_0-1} + 2, \ldots, a_{r_0-1} + k\}$ where $j = a_{r_0+1} - a_{r_0-1} - k$.

($a_0 = 0$, and $a_{r+1} = n + 1$)

Put $\mathbf{x}_t = (\cdot_i, \cdot_{N-(P^i \cup Q_t)}, 1_{P^i-\{i\}}, 0_{Q_t-\{i\}})$, $t = 1, 2, \ldots, j$.

Step 3. If P^i is the last minimal path sets that contains component i, then stop. Otherwise generate P^i, the next minimal path set containing component i and go to step 2.

Chapter 4

Non-Recursive Algorithms for System Reliability and Component Importance in Consecutive-k-out-of-n Systems

4.1 Introduction

In this chapter, the evaluation of reliability function and Vesely-Fussell measure of component importance and Birnbaum reliability measure of component importance in both a con|k|n:F system and a con|k|n:G system are considered.

In Section 2 using the minimal cut (path) sets of a con|k|n:G(F) system and the results of Chapter 3, we provide a nonrecursive algorithm for determining the system reliability with different component reliabilities. This is an efficient alternative to the inclusion-exclusion principle for evaluating system reliability in con|k|n:F(G) systems. It has no cancelling terms and number of terms equals the number of minimal cut (path) sets. We show that this algorithm can be simply used for determining the system reliability of a k-out-of-n system with different component reliabilities. We also show that this approach is applicable for a general case when the components of the system are not independent.

Section 3 considers the evaluation of the Vesely-Fussell reliability and structural measures of component importance in consecutive-k-out-of-n systems. We show that in case of a consecutive-k-out-of-n:F system, these measures can be computed easily. Using the Algorithm 3 of Chapter 3 we present an algorithm to compute Vesely-Fussell reliability and structural measures of component importance in a consecutive-k-out-of-n:G system.

In Section 4 we consider the problem of determining Birnbaum reliability measure of component importance in a consecutive-k-out-of-n:F system with different component reliabilities. We present an algorithm for this using Algorithm 4 of Chapter 3. This algorithm does not require computation of the reliability function.

In Section 5 we provide a simple and different recursive relationship for determining the reliability function of a consecutive-k-out-of-n:F system.

We use the following notations in the sequel.

n: number of components in a system.

k: minimum number of consecutive working(failed) components which cause a con$|k|n$:G(F) system works(fails), $k \leq n$.

p_i, q_i: reliability and unreliability of component i, $q_i = 1 - p_i$.

$\mathbf{x}$: $(x_1, x_2, \ldots, x_n)$ state vector of components.

$\mathbf{p}$: $(p_1, p_2, \ldots, p_n)$ reliability vector of components.

$\phi^{G(F)}(\mathbf{x})$: structure function of a con$|k|n$:G(F) system.

$h_k^{G(F)}(\mathbf{p}, n)$: reliability function of a con$|k|n$:G(F) system.

N: $\{1, 2, \ldots, n\}$ index set of components.

$I_{VF}^{G(F)}(i, \mathbf{p}), I_{VF}^{G(F),\phi}(i)$: reliability and structural Vesely-Fussell measure of importance for component i in a con$|k|n$:G(F) system.

$\mathbf{C}(i), \mathbf{P}(i)$: collection of all minimal cut sets and minimal path sets that contain component i.

$C_0(\mathbf{x}) = \{i : x_i = 0\}$, $C_1(\mathbf{x}) = \{i : x_i = 1\}$.

$|A|$: cardinality of set A.

$\bar{E}$: complement of event E.

We assume that each component and the system are either functioning or failed.

4.2 Non-Recursive Algorithm for System Reliability in General Case

Most of the literature on system reliability of a con|k|n:F system concentrates on i.i.d. components, for details see Bollinger [10], Bollinger and Salvia [11], Chen and Hwang [19], Chiang and Niu [21], Derman, Liberman and Ross [22], Hwang [32], Lambiris and Papastavridis [41], Peköz and Ross [48] and Ramamurthy [56]. In the case where the components of the system are independent but not identical it is expressed using recurrence relationships, see Shantikumar [63], Chan et *al.* [14] and Hwang [31]. Peköz and Ross [49] and Godbole [27] have given approximating expressions for computing system reliability and Ramamurthy [57] gives explicit formulae for determining the reliability function of the system only in special cases where $k \leq n \leq 6k + 4$.

In this Section we give an algorithm for direct computation of the reliability function of a con|k|n:G system that can be used for a con|k|n:F system with non i.i.d. components.

We know that a con|k|n:G system is a dual of a con|k|n:F system. Hence from Theorem 4 of Chapter 1, it follows that the collection of all minimal cut (path) sets in a con|k|n:G(F) system and the collection of all minimal path (cut) sets in a con|k|n:F(G) system are the same. Kuo *et al.* [40] studied the relationship between these two systems and applied the available results on one type of system to the other.

Suppose $C_1 \prec C_2 \prec \cdots \prec C_{n(k)}$ are all minimal cut sets of a con|k|n:G system arranged in lex ordering, where $n(k)$ is the number of minimal cut sets of the system. We note that

$$h_k^G(\mathbf{p}, n) = Pr\{\phi^G(\mathbf{X}) = 1\} = 1 - Pr\{\phi^G(\mathbf{X}) = 0\} = 1 - Pr\left\{\bigcup_{r=1}^{n(k)} E_r\right\}$$

where E_x is the event that all components of C_x are failed. We give a formula for determining $Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\}$ that contains only $n(k)$ terms. We have

$$Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\} = Pr\{E_1\} + Pr\{E_2 \cap \bar{E}_1\} + \cdots + Pr\{E_{n(k)} \cap \bar{E}_{n(k)-1} \cap \ldots \cap \bar{E}_1\}.$$

Now for a given x, $2 \leq x \leq n(k)$, we introduce a formula for calculating $Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\}$ which contains only one term. We note that $\bar{E}_x$ is the event that at least one component of C_x is working.

Let $C_x = \{a_{x,1}, a_{x,2}, \ldots, a_{x,r_x}\}$, $a_{x,1} < a_{x,2} < \cdots < a_{x,r_x}$, $1 \leq x \leq n(k)$ be a minimal cut set with cardinality r_x.

Definition 1. For $1 < x \leq n(k)$ we define

$$C_x^* = \{a_{x,r}+s \mid a_{x,r}+s-a_{x,r-1} \leq k,\ 1 \leq r \leq r_x,\ 1 \leq s \leq k-1 \text{ and } s \text{ is an integer}\}.$$

Note that $a_{x,r} \in \{1, 2, \ldots, n\}, r = 1, 2, \ldots, r_x$.

Theorem 1. C_x^* satisfies the following conditions:

(i) $C_x^* \subseteq N - C_x$

(ii) $C_x^* \subseteq \bigcup_{j=1}^{x-1} C_j$

(iii) $C_x^* \cap C_j \neq \emptyset$ for $j = 1, 2, \ldots, x-1$.

(iv) If C_x^{**} be a subset of N and satisfies (i) and (iii) then $C_x^* \subseteq C_x^{**}$.

(v) If C_x^{**} satisfies (i) and (iii) and $|C_x^{**}| = |C_x^*|$ then $C_x^{**} = C_x^*$.

Proof.

(i) In view of Theorem 1 of Chapter 3 and definition of C_x^*, we note that if $a_{x,r} + s \in C_x^*$ for some $1 \leq s \leq k-1$, then $a_{x,r} + s < a_{x,r+1}$, that is $a_{x,r} + s \notin C_x$. We also note that if $a_{x,r_x} + s \in C_x^*$ then $a_{x,r_x} + s \leq k + a_{x,r_x-1} \leq k + n - k = n$. Hence (i) is trivially true.

(ii) Suppose $y \in C_x^*$ then $\exists r, s,\ 1 \le r \le r_x,\ 1 \le s \le k-1$ such that $y = a_{x,r} + s$ and $y - a_{x,r-1} \le k$. We now show that for $y \in C_x^*$, there exists a minimal cut set that contains component y and is lexicographically less than C_x. We define the following set

$$\mathcal{C}_x = \begin{cases} C_x \cup \{y\} - \{a_{x,r}\} & \text{if } a_{x,r+2} - y \ge k+1,\ r < r_x \text{ or } r = r_x \\ C_x \cup \{y\} - \{a_{x,r}, a_{x,r+1}\} & \text{if } a_{x,r+2} - y \le k,\ r < r_x \end{cases}$$

We note that $\mathcal{C}_x \prec C_x$, since $y > a_{x,r}$. Consider the following cases.

(a) Suppose $a_{x,r+2} - y \ge k+1$ and $r < r_x$. We have

$y - a_{x,r-2} > a_{x,r} - a_{x,r-2} \ge k+1,\ \ a_{x,r+1} - y < a_{x,r+1} - a_{x,r} \le k,$ $a_{x,r+1} - a_{x,r-1} \ge k+1,\ \ y - a_{x,r-1} \le k$. Hence $y < a_{x,r-1}$. Therefore in view of Theorem 1 of Chapter 3, $\mathcal{C}_x$ is a minimal cut set of a con$|k|n$:G system.

(b) Suppose $r = r_x$. We have $y = a_{x,r_x} + s$. We showed that $y \le n$ in part (i). We also have

$y - a_{x,r_x-2} > a_{x,r_x} - a_{x,r_x-2} \ge k+1$ and therefore from Theorem 1 of Chapter 3, $\mathcal{C}_x$ is a minimal cut set.

(c) Suppose $a_{x,r+2} - y \le k$ and $r < r_x$. We have

$y - a_{x,r-1} \le k,\ \ y - a_{x,r-2} \ge a_{x,r} - a_{x,r-2} \ge k+1,\ \ a_{x,r+2} - y \le k,$

$a_{x,r+3} - y > a_{x,r+3} - a_{x,r+1} \ge k+1$, (if $r < r_x - 1$)

Again by Theorem 1 of Chapter 3, $\mathcal{C}_x$ is a minimal cut set. Hence We have $y \in \mathcal{C}_x$ therefore we get result $C_x^* \subseteq \bigcup_{j=1}^{x-1} C_j$.

Suppose $1 \le j \le x-1$ is given and let $C_j = \{a_{j,1}, a_{j,2}, \ldots, a_{j,r_j}\}$ we know that $C_j \prec C_x$ because we assumed that all minimal cut sets are arranged in lex ordering. Using Lemma 1 of Chapter 3, there exists ℓ, $1 \le \ell \le r_j$ such that $a_{j,\ell} > a_{x,\ell}$ and $a_{j,i} = a_{x,i},\ i = 1, 2, \ldots, \ell - 1$. We show that $a_{j,\ell} \in C_x^*$. We have:

$a_{j,\ell} > a_{x,\ell} \Rightarrow a_{j,\ell} = a_{x,\ell} + s$ for some $s \geq 1$. On the other hand we know that $a_{j,\ell} - a_{x,\ell-1} = a_{j,\ell} - a_{j,\ell-1} \leq k$, since C_j is a minimal cut set. Therefore $a_{x,\ell} + s - a_{x,\ell-1} \leq k \Rightarrow s \leq k-1$, that is $a_{j,\ell} \in C_x^*$ and hence $C_j \cap C_x^* \neq \emptyset$, $\forall 1 \leq j \leq x-1$.

) Suppose $C_x^{**} \subseteq \mathcal{N}$ and satisfies (i) and (iii).

We now show that $C_x^* \subseteq C_x^{**}$. Suppose not that is there exists $y \in C_x^*$ such that $y \notin C_x^{**}$. We have $y \in C_x^* \Rightarrow \exists$ r and s, $1 \leq r \leq r_x$, $1 \leq s \leq k-1$ such that $y = a_{x,r} + s$ and $y - a_{x,r-1} \leq k$. We consider $\mathcal{C}_x$ as defined in the proof of part (ii). We showed that $\mathcal{C}_x \prec C_x$ and $\mathcal{C}_x$ is a minimal cut set. We note that $C_x^{**} \cap \mathcal{C}_x = \emptyset$ (since C_x^{**} satisfies (i)). But this contradicts the assumption that C_x^{**} satisfies (iii). Therefore we have $y \in C_x^{**}$ and hence $C_x^* \subseteq C_x^{**} \Rightarrow |C_x^*| \leq |C_x^{**}|$.

Suppose C_x^{**} satisfies (i),(iii) and $|C_x^{**}| = |C_x^*|$. In the proof of part (iv), we showed that $C_x^* \subseteq C_x^{**}$. Therefore $C_x^* = C_x^{**}$. This completes the proof of the theorem. ■

now can provide a formula for the probability expression $Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\}$ which contains only one term.

1. We have

$$Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\} = Pr\{E_x \cap E_x^*\}$$

E_x^* is the event that all components of C_x^* are working.

. We show that two events; $E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1$ and $E_x \cap E_x^*$ are equivalent. obvious that $E_x \cap E_x^* \subseteq E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1$, because $C_x^* \cap C_j \neq \emptyset$, $1 \leq j \leq x-1$. Now suppose the event $E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1$ has occurred, that the event $E_x \cap E_x^*$ has also occurred. Suppose not, that is, there exists $y \in C_x^*$ such that component y is failed. We have $y \in C_x^* \Rightarrow \exists\ r$ and s, $1 \le r \le r_x$, $1 \le s \le k-1$ such that $y = a_{x,r} + s$ and $y - a_{x,r-1} \le k$. Once again we consider $\underset{\sim}{C}_x$ as defined before. We showed that $\underset{\sim}{C}_x \prec C_x$ and $\underset{\sim}{C}_x$ is a minimal cut set. We note that all components of $\underset{\sim}{C}_x$ are failed, because of the fact that the event E_x has occurred and component y is failed. But this contradicts the assumption that the event $\bar{E}_{x-1} \cap \bar{E}_{x-2} \cap \ldots \cap \bar{E}_1$ has occurred. That is at least one component from each $C_{x-1}, C_{x-2}, \ldots, C_1$ is working. Therefore we get $E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1 \subseteq E_x \cap E_x^*$ and hence these two events are equivalent. We then can write $Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\} = Pr\{E_x \cap E_x^*\}$. This completes the proof of the lemma.

Remark 1. We note that $C_x^* \subseteq N - C_x$ hence $C_x^* \cap C_x = \emptyset$, that is E_x and E_x^* are independent events. Therefore $Pr\{E_x \cap E_x^*\} = Pr\{E_x\}Pr\{E_x^*\}$. Now using Theorem 1 and Lemma 1, we have

$$1 - h_k^G(\mathbf{p}, n) = Pr\{\phi(\mathbf{X}) = 0\} = Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\}$$

$$= Pr\{E_1\} + \sum_{x=2}^{n(k)} Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\} = Pr\{E_1\} + \sum_{x=2}^{n(k)} Pr\{E_x \cap E_x^*\}$$

which is equal to

$$Pr\{E_1\} + \sum_{x=2}^{n(k)} Pr\{E_x\}Pr\{E_x^*\} = \prod_{i \in C_1} q_i + \sum_{x=2}^{n(k)} \prod_{i \in C_x} q_i \prod_{i \in C_x^*} p_i$$

Remark 2. Using inclusion-exclusion method, we know that for determining $Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\}$ we need to compute $2^{n(k)} - 1$ probability expressions but as per Remark 1 we need to compute only $n(k)$ probability expressions.

Now using Algorithm 1 of Chapter 3 and the results of Theorem 1 and Remark 1, we state an algorithm to compute the reliability function of a con$|k|n$:G system.

ALGORITHM 1

Input. Positive integers $n, k (n \geq k)$ and real numbers $p_1, p_2, \ldots, p_n$, $0 \leq p_i \leq 1$, $q_i = 1 - p_i$.

Output. Reliability function of a con$|k|n$:G system with components reliability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

Step 0. Put $x = 1$ and $\bar{R} = 0$. Go to step 1.

Step 1. Generate C_x. If $x = 1$ put $C_x^* = \emptyset$ and $\bar{P} = 1$, otherwise generate C_x^* and put $\bar{P} = \prod_{i \in C_x^*} p_i$ (C_x^* is defined in Definition 1). Go to step 2.

Step 2. Put $\bar{R} = \bar{R} + \prod_{i \in C_x} q_i * \bar{P}$. If $x = n(k)$ (that is C_x is the last minimal cut set of system) go to step 3, otherwise put $x = x + 1$ and then go to step 1.

Step 3. $1 - \bar{R}$ gives the reliability of a con$|k|n$:G system. Stop.

Remark 3. We know that a con$|k|n$:F system is a dual of a con$|k|n$:G system. Hence using Algorithm 1, we can obtain a nonrecursive formula for determining the reliability function of a con$|k|n$:F system as given by :

$$h_k^F(\mathbf{p}, n) = 1 - h_k^G(\mathbf{1} - \mathbf{p}, n).$$

Example 1. Consider a con|3|7:G system. Lex ordered collection of all minimal cut sets of this system is as follows :

x	C_x	C_x^*
1	3 6	
2	3 5	6
3	3 4 7	5 6
4	2 5	3
5	2 4 7	3 5
6	2 4 6	3 5 7
7	1 4 7	2 3
8	1 4 6	2 3 7
9	1 4 5	2 3 6 7

We have

$1 - h_3^G(\mathbf{p}, 7) = q_3q_6 + q_3q_5p_6 + q_3q_4q_7p_5p_6 + q_2q_5p_3 + q_2q_4q_7p_3p_5 + q_2q_4q_6p_3p_5p_7 + q_1q_4q_7p_2p_3 + q_1q_4q_6p_2p_3p_7 + q_1q_4q_5p_2p_3p_6p_7.$

We now consider a general case when the components of a con|k|n:G system are not independent. We note that in this case Lemma 1 still holds. Therefore we can write a direct formula for determining the unreliability of a con|k|n:G system with non-independent components as follows :

$$1 - h_k^G(\mathbf{p}, n) = Pr\{\phi(\mathbf{X}) = 0\} = Pr\left\{\bigcup_{x=1}^{n(k)} E_x\right\} =$$

$$Pr\{E_1\} + \sum_{x=2}^{n(k)} Pr\{E_x \cap \bar{E}_{x-1} \cap \ldots \cap \bar{E}_1\} = Pr\{E_1\} + \sum_{x=2}^{n(k)} Pr\{E_x \cap E_x^*\}$$

which is equal to

$$Pr\{X_i = 0,\ i \in C_1\} + \sum_{x=2}^{n(k)} Pr\{X_i = 0,\ i \in C_x,\ X_j = 1,\ j \in C_x^*\}. \quad (1)$$

For the special case of statistical dependency when the components lifetimes of a con$|k|n$:F system are *exchangeable*, Shantikumar [64], presented a formula for determining system reliability. The general case of reliability function of a con$|k|n$:F system was first studied by Kossow and Preuss [38]. Based on concepts given by Satyanarayana and Prabhakar [59], they presented an expression for determining the reliability of a con$|k|n$:F system using the concept of the *k-subgraph*. Sfakianakis and Papastavridis [62] introduced a different formula for the reliability of a linear and a circular con$|k|n$:F systems. They have shown that the number of terms in the reliability formula grows exponentially with n. Their formula is expressed by a summation over some subsets of components that will be generated recursively whereas formula (1) is a direct formula.

System Reliability of a k-out-of-n:F System

Here we show that the approach given in Algorithm 1, leads to a simple and explicit formula for determining the reliability function of a k-out-of-n:F system with different component reliabilities. Algorithm 1 can be applied using minimal cut sets as well as the minimal path sets of a k-out-of-n:F system. The number of terms in the reliability function equals to the number of minimal cut (path) sets of the system.

We know that a k-out-of-n:F system fails if and only if any k components of the system are failed. Hence the number of minimal cut sets of the system is $n_1 = \binom{n}{k}$ and the number of minimal path sets of the system is $n_2 = \binom{n}{k-1}$. It is easy to see that the number of minimal cut sets is less than the number of minimal path sets if and only if $n < 2k - 1$. Therefore we use minimal cut sets of the system if $n < 2k-1$, and we use minimal path sets the system if $n > 2k-1$. We assume that the collection of all minimal cut) sets of the system is arranged in ascending lex ordering.

ose $C_1 \prec C_2 \prec \cdots \prec C_{n_1}$ be the minimal cut sets of a k-out-of-n:F system anged in lex ordering.

$C_x = \{c_{x,1}, c_{x,2}, \ldots, c_{x,k}\}$, $c_{x,1} < c_{x,2} < \cdots < c_{x,k}$, $1 \le x \le n_1$, be a minimal t set of the system. We note that all minimal cut sets of the system are of k.

Lemma 2. Suppose $C_x^* = \{c_{x,1}, c_{x,1}+1, c_{x,1}+2, \ldots, n\} - C_x$. Then C_x^* satisfies heorem 1.

roof. It can be proved on the same lines as Theorem 1.

Therefore using Lemma 1 and Remark 1, we can obtain direct formula for determining reliability function of a k-out-of-n:F system.

Remark 4. When $n > 2k-1$, we use minimal path sets of the system. ppose $P_x = \{a_{x,1}, a_{x,2}, \ldots, a_{x,n-k-1}\}$, $1 \le x \le n_2$, be a minimal path set of a -out-of-n:F system. In this case $P_x^* = \{a_{x,1}, a_{x,1}+1, a_{x,1}+2, \ldots, n\} - P_x$ and $_x$ satisfies Theorem 1.

e note that when $x = 1$, C_x^* and P_x^* are empty sets.

Example 2. Consider a 2-out-of-6:F system. In this system we have $n_1 = 15$ minimal cut sets and $n_2 = 6$ minimal path sets. Therefore we use minimal ath sets to compute reliability function. Lex ordered minimal path sets of is system are as follows :

x	P_x					P^*_x
1	2	3	4	5	6	-
2	1	3	4	5	6	2
3	1	2	4	5	6	3
4	1	2	3	5	6	4
5	1	2	3	4	6	5
6	1	2	3	4	5	6

Reliability function is given by

$R_2(6, \mathbf{p}) = p_2p_3p_4p_5p_6 + p_1p_3p_4p_5p_6q_2 + p_1p_2p_4p_5p_6q_3 + p_1p_2p_3p_5p_6q_4 + p_1p_2p_3p_4p_6q_5 + p_1p_2p_3p_4p_5q_6.$

Remarks. Recently a new approach has been developed that can be used for efficient calculation of the reliability function of a consecutive-k-out-of-n:F system consisting of independent but not identical or even Markov dependent components (see e.g., Chao and Fu [16] and [17], Koutras [39] and Chao et al [18]).

These papers have efficiently described a wide class of reliability structures by finite Markov Chain. This approach provides an efficient tool not only for the system's reliability evaluation but also for the study of properties of the structure such as its asymptotic behaviour, generating function, development of reliability bounds, component importance and optimal arrangement problems, etc. Such systems can be described using imbedded finite Markov Chain and were introduced by Koutras [39]. They are called *Markov Chain Imbeddable Systems (MIS)*. In an MIS, a proper Markov chain with relatively small state space is available, and system's relibility is expressed in terms of a product of transition probability matrices. The MIS class is wide enough to accommodate many well-known systems. For a formal definition of MIS we refer to Koutras [39].

He has imbedded a consecutive-k-out-of-n:F system, in a finite Markov Chain as follows: Let $\{Y_i, i = 0, 1, \ldots, n\}$ be a finite Markov chain with the state space $S = \{0, 1, \ldots, k\}$ where k is an absorbing state and $Y_i = r$ if the number of failed components that follow the last working component in the system 1,2,...,i is exactly r $(0 \le r < k)$ and $Y_i = k$ if the system $1, 2, \ldots, i$ contains at least k consecutive failed components. It is easy to see that the transition probability matrix of this Markov Chain is given by:

$$M_i = \begin{bmatrix} p_i & q_i & 0 & \ldots & 0 & 0 \\ p_i & 0 & q_i & \ldots & 0 & 0 \\ \vdots & & & & & \\ p_i & 0 & 0 & \ldots & 0 & q_i \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \quad (k+1) \times (k+1)$$

where $M_i = (p_{rs}(i))$ and $p_{rs}(i) = Pr\{Y_i = s | Y_{i-1} = r\}$, $r, s = 0, 1, \ldots, k$. Using MIS approach the reliability function of a consecutive-k-out-of-n:F system is given by:

$$h_k^F(\mathbf{p}, n) = \pi_0 \left(\prod_{i=1}^{n} M_i \right) U\prime \tag{2}$$

where $\pi_0 = (1, 0, 0, \ldots, 0), 1 \times (k+1)$ vector, $U = (1, 1, \ldots, 1, 0), 1 \times (k+1)$ vector and M_i is transition matrix.

For illustration purpose, we compute the reliability function of a consecutive-3-out-4:F system, as given in the next example.

Example 3. Suppose $k = 3$ and $n = 4$. As per the formula (2) for determining of $h_3^F(\mathbf{p}, 4)$ we have:

$$h_3^F(\mathbf{p}, n) = \pi_0 \left(\prod_{i=1}^{4} M_i \right) U\prime = \pi_0 (M_1 \times M_2) \times (M_3 \times M_4) U\prime$$

In view of definition of M_i we have $M_1 \times M_2 = \begin{bmatrix} p_2 & p_1 q_2 & q_1 q_2 & 0 \\ p_2 & p_1 q_2 & 0 & q_1 q_2 \\ p_1 p_2 & p_1 q_2 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and $M_3 \times M_4 = \begin{bmatrix} p_4 & p_3q_4 & q_3q_4 & 0 \\ p_4 & p_3q_4 & 0 & q_3q_4 \\ p_3p_4 & p_3q_4 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Hence we get the result:

$$h_3^F(\mathbf{p},4) = (1,0,0,0) \begin{bmatrix} p_2 & p_1q_2 & q_1q_2 & 0 \\ p_2 & p_1q_2 & 0 & q_1q_2 \\ p_1p_2 & p_1q_2 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_4 & p_3q_4 & q_3q_4 & 0 \\ p_4 & p_3q_4 & 0 & q_3q_4 \\ p_3p_4 & p_3q_4 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

which is equal to

$p_4(p_2+p_1q_2+q_1q_2p_3)+p_3q_4(p_2+p_1q_2+q_1q_2)+p_2q_3q_4 = p_3+p_2q_3+p_1p_4q_2q_3.$
(after simplifications)

Using the approach given in Algorithm 1, lex ordering of minimal path sets of a consecutive-3-out-of-4:F system is as follows:

P_x	P_x^*
3	$\emptyset$
2	3
1 4	2 3

and therefore $h_3^F(\mathbf{p},4) = p_3 + p_2q_3 + p_1p_4q_2q_3$.

It seems the approach given in Algorithm 1 is easy to apply but not for large values of n as the number of minimal path sets of a consecutive-k-out-of-n:F system grows exponentially with n. For example using the combinatorial approach given in Chapter 5, it can be shown that for a linear consecutive-2-out-of-n:F system the number of minimal path sets of the system is the rounded value of the expression $\rho^n(1+\rho)^2/(2\rho+3)$ where $\rho = 1.324178$ is the unique real root of the cubic equation $x^3 - x - 1 = 0$. For a circular consecutive-2-out-of-n:F system, it is shown in Chapter 5 that the number of minimal path

sets of this system is the integer part of $\rho^n + 0.5$ for $n \geq 10$.
However it can be seen that for a given k, computational efforts of formula (2) grows linearly with n.

We now illustrate the application of Algorithm 1 for calculating of the reliability of a k-out-of-n:F system with non iid components. It can be simply shown that, for a k-out-of-n:F system, this approach leads to an explicit formula for determining the reliability function of the system with non iid components as follows:

$$R_k(n, \mathbf{p}) = p_k p_{k+1} \dots p_n + \sum_{r=1}^{k-1} \sum_{i_1 < \cdots < i_r} \prod_{s=1}^{r} q_{i_s} \prod_{\substack{j = k-r \\ j \neq i_1, \dots, i_r}}^{n} p_j \quad \text{for} \quad n \geq 2k-1$$

where $i_1, i_2, \dots, i_r \in \{k - r + 1, \dots, n\}$.
And for $n < 2k - 1$ we have

$$R_k(n, \mathbf{p}) = 1 - \prod_{j=k-1}^{n} q_j - \sum_{r=1}^{k-2} \sum_{i_1 < \cdots < i_r} \prod_{s=1}^{r} p_{i_s} \prod_{\substack{j = k-r-1 \\ j \neq i_1, \dots, i_r}}^{n} q_j$$

where $i_1, i_2, \dots, i_r \in \{k - r, \dots, n\}$.
Using these formulae, the Birnbaum reliability importance of component i, $I_B(i, \mathbf{p}) = \dfrac{\partial R_k(n, \mathbf{p})}{\partial p_i}$ can be computed easily.

4.3 Vesely Fussell Importance Measure

In this section we consider the evaluation of Vesely-Fussell measure of component importance in con$|k|n$:G(F) systems. Using Algorithm 3 of Chapter 3, we present a nonrecursive algorithm for determining Vesely-Fussell reliability and structural measures of component importance in a con$|k|n$:G system. We then show that in case of a con$|k|n$:F system these measures can be computed easily.

Vesely-Fussell Importance Measure in a con|k|n:G System

Vesely [66] and Fussell [25] proposed a measure for reliability and structural importance of component i respectively, as follows :

$$I_{VF}(i,\mathbf{p}) = Pr\{\exists\, C_j \in \mathbf{C}(i) \ s.t.\ C_j \subseteq C_0(\mathbf{X}) | \phi(\mathbf{X}) = 0\}$$

and

$$I^{\phi}_{VF}(i) = I_{VF}(i; 1/2, 1/2, \ldots, 1/2).$$

Here we present a method for computing Vesely-Fussell measure of component importance in a con|k|n:G system.

Suppose $C^i_1 \prec C^i_2 \prec \ldots \prec C^i_{n_k(i)}$ are all the minimal cut sets of a con|k|n:G system that contain component i and arranged in lex ordering, where $n_k(i)$ is the number of minimal cut sets containing component i. We note that

$$I^G_{VF}(i,\mathbf{p}) = \frac{Pr\{\exists\, C_j \in \mathbf{C}(i) \ s.t.\ C_j \subseteq C_0(\mathbf{X})\}}{Pr\{\phi^G(\mathbf{X}) = 0\}} = \frac{Pr\left\{\bigcup_{x=1}^{n_k(i)} E^i_x\right\}}{Pr\left\{\phi^G(\mathbf{X}) = 0\right\}}$$

where E^i_x is the event that all components of C^i_x are failed. In the previous section we proposed a formula for determining $Pr\{\phi^G(\mathbf{X}) = 0\} = 1 - h^G_k(\mathbf{p}, n)$. For the purpose of ranking components using Vesely-Fussell reliability measure, it is enough to compute $Pr\left\{\bigcup_{x=1}^{n_k(i)} E^i_x\right\}$.

We now give a formula for computing $Pr\left\{\bigcup_{x=1}^{n_k(i)} E^i_x\right\}$ containing only $n_k(i)$ terms. We have

$$Pr\left\{\bigcup_{x=1}^{n_k(i)} E^i_x\right\} = Pr\{E^i_1\} + \sum_{x=2}^{n_k(i)} Pr\{E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_2 \cap \bar{E}^i_1\}.$$

For a given x, $2 \leq x \leq n_k(i)$, we introduce an expression for determining $Pr\{E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_1\}$ which contains only one term.

Suppose $C_x^i = \{a_{x,1}^i, a_{x,2}^i, \ldots, a_{x,r_x^i}^i\}, a_{x,1}^i < a_{x,2}^i < \cdots < a_{x,r_x^i}^i$ and let $a_{x,r_0}^i = i$ for some integer r_0, $1 \le r_0 \le r_x^i$, since $i \in C_x^i$. We define $\hat{C}_x^i$ as follows.

Definition 2.

Let $\hat{C}_x^i = \{a_{x,r}^i + s \mid a_{x,r}^i + s - a_{x,r-1}^i \le k,\ 1 \le r \le r_x^i,\ r \ne r_0,\ 1 \le s \le k-1\}$, where r and s are integers. For the case $r = r_0 - 1$, we further assume that

(i) If $1 < r_0 < r_x^i$ and $a_{x,r_0-1}^i + s - a_{x,r_0-2}^i \le k$ and $a_{x,r_0+1}^i - (a_{x,r_0-1}^i + s) \le k$ then $a_{x,r_0-1}^i + s \notin \hat{C}_x^i$

(ii) If $r_0 = r_x^i$ and $a_{x,r_0-1}^i + s - a_{x,r_0-2}^i \le k$ and $a_{x,r_0-1}^i + s \le n-k$ then $a_{x,r_0-1}^i + s \in \hat{C}_x^i$

(iii) If $r_0 = r_x^i$ and $a_{x,r_0-1}^i + s - a_{x,r_0-2}^i \le k$ and $a_{x,r_0-1}^i + s > n-k$ then $a_{x,r_0-1}^i + s \notin \hat{C}_x^i$.

We also assume that $a_{x,0}^i = 0$ and $a_{x,r_x^i+1}^i = n+1$.

Theorem 2. $\hat{C}_x^i$ satisfies the following conditions :

(I) $\hat{C}_x^i \subseteq N - C_x^i$

(II) $\hat{C}_x^i \subseteq \bigcup_{j=1}^{x-1} C_j^i$

(III) $\hat{C}_x^i \cap C_j^i \neq \emptyset$ for $j = 1, 2, \ldots, x-1$.

(IV) If $\hat{\hat{C}}_x^i$ be a subset of N and satisfies (I) and (III) then $\hat{C}_x^i \subseteq \hat{\hat{C}}_x^i$.

(V) If $\hat{\hat{C}}_x^i$ satisfies (I) and (III) and $|\hat{\hat{C}}_x^i| = |\hat{C}_x^i|$ then $\hat{\hat{C}}_x^i = \hat{C}_x^i$.

Proof.

(I) It can be proved on the same lines as part (i) of Theorem 1.

(II) We consider two cases.

(a) Let $r_0 < r_x^i$.

Suppose $y \in \hat{C}_x^i$ we then show that there exists a minimal cut set containing i and y which is lexicographically less than C_x^i. We have

$\exists r, s$, $1 \le r \le r_x^i$, $r \ne r_0$, $1 \le s \le k-1$, $s.t.$ $y = a_{x,r}^i + s$ and $y - a_{x,r-1}^i \le k$. We define

$$\mathcal{C}_x^i = \begin{cases} C_x^i \cup \{y\} - \{a_{x,r}^i\} & \text{if } a_{x,r+2}^i - y \ge k+1 \text{ and } r < r_x^i \text{ or } r = r_x^i \\ C_x^i \cup \{y\} - \{a_{x,r}^i, a_{x,r+1}^i\} & \text{if } a_{x,r+2}^i - y \le k,\ r < r_x^i \end{cases}$$

We note that $\mathcal{C}_x^i \prec C_x^i$ since $y > a_{x,r}^i$. Using Theorem 1 of Chapter 3 and Theorem 4 of Chapter 1, it is easy to verify that $\mathcal{C}_x^i$ is a minimal cut set of a con/k/n:G system. We also note that $y \in \mathcal{C}_x^i$. We show that $i \in \mathcal{C}_x^i$. Consider the following two cases for $\mathcal{C}_x^i$.

(case 1) if $a_{x,r+2}^i - y \ge k+1$ and $r < r_x^i$ or $r = r_x^i$:

In this case it is obvious that $i \in \mathcal{C}_x^i$ since $i \in C_x^i$ and $r \ne r_0$, therefore $a_{x,r}^i \ne a_{x,r_0}^i = i$. Hence $i \in \mathcal{C}_x^i$.

(case 2) if $a_{x,r+2}^i - y \le k$, $r < r_x^i$:

In this case we note that $r+1 \ne r_0$. Because if $r = r_0 - 1$ then $a_{x,r_0-1}^i - a_{x,r_0-1}^i - s \le k$. By the first assumption of Definition 2, we have $y = a_{x,r_0-1}^i + s \notin \hat{C}_x^i$ and this contradicts the fact that $y \in \hat{C}_x^i$. Hence $r \ne r_0 - 1$ therefore $a_{x,r+1}^i \ne a_{x,r_0}^i = i$. We have $i \in \mathcal{C}_x^i$ and hence $\hat{C}_x^i \subseteq \bigcup_{j=1} C_j^i$.

(b) Let $r_0 = r_x^i$.

Suppose $y \in \hat{C}_x^i$. We have $y = a_{x,r}^i + s$ for some $1 \le r < r_x^i$, $1 \le s \le k-1$ and $y - a_{x,r-1}^i \le k$. We consider $\mathcal{C}_x^i$ as defined in case (a). We know that $\mathcal{C}_x^i \prec C_x^i$, $y \in \mathcal{C}_x^i$ and $\mathcal{C}_x^i$ is a minimal cut set. We now show that $i \in \mathcal{C}_x^i$. We have

$r \ne r_0$, therefore $a_{x,r}^i \ne a_{x,r_0}^i = i$. Suppose $a_{x,r+2}^i - y \le k$ and $r < r_x^i$. We show that $r \ne r_0 - 1$. Suppose not, that is $r = r_0 - 1$ we then have $a_{x,r+2}^i = a_{x,r_0+1}^i = a_{x,r_x^i-1}^i = n+1$ and $n+1-y \le k$. Therefore, $y = a_{x,r_0-1}^i + s \ge n-k+1 > n-k$. By the third condition of Definition 2, we get $y \notin \hat{C}_x^i$ resulting in a contradiction. Hence $r \ne r_0 - 1$ and $a_{x,r+1}^i \ne a_{x,r_0}^i = i$ therefore $i \in \mathcal{C}_x^i$. This completes the proof of part (II).

(III) Suppose for j, $1 \leq j \leq x-1$ and $C_j^i = \{a_{j,1}^i, a_{j,2}^i, \ldots, a_{j,r_j}^i\}$ is a minimal cut set that contains i and $C_j^i \prec C_x^i$. By Lemma 1 of Chapter 3, we have $\exists\, \ell$, $1 \leq \ell \leq r_j^i$ $s.t.$ $a_{j,\ell}^i > a_{x,\ell}^i$ and $a_{j,t}^i = a_{x,t}^i$, $t = 1, 2, \ldots, \ell - 1$. Here also we consider two cases:

case (i) Let $r_0 < r_x^i$.

We note that $\ell \neq r_0$. Because if $\ell = r_0$ then $a_{j,r_0}^i = a_{j,\ell}^i > a_{x,\ell}^i = a_{x,r_0}^i = i > a_{x,\ell-1}^i = a_{j,\ell-1}^i$. Hence $i \notin C_j^i$ and this contradicts our hypothesis that $i \in C_j^i$. Therefore $\ell \neq r_0$. We have

$a_{j,\ell}^i > a_{x,\ell}^i \Rightarrow \exists\; s \geq 1$ $s.t.$ $a_{j,\ell}^i = a_{x,\ell}^i + s$ and $a_{j,\ell}^i - a_{x,\ell-1}^i = a_{j,\ell}^i - a_{j,\ell-1}^i \leq k \Rightarrow\; s \leq k-1$.

If $\ell \neq r_0 - 1$ then in view of the first condition of Definition 2, we have $a_{j,\ell}^i \in \hat{C}_x^i$ and therefore $\hat{C}_x^i \cap C_j^i \neq \emptyset$.

Now suppose $\ell = r_0 - 1$. We show that $a_{x,r_0}^i = a_{j,r_0}^i = i$. It is enough to show that $a_{j,\ell}^i < i$. Suppose $a_{j,\ell}^i \geq i$ we then have

$a_{j,\ell}^i - a_{j,\ell-1}^i \geq i - a_{j,\ell-1}^i = a_{x,\ell+1}^i - a_{x,\ell-1}^i \geq k+1$ and this contradicts second condition of Theorem 1 of Chapter 3. Therefore $a_{j,\ell}^i = a_{j,r_0-1}^i < i$ and hence $a_{j,r_0}^i = i$.

If $a_{x,r_0+1}^i - a_{j,\ell}^i > k$ then obviously $a_{j,\ell}^i \in \hat{C}_x^i$. Now consider the case where

$$a_{x,r_0+1}^i - a_{j,\ell}^i \leq k. \tag{3}$$

In this case in view of the first condition of Definition 2, we note that $a_{j,\ell}^i \notin \hat{C}_x^i$. We now show that $a_{j,r_0+1}^i \in \hat{C}_x^i$. We note that $a_{j,r_0+1}^i > a_{x,r_0+1}^i$. Suppose not, i.e., $a_{j,r_0+1}^i \leq a_{x,r_0+1}^i$ then

$k+1 \leq a_{j,r_0+1}^i - a_{j,r_0-1}^i \leq a_{x,r_0+1}^i - a_{j,r_0-1}^i = a_{x,r_0+1}^i - a_{j,\ell}^i \leq k$

(by inequality (3)) and resulting in a contradiction. Therefore

$a_{j,r_0+1}^i > a_{x,r_0-1}^i \Rightarrow \exists\; s' \geq 1$ $s.t.$ $a_{j,r_0+1}^i = a_{x,r_0+1}^i + s'$.

On the other hand we have

$a_{j,r_0+1}^i - a_{j,r_0}^i = a_{j,r_0+1}^i - a_{x,r_0}^i = a_{x,r_0+1}^i + s' - a_{x,r_0}^i \leq k$. Hence $s' \leq k-1$.

It follows that $a^i_{j,r_0+1} \in \hat{C}^i_x$ and therefore $\hat{C}^i_x \cap C^i_j \neq \emptyset$. (We also note that $r_0 + 1 \leq r^i_x$).

case (ii) Now suppose $r_0 = r^i_x$.

In this case, we have $a^i_{x,r^i_x} = a^i_{j,r^i_j} = i$. In other words $\ell \neq r_0$.

We have

$C^i_j \prec C^i_x \Rightarrow \exists\, \ell,\ 1 \leq \ell < r^i_x$ *s.t.* $a^i_{j,\ell} > a^i_{x,\ell}$, $a^i_{j,t} = a^i_{x,t}$, $t = 1, 2, \ldots, \ell - 1$ and $\exists\, s,\ 1 \leq s \leq k-1$ *s.t.* $a^i_{j,\ell} = a^i_{x,\ell} + s$, $a^i_{j,\ell} - a^i_{x,\ell-1} \leq k$.

If $\ell < r_0 - 1$ then $a^i_{j,\ell} = a^i_{x,\ell} + s \in \hat{C}^i_x$.

Suppose $\ell = r_0 - 1$. We show that $a^i_{j,\ell} \leq n - k$.

Using second condition of Theorem 1 of Chapter 3, we have $n + 1 - a^i_{j,r_0-1} \geq k + 1$. Therefore $a^i_{j,r_0-1} \leq n - k$ and by second condition of Definition 2 we get $a^i_{j,r_0-1} = a^i_{x,r_0-1} + s \in \hat{C}^i_x$. Hence $\hat{C}^i_x \cap C^i_j \neq \emptyset$. This completes the proof of part (III).

(IV) Suppose $\hat{\hat{C}}{}^i_x \neq \hat{C}^i_x$ and $\hat{\hat{C}}{}^i_x$ satisfies (I) and (III). We show that $\hat{C}^i_x \subseteq \hat{\hat{C}}{}^i_x$. Suppose not, that is there exists $y \in \hat{C}^i_x$ such that $y \notin \hat{\hat{C}}{}^i_x$. By definition of $\hat{C}^i_x$ as stated in Definition 2 we have

$\exists\, r,\ 1 \leq r \leq r^i_x,\ r \neq r_0,\ 1 \leq s \leq k-1$ *s.t.* $y = a^i_{x,r} + s$ and $y - a^i_{x,r-1} \leq k$.

We consider $\underline{C}^i_x$ as defined in proof of part (II).

We showed that $\underline{C}^i_x \prec C^i_x$ and $\underline{C}^i_x$ is a minimal cut set that contains i and y. Now we note that $y \notin \hat{\hat{C}}{}^i_x$ and $\hat{\hat{C}}{}^i_x$ satisfies (I). Therefore we have $\hat{\hat{C}}{}^i_x \cap \underline{C}^i_x = \emptyset$ and this contradicts the fact that $\hat{\hat{C}}{}^i_x$ satisfies (III).

Hence $y \in \hat{\hat{C}}{}^i_x$ and then $\hat{C}^i_x \subseteq \hat{\hat{C}}{}^i_x$. Therefore we get $|\hat{C}^i_x| \leq |\hat{\hat{C}}{}^i_x|$. This completes the proof of part (IV).

(V) Suppose $\hat{\hat{C}}{}^i_x$ satisfies (I), (III) and (IV). In the proof of part (IV) we showed that $\hat{C}^i_x \subseteq \hat{\hat{C}}{}^i_x$ and $|\hat{C}^i_x| \leq |\hat{\hat{C}}{}^i_x|$. $\hat{\hat{C}}{}^i_x$ satisfies (IV) therefore $|\hat{C}^i_x| = |\hat{\hat{C}}{}^i_x|$. Hence we have $\hat{C}^i_x = \hat{\hat{C}}{}^i_x$ and this completes the proof of the theorem.

We now derive a formula for determining $Pr\{E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_1\}$ which contains only one term.

Lemma 3. $Pr\{E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_1\} = Pr\{E^i_x \cap \hat{E}^i_x\}$, where $\hat{E}^i_x$ is the event that all components of $\hat{C}^i_x$ are working.

Proof. We note that $E^i_x \cap \hat{E}^i_x \subseteq E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_1$, because $\hat{C}^i_x \cap C^i_j \neq \emptyset$, $j = 1, 2, \ldots, x-1$. We show that $E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_1 \subseteq E^i_x \cap \hat{E}^i_x$. Suppose not, then there exists $y \in \hat{C}^i_x$ such that component y is failed. By the definition of $\hat{C}^i_x$ (see Definition 2) we can find r and s, where
$1 \leq r \leq r^i_x$, $r \neq r_0$, $1 \leq s \leq k-1$ *s.t.* $y = a^i_{x,r} + s$ and $y - a^i_{x,r-1} \leq k$. We consider $\mathcal{C}^i_x$ as defined in the proof of part (II) of Theorem 2. We showed that $\mathcal{C}^i_x \prec C^i_x$ and $\mathcal{C}^i_x$ is a minimal cut set that contains i and y. We assumed that E^i_x has occurred and component y is failed, therefore all components of $\mathcal{C}^i_x$ are failed. But this contradicts the assumption that the event $\bar{E}^i_{x-1} \cap \bar{E}^i_{x-2} \cap \ldots \cap \bar{E}^i_1$ has occurred, hence at least one component from each of $C^i_{x-1}, C^i_{x-2}, \ldots, C^i_1$ is working. Thus $E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_1 \subseteq E^i_x \cap \hat{E}^i_x$, that is both these two events are equivalent. This completes the proof of the lemma. ∎

Remark 5. We note that $\hat{C}^i_x \subseteq N - C^i_x$ therefore $C^i_x \cap \hat{C}^i_x = \emptyset$ and hence the two events E^i_x and $\hat{E}^i_x$ are independent. So using Lemma 3, we can write

$$Pr\{E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_1\} = Pr\{E^i_x \cap \hat{E}^i_x\} = Pr\{E^i_x\}.Pr\{\hat{E}^i_x\} = \prod_{j \in C^i_x} q_j * \prod_{j \in \hat{C}^i_x} p_j \, .$$

We now can write a closed formula for determining $Pr\left\{\bigcup_{x=1}^{n_k(i)} E^i_x\right\}$ as follows :

$$Pr\left\{\bigcup_{x=1}^{n_k(i)} E^i_x\right\} = Pr\{E^i_1\} + \sum_{x=2}^{n_k(i)} Pr\{E^i_x \cap \bar{E}^i_{x-1} \cap \ldots \cap \bar{E}^i_1\}$$

ch is equal to

$$Pr\{E_1^i\} + \sum_{x=2}^{n_k(i)} Pr\{E_x^i \cap \hat{E}_x^i\} = Pr\{E_1^i\} + \sum_{x=2}^{n_k(i)} Pr\{E_x^i\} . Pr\{\hat{E}_x^i\}$$

$$= \prod_{j \in C_1^i} q_j + \sum_{x=2}^{n_k(i)} \prod_{j \in C_x^i} q_j \prod_{j \in \hat{C}_x^i} p_j.$$

Now using Algorithm 3 of Chapter 3, we state an algorithm to compute Vesely-Fussell reliability measure of component importance in a con$|k|n$:G system.

ALGORITHM 2.

Input. Positive integers n, k ($n \geq k$), reliability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $1 \leq i \leq n$.

Output. Vesely-Fussell reliability measure of importance for component i in a con$|k|n$:G system.

Step 0. Put $x = 1$ and $\bar{R}_i = 0$ and go to step 1.

Step 1. Generate C_x^i. If $x = 1$ put $\hat{C}_x^i = \emptyset$ and $\bar{P} = 1$; otherwise compute $\hat{C}_x^i$ and put $\bar{P} = \prod_{j \in \hat{C}_x^i} p_j$. ($\hat{C}_x^i$ is obtained as in Definition 2). Go to step 2.

Step 2. Put $\bar{R}_i = \bar{R}_i + \prod_{j \in C_x^i} q_j * \bar{P}$. If C_x^i is the last minimal cut set that contains the component i, that is $x = n_k(i)$, go to step 3; otherwise put $x = x + 1$ and then go to step 1.

Step 3. $\bar{R}_i / Pr\{\phi^G(\mathbf{X}) = 0\}$ gives the Vesely-Fussell reliability measure of importance for component i in a con$|k|n$:G system. Stop.

We note that $Pr\{\phi^G(\mathbf{X}) = 0\} = 1 - h_k^G(\mathbf{p}, n)$ can be determined by using Algorithm 1. It may be noted that for the purpose of ranking of components it is not necessary to compute $Pr\{\phi^G(\mathbf{X}) = 0\}$.

Example 4. Consider component 4 in Example 1. Minimal cut sets containing component 4 arranged in lex ordering are:

x	C_x^4			$\hat{C}_x^4$	
1	3	4	7		
2	2	4	7	3	
3	2	4	6	7	
4	1	4	7	2	3
5	1	4	6	2	7
6	1	4	5	6	7

Therefore using Remark 5, we have

$$I_{VF}^{G}(4,\mathbf{p}) = \frac{q_3q_4q_7 + q_2q_4q_7p_3 + q_2q_4q_6p_7 + q_1q_4q_7p_2p_3 + q_1q_4q_6p_2p_7 + q_1q_4q_5p_6p_7}{Pr\{\phi^G(\mathbf{X}) = 0\}}$$

Remark 6. Using Remark 1 and Remark 5, we can compute Vesely-Fussell structural importance of component i in a con$|k|n$:G system as follows :

$$I_{VF}^{G,\phi}(i) = I_{VF}^{G}(i;1/2,1/2,\ldots,1/2) = \frac{(1/2)^{|C_1^i|} + \sum_{x=2}^{n_k(i)} (1/2)^{|C_x^i|+|\hat{C}_x^i|}}{(1/2)^{|C_1|} + \sum_{x=2}^{n(k)} (1/2)^{|C_x|+|\hat{C}_x|}}.$$

C_x^i and $\hat{C}_x^i$ are as given in Definition 1 and Definition 2, respectively.

Vesely-Fussell Importance Measure in a con$|k|n$:F System

We now consider the problem of evaluation of the Vesely-Fussell measure of component importance in a con$|k|n$:F system. We know that a minimal cut set of a con$|k|n$:F system is of the form $D_x = \{x, x+1, \ldots, x+k-1\}$, $x = 1, 2, \ldots, n-k+1$. Hence we have

$$\mathbf{D}(i) = \begin{cases} \{D_1, D_2, \ldots, D_i\} & \text{if } 1 \le i \le k \\ \{D_{i-k+1}, D_{i-k+2}, \ldots, D_i\} & \text{if } k < i \le n-k+1 \\ \{D_{i-k+1}, D_{i-k+2}, \ldots, D_{n-k+1}\} & \text{if } n-k+1 < i \le n \end{cases}$$

where $\mathbf{D}(i)$ denotes the collection of all minimal cut sets that contain component i in a con$|k|n$:F system.

Vesely-Fussell reliability importance of component i in a con$|k|n$:F system is given by :

$$I_{VF}^{F}(i,\mathbf{p}) = \frac{Pr\left\{\bigcup_{x=1}^{m_k(i)} A_x^i\right\}}{Pr\left\{\phi^F(\mathbf{X})=0\right\}}$$

where A_x^i is the event that all components of the minimal cut set D_x^i are failed and $m_k(i)$ is the number of all minimal cut sets of a con$|k|n$:F system, that contain component i.

Lemma 4. $I_{VF}^{F}(i,\mathbf{p}) \propto Pr\left\{\bigcup_{x=1}^{m_k(i)} A_x^i\right\}$ and

$$Pr\left\{\bigcup_{x=1}^{m_k(i)} A_x^i\right\} = \begin{cases} \prod_{j=1}^{k} q_j & \text{if } i=1 \\ \prod_{j=1}^{k} q_j + \sum_{x=2}^{i} p_{x-1} \prod_{j=x}^{x+k-1} q_j & \text{if } 1<i\le k \\ \prod_{j=i-k+1}^{i} q_j + \sum_{x=i-k+2}^{i} p_{x-1} \prod_{j=x}^{x+k-1} q_j & \text{if } k<i\le n-k+1 \\ \prod_{j=i-k+1}^{i} q_j + \sum_{x=i-k+2}^{n-k+1} p_{x-1} \prod_{j=x}^{x+k-1} q_j & \text{if } n-k+1<i\le n-1 \\ \prod_{j=n-k+1}^{n} q_j & \text{if } i=n \end{cases}$$

Proof. Using the relation

$$Pr\left\{\bigcup_{x=1}^{m_k(i)} A_x^i\right\} = Pr\{A_1^i\} + \sum_{x=2}^{m_k(i)} Pr\{A_x^i \cap \bar{A}_{x-1}^i \cap \ldots \cap \bar{A}_2^i \cap \bar{A}_1^i\}$$

and the structure of $\mathbf{D}(i)$ the proof follows.

Remark 7. From Lemma 4, we note that

$$I_{VF}^{F}(1,\mathbf{p}) < I_{VF}^{F}(2,\mathbf{p}) < \cdots < I_{VF}^{F}(k,\mathbf{p}).$$

Similarly it can be verified that

$$I_{VF}^{F}(n,\mathbf{p}) < I_{VF}^{F}(n-1,\mathbf{p}) < \cdots < I_{VF}^{F}(n-k+1,\mathbf{p}).$$

Lemma 5. Vesely-Fussell structural importance of component i, in a con/k/n:F system is given by : $I_{VF}^{F,\phi}(i) = I_{VF}^{F}(i;\, 1/2,\ldots,1/2)$ and we have

$$I_{VF}^{F}(i;\,1/2,\ldots,1/2) \propto \begin{cases} (1/2)^k & \text{if } \; i=1 \\ (1/2)^k + (i-1)(1/2)^{k+1} = \frac{i+1}{2^{k+1}} & \text{if } \; 1<i\le k \\ (1/2)^k + (k-1)(1/2)^{k+1} = \frac{k+1}{2^{k+1}} & \text{if } \; k<i\le n-k+1 \\ (1/2)^k + (n-i)(1/2)^{k+1} = \frac{n-i+2}{2^{k+1}} & \text{if } \; n-k+1<i\le n-1 \\ (1/2)^k & \text{if } \; i=n \end{cases}$$

Proof. The proof follows from Lemma 4.

Remark 8. From Lemma 5, we have

$$I_{VF}^{F,\phi}(1) < I_{VF}^{F,\phi}(2) < \cdots < I_{VF}^{F,\phi}(k) = I_{VF}^{F,\phi}(k+1) = \cdots = I_{VF}^{F,\phi}(n-k+1)$$

$$I_{VF}^{F,\phi}(n) < I_{VF}^{F,\phi}(n-1) < \cdots < I_{VF}^{F,\phi}(n-k+1)$$

and

$$I_{VF}^{F,\phi}(i) = I_{VF}^{F,\phi}(n-i+1)\,,\; i=1,2,\ldots,n.$$

Remarks. Although using different structural importance measures, different importance patterns for components (ordering) can be established, but it does not seem to be case for the reliability importance measures, as the component reliabilities may vary. However under certain assumptions on component reliabilities, partial ordering of components can be obtained. Regarding the Vesely-Fussell reliability importance pattern of a consecutive-k-out-of-n:F system, we have obtained the following results as given in Lemma 6. First we assume that $p_i = p_{n-i+1}$, $i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$. In view of this and by using notations given in Lemma 4, it is easy to show that two events $\bigcup_{x=1}^{m_k(i)} A_x^i$ and $\bigcup_{x=1}^{m_k(n-i+1)} A_x^{n-i+1}$ are equivalent.

In other words, if $p_i = p_{n-i-1}$ then $I_{VF}^F(i, \mathbf{p}) = I_{VF}^F(n-i+1, \mathbf{p})$, $i = 1, 2, \ldots, m$. This means Vesely-Fussell reliability importance patterns among components $1, 2, \ldots, m$ includes analogous patterns for the remaining components. (It can be easily shown that this property also holds for the Birnbaum reliability importance measure if $p_i = p_{n-i+1}$, $i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$, which is considered in the next section).

Now using this and in view of Lemma 4, we have obtained the following results that are given in the next Lemma.

Lemma 6. $I_{VF}^F(i, \mathbf{p})$, the Vesely-Fussell importances for the components of a consecutive-k-out-of-n:F system satisfy the following patterns:

(a) If $p_1 < p_2 < \cdots < p_k$, $p_{k+1} = p_{k+2} = \cdots = p_{n-k} = p$, $p_i = p_{n-i+1}$,

$i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$ and $p < p_1$ then

$I_{VF}^F(1, \mathbf{p}) < I_{VF}^F(2, \mathbf{p}) < \cdots < I_{VF}^F(k, \mathbf{p}) < \cdots < I_{VF}^F(2k, \mathbf{p})$,

$I_{VF}^F(2k, \mathbf{p}) = I_{VF}^F(2k+1, \mathbf{p}) = \cdots = I_{VF}^F(n-2k+1, \mathbf{p})$

and $I_{VF}^F(n-2k+1, \mathbf{p}) > I_{VF}^F(n-2k+2, \mathbf{p}) > \cdots > I_{VF}^F(n, \mathbf{p})$.

$I_{VF}^F(i, \mathbf{p}) = I_{VF}^F(n-i+1, \mathbf{p}), i = 1, 2, \ldots, m.$

(b) If $p_1 < p_2 < \cdots < p_k$, $p_{k+1} = p_{k+2} = \cdots = p_{n-k} = p$, $p_i = p_{n-i-1}$,

$i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$ and $p > p_k$ then

$I^F_{VF}(1, \mathbf{p}) < I^F_{VF}(2, \mathbf{p}) < \cdots < I^F_{VF}(k, \mathbf{p})$,

$I^F_{VF}(k, \mathbf{p}) > I^F_{VF}(k+1, \mathbf{p}) > \cdots > I^F_{VF}(2k, \mathbf{p})$,

$I^F_{VF}(2k, \mathbf{p}) = I^F_{VF}(2k+1, \mathbf{p}) = \cdots = I^F_{VF}(n-2k+1, \mathbf{p})$,

$I^F_{VF}(n-2k+1, \mathbf{p}) < I^F_{VF}(n-2k+2, \mathbf{p}) < \cdots < I^F_{VF}(n-k+1, \mathbf{p})$

and $I^F_{VF}(n-k+1, \mathbf{p}) > I^F_{VF}(n-k+2, \mathbf{p}) > \cdots > I^F_{VF}(n, \mathbf{p})$.

$I^F_{VF}(i, \mathbf{p}) = I^F_{VF}(n-i+1, \mathbf{p}), i = 1, 2, \ldots, m$.

(c) If $p_1 = p_2 = \cdots = p_k = p$, $p_{k+1} < p_{k+2} < \cdots < p_m$, $p_i = p_{n-i+1}$,

$i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$ and $p < p_{k+1}$ then

$I^F_{VF}(1, \mathbf{p}) < I^F_{VF}(2, \mathbf{p}) < \cdots < I^F_{VF}(k, \mathbf{p})$,

$I^F_{VF}(k, \mathbf{p}) > I^F_{VF}(k+1, \mathbf{p}) > \cdots > I^F_{VF}(m-1, \mathbf{p}) > I^F_{VF}(m, \mathbf{p})$

and $I^F_{VF}(i, \mathbf{p}) = I^F_{VF}(n-i+1, \mathbf{p}), i = 1, 2, \ldots, m$.

(d) If $p_1 = p_2 = \cdots = p_k = p$, $p_{k+1} < p_{k+2} < \cdots < p_m$, $p_i = p_{n-i+1}$,

$i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$ and $p > p_m$ then

$I^F_{VF}(1, \mathbf{p}) < I^F_{VF}(2, \mathbf{p}) < \cdots < I^F_{VF}(k, \mathbf{p}) < I^F_{VF}(k+1, \mathbf{p})$,

$I^F_{VF}(2k, \mathbf{p}) > I^F_{VF}(2k+1, \mathbf{p}) > \cdots > I^F_{VF}(m-1, \mathbf{p}) > I^F_{VF}(m, \mathbf{p})$

and $I^F_{VF}(i, \mathbf{p}) = I^F_{VF}(n-i+1, \mathbf{p}), i = 1, 2, \ldots, m$.

(e) If $p_1 < p_2 < \cdots < p_m$ and $p_i = p_{n-i+1}, i = 1, 2, \ldots, m = \lfloor (n+1)/2 \rfloor$, then

$I^F_{VF}(1, \mathbf{p}) < I^F_{VF}(2, \mathbf{p}) < \cdots < I^F_{VF}(k, \mathbf{p})$,

$I^F_{VF}(k, \mathbf{p}) > I^F_{VF}(k+1, \mathbf{p}) > \cdots > I^F_{VF}(m-1, \mathbf{p}) > I^F_{VF}(m, \mathbf{p})$

and $I^F_{VF}(i, \mathbf{p}) = I^F_{VF}(n-i+1, \mathbf{p}), i = 1, 2, \ldots, m$.

roof. Using Lemma 4 it can be simply shown that for $i = k, k+1, \ldots, n-k$ we have:

$$I^F_{VF}(i+1,\mathbf{p}) - I^F_{VF}(i,\mathbf{p}) = p_i\left(\prod_{j=i+1}^{i+k} q_j\right) - p_{i+1}\left(\prod_{j=i-k+1}^{i} q_j\right).$$

Hence using this and in view of Remark 7, the above mentioned cases can be easily argued.

We note that, Remark 8 gives a complete ordering of structural Vesely-Fussell importance measure in a consecutive-k-out-of-n:F system. It also holds for reliability Vesely-Fussell importance measure in iid case ($p_i = p, i = 1, 2, \ldots, n$).

4.4 Birnbaum Importance Measure in a con|k|n:F System

This Section considers Birnbaum measure of component importance in a con|k|n:F system. Birnbaum [8], defined reliability and structural importance of component i respectively as follows :

$$I_B(i,\mathbf{p}) = Pr\{\phi(1_i,\mathbf{X}) > \phi(0_i,\mathbf{X})\} = Pr\{(\cdot_i,\mathbf{X}) \in \mathbf{B}(i)\}$$

and

$$I^{\phi}_B(i) = \frac{|\{(\cdot_i,\mathbf{x}):\ \phi(1_i,\mathbf{x}) > \phi(0_i,\mathbf{x})\}|}{2^{n-1}} = I_B(i, 1/2, 1/2, \ldots, 1/2)$$

where $\phi(\mathbf{x})$ is structure function of the system and $\mathbf{B}(i)$ is the collection of all critical vectors for component i. Recall that $(\cdot_i,\mathbf{x})$ is a critical vector for component i if and only if $\phi(1_i,\mathbf{x}) = 1$ and $\phi(0_i,\mathbf{x}) = 0$.

Using the approach presented in Section 2 and Algorithm 4 of Chapter 3, we provide a nonrecursive formula for determining Birnbaum reliability measure of importance for component i, in a con|k|n:F system. In this formula computation of reliability function is not required. Furthermore we use only *minimal critical vectors* for component i, and this reduces number of computational steps.

Let $(\cdot_i,\mathbf{x})$ be a critical vector for component i. In view of Theorem 7 of Chapter

3, we note that $(\cdot_i, \mathbf{x})$ is of the following form :

$$(\cdot_i, \mathbf{x}) = (\cdot_i, \mathbf{1}_{P^i-\{i\}}, \mathbf{0}_{Q^i-\{i\}}, \cdot_{N-(P^i \cup Q^i)})$$

where P^i and Q^i are a minimal path set and a minimal cut set of the system such that $P^i \cap Q^i = \{i\}$. Therefore we can write

$$I_B(i, \mathbf{p}) = Pr \left\{ \bigcup_{(\cdot_i, \mathbf{x}) \in \mathbf{B}(i)} W_i \right\}$$

where W_i is the event that all components of $P^i - \{i\}$ are working and all components of $Q^i - \{i\}$ are failed.

In order to generate all critical vectors for component i, in a con$|k|n$:F system it is enough to consider the case $k+1 \leq i \leq n-k$, since other cases for i can be obtained as special cases of this case (see Remark 8 of Chapter 3). Now suppose

$$P_x^i = \{a_{x,1}, a_{x,2}, \ldots, a_{x,r_0-1}, i, a_{x,r_0+1}, \ldots, a_{x,r_x^i}\} \text{ where } a_{x,r_0} = i$$

is a minimal path set of a con$|k|n$:F system that contains component i. In view of Theorem 7 of Chapter 3, the minimal cut sets corresponding to P_x^i are as follows :

$$Q_{x,s}^i = \{a_{x,r_0+1} - k - s + 1, a_{x,r_0+1} - k - s + 2, \ldots, a_{x,r_0+1} - s\} \text{ for } s = 1, 2, \ldots, n_x$$

where $n_x = a_{x,r_0+1} - a_{x,r_0-1} - k$. We note that $1 \leq n_x \leq k$ and these minimal cut sets are arranged in lex ordering. We also assume that all minimal path sets are arranged in lex ordering. Suppose W_x^i is the event that all components of $P_x^i - \{i\}$ are working and $F_{x,s}^i$ is the event that all components of $Q_{x,s}^i - \{i\}$ are failed. We know that for all $s = 1, 2, \ldots, n_x$ these two events are independent, since $P_x^i \cap Q_{x,s}^i = \{i\}$. Next lemma is required in the sequel.

Lemma 7.

$$Pr\left\{W_x^i \cap \left(\bigcup_{s=1}^{n_x} F_{x,s}^{i'}\right)\right\} = Pr\{W_x^i\} Pr\left\{\bigcup_{s=1}^{n_x} F_{x,s}^i\right\} = \left(\prod_{j\in P_x^i-\{i\}} p_j\right)\left(\sum_{s=1}^{n_x} p_m \prod_{t\in Q_{x,s}^i-\{i\}} q_t\right)$$

where $m = a_{x,r_0+1} - s + 1$ and we assume that $p_m = 1$ if $m = a_{x,r_0+1}$, that is if $s = 1$.

Proof. It is enough to show that

$$Pr\left\{\bigcup_{s=1}^{n_x} F_{x,s}^i\right\} = \sum_{s=1}^{n_x} p_m \prod_{t\in Q_{x,s}^i-\{i\}} q_t.$$

We know that

$$Pr\left\{\bigcup_{s=1}^{n_x} F_{x,s}^i\right\} = Pr\{F_{x,1}^i\} + \sum_{s=2}^{n_x} Pr\{F_{x,s}^i \cap \bar{F}_{x,s-1}^i \cap \bar{F}_{x,s-2}^i \cap \cdots \cap \bar{F}_{x,1}^i\}.$$

In view of the fact that $Q_{x,s}^i$'s are arranged in lex ordering and $n_x \leq k$, it is easy to verify that the two events $F_{x,s}^i \cap \bar{F}_{x,s-1}^i \cap \bar{F}_{x,s-2}^i \cap \cdots \cap \bar{F}_{x,1}^i$ and $F_{x,s}^i \cap \{X_m = 1\}$ are equivalent. Hence we can write

$$Pr\{F_{x,s}^i \cap \bar{F}_{x,s-1}^i \cap \bar{F}_{x,s-2}^i \cap \cdots \cap \bar{F}_{x,1}^i\} = Pr\left(F_{x,s}^i \cap \{X_m = 1\}\right) = Pr\{F_{x,s}^i\} Pr\{X_m = 1\}.$$

Using the assumption $p_m = 1$, when $s = 1$, we then have

$$Pr\left\{\bigcup_{s=1}^{n_x} F_{x,s}^i\right\} = \sum_{s=1}^{n_x} p_m \prod_{t\in Q_{x,s}^i-\{i\}} q_t$$

where $m = a_{x,r_0-1} - s + 1$. This completes the proof of the lemma. ∎

Again consider $P_x^i = \{a_{x,1}, a_{x,2}, \ldots, a_{x,r_0-1}, i, a_{x,r_0+1}, \ldots, a_{x,r_x^i}\}$.
We note that $m = a_{x,r_0+1} - s + 1$, for $s = 1, 2, \ldots, a_{x,r_0+1} - a_{x,r_0-1} - k$. Hence we have $a_{x,r_0-1} + k + 1 \leq m \leq a_{x,r_0+1}$, therefore $i < m \leq a_{x,r_0+1}$. Recall that $\hat{C}_x^i$, the set corresponding to P_x^i in Definition 2. As $i < m \leq a_{x,r_0+1}$ and the definition of $\hat{C}_x^i$, implies that $m \notin \hat{C}_x^i$. Using this and Lemma 7, we give a

direct formula for determining Birnbaum reliability measure of importance for component i, in a con$|k|n$:F system as follows.

Theorem 3. Birnbaum reliability measure of importance for component i, in a con$|k|n$:F system is given by

$$I_B(i,\mathbf{p}) = \sum_{x=1}^{n_k(i)} \left(\prod_{j \in P_x^i - \{i\}} p_j \right) \left(\sum_{s=1}^{n_x} p_m \prod_{l \in \left(Q_{x,s}^i \cup \hat{C}_x^i - \{i\}\right)} q_l \right)$$

where $n_k(i)$ is the number of minimal path sets of the system that contain component i.

Proof. We note that $I_B(i,\mathbf{p}) = Pr\left\{ \bigcup_{x=1}^{n_k(i)} \left(W_x^i \cap \left(\bigcup_{s=1}^{n_x} F_{x,s}^i \right) \right) \right\}$. Using the similar argument given in Lemma 3, it can be shown that two events

$$W_x^i \cap \left(\bigcup_{s=1}^{n_x} F_{x,s}^i \right) \cap \overline{\left\{ W_{x-1}^i \cap \left(\bigcup_{s=1}^{n_{x-1}} F_{x-1,s}^i \right) \right\}} \cap \cdots \cap \overline{\left\{ W_1^i \cap \left(\bigcup_{s=1} F_{1,s}^i \right) \right\}}$$

and $W_x^i \cap \left(\bigcup_{s=1}^{n_x} F_{x,s}^i \right) \cap \hat{W}_x^i$ are equivalent. Where $\hat{W}_x^i$ is the event that all components of $\hat{C}_x^i$ are failed. The proof of the theorem follows from Lemma 6.

■

Remark 9. For using Theorem 3 in the special cases $i \leq k$, (that is $r_0 = 1$) and $n - k < i$, (that is $r_0 = r_x^i$) we assume that $a_{x,r_0-1} = a_{x,0} = 0$ and $a_{x,r_0+1} = a_{x,r_x^i+1} = n + 1$, respectively.

In fact on the basis of Algorithm 4 of Chapter 3, Theorem 3 introduces an algorithm for determining Birnbaum reliability measure of component importance in a con$|k|n$:F system.

Remark 10. Griffith and Govindarajula [28], first considered the problem of calculating Birnbaum's measure of component importance in a con$|k|n$:F system. For the case where all componnents of the system are i.i.d., they introduced an approach for computing Birnbaum reliability measure of component importance in a con$|k|n$:F using Markov chain techniques.

Remarks. Chadjiconstantinidis and Koutras [13] have given both exact and recursive formulae for the evaluation of the Birnbaum importance of reliability structures belonging to the wide class of Markov Chain Imbeddable Systems (MIS). The general results obtained are used for studying the Birnbaum importance and the other two importance measures for a consecutive-k-out-of-n:F system. Some more results on importance ordering are also given.

On the basis of Birnbaum importance measure in iid case, Chang, Cui and Hwang [15] have given some partial orderings for the components of a consecutive-k-out-of-n:F system.

Using the Fibonacci sequence of order k, Lin, Kuo and Hwang [42], have given a closed formula for determining the Birnbaum structural importance of components in a consecutive-k-out-of-n:F system. For some values of n and k, and on the basis of Birnbaum structural importance measure, ordering of the components in a consecutive-k-out-of-n:F system are also given in [42].

Chadjiconstantinidis and Koutras [13], showed that Birnbaum reliability importance of component i in a consecutive-k-out-of-n:F system is given by:

$$I_B(i,\mathbf{p}) = \frac{1}{q_i}\{h_k(p_1,\ldots,p_{i-1},i-1)h_k(p_{i+1},\ldots,p_n,n-i) - h_k(\mathbf{p},n)\}$$

where $h_k(\mathbf{p},m)$ is the reliability of a consecutive-k-out-of-m:F system which is computed by the Markov Chain approach formula (2), given in Section 4.2. Here we see that, in order to evaluate the Birnbaum reliability importance measure of a component, we need to apply formula (2) for each component separately.

We now consider Theorem 3. It provides a closed formula for Birnbaum reliability measure of importance for component i, in a consecutive-k-out-of-n:F system. This Theorem requires the availability of critical vectors for component i, which can be obtained using Algorithm 4 of Chapter 3. In this algorithm we need to generate all minimal path sets containing a given component. Therefore Theorem 3 may be used but not for large values of n as the minimal path sets containing a given component are needed.

However we can compute Birnbaum reliability importance measure of components in a consecutive-k-out-of-n:F system using Algorithm 1 of Section 4.2.

Consider Example 1, where the reliability function of a consecutive-3-out-7:F system is given by:

$h_3^F(\mathbf{p},7) = p_3p_6 + p_3p_5q_6 + p_3p_4p_7q_5q_6 + p_2p_5q_3 + p_2p_4p_7q_3q_5 + p_2p_4p_6q_3q_5q_7 + p_1p_4p_7q_2q_3 + p_1p_4p_6q_2q_3q_7 + p_1p_4p_5q_2q_3q_6q_7$. Therefore we get the result:

$$
\begin{aligned}
I_B(1,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_1} &= p_4p_7q_2q_3 + p_4p_6q_2q_3q_7 + p_4p_5q_2q_3q_6q_7 \\
I_B(2,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_2} &= p_5q_3 + p_4p_7q_3q_5 + p_4p_6q_3q_5q_7 - p_1p_4p_7q_3 - p_1p_4p_6q_3q_7 \\
&\quad -p_1p_4p_5q_3q_6q_7 \\
I_B(3,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_3} &= p_6 + p_5q_6 + p_4p_7q_5q_6 - p_2p_5 - p_2p_4p_7q_5 - p_2p_4p_6q_5q_7 \\
&\quad -p_1p_4p_7q_2 - p_1p_4p_6q_2q_7 - p_1p_4p_5q_2q_6q_7 \\
I_B(4,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_4} &\quad p_3p_7q_5q_6 + p_2p_7q_3q_5 + p_2p_6q_3q_5q_7 + p_1p_7q_2q_3 + p_1p_6q_2q_3q_7 \\
&\quad +p_1p_5q_2q_3q_6q_7 \\
I_B(5,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_5} &= p_3q_6 - p_3p_4p_7q_6 + p_2q_3 - p_2p_4p_7q_3 - p_2p_4p_6q_3q_7 + p_1p_4q_2q_3q_6q_7 \\
I_B(6,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_6} &= p_3 - p_3p_5 - p_3p_4p_7q_5 + p_2p_4q_3q_5q_7 + p_1p_4q_2q_3q_7 - p_1p_4p_5q_2q_3q_7 \\
I_B(7,\mathbf{p}) = \frac{\partial h_3(7,\mathbf{p})}{\partial p_7} &= p_3p_4q_5q_6 + p_2p_4q_3q_5 - p_2p_4p_6q_3q_5 + p_1p_4q_2q_3 - p_1p_4p_6q_2q_3 \\
&\quad -p_1p_4p_5q_2q_3q_6
\end{aligned}
$$

Birnbaum Reliability Importance Measure in a k-out-of-n:F System

Using approach given in Algorithm 1 of Section 4.2 and in view of explicit formula for reliability function of a k-out-of-n:F system, Birnbaum reliability importance for component i of this system when $n \geq 2k-1$, is given by:

$$I_B(i, \mathbf{p}) = a_i + \sum_{r=1}^{k-1} \left\{ \sum_{i_1 < \cdots < i_r} \prod_{s=1}^{r} q_{i_s} \prod_{\substack{j = k-r \\ j \neq i, i_1, \ldots, i_r}}^{n} p_j \quad - \sum_{i_1 < \cdots < i_r} \prod_{\substack{s=1 \\ i_s \neq i}}^{n} q_{i_s} \prod_{\substack{j = k-r \\ j \neq i_1, \ldots, i_r}} p_j \right\}$$

where the first inner sum is over all $i_1, \ldots, i_r$ such that $i_1, \ldots, i_r \in \{k-r+1, \ldots, n\}$, $i \notin \{i_1, \ldots, i_r\}$ and the second inner sum is over all $i_1, \ldots, i_r$ such that $i_1, \ldots, i_r \in \{k-r+1, \ldots, n\}$, $i \in \{i_1, \ldots, i_r\}$ and

$$a_i = \begin{cases} \prod_{j=k, j \neq i}^{n} p_j & \text{if } i \in \{k, k+1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

4.5 Recursive Algorithm for System Reliability

In this section we examine the problem of evaluating the reliability function of a con$|k|n$:F system from a different angle. Shantikumar [63] introduced a recursive algorithm for computing the reliability of a con$|k|n$:F system. In his method $h_k^F(\mathbf{p}, n)$ is obtained by conditioning on subsystems of the original system. Another approach was given by Seth [60]. He used dual structure to obtain the reliability function of the original system. We give here a different approach using minimal cut sets of the system. We note that a minimal cut set of a con$|k|n$:F system is of the form :

$$D_x = \{x, x+1, \ldots, x+k-1\}, \; x = 1, 2, \ldots, n-k+1.$$

Let A_x denote the event that all components of minimal cut set D_x are failed.

Hence

$$\bar{h}_k^F(\mathbf{p},n) = 1 - h_k^F(\mathbf{p},n) = Pr\left\{\bigcup_{x=1}^{n-k+1} A_x\right\}$$

which is equal to

$$Pr\{A_1\} + \sum_{x=2}^{n-k+1} Pr\{A_x \cap \bar{A}_{x-1} \cap \bar{A}_{x-2} \cap \ldots \cap \bar{A}_1\}.$$

We have:

$$Pr\{A_x \cap \bar{A}_{x-1} \cap \bar{A}_{x-2} \cap \ldots \cap \bar{A}_1\} =$$

$$\begin{cases} \prod_{j=x}^{x+k-1} q_j p_{x-1} & \text{if } \ 2 \le x \le k+1 \qquad (\text{or } \ n \le 2k) \\ \prod_{j=x}^{x+k-1} q_j p_{x-1} \left(1 - Pr\left\{\bigcup_{j=1}^{x-k-1} A_j\right\}\right) & \text{if } \ k+2 \le x \le n-k+1 \quad (\text{or } \ n > 2k) \end{cases}$$

We note that $Pr\left\{\bigcup_{j=1}^{x-k-1} A_j\right\} = \bar{h}_k^F(\mathbf{p}, x-2)$. Hence

$$Pr\{A_x \cap \bar{A}_{x-1} \cap \ldots \cap \bar{A}_1\} = \begin{cases} \prod_{j=x}^{x+k-1} q_j p_{x-1} & \text{if } \ 2 \le x \le k+1 \\ \prod_{j=x}^{x+k-1} q_j p_{x-1} \left(1 - \bar{h}_k^F(\mathbf{p}, x-2)\right) & \text{if } \ k+2 \le x \le n-k+1 \end{cases}$$

If we assume that $\bar{h}_k^F(\mathbf{p}, x) = 0$, for $x = -1, 0, 1, \ldots, k-1$ we then have

$$Pr\left\{A_x \cap \bar{A}_{x-1} \cap \ldots \cap \bar{A}_1\right\} = \prod_{j=x}^{x+k-1} q_j p_{x-1} \left(1 - \bar{h}_k^F(\mathbf{p}, x-2)\right)$$

for $x = 1, 2, \ldots, n-k+1$. Therefore we have

$$\bar{h}_k^F(\mathbf{p}, n) = \sum_{x=1}^{n-k+1} \prod_{j=x}^{x+k-1} q_j p_{x-1} \left(1 - \bar{h}_k^F(\mathbf{p}, x-2)\right) \ \forall n \ge k \qquad (4)$$

where $p_0 = 1$.

Suppose $a_x = \prod_{j=x}^{x+k-1} q_j$ we then have $\bar{h}_k^F(\mathbf{p}, n) = \sum_{x=1}^{n-k+1} a_x p_{x-1} \left(1 - \bar{h}_k^F(\mathbf{p}, x-2)\right)$.

We note that $a_{x+1} = \frac{a_x}{q_x} q_{x+k}$. We now state an algorithm using the above formula to compute the reliability function of a con$|k|n$:F system.

ALGORITHM 3.

Input. Integers n, k $(n \geq k)$ and components reliabilities $p_1, p_2, \ldots, p_n$. $0 \leq p_i \leq 1$, $q_i = 1 - p_i$ $i = 1, 2, \ldots, n$.

Output. Reliability function of a con$|k|n$:F system.

Step 0. Put $\bar{R}_{-1} = \bar{R}_0 = \bar{R}_1 = \cdots = \bar{R}_{k-1} = 0$, $a_1 = \prod_{j=1}^{k} q_j$, $p_0 = 1$ and $S_0 = 0$.

Step 1. Do For $x = 1$ to $n - k + 1$. Put $R_x = a_x p_{x-1} \left(1 - \bar{R}_{x-2}\right)$, $S_x = S_{x-1} + R_x$. If $x = n - k + 1$ go to step 2. Otherwise put $a_{x+1} = \frac{a_x}{q_x} q_{x+k}$. If $x \geq k + 1$ put $\bar{R}_{x-1} = S_{x-k}$.

Step 2. $1 - S_{n-k+1}$ is the reliability function of a con$|k|n$:F system. Stop.

Remarks. For iid case recursive relation (4) reduces to the following equation as follows:

$$\bar{h}_k^F(\mathbf{p}, n) = \sum_{x=1}^{n-k+1} \prod_{j=x}^{x+k-1} q_j p_{j-1} h_k^F(\mathbf{p}, x-2)$$

where $p_0 = 1$ and $h_k^F(\mathbf{p}, x) = 1$ for $x = -1, 0, \ldots, k-1$. Hence if $p_i = p$, $i = 1, 2, \ldots, n$ we then have:

$$\bar{h}_k^F(p, n) = q^k + \sum_{x=2}^{n-k+1} pq^k h_k^F(p, x-2).$$

For $k \leq n \leq 2k$ it implies that $\bar{h}_k^F(p, n) = q^k + (n-k)pq^k = q^k(1 + (n-k)p)$. And for $n > 2k$ we have:

$$\bar{h}_k^F(p, n) = q^k + kpq^k + pq^k \sum_{x=k+2}^{n-k+1} h_k^F(p, x-2). \text{ Therefore}$$

$$h_k^F(p, n) = 1 - \bar{h}_k^F(p, n) = 1 - q^k - kpq^k - pq^k \sum_{x=k}^{n-k-1} h_k^F(p, x), \quad n > 2k. \qquad (5)$$

Papastavridis and Koutras [47], have given the lower and upper bounds for reliability function of a consecutive-k-out-of-n:F system with i.i.d. components as follows:

$$l = (1-q^k)^{n-k-1} \le h_k^F(p,n) \le (1-q^k+q^{k+1})^{n-k+1} = u.$$

Using l and u and in view of equation (5), we now give some new bounds for $h_k^F(p,n)$, when $n > 2k$. We have:

$$h_k^F(p,n) \le 1 - q^k - kpq^k - pq^k \sum_{x=k}^{n-k-1} (1-q^k)^{x-k+1} =$$
$$1 - q^k(1+kp) - p(1-q^k)\left(1-(1-q^k)^{n-2k}\right) = U.$$

On the other hand we have:

$$h_k^F(p,n) \ge 1 - q^k - kpq^k - pq^k \sum_{x=k}^{n-k-1} (1-q^k+q^{k+1})^{x-k+1} =$$
$$1 - q^k(1+kp) - (1-q^k+q^{k+1})\left(1-(1-q^k+q^{k+1})^{n-2k}\right) = L.$$

We note that $L = 1 - q^k(1+kp) - (1-pq^k)\left(1-(1-pq^k)^{n-2k}\right)$.

We also note that if $q \to 1$ then $h_k^F(p,n) \to 0$. In this case we also have $l \to 0$, $u \to 1$, $L \to 0$ and $U \to 0$. Therefore it seems u is not sharp for large values of q. The following numerical results, given in Table 1, show that the bounds L and U make good approximations for $h_k^F(p,n)$.

(Table 1.)

n	k	q	l	u	L	U	n	k	q	l	u	L	U
[illegible]0	2	0.1	.9135	.9219	.9197	.9199	10	2	0.5	.0751	.3007	.0177	.1917
[illegible]0	2	0.2	.6925	.7462	.7244	.7292	10	4	0.5	.6365	.8007	.7529	.7557
[illegible]0	3	0.3	.8033	.8584	.8442	.8457	10	4	0.7	.1463	.5926	.3429	.3754
[illegible]0	4	0.3	.9447	.9610	.9580	.9580	10	5	0.7	.3315	.7331	.5798	.5798
[illegible]0	2	0.1	.8262	.8422	.8385	.8396	10	4	0.8	.0250	.5497	.1185	.1858
[illegible]0	5	0.1	.9998	.9999	.9999	.9999	10	5	0.8	.0924	.6658	.3446	.3446
[illegible]0	2	0.2	.4604	.5390	.5033	.5277	10	5	0.9	.0047	.6941	.1143	.1143
[illegible]0	3	0.2	.8654	.8909	.8874	.8884	20	4	0.5	.3338	.5829	.5056	.5598
[illegible]0	2	0.1	.7472	.7694	.7644	.7671	20	5	0.5	.6017	.7773	.7472	.7589
[illegible]0	2	0.15	.5169	.5712	.5521	.5682	20	9	0.7	.6100	.8640	.8269	.8279
[illegible]0	2	0.05	.8846	.8900	.8894	.8897	20	10	0.7	.7296	.9106	.8870	.8870
[illegible]0	2	0.1	.6111	.6421	.6348	.6421	20	9	0.8	.1774	.7214	.5726	.5808
[illegible]0	12	0.8	.2585	.7688	.6878	.7016	20	10	0.8	.2866	.7876	.6779	.6779
[illegible]0	14	0.8	.4655	.8605	.8155	.8164	20	9	0.9	.0028	.6224	.1909	.2256
[illegible]0	15	0.8	.5638	.8932	.8593	.8593	20	10	0.9	.0089	.6768	.3026	.3026
[illegible]0	14	0.9	.0121	.6747	.4068	.4197	30	10	0.7	.5479	.8363	.8061	.8144
[illegible]0	15	0.9	.0250	.7169	.4853	.4853	30	14	0.7	.8908	.9660	.9607	.9607

Chapter 5

Number of Minimal Path Sets in a Consecutive-k-out-of-n:F System

5.1 Introduction

Although the minimal path sets of a consecutive-k-out-of-n:F system can be generated using the algorithms presented in chapter 3, but the combinatorial problem of determining the number of minimal path sets of the system still remains a difficult task. On the other hand we note that number of terms in the nonrecursive formula for determining the reliability of the system as given in Algorithm 1 of chapter 4, equals the number of minimal path sets of the system. In this chapter we consider this aspect of the system and study the problem of determining the number of minimal path sets of a con$|k|n$:F system. In Section 2 we present explicit formulae for determining the number of minimal path sets of a linear and a circular con$|2|n$:F systems. We show that the recurrence relationships for the number of minimal path sets of a linear and a circular con$|2|n$:F systems are the same but with different initial conditions. We also give closed form formulae for computing the number of minimal path sets of known size in both a linear and a circular con$|2|n$:F systems.

Section 3 considers the problem of determining the number of minimal path sets of a con$|3|n$:F system. We give the relationship between number of minimal path sets of a linear con$|3|n$:F system and that of a circular con$|3|n$:F system. In these two systems, recurrence relationships for the number of minimal path sets are also the same but with different initial conditions. A closed form formula for determining the number of minimal path sets of a linear con$|3|n$:F system is also given which is more complicated than that of a con|2|n:F system. This leads us to introduce a recursive relationship for determining the number of minimal path sets of a general con|k|n:F system, which is considered in Section 4. We shall use the following notations.

$\alpha_k(n)$: collection of all minimal path sets of a con|k|n:F system.

$p_n(k)$: number of minimal path sets of a con|k|n:F system.

$p_n^{r,L}$ ($p_n^{r,C}$) : number of minimal path sets of size r in a linear (circular) con|2|n:F system.

$\underline{r}_k(n)$ ($\bar{r}_k(n)$) : minimum (maximum) size of a minimal path set in a con|k|n:F system.

$p_n^i(k)$: number of minimal path sets of a con|k|n:F system that contain component i.

$|A|$: cardinality of set A.

5.2 Consecutive-2-out-of-n:F Systems

In this Section we consider the problem of determining the number of minimal path sets of a linear and a circular con|2|n:F systems.

Minimal Path Sets of a Linear con|2|n:F system

Suppose $p_n^L(2)$ denotes the number of minimal path sets of a linear con|2|n:F system. The following lemma is required in the sequel.

Lemma 1. $p_n^L(2) = p_{n-2}^L(2) + p_{n-3}^L(2)$ for $n \geq 3$.

Proof. Let $p_n^L(2) = |\alpha_2^L(n)|$ where $\alpha_2^L(n)$ is the collection of minimal path sets of a linear con|2|n:F system. We note that

$$\alpha_2^L(n) = \{P : P \in \alpha_2^L(n) \text{ and } n \in P\} \cup \{P : P \in \alpha_2^L(n) \text{ and } n \notin P\}$$

and the collections on the right hand side are disjoint. If $n \in P$ then $n-1 \notin P$

and $n-2 \in P$. And if $n \notin P$ then $n-1 \in P$. We have

$$\alpha_2^L(0) = \alpha_2^L(1) = \{\emptyset\},\ \alpha_2^L(2) = \{\{1\},\{2\}\},\ \alpha_2^L(3) = \{\{1,3\},\{2\}\} \text{ and}$$

$$\alpha_2^L(4) = \{\{1,3\},\{2,3\},\{2,4\}\}$$

It is easy to verify that for $n \geq 3$ we have

$$\{P :\ P \in \alpha_2^L(n) \text{ and } n \in P\} = \left\{P :\ P = T \cup \{n-2, n\} \text{ and } T \in \alpha_2^L(n-3)\right\}$$

and for $n \geq 2$ we have

$$\{P :\ P \in \alpha_2^L(n) \text{ and } n \notin P\} = \left\{P :\ P = T \cup \{n-1\} \text{ and } T \in \alpha_2^L(n-2)\right\}$$

Therefore we have

$$p_n^L(2) = |\alpha_2^L(n)| = |\{P :\ P \in \alpha_2^L(n) \text{ and } n \in P\}| + |\{P :\ P \in \alpha_2^L(n) \text{ and } n \notin P\}| =$$

$$|\{P :\ P = T \cup \{n-2, n\},\ T \in \alpha_2^L(n-3)\}| + |\{P :\ P = T \cup \{n-1\},\ T \in \alpha_2^L(n-2)\}|$$

That is, $p_n^L(2) = |\alpha_2^L(n-3)| + |\alpha_2^L(n-2)| = p_{n-2}^L(2) + p_{n-3}^L(2)$; $n \geq 3$. This completes the proof of the lemma. ∎

Suppose $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. We assume that $\binom{m}{r} = 0$ for $m < r$ or $r < 0$.

Theorem 1. $p_n^L(2) = \sum_{i=\lfloor n/3 \rfloor}^{\lfloor n/2 \rfloor} \sum_{j=n-2i-2}^{n-2i} \binom{i}{j}$

Proof. Let $g(x)$ denote the generating function of $p_n^L(2)$, the number of minimal path sets of a linear con|2|n:F system. Note that $p_0^L(2) = p_1^L(2) = 1, p_2^L(2) = 2$. We have

$$g(x) = p_0^L(2) + p_1^L(2)x + p_2^L(2)x^2 + p_3^L(2)x^3 + \cdots + p_n^L(2)x^n + \cdots$$

$$x^2 g(x) = \qquad p_0^L(2)x^2 + p_1^L(2)x^3 + \cdots + p_{n-2}^L(2)x^n + \cdots$$

$$x^3 g(x) = \qquad p_0^L(2)x^3 + \cdots + p_{n-3}^L(2)x^n + \cdots$$

Using Lemma 1, we get $(1 - x^2 - x^3)g(x) = 1 + x + x^2$. This implies that

$$g(x) = \frac{1 + x + x^2}{1 - x^2 - x^3} = \frac{1 + x + x^2}{1 - x^2(1 + x)}.$$

For sufficiently small x such that $|x^2(1+x)| < 1$, we have

$$g(x) = (1 + x + x^2)\left[\sum_{i=0}^{\infty} x^{2i}(1+x)^i\right] = (1 + x + x^2)\left[\sum_{i=0}^{\infty} x^{2i} \sum_{j=0}^{i} \binom{i}{j} x^j\right]$$

For a given i, the maximum value of power of x is $3i + 2$. Hence for getting coefficient of x^n, the minimum value of i is the nearest integer greater than or equal to $(n-2)/3$. That is $\lfloor n/3 \rfloor \leq i$. Similarly, the minimum value of power of x is $2i$. Hence $i \leq \lfloor n/2 \rfloor$. It is easy to see that coefficient of x^n in $g(x)$ is

$$p_n^L(2) = \sum_{=\lfloor n/3 \rfloor}^{\lfloor n/2 \rfloor} \sum_{j=n-2i-2}^{n-2i} \binom{i}{j}, \quad n \geq 0. \qquad \blacksquare$$

Minimal Path Sets with Known Size in a Linear con|2|n:F System

Here we first give a recursive relation for the number of minimal path sets with known size of a linear con|2|n:F system and then we derive a closed form expression for it. The next lemma is required in the sequel.

Lemma 2. Suppose $\bar{r}_k^L(n)$ and $\underline{r}_k^L(n)$ denote the maximum and minimum size of a minimal path set in a linear con|k|n:F system respectively. We then have

$$\bar{r}_k^L(n) = \begin{cases} 2\left\lfloor \frac{n}{k+1} \right\rfloor & \text{if } \frac{n+1}{k+1} > \left\lfloor \frac{n+1}{k+1} \right\rfloor \\ 2\left\lfloor \frac{n}{k+1} \right\rfloor + 1 & \text{if } \frac{n+1}{k+1} = \left\lfloor \frac{n+1}{k+1} \right\rfloor \end{cases} \quad \text{and} \quad \underline{r}_k^L(n) = \lfloor n/k \rfloor.$$

Proof. From Theorem 1 of Chapter 3, we note that $R_1 = \{k, 2k, 3k, \ldots, k\lfloor n/k \rfloor\}$ is a minimal path set of size $\lfloor n/k \rfloor$. Now suppose $A \subseteq N$ and A is a path set of a con$|k|n$:F system. We note that $C_s = \{sk+1, sk+2, \ldots, (s+1)k\}$, for $s = 0, 1, \ldots, \lfloor n/k \rfloor - 1$, are $\lfloor n/k \rfloor$ disjoint minimal cut sets. In view of Theorem 1 of Chapter 3, we have $|A \cap C_s| \geq 1 \ \forall s$. Hence

$$|A| = \left| \bigcup_{s=0}^{\lfloor n/k \rfloor - 1} (A \cap C_s) \cup \left\{ \left(N - \bigcup_{s=0}^{\lfloor n/k \rfloor - 1} C_s \right) \cap A \right\} \right|$$

which is greater than or equal to

$$\left| \bigcup_{s=0}^{\lfloor n/k \rfloor - 1} (A \cap C_s) \right| = \sum_{s=0}^{\lfloor n/k \rfloor - 1} |A \cap C_s| \geq \lfloor n/k \rfloor = |R_1|.$$

Hence R_1 is a minimal path set with minimum size.

We now consider two cases case (i) and case (ii) for $\bar{\bar{r}}_k^L(n)$.

(i) Let $\dfrac{n+1}{k+1} = \left\lfloor \dfrac{n+1}{k+1} \right\rfloor = t$. We note that $R_2 = \{1, k+1, k+2, 2k+2, 2k+3, \ldots, pk+p, pk+p+1\}$ is a minimal path set as it satisfies the conditions of Theorem 1 of Chapter 3, where $p = \left\lfloor \frac{n}{k+1} \right\rfloor$ and we have $|R_2| = 2p+1$. Suppose

$\bar{C}_1 = \{1, 2, \ldots, k\}$

$\bar{C}_2 = \{k+1, k+2, \ldots, 2k+1\}$

$\bar{C}_3 = \{2k+2, 2k+3, \ldots, 3k+2\}$

$\vdots$

$\bar{C}_t = \{(t-1)(k+1), \ldots, (t-1)(k+1)+k = n\}$

and Let P be a minimal path set of a linear con$|k|n$:F system. We note that $|P \cap \bar{C}_1| = 1$ and $|P \cap \bar{C}_j| \leq 2$ for $2 \leq j \leq t$. We also note that $P = \bigcup_{j=1}^{t} (P \cap \bar{C}_j)$ hence

$$|P| = \sum_{j=1}^{t} |P \cap \bar{C}_j| \leq 1 + 2(t-1) = 2t - 1 = 2\left\lfloor \frac{n+1}{k+1} \right\rfloor - 1$$

which is equal to

$$2(\frac{n+1}{k+1})-1 = 2\{\lfloor n/(k+1)\rfloor+1\}-1 = 2\lfloor n/(k+1)\rfloor+1 = 2p+1 = |R_2|.$$

Hence in this case R_2 is a minimal path set with maximum size.

(ii) If $\frac{n+1}{k+1} > \left\lfloor\frac{n+1}{k+1}\right\rfloor = t$ then using Theorem 1 of Chapter 3, we note that

$$R_3 = \{1, k+1, k+2, 2k+2, 2k+3, \ldots, (p-1)k+p, pk+p\}$$

is a minimal path set and $|R_3| = 2p = 2\lfloor n/(k+1)\rfloor$. We have
$n+1 = t(k+1)+s$ and $0 < s \leq k$. In this case we have
$\bar{C}_t = \{(t-1)(k+1), \ldots, (t-1)(k+1)+k = n-s\}$
We define $\bar{C}_{t+1} = \{n-s+1, n-s+2, \ldots, n\}$.
Suppose $P \subseteq N$ is a minimal path set. We note that $|P \cap \bar{C}_1| = 1$, $|P \cap \bar{C}_j| \leq 2$, for $j = 2, 3, \ldots, t$ and
$|P \cap \bar{C}_{t+1}| \leq 1$, since $\bar{C}_{t+1} \subseteq \{n-k+1, n-k+2, \ldots, n\}$ and $|P \cap \{n-k+1, n-k+2, \ldots, n\}| = 1$. We also note that $P = \bigcup_{j=1}^{t+1}(P \cap \bar{C}_j)$. Hence
$|P| \leq 1+2(t-1)+1 = 2t = 2\left\lfloor\frac{n+1}{k+1}\right\rfloor = 2\lfloor n/(k+1)\rfloor = 2p = |R_3|$.
Therefore in this case R_3 is a minimal path set with maximum size. This completes the proof of the lemma. ∎

Lemma 3. Let $p_n^{r,L}$ denote the number of minimal path sets of size r in a linear con|2|n:F system. We have: $p_n^{r,L} = p_{n-2}^{r-1,L} + p_{n-3}^{r-2,L}$, $n \geq 3$, $\bar{r}_1^L(n) \leq r \leq \bar{r}_2^L(n)$. We assume that

$$p_n^{r,L} = \begin{cases} 0 & \text{if } n < 0 \\ 0 & \text{if } n = 0,1 \text{ and } r \neq 0 \\ 1 & \text{if } n = 0,1 \text{ and } r = 0 \\ 0 & \text{if } n = 2 \text{ and } r \neq 1 \\ 2 & \text{if } n = 2 \text{ and } r = 1 \end{cases}$$

Proof. If $n = 3$ we have, $p_n^{r,L} = \begin{cases} 1 & \text{if } r = 1 \text{ or } r = 2 \\ 0 & \text{otherwise} \end{cases}$

and if $n = 4$ we have, $p_n^{r,L} = \begin{cases} 3 & \text{if } r = 2 \\ 0 & \text{otherwise.} \end{cases}$

Hence the lemma is trivially true for $n = 3$ and 4. Now consider the case when $n \geq 5$. Let $\mathbf{P}_n^r$ denote the collection of all minimal path sets of size r in a linear con|2|n:F system. We note that

$$\mathbf{P}_n^r = \{S \ : \ S \in \mathbf{P}_n^r \text{ and } n \in S\} \cup \{S \ : \ S \in \mathbf{P}_n^r \text{ and } n \notin S\}$$

and the collections on the right hand side are disjoint. We have

$|\{S \ : \ S \in \mathbf{P}_n^r \text{ and } n \in S\}| = |\mathbf{P}_{n-3}^{r-2}| = p_{n-3}^{r-2,L}$ (since $n-1 \notin S, n-2 \in S$)

$|\{S \ : \ S \in \mathbf{P}_n^r \text{ and } n \notin S\}| = |\mathbf{P}_{n-2}^{r-1}| = p_{n-2}^{r-1,L}$ (since $n-1 \in S$)

It follows that $p_n^{r,L} = |\mathbf{P}_n^r| = |\mathbf{P}_{n-3}^{r-2}| + |\mathbf{P}_{n-2}^{r-1}| = p_{n-2}^{r-1,L} + p_{n-3}^{r-2,L}$.

This completes the proof of the lemma. ∎

Theorem 2. $p_n^{r,L} = \binom{n-r+1}{2n-3r}$ for $\underline{r}_2^L(n) \leq r \leq \bar{r}_2^L(n)$ and $n \geq 2$.

Proof. Let $g(x, y)$ denote the generating function of $p_n^{r,L}$, the number of minimal path sets of size r in a linear con|2|n:F system. We have

$$g(x,y) = \sum_{n=0}^{\infty} \sum_{r=\underline{r}_2^L(n)}^{\bar{r}_2^L(n)} x^n y^r p_n^{r,L}$$

$$= 1 + x + 2x^2y + x^3(y+y^2) + 3x^4y^2 + x^5(y^2+3y^3) + x^6(4y^3+y^4) + \cdots$$

$$= 1 + x + 2x^2y + \sum_{n=3}^{\infty} x^n \sum_{r=\underline{r}_2^L(n)}^{\bar{r}_2^L(n)} y^r p_n^{r,L}$$

Using Lemma 3, we have

$$g(x,y) = 1 + x + 2x^2y + \sum_{n=3}^{\infty} x^n \sum_{r=\underline{r}_2^L(n)}^{\bar{r}_2^L(n)} y^r (p_{n-2}^{r-1,L} + p_{n-3}^{r-2,L})$$

$$= 1 + x + 2x^2y + x^2y\sum_{n=3}^{\infty} x^{n-2} \sum_{r=\underline{r}_2^L(n)}^{\bar{r}_2^L(n)} y^{r-1} p_{n-2}^{r-1,L} + x^3y^2\sum_{n=3}^{\infty} x^{n-3} \sum_{r=\underline{r}_2^L(n)}^{\bar{r}_2^L(n)} y^{r-2} p_{n-3}^{r-2,L}$$

$$= 1 + x + 2x^2y + x^2y\{g(x,y) - 1\} + x^3y^2\{g(x,y)\}$$

It implies that $\quad g(x,y) = \dfrac{1 + x + x^2y}{1 - x^2y - x^3y^2}.$

For sufficiently small x and y such that $|x^2y(1+xy)| < 1$, we have

$$g(x,y) = (1+x+x^2y)\left\{\sum_{i=0}^{\infty} x^{2i}y^i(1+xy)^i\right\} = (1+x+x^2y)\left\{\sum_{i=0}^{\infty} x^{2i}y^i \sum_{j=0}^{i}(xy)^j \binom{i}{j}\right\}.$$

For finding coefficient of x^n as in the proof of Theorem **1**, *we note that* $\lfloor n/3\rfloor \le i \le \lfloor n/2\rfloor$. We have

$$\sum_{r=\underline{r}_2^L(n)}^{\bar r_2^L(n)} y^r p_n^{r,L} = \sum_{i=\lfloor n/3\rfloor}^{\lfloor n/2\rfloor} y^i \left\{\sum_{j=n-2i-1}^{n-2i} y^j \binom{i}{j} + y^{n-2i-1}\binom{i}{n-2i-2}\right\}$$

Let $l = \lfloor n/2\rfloor - i$ then

$$\sum_{r=\underline{r}_2^L(n)}^{\bar r_2^L(n)} y^r p_n^{r,L} = \sum_{l=0}^{\lfloor n/2\rfloor - \lfloor n/3\rfloor} y^{\lfloor n/2\rfloor - l}\left\{\sum_{j=n-2\lfloor n/2\rfloor+2l-1}^{n-2\lfloor n/2\rfloor+2l} y^j \binom{\lfloor n/2\rfloor - l}{j} + \right.$$

$$\left. y^{n-2\lfloor n/2\rfloor+2l-1}\binom{\lfloor n/2\rfloor - l}{n-2\lfloor n/2\rfloor+2l-2}\right\}$$

For a given r, suppose $r = \lfloor n/2\rfloor + k'$ for some $k' = 0,1,\ldots,\bar r_2^L(n) - \lfloor n/2\rfloor$. Then the coefficient of $y^{\lfloor n/2\rfloor - k'}$ is

$$\binom{n-\lfloor n/2\rfloor - k' - 1}{2\lfloor n/2\rfloor - n + 2k' + 1} + \binom{n-\lfloor n/2\rfloor - k' - 1}{2\lfloor n/2\rfloor - n + 2k'} + \binom{n-\lfloor n/2\rfloor - k'}{2\lfloor n/2\rfloor - n + 2k'}$$

$$= \binom{n-r-1}{2r-n+1} + \binom{n-r-1}{2r-n} + \binom{n-r}{2r-n}$$

First and second binomial coefficients correspond to $l = 2\lfloor n/2\rfloor - n + k' + 1$ (if $k' \le n - \lfloor n/2\rfloor - \lfloor n/3\rfloor + 1$). We note that $l \ge 0$, the third binomial coefficient corresponds to $l = 2\lfloor n/2\rfloor - n + k'$ (if $k' \ge n - 2\lfloor n/2\rfloor$). We also note that

$l \leq 2\lfloor n/2 \rfloor - n + \bar{r}_2^L(n) - \lfloor n/2 \rfloor = \lfloor n/2 \rfloor - n + \bar{r}_2^L(n) \leq \lfloor n/2 \rfloor - \lfloor n/3 \rfloor$.

Therefore we have

$$p_n^{r,L} = \binom{n-r}{2r-n} + \binom{n-r-1}{2r-n+1} + \binom{n-r-1}{2r-n} = \binom{n-r+1}{2n-3r}$$

This completes the proof of the theorem. ∎

Remark 1.

(i) If n is odd then the minimal path set with minimum size ($r = \underline{r}_2^L(n) = \lfloor n/2 \rfloor$) is unique since in this case we have $p_n^{r,L} = 1$.

(ii) If $n/3 = \lfloor n/3 \rfloor$ then the minimal path set with maximum size ($r = \bar{r}_2^L(n)$) is unique because we have $(n+1)/3 > \lfloor (n+1)/3 \rfloor = n/3$. From Lemma 2, we have $\bar{r}_2^L(n) = 2\lfloor n/3 \rfloor = 2n/3$. It implies that

$$p_n^{\bar{r}_2^L(n),L} = \binom{n - \bar{r}_2^L(n) + 1}{2n - 3\bar{r}_2^L(n)} = \binom{n - 2n/3 + 1}{2n - 2n} = 1.$$

(iii) We now introduce another formula for $p_n^L(2)$, the number of minimal path sets of a linear con|2|n:F system as follows.

$$p_n^L(2) = \sum_{r=\underline{r}_2^L(n)}^{\bar{r}_2^L(n)} p_n^{r,L} = \sum_{r=\underline{r}_2^L(n)}^{\bar{r}_2^L(n)} \binom{n-r+1}{2n-3r}$$

Minimal Path Sets of a Circular con|2|n:F System

Here, we establish a relationship between the number of minimal path sets of a linear con|2|n:F system with that of a circular con|2|n:F system. Using this, we derive a closed form formula for $p_n^C(2)$, the number of minimal path sets of a circular con|2|n:F system, and $p_n^{r,C}$, the number of minimal path sets of a circular con|2|n:F system of size r. We further show that the same recursive relation holds in a linear con|2|n:F system as well as in a circular con|2|n:F system but with different initial values.

Lemma 4. $p_n^C(2) = 2p_{n-3}^L(2) + p_{n-6}^L(2)$; $n \geq 6$.

Proof. Suppose P_C is a minimal path set of a circular con|2|n:F system. We have the following three cases.

(i) Let $1 \in P_C$ and $2 \in P_C$.

We then have $n \notin P_C$ and $3 \notin P_C$. Hence $n-1 \in P_C$ and $4 \in P_C$. In this case P_C is a minimal path set for a circular con|2|n:F system if and only if $P_C \cap \{5, 6, \ldots, n-2\}$ is a minimal path set for a linear con|2|$n-6$:F subsystem with the component set $\{5, 6, \ldots, n-2\}$. Hence we have $p_n^C(2) = p_{n-6}^L(2)$. If $n = 6$ we know that the only minimal path set in this case is $P_C = \{1, 2, 4, 5\}$ and also we note that $p_n^C(2) = p_{n-6}^L(2) = 1$.

(ii) Let $1 \in P_C$ and $2 \notin P_C$.

We then have $3 \in P_C$. Similarly P_C is a minimal path set for a circular con|2|n:F system if and only if $P_C \cap \{4, 5, \ldots, n\}$ is a minimal path set for a linear con|2|$n-3$:F subsystem with the component set $\{4, 5, \ldots, n\}$. Hence we have $p_n^C(2) = p_{n-3}^L(2)$.

(iii) Let $1 \notin P_C$ and $2 \in P_C$.

We then have $n \in P_C$ and P_C is a minimal path set for a circular con|2|n:F system if and only if $P_C \cap \{3, 4, \ldots, n-1\}$ is a minimal path set for a linear con|2|$n-3$:F subsystem with the component set $\{3, 4, \ldots, n-1\}$. Hence we have $p_n^C(2) = p_{n-3}^L(2)$.

From these cases, we get the result $p_n^C(2) = p_{n-6}^L(2) + 2p_{n-3}^L(2)$ for $n \geq 6$.

Remark 2. We know that $p_n^L(2) = p_{n-2}^L(2) + p_{n-3}^L(2)$; $n \geq 3$ with $p_0^L(2) = p_1^L(2) = 1$ and $p_2^L(2) = 2$. We define

$p^L_{-1}(2) = 1$, $p^L_{-2}(2) = 0$, $p^L_{-3}(2) = 1$, $p^L_{-4}(2) = 0$, $p^L_{-5}(2) = 0$ and $p^L_{-6}(2) = 1$. Therefore the relation $p^L_n(2) = p^L_{n-2}(2) + p^L_{n-3}(2)$ holds true for all $n \geq -3$. Hence using Lemma 4, we have $p^C_0(2) = 2p^L_{-3}(2) + p^L_{-6}(2) = 3$, $p^C_1(2) = 0$, $p^C_2(2) = 2$, $p^C_3(2) = 3$, $p^C_4(2) = 2$, and $p^C_5(2) = 5$. Therefore Lemma 4 holds true for all $n \geq 0$.

Lemma 5. $p^C_n(2) = p^C_{n-2}(2) + p^C_{n-3}(2)$; $n \geq 3$.

Proof. From Lemma 4 and Remark 2, we have $p^C_n(2) = 2p^L_{n-3}(2) + p^L_{n-6}(2)$ for $n \geq 0$ and $p^L_n(2) = p^L_{n-2}(2) + p^L_{n-3}(2)$ for $n \geq -3$. Hence for $n \geq 3$ we can write

$p^C_n(2) = 2(p^L_{n-5}(2) + p^L_{n-6}(2)) + (p^L_{n-8}(2) + p^L_{n-9}(2)) =$

$(2p^L_{n-5}(2) + p^L_{n-8}(2)) + (2p^L_{n-6}(2) + p^L_{n-9}(2))$.

If we again use Lemma 4, we get the result $p^C_n(2) = p^C_{n-2}(2) + p^C_{n-3}(2)$. In other words, the same recursive relation holds in a circular con|2|n:F system as well as in a linear con|2|n:F system but with different initial conditions, $p^L_0(2) = p^L_1(2) = 1$, $p^L_2(2) = 2$ and $p^C_0(2) = 3$, $p^C_1(2) = 0$, $p^C_2(2) = 2$.

Remark 3. Using Lemma 4 and Remark 1(part (iii)) we can derive a closed form formula for $p^C_n(2)$.

We now use Lemma 5, to derive a simpler formula for $p^C_n(2)$ directly.

Let $g_c(x)$ denote the generating function of $p^C_n(2)$, the number of minimal path sets of a circular con|2|n:F system. By using lemma 5 we have

$$g_c(x) = \frac{p^C_0(2) + p^C_1(2)x + \left(p^C_2(2) - p^C_0(2)\right)x^2}{1 - x^2 - x^3} = \frac{3 - x^2}{1 - x^2 - x^3}.$$

For obtaining partial fraction expansion of $g_c(x)$, let

$$\frac{3 - x^2}{1 - x^2 - x^3} = \frac{a}{1 - \rho x} + \frac{b}{1 - \sigma x} + \frac{c}{1 - \bar{\sigma} x}$$

where ρ is the real root and σ and $\bar{\sigma}$ (conjugate of σ) are the complex roots of the cubic $x^3 - x - 1 = 0$. We note that $1/\rho$, $1/\sigma$ and $1/\bar{\sigma}$ are the roots of the equation $1 - x^2 - x^3 = 0$. It is easy to verify that

$\rho\sigma\bar{\sigma} = 1, \ \ \rho + \sigma + \bar{\sigma} = 0, \ \ R(\sigma) = -\rho/2, \ \ I^2(\sigma) = \dfrac{3-\rho}{4\rho}, \ |\rho - \sigma|^2 = \dfrac{2\rho + 3}{\rho}.$

We then have

$$a = \frac{3 - 1/\rho^2}{(1 - \sigma/\rho)(1 - \bar{\sigma}/\rho)} = \frac{3\rho^2 - 1}{(\rho - \sigma)(\rho - \bar{\sigma})} = \frac{3\rho^2 - 1}{|\rho - \sigma|^2} = \frac{\rho(3\rho^2 - 1)}{2\rho + 3} = \frac{3\rho^3 - \rho}{2\rho + 3} =$$

$$\frac{3(1+\rho) - \rho}{2\rho + 3} = 1.$$

$$b = \frac{3 - 1/\sigma^2}{(1 - \rho/\sigma)(1 - \bar{\sigma}/\sigma)} = \frac{3\sigma^2 - 1}{(\sigma - \rho)(\sigma - \bar{\sigma})} = -\frac{(\rho - \bar{\sigma})(3\sigma^2 - 1)}{2\sqrt{-1}I(\sigma)|\rho - \sigma|^2}.$$

And

$$c = \frac{3 - 1/\bar{\sigma}^2}{(1 - \rho/\bar{\sigma})(1 - \sigma/\bar{\sigma})} = \frac{3\bar{\sigma}^2 - 1}{(\bar{\sigma} - \rho)(\bar{\sigma} - \sigma)} = \frac{(\rho - \sigma)(3\bar{\sigma}^2 - 1)}{2\sqrt{-1}I(\sigma)|\rho - \sigma|^2}.$$

It follows that

$$p_n^C(2) = \rho^n - \frac{(\rho - \bar{\sigma})(3\sigma^2 - 1)}{2\sqrt{-1}I(\sigma)|\rho - \sigma|^2}\sigma^n + \frac{(\rho - \sigma)(3\bar{\sigma}^2 - 1)}{2\sqrt{-1}I(\sigma)|\rho - \sigma|^2}\bar{\sigma}^n.$$

Suppose $\sigma = r(\cos\theta + \sqrt{-1}\sin\theta)$, $c_1 = \dfrac{2+\rho}{2\rho+3}$ and $c_2 = \dfrac{-\sqrt{\rho}\rho^2}{(2\rho+3)\sqrt{3-\rho}}$

It is easy to verify that (see for example Spickerman [65])

$$p_n^C(2) = \rho^n + 3h_n - h_{n-2}, \quad \text{where} \quad h_n = r^n(c_1\cos n\theta + c_2 \sin n\theta)\ .$$

Theorem 3. $p_n^C(2)$, the number of minimal path sets of a circular con|2|n:F system for all $n \geq 10$ is given by $p_n^C = \lfloor \rho^n + 0.5 \rfloor$ where ρ is the unique real root of the cubic equation $x^3 - x - 1 = 0$.

Proof. From Lemma 5, we have $p_n^C(2) = p_{n-2}^C(2) + p_{n-3}^C(2)$, $n \geq 3$ with $p_0^C(2) = 3$, $p_1^C(2) = 0$, $p_2^C(2) = 2$.

Using this we get $p_{10}^C(2) = 17$, $p_{11}^C(2) = 22$ and $p_{12}^C(2) = 29$. Applying Cardan's formula to the cubic equation $x^3 - x - 1 = 0$, we have approximately $\rho = 1.324717958$. It can be verified that the theorem is trivially true for $n = 10, 11, 12$. Now suppose $n \geq 13$. We have already shown that $p_n^C(2) = \rho^n + 3h_n - h_{n-2}$. Therefore it is enough to show that $|3h_n - h_{n-2}| < 0.5$ for $n \geq 13$. We have

$$|3h_n - h_{n-2}| \leq 3|h_n| + |h_{n-2}| \leq 3r^n\sqrt{c_1^2 + c_2^2} + r^{n-2}\sqrt{c_1^2 + c_2^2} = (3r^n + r^{n-2})\sqrt{c_1^2 + c_2^2}.$$

Let $H_1(\rho) = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{2+\rho}{2\rho+3}\right)^2 + \frac{\rho^5}{(2\rho+3)^2(3-\rho)}} = \frac{2}{\sqrt{-2\rho^2 + 3\rho + 9}}$.

We note that $H_1(\rho)$ is an increasing function for $\rho > 3/4$.

Hence we have $H_1(\rho) < H_1(1.325) = 0.650127$, since we know that $\rho < 1.325$. We also note that $\rho\sigma\bar{\sigma} = 1$, hence $\rho r^2 = 1$ or $r = 1/\sqrt{\rho}$. If $n \geq 13$, we then have

$$3r^n + r^{n-2} \leq 3r^{13} + r^{11} = \frac{3}{\sqrt{\rho}\rho^6} + \frac{1}{\sqrt{\rho}\rho^5} = \frac{3}{\sqrt{\rho}(1+\rho)^2} + \frac{1}{\sqrt{\rho}(1+\rho)\rho^2} =$$

$$\frac{3}{\sqrt{\rho}(1+\rho)^2} + \frac{1}{\sqrt{\rho}(\rho^2+\rho+1)}, \text{ since } \rho^3 - \rho - 1 = 0.$$

Let $H_2(\rho) = \frac{3}{\sqrt{\rho}(1+\rho)^2} + \frac{1}{\sqrt{\rho}(\rho^2+\rho+1)}$. We note that $H_2(\rho)$ is a decreasing function for $\rho > 0$. Hence we have $3r^n + r^{n-2} \leq H_2(\rho) < H_2(1.324) = 0.696$, since $\rho > 1.324$. Therefore we have for all $n \geq 13$

$|3h_n - h_{n-2}| \leq H_1(\rho)H_2(\rho) < (0.65)(0.696) < 0.5$. This completes the proof of the theorem. ∎

We now consider the number of minimal path sets with known size in a circular con|2|n:F system. Suppose $\bar{\tau}_2^C(n)$ and $\underline{\tau}_2^C(n)$ denote the maximum and the minimum size of a minimal path set in a circular con|2|n:F system, respectively.

Lemma 6.

(i) $\bar{r}_2^C(n) = \bar{r}_2^L(n)$ (ii) $\bar{r}_2^C(n) = \begin{cases} \bar{r}_2^L(n) = \lfloor n/2 \rfloor & \text{if } n \text{ is even} \\ \bar{r}_2^L(n) + 1 = \lfloor n/2 \rfloor + 1 & \text{if } n \text{ is odd} \end{cases}$

Proof.

(i) We consider two cases as follows.

(a1) $n + 1 = 3s$ for some integer $s \geq 1$.

In view of Lemma 2, we note that

$R_1 = \{1, 3, 4, 6, 7, 9, \ldots, 3(s-1), 3(s-1)+1\}$ is a minimal path set with maximum size in a linear con|2|n:F system and $|R_1| = 2(s-1)+1 = 2s-1$. From Lemma 5 of Chapter 3, we note that R_1 is also a minimal path set of a circular con|2|n:F system. We show that size of R_1 in a circular con|2|n:F system is also maximum. Let

$C_1 = \{1, 2\}$

$C_2 = \{3, 4, 5\}$

$C_3 = \{6, 7, 8\}$

$C_s = \{3(s-1), 3(s-1)+1, 3(s-1)+2 = n\}$

and suppose P_C be a minimal path set of a circular con|2|n:F system. We note that $P_C \cap C_i$ is nonempty and we also note that $|P_C \cap C_i| \leq 2$ for $i = 1, 2, \ldots, s$. We show that there exists i^*, $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 1$. Suppose $|P_C \cap C_i| = 2\ \forall i$. We then have

$\{1, 2\} \subseteq P_C \Rightarrow 3 \notin P_C \Rightarrow \{4, 5\} \subseteq P_C \Rightarrow$

$6 \notin P_C \Rightarrow \cdots \Rightarrow 3(s-1) \notin P_C \Rightarrow 3(s-1)+1 = n-1 \in P_C$ and $n \in P_C$.

Hence we have $\{1, 2, n-1, n\} \subseteq P_C$ that is, P_C is not a minimal path set, resulting in a contradiction. Therefore there exists i^*, $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 1$. It implies that

$|P_C| = \left|\bigcup_{i=1}^{s}(P_C \cap C_i)\right| = \sum_{i=1}^{s} |P_C \cap C_i| \leq 1 + 2(s-1) = 2s - 1 = |R_1|.$

That is R_1 is a minimal path set with the maximum size in a circular con|2|n:F system.

(b1) $n+1=3s+t$, $s \geq 1$, $1 \leq t \leq 2$ and t and s are integers.

In view of Lemma 2, $R_2 = \{1,3,4,6,7,9,\ldots,3(s-1),3(s-1)+1,3s\}$ is a minimal path set with maximum size in a linear con|2|n:F system and $|R_2| = 2s$.

From Lemma 5 of Chapter 3, R_2 is also a minimal path set of a circular con|2|n|:F system. We have $C_s = \{3(s-1), 3(s-1)+1, 3(s-1)+2\} = \{n-t-2, n-t-1, n-t\}$. We define $C_{s+1} = \{n-t+1, n\}$. Let P_C be a minimal path set of a circular con|2|n:F system. We note that $|P_C \cap C_i| \leq 2$ for $i = 1,2,\ldots,s+1$.

If $t=1$ and $n \in P_C$ we then have $|P_C \cap C_1| = |P_C \cap \{1,2\}| = 1$. Hence we have

$$|P_C| = \left|\bigcup_{i=1}^{s+1}(P_C \cap C_i)\right| = \sum_{i=1}^{s+1}|P_C \cap C_i| \leq 1 + 2(s-1) + 1 = 2s = |R_2|.$$

Therefore R_2 is a minimal path set with the maximum size in a circular con|2|n:F system.

If $n \notin P_C$ we then have $|P_C| = \left|\bigcup_{i=1}^{s}(P_C \cap C_i)\right| = \sum_{i=1}^{s}|P_C \cap C_i| \leq 2s = |R_2|$ and the result is immediate.

Now suppose $t=2$. We show that there exist i^* and j^* such that $1 \leq i^* \leq s+1$, $1 \leq j^* \leq s+1$, $i^* \neq j^*$ and $|P_C \cap C_{i^*}| = |P_C \cap C_{j^*}| = 1$.

We note that $\{1,2,n-1,n\} \not\subset P_C$ that is $|P_C \cap C_1| = 1$ or $|P_C \cap C_{s+1}| = 1$.

Without loss of generality we assume that $|P_C \cap C_{s+1}| = 1$. We have $|P_C \cap C_i| = 2$ for $i = 1,2,\ldots,s$.

$\{1,2\} \subseteq P_C \Rightarrow 3 \notin P_C \Rightarrow \{4,5\} \subseteq P_C \Rightarrow 6 \notin P_C \Rightarrow \cdots \Rightarrow 3(s-1) = n-4 \notin P_C \Rightarrow \{n-3, n-2\} \subseteq P_C \Rightarrow n-1 \notin P_C \Rightarrow n \in P_C$.

Therefore we have $\{1,2,n\} \subset P_C$ resulting in a contradiction. Hence there exists i^*, $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 1$. We then have

$$|P_C| = \left|\bigcup_{i=1}^{s+1}(P_C \cap C_i)\right| = 1 + \sum_{i=1}^{s}|P_C \cap C_i| = 1 + \sum_{i \neq i^*}^{s}|P_C \cap C_i| + 1$$

which is less than or equal to $2 + 2(s-1) = 2s = |R_2|$. This completes the proof of part (i).

(ii) We consider two cases as follows.

(a2) $n = 2s$ for some integer $s \geq 1$.

Let $R_3 = \{2, 4, 6, \ldots, 2s\}$. From Lemma 2, we know that R_3 is a minimal path set with the minimum size in a linear con|2|n:F system and by Lemma 5 of Chapter 3, R_3 is also a minimal path set of a circular con|2|n:F system. We have $|R_3| = s = \lfloor n/2 \rfloor$. We show that size of R_3 is minimum. Suppose P_C is a minimal path set of a circular con|2|n:F system. We note that $C_i = \{2i-1, 2i\}$, $i = 1, 2, \ldots, \lfloor n/2 \rfloor$, are $\lfloor n/2 \rfloor$ disjoint minimal cut sets. We also note that $|P_C \cap C_i| \geq 1$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$. Hence $|P_C| = \left|\bigcup_{i=1}^{\lfloor n/2 \rfloor}(P_C \cap C_i)\right| \geq \lfloor n/2 \rfloor = |R_3|$. Therefore the result is immediate.

(b2) $n = 2s + 1$ for some integer $s \geq 1$

Let $R_4 = \{1, 2, 4, 6, 8, \ldots, 2s\}$. We know that R_4 is a minimal path set with the minimum size in a linear con|2|n:F system and is also a minimal path set of a circular con|2|n:F system. We now show that size of R_4 is minimum in a circular system. We have $|R_4| = s + 1 = [n/2] + 1$ and $|P_C \cap C_i| \geq 1$ for $i = 1, 2, \ldots, s$. Hence

$$|P_C| = \left|\bigcup_{i=1}^{s}(P_C \cap C_i) \cup \{P_C \cap \{n\}\}\right| \geq \left|\bigcup_{i=1}^{s}(P_C \cap C_i)\right| = \sum_{i=1}^{s}|P_C \cap C_i| \geq s.$$

If $n \in P_C$ then $|P_C| \geq s + 1 = |R_4|$ and the required result follows.

If $n \notin P_C$ we show that there exists i^*, $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 2$.

Suppose $|P_C \cap C_i| = 1$ for $i = 1, 2, \ldots, s$. We then have
$n \notin P_C \Rightarrow 1 \in P_C \Rightarrow 2 \notin P_C \Rightarrow 3 \in P_C \Rightarrow 4 \notin P_C \Rightarrow \cdots \Rightarrow$
$2s - 1 \in P_C \Rightarrow 2s \notin P_C \Rightarrow 2s + 1 = n \in P_C$ and this gives arise to a contradiction. Therefore, there exists i^*, $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 2$. Hence

$$|P_C| = \left|\bigcup_{i=1}^{s}(P_C \cap C_i)\right| = \sum_{i=1}^{s} |P_C \cap C_i| \geq (s-1) + 2 = s + 1 = |R_4|.$$

This completes the proof of the lemma. ∎

Lemma 7. $p_n^{r,C} = 2p_{n-3}^{r-2,L} + p_{n-6}^{r-4,L}$; $n \geq 6$, $\underline{r}_2^C(n) \leq r \leq \bar{r}_2^C(n)$.
In view of Lemma 4, the proof of this lemma is easy and omitted.

Lemma 8. $p_n^{r,C} = p_{n-2}^{r-1,C} + p_{n-3}^{r-2,C}$; $n \geq 3$, $\underline{r}_2^C(n) \leq r \leq \bar{r}_2^C(n)$.
Proof. Using Lemma 4 and Lemma 7, the proof follows.

Therefore the recurrence relation for the number of minimal path sets of given size r in both a linear and a circular con|2|n:F system is the same.

Now using Theorem 2 and Lemma 7 we give a closed form formula for $p_n^{r,C}$. We have

$$p_n^{r,C} = 2p_{n-3}^{r-2,L} + p_{n-6}^{r-4,L} = 2\binom{n-3-(r-2)+1}{2(n-3)-3(r-2)} + \binom{n-6-(r-4)+1}{2(n-6)-3(r-4)}$$

$$= 2\binom{n-r}{2n-3r} + \binom{n-r-1}{2n-3r} = \frac{n}{2r-n}\binom{n-r-1}{2n-3r}$$

5.3 Consecutive-3-out-of-n:F System

In this section, we confine our attention to the minimal path sets of a con|3|n:F system. We show that $p_n^L(3)$, the number of minimal path sets of a linear con|3|n:F system satisfies the recursive relation $p_n^L(3) = p_{n-2}^L(3) + p_{n-3}^L(3) + p_{n-4}^L(3) - p_{n-6}^L(3)$, $n \geq 6$. We also show that the same relation holds for $p_n^C(3)$, the number of minimal path sets of a circular con|3|n:F system, but with different initial values. We give a direct formula for determining $p_n^L(3)$.

Lemma 9. Let $\alpha_3^L(n)$ denote the collection of all minimal path sets of a linear con|3|n:F system. We then have for $n \geq 4$

$$\{S : S \in \alpha_3^L(n) \text{ and } n \in S\} = \left\{S : S = T \cup \{n-3, n\} \text{ and } T \in \alpha_3^L(n-4)\right\}.$$

Proof. It is easy to verify that
$\alpha_3^L(4) = \{\{1,4\},\{2\},\{3\}\}$, $\alpha_3^L(5) = \{\{1,4\},\{2,5\},\{2,4\},\{3\}\}$ and also $\alpha_3^L(6) = \{\{1,4\},\{2,5\},\{2,4\},\{3,6\},\{3,5\},\{3,4\}\}$. We assume that $\alpha_3^L(0) = \alpha_3^L(1) = \alpha_3^L(2) = \{\emptyset\}$. Then the lemma is trivially true for $n = 4, 5$ and 6. Now consider the case where $n \geq 7$. Let $S \in \alpha_3^L(n)$ and $n \in S$. We note that $\{n-1, n-2\} \cap S = \emptyset$. Hence $n-3 \in S$. It is easy to see that $T = S - \{n-3, n\}$ is a minimal path set of a linear con|3|$n-4$:F system, and hence $T \in \alpha_3^L(n-4)$. Conversely let $T \in \alpha_3^L(n-4)$ and $S = T \cup \{n-3, n\}$. In view of Theorem 1 of Chapter 3, S is a minimal path set of a linear con|3|n:F system. Therefore $S \in \alpha_3^L(n)$. This completes the proof of the lemma. ■

Lemma 10. For $n \geq 3$, we have

$$\{S : S \in \alpha_3^L(n) \text{ and } n-2 \in S\} = \left\{S : S = T \cup \{n-2\} \text{ and } T \in \alpha_3^L(n-3)\right\}.$$

Lemma 11. For $n \geq 3$, we have

$$\{S : S \in \alpha_3^L(n) \text{ and } n-1 \in S\}$$

$$= \left\{S : S = T \cup \{n-1\} \text{ where } T \in \alpha_3^L(n-2) \text{ and } n-2 \notin T\right\}.$$

Lemma 10 and Lemma 11 can be easily verified.

Lemma 12. In a linear con|3|n:F system we have

(i) $| \{S : S \in \alpha_3^L(n), 1 \in S\} | = | \{S : S \in \alpha_3^L(n), n \in S\} | = p_{n-4}^L(3)$ for $n \geq 4$.

(ii) $| \{S : S \in \alpha_3^L(n)$ and $1 \in S, n \in S\} | = p_{n-8}^L(3)$ for $n \geq 8$.

(iii) $| \{S : S \in \alpha_3^L(n)$ and $1 \notin S, n \notin S\} | = p_n^L(3) - 2p_{n-4}^L(3) + p_{n-8}^L(3)$ for $n \geq 8$.

Proof. Using Lemma 9, the proof of this Lemma follows.

Theorem 4. $p_n^L(3) = p_{n-2}^L(3) + p_{n-3}^L(3) + p_{n-4}^L(3) - p_{n-6}^L(3)$ for $n \geq 6$ with $p_0^L(3) = p_1^L(3) = p_2^L(3) = 1$, $p_3^L(3) = p_4^L(3) = 3$ and $p_5^L(3) = 4$.

Proof. We know that

$$\alpha_3^L(n) = \{S : S \in \alpha_3^L(n) \text{ and } n \in S\} \cup \{S : S \in \alpha_3^L(n) \text{ and } n-1 \in S\} \cup$$

$$\{S : S \in \alpha_3^L(n) \text{ and } n-2 \in S\}$$

and the collections on the right hand side are disjoint. From Lemma 9, we have $| \{S : S \in \alpha_3^L(n)$ and $n \in S\} | = | \alpha_3^L(n-4) | = p_{n-4}^L(3)$. Using Lemma 10, we have $| \{S : S \in \alpha_3^L(n)$ and $n-2 \in S\} | = | \alpha_3^L(n-3) | = p_{n-3}^L(3)$. And in view of Lemma 11 and Lemma 9, we have

$$| \{S : S \in \alpha_3^L(n) \text{ and } n-1 \in S\} | = | \alpha_3^L(n-2) | - | \alpha_3^L(n-6) | = p_{n-2}^L(3) - p_{n-6}^L(3).$$

Therefore we get the result $p_n^L(3) = |\alpha_3^L(n)| = p_{n-4}^L(3) + p_{n-3}^L(3) + p_{n-2}^L(3) - p_{n-6}^L(3)$. This completes the proof of the theorem. ∎

Minimal Path Sets of a Circular con|3|n:F System

Let $\alpha_3^C(n)$ denote the collection of all minimal path sets of a circular con|3|n:F system and $p_n^C(3) = | \alpha_3^C(n) |$. Here we show that

$$p_n^C(3) = p_{n-2}^C(3) + p_{n-3}^C(3) + p_{n-4}^C(3) - p_{n-6}^C(3), \; n \geq 5$$

with $p_{-1}^C(3) = 0, p_0^C(3) = 6, p_1^C(3) = 0, p_2^C(3) = 2, p_3^C(3) = 3$ and $p_4^C(3) = 6$.

We assume that $\alpha_3^L(m) = \{\emptyset\}$ for $0 \leq m < k$. The next Lemmae are required in the sequel.

Lemma 13. For $n \geq 8$ we have

$$\{S : S \in \alpha_3^C(n) \text{ and } 1 \in S, 2 \in S\} = \{S : S = \{1, 2, 5, n-2\} \cup T\}$$

where T is a minimal path set of a linear con|3|$n-8$:F subsystem that consists of the component $\{6, 7, 8, \ldots, n-3\}$ of the original system.

Proof. For $n = 8, 9$ and 10 the lemma is trivially true. Now consider $n \geq 11$. Let $S \in \alpha_3^C(n)$ and $\{1, 2\} \subseteq S$. We note that $\{3, 4, n-1, n\} \cap S = \emptyset$ and $\{5, n-2\} \subseteq S$. Using Theorem 1 of Chapter 3, it can be shown that $T = S - \{1, 2, 5, n-2\}$ is a minimal path set of a linear con|3|$n-8$:F subsystem with the component set $\{6, 7, \ldots, n-3\}$.
Conversely if T is a minimal path set of this subsystem then obviously $S = \{1, 2, 5, n-2\} \cup T$ is a minimal path set of a circular con|3|n:F system (see Lemma 4 of Chapter 3). This completes the proof of the lemma. ∎

Lemma 14. For $n \geq 8$ we have

$$\{S : S \in \alpha_3^C(n) \text{ and } 2 \in S, 3 \in S\} = \{S : S = \{2, 3, 6, n-1\} \cup T\}$$

where T is a minimal path set of a linear con|3|$n-8$:F subsystem with the component set $\{7, 8, \ldots, n-2\}$.
The proof of this lemma is similar to that of Lemma 13 and hence omitted.

Remark 4. Using Lemma 13 and Lemma 14 we have for $n \geq 8$

$$\left|\{S : S \in \alpha_3^C(n) \text{ and } 1 \in S, 2 \in S\}\right| = \left|\{S : S \in \alpha_3^C(n) \text{ and } 2 \in S, 3 \in S\}\right| = p_{n-8}^L(3)$$

Lemma 15. For $n \geq 4$ we have
$\left\{S : S \in \alpha_3^C(n) \text{ and } \{1,3\} \subseteq S\right\} = \{S : S = \{1,3\} \cup T\}$ where T is a minimal path set of a linear con|3|$n-3$:F subsystem with the component set $\{4, 5, \ldots, n\}$, such that $4 \notin T$ and $n \notin T$.

Proof. For $n = 4, 5$, the lemma is trivially true. Let $S \in \alpha_3^C(n)$ and $1 \in S$, $3 \in S$. We note that $4 \notin S, n \notin S$. In view of Theorem 1 of Chapter 3, $T = S - \{1,3\}$ is a minimal path set of a linear con|3|$n-3$:F subsystem. Conversely, if T is a minimal path set of this linear subsystem such that $4 \notin T, n \notin T$, then $S = \{1,3\} \cup T$ satisfies the conditions of Lemma 4 of Chapter 3. Therefore $S \in \alpha_3^C(n)$. This completes the proof of the lemma. ∎

Remark 5. Lemma 15 and part (iii) of Lemma 12 imply that $\left|\left\{S : S \in \alpha_3^C(n) \text{ and } \{1,3\} \subseteq S\right\}\right| = p_{n-3}^L(3) - 2p_{n-7}^L(3) + p_{n-11}^L(3)$, $n \geq 11$.

Lemma 16. For $n \geq 4$ we have

$$\{S : S \in \alpha_3^C(n) \text{ and } 1 \in S, 2 \notin S, 3 \notin S\} = \{S : S = \{1,4\} \cup T\}$$

where T is a minimal path set of a linear con|3|$n-4$:F subsystem with the component set $\{5, 6, \ldots, n\}$.

Lemma 17. For $n \geq 3$ we have

$$\{S : S \in \alpha_3^C(n) \text{ and } 1 \notin S, 2 \notin S, 3 \in S\} = \{S : S = \{3,n\} \cup T\}$$

where T is a minimal path set of a linear con|3|$n-4$:F subsystem with the component set $\{4, 5, \ldots, n-1\}$.

Lemma 16 and Lemma 17 can be easily verified.

Remark 6. Using Lemma 16 and Lemma 17 we have for $n \geq 4$

$$\left|\{S : S \in \alpha_3^C(n) \text{ and } 1 \in S, 2 \notin S, 3 \notin S\}\right| =$$

$$\left|\{S : S \in \alpha_3^C(n) \text{ and } 1 \notin S, 2 \notin S, 3 \in S\}\right| = p_{n-4}^L(3).$$

Lemma 18. For $n \geq 3$ we have

$$\{S : S \in \alpha_3^C(n) \text{ and } 1 \notin S, 2 \in S, 3 \notin S\} = \{S : S = \{2\} \cup T\}$$

where T is a minimal path set of a linear con$|3|n-1$:F system with the component set $\{3, 4, \cdots, n, n+1\}$, such that $T \cap \{3, n+1\} = \emptyset$.

Proof. Let $S \in \alpha_3^C(n)$ and $1 \notin S$, $2 \in S$, $3 \notin S$. Using Theorem 1 of Chapter 3, $T = S - \{2\}$ is a minimal path seth of a linear con$|3|n-1$:F system with the component set $\{3, 4, \cdots, n+1\}$, such that $\{3, n+1\} \cap T = \emptyset$. Conversely if T is a minimal path set of a linear con$|3|n-1$:F system with the component set $\{3, 4, \cdots, n, n+1\}$, such that $\{3, n+1\} \cap T = \emptyset$, then $S = \{2\} \cup T$ satisfies the conditions of Lemma 4 of Chapter 3. This completes the proof. ∎

Remark 7. Using Lemma 18 and part (iii) of Lemma 12, we then have $\left|\{S : S \in \alpha_3^C(n) \text{ and } 1 \notin S, 2 \in S, 3 \notin S\}\right| = p_{n-1}^L(3) - 2p_{n-5}^L(3) + p_{n-9}^L(3)$ for $n \geq 9$.

Theorem 5. For $n \geq 6$ we have

$$p_n^C(3) = p_{n-1}^L(3) + p_{n-3}^L(3) + 2p_{n-4}^L(3) - 2p_{n-5}^L(3) - 2p_{n-7}^L(3) + 2p_{n-8}^L(3) +$$

$p_{n-9}^L(3) + p_{n-11}^L(3)$ with $p_{-1}^L(3) = 1, p_{-2}^L(3) = p_{-3}^L(3) = 0, p_{-4}^L(3) = 1$ and $p_{-5}^L(3) = 0$.

Proof. We note that, Lemma 13 to Lemma 18, cover all possible and disjoint cases of a minimal path set in a circular con$|3|n$:F system. In view of this and using Remarks 4 to 7, we then have $p_n^C(3) = 2p_{n-8}^L(3) + p_{n-3}^L(3) - 2p_{n-7}^L(3) + p_{n-11}^L(3) + 2p_{n-4}^L(3) + p_{n-1}^L(3) - 2p_{n-5}^L(3) + p_{n-9}^L(3) = p_{n-1}^L(3) + p_{n-3}^L(3) + 2p_{n-4}^L(3) - 2p_{n-5}^L(3) - 2p_{n-7}^L(3) + 2p_{n-8}^L(3) + p_{n-9}^L(3) + p_{n-11}^L(3)$. ∎

Theorem 6. $p_n^C(3) = p_{n-2}^C(3) + p_{n-3}^C(3) + p_{n-4}^C(3) - p_{n-6}^C(3), n \geq 5$ with

$p^C_{-1}(3) = 0$, $p^C_0(3) = 6$, $p^C_1(3) = 0$, $p^C_2(3) = 2$, $p^C_3(3) = 3$ and $p^C_4(3) = 6$.

Proof. The proof follows using Theorem 4 and Theorem 5.

From Theorem 6, we note that the recurrence relations for $p^C_n(3)$ and $p^L_n(3)$ are the same but with different initial values.

We now consider direct computation of $p^L_n(3)$, the number of minimal path sets of a linear con|3|n:F system. In Theorem 4, we showed that for $n \geq 6$

$$p^L_n(3) = p^L_{n-2}(3) + p^L_{n-3}(3) + p^L_{n-4}(3) - p^L_{n-6}(3)$$

with $p^L_0(3) = p^L_1(3) = p^L_2(3) = 1$, $p^L_3(3) = p^L_4(3) = 3$ and $p^L_5(3) = 4$.

Let $g_L(x)$ denote the generation function of $p^L_n(3)$, that is $g_L(x) = \sum_{n=0}^{\infty} p^L_n(3)x^n$.

In view of Theorem 4, we then have $g_L(x)(1-x^2-x^3-x^4+x^6) = 1+x+x^3-x^5$.

Hence we get the result $g_L(x) = \dfrac{1+x+x^3-x^5}{1-x^2-x^3-x^4+x^6}$.

For obtaining partial fraction expansion of $g_L(x)$, we need to find the roots of the equation $1-x^2-x^3-x^4+x^6 = 0$. We know that, this equation has two positive real roots and four complex roots. We also note that if x be a root of this equation then $\frac{1}{x}$ is also another root. Hence we denote x_1 and $\frac{1}{x_1}$ as the real roots and $x_2, \frac{1}{x_2}, x_3$ and $\frac{1}{x_3}$ as the complex roots.

We have $1-x^2-x^3-x^4+x^6 = (x-x_1)(x-\frac{1}{x_1})(x-x_2)(x-\frac{1}{x_2})(x-x_3)(x-\frac{1}{x_3})$.

It implies that

$$(x_3+\frac{1}{x_3})+(x_2+\frac{1}{x_2}+x_1+\frac{1}{x_1})=0$$

$$3+(x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2})(x_3+\frac{1}{x_3})+(x_1+\frac{1}{x_1})(x_2+\frac{1}{x_2})=-1$$

$$-(x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2})-\left[2+(x_1+\frac{1}{x_1})(x_2+\frac{1}{x_2})\right](x_3+\frac{1}{x_3})-(x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2})=-1$$

From the first equation, we get $x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2}=-(x_3+\frac{1}{x_3})$. Using this and the second and third equations, we get

$$\begin{cases} 3+(x_2+\frac{1}{x_2})(x_3+\frac{1}{x_3})-\left(x_1+\frac{1}{x_1}\right)^2=-1 \\ -(x_1+\frac{1}{x_1})(x_2+\frac{1}{x_2})(x_3+\frac{1}{x_3})=-1 \Rightarrow (x_2+\frac{1}{x_2})(x_3+\frac{1}{x_3})=\dfrac{1}{x_1+\frac{1}{x_1}} \end{cases}$$

It implies that

$$3+\frac{1}{x_1+\dfrac{1}{x_1}}-(x_1+\frac{1}{x_1})^2=-1. \text{ Therefore}$$

$$(x_1+\frac{1}{x_1})^3-4(x_1+\frac{1}{x_1})-1=0 \qquad (1)$$

Similarly we can obtain

$$(x_2+\frac{1}{x_2})^3-4(x_2+\frac{1}{x_2})-1=0 \qquad (2)$$

$$(x_3+\frac{1}{x_3})^3-4(x_3+\frac{1}{x_3})-1=0 \qquad (3)$$

Let $s=x+\frac{1}{x}$ where x is a root of equation $1-x^2-x^3-x^4+x^6=0$. From equations (1),(2) and (3) we have $s^3-4s-1=0$. We note that the cubic equation $s^3-4s-1=0$ has three real roots.

Using Cardan formula we have $s_1 \simeq 2.114907542$, $s_2 \simeq -1.86080585$ and $s_3 \simeq -0.25410168$. We note that $s_1 = x_1 + \frac{1}{x_1}$, $s_2 = x_2 + \frac{1}{x_2}$ and $s_3 = x_3 + \frac{1}{x_3}$. For $s = s_1$, x_1 and $\frac{1}{x_1}$ can then be obtained from the quadratic equation $x^2 - s_1 x + 1 = 0$, as follows : $x_1 = \frac{s_1 + \sqrt{s_1^2 - 4}}{2} \simeq 0.713639174$ and $x_2 = \frac{1}{x_1} = \frac{s_1 + \sqrt{s_1^2 - 4}}{2} \simeq 1.401268368$.

For $s = s_2$ we have $x^2 - s_2 x + 1 = 0$. This equation has two complex roots, x_3 and $x_4 = \frac{1}{x_3}$, on the unit circle as follows :

$$x_3 = \frac{s_2 + \sqrt{s_1^2 - 4}}{2} = \frac{s_2 + i\sqrt{4 - s_1^2}}{2} \simeq 0.93040292 + i\, 0.36653897$$

where $i^2 = -1$ and $x_4 = \frac{1}{x_3} = \bar{x}_3$ (conjugate of x_3).

Suppose $x_3 = cis(\theta_3) = \cos\theta_3 + i\ \sin\theta_3$, where $\theta_3 = Arctg\left(\frac{I(x_3)}{R(x_3)}\right) + \pi \simeq 158^0, 49'$.

It implies that $x_4 = \bar{x}_3 = cis(\theta_4)$, where $\theta_4 = Arctg\left(\frac{I(x_4)}{R(x_4)}\right) + \pi = -Arctg\left(\frac{I(x_3)}{R(x_3)}\right) + \pi = 2\pi - \theta_3 \simeq 201^\circ, 51'$.

Similarly for $s = s_3$, we have $x^2 - s_3 x + 1 = 0$ which implies that $x_5 \simeq -0.127050844 + i\ 20.991896206$, and $x_6 = \frac{1}{x_6} = \bar{x}_5$. Suppose $x_5 = cis(\theta_5)$, where $\theta_5 = Arctg\left(\frac{I(x_5)}{R(x_5)}\right) + \pi \simeq 97^\circ, 29'$. And $x_6 = cis(\theta_6)$ with $\theta_6 = 2\pi - \theta_5 \simeq 262^\circ, 71'$.

Recall that, $g_L(x) = \sum_{n=0}^{\infty} p_n^L(3) x^n = \frac{1 + x + x^3 - x^5}{1 - x^2 - x^3 - x^4 + x^6} = \frac{U(x)}{V(x)}$. It is known that

$$p_n^L(3) = \frac{\rho_1}{{x_1}^{n+1}} + \frac{\rho_2}{{x_2}^{n+1}} + \cdots + \frac{\rho_6}{{x_6}^{n+1}}$$

where $\rho_i = \frac{-U(x_i)}{V'(x_i)}$, $i = 1, 2, \ldots, 6$. We also note that $\rho_4 = \bar{\rho}_3$ and $\rho_6 = \bar{\rho}_5$.

Now suppose $\rho_3 = a_3 + i\, b_3$, $\rho_5 = a_5 + i\, b_5$, $e_{n,3} = \frac{\rho_3}{x_3^{n+1}} + \frac{\rho_4}{x_4^{n+1}}$ and $e_{n,5} = \frac{\rho_5}{x_5^{n+1}} + \frac{\rho_6}{x_6^{n+1}}$.

We have $e_{n,3} = \frac{\rho_3}{{x_3}^{n+1}} + \frac{\bar{\rho}_3}{\bar{x}_3^{n+1}}$ and $e_{n,5} = \frac{\rho_5}{{x_5}^{n+1}} + \frac{\bar{\rho}_5}{\bar{x}_5^{n+1}}$. Therefore

$e_{n,3} = (a_3 + i\, b_3)[cis(-n-1)\theta_3] + (a_3 - i\, b_3)[cis(n+1)\theta_3] = 2[a_3 \cos(n+1)\theta_3 + b_3 \sin(n+1)\theta_3]$.

Similarly $e_{n,5} = 2[a_5 \cos(n+1)\theta_5 + b_5 \sin(n+1)\theta_5]$.

Theorem 7. For $n \geq 0$ we have $|e_{n,3} + \frac{\rho_2}{{x_2}^{n+1}}| < 1$ and $|e_{n,5} + \frac{\rho_2}{{x_2}^{n+1}}| < 1$. Hence $p_n^L(3) = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + e_{n,3} + 0.5 \right\rfloor = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + e_{n,5} + 0.5 \right\rfloor$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and $|x|$ is absolute value of x.

Proof. Using values of x_3 and x_5 we can compute $\rho_3 = \frac{-U(x_3)}{V'(x_3)}$ and $\rho_5 = \frac{-U(x_5)}{V'(x_5)}$ and then show that

$$\left| e_{n,3} + \frac{\rho_2}{{x_2}^{n+1}} \right| = \left| 2[a_3 cos(n+1)\theta_3 + b_3 sin(n+1)\theta_3] + \frac{\rho_2}{{x_2}^{n+1}} \right|$$

is less than or equal to $2(|a_3| + |b_3|) + \frac{\rho_2}{x_2} \leq 0.39 + 0.014 < 1$

where $\rho_3 = a_3 + i\, b_3$ and $\rho_2 = \frac{-U(x_2)}{V'(x_2)}$. Similarly we have

$$\left| e_{n,5} + \frac{\rho_2}{{x_2}^{n+1}} \right| = \left| 2[a_5 \cos(n+1)\theta_5 + b_5 \sin(n+1)\theta_5] + \frac{\rho_2}{{x_2}^{n+1}} \right|$$

$$\leq 2(|\, a_5 \,| + |\, b_5 \,|) + \frac{\rho_2}{x_2} \leq 0.73 + 0.014 < 1.$$

Therefore we get the result

$$p_n^L(3) = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + \frac{\rho_3}{{x_3}^{n+1}} + \frac{\rho_4}{{x_4}^{n+1}} + 0.5 \right\rfloor = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + \frac{\rho_5}{{x_5}^{n+1}} + \frac{\rho_6}{{x_6}^{n+1}} + 0.5 \right\rfloor.$$

That is

$$p_n^L(3) = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + e_{n,3} + 0.5 \right\rfloor = \left\lfloor \frac{\rho_1}{{x_1}^{n-1}} + e_{n,5} + 0.5 \right\rfloor.$$

This completes the proof of the theorem. ■

Remark 8. Using the approach described and Theorem 6 we can obtain an expression for $p_n^C(3)$, the number of minimal path sets of a circular con|3|n:F system. This approach becomes cumbersome for $k \geq 4$ and leads us to introduce a general recursive relationship for determining the number of minimal path sets of a con|k|n:F system, which is considered in the next section.

5.4 Consecutive-k-out-of-n:F System

In this Section we give a recursive relationship for determining the number of minimal path sets of a linear con|k|n:F system. For some special cases we give closed form formulae. We may add that direct computation of the number of minimal path sets of a con|k|n:F system still remains a difficult task.

Let $\alpha_k(n)$ denote the collection of all minimal path sets of a con|k|n:F system and suppose $P \in \alpha_k(n)$ is a minimal path set of the system. In view of Theorem 1 of Chapter 3, we note that $|P \cap \{1, 2, \ldots, k\}| = 1$. Therefore we can partition $\alpha_k(n)$ into k disjoint subcollections, that is $\alpha_k(n) = \bigcup_{i=1}^{k} \alpha_k^i(n)$, where $\alpha_k^i(n)$ is the collection of all minimal path sets of a con|k|n:F system that contain component i, $i = 1, 2, \ldots, k$.

Suppose $p_n(k) = |\alpha_k(n)|$ and $p_n^i(k) = |\alpha_k^i(n)|$. We then have $p_n(k) = \sum_{i=1}^{k} p_n^i(k)$. The following lemmae are required in the sequel.

Lemma 19. $p_n^i(k) = p_n^{n-i+1}(k)$ for all $i = 1, 2, \ldots, \left\lfloor \dfrac{n+1}{2} \right\rfloor$.

Proof. Consider another con|k|n:F system in which the components are numbered in the reverse order. Hence the proof follows. ■

Component $n-i+1$ is called the *mirror image* of component i.

Remark 9. We note that for $1 \leq n < k$, a con$|k|n$:F system is always working, irrespective of the states of the components. In this case $\alpha_k(n) = \{\emptyset\}$. Hence we assume that $p_n(k) = 1$ for $n = 1, 2, \ldots, k-1$. We also assume that $p_0(k) = 1$.

Lemma 20. For $1 \leq i \leq k$, we have

$$\{S : S \in \alpha_k^i(n)\} = \{S : S = \{i\} \cup T\}$$

where T is a minimal path set of a con$|k|n-i$:F subsystem that consists of last $n-i$ components of the original system such that $T \cap \{i+1, i+2, \ldots, k\} = \emptyset$. We assume that $\{i+1, i+2, \ldots, k\} = \emptyset$ if $i = k$.

Proof. The proof follows from the argument given in Section 4 of Chapter 4.

Theorem 8. For $1 \leq i \leq k$, we have

$$p_n^i(k) = p_{n-i}(k) - \sum_{x=1}^{k-i} p_{n-i}^x(k) = \sum_{x=k-i+1}^{k} p_{n-i}^x(k)$$

we assume that $\sum_{x=1}^{k-i} p_{n-i}^x(k) = 0$ if $i = k$.

Proof. Note that $p_{n-i}(k) = |\alpha_k(n-i)| = \left|\bigcup_{x=1}^{k} \alpha_k^x(n-i)\right| = \sum_{x=1}^{k} |\alpha_k^x(n-i)|$.
Hence we have

$$|\{T : T \in \alpha_k(n-i),\ T \cap \{1, 2, \ldots, k-i\} = \emptyset\}|$$

$$= \sum_{x=k-i+1}^{k} |\alpha_k^x(n-i)| = p_{n-i}(k) - \sum_{x=1}^{k-i} |\alpha_k^x(n-i)|.$$

On the other hand in view of Lemma 20, we have
$|\{T : T \in \alpha_k(n-i),\ T \cap \{1, 2, \ldots, k-i\} = \emptyset\}| = \sum_{x=k-i+1}^{k} p_{n-i}^x(k)$. Therefore

we get the result

$$p_n^i(k) = p_{n-i}(k) - \sum_{s=1}^{k-i} |a_k^s(n-i)| = p_{n-i}(k) - \sum_{s=1}^{k-i} p_{n-i}^s(k) = \sum_{s=k-i+1}^{k} p_{n-i}^s(k).$$

This completes the proof of the theorem. ∎

Remark 10. We note that in view of Theorem 8 to get $p_n^i(k)$ for $i = 1, 2, \ldots, k$, we first compute $p_n^k(k) = p_{n-k}(k)$. Using this, compute $p_n^1(k) = p_{n-1}^k(k) = p_{n-k-1}(k)$ and using this compute $p_n^{k-1}(k) = p_{n-k+1}(k) - p_{n-k+1}^1(k) = p_{n-k+1}(k) - p_{n-2k}(k)$ and then compute $p_n^2(k)$ and so on. We note that the last term is $p_n^{\bar{r}}(k)$ where $\bar{r} = \lfloor (k+1)/2 \rfloor$.

Example 1.

(a) If $k = 2$, we have $p_n^2(2) = p_{n-2}(2)$ and $p_n^1(2) = p_{n-1}^2(2) = p_{(n-1)-2}(2) = p_{n-3}(2)$. We get $p_n(2) = p_n^1(2) + p_n^2(2) = p_{n-2}(2) + p_{n-3}(2)$, as given in Section 2.

(b) If $k = 3$, we have $p_n^3(3) = p_{n-3}(3)$, $p_n^1(3) = p_{n-1}^3(3) = p_{n-4}(3)$ and $p_n^2(3) = p_{n-2}(3) - p_{n-2}^1(3) = p_{n-2}(3) - p_{n-6}(3)$. Hence we get $p_n(3) = p_n^1(3) + p_n^2(3) + p_n^3(3) = p_{n-2}(3) + p_{n-3}(3) + p_{n-4}(3) - p_{n-6}(3)$. As given in Section 3.

(c) If $k = 4$, we have $p_n^4(4) = p_{n-4}(4)$, $p_n^1(4) = p_{n-5}(4)$, $p_n^3(4) = p_{n-3}(4) - p_{n-8}(4)$ and $p_n^2(4) = p_{n-5}(4) - p_{n-10}(4) + p_{n-6}(4)$. Using these we have

$$p_n(4) = \sum_{s=1}^{4} p_n^s(4)$$

$$= p_{n-3}(4) + p_{n-4}(4) + 2p_{n-5}(4) + p_{n-6}(4) - p_{n-8}(4) - p_{n-10}(4).$$

For $k = 5, 6$ and 7 we have the following recursive relations.

$p_n(5) = p_{n-3}(5) + p_{n-4}(5) + p_{n-5}(5) + 2p_{n-6}(5) + p_{n-7}(5) - 2p_{n-9}(5) - 2p_{n-10}(5) - p_{n-12}(5) + p_{n-15}(5)$.

$p_n(6) = p_{n-4}(6)+p_{n-5}(6)+p_{n-6}(6)+3p_{n-7}(6)+2p_{n-8}(6)+p_{n-9}(6)-2p_{n-11}(6)-2p_{n-12}(6) - 3p_{n-14}(6) - 2p_{n-15}(6) + p_{n-18}(6) + p_{n-21}(6)$.

$p_n(7) = p_{n-4}(7)+p_{n-5}(7)+p_{n-6}(7)+p_{n-7}(7)+3p_{n-8}(7)+2p_{n-9}(7)+p_{n-10}(7)-3p_{n-12}(7)-4p_{n-13}(7)-3p_{n-14}(7)-3p_{n-16}(7)-2p_{n-17}(7)+3p_{n-20}(7)+3p_{n-21}(7)+p_{n-24}(7) - p_{n-28}(7)$.

Using Theorem 8, similar expressions for $p_n(k)$ with $k \geq 8$, can be obtained.

Remark 11. In view of Theorem 8 and Remark 10, $p_n^i(k)$ is of the form $\sum_{x=1}^{d_i} c_i(x) p_{n-n_i(x)}(k)$ where d_i, $c_i(x)$'s and $n_i(x)$'s are integers such that $0 < n_i(1) < n_i(2) < \cdots < n_i(d_i) = \bar{n}_i$ and $d_i > 0$. We note that this expression holds true if $n \geq \bar{n}_i$. For example we have $p_n^k(k) = p_{n-k}(k)$ if $n \geq k$, $p_n^1(k) = p_{n-k-1}(k)$ if $n \geq k+1$, $p_n^{k-1}(k) = p_{n-k+1}(k) - p_{n-2k}(k)$ if $n \geq 2k$ and $p_n^2(k) = p_{n-k-1}(k) + p_{n-k-2}(k) - p_{n-2k-2}(k)$ if $n \geq 2k+2$. We get $\bar{n}_k = k$, $\bar{n}_1 = k+1$, $\bar{n}_{k-1} = 2k$ and $\bar{n}_2 = 2k+2$. Using Remark 10, it is easy to verify that $\bar{n}_k < \bar{n}_1 < \bar{n}_{k-1} < \bar{n}_2 < \cdots < \bar{n}_{\bar{r}}$ where $\bar{r} = \lfloor (k+1)/2 \rfloor$. Therefore the recursive relation $p_n(k) = \sum_{i=1}^{k} p_n^i(k)$ holds true if $n \geq \bar{n}_{\bar{r}}$.

Lemma 21. $\bar{n}_{\bar{r}} = \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$

Proof. If $k = 2s+1$, for some integer s, $0 \leq s \leq \lfloor k/2 \rfloor$, we then have $\bar{r} = \lfloor (k+1)/2 \rfloor = s+1$. On the other hand we have $\bar{n}_k = k$, $\bar{n}_1 = k+1$, $\bar{n}_{k-1} = 2k$, $\bar{n}_2 = 2k+2$, $\bar{n}_{k-2} = 3k$, $\bar{n}_3 = 3k+3$, $\ldots$, $\bar{n}_{k-(s-1)} = sk$, $\bar{n}_s = sk+s$ and $\bar{n}_{k-s} = (s+1)k$. Note that $\bar{n}_{k-s} = \bar{n}_{s+1} = \bar{n}_{\bar{r}} = (s+1)k = (k+1)k/2 = \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$.

If $k = 2s$, we have $\bar{n}_{k-(s-1)} = sk$, $\bar{n}_s = sk+s$. In this case $\bar{r} = \lfloor (k+1)/2 \rfloor = s$.

Hence $\bar{n}_p = \bar{n}_s = sk + s = s(k+1) = k(k+1)/2 = \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$. This completes the proof of the lemma. ∎

Remark 12. We have $p_n(k) = 1$ for $0 \leq n \leq k-1$ and $p_k(k) = k$. We note that we first need to find $p_n(k)$ for $k < n < \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$. We then can use the recurrence equation $p_n(k) = \sum_{i=1}^{k} p_n^i(k)$ to find $p_n(k)$ for $n \geq \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$, where $p_n^i(k)$ can be obtained using Theorem 8.

In the remainder of this section we consider some special cases.

Lemma 22. If $k \leq n \leq 2k$ then $p_n(k) = 2k - n + \frac{(n-k)(n-k+1)}{2}$.

Proof. Suppose $n = k + t$ for some integer t, $0 \leq t \leq k$. In view of Lemma 2, we have $r_k(n) \geq 1$ and $\bar{r}_k(n) \leq 2$. If $t < k$ then minimal path sets of size 1 are given by $\{k\}$, $\{k-1\}$, …, $\{t+1\}$. If $t > 0$, then minimal path sets of size 2 are given by :

$\{1, k+1\}$

$\{2, k+2\}$, $\{2, k+1\}$

$\{t, k+t\}$, $\{t, k+t-1\}$, … , $\{t, k+1\}$.

Therefore the number of minimal path sets of size 1 equals to $k - t$ and the number of minimal path sets of size 2 equals to $1 + 2 + \cdots + t = t(t+1)/2$. Hence $p_n(k) = k - t + \frac{t(t+1)}{2}$, where $t = n - k$. This completes the proof of the lemma. ∎

Theorem 9. If $2k \leq n \leq 3k+1$ then $p_n(k)$ is given by

$$\frac{(3k-n)(3k-n+1)}{2} + \frac{(n-2k)\left[3k(k+1) - 2\left((n-2k)^2 - 1\right)\right]}{6}$$

$$+\frac{(n-2k)^2\left[(n-2k)^2-1\right]}{12}.$$

Proof. If $2k+1 \le n \le 3k+1$, then in view of Lemma 2, we have $\bar{r}_k(n) \ge 2$ and $\bar{\bar{r}}_k(n) \le 4$.

Using a simple enumeration process, it can be shown that the numbe of minimal path sets of size 2 is

$$1+2+3+\cdots+3k-n=\frac{(3k-n)(3k-n+1)}{2}. \tag{4}$$

The number of minimal path sets of size 3 is given by

$$\sum_{t=1}^{s}\left[t(k-t+1)+(k-t)+(k-t-1)+\cdots+(n-2k-t+1)\right]=$$

$$\frac{s\left[3k(k+1)-2(s^2-1)\right]}{6} \tag{5}$$

where $s=n-2k$.

And finally the number of minimal path sets of size 4 is given by

$$\sum_{j=2}^{s}\left[1+2^2+3^2+\cdots+(j-1)^2\right]=\sum_{j=2}^{s}\sum_{i=1}^{j-1}i^2=\frac{s^2(s^2-1)}{12} \tag{6}$$

Now using (4), (5) and (6) the proof follows. ∎

Remark 13. If $3k \le n \le 4k+1$, we note that $\bar{r}_k(n) \ge 3$ and $\bar{\bar{r}}_k(n) \le 6$. It can be verified that in this case, the number of minimal path sets of size 3, is given by

$$\sum_{j=1}^{k-s}\sum_{i=1}^{j}i=\sum_{j=1}^{k-s}\frac{j(j+1)}{2}=\frac{(k-s)(k-s+1)(k-s+2)}{6}$$

$$=\frac{(4k-n)(4k-n+1)(4k-n+2)}{6}, \text{ where } s=n-3k.$$

Conjectures

In view of the argument given in this chapter we make folldwing conjectures.

Using Lemma 22, we note that if $k \leq n \leq 2k$ then the number of minimal path sets of size 1 is $2k - n = \binom{2k-n}{1}$

Using formula (4), as given in the proof of Theorem 9, we note that if $2k \leq n \leq 3k+1$ then the number of minimal path sets of size 2 is

$$\frac{(3k-n)(3k-n+1)}{2} = \binom{3k-n+1}{2}$$

And in view of Remark 13, we note that if $3k \leq n \leq 4k+1$ then the number of minimal path sets of size 3 is

$$\frac{(4k-n)(4k-n+1)(4k-n+2)}{6} = \binom{4k-n+2}{3}$$

Now let $p_n^{r_k(n)}(k)$ denote the number of minimal path sets with minimum size $r_k(n) = \lfloor n/k \rfloor$, in a con$|k|n$:F system. We then have

Conjecture 1. $p_n^{r_k(n)}(k) = \binom{(r_k(n)+1)k - n + r_k(n) - 1}{r_k(n)}$

Note that the number of minimal path sets with minimum size and the number of path sets with minimum size in a con$|k|n$:F system are the same. Therefore we can use the results of Section 2 of Chapter 2, to compute $p_n^{r_k(n)}(k)$, but the expression as given in Conjecture 1, is simpler.

Conjecture 2. We have shown in Section 2 and Section 3 that the recurrence relations for $p_n^C(k)$ and $p_n^L(k)$, for the cases $k = 2$ and $k = 3$ are the same but with different initial conditions. Hence we conjecture that this property may also hold for a general con$|k|n$:F system, that is for $k \geq 4$.

We may add that direct computation for determining $p_n(k)$, the number of minimal path sets of a con$|k|n$:F system still remains a difficult task.

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