CONSTRUCTION OF SOME COMBINATORIAL DESIGNS ARISING OUT OF STATISTICAL EXPERIMENTS

By

Tridib Kumar Dutta

Under the supervision
of

Dr. Bimal Kr. Roy
Indian Statistical Institute
Calcutta

Thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the degree of Doctor of Philosophy

September, 1997.

CONSTRUCTION OF SOME COMBINATORIAL DESIGNS ARISING OUT OF STATISTICAL EXPERIMENTS

By

Tridib Kumar Dutta

Under the supervision
of

Dr. Bimal Kr. Roy
Indian Statistical Institute
Calcutta

Thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ACKNOWLEDGEMENT

I take this opportunity to express my deep gratitude to Dr. Bimal K. Roy, the supervisor of my research work, for his able guidance, constructive criticism and constant supervision at all stages of my work. This thesis would not have seen the light of the day but for his continuous interest and inspiration.

I owe a special debt of gratitude to Dr. A. C. Mukhopadhyay for his valuable suggestions, encouragement and critical comments, which has gone a long way in shaping and completion of this work. He was kind enough to oblige me readily whenever I approached him for comments and suggestions on my work.

I thank Prof. S. B. Rao, the Director of the Indian Statistical Institute, Mr. A. Majumder, Head of the SQC and OR Division, and Dr. S. P. Mukherjee, Head of the SQC-OR (T&P) Unit for kindly giving me the opportunity of carrying out the research work and providing the necessary help whenever requested for.

Last but not the least, I appreciate the readiness and cooperation provided by the stuff of SQC-OR (T&P) Unit in general and Shri J. Sharma in particular during the course of this work.

Tridib K Dutta

CONTENTS

1. INTRODUCTION

This dissertation considers construction of two kinds of combinatorial designs as used by statisticians: repeated measurements designs (RMDs) and symmetric balanced squares (SBSs).

1.1. REPEATED MEASUREMENTS DESIGNS

The researchers need to perform experiments where each experimental unit receives some or all of the treatments in an appropriate sequence over a number of successive periods. These designs are known by several names in the statistical literature: repeated measurements designs, crossover or changeover designs, (multiple) time series designs, and before-after designs. If there are n experimental units $1, 2, \ldots, n$, t treatments and p periods $0, 1, \ldots, p-1$, over which these treatments are to be applied, then an RMD(t,n,p) is an $n \times p$ array, say $D = ((d_{ij}))$ each element of which is one of the t treatments. The ith row of D gives the sequence of treatments applied to the ith unit over different periods. Generally in statistical literature transpose of the matrix D, D^T is defined as RMD(t,n,p), however, we will use D and not D^T in subsequent sections.

The applications of these designs are not limited to any single field of study but are gaining importance over such diverse fields as agriculture, medicine, pharmacology, industry, social sciences, animal husbandry, psychology and education. The designs have proved to be attractive because of their economic use of experimental units and because of the more sensitive treatment comparisons that result from elimination of inter-unit variation. The practical necessity of the experimental setup may also force us to use RMDs. These are the only options to an experimenter in studies to evaluate the effect of different sequences of drugs or nutrients or learning experiences. For details of models, practical applicability and examples, one may refer to Hedayat and Afsarinejad (1975), Hedayat (1981), Afsarinejad (1990), Patterson (1950, 51, 52), Patterson and Lucas (1962), Davis and Hall (1969) and Atkinson (1966).

The application of a sequence of treatments to the same unit in RMDs, however, has the potential of producing residual or carryover treatment effects in the periods following the application of the treatment. A residual effect which persists in the ith period after its application is called a residual effect of the ith order. In most of the work done, till date, it has been assumed that second- and higher-order residual effects are negligible. Consequently, most of the designs developed so far, permit only the estimation of first order residual effects along with the treatment effects. In this discourse, we also restrict our attention mostly to the first order residual effects except in section 3.3 where we have constructed designs considering second-order residual effects.

Residual effects are inherent to RMDs. In some cases, it might be desirable to measure such residual effects. In those cases where residual effects are undesirable but cannot be avoided due to inherent nature of the experimental requirement, they act as nuisance parameters and may need to be eliminated or measured by a proper design. So, RMDs gain importance over other designs in such situations too.

Before proceeding to literature survey on the subject, let us formally define some useful concepts.

Definition 1.1.1. An RMD is called uniform on periods if in each period the same number of units is assigned to each treatment.

Definition 1.1.2. An RMD is said to be uniform on units if on each unit each treatment appears in the same number of periods.

Definition 1.1.3. An RMD is said to be uniform if it is uniform on periods and experimental units simultaneously.

Definition 1.1.4. The underlying statistical model is called circular if in each unit the residuals in the initial period are incurred from the last period.

DEFINITION 1.1.5. The underlying statistical model is said to be linear or without preperiods or non-circular if in each unit the residuals in the initial period are zero.

DEFINITION 1.1.6. Under the non-circular model an RMD is called balanced if the collection of ordered pairs (d_{ij}, d_{ij+1}), $1 \le i \le n$, $0 \le j \le p-2$ contains each distinct ordered pair of treatments the same number of times and does not contain the pairs of identical treatments at all.

DEFINITION 1.1.7. Under the non-circular model an RMD is called strongly balanced if the collection of ordered pairs (d_{ij}, d_{ij+1}), $1 \le i \le n$, $0 \le j \le p-2$ contains each ordered pair of treatments, distinct or identical, the same number of times.

DEFINITION 1.1.8. An RMD is called circular balanced if the collection of ordered pairs (d_{ij}, d_{ij+1}), $1 \le i \le n$, $0 \le j \le p-1$ (operation on the second suffix is modulo p) contains each ordered pair of distinct treatments the same number of times and does not contain the pair of identical treatments at all.

DEFINITION 1.1.9. Under the circular model an RMD is called circular strongly balanced if the collection of ordered pairs (d_{ij}, d_{ij+1}), $1 \le i \le n$, $0 \le j \le p-1$ (operation on the second suffix is modulo p), contains each ordered pair of treatments, distinct or identical, the same number of times.

DEFINITION 1.1.10. In case $p < t$ we call an RMD balanced (strongly balanced) if it is uniform on periods, balanced (strongly balanced) for residuals and each experimental unit is administered distinct treatments, viz., no treatment is repeated on the same experimental unit.

DEFINITION 1.1.11. An RMD(t, n, t) is called a Williams design if it is uniform and the number of times treatment i appears adjacent to treatment j ($i \ne j$), on the same unit, are equal for all $i, j = 1, 2, \ldots, t$. [Treatment i is said to be adjacent to treatment j if either treatment i precedes or follows treatment j on the same experimental unit].

1.2 SYMMETRIC BALANCED SQUARES

Latin squares and symmetric latin squares are used in designing experiments, requiring two-way elimination of heterogeneity in the absence of interaction in the model. The book by Denes and Keedwell (1974) contains an excellent presentation of the subject. The combinatorial and algebraic features of the subject are covered, also applications to statistics and information theory are emphasized in it. Gomez and Gomez (1984) have pointed out the limitations of latin square designs. The requirement that all treatments appear in each row-block and in each column block is too stringent. As a result, when the number of treatments is large, the design becomes impractical, because the number of plots or experimental units is the square of the number of treatments. For practical purposes its use is restricted to trials with fewer than ten treatments. A more general class of squares, where such restriction is removed, is the class of balanced squares (BSs).

A BS of size n in v elements is an $n \times n$ array, $D = ((d_{ij}))$, where d_{ij} denotes the element assigned to the jth column in the ith row, $i, j = 1, 2, \ldots, n$, satisfying the following conditions:

i) each element occurs r or $r + 1$ times in every row and in every column (viz., rows and columns are balanced), where $r = [n/v]$.

ii) each element occurs f or $f + 1$ times in the array (viz., array is balanced), where $f = [n^2/v]$.

A balanced square of size n in v elements is abbreviated as BS (n, v). It is said to be symmetric if $d_{ij} = d_{ji}$, $i, j = 1, 2, \ldots, n$. A symmetric BS (n, v) is abbreviated as SBS (n, v).

It may be noted that in SBS(n, v), the only properties of a Latin Square extended are those given by (i) and (ii) above. A Latin Square also satisfies other important properties like perfect pairwise balance of elements in rows and columns. Those properties may no longer be preserved even in the extended sense in the SBS(n, v).

A generalized notion of balanced squares is balanced rectangles (BRs). A BR of size $n\times c$ in v elements, abbreviated as BR(n, c, v), is an $n\times c$ array containing v elements where

i) each element occurs either r or $r + 1$ times in every column, where $r = [n/v]$.

ii) each element occurs s or $s+1$ times in every row, where $s = [c/v]$.

iii) every element appears either f or $f + 1$ times in the array, where $f = [nxc/v]$.

We are not aware of much work in this area except for the pioneering work done by P J Schellenberg, who has also introduced the problem to us.

1.3 LITERATURE SURVEY

A number of papers in the statistical literature in recent years have considered the structure of designs with certain desirable statistical properties. As this dissertation considers only construction of designs, so a survey of literature on construction of RMDs is undertaken and a summery of important work done is presented in subsequent sections. Different authors have used various methods of construction, like cyclic arrangements of the treatments when the number of periods is less than that of treatments, and construction based on finite fields and sequenceable non-abelian groups for generating uniform and balanced designs.

Cochran, Autrey and Cannon (1941) were probably the first to point out that the classical designs are not suitable for estimation of direct and residual effects in their dairy cattle feeding experiment. Williams (1949,50) introduced and constructed two families of balanced RMDs when number of periods is equal to number of treatments. Algorithms for construction based on Williams' method are given by Bradley (1958) for even number of treatments while the same by Sheehe and Bross (1961) is for odd number of treatments.

Williams, and Sheehe and Bross basically used cyclic groups for construction of RMDs and failed to construct balanced RMDs for an odd number of treatments having a minimal number of experimental units. Houston (1966) showed that construction of such minimal balanced RMDs is not possible based on a cyclic group when number of treatments are odd. However, Williams, and Sheehe and Bross gave methods for constructing balanced RMD($t,2t,t$) for all odd t. Later Medelshon (1968) constructed a balanced minimal RMD for 21 treatments based on a non cyclic group. Balanced minimal RMDs do not exist for t = 3, 5, or 7 while they exist for t = 27, 39, 55, 57 (Denes and Keedwell, 1974). Hedayat and Afsarinejad (1978) list such designs for t = 9 and 15 attributing them to K.B. Mertz and E. Sonnemann respectively.

Scattered results on construction are available prior to 1970. Many examples can be found in works by Patterson (1950,51,52), Patterson and Lucas (1962), Atkinson (1966) and Davis and Hall (1969). The paper by Hedayat and Afsarinejad (1975) puts the subject in proper research perspective. In this paper, they have given method of construction of minimal (viz., requiring minimum number of experimental units for given values of number of treatments) balanced RMD($t,2t,p$) with $p < t$ whenever t is a prime power and also claim to have done the same for any odd integer t. Here they have also provided a method of construction of the universally optimal RMD with $p < t$ whenever t is a prime power and is congruent to 3 modulo 4. Patterson (1952) has also considered the case in which the number of periods is less than the number of treatments and constructed a series of minimal balanced RMDs. Patterson and Lucas (1962), Davis and Hall (1969) and Atkinson (1966) have also constructed some families of balanced minimal RMDs. Constantine and Hedayat (1982), and Afsarinejad (1983) have constructed by two different methods, balanced minimal RMDs whenever $p < t$ and the divisibility conditions allow for their existence. Afsarinejad, in the same paper, has also constructed strongly balanced RMDs whenever $p < t$.

If $t^2 \mid n$ and p / t is an even integer then Cheng and Wu (1980) showed the existence of a strongly balanced uniform RMD and give a method to

construct such designs. Berenblut (1964), Patterson (1970,73), and Kok and Patterson (1976) have constructed similar designs. Sen and Mukherjee (1987) have given a method of construction of such designs when $t^2 \mid n$ and pt^{-1} is an odd integer, using MOLS of order t. Consequently, the method fails for $t = 6$. However, they have listed a different method of construction for the same design which works for $t = 6$. But these later designs lacks some statistical properties compared to the ones based on MOLS.
Sonnemann, quoted in Kunert (1985a) gives a method of construction for circular balanced uniform RMDs with a minimum number of experimental units whenever $t > 2$ is an even integer. Afsarinejad (1990), using disjoint directed Hamiltonian cycles, constructs circular balanced uniform RMDs with minimum number of experimental units whenever t is an odd number.

Sharma (1982) constructs circular balanced uniform RMDs whenever $n = t$ and pt^{-1} is an even integer. Roy (1988), and Dutta and Roy (1992) have constructed circular balanced uniform RMDs, using different methods, when $t \nmid n$, pt^{-1} is an odd integer and $t = 0, 1, 3 \pmod 4$.

The problem of finding optimal experimental designs for comparing t treatments with a control in repeated measurements model has recently been considered by Pigeon (1984), Pigeon and Raghavarao (1987) and Majumder (1988). Pigeon and Raghavarao have constructed several families of efficient balanced RMDs for comparing test treatments with a control. Majumder provides a method of construction for certain optimal designs for direct treatment effects for comparing ω^2 test treatments with a control in the collection of such RMDs for a model without preperiods.

In situations with correlated errors, the optimal designs usually prove to be variants of the designs constructed by Willams (1949) and called Williams designs by Kunert (1985a). For $t = 2 \pmod 4$ Street (1988a) gives a method of construction of Williams design with a circular structure. Matthews (1987) obtains optimal designs under a linear fixed effects model with auto-correlated errors for three- and four-periods designs.

An extreme form of an RMD is the one in which the entire experiment is planned on a single experimental unit. Details on this can be found in Williams (1952), Finney and Outhwaite (1956) and Kiefer (1960).

For a general survey of RMDs one can refer to Hedayat and Afsarinejad (1975), Hedayat (1981), Bishop and Jones (1984), Street (1989) and Afsarinejad (1990). Hedayat and Afsarinejad (1975) and Afsarinejad (1990) have extensive bibliography on the subject. For structure of optimal designs when the treatments to be applied have a factorial structure, the interested reader is referred to Fletcher and John (1985) and Fletcher (1987).

1.4 SUMMARY OF WORK DONE

All the designs, discussed in subsequent chapters, arise out of statistical considerations. However in this dissertation we only consider the problem of construction of these designs based on combinatorial arrangements. We are solely concerned with the combinatorics and not the statistical properties of these designs. For construction of RMDs different adaptations of R.C. Bose's method of "symmetrically repeated differences" (1939) have been used under the heading of "method of differences". The method of construction employed to construct symmetric balanced squares are 1-factorizations or near 1-factorizations of a complete graph and Hall's matching theorem together with Fulkerson's (1959) theorem on the existence of a feasible flow in a network with bounds on flow leaving the sources and entering sinks.

1.4.1. In Chapter 2, section 1, we assume the underlying statistical model to be circular for construction of strongly balanced uniform repeated measurements designs and introduce the following notion of the method of differences. If G is a group with t elements and operation +, B is a p-tuple $(a_0, a_1, \ldots, a_{p-1})$ of the elements of G, where each element of G occurs exactly s ($s = p/t$) times in B and $\{a_i - a_{i+1} : i = 0, 1, \ldots, p-1\}$ (operation on the suffixes is modulo p) contains each element of G precisely s times, then $\{B + g : g \in G\}$ arranged in t rows, forms SBURMD(t,t,p). The tuple $B + g$ is defined as $(a_0 + g, a_1 + g, \ldots, a_{p-1} + g)$.

To simplify the construction of the designs, it is observed that an SBURMD(t,n,p) may be constructed by repeating $n \cdot t^{-1}$ SBURMD(t,t,p)'s vertically and, if SBURMD($t,t,2t$) exists then SBURMD(t,t,mt), where m is even, can be constructed by repeating SBURMD($t,t,2t$)'s horizontally. Also we note that if SBURMD($t,t,2t$) and SBURMD($t,t,3t$) both have two consecutive columns of the form $(0\ 1\ 2\ \ldots\ t-1)^T$, then SBURMD(t,t,mt), where m is odd and $m \geq 3$, can be constructed by taking SBURMD($t,t,3t$) followed by SBURMD($t,t,2t$)'s horizontally. These conditions are ensured if SBURMD's are constructed using difference vectors. We then attempt to get difference vectors for SBURMD($t,t,3t$). The contents of this chapter has been published (Dutta and Roy, 1992).

The cases t odd and even are dealt with separately.

When t is an odd integer we define the $2k$-tuple D to be:

$(0, 2k, 1, 2k-1, \ldots, k-1, k+1, k, k, k+1, k-1, \ldots, 2k-1, 1, 2k, 0)$.

Next, one occurrence of i is replaced by the triplet (i, $2k - i$, i) for $i = 0, 1, \ldots, k-1$ and one occurrence of k is replaced by the ordered pair (k,k). The resulting ordered $3t$-tuple is called D^*. It is then shown that $\{D^* + i : i = 0, 1, \ldots, 2k\}$ is an SBURMD($t,t,3t$). In view of this and the earlier observations, we obtain that for t an odd integer if $t|n$, $t|p$ *and* $p > t$ then SBURMD(t,n,p) always exists.

Next we consider the case $t = 2k$ and n an even multiple of t and define the following notation:

If A is an n-tuple $(a_1, a_2, \ldots, a_n)$ then A' is the tuple $(a_n, \ldots, a_2, a_1)$. If $A : (a_1, \ldots, a_n)$ and $B : (b_1, \ldots, b_m)$ then by AB we mean the $(n+m)$-tuple $(a_1, \ldots, a_n, b_1, \ldots, b_m)$.

Next on $\mathbb{Z}_{2k}$ we define $D_1 = (a_1, a_2, \ldots, a_k)$ and $D_2 = (a_{k+1}, a_{k+2}, \ldots, a_{2k})$ where

$$a_i = \begin{cases} (i-1)/2 & \text{if } i \text{ is odd} \\ 2k - i/2 & \text{otherwise} \end{cases}$$

Considering $D = D_1D_2$, $B = D\,D'D'$ and $C = D_1D_1'D_2'D_2D'$, it is proved that if $\mathcal{D}_1 = \{ B + i : i \in \mathbb{Z}_{2k} \}$, $\mathcal{D}_2 = \{ C + i : i \in \mathbb{Z}_{2k}\}$ and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, then $\mathcal{D}$ is an SBURMD$(t, 2t, 3t)$. So it follows that if t is an even integer and if $t|n$, $t|p$, $p > t$ and nt^{-1} is even then SBURMD(t,n,p) always exists.

Construction of SBURMD$(t,t,3t)$ for t an even integer is considered separately for $t \equiv 0 \pmod 4$ and $t \equiv 2 \pmod 4$.

For $t \equiv 0 \pmod 4$ we define

$$D : (1,2,3,\ldots,4k-2,4k-1,0,1,2,\ldots,2k,0,2k,0)$$

and tuples A_i's as:

$A_i \;:\; (i, 2k-i, i)$, $\quad i = 1,2,\ldots,k-1$

$A_k \;:\; (k,k)$

$A_i \;:\; (i, 2k-i, i, 2k-i, i)$, $\quad i = 2k+1, 2k+2, \ldots, 3k-1$

$A_{3k} \;:\; (3k,3k,3k)$.

A tuple D^* from D is obtained by replacing *one* occurrence of i by A_i for $i = 1,2,\ldots,k$ and $i = 2k+1,\ldots,3k$. Now we define a new operation $\pm$. This method of construction do not use the usual notion of difference technique. We term this technique as method of sum-difference. Such a concept is entirely new.

If $A = (a_1, a_2, \ldots, a_n)$, $a_i \in \mathbb{Z}_N$ and $k \in \mathbb{Z}_N$, then $A \pm k$ is defined to be the tuple $C = (C_1, C_2, \ldots, C_n)$, where

$$C_i = \begin{cases} a_i + k & \text{if } a_i \text{ is even} \\ a_i - k & \text{otherwise} \end{cases}$$

We now show that $\{D^* \pm i : i \in \mathbb{Z}_{4k}\}$ is an SBURMD$(t,t,3t)$ where $t = 4k$. In view of this and the earlier observations we have for $t \equiv 0 \pmod 4$, if $t \mid n$, $t \mid p$ and $p > t$ then an SBURMD(t,n,p) always exists.

When $t \equiv 2 \pmod 4$ and $n.t^{-1}$ is odd, the existence problem of SBURMD(t,n,p) is still unresolved. The method of differences for constructing SBURMD$(t,t,3t)$ works only if the group with t elements has the property that the sum of all the elements in that group is the identity element. We could not get a group with $(4k + 2)$ elements having the above mentioned property. The method of linked sum-difference uses the fact that $t \equiv 0 \pmod 4$ and could not be extended to the case $t \equiv 2 \pmod 4$.

1.4.2. In Chapter 2, section 2, strongly balanced uniform repeated measurements designs when number of periods is an odd multiple of number of treatments have been constructed under a non-circular model.

First it is shown here that if $t^2 | n$ and $p = 3t$ then there exists a SBURMD(t, n, p). It is further observed that this method of construction works for $p = 2t$ also.

Next using the previous result we prove that if $t^2 | n$ and p/t is an odd integer greater than one then there exists a SBURMD(t, n, p).

The construction method suggested is for obtaining SBURMDs when the underlying statistical model is non-circular. But for the same set of parameters, the constructed designs are strongly balanced uniform repeated measurements designs under circular model too.

1.4.3. A strongly balanced design is not possible for all combinations of the parameter values. So, in Chapter 3, section 1, we present methods of construction for a wide class of circular nearly balanced uniform repeated measurements designs (NBURMDs) when number of experimental units (n) equals the number of treatments (t). This work also suggests the existence of such designs for $n < t$, when number of periods (p) is less than number of treatments. In this section constructions of nearly balanced uniform RMD's are discussed when the underlying statistical model is circular. These designs are uniform on periods but not on experimental units. Here near uniformity on units is achieved, viz., on each experimental unit the frequencies of administration of treatments differ by at most one.

The notion of method of differences, adopted in the previous chapter is modified here. The modified notion is as given subsequently.

If G is a group with t elements and an operation $+$, B is a p-tuple $(a_0, a_1, \ldots, a_{p-1})$ of elements of G, where each element of G occurs s or $s+1$ ($s = [\, p/t \,]$) times in B and $C = \{a_{i+1} - a_i : i = 0, 1, \ldots, p-1\}$ (operation on the suffixes is modulo p) contains each element of G either s or $s+1$ times then $\{ B + g : g \in G \}$ arranged in t rows, forms NBURMD(t, t, p).

It is observed that NBURMD($t, t, t-1$) where t is an even integer, NBURMD($t, t, t-2$) and NBURMD($t, t, t+1$) where t is an odd integer cannot be constructed using the method of difference over $\mathbb{Z}_t$ and using the usual operation of addition modulo t.

The cases $p < t$ and $p > t$ are dealt with separately. The case $p < t$ has two sub cases, one for p even and the other for p odd. Similarly in case $p > t$ we deal separately for t even and t odd.

For the case $p < t$ and p even, we define

$$\begin{aligned} a_{i-1} &= (i - 1) / 2 && \text{for odd } i \text{ and } 1 \le i \le k_1 \\ &= t - i / 2 && \text{for even } i \text{ and } 1 \le i \le k_1 \\ &= i / 2 && \text{for even } i \text{ and } k_1+1 \le i \le p \\ &= p - (i - 3) / 2 && \text{for odd } i \text{ and } k_1+1 \le i \le p \end{aligned}$$

where k_1 and k_2 are odd positive integers such that $k_1 + k_2 = p$ and $0 \le k_1 - k_2 \le 2$.

It is proved that $B : (a_0, a_1, a_2, \ldots, a_{p-1})$ is a difference vector and when developed over $\mathbb{Z}_t$ produces a NBURMD(t, t, p).

Next for the case $p < t$ and p odd, it has been observed that if $p = t - 2$ for t an odd integer or $p = t - 1$ for t even then using the method of differences does not produce a NBURMD(t, t, p). For p not equal to these two values, the construction of NBURMD(t, t, p) for $p < t$ and p odd has been discussed. If $p' = p + 1$, then p' is even. Following the steps for the case p even, we construct a difference vector for NBURMD(t, t, p') where $k_1 + k_2 = p'$, k_1 and k_2 are odd integers and $0 \le k_1 - k_2 \le 2$. If $k_1 = k_2$ then we drop a_{k_1} else if $k_1 = k_2 + 2$ then $a_{k_1 - 1}$ is deleted from this difference vector and we call this resultant p-tuple as B.

In view of the preceding discussions, and the earlier result for p even, we show that the p-tuple B is a difference vector and when developed over $\mathbb{Z}_t$ gives a NBURMD(t, t, p).

It is observed that if some of the rows in the previous designs are deleted then they still continues to be a NBURMD. So NBURMD(t, n, p) can be constructed for $n \le t$, $p < t$ and t not equal to $p + 1$ or $p + 2$ for odd p.

Next we have shown that if $n \le t$, $p < t$ and, t is not equal to $p+1$ or $p+2$ for p an odd integer, then an NBURMD(t, n, p) exists.

The case $p > t$ is dealt with separately for t even and t odd.

For t even, we use the same notation as defined in section 1.4.1. We consider $\mathbb{Z}_{2k}$ and the t-tuple D : $(0\ 2k-1\ 1\ 2k-2\ .\ .\ .\ k+1\ k-1\ k)$, and observe that if the model is circular then $D_1 = D\ D'$ is a difference vector for construction of SBURMD$(t, t, 2t)$.

Next we defined tuples A_i's as:

$$
\begin{aligned}
A_i &: (i,\ 2k-1-i,\ i) &,\quad & i = 0,\ 1, \dots, [k/2]-1 \\
&: (i,\ 2k-2-i,\ i) &,\quad & i = [k/2], [k/2]+1, \dots, k-2 \\
&: (k-1, k-1) &,\quad & i = k-1 \\
&: (k+[(k-1)/2],\ k+[(k-1)/2]) &,\quad & i = k.
\end{aligned}
$$

For $t < p < 2t$, considering the t-tuple D, We replace an occurrence of i by the tuple A_i, $0 \le i \le (p-t-1)/2 - 1$ for $(p-t-1)/2$ elements of D and replace the element $k-1$ by A_{k-1} if $p-t$ is an odd integer. If $p-t$ is an even integer then replace i by A_i for $(p-t-2)$ elements of D, $0 \le i \le (p-t-2)/2 - 1$ and the elements $k-1$ and $k+[(k-1)/2]$ by A_{k-1} and A_k respectively. We also note that the procedure gives a difference vector for $p = 2t$ too which on developing over $\mathbb{Z}_t$ gives an SBURMD$(t, t, 2t)$.

For $2t < p < 3t$, we follow a similar procedure on the $2t$-tuple D_1. Here, if $p-2t$ is an odd integer then replace i by A_i for $(p-2t-1)/2$ elements of D_1, $0 \le i \le (p-2t-1)/2 - 1$ and the element $k-1$ of D_1 by A_{k-1}. Else, if $p-2t$ is even then replace an occurrence of i by the triplet A_i for

$(p-2t)/2$ elements of D_1, $0 \le i \le (p-2t)/2 - 1$.

We denote the augmented t-tuple D or $2t$-tuple D_1 by the p-tuple E and then we obtain that $\{ E + i : i = 0,1,\dots,t-1 \}$ is an NBURMD(t,t,p) when t is an even integer, $t \nmid p$ and $t < p < 3t$.

We further note that if SBURMD($t,t,p_1 t$) has two consecutive columns of the form $(0\ 1\ 2\ \dots\ t-1)^T$ and NBURMD(t,t,p_2) has a column of the form $(0\ 1\ 2\ \dots\ t-1)^T$ then NBURMD($t,t,p_1 t+p_2$) can be constructed by taking SBURMD($t,t,p_1 t$) followed by NBURMD(t,t,p_2). If SBURMD's and NBURMD's are constructed using difference vectors then this condition is trivially satisfied.

For any t and p such that $p > t$ we can express p as $p_1 + p_2$ where p_1 is an even integer and $t < p_2 < 3t$. Since SBURMD(t,t,p_1) and NBURMD(t,t,p_2) exist, we obtain that if t is an even integer and p is any integer such that $t \nmid p$ and $p > t$ then NBURMD(t,t,p) always exists.

If t is odd and $p > (t+1)$ [for $p = t+1$ NBURMD(t,t,p) cannot be constructed using the method of modified differences] define tuples A_i's as:

A_i : $(i,\ 2k-i,\ i)$, $i = 0,1,\dots,k-1$

: $(k,\ k)$, $i = k$, and

A : $(k+1,\ k+1)$.

The construction of NBURMD(t,t,p) for $t+1 < p < 2t$ and for $p > 2t$ is dealt with separately.

For $t+1 < p < 2t$, let D be a difference vector for a NBURMD($t,t,t-1$) and let $p = (t-1) + j$. Note $j \ge 3$.

For construction of difference vector when j is an odd integer we replace the elements $k-1$ and k by the tuples A_{k-1} and A_k respectively. Also we replace an occurrence of i by A_i, $0 \le i \le (j-3)/2 - 1$ for $(j-3)/2$ elements of D. If j is an even integer, then we replace the elements $k-1$ and k by the tuples A_{k-1} and A_k respectively. Next in this augmented

$(t+2)$-tuple, say D_2, we replace the element $k+1$ by the ordered pair A and replace an occurrence of i by A_i, $0 \le i \le (j-4)/2 - 1$ for $(j-4)/2$ elements of D_2.

For $p > 2t$, let $p = rt + p_1$, where $r\ (\ge 2)$ is a positive integer and $p_1 < t$. Suppose D_1 is a difference vector for a SBURMD(t, t, rt). If p_1 is an odd integer then we replace an occurrence of i by A_i for $(p_1 - 1)/2$ elements of D_1, $0 \le i \le (p_1 - 1)/2 - 1$ and k by A_k. Else if p_1 is even then we replace i in D_1 by A_i, $0 \le i \le p_1 - 1$.

The p-tuple obtained by augmenting $(t-1)$-tuple D or rt-tuple D_1 is called E. In view of the preceding discussions it follows that if t is an odd integer and p is any integer such that $t \nmid p$ and $p > t+1$ then NBURMD(t, t, p) always exists and is obtained by developing E over $\mathbb{Z}_t$.

When p is an odd integer and $t = p + 1$ or $t = p + 2$ and when $t = p - 1$ for an even p, the existence problem of NBURMD(t, n, p) is still unresolved. Our inability to construct these designs is basically because of the same reason as mentioned in the earlier section. Also it would be interesting to find such designs when $t \nmid n$ and $t \nmid p$ and $n > t$ and / or $p > t$.

1.4.4. Chapter 4 deals with two problems on construction of repeated measurements designs involving t treatments. In the first, balanced uniform repeated measurements designs have been constructed when the number of experimental unit is 1, which can be sliced or partitioned into blocks of k distinct elements each. Construction algorithm is given when $t(t-1)$ divides p, and k divides p for odd t while k divides t or $t-1$ for even t, p being the number of periods.

In the second problem, we consider the class of SBURMD's and give constructions for SBURMD's where the collection of ordered triplets $(d_{ij}, d_{i\,\overline{j+1}}, d_{i\,\overline{j+2}})$, $1 \le i \le n$, $0 \le j \le p-1$ (operation on the second suffix is modulo p), contains each ordered triplet of treatments, distinct or not, the same number of times, that is, np/t^3 times. This design is referred to as a second order SBURMD(t, n, p). The problem is solved when t is an odd prime and partially solved when t is a power of 2.

1.4.4.1. In section 2 of chapter 4 we consider the first problem. Here the number of experimental unit is one and we look for balanced uniform design in t elements over p periods which can be sliced or partitioned into blocks of k distinct elements. Such designs are denoted by BURMD($t, 1, p, k$). A fourth parameter k is introduced due to slicing requirement and absence of this fourth parameter, viz., a BURMD($t, 1, p$) indicates a balanced uniform repeated measurements design on which there is no slicing requirement. Since the design is balanced, so $t(t-1)|p$. We note that if BURMD($t, 1, t(t-1), k$) exists then BURMD($t, 1, mt(t-1), k$) where m is any positive integer, can be constructed repeating BURMD($t, 1, t(t-1), k$)'s horizontally m times. So we claim that if BURMD($t, 1, t(t-1), k$) exists then BURMD($t, 1, p, k$) exists where $p = mt(t-1)$, m a positive integer.

Therefore we consider designs with $p = t(t-1)$. Additional requirement on blocks requires slicing or partitioning of the design, viz., the sequence of $t(t-1)$ treatments into subsequences, each of k distinct elements. Therefore, it is observed that it is necessary for a BURMD($t, 1, t(t-1), k$) to exists that $k \le t$ and $k | t(t-1)$.

Further we note that if BURMD($t, 1, t(t-1), k$) exists for $k = t$ and $k = t-1$ then BURMD($t, 1, t(t-1), k$) exists for all k, where k divides t or else $t-1$.

In construction of BURMD's for $k = t$ or $t-1$, the following adaptation of the method of differences is used.

If G is a group with t elements and operation +, $B = (b_0, b_1, \ldots, b_{n-1}, b_0)$ is any circuit on G, $e_i = (b_i, b_{i+1})$, $i = 0, 1, \ldots, n-1$, (operation on suffixes is modulo n) and all elements of the collection $D = \{ b_{i+1} - b_i, i = 0, 1, \ldots, n-1 \}$ are distinct, then the circuits $\{B + g : g \in G\}$ are arc disjoint.

B is referred to as a difference vector and D as the set of linked differences in B. The construction of BURMD($t,1,t-1,k$) is taken up separately for $k = t-1$ and $k = t$, and for t odd or even, by applying the method of differences with $G = z_{t-1}$ or z_{t-2} and denoting the vertices of the complete directed graph with n vertices by the elements of z_n.

In case of $k = t - 1$ and t even, we consider $G_1 = z_{t-1} \cup \{\infty\}$, where ∞ is such that $x + \infty = \infty + x = \infty$ for all $x \in z_{t-1}$. Let k_1 and k_2 be odd positive integers such that $k_1 + k_2 = t - 2$ and $0 \le k_1 - k_2 \le 2$ and we define

$$
\begin{aligned}
b_i &= (i-1)/2 & , \quad 1 \le i \le k_1 & \quad \text{and} \quad i \text{ odd,} \\
&= (t-1) - i/2 & , \quad 1 \le i \le k_1 & \quad \text{and} \quad i \text{ even,} \\
&= (t-1) - (i-1)/2 & , \quad k_1 + 1 \le i \le t-2 & \quad \text{and} \quad i \text{ odd,} \quad \text{and} \\
&= i/2 & , \quad k_1 + 1 \le i \le t-2 & \quad \text{and} \quad i \text{ even.}
\end{aligned}
$$

If $B = (b_{k_1+1}, b_{k_1+2}, \ldots, b_{t-2}, b_1, b_2, \ldots, b_{k_1}, \infty)$ then differences on edges of B contain all non-zero residues modulo $t-1$ except 1, exactly once. So, developing B according to z_{t-1} provides a total of $t-1$ blocks of size $t-1$ each. If one additional block $C = (b_{k_1+1}, b_{k_1+1} + 1, \ldots, b_{k_1})$ is adjoined to these blocks,a total of t blocks of size $t-1$ is obtained which is a BURMD($t,1,t(t-1),t-1$).

For the case $k = t-1$ and t odd, we consider $G_1 = z_{t-1} \cup \{\infty\}$ as before and the following directed elementary path

$$0 \quad \overline{t-2} \quad 1 \quad \overline{t-3} \quad 2 \quad \overline{t-4} \ \ldots \ \overline{(t-1)/2 - 1} \quad (t-1)/2$$

The differences among edges of the above path contain all non-zero residues modulo $t-1$, exactly once. $B = (0, t-2, 1, t-3, 2, \ldots, (t-1)/2-1, \infty)$ is obtained by replacing $(t-1)/2$ by ∞. Now by developing B according to z_{t-1}, as before, we get a total of $t-1$ blocks of size $t-1$ each. To this we adjoin one additional block $C = (0, 1, 2, \ldots, t-2)$ and obtain the required t blocks of size $t-1$ for a BURMD($t,1,t(t-1),t-1$).

Next we have considered the case $k = t$ and t odd. For $G_1 = z_{t-1} \cup \{\infty\}$, the same directed elementary path is considered as in case $k = t-1$, viz.,

$$0 \quad \overline{t-2} \quad 1 \quad \overline{t-3} \quad 2 \quad \overline{t-4} \ \ldots \ \overline{(t-1)/2 - 1} \quad (t-1)/2.$$

If $B = (0, t-2, 1, t-3, \ldots, (t-1)/2, \infty)$ then developing B according to z_{t-1} produces a total of $t-1$ blocks of size t each for a BURMD$(t, 1, t(t-1), t)$.

For $k = t$ and t even, we consider $V = z_{t-2} \cup \{\infty\} \cup \{\infty_1\}$ where ∞_1 is defined in the same way as ∞ and, define a directed elementary path E by $E = (0, \infty_1, t-3, 1, t-4, 2, t-5, \ldots, (t-2)/2+1, (t-2)/2-1, (t-2)/2, \infty)$. Since among the differences on the edges of E, all non-zero residues modulo $t-2$ except -1 occur precisely once, developing E according to z_{t-2} produces a total of $t-2$ blocks of size t. We adjoin to this an additional block $C = (\infty, \infty, 0, t-3, t-4, t-5, \ldots, 2, 1)$, and obtain the required BURMD$(t, 1, t(t-1), t)$ when t is even.

Following the construction of BURMD$(t, 1, t(t-1), t)$ for t odd, it is noted that any sequence of less than or equal to $t-2$ elements contains all distinct treatments. So BURMD$(t, 1, p, k)$ where $k \le t-2$ and $k|p$, exists. Also we have given construction procedure for BURMD$(t, 1, t(t-1), k)$ for $k = t$ and $k = t-1$. Hence we obtain that if t is odd, $k \le t$, $t(t-1)|p$ and $k|p$ then BURMD$(t, 1, p, k)$ exists.

Also we have proved that BURMD$(t, 1, p, k)$ exists for even t if $k \le t$, $t(t-1)|p$ and k divides t or else $t-1$.

1.4.4.2. In chapter 4, section 3 we have considered the second problem discussed in section 1.4.4.1. earlier, viz., construction of second order strongly balanced uniform repeated measurements design.

Given a set $T = \{a_1, a_2, \ldots, a_t\}$ of t treatments, the problem considered here is to construct an RMD, viz., an array $D_{n \times p} = ((d_{ij}))$, where each $d_{ij} \in T$, $t|n$, $t|p$ and $t^3|np$ such that

(i) each treatment appears exactly p/t times in a row,

(ii) each treatment appears exactly n/t times in a column,

(iii) the design is strongly balanced, and

(iv) if $A_i = \{ (d_{ij}, d_{ij+1}, d_{ij+2}) \mid 0 \le j \le p-1$, addition of the second suffix is modulo p $\}$, $1 \le i \le n$ then $\bigcup_{i=1}^{n} A_i$ contains each ordered triplet of treatments (a_i, a_j, a_k); $a_i, a_j, a_k \in T$, exactly the same number of times, that is, np/t^3 times.

We note that requirement (iv) implies requirement (iii), that is, a second order URMD is also strongly balanced. Without loss of generality, we denote the treatments by the elements of z_t.

The notion of the method of differences, adopted in this section, is same as the one given in section 1.4.1. The definitions of difference vector and the set of linked differences also remain the same.

The cases for t an odd prime and t a power of 2 are dealt with differently.

For the case t an odd prime we define for $1 \le i \le [t/2]$ and $0 \le j \le 2t-1$,

$$\begin{aligned} b_j(i) &= (i.j\ /\ 2 \mod t) \quad \text{for } j \text{ even, and} \\ &= (-i.\overline{j+1}\ /\ 2 \mod t) \quad \text{for } j \text{ odd.} \end{aligned}$$

Considering the $2t$-tuple $B_i = (b_0(i), b_1(i), \ldots, b_{2t-1}(i))$, $1 \le i \le [t/2]$. B_i is a difference vector over z_t for each $i = 1, 2, \ldots, [t/2]$. We now modify B_1 and replace the ordered pairs $(\ell-1, t-\ell)$ by the quadruplets $(\ell-1, t-\ell, \ell-1, t-\ell)$, $\ell = 1, 2, \ldots, t-3/2$, and the ordered pair $(\overline{t-1}/2, \overline{t-1}/2)$ by the triplet $(\overline{t-1}/2, \overline{t-1}/2, \overline{t-1}/2)$. The modified B_1 is termed as B_1'. It is now proved that $B = B_1' B_2 \ldots B_{[t/2]}$ is a difference vector over z_t for t an odd prime number and when developed over z_t produces a second order SBURMD(t, t, t^2).

Since we have shown that a second order SBURMD(t, t, t^2) can be constructed for t an odd prime number, so second order SBURMD(t, n, p) can also be constructed for $n = mt$ and $p = rt^2$, both m and r are positive integers, by first writing m SBURMD(t, t, t^2)'s vertically and then by repeating this configuration r times horizontally. Thus we obtain that if t is an odd prime number and, $t|n$ and $t^2|p$ then a second order SBURMD(t, n, p) exists.

For the case t a power of 2, we consider z_t and define for $1 \le i \le t/2$, $0 \le j \le 2t-1$,

$$\begin{aligned} b_j(i) &= ((2i-3).(j/2) \mod t), \quad \text{for } j \text{ even, and} \\ &= ((i-1).j \mod t), \quad \text{for } j \text{ odd.} \end{aligned}$$

Next we consider $B_i = (b_j(i) | 0 \le j \le 2t-1)$, $i = 1,2,\ldots,t/2$ and replace $(t-i+1,0)$ in B_1 by $(t-i+1|B_i|0)$ for $i = 2,3,\ldots,t/2$. The resultant configuration is termed as B'. Here the ordered pair (1,0) appears at the end of B', and $b_{t-1}(t/2) \equiv t/2 + 1$ and $b_t(t/2) \equiv t/2$. So $B'+\{t/2\}$ contains the ordered pair (1,0). This new array $B'+\{t/2\}$ is so arranged that (1,0) appears at the end and this rearranged $B'+\{t/2\}$ is termed as D. Now $(B'|D)$ developed over z_t is a second order SBURMD$(t,t,2t^2)$. Thus we prove that if t is a power of 2, $t|n$ and $2t^2|p$ then a second order SBURMD(t,n,p) exists.

1.4.5. In Chapter 5 symmetric balanced squares for different sizes of array and for different numbers of treatments have been constructed. Algorithms, easily implementable on computers, have been developed for construction of such squares whenever the parameters satisfy the necessary conditions for existence of the square. The method of construction employs 1-factorizations of a complete graph or near 1-factorization of a complete graph, depending on whether the size of the array is even or odd, respectively. A paper based on the results of this chapter has been accepted for publication (Dutta and Roy, 1997).

First, it is shown in this chapter that a BS(n,v) always exists. It is further noted that this result is easily extended to balanced rectangles with n rows, c columns and v elements. The same construction procedure as with BS's works as well with BR's.

Further, we note in this chapter that the necessary conditions for existence of an SBS(n,v), $n > 1$ are $v \le n(n+1)/2$, and the number of elements of odd frequency in the square should not exceed the size of the array, n. Next, it is proved that if SBS(n,v) exists for $v \le n < 2v$ then SBS(n,v) exist for all $n \ge 2v$. Also we note that an SBS(n,n) is a symmetric latin square of order n which always exists.

Following these results, we define 1-factorization and near 1-factorization of a complete graph and proceed to provide a construction of these factorizations which are used for construction of SBS's.

The construction of these squares are considered separately for $n < v$ and for $v < n < 2v$. For $n < v$, the cases corresponding to n odd and n even are dealt differently. For $n < v$ and n odd our algorithm on construction is based directly on near 1-factorization of K_n.

The case $n < v$ and n even is divided into subcases - f even and f odd, where f is the integral part of n^2/v. The construction algorithm for f even, uses a 1-factorization of K_n along with Hall's matching theorem. For the case f odd, we use Hall's matching theorem together with Fulkerson's (1959) theorem on the existence of a feasible flow in a network with bounds on flow leaving the sources and entering the sinks, to obtain the required SBS.

When $v < n < 2v$, a simple approach exploiting the arrays for $n < v$ is proposed for construction of SBS's.

Finally we conclude with the result that necessary and sufficient conditions for the existence of an SBS(n,v) is $v \leq n(n+1)/2$, and the number of elements with odd frequency is at most n.

2. CONSTRUCTION OF STRONGLY BALANCED UNIFORM REPEATED MEASUREMENTS DESIGNS

2.1. UNDER A CIRCULAR MODEL

2.1.1. INTRODUCTION

In this section we give construction of SBURMD(t, n, p) for $t \equiv 0$, 1 or 3 (mod 4) assuming the underlying statistical model to be circular. Sharma (1982) and Sen and Mukherjee (1987) give method for constructing SBURMD(t, n, p) if $t \mid n$ and pt^{-1} is an even integer. Roy (1988) gives a method of construction for $t \equiv 0$, 1 or 3 (mod 4). The method used here for construction is considerably simpler than that of Roy. Especially the case $t = 0$ (mod 4) is dealt by a completely new method. Also we give a method of construction of SBURMD(t, n, p) when $t \equiv 0$ or 2 (mod 4) and nt^{-1} is even. The methods used are method of linked differences and method of linked sum-differences.

2.1.2. METHOD OF DIFFERENCES

Let G be a group with operation +, B be a subset of G and $g \in G$. Then $B + g$ is defined as follows:

$B + g = \{ b + g : b \in B \}$.

The proof of the following theorem which is an adaptation of R.C. Bose's method of "symmetrically repeated differences" (1939), being trivial, is omitted.

Theorem 2.1.1. *Let G be a group with t elements. Consider the p-tuple B : $(a_0, a_1, \ldots, a_{p-1})$ where $a_i \in G$ $\forall$ $i = 0, 1, \ldots, p-1$; each element of G occurs exactly s ($s = p/t$) times in B and $C = \{a_i - a_{i+1} : i = 0, 1, \ldots, p-1\}$ (operation on the suffixes is modulo p) contains each element of G precisely s times. Then $\{B + g : g \in G\}$ arranged in t rows, forms SBURMD(t, t, p).* □

B will be referred to as a difference vector and C as the set of linked differences in B. It is easy to see that the following are true.

NOTE **2.1.2.** An SBURMD(t,n,p) may be constructed by repeating $n \cdot t^{-1}$ SBURMD(t,t,p)'s vertically.

NOTE **2.1.3.** If SBURMD$(t,t,2t)$ exists then SBURMD(t,t,mt), where m is even, can be constructed by repeating SBURMD$(t,t,2t)$'s horizontally.

NOTE **2.1.4.** If SBURMD$(t,t,2t)$ and SBURMD$(t,t,3t)$ both have an identical column then SBURMD(t,t,mt), where m is an odd integer and $m \geq 3$, can be constructed easily. Since the underlying statistical model is circular, we can always assume, w.l.g., that this identical column is the first column for both these designs. Consequently, SBURMD(t,t,mt), where m is odd and $m \geq 3$, can be constructed by taking SBURMD$(t,t,3t)$ followed by SBURMD$(t,t,2t)$'s.

If SBURMD's are constructed using difference vectors, then the condition of Note 2.1.4 holds trivially.

We now attempt to get difference vectors for SBURMD$(t,t,3t)$.

2.1.3. THE CASE $t = 2k + 1$

Consider the difference vector

D : $(0,2k,1,2k-1,\ldots,k-1,k+1,k,k,k+1,k-1,\ldots,2k-1,1,2k,0)$

as constructed by Sen and Mukherjee (1987).

We replace *one* occurrence of i by the triplet $(i,2k-i,i)$ for $i = 0, 1, \ldots, k-1$ and replace *one* occurrence of k by the ordered pair (k,k).

Consider this modification on D and call the resulting ordered $3t$-tuple D^{*}.

LEMMA 2.1.5. *$\{D^* + i : i = 0, 1, \ldots, 2k\}$ is an SBURMD($t, t, 3t$).*

Proof: Sen and Mukherjee (1987) have shown that D is a $2k$-tuple, each element of $\mathbb{Z}_{2k+1}$ occurs twice and the collection of linked differences in D contains each element of $\mathbb{Z}_{2k+1}$ precisely twice. Now introduction of $(i, 2k - i, i)$ instead of an i contributes the elements $2k - i$, i to D^* and, $2k - 2i$ and $2i + 1$ in the collection of linked differences in D^*, $i = 0, 1, \ldots, k - 1$.

It is easy to observe that $\bigcup_{i=0}^{k-1} \{2k - i, i\} \cup \{k\} = \mathbb{Z}_{2k+1}$ and also $\bigcup_{i=0}^{k-1} \{2k-2i, 2i+1\} \cup \{0\} = \mathbb{Z}_{2k+1}$.

Hence D^* contains each element of $\mathbb{Z}_{2k+1}$ thrice and the collection of linked differences in D^* also contains each element of $\mathbb{Z}_{2k+1}$ thrice. Thus D^* is a difference vector for SBURMD($t, t, 3t$). □

Illustration: $k = 3$, $t = 7$ and

$$D : (0\ 6\ 1\ 5\ 2\ 4\ 3\ 3\ 4\ 2\ 5\ 1\ 6\ 0)$$

One 0 is replaced by (0 6 0), one 1 is replaced by (1 5 1), one 2 is replaced by (2 4 2), one 3 is replaced by (3 3). Thus

$$D^* : (0\ 6\ 0\ 6\ 1\ 5\ 1\ 5\ 2\ 4\ 2\ 4\ 3\ 3\ 3\ 4\ 2\ 5\ 1\ 6\ 0).$$

In view of the previous lemma and notes the following theorem is immediate.

THEOREM 2.1.6. *Let $t|n$, $t|p$ and $p > t$. If t is an odd integer, then SBURMD(t, n, p) always exists.* □

2.1.4. THE CASE $t = 2k$ AND n AN EVEN MULTIPLE OF t

We define the following notations:

If A is an n-tuple $(a_1, a_2, \ldots, a_n)$ then A' is the tuple $(a_n, \ldots, a_2, a_1)$. If

$A : (a_1, \dots, a_n)$ and $B : (b_1, \dots, b_m)$ then by AB we mean the $(n + m)$-tuple $(a_1, \dots, a_n, b_1, \dots, b_m)$. Consider $\mathbb{Z}_{2k}$.

Let $D_1 = (a_1, a_2, \dots, a_k)$ and $D_2 = (a_{k+1}, a_{k+2}, \dots, a_{2k})$ where

$$a_i = \begin{cases} (i - 1)/2 \text{ if } i \text{ is odd} \\ 2k - i/2 \quad \text{otherwise} \end{cases}$$

Let $D = D_1 D_2$. Then $D' = D_2' D_1'$. Let $B = D\ D' D'$ and $C = D_1 D_1' D_2' D_2 D'$.

LEMMA 2.1.7. *Let* $\mathcal{D}_1 = \{\ B + i : i \in \mathbb{Z}_{2k}\ \}$, $\mathcal{D}_2 = \{\ C + i : i \in \mathbb{Z}_{2k}\}$ *and* $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. *Then* $\mathcal{D}$ *is an SBURMD*$(t, 2t, 3t)$.

Proof: (i) The fact that both B and C have all the elements of $\mathbb{Z}_{2k}$ thrice follows from the observation that the set of elements in $\mathcal{D}$ is $\mathbb{Z}_{2k}$.

(ii) One can easily check that the collection of linked differences in B contains the element k four times, 0 twice and all other elements in $\mathbb{Z}_{2k}$ thrice.

Also the collection of linked differences in C contains the element k twice, element 0 four times and all other elements in $\mathbb{Z}_{2k}$ thrice.

Hence, a pair (a, a), $a \in \mathbb{Z}_{2k}$, appears twice in $\mathcal{D}_1$ and four times in $\mathcal{D}_2$, thus appearing six times in $\mathcal{D}$.

A pair (a, b), where $a - b = k$, appears four times in $\mathcal{D}_1$ and twice in $\mathcal{D}_2$, thus appearing six times in $\mathcal{D}$.

Any other ordered pair appears three times in $\mathcal{D}_1$ as well as in $\mathcal{D}_2$, thus appearing six times in $\mathcal{D}$.

Hence $\mathcal{D}$ is an SBURMD$(t, 2t, 3t)$ where $t = 2k$. □

Illustration: $k = 5$ i.e. $t = 10$,

$D_1 = (0\ 9\ 1\ 8\ 2)$,
$D_2 = (7\ 3\ 6\ 4\ 5)$.

Then

$B = (0\ 9\ 1\ 8\ 2\ 7\ 3\ 6\ 4\ 5\ 5\ 4\ 6\ 3\ 7\ 2\ 8\ 1\ 9\ 0\ 5\ 4\ 6\ 3\ 7\ 2\ 8\ 1\ 9\ 0)$

$C = (0\ 9\ 1\ 8\ 2\ 2\ 8\ 1\ 9\ 0\ 5\ 4\ 6\ 3\ 7\ 7\ 3\ 6\ 4\ 5\ 5\ 4\ 6\ 3\ 7\ 2\ 8\ 1\ 9\ 0)$

Observe that SBURMD$(t,2t,3t)$ constructed in the above method satisfies the condition of note 2.1.4. Hence we have the following theorem.

Theorem 2.1.8. *Let $t \mid n$, $t \mid p$, $p > t$, nt^{-1} be even and t be an even integer. Then SBURMD(t,n,p) exists.* □

2.1.5. THE CASE $t = 4k$

Consider the tuple

$$D : (1,2,3,\ldots,4k-2,4k-1,0,1,2,\ldots,2k,0,2k,0)$$

Define tuples A_i's as follows:

$$\begin{aligned} A_i &: (i,2k-i,i), && i = 1,2,\ldots,k-1 \\ A_k &: (k,k) \\ A_i &: (i,2k-i,i,2k-i,i), && i = 2k+1,2k+2,\ldots,3k-1 \\ A_{3k} &: (3k,3k,3k). \end{aligned}$$

Form a tuple D^{*} from D by replacing *one* occurrence of i by A_i, $i = 1,2,\ldots,k$ and $i = 2k+1,\ldots,3k$.

Let us define an operation $\pm$ as follows:

Let $A = (a_1,a_2,\ldots,a_n)$, $a_i \in \mathbb{Z}_N$ and $k \in \mathbb{Z}_N$. Then $A \pm k$ is defined to be the tuple $C = (C_1,C_2,\ldots,C_n)$, where

$$C_i = \begin{cases} a_i + k & \text{if } a_i \text{ is even} \\ a_i - k & \text{otherwise} \end{cases}$$

Lemma 2.1.9. *$\{D^{*} \pm i : i \in \mathbb{Z}_{4k}\}$ is an SBURMD$(t,t,3t)$ where $t = 4k$.*

Proof: (1) In D the elements 0 and $2k$ have occurred thrice, the elements $1,2,\ldots,2k-1$ have occurred twice and the elements $2k+1$, $2k+2,\ldots,4k-1$ have occurred once. Observe that the collection of elements in $\{\{2k-i,i\},\ i = 1,2,\dots,k-1\}$ is $\{1,2,\dots,k-1,k+1,\dots,2k-1\}$, the collection of elements in $\{\{2k-i,i,2k-i,i\} : i = 2k+1,\dots,3k-1\}$ is $2.\{2k+1,\dots,3k-1,3k+1,\dots,4k-1\}$ where if X is a collection of elements then $2X$ denotes the collection of elements in X, each occurring twice. Also we replace k by (k,k) and $3k$ by $(3k,3k,3k)$. Thus in D^* each element of $\mathbb{Z}_{4k}$ occurs thrice.

(ii) The collection of linked sums in D contains all the odd numbers in $\mathbb{Z}_{4k}$ and the number $2k$ precisely thrice.

(iii) The collection of linked differences in A_i, $i = 1, 2,\dots, k$ and $i = 2k + 1,\dots, 3k$, contains all the even numbers in $\mathbb{Z}_{4k}$ except $2k$, precisely thrice.

(iv) The sum and difference of 0 and $2k$ are both $2k$.

(v) i and $2k - i$ are either both odd or both even.

(vi) Let us see that a pair (a,b), $a,b \in \mathbb{Z}_{4k}$ occurs in $\{D^* \pm i : i \in \mathbb{Z}_{4k}\}$ precisely thrice.

Case 1 : $b = a$.
(a,a) occurs precisely once in $D^* \pm (a - k)$ and twice in $D^* \pm (a - 3k)$ if k is even and precisely once in $D^* \pm (k - a)$ and twice in $D^* \pm (3k - a)$ if k is odd.

Case 2 : $a - b = 2k$.
(a,b) occurs precisely twice in $D^* \pm (a - k)$ and once in $D^* \pm a$.

Case 3 : $a + b$ is odd or equivalently $a - b$ is odd.
In D, the collection of linked sums contains $a + b$ precisely thrice, say the corresponding occurrences in D^* are (u,v), (w,x) and (y,z). Then

$$(a,b) \text{ occurs once in } \begin{cases} D^* \pm (a-u) & \text{if } u \text{ is even} \\ D^* \pm (u-a) & \text{if } u \text{ is odd.} \end{cases}$$

Similarly (a,b) occurs once corresponding to (w,x) and (y,z).

Case 4 : $a - b$ is even or equivalently $a + b$ is even and $a - b \neq 2k$.

Due to (iii), we can find pairs (u,v), (w,x) and (y,z) in D^* such that $u-v = w-x = y-z = a-b$. Then corresponding to the pair (u,v),

$$(a,b) \text{ occurs once in } \begin{cases} D^* \pm (a - u) \text{ if } u \text{ is even} \\ D^* \pm (u - a) \text{ if } u \text{ is odd.} \end{cases}$$

Similarly (a,b) occurs once corresponding to (w,x) and (y,z). □

Illustration : $k = 2$, $t = 8$.

D : (1 2 3 4 5 6 7 0 1 2 3 4 0 4 0)

A_1 : (1 3 1)

A_2 : (2 2)

A_5 : (5 7 5 7 5)

A_6 : (6 6 6)

The method described gives the following SBURMD(8,8,24).

1	3	1	2	2	3	4	5	7	5	7	5	6	6	6	7	0	1	2	3	4	0	4	0
0	2	0	3	3	2	5	4	6	4	6	4	7	7	7	6	1	0	3	2	5	1	5	1
7	1	7	4	4	1	6	3	5	3	5	3	0	0	0	5	2	7	4	1	6	2	6	2
6	0	6	5	5	0	7	2	4	2	4	2	1	1	1	4	3	6	5	0	7	3	7	3
5	7	5	6	6	7	0	1	3	1	3	1	2	2	2	3	4	5	6	7	0	4	0	4
4	6	4	7	7	6	1	0	2	0	2	0	3	3	3	2	5	4	7	6	1	5	1	5
3	5	3	0	0	5	2	7	1	7	1	7	4	4	4	1	6	3	0	5	2	6	2	6
2	4	2	1	1	4	3	6	0	6	0	6	5	5	5	0	7	2	1	4	3	7	3	7

One can see that the condition of note 2.1.4 is satisfied by the design constructed by the method described in this section. Hence, we have the following theorem.

THEOREM **2.1.10.** *Let* $t \mid n$, $t \mid p$, $t < p$ *and* $t \equiv 0 \pmod 4$. *Then an SBURMD(t,n,p) always exists.* □

2.1.6. CONCLUDING REMARKS

When $t \equiv 2 \pmod 4$ and $n.t^{-1}$ is odd, the existence problem of SBURMD(t, n, p) is still unresolved. It has been shown by Roy (1988) that SBURMD(2,2,6) does not exist. The method of differences for constructing SBURMD($t, t, 3t$) works only if the group with t elements has the property that the sum of all the elements in that group is the identity element. We could not get a group with ($4k + 2$) elements having the above mentioned property. The method of linked sum-difference as used in sub-section 2.1.5 uses the fact that $t \equiv 0 \pmod 4$ and could not be extended to the case $t \equiv 2 \pmod 4$.

2.2 UNDER A NON-CIRCULAR MODEL

2.2.1. INTRODUCTION

Here a method of construction of strongly balanced uniform repeated measurements designs for t treatments, n experimental units and p periods is given under a non-circular model, when $t^2 | n$ and p/t is an odd integer greater than 1. Cheng and Wu (1980) show that $t^2 | n$ and $t | p$ are necessary conditions for the existence of an SBURMD(t, n, p) and give a method for construction when $t^2 | n$ and p/t is an even integer. We basically mimic this method to arrive at the required design. Sen and Mukherjee (1987) too provide method of construction for such designs but our method is much simpler.

2.2.2. SOLUTION

Let G be a group of t elements with operation + (i.e. addition modulo t) and let $A = ((a_{ij}))$ be an $n \times p$ array with elements from G. For $g \in G$, define $A + g$ as an $n \times p$ array $((a_{ij} + g))$.

LEMMA 2.2.1. *If* $t^2 | n$ *and* $p = 3t$ *then there exists an SBURMD*(t, n, p).

Proof: Let A be an orthogonal array of size t^2 in 3 constraints, t levels and is of strength 2. This array always exists; a latin square design of order t is an example of such an A. Note that between any two consecutive columns, every ordered pair of treatments appear exactly once. Consider the $n \times 3t$ array B constructed as :

$$B = [\ A | A+1 | A+2 | \dots | A+(t-1)]$$

Since A is an orthogonal array of strength 2, so are $A+1, A+2, \dots, A+(t-1)$. Note that every ordered pair of treatments occur exactly once in the $n \times 2$ submatrix constructed using the third column of $A+g$ and first column of $A+(g+1)$; $0 \le g \le t-2$. Thus B is obviously an SBURMD$(t, t^2, 3t)$.

An SBURMD$(t, \mu t^2, 3t)$ with μ being any positive integer is obtained by piecing μ copies of B together as

$$\begin{bmatrix} B \\ \hline B \\ \hline \vdots \\ \hline B \end{bmatrix}$$

□

NOTE 2.2.2. *This method of construction may also be followed when* $p=2t$. *It is easy to see that the treatment combinations as suggested by Cheng and Wu (1980) for the first two periods is really an OA*$(n, 2, t, 2)$.

THEOREM 2.2.3. *If* $t^2 | n$ *and* p/t *is an odd integer greater than one then there exists an SBURMD*(t, n, p).

Proof: Suppose $p = \alpha t$, where α is an odd integer greater than 1. Let A be an SBURMD$(t, n, 2t)$ and B be an SBURMD$(t, n, 3t)$ constructed as above. Clearly we can obtain C, an SBURMD(t, n, p) by piecing $(\alpha - 3)/2$ numbers of A with B as

$$C = [\ A | A | \dots | A | B\ ].$$

□

Illustration : $t = 3$, $n = 9$, $p = 9$. The following is an SBURMD(3,9,9):

	periods								
	0	0	0	1	1	1	2	2	2
	0	1	1	1	2	2	2	0	0
	0	2	2	1	0	0	2	1	1
	1	0	1	2	1	2	0	2	0
units	1	1	2	2	2	0	0	0	1
	1	2	0	2	0	1	0	1	2
	2	0	2	0	1	0	1	2	1
	2	1	0	0	2	1	1	0	2
	2	2	1	0	0	2	1	1	0

2.2.3 CONCLUDING REMARKS

The preceding construction method is suggested when the underlying model is non-circular. But for the same set of parameters, the constructed designs are strongly balanced uniform repeated measurements designs under circular model too.

3. CONSTRUCTION OF CIRCULAR NEARLY BALANCED UNIFORM REPEATED MEASUREMENTS DESIGNS

3.1. INTRODUCTION

In this chapter construction of nearly balanced uniform RMD's are discussed when the underlying statistical model is circular. Such designs are constructed when the number of experimental unit (n) is equal to the number of treatments (t). These designs are uniform on periods but not on experimental units. Here near uniformity on units are achieved, viz., on each experimental unit the frequency of administration of each treatment differ by at most one. An RMD(t,n,p) is called circular nearly balanced if the collection of ordered pairs (d_{ij}, d_{ij+1}), $1 \le i \le n$, $0 \le j \le p-1$ (operation on the second suffix is modulo p), contains each ordered pair of treatments, distinct or not, either s or $s+1$ times, where $s = [\ p/t\ ]$, the integral part of p/t. A circular nearly balanced RMD(t,n,p) which is uniform on periods and near uniform on experimental units is termed as circular nearly balanced uniform repeated measurements design and is abbreviated as NBURMD(t,n,p). Note that for such designs $t|n$ and $t \nmid p$. If $t|p$ then circular SBURMD(t,n,p) exists except when pt^{-1} is odd and $t \equiv 2$ (mod 4) [, refer Sharma(1982), Sen and Mukherjee (1987), and Dutta and Roy (1992)]. In this section SBURMD's are used to construct NBURMD's for $p > 2t$. Also the method of differences, as presented in section 2.1.2, is modified here for such constructions.

3.2. MODIFIED METHOD OF DIFFERENCES

Let G be a group with operation +, B be a subset of G and $g \in G$. Then $B+g$ is defined as in section 2.1.2, viz., $B + g = \{\ b + g\ :\ b \in B\ \}$. The proof of the following theorem being trivial, is omitted.

Theorem 3.2.1. *Let G be a group with t elements. Consider the p-tuple $B : (a_0, a_1, \ldots, a_{p-1})$ where $a_i \in G \ \forall\ i = 0, 1, \ldots, p-1$; each element of G occurs s or $s+1$ ($s = [\, p/t \,]$) times in B and $C = (a_{i+1} - a_i : i = 0, 1, \ldots, p-1)$ (operation on the suffixes is modulo p) contains each element of G either s or $s+1$ times. Then $\{ B + g : g \in G \}$ arranged in t rows, forms NBURMD(t, t, p).*

B will be referred to as a difference vector and C as the collection of linked differences in B.

In B and C, the frequency of different elements of $\mathbb{Z}_t$ differ by atmost one and the sum of the elements of C is zero. So in a difference vector B for an NBURMD(t, t, p), where $p < t$, no element can repeat and consequently the element 0 does not belong to C. Also note that the sum of elements of $\mathbb{Z}_t$ is $t/2$ if t is an even integer and is 0 if t is odd. Therefore C can neither be a $\overline{t-1}$ element set for t even and nor a $\overline{t-2}$ element set for t an odd integer. Also for $p = t+1$ when t is odd, the existence of a difference vector B will imply C contains one non-zero element of $\mathbb{Z}_t$ twice and the rest once. Since the sum of the elements of such a C is non-zero, it cannot be a set of linked differences. Thus in view of these discussions, the following is easy to note.

Note 3.2.2. NBURMD$(t, t, t-1)$ where t is an even integer, NBURMD$(t, t, t-2)$ and NBURMD$(t, t, t+1)$ where t is an odd integer cannot be constructed using the method of difference over $\mathbb{Z}_t$ and using the usual operation of addition modulo t. For these cases no construction procedure is provided in this chapter.

3.3.(i). CASE $p < t$ AND p EVEN

Define,
$$
\begin{aligned}
a_{i-1} &= (i-1)/2 && \text{for odd } i \text{ and } 1 \le i \le k_1 \\
&= t - i/2 && \text{for even } i \text{ and } 1 \le i \le k_1 \\
&= i/2 && \text{for even } i \text{ and } k_1+1 \le i \le p \\
&= p - (i-3)/2 && \text{for odd } i \text{ and } k_1+1 \le i \le p
\end{aligned}
$$

where k_1 and k_2 are odd positive integers such that $k_1 + k_2 = p$ and $0 \le k_1 - k_2 \le 2$.

LEMMA 3.3.3. *$B : (a_0, a_1, a_2, \dots, a_{p-1})$ is a difference vector and when developed over $\mathbb{Z}_t$ produces an NBURMD(t, t, p).*

Proof: It is easily seen that a_i's are all distinct elements of $\mathbb{Z}_t$. Let $d_i = a_i - a_{i-1}$ for $i = 1, 2, \dots, p$ where $a_p = a_0$. Then the following can be easily noted:

(i)

i	Range of i	d_i	Range of d_i
Odd	$1 \le i \le k_1 - 2$	$t - i$	$t - k_1 + 2 \le d_i \le t - 1$
Even	$2 \le i \le k_1 - 1$	i	$2 \le d_i \le k_1 - 1$
Odd	$i = k_1$	1	$d_i = 1$
Even	$k_1 + 1 \le i \le p-2$	$p + 1 - i$	$3 \le d_i \le k_2$
Odd	$k_1 + 2 \le i \le p-1$	$t - p - 1 + i$	$t - k_2 + 1 \le d_i \le t - 2$
Even	$i = p$	$t - p / 2$	$d_i = t - p/2$

(ii) For even integers i, d_i's are even for $i \le k_1 - 1$ and are odd for $i \ge k_1 + 1$. So d_i's are distinct and $2 \le d_i \le \max(k_1 - 1, k_2) \le p/2$.

(iii) For odd integers i, if d_i's are even (odd) for $i \le k_1 - 2$ then d_i's are odd (even) for $i \ge k_1 + 2$. So, in this case too, d_i's are distinct and $t - p/2 + 1 = \min(t - k_1 + 2,\ t - k_2 + 1) \le d_i \le t - 1$.

(iv) For $i \ne k_1$, $d_i > 1\ (= d_{k_1})$ and $p/2 < t - p/2\ (= d_p) < t - p/2 + 1$.

Hence, the linked differences in B are all distinct elements of $\mathbb{Z}_t$ and from the modified method of differences it follows that $(a_0, a_1, \dots, a_{p-1})$ is a difference vector and when developed over $\mathbb{Z}_t$ produces an NBURMD(t, t, p). □

Illustration : $p = 10$, $t = 15$.

Note $k_1 = k_2 = 5$. Thus the difference vector is B : (0 14 1 13 2 3 8 4 7 5) and the set of linked differences is $C = \{14, 2, 12, 4, 1, 5, 11, 3, 13, 10\}$. So, $\{B + i : i \in \mathbb{Z}_{15}\}$ is an NBURMD(15, 15, 10).

3.3.(ii). CASE $p < t$ AND p ODD

It has been observed in note 3.2.2 that if $p = t - 2$ for t an odd integer or $p = t - 1$ for t even then using the method of differences does not produces an NBURMD(t, t, p). For p not equal to these two values, the construction is discussed. Let $p' = p + 1$. Note p' is even. Following the steps in 3.3.(i), construct a difference vector for NBURMD(t, t, p') where $k_1 + k_2 = p'$, k_1 and k_2 are odd integers and $0 \le k_1 - k_2 \le 2$. It is easily seen that $p'/2 + 1$ does not appear as a linked difference in this difference vector. So, if $k_1 = k_2$ then drop a_{k_1} else if $k_1 = k_2 + 2$ then delete $a_{k_1 - 1}$ from this difference vector.

Call this resultant p-tuple as B.

In view of the preceding discussions and lemma 3.3.3, the following is immediate.

LEMMA 3.3.4. *The p-tuple B is a difference vector and when developed over $\mathbb{Z}_t$ gives an NBURMD(t, t, p).* □

Illustration : $p = 9$, $t = 15$. So $p' = p + 1 = 10$.
Here $k_1 = k_2 = 5$. Thus the difference vector B' for NBURMD(15, 15, $p' = 10$) is same as that of B obtained in the illustration in 3.3. (i), viz., $B' = (0\ 14\ 1\ 13\ 2\ 3\ 8\ 4\ 7\ 5)$. Since $k_1 = k_2$ so we drop $a_5 = 3$ and obtain B as $(0\ 14\ 1\ 13\ 2\ 8\ 4\ 7\ 5)$ and C as $\{14, 2, 12, 4, 6, 11, 3, 13, 10\}$.

NOTE 3.3.5. *It is observed that if some of the rows in the previous designs are deleted then they still continues to be an NBURMD. So NBURMD(t, n, p) can be constructed for $n \le t$, $p < t$ and t* not equal to *$p + 1$ or $p + 2$ for odd p.*

In view of the preceding discussions the following theorem is immediate.

Theorem 3.3.6. *If $n \le t$, $p < t$ and, t is not equal to $p+1$ or $p+2$ if p is an odd integer, then an NBURMD(t,n,p) exists.* □

3.4 CASE $t = 2k$ and $p > t$

We define the following notations:

If A is an n-tuple $(a_1, a_2, \ldots, a_n)$ then A' is the n-tuple $(a_n, \ldots, a_2, a_1)$. If A is an n-tuple $(a_1, a_2, \ldots, a_n)$ and B is an m-tuple $(b_1, b_2, \ldots, b_m)$ then by AB we mean the $(n+m)$-tuple $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$. Consider $\mathbb{Z}_{2k}$ and the t-tuple D : $(0 \; 2k-1 \; 1 \; 2k-2 \; . \; . \; . \; k+1 \; k-1 \; k)$. Note in D each element of $\mathbb{Z}_{2k}$ occurs exactly once and, the collection of linked differences in D contains each element of $\mathbb{Z}_{2k}$ precisely once except for the elements 0 and k. Assuming the model to be circular, the set of linked differences in D contains the element k twice and the element 0 does not figure at all. Sharma (1982) and, Sen and Mukherjee (1987) have shown $D_1 = D \; D'$ to be a difference vector for construction of SBURMD$(t,t,2t)$, viz., D_1 and the associated set of linked differences in D_1 contain each element of $\mathbb{Z}_t$ twice.

Define tuples A_i's as:

A_i : $(i,\ 2k-1-i,\ i)$	,	$i = 0,\ 1, \ldots, [k/2]-1$
: $(i,\ 2k-2-i,\ i)$	,	$i = [k/2], [k/2]+1, \ldots, k-2$
: $(k-1, k-1)$	,	$i = k-1$
: $(k+[(k-1)/2],\ k+[(k-1)/2])$	,	$i = k$.

Let A be an ordered triplet (i,j,i) or an ordered pair (i,i) then by $\{A\}$ we will denote the set $\{i,j\}$ or $\{i\}$ respectively. The collection of linked differences in A will mean the set $\{j-i,\ i-j\}$ or $\{0\}$ respectively and will be denoted by $\{d(A)\}$. Note that $\bigcup_{i=0}^{k}\{A_i\}$ is $\mathbb{Z}_{2k}$ and $\bigcup_{i=0}^{k}\{d(A_i)\}$ contains each element of $\mathbb{Z}_{2k}$ exactly once except for the elements 0 and k. The difference 0 occurs twice while the difference k does not appear in the collection.

For $t < p < 2t$, consider the t-tuple D. Now replace an occurrence of i by the tuple A_i, $0 \le i \le (p-t-1)/2 - 1$ for $(p-t-1)/2$ elements of D and replace the element $k-1$ by A_{k-1} if $p-t$ is an odd integer. If $p-t$ is an even integer then replace an occurrence of i by A_i for $(p-t-2)$ elements of D, $0 \le i \le (p-t-2)/2 - 1$ and the elements $k-1$ and $k+[(k-1)/2]$ by A_{k-1} and A_k respectively. Note the procedure also gives a difference vector for $p = 2t$, which on developing over $\mathbb{Z}_t$ gives an SBURMD$(t,t,2t)$.

For $2t < p < 3t$, we follow a similar procedure on the $2t$-tuple D_1. Here, if $p-2t$ is an odd integer then replace an occurrence of i by A_i for $(p-2t-1)/2$ elements of D_1, $0 \le i \le (p-2t-1)/2 - 1$ and the element $k-1$ of D_1 by A_{k-1}. Else, if $p-2t$ is even then replace an occurrence of i by the triplet A_i for $(p-2t)/2$ elements of D_1, $0 \le i \le (p-2t)/2 - 1$.

Let us denote the augmented t-tuple D or $2t$-tuple D_1 by the p-tuple E. In light of the preceding discussions the following lemma holds.

Lemma 3.4.7. *$\{ E + i : i = 0,1,\ldots,t-1 \}$ is an NBURMD(t,t,p) when t is an even integer, $t \nmid p$ and $t < p < 3t$.* □

Note 3.4.8. If SBURMD(t,t,p_1t) and NBURMD(t,t,p_2) both have an identical column then NBURMD$(t,t,p_1t + p_2)$ can be constructed easily. Since the underlying statistical model is circular, we can always assume that the identical column is the first column for both these designs. Consequently by taking SBURMD(t,t,p_1t) followed by NBURMD(t,t,p_2) we obtain an NBURMD(t,t,p_1t+p_2).

If SBURMD's and NBURMD's are constructed using difference vectors then the condition of the note is trivially satisfied.

For any t and p such that $p > t$ we can express p as $p_1 + p_2$ where p_1 is an even integer and $t < p_2 < 3t$. Sen and Mukherjee (1987) showed the existence of SBURMD(t,t,p_1) using difference vectors and we have given method of construction of NBURMD(t,t,p_2). In view of the preceding discussions, lemma 3.4.7. and note 3.4.8, the following theorem is true.

THEOREM 3.4.9. *If t is an even integer and p is any integer such that $t \nmid p$ and $p > t$ then NBURMD(t,t,p) always exists.* □

Illustration : $t = 10$ and $p = 15$.

D : (0 9 1 8 2 7 3 6 4 5)

A_0: (0 9 0), A_1: (1 8 1) , A_2: (2 6 2), A_3: (3 5 3), A_4: (4 4), A_5: (7 7). Replace an occurrence of i in D by A_i for $i = 0,1$ and replace 4 in D by (4 4). The resultant is the difference vector

E : (0 9 0 9 1 8 1 8 2 7 3 6 4 4 5)

and the set of linked difference in E is

$C = \{ 9, 1, 9, 2, 7, 3, 7, 4, 5, 6, 3, 8, 0, 1, 5\}$.

Now developing E over $\mathbb{Z}_{10}$ produces an NBURMD(10,10,15).

3.5 CASE $t = 2k + 1$ AND $p > (t+1)$

Define tuples a_i's as:

A_i : $(i,\ 2k-i,\ i)$, $i = 0,1,\dots,k-1$

: $(k,\ k)$, $i = k$, and

A : $(k+1,\ k+1)$.

The construction of NBURMD(t,t,p) for $t+1 < p < 2t$ and for $p > 2t$ will be dealt with seperately.

For $t+1 < p < 2t$, let D be the difference vector for an NBURMD$(t,t,t-1)$ constructed following the steps given in 3.3.(i) and let $p = (t-1) + j$. Note $j \geq 3$. In D, the element $k+1$ does not occur while the collection of linked differences in D does not contain the element 0. Note $\bigcup_{i=0}^{k}\{A_i\} = \mathbb{Z}_t$ and $\bigcup_{i=0}^{k}\{d(A_i)\} = \mathbb{Z}_t$. For construction of difference vector when j is an odd integer replace the elements $k-1$ and k by the tuples A_{k-1} and A_k respectively. Also replace an occurrence of i by $A_i, 0 \leq i \leq (j-3)/2 - 1$ for $(j-3)/2$ elements of D. If j is an even integer, then as before, replace the elements $k-1$ and k by the tuples A_{k-1} and A_k respectively. Next in this augmented $(t+2)$-tuple, say D_2, replace the element $k+1$ by the ordered pair A and replace an occurrence of i by A_i, $0 \le i \le (j-4)/2 - 1$ for $(j-4)/2$ elements of D_2.

For $p > 2t$, let $p = rt + p_1$, where r (≥ 2) is a positive integer and $p_1 < t$. Let D_1 be the difference vector for an SBURMD(t,t,rt) constructed following Dutta and Roy (1992). If p_1 is an odd integer then replace an occurrence of i by A_i for $(p_1-1)/2$ elements of D_1, $0 \le i \le (p_1-1)/2 - 1$ and k by A_k. Else if p_1 is even then replace i in D_1 by A_i, $0 \le i \le p_1-1$.

Let the p-tuple obtained by augmenting $(t-1)$-tuple D or rt-tuple D_1 be called E. In view of the preceding discussions the following theorem is immediate.

THEOREM 3.5.10. *If t is an odd integer and p is any integer such that $t \nmid p$ and $p > t+1$ then NBURMD(t,t,p) always exists.* □

Illustration : $t = 9$, $p = 16$. Note $p = (t-1) + 8$ and $t + 1 < p < 2t$. Let D be the difference vector for an NBURMD(9,9,8) where

$$D : (0\ 8\ 1\ 7\ 2\ 3\ 6\ 4)$$

The tuples A_i's are:

A_0: (0 8 0), A_1: (1 7 1), A_2: (2 6 2), A_3: (3 5 3), A_4: (4 4), A : (5 5).

Since $j = 8$, we replace the elements 3 and 4 in D by A_3 and A_4 respectively and the element 5 by A. Also the elements 0 and 1 are replaced by the tuples A_0 and A_1 respectively. The resultant difference vector is

$$E : (0\ 8\ 0\ 8\ 1\ 7\ 1\ 7\ 2\ 3\ 5\ 5\ 3\ 6\ 4\ 4)$$

and the collection of linked differences in E is

$$\{\ 8,\ 1,\ 8,\ 2,\ 6,\ 3,\ 6,\ 4,\ 1,\ 2,\ 0,\ 7,\ 3,\ 7,\ 0,\ 5\ \}$$

In developing E over $\mathbb{Z}_9$ the following NBURMD(9,9,16) is obtained.

0	8	0	8	1	7	1	7	2	3	5	5	3	6	4	4
1	0	1	0	2	8	2	8	3	4	6	6	4	7	5	5
2	1	2	1	3	0	3	0	4	5	7	7	5	8	6	6
3	2	3	2	4	1	4	1	5	6	8	8	6	0	7	7
4	3	4	3	5	2	5	2	6	7	0	0	7	1	8	8
5	4	5	4	6	3	6	3	7	8	1	1	8	2	0	0
6	5	6	5	7	4	7	4	8	0	2	2	0	3	1	1
7	6	7	6	8	5	8	5	0	1	3	3	1	4	2	2
8	7	8	7	0	6	0	6	1	2	4	4	2	5	3	3

3.6 CONCLUDING REMARKS

When p is an odd integer and $t = p + 1$ or $t = p + 2$ and when $t = p - 1$ for an even p, the existence problem of NBURMD(t, n, p) is still unresolved. Our inability to construct these designs is basically because of the same reason as mentioned in section 2.1.6. Also it would be interesting to find such designs when $t \nmid n$ and $t \nmid p$, and $n > t$ and/or $p > t$.

4. TWO VARIANTS OF REPEATED MEASUREMENTS DESIGNS

4.1 INTRODUCTION

This chapter deals with two problems on construction of repeated measurements designs (RMDs) involving t treatments. Both the problems were posed by Bose (1995). In the first, balanced uniform repeated measurements designs have been constructed when the number of experimental unit is 1, which can be sliced or partitioned into blocks of k distinct elements each. Construction algorithm is given when $t(t-1)$ divides p, and k divides p for odd t while k divides t or $t-1$ for even t, p being the number of periods. It is easily noted that for such a design to have the same number of units assigned to each treatment in each period is infeasible as there is a single experimental unit. So in light of the earlier definition, the design lacks uniformity over periods. This property of uniformity is required to eliminate the heterogeneity of the units in estimating the treatment effects. But when there is only one unit this aspect is taken care of. So for $n = 1$ we term an RMD to be uniform if on the unit each treatment appears in the same number of periods.

In the second problem, we consider the class of SBURMD's and give constructions for SBURMD's where the collection of ordered triplets $(d_{ij}, d_{i\overline{j+1}}, d_{i\overline{j+2}})$, $1 \le i \le n$, $0 \le j \le p-1$ (operation on the second suffix is modulo p), contains each ordered triplet of treatments, distinct or not, the same number of times, that is, np/t^3 times. Such a design is considered by Bose (1995) which is optimal if treatment effect from two previous periods are assumed in the model. Henceforth, such a design will be referred to as a second order SBURMD(t,n,p). The problem is solved when t is an odd prime and partially solved when t is a power of 2.

4.2. BURMD $(t,1,p)$ PARTITIONED IN BLOCKS OF k DISTINCT ELEMENTS

Here the number of experimental unit is one and we are looking for a balanced uniform design in t elements over p periods which can be sliced or partitioned into blocks of k distinct elements. Such a design will be denoted by BURMD($t,1,p,k$). Notice that a fourth parameter k is introduced due to slicing requirement and absence of this fourth parameter, viz., a BURMD($t,1,p$) indicates a balanced uniform repeated measurements design on which there is no slicing requirement. Since the design is balanced, so $t(t-1)|p$. Also one can note that if BURMD($t,1,t(t-1),k$) exists then BURMD($t,1,mt(t-1),k$) where m is any positive integer, can be constructed repeating BURMD($t,1,t(t-1),k$)'s horizontally m times. In view of these preceding observations, the following lemma is immediate.

LEMMA 4.2.0.1. *If BURMD($t,1,t(t-1),k$) exists then BURMD($t,1,p,k$) exists where $p = mt(t-1)$, m a positive integer.* □

Therefore to start with we consider designs with $p = t(t-1)$.

Additional requirement on blocks requires slicing or partitioning of the design, viz., the sequence of $t(t-1)$ treatments into subsequences, each of k distinct elements. Therefore, $k \le t$ and $k|t(t-1)$.

A BURMD($t,1,t(t-1)$) is an eulerian circuit in K_t^* which exists for all t, (refer Berge, 1985). But for the problem concerned an eulerian circuit which can be sliced into blocks of k distinct elements is required. For example, if $t = 4$ and $k = 3$ then 1 0 3 | 2 1 3 | 0 2 3 | 1 2 0 is a BURMD(4,1,12,3). It is easy to see that the following lemmas are true.

LEMMA 4.2.0.2. *It is necessary for a BURMD($t,1,t(t-1),k$) to exist that $k \le t$ and $k|t(t-1)$.* □

LEMMA 4.2.0.3. *If BURMD($t,1,t(t-1),k$) exists for $k = t$ and $k = t-1$ then BURMD($t,1,t(t-1),k$) exists for all k, where k divides t or else $t-1$.* □

The proof of lemma 4.2.0.3 is trivial since we have to put a '|' mark after every k elements in a BURMD($t,1,t(t-1),t$) or in a BURMD($t,1,t(t-1),t-1$) depending on whether k divides t or divides $t-1$ respectively.

In construction of BURMD's for $k = t$ or $t-1$, the following adaptation of the method of differences is used.

4.2.1 METHOD OF DIFFERENCES

Let G be a group with operation +, $B = (b_0, b_1, \ldots, b_{n-1}, b_0)$ be a circuit, $b_i \in G$, for $i = 0, 1, \ldots, n-1$, then $B + g$, $g \in G$, is defined as the circuit $(b_0 + g, b_1 + g, \ldots, b_{n-1} + g, b_0 + g)$. The proof of the following lemma, being trivial, is omitted.

Lemma 4.2.1.4. *Let G be a group with t elements. Let B be any circuit on G and $e_i = (b_i, b_{i+1})$, $i = 0, 1, \ldots, n-1$, (operation on suffix is modulo n). If all elements of the collection $L = \{ b_{i+1} - b_i,\ i = 0, 1, \ldots, n-1 \}$ are distinct, then the circuits $\{B + g : g \in G\}$ are arc disjoint.* □

B will be referred to as a difference vector and L as the set of linked differences in B. We now attempt to construct BURMD$(t, 1, \overline{t-1}, k)$ for $k = t$ and $k = t-1$ by applying the lemma with $G = \mathbb{Z}_{t-1}$ or $\mathbb{Z}_{t-2}$ and denoting the vertices of the complete directed graph with n vertices by the elements of $\mathbb{Z}_n$.

4.2.2. CASE $k = t - 1$ AND t EVEN

Consider $G_1 = \mathbb{Z}_{t-1} \cup \{\infty\}$, where ∞ is such that $x + \infty = \infty + x = \infty$ for all $x \in \mathbb{Z}_{t-1}$. Let k_1 and k_2 be odd positive integers such that $k_1 + k_2 = t - 2$ and $0 \le k_1 - k_2 \le 2$.

Define

$$
\begin{aligned}
b_i &= (i-1)/2, & 1 \le i \le k_1 &\text{ and } i \text{ odd},\\
&= (t-1) - i/2, & 1 \le i \le k_1 &\text{ and } i \text{ even},\\
&= (t-1) - (i-1)/2, & k_1 + 1 \le i \le t-2 &\text{ and } i \text{ odd, and}\\
&= i/2, & k_1 + 1 \le i \le t-2 &\text{ and } i \text{ even}.
\end{aligned}
$$

Note $b_1, b_2, \ldots, b_{t-2}, b_1$ is a directed elementary circuit of length $t-2$ ($\overline{t-2}$-circuit), and contains all the differences from 1 to $t-2$ and all b_i's are distinct. Also note $b_{k_1+1} - b_{k_1} \equiv 1 \pmod{t-1}$.

Let $B = (b_{k_1+1}, b_{k_1+2}, \ldots, b_{t-2}, b_1, b_2, \ldots, b_{k_1}, \infty)$. Differences on edges of B contain all non-zero residues modulo $t-1$ except 1, exactly once. So, developing B according to $\mathbb{Z}_{t-1}$ provides a total of $t-1$ blocks of size $t-1$ each. If one additional block $C = (b_{k_1+1}, b_{k_1+1}+ 1, \ldots, b_{k_1})$ is adjoined to these blocks, a total of t blocks of size $t-1$ is obtained which is a BURMD($t, 1, t(t-1), t-1$).

Illustration : $t = 8$, $k = 7$, and $G_1 = \mathbb{Z}_7 \cup \{\infty\}$.
Therefore, $k_1 = 3$ and $k_2 = 3$, and
$(b_1, b_2, \ldots, b_6, b_1) = (0\ \ 6\ \ 1\ \ 2\ \ 5\ \ 3\ \ 0)$.
So, $B = (2\ \ 5\ \ 3\ \ 0\ \ 6\ \ 1\ \ \infty)$,
$B + i = (2+i\ \ 5+i\ \ 3+i\ \ i\ \ 6+i\ \ 1+i\ \ \infty)$, $i \in \mathbb{Z}_7$.
Therefore, the blocks are

$$\begin{aligned} B+0 &= (2\ \ 5\ \ 3\ \ 0\ \ 6\ \ 1\ \ \infty), \\ B+1 &= (3\ \ 6\ \ 4\ \ 1\ \ 0\ \ 2\ \ \infty), \\ B+2 &= (4\ \ 0\ \ 5\ \ 2\ \ 1\ \ 3\ \ \infty), \\ B+3 &= (5\ \ 1\ \ 6\ \ 3\ \ 2\ \ 4\ \ \infty), \\ B+4 &= (6\ \ 2\ \ 0\ \ 4\ \ 3\ \ 5\ \ \infty), \\ B+5 &= (0\ \ 3\ \ 1\ \ 5\ \ 4\ \ 6\ \ \infty), \\ B+6 &= (1\ \ 4\ \ 2\ \ 6\ \ 5\ \ 0\ \ \infty), \text{ and} \\ C &= (2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 0\ \ 1). \end{aligned}$$

Thus we have constructed a BURMD(8,1,56,7) as:

$$B+0 \mid B+1 \mid B+2 \mid B+3 \mid B+4 \mid B+5 \mid B+6 \mid C.$$

4.2.3. CASE $k = t-1$ AND t ODD

Consider $G_1 = \mathbb{Z}_{t-1} \cup \{\infty\}$ as before and the following directed elementary path

$$0\ \ \overline{t-2}\ \ 1\ \ \overline{t-3}\ \ 2\ \ \overline{t-4} \cdots \underbrace{(t-1)/2 - 1\ \ \ (t-1)/2}_{d_e = 1}$$

The differences among edges of the above path contain all non-zero residues modulo $t-1$, exactly once. Let $B = (0, t-2, 1, t-3, 2, t-4, \ldots, (t-1)/2-1, \infty)$ obtained by replacing $(t-1)/2$ by ∞. Developing B according to $\mathbb{Z}_{t-1}$, as before, produces a total of $t-1$ blocks of size $t-1$ each. If this is adjoined by one additional block $C = (0, 1, 2, \ldots, t-2)$, we obtain the required t blocks of size $t-1$ for a BURMD$(t, 1, t(t-1), t-1)$.

Illustration : $t = 7$, $k = 6$, and $G_1 = \mathbb{Z}_6 \cup \{\infty\}$. So $B = (0\ 5\ 1\ 4\ 2\ \infty)$. Hence the blocks for BURMD(7,1,42,6) are

$B+0 = (0\ 5\ 1\ 4\ 2\ \infty)$,
$B+1 = (1\ 0\ 2\ 5\ 3\ \infty)$,
$B+2 = (2\ 1\ 3\ 0\ 4\ \infty)$,
$B+3 = (3\ 2\ 4\ 1\ 5\ \infty)$,
$B+4 = (4\ 3\ 5\ 2\ 0\ \infty)$,
$B+5 = (5\ 4\ 0\ 3\ 1\ \infty)$, and
$C = (0\ 1\ 2\ 3\ 4\ 5)$.

The required design is:

$B+0 \mid B+1 \mid B+2 \mid B+3 \mid B+4 \mid B+5 \mid C.$

4.2.4 CASE $k = t$ AND t ODD

Consider $G_1 = \mathbb{Z}_{t-1} \cup \{\infty\}$. Consider the same directed elementary path as in section 4.2.3, viz.,

$$0\ \ \overline{t-2}\ \ 1\ \ \overline{t-3}\ \ 2\ \ \overline{t-4}\ \ \ldots\ \ \overline{(t-1)/2-1}\ \ (t-1)/2.$$

Let $B = (0, t-2, 1, t-3, \ldots, (t-1)/2, \infty)$. Now developing B according to $\mathbb{Z}_{t-1}$ produces a total of $t-1$ blocks of size t each for a BURMD$(t, 1, t(t-1), t)$.

Illustration : $t = 7$, $k = 7$ and $G_1 = \mathbb{Z}_6 \cup \{\infty\}$ as before. So $B = (0,5,1,4,2,3,\infty)$. Hence the blocks of size 7 for BURMD(7,1,42,7) are:

$B+0 \mid B+1 \mid B+2 \mid B+3 \mid B+4 \mid B+5.$

4.2.5 CASE $k = t$ AND t EVEN

Let $V = \mathbb{Z}_{t-2} \cup \{\infty\} \cup \{\infty_1\}$ where ∞_1 is defined in the same way as ∞. Define a directed elementary path E by $E = (0, \infty_1, t-3, 1, t-4, 2, t-5, \ldots, (t-2)/2+1, (t-2)/2-1, (t-2)/2, \infty)$. Since among the differences on the edges of E, all non-zero residues modulo $t-2$ except -1 occur precisely once, developing E according to $\mathbb{Z}_{t-2}$ produces a total of $t-2$ blocks of size t. Adjoin to this an additional block $C = (\infty_1, \infty, 0, t-3, t-4, t-5, \ldots, 2, 1)$, to obtain the required BURMD$(t,1,t(t-1),t)$ when t is even.

Illustration : $t = 8$, $k = 8$, and $V = \mathbb{Z}_6 \cup \{\infty\} \cup \{\infty_1\}$.
Define $E = (0,\infty_1,5,1,4,2,3,\infty)$. Therefore, the required BURMD(8,1,56,8) is $E+0|E+1|E+2|\ldots|E+5|C$, where $C = (\infty_1,\infty,0,5,4,3,2,1)$.

Following the construction of BURMD$(t,1,t(t-1),t)$ for t odd (section 4.2.4), it may be noted that any sequence of less than or equal to $t-2$ elements contains all distinct treatments. So BURMD$(t,1,p,k)$ where $k \le t-2$ and $k|p$, exists. Also it is observed that BURMD$(t,1,t(t-1),k)$ for $k = t$ and $k = t-1$ exists. Hence we have the following theorems.

THEOREM 4.2.5.5. *If t is odd, $k \le t$, $t(t-1)|p$ and $k|p$ then BURMD$(t,1,p,k)$ exists.* □

THEOREM 4.2.5.6. *A BURMD$(t,1,p,k)$ exists for even t if $k \le t$, $t(t-1)|p$ and k divides t or else $t-1$.* □

4.2.6. CONCLUDING REMARKS

If t is odd, $k \le t$, $t(t-1)|p$ and $k|p$ then the existence problem of BURMD$(t,1,p,k)$ is fully resolved. But the same is not true when t is an even integer. For even t, an additional requirement, that is $k|t$ or else $k|\overline{t-1}$ is imposed on k to construct BURMD$(t,1,p,k)$'s.

4.3. CONSTRUCTION OF SECOND ORDER SBURMDs

Given a set $T = \{a_1, a_2, \ldots, a_t\}$ of t treatments, the problem is to construct an RMD, viz., an array $D_{n\times p} = ((d_{ij}))$, where each $d_{ij} \in T$, $t|n$, $t|p$ and $t^3|np$ such that

(i) each treatment appears exactly p/t times in a row,
(ii) each treatment appears exactly n/t times in a column,
(iii) the design is strongly balanced, and

(iv) if $A_i = \{ (d_{ij}, d_{i\,\overline{j+1}}, d_{i\,\overline{j+2}}) \mid 0 \le j \le p-1$, addition of the second suffix is modulo p $\}$, $1 \le i \le n$ then $\bigcup_{i=1}^{n} A_i$ contains each ordered triplet of treatments (a_i, a_j, a_k); $a_i, a_j, a_k \in T$, exactly the same number of times, that is, np/t^3 times.

It may be noted that requirement (iv) implies requirement (iii), that is, a second order URMD is also strongly balanced. Without loss of generality, we shall denote the treatments by the elements of $\mathbb{Z}_t$.

The notion of the method of differences, adopted in this section, is same as that of section 2.1 (Theorem 2.1.2). The definitions of difference vector and the set of linked differences also remain the same.

4.3.1. CASE t AN ODD PRIME

Given t, an odd prime number, define for $1 \le i \le [t/2]$ and $0 \le j \le 2t-1$,

$$b_j(i) = (i.j\ /\ 2 \mod t) \quad \text{for } j \text{ even, and}$$
$$= (-i.\overline{j+1}\ /\ 2 \mod t) \quad \text{for } j \text{ odd.}$$

Note, the $2t$-tuple $B_i = (b_0(i), b_1(i), \ldots, b_{2t-1}(i))$, $1 \le i \le [t/2]$, contains each element of $\mathbb{Z}_t$ twice.

On B_i, let us define e_i as the ordered pair$(b_j(i), b_{j+1}(i))$. Also, note for e_i

$$d_{e_i} = b_{j+1}(i) - b_j(i) = (-i.\overline{j+1} \mod t) \quad \text{for } j \text{ even, and}$$
$$= (i.\overline{j+1} \mod t) \quad \text{for } j \text{ odd,}$$

where addition on suffix of b is modulo $2t$. So the collection $\{d_{e_i} : e_i \in B_i\}$, $1 \le i \le [t/2]$, contains each element of $\mathbb{Z}_t$ exactly twice. Therefore, B_i is a difference vector over $\mathbb{Z}_t$ for each $i = 1, 2, \ldots, [t/2]$.

It is easily checked that for the difference vector B_i, sum of any two consecutive difference is $\pm i$ over $\mathbb{Z}_t$, $1 \le i \le [t/2]$. So, among the pair of differences in the preceding $[t/2]$ difference sets, missing are the ones whose sum is zero. We now modify B_1 and replace the ordered pairs $(\ell-1, t-\ell)$ by the quadruplets $(\ell-1, t-\ell, \ell-1, t-\ell)$, $\ell = 1, 2, \ldots, \overline{t-3}/2$, and the ordered pair $(\overline{t-1}/2, \overline{t-1}/2)$ by the triplet $(\overline{t-1}/2, \overline{t-1}/2, \overline{t-1}/2)$. Let the modified B_1 be termed as B_1'.

LEMMA 4.3.1.7. $B = B_1'B_2 \cdots B_{[t/2]}$ *is a difference vector over* $\mathbb{Z}_t$ *for* t *an odd prime number and when developed over* $\mathbb{Z}_t$ *produces a second order* *SBURMD*(t, t, t^2).

Proof: In B each element of $\mathbb{Z}_t$ occurs t times and the collection of differences over the ordered pairs of B, viz., the set of linked differences, contains each element of $\mathbb{Z}_t$ exactly t times. Also, it may be observed that each ordered pair of elements of $\mathbb{Z}_t$ appears exactly once in the collection of linked differences over the ordered pairs of B. So, developing B over $\mathbb{Z}_t$ produces a second order SBURMD(t, t, t^2).

For any triplet (i, j, k), $i, j, k \in \mathbb{Z}_t$, let $d_1 = j-i$, $d_2 = k-j$ and $d = d_1 + d_2$. If d is 0, +1 or −1 then the triplet can be found among the triplets developed from B_1'. Otherwise, if $d = \pm\ell$, $2 \le \ell \le [t/2]$, the triplet can be located among the triplets developed from B_ℓ. □

Illustration : $t = 7$.

B_1 = (0 6 1 5 2 4 3 3 4 2 5 1 6 0)
B_2 = (0 5 2 3 4 1 6 6 1 4 3 2 5 0)
B_3 = (0 4 3 1 6 5 2 2 5 6 1 3 4 0).

Note sum of two consecutive differences is $\pm i$ for B_i. Those pair of differences whose sum are zero, are missing. These triplets are 6 0 6, 5 1 5, 4 2 4, 3 3 3, 2 4 2, 1 5 1, and 0 6 0. Consider B_1, and modify it as follows:

B_1' = (0 6 0 6 1 5 1 5 2 4 2 4 3 3 3 4 2 5 1 6 0).

$B = B_1' B_2 B_3$ is a difference vector for a second order SBURMD(7,7,49) and developing B according to $\mathbb{Z}_7$ produces the required design.

Searching for a triplet:
Let us locate the triplet (1,2,3). Note the sum of the differences (d) is 2 and the first difference (d_1) is 1. The triplet (2,3,4) is a similar triplet in B_2. It is easy to see (2,3,4) + 6 = (1,2,3) over $\mathbb{Z}_7$. So the triplet (1,2,3) can be located in the 7*th* row of the said SBURMD(7,7,49).

Since we have shown that a second order SBURMD(t, t, t^2) can be constructed for t an odd prime number, so second order SBURMD(t,n,p) can also be constructed for $n = mt$ and $p = rt^2$, both m and r are positive integers, by first writing m SBURMD(t,t,t^2)'s vertically and then by repeating this configuration r times horizontally. Thus the following theorem is true.

Theorem 4.3.1.8. *If t is an odd prime number and, $t|n$ and $t^2|p$ then a second order SBURMD(t,n,p) exists.* □

4.3.2. CASE t A POWER OF 2

Given t, a power of 2, let us consider $\mathbb{Z}_t$ and define for $1 \le i \le t/2$, $0 \le j \le 2t-1$,

$$b_j(i) = ((2i-3).(j/2) \bmod t), \quad \text{for } j \text{ even, and}$$
$$= ((i-1).j \bmod t), \quad \text{for } j \text{ odd.}$$

Let $d_{ij} = b_{j+1}(i) - b_j(i)$, $D_i = (d_{ij} | 0 \le j \le 2t-1)$, and $D_i^2 = (d_{ij} + d_{i\,\overline{j+1}} | 0 \le j \le 2t-1)$ (operation on suffix j is modulo $2t$). It can be observed that

i) D_i contains each element of $\mathbb{Z}_t$ twice. So $\bigcup_{i=1}^{t/2} D_i$ contains each element of $\mathbb{Z}_t$ t times.

ii) $d_j(i) = d_{j'}(i)$ where $j' = 2t-(j+1)$. This implies no ordered pair of elements of $\mathbb{Z}_t$ can repeat in D_i, $i = 1,2,\ldots,t/2$, $j = 0,1,\ldots,2t-1$.

iii) D_i^2 contains $2(i-1)$ and $2(i-1)-1$ (computations are modulo t) of $\mathbb{Z}_t$, t times each and also note that no other difference appears in D_i^2. So, any ordered pair of elements of $\mathbb{Z}_t$ can appear in only one of the D_i's, $i = 1,2,\ldots,t/2$.

iv) $\bigcup_{i=1}^{t/2}\{(d_{ij}, d_{i\,\overline{j+1}}) | 0 \le j \le 2t-1$, operation on j is modulo $2t\}$ contains each ordered pair of elements of $\mathbb{Z}_t$ exactly once.

Now let us count the frequency of each element of $\mathbb{Z}_t$ in B where

$$B = \bigcup_{i=1}^{t/2} \{b_j(i) \mid 0 \le j \le 2t-1\}.$$

Note $\{b_j(i) \mid 0 \le j \le 2t-1,\ j \text{ even}\} = \mathbb{Z}_t$, since t is a power of 2. Therefore, $\bigcup_{i=1}^{t/2}\{b_j(i) \mid 0 \le j \le 2t-1,\ j \text{ even}\}$ contains each element of $\mathbb{Z}_t$ $t/2$ times.

Now for the case where j is odd, consider the collection:

$$C_j = \bigcup_{i=1}^{t/2} \{(b_j(i), b_{2t-j}(i))\}, \quad j = 1,3,5,\ldots,t-1.$$

It is easy to note that

i) $b_j(i) + b_{2t-j}(i) \equiv 0 \bmod t$,

ii) $b_j(i) = b_{2t-j}(i')$ implies $i = i' = 1$, and $b_j(1) = 0$ for all $j = 1,3,\ldots,2t-1$. So pair of the type (a,a), $a \in \mathbb{Z}_t$ does not belong to C_j for nonzero a.

iii) $b_j(i) + b_j(i') \equiv 0 \bmod t$ implies $i = i' = 1$. So in the same column we cannot have elements whose sum is zero. Therefore, for $a, b \in \mathbb{Z}_t$ such that $a \neq b$, it is impossible to have the ordered pairs (a,b) and (b,a) both in C_j.

iv) All the elements in $\{b_j(i) \mid j \text{ odd}, 1 \le i \le t/2\}$ are distinct since t is a power of 2.

v) C_j contains the pair (0,0) and all pairs sum to zero, and either (a,b) or else (b,a) belongs to C_j for $a, b \in \mathbb{Z}_t \ni a \neq b$.

In view of the above observations, it is concluded that among the elements of $\mathbb{Z}_t$, 0 appears twice, $t/2$ does not appear and rest appears once in C_j. Therefore $\bigcup_{i=1}^{t/2}\{b_j(i) \mid 0 \le j \le 2t-1,\ j \text{ odd}\}$ contains the element 0 t times, the elements other than 0 and $t/2$ exaqctly $t/2$ times, and the element $t/2$ does not appear.

In view of the preceding discussions, it is noted that the frequency of different elements of $\mathbb{Z}_t$ in B is as follows:

the element 0 appears $3t/2$ times
the element $t/2$ appears $t/2$ times, and
rest appears t times each.

Let $B_i = (b_j(i) \mid 0 \le j \le 2t-1)$, $i = 1,2,\ldots,t/2$. Note $B_1 = (0,0,t-1,0,t-2,\ldots,t-j+1,0,\ldots,1,0)$, j even, and $B_i = (0,\ldots,\ldots,\ldots,\ldots,t-i+1)$, for $i \neq 1$.

Now replace $(t-i+1,0)$ in B_1 by $(t-i+1|B_i|0)$ for $i = 2,3,\ldots,t/2$, and the resultant configuration be termed as B'. Note the ordered pair $(1,0)$ appears at the end of B', and further note that $b_{t-1}(t/2) \equiv t/2 + 1$ and $b_t(t/2) \equiv t/2$. So $B'+\{t/2\}$ will contain the ordered pair $(1,0)$. Let this new array $B'+\{t/2\}$ be rearranged such that $(1,0)$ appears at the end and this rearranged $B'+\{t/2\}$ be termed as D. Now $(B'|D)$ developed over $\mathbb{Z}_t$ is a second order SBURMD$(t,t,2t^2)$. Following the procedure mentioned in section 4.3.1, one can easily construct a second order SBURMD$(t,mt,2rt^2)$, where both m and r are positive integers, and t is a power of 2. Thus the following theorem is true.

Theorem 4.3.2.9. *If t is a power of 2, $t|n$ and $2t^2|p$ then a second order SBURMD(t,n,p) exists.* □

4.3.3. CONCLUDING REMARKS

The existence problem of the second order SBURMD(t,n,p)'s, where we assume that in each unit the residuals in a period are incurred from the two previous periods, is not fully resolved. If t is neither an odd prime nor a power of 2 then the problem remains unresolved. Also for t a power of 2, the problem has a partial solution.

5. CONSTRUCTION OF SYMMETRIC BALANCED SQUARES

5.1. INTRODUCTION

In this chapter a constructive proof of existence of symmetric balanced squares (SBSs) for different sizes of array and for different numbers of treatments is given. An algorithm, easily implementable on computers, has been developed for construction of such squares whenever the parameters satisfy the necessary conditions for existence of the square. The method of construction employs 1-factorizations of a complete graph or near 1-factorizations of a complete graph, depending on whether the size of the array is even or odd, respectively. For odd sized squares the method provides a solution directly based on the near 1-factorization. In the case of the squares being of even size, we use Hall's matching theorem along with a 1-factorization if $[n^2/v]$ is even, otherwise, Hall's matching theorem together with Fulkerson's (1959) theorem, on the existence of a feasible flow in a network with bounds on flow leaving the sources and entering the sinks, lead to the required solution.

Let us recall the definition of a balanced square BS(n,v) as given in section 1.2. Let $r = [n/v]$, the integral part of n/v, and let $f = [n^2/v]$. From the definition it follows that a BS of size n in v elements is an nxn array, $D=((d_{ij}))$, where d_{ij} denotes the element assigned to the jth column in ith row, $i,j = 1,2, \ldots, n$ satisfy the following conditions:

(i) each element occurs r or $r+1$ times in every row and column (viz., rows and columns are balanced),

(ii) each element occurs f or $f+1$ times in the array (viz., array is balanced).

Construction of such a square is quite simple. The elements are written sequentially over a row. A new row starts with the element that follows the last element of the previous row. If the element already appears in the first position of one of the previous rows, and if not all the elements have appeared in the first column, then take an element which does not appear and repeat the process. Here is an example of BS(6,4). Note $r=1$ and $f=9$.

1	2	3	4	1	2
3	4	1	2	3	4
2	3	4	1	2	3
4	1	2	3	4	1
2	3	4	1	2	3
4	1	2	3	4	1

Table 5.1: BS(6,4)

A BS (n, v) is said to be symmetric if $d_{ij} = d_{ji}$, $i, j = 1, 2, \ldots, n$. A symmetric BS (n, v) is abbreviated as SBS (n, v).

If the number of elements (v) is the same as that of the size of the array (n), then an SBS (n, n) is a symmetric latin square of order n. The addition table of Z_n modulo n is such an example.

A generalized notion of balanced squares is balanced rectangles (BRs). A BR of size $n \times c$ in v elements, abbreviated as BR(n, c, v), is an $n \times c$ array containing v elements where

i) each element occurs either r or $r + 1$ times in every column, where $r = [n/v]$.

ii) each element occurs s or $s+1$ times in every row, where $s = [c/v]$.

iii) every element appears either f or $f + 1$ times in the array, where $f = [n \times c/v]$.

The same construction procedure as with BS's works as well with BR's.

1	2	3	4	1	2
3	4	1	2	3	4
2	3	4	1	2	3
4	1	2	3	4	1
1	2	3	4	1	2
3	4	1	2	3	4
2	3	4	1	2	3

Table 5.2: BR(7,6,4)

5.2. DEFINITIONS AND SOME OBSERVATIONS

Let $r=[n/v]$, the integral part of n/v, and let $f = [n^2/v]$, the integral part of n^2/v. It can be easily seen that

$$n = (n-rv)(r+1) + (\overline{r+1}\ v-n)r \tag{5.2.1}$$

$$n^2 = (n^2-fv)(f+1) + (\overline{f+1}v-n^2)f \tag{5.2.2}$$

Thus an SBS (n, v) has $(\overline{r+1}v-n)$ elements in r cells each and $(n-rv)$

elements in $r+1$ cells each in any row or column. Also, there are $(\overline{f+1}v-n^2)$ elements in f cells each and (n^2-fv) elements in $f+1$ cells each in the square.

An SBS being symmetric, any element with odd frequency must occupy a position on the main diagonal. So a necessary condition for the array to exist is that the number of elements with odd frequency must not exceed the size of the square, n. It is also easy to note that the maximum number of distinct elements that a symmetric square can accommodate is $n(n+1)/2$. The existence of SBS $(1,v)$ is a triviality. These observations imply the following lemma.

LEMMA 5.2.1. *The necessary conditions for existence of an SBS(n,v), $n>1$ are (i) $v \le n(n+1)/2$ and (ii) the number of elements of odd frequency in the square should not exceed the size of the array, n.* □

It can be seen that if SBS (n,v) exist for $v \le n < 2v$ then SBS (n,v) exist for all $n \ge 2v$. Let $n = rv + s$, $0 \le s < v-1$, $r \ge 2$. An SBS (n,v), $n \ge 2v$, can then be generated as

A	A	⋯	A	A	B
A	A	⋯	A	A	B
⋮	⋮	⋮	⋮	⋮	⋮
A	A	⋯	A	A	B
A	A	⋯	A	C	
B^T	B^T	⋯	B^T		

where A is any symmetric latin square of order v, B is the first s columns of A and C is an SBS $(v+s, v)$.

LEMMA 5.2.2. *If SBS (n,v) exist for $v \le n < 2v$ then SBS (n,v) exist for all $n \ge 2v$.* □

In view of the preceding lemma, we will restrict our construction algorithm to n and v such that $n < 2v$. The proposed construction algorithm depends on 1-factorization/near 1-factorization of K_n depending on whether the value of n is even or odd, respectively. So we define and discuss it in subsequent sections.

Definition 5.2.3. Let n be even. A spanning subgraph of K_n consisting of $n/2$ vertex-disjoint edges is called a 1-factor of K_n. A decomposition of K_n into $(n-1)$ disjoint 1-factors is called a 1-factorization of K_n.

Definition 5.2.4. Let n be odd. A spanning subgraph of K_n consisting of an isolated vertex and $(n-1)/2$ vertex-disjoint edges is defined as a near 1-factor of K_n. A decomposition of K_n into n disjoint near 1-factors is called a near 1-factorization of K_n.

These factorizations exist for all n. Now we proceed to provide one such construction which will be used in subsequent sections for construction of SBS's.

Let the vertices of K_{2n} be denoted by $0, 1, \ldots, 2n-2, \infty$. Define for $i=1,2, \ldots, 2n-1$, the ordered set of edges

$$S_i = \{(\infty, \overline{i-1})\} \cup \{(\overline{i-1+j}, \overline{i-1-j}), j=1,2, \ldots, n-1\}$$

where each of the vertices $i-1+j$ and $i-1-j$ is expressed as one of the numbers $0, 1, 2, \ldots, 2n-2$ modulo $2n-1$. Clearly, the collection $\{S_i, i=1,2, \ldots, 2n-1\}$ is a 1-factorization of K_{2n}.

If the number of vertices is odd say $2n-1$, then we use a near 1-factorization. This is obtained by just replacing the edge $(\infty\ \overline{i-1})$ in the ith factor of K_{2n}, as defined above, by the vertex $\{i-1\}$ to obtain a near 1-factor of K_{2n-1}.

The above edge-decomposition of K_n will be termed as an array of edges associated with the corresponding 1-factorization or near 1-factorization. Let there be an $n\times n$ array $D_n = ((d_{ij}))$ associated with this array of edges, whose row and column are labelled with the vertices of K_n. The edge (i, j) represents cells (i, j) and (j, i) of D. We say an element x is placed in the edge (i, j), if and only if, $d_{ij} = d_{ji} = x$; furthermore, we say that x is placed in vertex i of the near 1-factor S_{i+1} if and only if $d_{ii}=x$.

We say m edges are *consecutive* if either they are consecutive edges in the same row of the array of edges, or are divided between rows as consecutive edges at the left most end of a row and the right most end of the subsequent row. To be more precise, the factors are considered in the order

$$S_{2n-1}S_{2n-2}\ \cdot\ \cdot\ \cdot\ S_2S_1$$

and the edges are scanned from right to left. The consecutiveness of edges discussed above implies these edges are consecutive edges in this ordered arrangement of factors. For example let us consider the following 1-factorization of K_8.

consecutive

$(\infty\,0)$	$(1\,6)$	$(2\,5)$	$(3\,4)$
$(\infty\,1)$	$(2\,0)$	$(3\,6)$	$(4\,5)$ consecutive
$(\infty\,2)$	$(3\,1)$	$(4\,0)$	$(5\,6)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\infty\,6)$	$(0\,5)$	$(1\,4)$	$(2\,3)$

Here edges (2 5) and (3 4) are consecutive edges, so are the edges $(\infty\ 0)$ and (4 5).

It is easy to note that any set of $n-1$ consecutive edges, either in the 1-factorization of K_{2n} or in the near 1-factorization of K_{2n+1}, contains $2(n-1)$ distinct vertices of the corresponding complete graph. Any complete row contains all the vertices. This observation is instrumental in providing an algorithm for the construction of SBS's.

5.3. CASE $n < v$ AND n ODD

Let n and v satisfy the two necessary conditions for existence of an SBS(n, v). Let r and f satisfy the equations 5.2.1 and 5.2.2 respectively.

If f is odd then $\overline{f+1}v - n^2$ elements are to be placed in the array of edges using $\frac{1}{2}(f-1)$ edges each and $n^2 - fv$ elements using $\frac{1}{2}(f+1)$ edges each, provided all can be accommodated in the array without using the isolated vertices. Total number of edges required will be $n^2 - v(f+1)/2$ while there are $n(n-1)/2$ edges in the array. Thus, there will be a shortfall of $[n-(\overline{f+1}v-n^2)]/2$ edges, which are to be obtained from the diagonal cells, replacing each edge by two vertices, viz., two diagonal cells The number of cells left on the diagonal after this allocation will be $\overline{f+1}\ v - n^2$, meant for allocating elements with odd frequency.

If f is even then all v elements are to be placed in the array of edges allocating $f/2$ edges each for every element. Thus we fall short by $[n-(n^2 - fv)]/2$ edges. These are obtained again from the main diagonal in the same way as before. The remaining n^2-fv positions left on the diagonal accommodate the elements with odd frequency.

In view of the above observations we propose a simple algorithm to construct an SBS (n, v) when n is odd. Before we proceed with the general algorithm, let us deal with two special cases, namely, $f=n-1$ and $f=n-2$.

Case $f=n-1$. Let $v = n + \Delta$, where $\Delta \geq 1$ as $n < v$. Note, $n^2 = (n-1)(n + \Delta) + [\Delta - (\Delta - 1)\ n]$. Now $f = n-1$ implies $0 \leq \Delta - (\Delta - 1)n < v$, viz., $0 < \Delta \leq 1 + 1/(n - 1)$. Δ being integer, we have $\Delta = 1$.

So, an SBS $(n, n+1)$ has to be constructed. This is easily achieved by considering a symmetric latin square of order $n+1$, and then dropping any of the $n+1$ rows and the same column from the square, viz., if we drop ith row then ith column has to be dropped to get an SBS $(n, n+1)$.

Case $f = n-2$. Note $n^2 = (n-2)\ (n+ \Delta) + [2\ \Delta - (\Delta -2)n]$, where $v=n + \Delta$, $\Delta \geq 1$. Now, $f = n - 2$ implies $2 \leq \Delta \leq 2 + 4/(n-2)$. So for $n > 6$, we have $\Delta = 2$, and for $n = 5$, $\Delta = 2$ or 3. But, if $\Delta = 3$, viz., $v = 8$, the necessary condition for existence of SBS is violated since the number of odd frequency elements is greater than the array size. So if $n=5$ then $\Delta = 2$. The only other case to be looked into is that for $n=3$.

If $n=3$ then $2 \leq \Delta \leq 6$. For SBS to exist it is required that $v \leq n(n+1)/2 = 6$, this implies $2 \leq \Delta \leq 3$. Let us deal with the case $\Delta = 3$ separately. The following is an SBS (3, 6).

1	4	5
4	2	6
5	6	3

Table 5.3: SBS (3,6)

Now the remaining case, viz., construction of SBS $(n, n+2)$ is discussed. Since n is odd, consider the near 1-factorization of K_n. Allocate $n+2$ elements in the array of edges, each placed in $(n - 1)/2$ consecutive edges or vertices, starting with the right most edge in the first row. The isolated vertex, $\{n-1\}$, remains unallocated. During this allocation process, 3 elements are placed in $(n-1)/2$ consecutive edges only; $n-1$ elements are placed in $(n-3)/2$ consecutive edges and a vertex. Three of these latter $n-1$ elements do not contain the vertex $\{n-1\}$. So, attach to any of these three elements the vertex $\{n-1\}$. Thus an SBS $(n, n+2)$ has been constructed.

Example 5.3.5. $n=7$ and $v=9$. Thus $r=0$ and $f=5$. This implies 4 elements occur with frequency $f+1 = 6$ and the remaining 5 with frequency $f = 5$. Consider the near 1-factorization of K_7.

0	(1 6)	(2 5)	(3 4)
1	(2 0)	(3 6)	(4 5)
→2	(3 1)	(4 0)	(5 6)
3	(4 2)	(5 1)	(6 0)
4	(5 3)	(6 2)	(0 1)
5	(6 4)	(0 3)	(1 2)
→6	(0 5)	(1 4)	(2 3)

	0	1	2	3	4	5	6
0	2	6	3	8	4	9	5
1		3	7	4	9	5	1
2			4	9	5	1	6
3				6	1	7	2
4					7	2	8
5						8	3
6							4

Table 5.4: SBS (7,9)

The above is an example of an SBS (7, 9) with frequency of elements 1,4,5 and 9 being 6 and the frequency of the remaining elements being 5.

ALGORITHM 5.3.6. (Case $n < v$, n odd and $f < n-2$)

Step 1. Consider the near 1-factorization of K_n. Calculate the number of positions to be filled on the diagonal to account for the even parts. [An even part of an element with frequency f is $f-1$ if f is odd and is f if f is even.] This is $n-(\overline{f+1}\,v-n^2)$ or $n-(n^2-fv)$ depending on whether f is odd or even respectively.

Occupy these vertices by placing elements necessarily with even frequency, so that no vertex is repeated for the element placed using the diagonals and the left most consecutive edges of the last row in the array of edges. A simple algebraic manipulation will give that number of elements of even frequency that occupy vertices of K_n is r, and p vertices of K_n are occupied by an element of even frequency where

$$
\begin{aligned}
p &= n-(\overline{f+1}v - n^2) && \text{if } f \text{ odd,} \\
&= n-(n^2-fv) && \text{if } f \text{ even,} \quad \text{and} \\
r &= \lceil p/f \rceil && \text{if } f \text{ even,} \\
&= \lceil p/(f+1) \rceil && \text{if } f \text{ odd.}
\end{aligned}
$$

Step 2. Place elements, one by one, starting from right most end of first row, in the required number of consecutive edges and vertices of the near 1-factorization of K_n. During this allocation of elements, whenever an occupied vertex of K_n (that is, a vertex which already has an element placed in it) is encountered, it is simply passed over and the next edge in the sequence is considered. The elements placed in a vertex of the array of edges are the ones having odd frequency.

One can easily note that the construction has proceeded by maintaining the requirement on frequency of the elements, and as $f < n-2$, so the vertices occupied by each element are distinct. Thus the following lemma follows.

LEMMA 5.3.7. *The algorithm* 5.3.6 *produces an SBS* (n, v), *when* $n < v$, n *is odd and* $f < n-2$. □

Example 5.3.8. $n = 7$ and $v = 11$. Thus $r = 0$ and $f = 4$. So, there will be 5 elements with frequency 5 and 6 elements with frequency 4. Now consider the near 1-factorization of K_7. Total number of edges required to accommodate the even parts is 5×2+6×2 = 22. The total number of available edges is 3×7 = 21. So we fall short by 1 edge, which is accounted for by 2 diagonal positions, viz., 11*th* element contains the last edge (0 5) and two diagonal cells from amongst 1,2,3,4 and 6.

0	(1 6)	(2 5)	(3 4)
1	(2 0)	(3 6)	(4 5)
2	(3 1)	(4 0)	(5 6)
3	(4 2)	(5 1)	(6 0)
→4	(5 3)	(6 2)	(0 1)
5	(6 4)	(0 3)	(1 2)
→6	(0 5)	(1 4)	(2 3)

	0	1	2	3	4	5	6
0	2	7	3	9	4	11	5
1		3	8	5	10	6	2
2			5	10	6	1	7
3				6	1	8	3
4					11	2	9
5						9	4
6							11

Table 5.5: SBS (7,11)

We allocated vertices 6 and 4 to the 11*th* element along with the edge (0,5). For the rest we start from (3 4), allocate 2 edges for elements not occupying a vertex and two edges plus a vertex for those occupying one. Thus the above is an SBS (7,11) where elements 1,4,7,8,10 and 11 appear 4 times and 2,3,5,6 and 9 appear 5 times in the array. No element appears twice in a row or column.

5.4. CASE $n < v$ AND n EVEN

Let n and v satisfy the necessary conditions for existence, and let r and f satisfy equations 5.2.1 and 5.2.2.

Case f even. Since f+1 is odd, any feasible solution must contain $n^2 - fv$ elements, those with frequency $f+1$ in the array, appearing on the main diagonal an odd number of times, and the remaining elements either do not appear, or appear even number of times, on the main diagonal.

ALGORITHM 5.4.9. (Case $n < v$, n even and f even)

Step 1. Consider the 1-factorization of K_n. Starting from the right hand end of the first row of the factorization, place elements one by one, in $f/2$ consecutive edges, until all the edges are exhausted or the left over edges are fewer than $f/2$.

Step 2. Let X be the set of vertices of K_n, Y be the set of elements allocated in step 1, and let E be the edges (x, y) such that the vertex x is not allocated to the element y. Consider the bipartite graph $G(X,Y,E)$. Get a matching of X into Y. Place the matched elements of Y in the corresponding main diagonal cells as represented by the matched vertices.

Step 3. Consider the next element not allocated in step 1, if any. If there are unallocated edges in the last row, allot them to this element. Continue the allocation on diagonal cells, counting two diagonal cells as one edge equivalent, so that no vertex is repeated.

Step 4. If there are unallocated elements, then allocate them on diagonals, at f positions each, replacing the matched elements of step 2. The positions occupied by the element placed in step 3 should not be disturbed. Continue this until all v elements are placed.

REMARK 5.4.10. *At least n+1 elements will be allocated f/2 edges each, in step 1.*

REMARK 5.4.11. *The matching, discussed in step 2, exists.*

Proof: $|X| = n < n+1 \le |Y|$, by remark 5.4.10. $\deg(y) = n-f$ for all $y \in Y$.

Note any vertex x will be contained in either $n-1$ or $n-2$ of the elements depending on whether or not the vertex stands allocated in the last factor S_{n-1}, viz., last row.

So, $\deg(x) = u$ or $u+1$, where $u = |Y| - (n-1)$.

Therefore, $\min_{x \in X} \deg(x) = u \ge n-f = \max_{y \in Y} \deg(y)$. Thus it follows from Hall's theorem on matching that X can be matched into Y. □

LEMMA 5.4.12. *The algorithm 5.4.9 constructs an SBS(n,v) when n < v and, both n and f are even.*

Proof: Let $t = \lceil n(n-1)/f \rceil$, then in step 1 $y_1, y_2, \ldots, y_t$ elements are allocated to $f/2$ edges. Consequently, they have a frequency of f in the array. At step 2, for n of these t elements, the frequency in the array is increased to $(f+1)$.

At step 3 and 4, remaining elements are entered on the diagonal and/or left over edges with an individual frequency of f in the array. Some of the n elements placed on the diagonal at step 2 are reduced to frequency f in the array.

Observe that the frequency of the elements in the array is f or $f+1$. The algorithm allocates elements in such a way that no vertex is repeated for any element. Thus the frequency of an element in any row or column is at most 1. Therefore, the algorithm yields an SBS(n,v). □

Example **5.4.13.** n=8 and v=10. Thus r=0 and f=6 and there are 4 elements in 7 cells each and 6 elements in 6 cells each in the array. An element appears at most once in any row or column. Consider the 1-factorization of K_8.

(∞0)	(1 6)	(2 5)	(3 4)
(∞1)	(2 0)	(3 6)	(4 5)
(∞2)	(3 1)	(4 0)	(5 6)
(∞3)	(4 2)	(5 1)	(6 0)
(∞4)	(5 3)	(6 2)	(0 1)
(∞5)	(6 4)	(0 3)	(1 2)
(∞6)	(0 5)	(1 4)	(2 3)

	∞	0	1	2	3	4	5	6
∞	9	2	3	4	6	7	8	10
0		1	6	3	8	4	9	5
1			2	7	4	9	5	1
2				$\not{8}_{10}$	9	5	1	6
3					$\not{5}_{10}$	1	7	2
4						$\not{3}_{10}$	2	8
5							$\not{6}_{10}$	3
6								7

Table 5.6: SBS (8,10).

X	:	∞	0	1	2	3	4	5	6
Y	:	1	①	②	2	3	③	4	4
		5	7	8	⑧	⑤	6	⑥	⑦
		⑨							9

Table 5.7: A matching of the diagonals.

The 10th element is placed in the edge (∞ 6) and 4 diagonal positions 2,3,4 and 5. Thus an SBS(8,10) with 4 elements 1,2,7, and 9 appearing in 7 cells each and the remaining 6 elements in 6 cells each in the array. No element occurs more than once in any row or column.

Case f odd. For any feasible SBS, the $\overline{f+1}\ v-n^2$ elements that occur with frequency f in the array, must appear an odd number of times in cells of the main diagonal, the remaining $n^2 - fv$ elements either do not appear or appear even number of times on the main diagonal.

ALGORITHM 5.4.14. (Case $n < v$, n even and f odd).

Step 1. Set $t=(f + 1)\ v - n^2$. As n and v satisfies the necessary conditions for existence, so $n-t \geq 0$ and $2|(n-t)$.

Let $\ell=t+(n-t)/2 = (n+t)/2$. Consider the 1-factorization of K_n.

Step 2. Place ℓ elements in $(f-1)/2$ consecutive edges as before.

Step 3. Place the remaining $v-\ell$ elements in $(f+1)/2$ consecutive edges, starting from the unoccupied edge consecutive to the last edge of the ℓth element.

Step 4. Let X be the set of vertices of K_n, Y be the set of first ℓ elements allocated in step 2, and let E be the set of edges $(y\ x)$ such that the vertex x is not allocated to the element y. Consider the bipartite graph $G(Y, X, E)$. Get a minimal vertex covering, C, of G, viz., t of the elements of Y have degree 1 in C and the remaining $\ell-t$ elements of Y have degree 2 in C, and each vertex of X has degree 1 in C. Place the incident elements of Y in the corresponding diagonals as represented by the matched vertices.

Remark 5.4.15. *All edges of the 1-factorization of K_n are occupied after the allocation of elements in step 3.*

Remark 5.4.16. *The minimal vertex covering, discussed in step 4, exists.*

Proof: Let X and Y be as defined in step 4 and let E be the set of edges (y, x) such that the vertex x is not allocated to the element y. Consider the network $[Y \cup X,\ E]$ with the capacity function c defined as $c(y, x)=1$ for all $(y, x) \in E$, where Y is the set of sources and X is the set of sinks.

Let there be associated with each $y \in Y$, two non-negative numbers $a(y)$ and $a'(y)$, and with each $x \in X$, a non-negative number $b(x)$, defined by $a(y)=1$, $a'(y)=2$ for all $y \in Y$ and $b(x)=1$ for all $x \in X$.

Finding a minimal vertex covering in step 4 is equivalent to finding a feasible flow f satisfying

$$\begin{aligned} a(y) \le f(y, Y\cup X) - f(Y\cup X, y) &\le a'(y), & y \in Y \\ f(Y\cup X, x) - f(x, Y\cup X) &= b(x), & x \in X \\ 0 \le f(y, x) &\le c(y, x), & (y, x) \in E \end{aligned} \tag{5.4.3}$$

Fulkerson (1959) had shown that the constraint set (5.4.3) is feasible if and only if each of the constraint sets

$$\begin{aligned} a(y) \le f(y, Y\cup X) - f(Y\cup X, y)\ , &\quad y \in Y \\ f(Y\cup X, x) - f(x, Y\cup X) \le b(x), &\quad x \in X \\ 0 \le f(y,x) \le c(y,x), &\quad (y,x) \in E. \end{aligned} \tag{5.4.4}$$

and

$$\begin{aligned} f(y, Y\cup X) - f(Y\cup X, y) \le a'(y), &\quad y \in Y \\ b(x) \le f(Y\cup X, x) - f(x, Y\cup X), &\quad x \in X \\ 0 \le f(y,x) \le c(y,x), &\quad (y,x) \in E. \end{aligned} \tag{5.4.5}$$

is feasible. So we proceed to show that the constraint sets (5.4.4) and (5.4.5) are feasible.

Note $|Y| = \ell \le n = |X|$. deg $(y) = n-(f-1)$ for all $y \in Y$. Let $r = \lceil \overline{\ell(f-1)}/n \rceil$. Note r 1-factors are needed to accommodate ℓ elements, as discussed in step 2. Any vertex x appears r times in these r 1-factors. So, either r of them or r-1 of them stand allocated, depending on whether it stands allocated in S_r. This implies, either $\ell-r$ or $\ell-r+1$ elements of Y are not allocated to vertex x. So $\deg(x) = u$ or $u+1$ where $u = \ell - r$. Therefore, $\max_{x\in X} \deg(x) = u+1 \le n-(f-1) = \min_{y\in Y} \deg(y)$.

By Hall's matching theorem a matching of Y into X exists. Take any such matching $M \subseteq E$ and define $f(y,x) = 1$ if $(y,x) \in M$ and zero otherwise. Clearly f satisfies (5.4.4).

Now, Y', a set of 2ℓ elements, is defined from Y as

$$\begin{aligned} y'_i &= y_i, &\quad 1 \le i \le \ell, \\ y'_i &= y_{i-\ell}, &\quad (\ell+1) \le i \le 2\ell. \end{aligned}$$

Extend E to E' by adding $|E|$ edges by connecting $y'_{i+\ell}$ to all those x's to which y_i is connected, $i=1,2,\ldots,\ell$.

Note $|Y'| = 2\ell \ge n = |X|$. $\deg(y') = n-(f-1)$ for all $y' \in Y'$, and $\deg(x) = 2u$ or $2(u+1)$ for all $x \in X$. It can easily be shown that $2u \ge n-(f-1)$. Thus, $\max_{y'\in Y} \deg(y') = n-(f-1) \le 2u = \min_{x\in X} \deg(x)$. Again, by Hall's theorem a matching of X into Y' exists. Take any such matching $M' \subseteq E'$ and define f' from Y to X by

$$
\begin{aligned}
f'(y_i, x) &= 2 \quad \text{if } (y_i, x) \text{ and } (y_{i+\ell}, x) \in M' \\
&= 1 \quad \text{if } (y_i, x) \text{ or else } (y_{i+\ell}, x) \in M' \\
&= 0 \quad \text{otherwise.}
\end{aligned}
$$

Clearly f' satisfies the constraint set (5.4.5).

Thus, there exists a feasible flow f satisfying the constraint set (5.4.3) which provides a minimal vertex covering as discussed in step 4. □

Lemma 5.4.17. *The algorithm* 5.4.14 *produces an* $SBS(n, v)$ *when* $n < v$, n *is even and* f *is odd.* □

The proof is simple and follows in the same way as that of lemma 5.4.12, and thus is omitted.

Example **5.4.18.** n=8 and v=11. Thus f=5, t=2 and ℓ=5. So, there will be 9 elements in 6 cells each and 2 elements in 5 cells each in the array SBS(8, 11). Consider the 1-factorization of K_8. As suggested in step 2, ℓ=5 elements, 1,2,. . . .,5, say, are first placed in the array, each in 2 consecutive edges. Following step 3, elements 6, ..., 11 are placed in the array, each in 3 consecutive edges.

(∞0)	(1 6)	(2 5)	(3 4)
(∞1)	(2 0)	(3 6)	(4 5)
(∞2)	(3 1)	(4 0)	(5 6)
(∞3)	(4 2)	(5 1)	(6 0)
(∞4)	(5 3)	(6 2)	(0 1)
(∞5)	(6 4)	(0 3)	(1 2)
(∞6)	(0 5)	(1 4)	(2 3)

	∞	0	1	2	3	4	5	6
∞	3	2	4	6	7	9	10	11
0		1	8	4	9	5	11	6
1			5	9	6	11	7	2
2				3	10	7	1	8
3					5	1	8	3
4						4	3	10
5							2	5
6								4

Table 5.8: SBS (8,11)

X :	∞	0	1	2	3	4	5	6
Y :	1	①	1	2	2	2	②	1
	③	3	3	③	4	④	4	④
	5		⑤	5	⑤			

Table 5.9: A covering of the diagonals.

So, an SBS(8, 11) is constructed, where 2 elements, 1 and 2, appear in 5 cells each, while the remaining elements appear in 6 cells each. Each element occurs at most once in any row or column.

5.5. CASE $v < n < 2v$

Here, the necessary conditions for existence of an SBS(n, v) are satisfied for all v and n. If an SBS ($n-v, v$) exists, then SBS(n, v) can be generated as

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

where A is any symmetric latin square of order v, B is the first $n-v$ columns of A, B^T is the transpose of B, and C is an SBS($n-v, v$).

If an SBS($n-v, v$) does not exist then it implies a violation of either of the necessary conditions for existence. If the condition on the feasible number of elements is violated, viz., if $v > \frac{1}{2}(n-v)(n-v+1)$, then $n-v$ of the v elements occupying the main diagonal occur with frequency 1. Of the remaining elements, $(n-v)(n-v-1)/2$ occur in the off-diagonal positions, each twice and the rest do not appear. So the array will not be balanced. An extended concept is defined in such situations. A symmetric square of size m is said to be near balanced if the rows and columns are balanced, and $|f_i - f_j| \le 2$, $i, j = 1,2, \ldots, v$, where f_i and f_j are frequencies of ith and jth elements respectively in the array. Such an array will be abbreviated as NSBS(m, v). The above array is thus an NSBS($n-v, v$).

The other case of non-existence of an SBS($n-v, v$) occurs when the number of elements with odd frequency is more than $n-v$. In such situations too, a near balance is attempted.

One can easily check that m^2 can be expressed as $m^2 = x_1(f'-1)+mf'+x_2(f'+1)$ where $x_1+x_2=v-m$ and f' is odd. So, to achieve near balance x_1, m and x_2 elements are to be placed in $f'-1$, f' and $f'+1$ cells each respectively. This can easily be done by placing elements one by one, in the 1-factorization or near 1-factorization of K_m depending on m. The procedure is similar to the ones discussed in algorithms 5.3.6 and 5.4.9, and then finding a matching of the elements to the diagonal if m is even.

Example 5.5.19. $m=7$ and $v=10$. Thus $f=4$ and this implies there has to be 1 element with frequency 4 and 9 elements with frequency 5 in a 7×7 symmetric array, which is infeasible. But $7^2 = 2.4 + 7.5 + 1.5$, viz., $f' = 5$, $x_1 = 2$ and $x_2 = 1$. So we find an NSBS(7,10).

0	(1 6)	(2 5)	(3 4)
1	(2 0)	(3 6)	(4 5)
2	(3 1)	(4 0)	(5 6)
3	(4 2)	(5 1)	(6 0)
4	(5 3)	(6 2)	(0 1)
5	(6 4)	(0 3)	(1 2)
6	(0 5)	(1 4)	(2 3)

2	6	3	8	4	10	5
	3	8	4	10	5	1
		4	9	6	1	7
			6	1	7	2
				7	2	9
					9	3
						10

Table 5.10: An NSBS (7, 10)

Lemma 5.5.20. *An SBS(n,v) exists for $v < n < 2v$.*

Proof: let B be a $v \times \overline{n-v}$ matrix whose first column is b_1 where $b_1^T = (0,2,4, \ldots, 2[\overline{v-1}/2], 1,3,5, \ldots, 2[v/2]-1)$. Generate the $i+1$ th column of B from the ith column by moving the elements of the column up one position and placing the first element in the last position. In B, do not allow the column with 1 in the first position. In such a situation generate $n-v+1$ columns and discard the offending one.

Let C be an NSBS$(n-v,v)$ where $x_1, n-v, x_2$ elements have frequency $f'-1$, f' and $f'+1$ respectively. Let $x = \min(x_1, x_2)$. Without loss of generality, elements $1,3,5, \ldots, 2x-1$ occur with frequency $f'-1$ and elements $0,2,4, \ldots, 2x-2$ occur with frequency $f'+1$. Consider the $n \times n$ array D,

$$D = \begin{pmatrix} \underset{v\times v}{A} & \underset{v\times\overline{n-v}}{B} \\ \underset{\overline{n-v}\times v}{B^T} & \underset{\overline{n-v}\times\overline{n-v}}{C} \end{pmatrix}$$

where A is the symmetric latin square generated as the addition table of Z_v. Note ith row of D contains $2(i-1)$ twice and $2(i-1)+1$ only once, $i=1,2, \ldots, [v/2]$. So by dropping the elements $0,2,4, \ldots$, from the diagonal of A and replacing them by elements $1,3,5, \ldots$, the balance in D is restored. So D is an SBS(n,v) for $v < n < 2v$ after the modification.

□

Example 5.5.21. n=17, v=10.

Solution: Consider the NSBS(7,10) constructed in the example 5.5.19. Let us renumber the elements in it as 1 ⟶ 0, 2 ⟶ 2, 3 ⟶ 4, 4 ⟶ 5, 5 ⟶ 1,6 ⟶ 6, 7 ⟶ 7, 8 ⟶ 3,9 ⟶ 8 and 10 ⟶ 9.

$D =$

				$A =$									$B =$			
[1] ≠ [0]	1	2	3	4	5	6	7	8	9	0	2	4	6	8	3	5
1	2	3	4	5	6	7	8	9	0	2	4	6	8	1	5	7
2	3	4	5	6	7	8	9	0	1	4	6	8	1	3	7	9
3	4	5	6	7	8	9	0	1	2	6	8	1	3	5	9	0
4	5	6	7	8	9	0	1	2	3	8	1	3	5	7	0	2
5	6	7	8	9	0	1	2	3	4	1	3	5	7	9	2	4
6	7	8	9	0	1	2	3	4	5	3	5	7	9	0	4	6
7	8	9	0	1	2	3	4	5	6	5	7	9	0	2	6	8
8	9	0	1	2	3	4	5	6	7	7	9	0	2	4	8	1
9	0	1	2	3	4	5	6	7	8	9	0	2	4	6	1	3
										2	6	4	3	5	9	1
										6	4	3	5	9	1	0
										4	3	5	8	6	0	7
				B^T						3	5	8	6	0	7	2
										5	9	6	0	7	2	8
										9	1	0	7	2	8	4
										1	0	7	2	8	4	9
													$= C$			

Table 5.11: An SBS(17,10)

Note, frequency of 0 in the NSBS(7,10) is 6 while that of 1 and 3 are 4. So, from the diagonal of the first row of the 17 × 17 matrix, 0 is removed and 1 is introduced. The modified array is then an SBS(17,10).

5.6. A THEOREM ON EXISTENCE

Based on the previous discussions, the following theorem follows.

Theorem 5.6.22. *An SBS(n,v) exists if and only if*

i) $v \le n(n+1)/2$, *and*

ii) *the number of elements with odd frequency is at most* n. □

5.7. CONCLUDING REMARKS

It may be noted that in SBS(n, v), the only properties of a Latin Square extended are those of balance or near balance of rows, of columns and of the array. A latin square also satisfies other important properties like perfect pairwise balance of elements in rows and columns. Our construction method does not necessarily provide this pairwise balance of elements even in the extended sense. The corresponding combinatorial problem is much more complicated and also cannot be solved, in general, with respect to pairwise balance. Though the problem is interesting and challenging, has not been considered in the present thesis.

6. CONCLUDING REMARKS

For $t \equiv 2 \pmod 4$ and $n.t^{-1}$ odd, the existence problem of SBURMD(t, n, p) is still open. It has only been shown by Roy (1988) that SBURMD(2,2,6) does not exist.

In the class of nearly balanced uniform RMD's, our constructive proof of existence is limited to the case $n = t$. Even in this class of designs, when p is an odd integer and $t = p + 1$ or $t = p + 2$, and when $t = p - 1$ for an even p, the existence problem of NBURMD(t, t, p) is still unresolved. It would also be interesting to study near balanced uniform RMD's when $n \neq t$. We have only shown that when $n < t$ and $p < t$, such designs exist.

Our search for an eulerian circuit in K_t^* which can be "sliced" or partitioned into blocks of k distinct elements, is completely successful when t is an odd integer. It may be noted that for such a circuit to exist, it is necessary that $k \leq t$ and $k \mid \overline{t(t-1)}$. For an odd t, we always have a circuit with the above mentioned property whenever k satisfies these two necessary conditions. But if t is even then in addition to these two we also need that $k \mid t$ or else $k \mid (t-1)$. So the problem of obtaining an eulerian circuit for even t which can be sliced into blocks of k distinct elements is partially resolved. A complete solution for even t is still an open problem if $k \nmid (t-1)$, $k \nmid t$ but $k \mid \overline{t(t-1)}$.

The existence problem of second order SBURMD when t is neither an odd prime nor a power of 2 is still unresolved. Also for t a power of 2, our design is not a minimal design in the sense that it requires twice the number of periods that is theoretically necessary for such a design to exist. So we could get a partial solution in this case.

For symmetric balanced squares (SBSs), in addition to the type of balance considered by us, the concept of pairwise balance (in rows and/or columns) is also a useful concept which the statisticians are very fond of. This property is not guaranteed following our method of construction, even in an extended sense. The corresponding combinatorial problem is much more complicated and also cannot be solved, in general, with respect to pairwise balance. This would be a challenging problem though not considered by us in the present thesis.

BIBLIOGRAPHY

Afsarinejad, K. (1983). Balanced repeated measurements designs. *Biometrika 70*, 199-204.

Afsarinejad, K. (1990). Repeated measurements designs - a review. *Commun. Statist.- Theory Meth. 19(11)*, 3985-4028.

Atkinson, G.F. (1966). Designs for sequences of treatments with carry-over effects. *Biometrics 22*, 292-309.

Berenblut, I.I. (1964). Change-over designs with complete balance for first residual effects. *Biometrics 20*, 707-712.

Berge, Claude (1985). *Graphs*. North-holland, Amsterdam.

Bermond, J.C. and Faber, V. (1976). Decomposition of the complete directed graph into *k*-circuits. *J. Comb. Theor., B 21*, 146-155.

Bishop, S.H. and Jones, B. (1984). A review of higher order crossover designs. *J. Appl. Statist. 11*, 29-50.

Bose, M. (1995). Some incomplete block sequences. Technical Report No. ASC/95/1, Indian Statist. Inst.

Bose, R.C. (1939). On the construction of balanced incomplete block designs. *Ann. Eugenics 9*, 353-399.

Bradley, J.V. (1958). Complete counterbalancing of immediate sequential effects in a Latin square design. *J. Amer. statist. Ass. 53*, 525-528.

Cheng, C.S. and Wu, C.F. (1980). Balanced repeated measurements designs. *Ann. Statist. 8*, 1272-1283. Corrigenda (1983), 11, 349.

Cochran, W.G., Autrey, K.M. and Cannon, C.Y. (1941). A double change-over design for dairy cattle feeding experiments. *J. Dairy Sci. 24*, 937-951.

Constantine, G. and Hedayat, A. (1982). A construction of repeated measurements designs with balance for residual effects. *J. Statist. Planning Infer. 6*, 153-164.

Davis, A.W. and Hall, W.B. (1969). Cyclic change-over designs. *Biometrika 56*, 283-293.

Denes, J. and Keedwell, A. D. (1974). *Latin Squares and their Applications*. Academic Press, New York.

Dutta, T.K. and Roy, B.K. (1992). Construction of strongly balanced uniform repeated measurements designs: A new approach. *Sankhya, Special Volume 54*, 147-153.

Dutta, T.K. and Roy, B.K. (1995). Construction of some repeated measurements designs. Technical Report No.ASC/95/8, Indian Statist. Inst.

Dutta, T.K. and Roy, B.K. (1997). Construction of symmetric balanced squares. *Ars Combin.*, (to appear).

Finney, D.J. and Outhwaite, A.D. (1955). Serially balanced sequences. *Nature 176*, 748.

Finney, D.J. and Outhwaite, A.D. (1956). Serially balanced sequences in bioassay. *Proc. Roy. Soc., B 145*, 493-507.

Fletcher, D.J. (1987). A new class of change-over designs for factorial experiments. *Biometrika 74*, 649-654.

Fletcher, D.J. and John, J.A. (1985). Changeover designs and factorial structure. *J. Roy. Statist. Soc. B 47*, 117-124.

Ford, L.R., Jr., and Fulkerson, D.R. (1957). A simple algorithm for finding maximal network flows and an application to the Hitchcock problem. *Can. J. Math. 9*, 210-218.

Fulkerson, D.R. (1959). A network flow feasibility theorem and combinatorial applications. *Can. J. Math. 11*, 440-451.

Gomez, K. A. and Gomez, A. A. (1984). *Statistical Procedures for Agricultural Research*. John Wiley and Sons, New York.

Hedayat, A. (1981). Repeated measurements designs, IV: Recent advances. *Bull. Int. Statist. Inst. XLIX*, 591-610.

Hedayat, A. and Afsarinejad, K. (1975). Repeated measurements designs, I. In *A Survey of Statistical Designs and Linear Models* (J. N. Srivastava, ed.). North-Holland, Amsterdam, 229-242.

Hedayat, A. and Afsarinejad, K. (1978). Repeated measurements designs, II. *Ann. Statist. 6*, 619-628.

Houston, T.P. (1966). Sequential counterbalancing in latin squares. *Ann. Math. Statist. 37*, 741-743.

Keedwell, A.D. (1981). On the sequenceability of non-abelian groups of order pq. *Discrete Math. 37*, 203-216.

Keedwell, A.D. (1982). On the sequenceability of Dihedral groups. *Ann. Discrete Math. 15*, 253-258.

Keedwell, A.D. (1983a). On the existence of super p-groups. *J. Comb. Theory, A 35*, 89-97.

Keedwell, A.D. (1983b). On *R*-sequenceability and R_h-sequenceability of groups. *Ann. Discrete math. 18*, 535-548.

Keedwell, A.D. (1984). More super p-groups. *Discrete Math. 49*, 205-207.

Kiefer, J. (1960). Optimum experimental designs V, with applications to systematic and rotatable designs. *Proc. of the Fourth Berkeley Symp. 2*, 381-405.

Kok, K.L. and Patterson, H.D. (1976). Algebraic results in the theory of serial factorial design. *Biometrika 63*, 559-565.

Kunert, J. (1985a). Optimal repeated measurements designs for correlated observations and analysis by weighted least squares. *Biometrika 72*, 375-389.

Kunert, J. (1985b). Optimal experimental design when errors are assumed to be correlated. *Statist. and dec. Supplemented Issue No. 2*, 287-298.

Magda, C.G. (1980). Circular balanced repeated measurements designs. *Commun. Statist.- Theory Meth. A9*, 1901-1918.

Majumder, D. (1988). Optimal repeated measurements designs for comparing test treatments with a control. *Commun. Statist.- Theory Meth. 17*, 3687-3703.

Matthews, J.N.S. (1987). Optimal crossover designs for the comparison of two treatments in the presence of carryover effects and autocorrelated errors. *Biometrika* *74*, 311-320.

Mendelsohn, N.S. (1968). Hamiltonian decomposition of the complete directed *n*-graph. Theory of graphs: *Proc. of the colloquium held at Tihany, Hungary, Sept. 1966* (P. Erdos and G. Katona ed.). North Holland, Amsterdam, 237-241.

Mukhopadhyay, A.C. and Saha, R. (1983). Repeated measurements designs. *Calcutta Statist. Ass. Bull.* *32*, 153-168.

Patterson, H.D. (1950). The analysis of change-over trials. *J. Agri. Sci.* *40*, 375-380.

Patterson, H.D. (1951). Change-over trials. *J. Roy. Statist. Soc. Ser. B* *13*, 256-271.

Patterson, H.D. (1952). The construction of balanced designs for experiments involving sequences of treatments. *Biometrika* *39*, 32-48.

Patterson, H.D. (1970). Nonadditivity in change-over designs for a quantitative factor at four levels. *Biometrika* *57*, 537-549.

Patterson, H.D. (1973). Quenouille's changeover designs. *Biometrika* *60*, 33-45.

Patterson, H.D. and Lucas, H.L. (1962). Change-over designs. North Carolina Agri. Exp. Station Tech. Bull. No. 147.

Pigeon, J.G. and Raghavarao, D. (1987). Crossover designs for comparing treatments with a control. *Biometrika* *74*, 321-328.

Raghavarao, D. and Blaisdell, E.A. (1985). Efficiency bounds for partially balanced change-over designs based on M-associate class PBIB designs. *J. Roy. Statist. Soc. B 47*, 132-135.

Roy, B.K. (1988). Construction of strongly balanced uniform repeated measurements designs. *J. Statist. Planning Infer. 19*, 341-348.

Schellenberg, P.J. (1993). Symmetric balanced squares. *Private Communication.*

Sen, M. and Mukherjee, R. (1987). Optimal repeated measurements designs under interaction. *J. Statist. Planning Infer. 17*, 81-91.

Sen, M. and Sinha, B.K. (1986). A statistical analysis of serially balanced sequences: first order residuals proportional to direct effects. *Calcutta Statist. Ass. Bull. 53*, 31-36.

Sharma, V.K. (1982). Extra-period balanced changeover designs. *Sankhya B 44*, 167-173.

Sheehe, P.R. and Bross, D.J. (1961). Latin squares to balance immediate residual and other order effects. *Biometrics 17*, 405-414.

Street, D.J. (1988a). Some repeated measurements designs. *Commun. Statist.- Theory Meth. 17*, 87-104.

Street, D.J. (1988b). Some construction methods for optimal designs when the errors are assumed correlated. *Ars Combin. 26A*, 193-196.

Street, D.J. (1989). Combinatorial problems in repeated measurements designs. *Discrete Math. 77*, 323-343.

Street, A.P. and Street, D.J. (1987). *Combinatorics of experimental designs*. Oxford: Clarendon press.

Williams, E.J. (1949). Experimental designs balanced for the estimation of residual effects of treatments. *Austral. J. Sci. Res. A2*, 149-168.

Williams, E.J. (1950). Experimental designs balanced for pairs of residual effects. *Austral. J. Sci. Res. A3*, 351-363.

Williams, R.M. (1952). Experimental designs for serially correlated observations. *Biometrika 39*, 151-167.