Topological conjugacy and rigidity of affine actions

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by

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Declaration

The work presented in this thesis has been carried out by me under the guidance of Professor S.G Dani at School of Mathematics, Tata Institute of Fundamental Research, Mumbai.

The work reported in this thesis is original and has not been submitted in part or in full for a Degree, a Diploma or a Fellowship in any other university or institution.

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Statement regarding joint work

The work presented in this thesis is my own and is not, in part or whole, joint work with any person.

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Contents

Introduction

The main theme of this thesis is topological classification of affine flows on homogeneous spaces and rigidity of equivariant continuous maps between such flows. Both these aspects have been extensively studied in the literature for subgroup actions (cf. [Be], [B-D] and [Wi]) and for automorphism flows of the cyclic group (cf. [Ar], [K-R] and [C-S]). We will consider similar questions in more general situations. A detailed outline is given below.

For a topological group Γ, by a Γ-flow we mean pair (X, ρ), where X is a topological space and ρ is a continuous action of Γ on X. For any two Γ-flows (X, ρ) and (X', σ), a continuous map $f : X \to X'$ is said to be Γ-*equivariant* if $f \circ \rho(\gamma) = \sigma(\gamma) \circ f, \quad \forall \gamma \in \Gamma$. Two Γ-flows (X, ρ) and (X', σ) are said to be *topologically conjugate* if there exists a Γ-equivariant homeomorphism $f : X \to X'$, and they are said to be *orbit equivalent* if there exists a homeomorphism $f : X \to X'$ which takes orbits under ρ to orbits under σ.

If G is a locally compact topological group and $H \subset G$ is a closed subgroup, then the quotient space $X = G/H$ is called a homogeneous space. If $X_1 = G_1/H_1$ and $X_2 = G_2/H_2$ are homogeneous spaces then a continuous map from X_1 to X_2 is said to be a homomorphism if it is induced by a continuous homomorphism from G_1 to G_2 which maps H_1 into H_2. Isomorphisms and automorphisms of homogeneous spaces are defined similarly. A continuous map f from X_1 to X_2 is said to be affine if there exists an element g_0 of G_2 and a continuous homomorphism θ

from G_1 to G_2 such that $\theta(H_1) \subset H_2$ and $f(gH_1) = g_0\theta(g)H_2 \quad \forall g \in G_1$. A Γ-flow (X, ρ) is said to be affine if for all γ in Γ, $\rho(\gamma)$ is an affine map. (X, ρ) is said to be an *automorphism flow* (resp. a *translation flow*) if each $\rho(\gamma)$, $\gamma \in \Gamma$, is an automorphism of X (resp. a translation on X). Two Γ-flows (X_1, ρ) and (X_2, σ) are said to be *algebraically conjugate* if there exists a Γ-equivariant isomorphism from X_1 to X_2.

In Chapter 1 we set up various definitions, notations etc. and discuss the background material. In Chapter 2 we consider flows on compact connected metrizable abelian groups. We give various sufficient conditions for 'rigidity' of $\Gamma-$equivariant continuous maps in this situation. In particular we prove that if G and H are compact connected metrizable abelian groups and ρ, σ are affine actions of a discrete group Γ on G and H respectively such that (H, σ) is expansive then every Γ-equivariant continuous map $f : (G, \rho) \to (H, \sigma)$ is an affine map. We also classify certain classes of translation flows on such groups up to orbit equivalence and topological conjugacy. For $\Gamma = \mathbb{R}$, we prove that two one-parameter translation flows on such groups are orbit equivalent if and only if they are algebraically conjugate after a change of scale. This generalizes an earlier result in the case of tori (see [Be]). Also for any topological group Γ, we prove that two translation flows of Γ on compact connected metrizable abelian groups are topologically conjugate if and only if they are algebraically conjugate.

In Chapter 3 we study rigidity of continuous equivariant maps between automorphism flows of a discrete group Γ on nilmanifolds. When $\Gamma = \mathbb{Z}$, (X_2, σ) is a factor of (X_1, ρ) and ρ, σ are generated by affine transformations, this phenomenon has been studied in [AP], [Wa1] and [Wa2]. In this case a necessary and sufficient condition for existence of a non-affine Γ-equivariant map is given in [Wa2]. Our methods are however different and applicable in more general situations.

In Section 1 we consider arbitrary automorphism flows of a discrete group Γ on nilmanifolds. We give several necessary conditions for existence of Γ-equivariant non-affine maps between such flows. In particular when (X_1, ρ) is ergodic or

(X_2, σ) is expansive we prove that every Γ-equivariant continuous map from X_1 to X_2 is an affine map. In Section 2 we give a necessary and sufficient condition for rigidity of Γ-equivariant continuous maps when X_1 is a torus. In Section 3 we consider the case when Γ is abelian. In the special case when $(X_1, \rho) = (X_2, \sigma)$ and we prove that there exists a non-affine Γ-equivariant continuous map from (X_1, ρ) to (X_2, σ) if and only if (X_1, ρ) is not ergodic.

In Chapter 4 we classify one-parameter automorphism flows on certain compact connected Lie groups up to topological conjugacy. In particular we prove that if $G = SO(2n+1)$, $\mathrm{Spin}(2n+1)$ or $\mathrm{Sp}(n)$ for some $n \geq 1$ then two one-parameter automorphism flows on G are topologically conjugate if and only if they are algebraically conjugate.

Chapter 1

Preliminaries

In this Chapter we introduce notations and definitions and discuss some background material.

1.1 Locally compact abelian groups

In this Section we recall some basic results from duality theory of locally compact abelian groups. For details the reader is referred to [Mo].

For a locally compact abelian group G, by $\widehat{G}$ we denote the set of all continuous homomorphisms from G to S^1. It is known that $\widehat{G}$ is a locally compact abelian group under pointwise multiplication and the compact-open topology. If G and H are two locally compact abelian groups and if $\theta : G \to H$ is a continuous homomorphism then $\widehat{\theta}$ will denote the continuous homomorphism from $\widehat{H}$ to $\widehat{G}$ defined by

$$\widehat{\theta}(\phi) = \phi \circ \theta \;\; \forall \phi \in \widehat{H}.$$

It is easy to see that a locally compact abelian group G is compact if and only if $\widehat{G}$ is discrete. The following proposition characterizes various topological properties of a compact abelian group G in terms of the dual group $\widehat{G}$.

Proposition 1.1.1 : *Let G be a compact abelian group. Then*

a) G is connected if and only if $\widehat{G}$ is torsion free.

b) G is metrizable if and only if $\widehat{G}$ is countable.

c) G is finite dimensional if and only if $\widehat{G}$ has finite rank.

In particular a compact abelian group G is connected. metrizable and finite dimensional if and only if there exists $n > 0$ and a subgroup H of $\mathbb{Q}^n$ such that $\mathbf{Z}^n \subset H$ and H is isomorphic to $\widehat{G}$. We recall also the following results concerning dual groups.

Theorem 1.1.2 (Pontryagin Duality Theorem) : *Let G be a locally compact abelian group and $\alpha : G \to \widehat{\widehat{G}}$ be the homomorphism defined by*

$$\alpha(g)(\phi) = \phi(g) \quad \forall g \in G, \phi \in \widehat{G}.$$

Then α is an isomorphism of the topological groups.

In particular $\widehat{G}$ separates points of G i.e. for any $g, h \in G$, $g \neq h$, there exists a $\phi \in \widehat{G}$ such that $\phi(g) \neq \phi(h)$.

Proposition 1.1.3 : *Let G, H be locally compact abelian groups and let f be a continuous homomorphism from $\widehat{G}$ to $\widehat{H}$. Then there exists a unique homomorphism $\theta : H \to G$ such that $f = \widehat{\theta}$.*

For a compact abelian group G we denote by λ_G the normalized Haar measure on G. If G is a discrete abelian group then λ_G will denote the counting measure on G. For any compact abelian group G and a function f in $L^2(G, \lambda_G)$ we denote by $\widehat{f} : \widehat{G} \to \mathbb{C}$ the Fourier transform of f. namely the function defined by

$$\widehat{f}(\phi) = \int_G f(g)\overline{\phi(g)}\, d\lambda_G.$$

Proposition 1.1.4 : *Let G be a compact abelian group. Then the elements of $\widehat{G}$ form an orthonormal basis for $L^2(G, \lambda_G)$. Furthermore the map $f \mapsto \widehat{f}$ is an isometry from $L^2(G, \lambda_G)$ onto $L^2(\widehat{G}, \lambda_{\widehat{G}})$.*

We will also use the following result due to Van Kampen; for a proof see [Ar],[Va].

Theorem 1.1.5 (Van Kampen): *Let G be a compact connected metrizable abelian group and $f : G \to S^1$ be a continuous map. Then there exist $c \in S^1$, $\phi \in \widehat{G}$ and a continuous map $h : G \to \mathbb{R}$ such that*

$$h(0) = 0 \ , \ f(x) = c\ \phi(x)\ e^{2\pi i h(x)} \quad \forall x \in G.$$

Moreover c, ϕ and h are uniquely defined.

1.2 Structure of nilmanifolds

For a Lie group G, we denote the Lie algebra of G by $L(G)$. We will denote the set of all linear automorphisms of $L(G)$ by $\mathrm{Aut}(L(G))$. The standard exponential map from $L(G)$ to G will be denoted by exp. We denote by Ad the adjoint representation of G into $\mathrm{Aut}(L(G))$ and by ad the differential of Ad.

Let G be a connected simply connected nilpotent Lie group and let H be a closed subgroup of G such that G/H is compact. Then the quotient space $X = G/H$ is called a nilmanifold. In this Section we recall a few basic results concerning the structure of nilmanifolds. For details the reader is referred to [Ma] and [Ra].

Let $X = G/H$ be nilmanifold and H_0 be the connected component of H containing the identity. Then H_0 is a closed normal subgroup of G. Furthermore if $G' = G/H_0$ and $D = H/H_0$ then we have:

a) $X = G/H$ is homeomorphic to $X' = G'/D$

b) G' is a connected simply connected nilpotent Lie group.

c) D is a discrete subgroup of G'.

Henceforth we will restrict the discussion of nilmanifolds to the case when $X = G/D$, where G is a connected simply connected nilpotent Lie group and D is a discrete subgroup of G such that G/D is compact.

Proposition 1.2.1 : *Let $X = G/D$ be a nilmanifold and let $\pi : G \to G/[G,G]$*

be the projection map. Then $[G,G] \cap D$ *is a closed uniform subgroup of* $[G,G]$ *and* $\pi(D)$ *is a closed uniform subgroup of* $G/[G,G]$.

Since $G/[G,G]$ is an abelian Lie group, this implies that for any nilmanifold $X = G/D$, $X^0 = G/([G,G] \cdot D)$ is a torus.

Theorem 1.2.2 (cf. [Ma]): *Let* $X_1 = G_1/D_1$ *and* $X_2 = G_2/D_2$ *be nilmanifolds and let* ϕ *be a homomorphism from* D_1 *to* D_2. *Then there exists a unique continuous homomorphism* $\bar{\phi}$ *from* G_1 *to* G_2 *such that* $\bar{\phi}|_{D_1} = \phi$.

We recall the following fact (cf. [Au], pp. 54).

Proposition 1.2.3 : *Let* $X = G/D$ *be a nilmanifold. For* $a = (a_1, \ldots, a_n)$ *in* $\mathbb{R}^n$ *let* I_a *denote the set defined by*

$$I_a = \{ \ x \mid a_i \leq x_i \leq a_i + 1 \ \forall i = 1, \ldots, n \ \}.$$

Then there exists an invertible linear map T *from* $\mathbb{R}^n$ *to* $L(G)$ *such that for all* a *in* $\mathbb{R}^n$, *the set* $\exp \circ T(I_a)$ *is a fundamental domain for* $X = G/D$.

The preceeding result in particular implies the following.

Proposition 1.2.4 : *Let* $X = G/D$ *be a nilmanifold and let* ρ *be an automorphism action of a discrete group* Γ *on* X. *Let* $B \subset X$ *denote the set of points with finite* ρ*-orbit. Then* B *is a dense subset of* X.

Proof : We define $A, A_1, A_2, \ldots \subset L(G)$ by

$$A_k = \{v \mid \exp(kv) \in D\} \ , \quad A = \textstyle\bigcup_1^\infty A_k.$$

If $\pi : G \to G/D$ denotes the projection map then we define $B, B_1, B_2, \ldots \subset X$ by

$$B_k = \pi \circ \exp(A_k) \ , \quad B = \textstyle\bigcup_1^\infty B_k.$$

From Proposition 1.2.3 it follows that each B_k is a finite subset of X and B is dense in X. Also it is easy to see that each B_k is invariant under the action ρ. Therefore for any element x in B, the ρ-orbit of x is finite. □

The following theorem is an analogue of Theorem 1.1.5 (cf. [Wa2]).

Theorem 1.2.5 : *Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and let f be a continuous map from X_1 to X_2. Let $F : G_1 \to G_2$ be a lift of f. Then there exist a $g_0 \in G_2$, a continuous homomorphism $\theta : G_1 \to G_2$ and a continuous map $P : G_1 \to G_2$ such that*

a) $P(e_1) = e_2$, $P(g \cdot \gamma) = P(g)$ $\forall g \in \Gamma$.

b) $F(g) = P(g) \cdot g_0 \cdot \theta(g)$ $\forall g \in \Gamma$.

Moreover θ and P are unique.

1.3 Compact Lie groups

In this Section we recall a few basic results about structure of compact connected Lie groups. For details the reader is referred to [B-T].

If G is a compact connected Lie group then a subgroup $T \subset G$ is called a maximal torus if it is a maximal connected abelian subgroup of G. It is easy to see that such a subgroup is closed and hence topologically isomorphic to a torus.

Theorem 1.3.1 : *Any two maximal tori in a compact connected Lie group are conjugate and every element of G is contained in a maximal torus.*

If G is a compact connected Lie group and $T \subset G$ is a maximal torus then $L(T)$ is a maximal abelian subalgebra of $L(G)$. Restricting the Lie algebra homomorphism ad : $L(G) \to \text{End}(L(G))$ to $L(T)$ we obtain an action of $L(T)$ on $L(G)$ and this induces an action of $L(T)$ on $L(G) \otimes \mathbb{C}$ which will also be denoted by ad.

It is known that if $\dim(T) = r$ then there exists a basis $\{x_1, \ldots, x_r, y_1, \ldots, y_k\}$ of $L(G)\otimes\mathbb{C}$ and homomorphisms $R_1, \ldots, R_k : L(T) \to \mathbb{R}$ such that for all $w \in L(T)$,

$$\mathrm{ad}(w)(x_j) = 0 \qquad \text{for } j = 1, \ldots, r,$$
$$\mathrm{ad}(w)(y_j) = 2\pi i R_j(w) y_j \qquad \text{for } j = 1, \ldots, k.$$

$R_1, \ldots, R_k$ are called the real roots of G. For each real root α we define a hyperplane $H_\alpha \subset L(T)$ by

$$H_\alpha = \ker(\alpha) = \{ w \mid \alpha(w) = 0 \}.$$

Each connected component of the set

$$W = L(T) - \cup H_\alpha$$

is called a *Weyl chamber* of $L(T)$.

Proposition 1.3.2 : *Let G be a compact connected Lie group, T be a maximal torus of G and C be a Weyl chamber of $L(T)$. Then for any w in $L(G)$, there exists a $g \in G$ such that $\mathrm{Ad}(g)(w)$ lies in the closure of C.*

1.4 Ergodicity of automorphism actions

Let (X, λ) be a finite measure space and let ρ be a measure preserving action of a group Γ on (X, λ). Then ρ is said to be *ergodic* if every Γ-invariant function f in $L^2(X, \lambda)$ is a constant almost everywhere. If G is a compact abelian group then it is easy to see that any continuous automorphism of G preserves the Haar measure on G. An automorphism action ρ of a discrete group Γ on a compact abelian group G is said to be ergodic if ρ is ergodic with respect to the Haar measure λ_G. The following theorem of Halmos gives a necessary and sufficient condition for ergodicity of automorphism actions on compact abelian groups.

Theorem 1.4.1 : *Let G be a compact abelian group and let ρ be an automorphism action of a discrete group Γ on G. Then ρ is ergodic if and only if the dual action $\widehat{\rho}$ has no nontrivial finite orbit in $\widehat{G}$.*

In particular if A is an element of $GL(n, \mathbb{Z})$ and ρ is the induced automorphism action of the cyclic group on T^n then ρ is ergodic if and only if the spectrum of A does not contain any root of unity.

If $X = G/D$ is a nilmanifold then by X^0 we denote the torus $G/([G,G] \cdot D)$ and by π^0 we denote the projection map from G onto X^0. If ρ is an automorphism action of a discrete group Γ on X then ρ^0 will denote the automorphism action of Γ on X^0 induced by ρ.

We will use the following two results.

Theorem 1.4.2 (see [Ber], Theorem 5.1): *Let Γ be a discrete abelian group, T^n be the n-torus and ρ be an ergodic automorphism action of Γ on T^n. Then there exists an element γ_0 of Γ such that $\rho(\gamma_0)$ is an ergodic automorphism.*

Theorem 1.4.3 (see [Pa]): *Let $X = G/D$ be a nilmanifold and θ be an automorphism of X such that θ induces an ergodic automorphism on the torus $G/([G,G] \cdot D)$. Then θ is an ergodic automorphism of X.*

We note the following simple consequence of the above mentioned results.

Proposition 1.4.4 : *Let Γ be a discrete abelian group, $X = G/D$ be a nilmanifold and ρ be an automorphism action of Γ on X. Then (X, ρ) is ergodic if and only if (X^0, ρ^0) is ergodic.*

Proof : Let $q : X \to X^0$ denote the projection map. Then it is easy to check that q is a measure preserving Γ-equivariant map from (X, ρ) to (X^0, ρ^0). Therefore ergodicity of (X, ρ) implies ergodicity of (X^0, ρ^0). On the other hand if (X^0, ρ^0) is ergodic then by Theorem 1.4.2 there exists a γ_0 in Γ such that $\rho^0(\gamma_0)$ is an ergodic automorphism of X^0. Applying Theorem 1.4.3 we see that $\rho(\gamma_0)$ is an ergodic automorphism of X i.e. (X, ρ) is ergodic. □

Chapter 2

Affine actions on compact abelian groups

In this Chapter we study continuous affine actions of a group Γ on compact connected metrizable abelian groups. We show that if Γ is a discrete group and (G,ρ), (H,σ) are affine Γ-flows on such groups then under certain conditions every Γ-equivariant continuous map from (G,ρ) to (H,σ) is an affine map. We also classify certain classes of translation flows on such groups up to orbit equivalence and topological conjugacy.

2.1 Structure of continuous maps

In this Section we study the structure of continuous maps between compact connected metrizable abelian groups. For a compact connected metrizable abelian group G, we denote by $L(G)$ the topological vector space consisting of all homomorphisms from $\widehat{G}$ to $\mathbb{R}$ under pointwise addition and scalar multiplication and the topology of pointwise convergence.

If G and H are compact connected metrizable abelian groups and θ is a continuous homomorphism from G to H then by $D\theta$ we denote the map from $L(G)$ to $L(H)$ defined by

$$D\theta(p)(\phi) = p(\phi\circ\theta) \quad \forall p\in L(G), \phi\in\widehat{H}.$$

It is easy to see that this defines a functor from the category of compact connected metrizable abelian groups to the category of topological vector spaces. For any compact connected metrizable abelian group G, we define a map E from $L(G)$ to G by the condition

$$(\phi \circ E)(p) = e^{2\pi i p(\phi)} \quad \forall p \in L(G), \phi \in \widehat{G}.$$

Since for a fixed p in $L(G)$, the map $\phi \to e^{2\pi i p(\phi)}$ is a continuous homomorphism from $\widehat{G}$ to S^1, by the duality theorem the map E is well defined and unique. We note the following properties of the map E.

Proposition 2.1.1 : *For any compact connected metrizable abelian group G, the map $E : L(G) \to G$ is a continuous homomorphism and the kernel of E is a totally disconnected subgroup of $L(G)$. Furthermore if G and H are compact connected metrizable abelian groups and θ is a continuous homomorphism from G to H then $\theta \circ E = E \circ D\theta$.*

Proof : From the defining condition it is easy to see that E is a homomorphism. Suppose $p_\alpha \to p$ pointwise. Then $\phi \circ E(p_\alpha) \to \phi \circ E(p)$ for all ϕ in $\widehat{G}$ i.e. $E(p_\alpha) \to E(p)$. Therefore E is a continuous homomorphism from $L(G)$ to G. Clearly the kernel of E can be identified with the set of all homomorphism from $\widehat{G}$ to $\mathbb{Z}$. We note that for any ϕ in $\widehat{G}$, $p \mapsto p(\phi)$ is a continuous function from $L(G)$ to $\mathbb{R}$. Since $\mathbb{Z}$ is a totally disconnected subset of $\mathbb{R}$ it follows that the kernel of E is totally disconnected. To prove the last assertion we note that for any ϕ in $\widehat{H}$ and p in $L(G)$,

$$\phi \circ \theta \circ E(p) = e^{2\pi i p(\phi \circ \theta)} = e^{2\pi i D\theta(p)(\phi)} = \phi \circ E \circ D\theta(p).$$

Since $\widehat{H}$ separates points of H, it follows that $\theta \circ E = E \circ D\theta$. □

Note that using the duality theorem we can realize $L(G)$ as the set of all one-parameter subgroups of G, and E as the map $\alpha \mapsto \alpha(1)$. In particular when G is a torus, $L(G)$ can be identified with $\mathbb{R}^n$, the Lie algebra of G, and E can be identified with the usual exponential map. However the following example shows that in general E is not surjective.

Example 2.1.2 : Let $G = \widehat{\mathbb{Q}}$. where $\mathbb{Q}$ is the additive group of rational numbers equipped with the discrete topology. Then it is easy to check that $L(G)$ is isomorphic to $\mathbb{R}$. We claim that in this case the map $E : L(G) \to G$ is not surjective. Suppose otherwise. Then G is the image of $\mathbb{R}$ under the continuous homomorphism E. Since G is compact, this implies that G is isomorphic to S^1. Since $\widehat{S^1} \cong \mathbb{Z}$ and $\widehat{G} \cong \mathbb{Q}$, we get a contradiction; this proves the claim.

For any two groups G, H and any continuous map $f : G \to H$ we will define a continuous homomorphism $\theta(f) : G \to H$ as follows. For each character ϕ of H let $c_\phi \in S^1$, $\widehat{\theta}(\phi) \in \widehat{G}$ and $f_\phi : G \to \mathbb{R}$ be such that

$$f_\phi(0) = 0 \ , \ \phi \circ f(x) = c_\phi \, \widehat{\theta}(\phi)(x) \, e^{2\pi i f_\phi(x)} \quad \forall x \in G;$$

by Theorem 1.1.5 there exist $c_\phi, \widehat{\theta}(\phi)$ and f_ϕ satisfying the conditions, and they are unique. From the uniqueness one can deduce that $\phi \mapsto \widehat{\theta}(\phi)$ is a homomorphism from $\widehat{H}$ to $\widehat{G}$. By the duality theorem there exists a continuous homomorphism $\theta(f) : G \to H$ such that

$$\widehat{\theta}(\phi) = \phi \circ \theta(f) \quad \forall \phi \in \widehat{H}.$$

Using the uniqueness part of Theorem 1.1.5 it is easy to see that

i) If f is a continuous homomorphism then $\theta(f) = f$.

ii) If $f : G_1 \to G_2$ and $g : G_2 \to G_3$ be two continuous maps then $\theta(g \circ f) = \theta(g) \circ \theta(f)$.

The following generalizes VanKampen's theorem.

Theorem 2.1.3 : *Let G, H be two compact connected metrizable abelian groups and f be a continuous map from G to H. Then there exist $c \in H$, a continuous homomorphism $\theta : G \to H$ and a continuous map $S : G \to L(H)$ such that $S(0) = 0$ and $f(x) = c\theta(x)(E \circ S)(x)$. $\forall x \in G$. Moreover c, θ and S are unique.*

Proof : For each character ϕ of H define $c_\phi \in S^1, \theta'(\phi) \in \widehat{G}$ and $f_\phi : G \to \mathbb{R}$ by the condition

$$f_\phi(0) = 0 \ , \ \phi \circ f(x) = c_\phi \ \theta'(\phi)(x) \ e^{2\pi i f_\phi(x)} \quad \forall x \in G;$$

we note that by Theorem 1.1.5 there exist uniquely defined $c_\phi, \theta'(\phi)$ and f_ϕ satisfying the condition. From the uniqueness it follows that

$$f_{\phi\psi} = f_\phi + f_\psi \ , \ f_{\phi^{-1}} = -f_\phi.$$

Define $S : G \to L(H)$ by $S(x)(\phi) = f_\phi(x)$. Since each f_ϕ is continuous, S is continuous. Similarly using uniqueness of c_ϕ we see that the map $\phi \mapsto c_\phi$ is a homomorphism from $\widehat{H}$ to S^1. By the duality theorem there exists $c \in H$ such that $c_\phi = \phi(c) \cdot \forall\phi \in \widehat{H}$. Also, putting $\theta = \theta(f)$ we see that $\theta'(\phi) = \phi \circ \theta$, $\forall \phi \in \widehat{H}$. Hence for all $x \in G$ and $\phi \in \widehat{H}$, we have

$$\phi \circ f(x) = c_\phi \ \theta'(\phi)(x) \ e^{2\pi i f_\phi(x)} = \phi(c) \ (\phi \circ \theta)(x) \ (\phi \circ E \circ S)(x).$$

Since characters separate points, we get $f(x) = c\,\theta(x)(E \circ S)(x)$, $\forall x \in G$. Using Theorem 1.1.6 we see that for a fixed $\phi \in \widehat{H}$, ~~$\phi(c)$~~, $\phi \circ \theta$ and the map $x \mapsto S(x)(\phi)$ are determined by the equation

$$(\phi \circ f)(x) = \phi(c) \ (\phi \circ \theta)(x) \ e^{2\pi i S(x)(\phi)}.$$

Hence c, θ and S are unique. □

2.2 Rigidity of affine actions

Suppose Γ be a discrete group and (G, ρ) be an affine Γ-flow on G. Note that ρ induces an automorphism flow ρ_a and a map $\rho_t : \Gamma \to G$ defined by the condition

$$\rho(\gamma)(x) = \rho_a(\gamma)(x) \ \rho_t(\gamma) \ \ \forall x \in G \text{ and } \gamma \in \Gamma.$$

We define an automorphism action ρ_* of Γ on $L(G)$ by

$$\rho_*(\gamma)(p)(\phi) = p(\phi \circ \rho_a(\gamma)) \ \ \forall \phi \in \widehat{G}, \gamma \in \Gamma.$$

Theorem 2.2.1 : *Let Γ be a discrete group and G and H be compact connected metrizable abelian groups. Let ρ, σ be affine actions of Γ on G and H respectively. Let $f : G \to H$ be a Γ-equivariant continuous map. Then there exist $c \in H$, a continuous homomorphism $\theta : G \to H$ and a continuous map $S : G \to L(H)$ such that*

a) $S(e) = 0$ and for all x in G, the orbit of $S(x)$ under σ_* is bounded.

b) $f(x) = c\,\theta(x)(E \circ S)(x)$, $\forall x \in G$.

Moreover, if ρ and σ are automorphism actions then S is a Γ-equivariant map from (G, ρ) to $(L(H), \sigma_*)$.

Proof : Suppose $f = c\,\theta(E \circ S)$ where c, θ and S are as in Theorem 2.1.3. Fix any $\gamma \in \Gamma$. Note that for all $x \in G$,

$$f \circ \rho(\gamma)(x) = c_1\,\theta_1(x)(E \circ S_1)(x),$$

where $c_1 = f \circ \rho(\gamma)(e)$, $\theta_1 = \theta \circ \rho_a(\gamma)$ and $S_1(x) = S \circ \rho(\gamma)(x) - S \circ \rho(\gamma)(e)$. Also for all $x \in G$,

$$\sigma(\gamma) \circ f(x) = c_2\,\theta_2(x)(E \circ S_2)(x),$$

where $c_2 = \sigma(\gamma) \circ f(e)$, $\theta_2 = \sigma_a(\gamma) \circ \theta$ and $S_2 = \sigma_*(\gamma) \circ S$. From the uniqueness part of Theorem 2.1.3 it follows that $S_1 = S_2$ i.e.

$$S \circ \rho(\gamma)(x) - S \circ \rho(\gamma)(e) = \sigma_*(\gamma) \circ S(x), \quad \forall x \in G.$$

Since for a fixed $x \in G$ the left hand side is contained in a bounded subset of $L(H)$, it follows that for all x in G, the σ_*-orbit of $S(x)$ is bounded. Also it is easy to see from the previous identity that when ρ and σ are automorphism actions, S is a Γ-equivariant map. □

Let (X, d) be a compact metric space and ρ be a continuous action of a group Γ on X. Then (X, ρ) is said to be *expansive* if there exists $\epsilon > 0$ such that for any two distinct points x, y in X,

$$\text{Sup } \{ d(\rho(\gamma)(x), \rho(\gamma)(y)) \mid \gamma \in \Gamma \} \geq \epsilon;$$

an ϵ for which this holds is called an expansive constant of (X, ρ). It is easy to check that the notion of expansiveness is independent of the metric d. When X is a topological group, (X, ρ) is expansive if there exists a neighborhood U of the identity such that for any two distinct elements $x, y \in X$ there exists a $\gamma \in \Gamma$ such that $\rho(\gamma)(x)\ \rho(\gamma)(y)^{-1}$ is not contained in U; such a neighborhood is called an expansive neighborhood. For various characterizations of expansiveness of automorphism actions on compact abelian groups the reader is referred to [K-S] and [Sc].

Corollary 2.2.2 : *Let Γ be a discrete group and G and H be compact connected metrizable abelian groups. Let ρ, σ be affine actions of Γ on G and H respectively and suppose that (H, σ) is expansive. Then every Γ-equivariant continuous map f from (G, ρ) to (H, σ) is an affine map.*

Proof : Since σ is expansive, σ_a is an expansive automorphism action on H. We claim that for every nonzero point $p \in L(H)$, the orbit of p under σ_* is unbounded. Suppose this is not the case. Choose a non-zero p and a compact set $C \subset L(H)$ such that orbit of p under σ_* is contained in C. Since kernel of E is totally disconnected, there exists a sequence $\{t_i\}$ such that $t_i \to 0$ as $i \to \infty$ and $E(t_i p) \neq e\ \forall i$. Let U be an expansive neighborhood of e in H. Since e is fixed by σ_a, e is the only element in G whose orbit under σ_a is contained in U. Since $t_i \to 0$, it is easy to see that $\cup\, t_i^{-1} E^{-1}(U) = L(H)$. From the compactness of C it follows that there exists n such that $t_n C \subset E^{-1}(U)$ i.e. $E(t_n C) \subset U$. Since the orbit of $t_n p$ under σ_* is contained in $t_n C$ and $E \circ \sigma_*(\gamma) = \sigma_a(\gamma) \circ E\ \forall \gamma$, this implies that orbit of $E(t_n p)$ under σ is contained in U. This contradicts the fact that $E(t_n p) \neq e$.

Now suppose $f = c\,\theta(E \circ S)$, where c, θ and S are as in Theorem 2.2.1. Then from Theorem 2.2.1 and the preceding observation it follows that $S = 0$, i.e. f is an affine map. □

Corollary 2.2.3 : *Let Γ be a discrete group, G and H be compact connected metrizable abelian groups and ρ, σ be automorphism actions of Γ on G and H*

respectively. Suppose that H is finite-dimensional and (G,ρ) is ergodic. Then every continuous Γ-equivariant map from (G,ρ) to (H,σ) is an affine map.

Proof : Suppose $f = c\,\theta(E \circ S)$ is a continuous Γ-equivariant map from (G,ρ) to (H,σ) where c, θ and S are as in Theorem 2.2.1. Clearly to prove the stated assertion it is enough to prove that $S = 0$. Since H is finite-dimensional, applying Proposition 1.1.1 we see that $L(H)$ is a finite dimensional vector space over $\mathbb{R}$. Let W denote the subspace of $L(H)$ consisting of all points with bounded σ_*-orbit. Then from Theorem 2.2.1 it follows that the image of S is contained in W. We fix a norm $||.||$ on W and define a map $N : W \to \mathbb{R}$ by

$$N(w) = \text{Sup } \{||\sigma_*(\gamma)(w)|| \mid \gamma \in \Gamma\}.$$

It is easy to see that N defines another norm on W. Since any two norms on a finite dimensional vector space over $\mathbb{R}$ are equivalent, N is a continuous map from W to $\mathbb{R}$. Since S is Γ-equivariant, $h = N \circ S$ is a continuous Γ-invariant map from (G,ρ) to $\mathbb{R}$. From the ergodicity of (G,ρ) it follows that h is a constant map. Since the image of h contains 0, this implies that S is identically zero. □

For a set A we denote by $|A|$ the cardinality of A.

Corollary 2.2.4 : *Let Γ be a discrete group and G and H be compact connected metrizable abelian groups. Let ρ, σ be automorphism actions of Γ on G and H respectively such that*

a) $\{g \in G \mid |\rho(\Gamma)(g)| < \infty\}$ is dense in G.

b) for any finite-index subgroup $\Gamma_0 \subset \Gamma$, the set of points of H which are fixed by $\sigma(\Gamma_0)$ is totally disconnected.

Then every continuous Γ-equivariant map $f : (G,\rho) \to (H,\sigma)$ is an affine map.

Proof : Suppose $f = c\,\theta(E \circ S)$, where c, θ and S are as in Theorem 2.1.3. Let $g \in G$ be such that the orbit of g under ρ is finite. Let $\Gamma_0 \subset \Gamma$ be the stabilizer of g. Since S is Γ-equivariant by Theorem 2.2.1, $S(g)$ is fixed by Γ_0 under the action σ_*. Since σ_* is a linear action on $L(H)$ and E is a Γ-equivariant map from $(L(H), \sigma_*)$ to (H, σ), it follows that for all $t \in \mathbb{R}$ the stabilizer of $E(tS(g))$ under σ contains Γ_0. Now from b) it follows that $S(g) = 0$. Since $\{g \in G \mid |\rho(\Gamma)(g)| < \infty\}$ is dense in G, $S = 0$ i.e. f is an affine map. □

Remark 2.2.5 : It is easy to see that condition (a) as in the previous corollary, holds when $G = T^n$ for some n. Various other conditions under which the set of periodic orbits of an automorphism action on a compact abelian group is dense, viz condition (a) as in corollary 2 holds, are described in [K-S]. Condition (b) holds in the case of $H = T^n$ if Γ contains an element acting ergodically; more generally this holds for any finite dimensional compact abelian group H.

2.3 Classification of translation flows

We begin with the following observation.

Lemma 2.3.1 : *Let G be a compact connected metrizable abelian group and α be a one-parameter subgroup of G. Then there exists a $p \in L(G)$ such that $E(tp) = \alpha(t)$ $\forall t \in \mathbb{R}$.*

Proof : For each $\phi \in \widehat{G}$, we define $\alpha_\phi \in \mathbb{R}$ by

$$\phi \circ \alpha(t) = e^{2\pi i \alpha_\phi t} \ \forall t \in \mathbb{R}.$$

Since $\phi \circ \alpha$ is a continuous homomorphism from $\mathbb{R}$ to S^1, α_ϕ is well defined. We define $p \in L(G)$ by $p(\phi) = \alpha_\phi$ $\forall \phi \in \widehat{G}$. Fix $t \in \mathbb{R}$. From the defining equation of E it follows that for all $\phi \in \widehat{G}$,

$$\phi \circ E(tp) = e^{2\pi i t p(\phi)} = e^{2\pi i \alpha_\phi t} = \phi \circ \alpha(t).$$

Since characters separate points, it follows that $\alpha(t) = E(tp)$ $\forall t \in \mathbb{R}$. □

The following lemma is needed to prove Theorem 2.3.3. The main idea of the proof is derived from [Be].

Lemma 2.3.2 : *Let G be a compact connected metrizable abelian group and $p, q \in L(G)$, with $p \neq 0$. Let $f : \mathbb{R} \to L(G)$ be a bounded continuous function such that*

$$f(0) = 0 \text{ and } \{E(tp + f(t)) \mid t \in \mathbb{R}\} = \{E(tq) \mid t \in \mathbb{R}\}.$$

Then $p = cq$ for some nonzero $c \in \mathbb{R}$.

Proof : First we will prove the special case when $G = T^2$, the two-dimensional torus. After suitable identifications we have

$$L(G) = \mathbb{R}^2 \ , \ E(x_1, x_2) = \exp(x_1, x_2) = (e^{2\pi i x_1}, e^{2\pi i x_2}).$$

Define a function $d : \mathbb{R}^2 \to \mathbb{R}^+$ by

$$\begin{aligned} d(x) &= \text{distance between the point } x \text{ and the line } \{tq \mid t \in \mathbb{R}\} \\ &= \inf \{\|y\| \mid x + y = tq \text{ for some } t \in \mathbb{R}\}. \end{aligned}$$

By our hypothesis, for all $t \in \mathbb{R}$, $tp + f(t) = z + t'q$ for some $t' \in \mathbb{R}, z \in \mathbb{Z}^2$. This implies

$$\{d(tp + f(t)) \mid t \in \mathbb{R}\} \subset \{d(z) \mid z \in \mathbb{Z}^2\}.$$

Since the map $t \mapsto d(tp + f(t))$ is continuous, the left hand side is a connected subset of $\mathbb{R}$ containing 0. Since the right hand side is countable, $d(tp + f(t)) = 0$, for all t. Since f is bounded this implies that $d(tp)$ is bounded by a constant M, for all $t \in \mathbb{R}$. Since distinct lines in $\mathbb{R}^2$ diverge from each other we conclude that $p = cq$ for some $c \neq 0$.

To prove the general case choose ϕ such that $p(\phi) \neq 0$. For each $\psi \in \widehat{G}$ define $h : G \to T^2$ and $h^* : L(G) \to \mathbb{R}^2$ by

$$h(x) = (\phi(x), \psi(x)) \ , \quad h^*(r) = (r(\phi), r(\psi)).$$

Now for all $r \in L(G)$,

$$h \circ E(r) = (\phi \circ E(r), \psi \circ E(r)) \text{ and } \exp \circ h^*(r) = (e^{2\pi i r(\phi)}, e^{2\pi i r(\psi)}).$$

From the defining equation of E it follows that $h \circ E = \exp \circ h^*$. Define $p', q' \in \mathbb{R}^2$ and $f' : G \to \mathbb{R}^2$ by $p' = h^*(p)$, $q' = h^*(q)$ and $f' = h^* \circ f$. From our hypothesis it follows that

$$\{\exp(tp' + f'(t)) \mid t \in \mathbb{R}\} = \{\exp(tq') \mid t \in \mathbb{R}\}$$

Now applying the special case we see that $(p(\phi), p(\psi)) = c\,(q(\phi), q(\psi))$ for some nonzero real number c. Therefore $q(\phi) \neq 0$ and, since ψ is arbitrary, $p = c_0 q$ where $c_0 = p(\phi)/q(\phi)$. □

Theorem 2.3.3 : *Let G and H be compact connected metrizable abelian groups and α and β be one-parameter subgroups of G and H respectively. Then the translation flows induced by α and β are orbit equivalent if and only if there exist a continuous isomorphism $\theta : G \to H$ and a nonzero $c \in \mathbb{R}$ such that $\theta\alpha(t) = \beta(ct) \quad \forall t \in \mathbb{R}$.*

Proof : Let f be an orbit equivalence between the translation flows induced by α and β. Define $h : G \to H$ by $h(x) = h(x)h(e)^{-1}$. Then $h(e) = e$ and it is easy to check that h is also an orbit equivalence. Suppose $h = \theta(E \circ S)$, where θ and S are as in Theorem 2.1.3. By Lemma 2.3.1 there exists p, q in $L(H)$ such that $E(tq) = \beta(t)$ and $E(tp) = \theta(\alpha(t))$. Since h is an orbit equivalence,

$$\{h(\alpha(t)) \mid t \in \mathbb{R}\} = \{\beta(t) \mid t \in \mathbb{R}\}.$$

Now for all $t \in \mathbb{R}$, $\beta(t) = E(tq)$ and

$$h(\alpha(t)) = (\theta \circ \alpha)(t)(E \circ S \circ \alpha)(t) = E(tp + S \circ \alpha(t)).$$

Since h is a homeomorphism, θ is an isomorphism. Hence $\theta(\alpha) \neq 0$, i.e. $p \neq 0$. By applying Lemma 2.3.2 we see that $p = cq$ for some nonzero c. Therefore for some $c \neq 0$, $\theta(\alpha(t)) = \beta(ct)$, $\forall t \in \mathbb{R}$. This proves the theorem. □

Theorem 2.3.4 : *Let G, H be two compact connected metrizable abelian groups and ρ, σ be two translation flows of a topological group Γ on G and H respectively. Then (G, ρ) and (H, σ) are topologically conjugate if and only if they are algebraically conjugate.*

Proof : Let f be a topological conjugacy between the induced translation flows. Suppose $f = c\,\theta\,(E \circ S)$, where c, θ and S are as in Theorem 2.1.3. Since f is a homeomorphism, θ is an isomorphism. We claim that θ is an algebraic conjugacy between (G, ρ) and (H, σ). To see this fix any $\gamma \in \Gamma$. Define $x_0 \in G$, $y_0 \in H$ by

$$\rho(\gamma)(x) = x_0 x \text{ and } \sigma(\gamma)(y) = y_0 y \quad \forall x \in G,\ y \in H.$$

Then for all $x \in G$,

$$f(x_0 x) = c\,\theta(x_0 x)(E \circ S)(x_0 x) = c_1\,\theta(x)(E \circ S_1)(x),$$

where $c_1 = c\,\theta(x_0)(E \circ S)\ (x_0)$, $S_1(x) = S(x_0 x) - S(x_0)$. Also for all $x \in G$,

$$y_0 f(x) = c_2\,\theta(x)(E \circ S)(x),$$

where $c_2 = c\ y_0$. Since $f(x_0 x) = y_0 f(x)$, from the uniqueness part of Theorem 2.1.3 it follows that $S_1 = S$ i.e.

$$S(x_0 x) = S(x) + S(x_0), \quad \forall x \in G.$$

Putting $x = x_0, x_0^2, \ldots$ and using the above recursion relation we obtain

$$S(x_0^n) = nS(x_0), \quad \forall n \in \mathbb{Z}^+.$$

Since the left hand side is contained in the image of S, which is compact, it follows that $S(x_0) = 0$. Since $c_1 = c_2$, this implies $\theta(x_0) = y_0$. Hence $\theta \circ \rho(\gamma) = \sigma(\gamma) \circ \theta$.□

Chapter 3

Rigidity of actions on nilmanifolds

In this Chapter we study rigidity of equivariant continuous maps between automorphism actions of discrete groups on nilmanifolds. Throughout this Chapter for $i = 1, 2$, $X_i = G_i/D_i$ will denote a nilmanifold, e_i will denote the identity element of G_i and $\bar{e}_i$ will denote the image of e_i in G_i/D_i. Also if ρ is an automorphism action of a discrete group Γ on a nilmanifold $X = G/D$ then $\bar{\rho}$ will denote the induced automorphism action of Γ on G and ρ_e will denote the induced Γ-action on $L(G)$ defined by

$$\rho_e(\gamma) = d\rho(\gamma) \quad \forall \gamma \in \Gamma.$$

Definition : Suppose X_1, X_2 are nilmanifolds and ρ, σ are continuous actions of a discrete group Γ on X_1 and X_2 respectively. Then a continuous map $f : X_1 \to X_2$ is said to be *almost equivariant* if there exists a finite-index subgroup $\Gamma_0 \subset \Gamma$ such that

$$f \circ \rho(\gamma) = \sigma(\gamma) \circ f \quad \forall \gamma \in \Gamma_0.$$

In Section 1 we give a necessary and sufficient condition for existence of a non-affine almost equivariant continuous map between such actions. As a consequence we prove that if either (X_1, ρ) is ergodic or (X_2, σ) is expansive then every Γ-equivariant continuous map is an affine map. In Section 2 we consider the case when X_1 is a torus. We give a necessary and sufficient condition for existence of a non-affine Γ-equivariant continuous map in this case. In the final Section we consider the case when Γ is abelian and (X_2, σ) is a factor of (X_1, ρ).

3.1 Almost equivariant maps

Let $X_1 = G_1/D_1$ and $X_2 = G_2/D_2$ be nilmanifolds and let f be a continuous map from X_1 to X_2. Then by Theorem 1.2.5 there is a unique continuous map $P(f) : G_1 \to G_2$ satisfying the following conditions.

1) $P(f)(e_1) = e_2,\ P(f)(g \cdot \gamma) = P(g) \ \ \forall g \in \Gamma.$

2) For some affine map $A : G_1 \to G_2$, we have the following commutative diagram.

$$\begin{array}{ccc} G_1 & \xrightarrow{P(f)\cdot A} & G_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

Suppose θ_1 is an automorphism of X_1 and $\overline{\theta}_1$ is the induced automorphism on G_1. Then we have the following commutative digram.

$$\begin{array}{ccccc} G_1 & \xrightarrow{\overline{\theta_1}} & G_1 & \xrightarrow{P(f)\cdot A} & G_2 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ X_1 & \xrightarrow{\theta_1} & X_1 & \xrightarrow{f} & X_2 \end{array}$$

Since $(P(f) \cdot A) \circ \overline{\theta_1} = (P(f) \circ \overline{\theta_1}) \cdot (A \circ \overline{\theta_1})$, from the uniqueness part of Theorem 1.2.5 it follows that $P(f \circ \theta_1) = P(f) \circ \overline{\theta}_1$. Similarly if θ_2 is an automorphism of X_2 and $\overline{\theta}_2$ is the induced automorphism on G_2 then $\overline{\theta_2} \circ P(f) = P(\theta_2 \circ f)$. Therefore if (X_1, ρ) and (X_2, σ) are automorphism actions of a discrete group Γ and if $f : X_1 \to X_2$ is a Γ-equivariant continuous map then $P(f)$ is a Γ-equivariant continuous map from $(G_1, \overline{\rho})$ to $(G_2, \overline{\sigma})$.

Lemma 3.1.1 : *Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and ρ, σ be automorphism actions of a discrete group Γ on X_1 and X_2 respectively. Then there exists a non-affine continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) if and only if there exists a nonzero continuous Γ-equivariant map S from (X_1, ρ) to $(L(G_2), \sigma_e)$ such that $S(\bar{e_1}) = 0$.*

Proof : Suppose there exists a non-affine Γ-equivariant continuous map f from (X_1, ρ) to (X_2, σ). Let $P = P(f) : G_1 \to G_2$ be as defined above. Since $P(e_1) = e_2$ and $P(g \cdot \gamma) = P(g)$ for all $g \in \Gamma$, there exists a unique continuous map Q from X_1 to G_2 such that $Q(\overline{e_1}) = e_2$ and we have the following commutative diagram.

$$\begin{array}{ccc} G_1 & \xrightarrow{P} & G_2 \\ {\scriptstyle \pi_1}\downarrow & \nearrow_{Q} & \\ X_1 & & \end{array}$$

It is easy to see that Q is a Γ-equivariant map from (X_1, ρ) to $(G_2, \overline{\sigma})$. Note that since G_2 is a connected simply connected nilpotent Lie group, the map $\exp : L(G_2) \to G_2$ is a diffeomorphism. Hence there is a unique map $S : X_1 \to L(G_2)$ such that the following diagram is commutative.

$$\begin{array}{ccc} & & G_2 \\ & \nearrow^{Q} & \uparrow{\scriptstyle \exp} \\ X_1 & \xrightarrow[S]{} & L(G_2) \end{array}$$

Then $S(\overline{e_1}) = 0$ and since exp is a Γ-equivariant map from $(L(G_2), \sigma_e)$ to $(G_2, \overline{\sigma})$ it is easy to see that S is a Γ-equivariant map from (X_1, ρ) to $(L(G_2), \sigma_e)$. Since f is a non-affine map, $P(f) = \exp \circ S \circ \pi_1$ is non-constant i.e. S is a nonzero map.

Now suppose there exists a non-zero Γ-equivariant continuous map S from (X_1, ρ) to $(L(G_2), \sigma_e)$ such that $S(\bar{e_1}) = 0$. Define a map $f : X_1 \to X_2$ by

$$f(x) = \pi_2 \circ \exp \circ S(x) \; \forall x \in X_1.$$

It is easy to check that f is a Γ-equivariant map from (X_1, ρ) to (X_2, σ) and $P(f) = \exp \circ S \circ \pi_1$. Since the map $\exp \circ S$ is non-constant, so is P. Now from the uniqueness part of Proposition 1.2.5 it follows that f is a non-affine map. □

Lemma 3.1.2 : *Let $X = G/D$ be a nilmanifold and V be a finite dimensional vector space over $\mathbb{R}$. Let Γ be a discrete group and ρ, σ be automorphism actions of Γ on X and V respectively. Then for any Γ-equivariant map $S : X \to V$ there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ such that $S \circ \rho(\gamma) = S \quad \forall \gamma \in \Gamma_0$.*

Proof : Let W denote the subspace of V which consists of all elements of V whose σ-orbit is finite and let $B \subset X$ denote the set of points in X with finite ρ-orbit. Then from Proposition 1.2.4 it follows that B is a dense subset of X. Since S is Γ-equivariant, it follows that the image of S is contained in W. Now choose a basis $\{w_1, w_2, \ldots w_l\}$ of W. Define $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ and Γ_0 by

$$\Gamma_i = \{\gamma \in \Gamma \mid \sigma(\gamma)(w_i) = w_i\}, \quad \Gamma_0 = \bigcap \Gamma_i.$$

Since each $\Gamma_i \subset \Gamma$ is a subgroup of finite index, so is Γ_0. Since Γ_0 acts trivially on W and image of S is contained in W, it follows that S is a Γ_0-invariant map. □

Theorem 3.1.3 : *Let Γ be a discrete group and $X_1 = G_1/D_1$, $X_2 = G_2/D_2$ be nilmanifolds. Let ρ and σ be automorphism actions of Γ on X_1 and X_2 respectively. Then there exists a non-affine almost equivariant continuous map from (X_1, ρ) to (X_2, σ) if and only if the following two conditions are satisfied.*

a) There exists a non-constant Γ-invariant continuous function from X_1 to $\mathbb{R}$.

b) There exists a nonzero vector v in $L(G_2)$ such that the σ_e-orbit of v is finite.

Proof : Suppose there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ and a non-affine continuous map f from (X_1, ρ) to (X_2, σ) which is Γ_0-equivariant. Then applying Lemma 3.1.1 we see that there exists a nonzero continuous Γ_0-equivariant map S from (X_1, ρ) to $(L(G_2), \sigma_e)$ such that $S(\bar{e_1}) = 0$. Let W denote the subspace of $L(G_2)$ which consists of all elements of $L(G_2)$ whose σ_e-orbit is finite. Since Γ_0 is a finite-index subgroup of Γ, it is easy to see that, if v is a vector in $L(G_2)$ with finite Γ_0-orbit under the action σ_e then $v \in W$. Since S is Γ_0-equivariant, from Proposition 1.2.4 it follows that the image of S is contained in W. Since S is non-zero, there exists a nonzero vector v in $L(G_2)$ such that the σ_e-orbit of v is finite. To prove a) we choose a norm $||.||$ on W and define a function $N : W \mapsto \mathbb{R}$ by

$$N(w) = \text{Sup } \{||\sigma_e(\gamma)(w)|| \mid \gamma \in \Gamma \}.$$

Then it is easy to see that N defines a norm on W and the map $N \circ S$ is a non-constant continuous Γ_0-invariant function from X_1 to $\mathbb{R}$. We choose a finite set $A \subset \Gamma$ which contains exactly one element of each right coset of Γ_0 and define a function $q_A : X_1 \mapsto \mathbb{R}$ by

$$q_A(x) = \sum_{\gamma \in A} N \circ S \circ \rho(\gamma)(x).$$

Since $N \circ S$ is Γ_0-invariant, it is easy to see that for any two elements γ_1 and γ_2 in the same right coset of Γ_0,

$$N \circ S \circ \rho(\gamma_1) = N \circ S \circ \rho(\gamma_2).$$

Hence if $B \subset \Gamma$ is another finite set containing exactly one element of each right coset of Γ_0 then $q_A = q_B$. This implies that for any γ in Γ,

$$q_A \circ \rho(\gamma) = q_{A\gamma} = q_A.$$

Since $N \circ S$ is a non-negative function which is not identically zero, it follows that q_A is a non-constant continuous Γ-invariant function from X_1 to $\mathbb{R}$.

Now suppose both the conditions a) and b) are satisfied. Then W, as defined above, is a nonzero subspace of $L(G_2)$ and there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ such that the σ_e-action of Γ_0 on W is trivial. Furthermore there exists a non-constant continuous Γ-invariant function q from X_1 to $\mathbb{R}$. We choose a continuous map $h : \mathbb{R} \mapsto W$ such that the map $h \circ q$ is nonzero and $h \circ q(\bar{e}_1) = 0$. Since the σ_e-action of Γ_0 on W is trivial, $S = h \circ q$ is a nonzero continuous Γ_0-equivariant map from (X_1, ρ) to $(L(G_2), \sigma_e)$ and $S(\bar{e}_1) = 0$. Applying Lemma 3.1.1 we see that there exists a non-affine continuous Γ_0-equivariant map from (X_1, ρ) to (X_2, σ).□

Corollary 3.1.4 : *Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and ρ, σ be automorphism actions of a discrete group Γ on X_1 and X_2 respectively. Suppose that either (X_1, ρ) is ergodic or (X_2, σ) is expansive. Then every continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) is an affine map.*

Proof: If (X_1, ρ) is ergodic then there is no non-constant Γ-invariant continuous function from (X_1, ρ) to $\mathbb{R}$. Applying Theorem 3.1.3 we see that there exists no non-affine continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ).

Now suppose that (X_2, σ) is expansive. Choose a metric d on X_2 and an expansive constant $\epsilon > 0$ with respect to d. Define open sets $U \subset X_2$ and $V \subset L(G_2)$ by

$$U = \{x \mid d(\bar{e}_2, x) < \epsilon\} \text{ and } V = (\pi_2 \circ \exp)^{-1}(U).$$

We claim that for every nonzero vector v in $L(G_2)$, the σ_e-orbit of v is infinite. To see this suppose $v_0 \in L(G_2)$ is such that the σ_e-orbit of v_0 is finite. Choose $\alpha > 0$ sufficiently small so that the σ_e-orbit of αv_0 is contained in V and does not intersect the set $\exp^{-1}(D_2) - \{0\}$. Then the σ-orbit of the element $x_0 = \pi_2 \circ \exp(\alpha v_0)$ is contained in U. Since $\bar{e}_2$ is fixed by the action σ, it follows that $x_0 = \bar{e}_2$ i.e. $v_0 = 0$, thus proving the claim. Now from Theorem 3.1.3 it follows that every continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) is an affine map. □

3.2 Automorphism actions on tori

In this Section we will consider automorphism actions of discrete groups on tori. Suppose ρ is an automorphism action of a discrete group Γ on T^m. Then $\widehat{\rho}$ will denote the automorphism action of Γ on the dual of T^m defined by

$$\widehat{\rho}(\gamma)(\chi) = \chi \circ \rho(\gamma) \quad \forall \chi \in \widehat{T^m}, \gamma \in \Gamma.$$

From Theorem 1.4.1 it follows that (T^m, ρ) is ergodic if and only if $\widehat{\rho}$ has no nontrivial finite orbit. If (T^m, ρ) is not ergodic then $F_\rho \subset \widehat{T}^m$ will denote the subgroup consisting of all elements with finite $\widehat{\rho}$-orbit and $\Gamma_\rho \subset \Gamma$ will denote the subgroup defined by

$$\Gamma_\rho = \{\gamma \mid \chi \circ \rho(\gamma) = \chi \ \forall \chi \in F_\rho \}.$$

Since F_ρ is a finitely generated group, it follows that $\Gamma_\rho \subset \Gamma$ is a subgroup of finite index.

Lemma 3.2.1 : *Suppose Γ, ρ and Γ_ρ are as above. Then there exists $\chi_0 \in \widehat{T}^m$ such that*

$$\Gamma_\rho \ = \ \{\gamma \mid \chi_0 \circ \rho(\gamma) = \chi_0\}.$$

Proof : For any χ in F_ρ let $\Gamma_\chi \subset \Gamma$ denote the stabilizer of χ under the Γ-action $\widehat{\rho}$. We claim that for any χ_1, χ_2 in F_ρ, there exists a χ' in F_ρ such that $\Gamma_{\chi'} = \Gamma_{\chi_1} \cap \Gamma_{\chi_2}$. To see this, for $i = 1, 2$ define $A_i \subset \widehat{T}^m$ by

$$A_i \ = \ \{\chi_i \circ \rho(\gamma) - \chi_i \mid \gamma \in \Gamma\}.$$

Since χ_1, χ_2 are elements of F_ρ, both A_1 and A_2 are finite. Choose n large enough so that $nA_1 \cap A_2 = \{0\}$. Define $\chi' = n\chi_1 - \chi_2$. Clearly $\Gamma_{\chi_1} \cap \Gamma_{\chi_2}$ is contained in $\Gamma_{\chi'}$. On the other hand if $\gamma \in \Gamma_{\chi'}$ then

$$n(\chi_1 \circ \rho(\gamma) - \chi_1) = \chi_2 \circ \rho(\gamma) - \chi_2.$$

Since $nA_1 \cap A_2 = \{0\}$, this implies that $\gamma \in \Gamma_{\chi_1} \cap \Gamma_{\chi_2}$.

Suppose $\chi_1, \ldots, \chi_d$ is a finite set of generators of F_ρ. From the above claim it follows that there exists a χ_0 in F_ρ such that

$$\Gamma_{\chi_0} = \Gamma_{\chi_1} \cap \cdots \cap \Gamma_{\chi_d} = \Gamma_\rho.$$

□

Lemma 3.2.2 : *Let $\Gamma_1 \subset \Gamma$ be a subgroup of finite index and f be a Γ_1-invariant continuous map from (T^m, ρ) to a metric space (Y, d). Then f is Γ_ρ-invariant.*

Proof : First let us consider the case when (Y, d) is the complex plane with the usual metric. Let $\widehat{f} : \widehat{T}^m \to \mathbb{C}$ be the Fourier transform of f. Then f is invariant under a subgroup $\Gamma_2 \subset \Gamma$ if and only if $\widehat{f}$ is Γ_2-invariant under the action $\widehat{\rho}$ (see Proposition 1.1.4). Since Γ_ρ acts trivially on F_ρ under the action $\widehat{\rho}$, to prove Γ_ρ-invariance of f it is sufficient to show that $\widehat{f} = 0$ on $\widehat{T}^m - F_\rho$.

Let ϕ be an element of $\widehat{T}^m - F_\rho$. Since Γ_1 is a subgroup of finite index, the Γ_1-orbit of ϕ is infinite. Since f is Γ_1-invariant, $\widehat{f}$ is constant on the Γ_1-orbit of ϕ. Also if $A \subset \widehat{T}^m$ denotes the Γ_1-orbit of ϕ then applying Proposition 1.1.4 we see that

$$\sum_{\chi \in A} |\widehat{f}(\chi)|^2 \leq \sum_{\chi \in \widehat{T}^m} |\widehat{f}(\chi)|^2 = \int_G |f(g)|^2 \, d\lambda_G < \infty,$$

Since Γ_1-orbit of ϕ is infinite, this implies that $\widehat{f}(\phi) = 0$. Thus $\widehat{f} = 0$ on $\widehat{T}^n - F_\rho$.

Now let (Y, d) be any arbitrary metric space and f be a continuous Γ_1-invariant map from T^m to Y. If $C(Y, \mathbb{C})$ denotes the set of all continuous functions from Y to $\mathbb{C}$, then for each g in $C(Y, \mathbb{C})$ the map $g \circ f$ is Γ_1-invariant. Hence from the previous argument it follows that for each g in $C(Y, \mathbb{C})$ the map $g \circ f$ is Γ_ρ-invariant. Since $C(Y, \mathbb{C})$ separates points of Y, we conclude that f is Γ_ρ-invariant. □

Theorem 3.2.3 : *Let Γ be a discrete group, T^m be the m-torus and $X = G/D$ be a nilmanifold. Let ρ, σ be automorphism actions of Γ on T^m and X respectively. Then there exists a non-affine continuous Γ-equivariant map from (T^m, ρ) to (X, σ) if and only if the following two conditions are satisfied.*

a) (T^m, ρ) is not ergodic.

b) There exists a nonzero vector v in $L(G)$ which is fixed by Γ_ρ under the action σ_e.

Proof : Suppose there exists a non-zero continuous Γ-equivariant map from (T^m, ρ) to (X, σ). Then the condition a) follows from Corollary 3.1.4. Also from Lemma 3.1.1 it follows that there exists a non-zero continuous Γ-equivariant map

S from (T^m, ρ) to $(L(G), \sigma_e)$. Applying Lemma 3.1.2 and Lemma 3.2.2 we see that S is Γ_ρ-invariant. This implies that the σ_e-action of Γ_ρ on the image of S is trivial. Now the condition b) follows from the fact that S is nonzero.

Now suppose the conditions a) and b) are satisfied. Fix a finite subset $A = \{\gamma_1, \dots, \gamma_d\}$ of Γ containing exactly one element of each right coset of Γ_ρ. Let W denote the subspace of $L(G)$ which is fixed by $\sigma_e(\gamma)$ for all γ in Γ_ρ. For any Γ_ρ-invariant map $h : T^m \to W$, we define a map $h_A : T^m \to L(G)$ by

$$h_A = \sum_{\gamma \in A} \sigma_e(\gamma^{-1}) \circ h \circ \rho(\gamma).$$

Let α and $\beta = \gamma_0 \alpha$ be two elements of Γ belonging to the same right coset of Γ_ρ. Since h is Γ_ρ-invariant and Γ_ρ-action on W is trivial, it is easy to see that

$$\sigma_e(\beta^{-1}) \circ h \circ \rho(\beta) = \sigma_e(\alpha^{-1}) \circ h \circ \rho(\alpha).$$

Therefore if B is another set containing exactly one element of each right coset of Γ_ρ then $h_A = h_B$. Now it is easy to verify that for all γ in Γ,

$$h_A \circ \rho(\gamma) = \sigma_e(\gamma) \circ h_{A\gamma} = \sigma_e(\gamma) \circ h_A.$$

Hence h_A is a Γ-equivariant map from (T^m, ρ) to $(L(G), \sigma_e)$. We will show that for a suitable choice of h, h_A is nonzero and $h_A(e) = 0$.

Let $\chi_0 \in \widehat{T^m}$ be as in Lemma 3.2.1 and $x_0 \in T^m$ be an element generating a dense subgroup of T^m. Define $c_0, c_1, \dots, c_d \in S^1$ by

$$c_0 = 1, \quad c_i = \chi_0 \circ \rho(\gamma_i)(x_0) \quad \text{for } i = 1, \dots, d.$$

Since the cyclic subgroup generated by x_0 is dense in T^m and $\gamma_1, \dots, \gamma_d$ belong to different right cosets of Γ_ρ, it follows that $1, c_1, \dots, c_d$ are distinct. Let g be a continuous map from S^1 to W such that

$$g(c_d) \neq 0, \quad g(c_i) = 0, \quad i = 0, \dots, d-1.$$

Since the map $g \circ \chi_0 : T^m \to W$ is Γ_ρ-invariant, from the previous argument it follows that the map $S = (g \circ \chi_0)_A$ is a Γ-equivariant map from (T^m, ρ) to $(L(G), \sigma_e)$. Also it is easy to see that $S(e) = 0$ and

$$S(x_0) = \sigma_e(\gamma_d^{-1}) \circ g(c_d) \neq 0.$$

Now applying Lemma 3.1.1 we see that there exists a non-affine Γ-equivariant continuous map from (T^m, ρ) to (X, σ). □

The following corollary generalizes earlier results of [AP] and [Wa1].

Corollary 3.2.4 : *Let A and B be elements of $GL(m,\mathbb{Z})$ and $GL(n,\mathbb{Z})$ respectively. Let k_A be the smallest positive integer k such that A^k has no eigenvalue which is a root of unity other than 1. Then the following two statements are equivalent.*

a) There exists a continuous non-affine map $f : T^m \to T^n$ satisfying $f \circ A = B \circ f$.

b) 1 is an eigenvalue of B^{k_A}.

Proof : Let Γ be the cyclic group, ρ be the Γ-action on T^m generated by A and σ be the Γ-action on T^n generated by B. Then after suitable identifications we have,

$$\widehat{T^m} = \mathbb{Z}^m \text{ and } F_\rho = \{z \in \mathbb{Z}^m \mid A^i(z) = z \text{ for some } i\}.$$

Since no eigenvalue of A^{k_A} is a root of unity other than 1, it follows that A^{k_A} leaves F_ρ pointwise fixed. Furthermore if j is another positive integer such that A^j acts trivially on F_ρ then j is a multiple of k_A. Therefore $\Gamma_\rho = k_A\mathbb{Z}$. It is easy to see that the action $\sigma|_{\Gamma_\rho}$ has a nonzero fixed point in $L(\mathbb{R}^n)$ if and only if 1 is an eigenvalue of B^{k_A}. Now the given assertion follows from Theorem 3.2.3. □

3.3 Rigidity of factor maps

In this Section we will consider the case when Γ is abelian and X_2 is a topological factor of X_1 i.e. there exists a continuous Γ-equivariant map from X_1 onto X_2. Recall that if $X = G/D$ is a nilmanifold then X^0 denotes the torus $G/([G,G]\cdot D)$ and π^0 denotes the projection map from G onto X^0. Also if ρ is an automorphism action of a discrete group Γ on X then ρ^0 will denote the automorphism action of Γ on X^0 induced by ρ.

Proposition 3.3.1 : *Let Γ be an abelian group and V be a finite dimensional vector space over $\mathbb{R}$. Let $\rho : \Gamma \to GL(V)$ be an automorphism action of Γ on V such that the induced Γ-action on the dual of V has a nontrivial fixed point. Then ρ has nontrivial fixed point in V.*

Proof : By passing to the complexification we see that it is enough to prove the analogous statement when V is a finite dimensional vector space over $\mathbb{C}$. In that case after suitable identifications we can assume that $V = V^* = \mathbb{C}^n$, $\rho : \Gamma \to GL(n, \mathbb{C})$ is a homomorphism and $\rho^* : \Gamma \to GL(n, \mathbb{C})$ is the homomorphism defined by $\rho^*(\gamma) = \rho(\gamma^{-1})^T$. Let us consider the special case when with respect to some basis in $\mathbb{C}^n$ each $\rho(\gamma)$ is given by an upper triangular matrix with equal diagonal entries. In this case it is easy to verify that ρ or $\rho*$ has a nonzero fixed vector in $\mathbb{C}^n$ if and only if for any γ in Γ all the diagonal entries of $\rho(\gamma)$ are equal to 1. To prove the general case we note that since Γ is abelian, there exist subspaces $V_1, V_2, \dots, V_k$ of $\mathbb{C}^n$ and homomorphisms $\rho_i : \Gamma \to GL(V_i)$; $i = 1, \dots, k$ such that $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_k$, $\rho = \rho_1 \oplus \cdots \oplus \rho_k$ and each ρ_i satisfies the above condition (see [Ja], pp. 134). □

Proposition 3.3.2 : *Let σ be an automorphism action of a discrete abelian group Γ on a torus T^n. Then (T^n, σ) is ergodic if and only if there is no nonzero element in $L(T^n)$ with finite σ_e-orbit.*

Proof : Since $T^n = \mathbb{R}^n/\mathbb{Z}^n$, σ_e can be realized as a homomorphism from Γ to $GL(n, \mathbb{Z})$, the dual action σ_e^* can be realized as the homomorphism from Γ to $GL(n, \mathbb{Z})$ which takes γ to $\sigma_e(\gamma^{-1})^T$ and $\widehat{\sigma}$ can be identified with $\sigma_e^*|_{\mathbb{Z}^n}$. Suppose (T^n, σ) is ergodic. Let $\Gamma_0 \subset \Gamma$ be a subgroup of finite index. Since $\widehat{\sigma} = \sigma_e^*|_{\mathbb{Z}^n}$, applying Theorem 1.4.1 we see that no element of $\mathbb{Z}^n$ is fixed by Γ_0 under the action σ_e^*. Since $\sigma_e^*(\gamma) \in GL(n, \mathbb{Z})$ for all γ, this implies that no nonzero element of $\mathbb{R}^n$ is fixed by Γ_0 under the action σ_e^*. Applying Proposition 3.3.1 we see that no nonzero element of $\mathbb{R}^n$ is fixed by Γ_0 under the action σ_e. Now suppose (T^n, σ) is not ergodic. Then there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ and a nonzero point z in $\mathbb{Z}^n$ such that z is fixed by Γ_0 under the action σ_e^*. Now from

Proposition 3.3.1 we conclude that there exists a nonzero element in $\mathbb{R}^n$ which is fixed by Γ_0 under the action σ_e. □

Theorem 3.3.3 : *Let X_1, X_2 be nilmanifolds and ρ, σ be automorphism actions of a discrete abelian group Γ on X_1 and X_2 respectively. Suppose that (X_2, σ) is a factor of (X_1, ρ) and either $(X_1, \rho) = (X_2, \sigma)$ or X_2 is a torus. Then there is a non-affine continuous Γ-equivariant map from X_1 to X_2 if and only if (X_2, σ) is not ergodic.*

Proof : Suppose (X_2, σ) is not ergodic. By our assumption there exists a continuous Γ-equivariant map f from (X_1, ρ) onto (X_2, σ). If f is non-affine then there is nothing to prove. Therefore we may assume that there exists a $g_0 \in G_2$ and a continuous homomorphism $\theta : G_1 \to G_2$ such that $f(gD_1) = g_0\theta(g)D_2$ for all g in G_1. Since f is surjective and Γ-equivariant, so is θ. Let θ^0 denote the homomorphism from X_1^0 to X_2^0 induced by θ. Then θ^0 is surjective. Since (X_2, σ) is not ergodic, from Proposition 1.4.4 it follows that (X_2^0, σ^0) is not ergodic. Let ϕ be an element of X_2^0 such that $\widehat{\sigma^0}$-orbit of ϕ is finite. Since θ^0 is Γ-equivariant, it follows that $\widehat{\rho^0}$-orbit of $\phi \circ \theta^0$ is also finite, which implies that (X_1^0, ρ_0) is not ergodic. Also for any γ in Γ_ρ,

$$\phi \circ \sigma^0(\gamma) \circ \theta^0 = \phi \circ \theta^0 \circ \rho^0(\gamma) = \phi \circ \theta^0.$$

Since θ^0 is a surjective map, this implies that $\phi \circ \sigma^0(\gamma) = \phi$ for all γ in Γ_ρ. Let π_2^0 denote the projection map from G_2 onto X_2^0 and let q denote the map $\phi \circ \pi_2^0 \circ \exp$. Then $dq : L(G_2) \to \mathbb{R}$ is an element of the dual of $L(G_2)$ such that $dq \circ \sigma_e(\gamma) = dq$ for all γ in Γ_ρ. Now from Proposition 3.3.1 it follows that there exists a nonzero point in $L(G_2)$ which is fixed by Γ_ρ under the action σ_e. Applying Theorem 3.2.3 we see that there exists a continuous non-affine Γ-equivariant map h from (X_1^0, ρ^0) to (X_2, σ). If π_1^0 denotes the projection map from X_1 to X_1^0 then it is easy to see that $h \circ \pi_1^0$ is a continuous non-affine Γ-equivariant map h from (X_1, ρ) to (X_2, σ).

Now suppose (X_2, σ) is ergodic. Since by our assumption either $(X_1, \rho) = (X_2, \sigma)$ or X_2 is a torus, from Proposition 3.3.2 it follows that either (X_1, ρ) is ergodic or there is no non-zero element in $L(G_2)$ whose σ_e-orbit is finite. Applying Theorem 3.1.3 we conclude that every continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) is an affine map. □

The following examples show that Theorem 3.3.3 does not hold if any of the assumptions in the hypothesis is dropped.

Example 3.3.4 : Let Γ be the cyclic group and ρ, σ be the automorphism actions of Γ on $\mathbb{R}/\mathbb{Z}$ generated by the identity automorphism and the automorphism $x \to -x$ respectively. Then it is easy to see that $\Gamma_\rho = \Gamma$ and no nonzero element of $L(\mathbb{R})$ is fixed by Γ_ρ under the action σ_e. Now applying Theorem 3.2.3 we conclude that there is no non-affine continuous Γ-equivariant map from (S^1, ρ) to (S^1, σ). Note that in this case Γ is abelian and neither of the two actions is ergodic.

Example 3.3.5 : Fix $n \geq 3$ and define a subgroup Γ of $GL(n, \mathbb{Z})$ by

$$\Gamma = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL(n-1, \mathbb{Z}), b \in \mathbb{Z}^{n-1} \right\}.$$

Let ρ denote the natural action of Γ on $\mathbb{R}^n/\mathbb{Z}^n$. Then it is easy to see that for any $x = (x_1, \ldots, x_n)$ in $L(\mathbb{R}^n)$, the ρ_e-orbit of x is given by the set

$$\{Ay + x_n b \mid A \in GL(n-1, \mathbb{Z}), b \in \mathbb{Z}^{n-1}\} \text{ where } y = (x_1, \ldots, x_{n-1}).$$

Hence for every nonzero x in $L(\mathbb{R}^n)$, the ρ_e-orbit of x is unbounded. Applying Theorem 3.1.3 we see that there is no non-affine continuous Γ-equivariant map from (T^n, ρ) to (T^n, ρ). Note that in this case (T^n, ρ) is not ergodic since the vector $x_0 = (0, \ldots, 0, 1)$ is fixed by the dual action ρ^*.

Example 3.3.6 : Suppose $X = G/D$ where G and D are defined by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}, \quad D = \left\{ \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \mid p, q, r \in \mathbb{Z} \right\}.$$

Let A be an ergodic automorphism of G/D. If G_0 denotes the center of G then it is easy to see that $G_0/(G_0 \cap D)$ is isomorphic to S^1. Hence replacing A by A^2

if necessary we may assume that A acts trivially on G_0. Define a nilmanifold X_1 and an automorphism A_1 of X_1 by $X_1 = X \times S^1$, $A_1 = A \times \mathrm{Id}$. Let ρ_1 and ρ denote the automorphism actions of $\mathbb{Z}$ on X_1 and X generated by A_1 and A respectively. Then (X, ρ) is a factor of (X_1, ρ_1). Let $\pi : X_1 \to S^1$ be the projection map and $h : S^1 \to L(G_0)$ be any nonzero continuous map such that $h(e) = 0$. Then $h \circ \pi(\overline{e_1}) = 0$ and $h \circ \pi$ is a nonzero Γ-equivariant map from X_1 to $L(G)$. Applying Lemma 3.1.1 we see that there exists a non-affine continuous Γ-equivariant map from X_1 to X.

Chapter 4

Conjugacy of flows on compact Lie groups

In this Chapter our goal is to classify one-parameter automorphism flows on compact connected Lie groups up to topological conjugacy.

Definition : Let G be a Lie group and ϕ, ϕ' be two one-parameter subgroups of $\mathrm{Aut}(G)$. We say that ϕ and ϕ' are *topologically conjugate at the identity* if there exists an invertible linear transformation $S : L(G) \to L(G)$ such that

$$d\phi_t \circ S = S \circ d\phi'_t \quad \forall t \in \mathbb{R}.$$

In Section 1 we classify one-parameter automorphism flows on compact connected Lie groups up to topological conjugacy at the identity. In Section 2 we prove that on certain compact Lie groups two one-parameter automorphism flows are topologically conjugate at the identity if and only if they are algebraically conjugate. In Section 3 applying results of the previous Sections we prove that if $G = \mathrm{Sp}(n), SO(n+1)$ or $\mathrm{Spin}(n)$ for some $n \geq 1$ then two one-parameter automorphism flows on G are topologically conjugate if and only if they are algebraically conjugate.

4.1 Topological conjugacy at the identity

For a Lie group G, $\sigma : G \to \mathrm{Int}(G)$ will denote the homomorphism defined by $\sigma(g)(x) = gxg^{-1}$ for all $g, x \in G$. Also for any v in $L(G)$, ϕ^v will denote the one-parameter subgroup of $\mathrm{Aut}(G)$ defined by $\phi_t^v = \sigma(\exp(tv))$.

Proposition 4.1.1 : *Let G be a compact connected Lie group and ϕ be a one-parameter subgroup of $\mathrm{Aut}(G)$. Then $\phi = \phi^v$ for some v in $L(G)$.*

Proof : Since G is compact, $\mathrm{Int}(G)$ is the connected component of $\mathrm{Aut}(G)$ containing the identity. Hence the image of ϕ is contained in $\mathrm{Int}(G)$. Since σ is surjective, there exists v in $L(G)$ such that $\phi_t = \sigma(\exp(tv)) = \phi_t^v$ for all t in $\mathbb{R}$. □

Proposition 4.1.2 : *Let G be a compact connected Lie group and ϕ^v, ϕ^w be one-parameter subgroups of $\mathrm{Aut}(G)$. Then ϕ^v and ϕ^w are topologically conjugate at the identity if and only if $\mathrm{ad}(v)$ and $\mathrm{ad}(w)$ are linearly conjugate.*

Proof : For any x in $L(G)$ we have

$$d\phi_t^x = d\sigma(\exp tx) = \mathrm{Ad}(\exp tx) = \exp(t\,\mathrm{ad}(x)).$$

Hence ϕ^v and ϕ^w are topologically conjugate at the identity if and only if there exists an invertible linear map $P : L(G) \mapsto L(G)$ such that

$$P \circ \exp(t\,\mathrm{ad}(v)) = \exp(t\,\mathrm{ad}(w)) \circ P \quad \forall t \in \mathbb{R}.$$

Differentiating at $t = 0$ we see that the last identity holds if and only if $\mathrm{ad}(v)$ and $\mathrm{ad}(w)$ are linearly conjugate. Conversely if P is a linear conjugacy between $\mathrm{ad}(v)$ and $\mathrm{ad}(w)$ then it is easy to see that $P \circ d\phi_t^v = d\phi_t^w \circ P$ for all t in $\mathbb{R}$. □

Recall that for a finite set A, $|A|$ denotes the cardinality of A. We will denote the group of all permutations on $\{1, \dots, n\}$ by S_n. If $\pi \in S_n$ and $x \in \mathbb{R}^n$ then $\pi \cdot x$ will denote the vector in $\mathbb{R}^n$ defined by $(\pi \cdot x)_i = x_{\pi(i)}$ $\forall i = 1, \dots, n$.

Lemma 4.1.3 : *Let $x = (x_1, \dots x_n)$ and $y = (y_1, \dots y_n)$ be two vectors in $\mathbb{R}^n$ such that* *a)* $|\{ i \mid tx_i \in \mathbb{Z} \}| = |\{ i \mid ty_i \in \mathbb{Z} \}| \quad \forall t \in \mathbb{R}$ and

b) there exist $\pi_1, \pi_2 \in S_n$ *such that* $\pi_1 \cdot x = -x$ *and* $\pi_2 \cdot y = -y$.

Then there exists $\pi \in S_n$ *such that* $\pi \cdot x = y$.

Proof : For each $\alpha \in \mathbb{R}$ let n_α and n'_α be the multiplicities of α in $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. Also, for all α let $d_\alpha = n_\alpha - n'_\alpha$. Then it is easy to see that for any nonzero α in $\mathbb{R}$

$$\begin{aligned} \sum_j n_{j\alpha} &= \sum_j |\{ i \mid x_i = j\alpha \}| \\ &= |\{ i \mid x_i/\alpha \in \mathbb{Z} \}| \\ &= |\{ i \mid y_i/\alpha \in \mathbb{Z} \}| \\ &= \sum_j n'_{j\alpha}. \end{aligned}$$

Hence for all nonzero α in $\mathbb{R}$,

$$\sum_j d_{j\alpha} = 0.$$

Since $d_\alpha = 0$ for all but finitely many α's, there exists a $\alpha_0 \in \mathbb{R}$ such that $d_{j\alpha_0} = 0$ whenever $j \neq 0$. Putting $\alpha = \alpha_0$ in the previous identity we see that $d_0 = 0$. Now to prove the given assertion it is enough to prove that $d_\alpha = 0$ for all α. Suppose otherwise. Since $d_\alpha = 0$ for all but finitely many α's, there exists a $\alpha_0 \in \mathbb{R}$ such that $d_{\alpha_0} \neq 0$ and $d_\beta = 0$ whenever $|\beta| > |\alpha_0|$. Applying the previous identity we obtain that

$$d_{\alpha_0} + d_0 + d_{-\alpha_0} = 0.$$

From condition b) it follows that for all $\alpha \in \mathbb{R}$, $n_\alpha = -n_{-\alpha}$ and $n'_\alpha = -n'_{-\alpha}$. Hence $d_{\alpha_0} = d_{-\alpha_0}$ and since $d_0 = 0$, this implies that $2d_{\alpha_0} = 0$. This gives a contradiction. □

Theorem 4.1.4 : *Let G be a compact connected Lie group and let ϕ_1 and ϕ_2 be two topologically conjugate one-parameter automorphism flows on G. Then ϕ_1 and ϕ_2 are topologically conjugate at the identity.*

Proof : Using Proposition 4.1.1 we see that there exist $v_1, v_2 \in L(G)$ such that $\phi_1 = \phi^{v_1}$ and $\phi_2 = \phi^{v_2}$. From Proposition 4.1.2 it follows that ϕ_1 and ϕ_2 are topologically conjugate at the identity if and only if $\mathrm{ad}(v_1)$ and $\mathrm{ad}(v_2)$ are linearly conjugate endomorphisms of $L(G)$. Now for $j = 1, 2$ and for all $t \in \mathbb{R}$ we define a closed subgroup H_j^t of G by

$$H_j^t = \{\ g \in G \mid \phi_j(t)(g) = g\ \}.$$

Then the Lie algebra of H_j^t is given by

$$L(H_j^t) = \{w \in L(G) \mid \mathrm{Ad}(\exp tv_j)(w) = w\ \}.$$

If $i\lambda_1^j, \ldots, i\lambda_k^j$ denote the eigenvalues of $\mathrm{ad}(v_j)$ then the eigenvalues of $\mathrm{Ad}(\exp tv_j)$ are given by $e^{2\pi i t\lambda_1^j}, \ldots, e^{2\pi i t\lambda_k^j}$. Hence for all t in $\mathbb{R}$,

$$\begin{aligned}\dim(H_j^t) &= \dim(L(H_j^t)) \\ &= \dim(\{w \in L(G) \mid \mathrm{Ad}(\exp tv_j)(w) = w\ \}) \\ &= |\{\ i \mid t\lambda_i^j \in \mathbb{Z}\ \}|.\end{aligned}$$

Since ϕ_1 and ϕ_2 are topologically conjugate it follows that for all t in $\mathbb{R}$, H_1^t and H_2^t are homeomorphic. Hence for all t in $\mathbb{R}$, $\dim(H_1^t) = \dim(H_2^t)$ i.e.

$$|\{\ i \mid t\lambda_i^1 \in \mathbb{Z}\ \}| = \{\ i \mid t\lambda_i^2 \in \mathbb{Z}\ \}|.$$

Define $x_1,\ x_2 \in \mathbb{R}^k$ by $x_1 = (\lambda_1^1, \ldots, \lambda_k^1)$ and $x_2 = (\lambda_1^2, \ldots, \lambda_k^2)$. We fix an Ad-invariant inner product on $L(G)$. Note that $\mathrm{ad}(x)$ is skew symmetric with respect to this inner product for all x in $L(G)$. This implies that for all x in $L(G)$ and α in $\mathbb{R}$ the multiplicities of $i\alpha$ and $-i\alpha$ in the spectrum of $\mathrm{ad}(x)$ are the same. Hence there exists $\pi_1,\ \pi_2 \in S_n$ such that $\pi_1 \cdot x_1 = -x_1$ and $\pi_2 \cdot x_2 = -x_2$. Now applying Lemma 4.1.3 we see that $\lambda_1^1, \ldots, \lambda_k^1$ are same as $\lambda_1^2, \ldots, \lambda_k^2$ up to a permutation. Since $\mathrm{ad}(x_1)$ and $\mathrm{ad}(x_2)$ are skew symmetric with respect to an inner product on $L(G)$, it follows that $\mathrm{ad}(x_1)$ and $\mathrm{ad}(x_2)$ are linearly conjugate endomorphisms of $L(G)$. □

Now let G be a compact connected Lie group and T be a maximal torus of G. Let $R_1, \ldots, R_k : L(T) \to \mathbb{R}$ be the real roots of G with respect to T. Then for any

$v, w \in L(G)$, $\mathrm{ad}(v)$ and $\mathrm{ad}(w)$ are conjugate if and only if $R_1(v), \ldots, R_k(v)$ are same as $R_1(w), \ldots, R_k(w)$ up to a permutation. Theorem 4.1.4 therefore implies the following.

Corollary 4.1.5 : *Let G be a compact connected Lie group and T be a maximal torus of G. Let v, w be two elements of $L(T)$ such that ϕ^v and ϕ^w are topologically conjugate. Then $R_1(v), \ldots, R_k(v)$ are same as $R_1(w), \ldots, R_k(w)$ up to a permutation, where $R_1, \ldots, R_k$ are the real roots of G with respect to T.*

The following example shows that Theorem 4.1.4 does not hold for noncompact groups.

Example 4.1.6 : Suppose $G = \mathbb{R}$ and ϕ, ϕ' are one-parameter subgroups of $\mathrm{Aut}(G)$ defined by

$$\phi_t(x) = e^t x, \;\; \phi_t'(x) = e^{2t} x.$$

Then after suitable identifications we have, $L(G) = \mathbb{R}$ and $d\phi_t = \phi_t$, $d\phi_t' = \phi_t'$ for all t in $\mathbb{R}$. Since ϕ and ϕ' are not linearly conjugate, the automorphism flows on $\mathbb{R}$ induced by ϕ and ϕ' are not topologically conjugate at the identity. On the other hand it is easy to see that the map $x \mapsto x^2$ is a topological conjugacy between these two flows.

4.2 Algebraic conjugacy

In this Section we prove that two one-parameter automorphism flows on certain compact Lie groups are topologically conjugate at the identity only if they are algebraically conjugate. We begin with the following lemma.

Lemma 4.2.1 : *Let G be a connected Lie group and Z be the center of G. Then for $v, w \in L(G)$ the automorphism flows ϕ^v and ϕ^w are algebraically conjugate if and only if $w = w_0 + d\tau(v)$ for some $w_0 \in L(Z)$ and $\tau \in \mathrm{Aut}(G)$.*

Proof : Suppose $w = w_0 + d\tau(v)$ for some $w_0 \in L(Z)$ and $\tau \in \mathrm{Aut}(G)$. Then for all t in $\mathbb{R}$,

$$\begin{aligned}\exp(tw) &= \exp(tw_0)\cdot\exp(d\tau(v)) \\ &= \exp(tw_0)\cdot\tau(\exp tv).\end{aligned}$$

Hence for all $t \in \mathbb{R}$,

$$\begin{aligned}\phi_t^w \circ \tau &= \sigma(\exp tw)\circ\tau \\ &= \sigma(\exp tw_0)\circ\sigma(\tau(\exp tv))\circ\tau.\end{aligned}$$

Since $w_0 \in L(Z)$, $\sigma(\exp tw_0)$ is the identity map for all $t \in \mathbb{R}$. Also it is easy to check that for any $\tau \in \mathrm{Aut}(G)$ and $g \in G$, $\sigma(\tau(g))\circ\tau = \tau\circ\sigma(g)$. Therefore from the previous identity it follows that

$$\phi_t^w \circ \tau = \sigma(\tau(\exp tv))\circ\tau = \tau\circ\phi_t^v \quad \forall t \in \mathbb{R},$$

i.e. ϕ^v and ϕ^w are algebraically conjugate.

Now suppose that there exists a τ in $\mathrm{Aut}(G)$ such that, $\tau\circ\phi_t^v\circ\tau^{-1} = \phi_t^w$ for all t in $\mathbb{R}$. Then it is easy to see that for all $t \in \mathbb{R}$,

$$\begin{aligned}\sigma(\exp tw) &= \tau\circ\phi_t^v\circ\tau^{-1} \\ &= \tau\circ\sigma(\exp tv)\circ\tau^{-1} \\ &= \sigma(\tau(\exp tv))\circ\tau\circ\tau^{-1} \\ &= \sigma(\tau(\exp tv)).\end{aligned}$$

We define $\alpha, \beta, \gamma : \mathbb{R} \to G$ by

$$\alpha(t) = \tau(\exp tv) = \exp(t\, d\tau(v)),\ \ \beta(t) = \exp tw,\ \text{ and } \gamma(t) = \alpha(t)\cdot\beta(t)^{-1}.$$

Since the kernel of σ is Z, it follows that that the image of γ lies in Z. Differentiating γ at $t = 0$ we conclude that $d\tau(v) - w = \gamma'(0) \in L(Z)$. □

Let G be a compact connected Lie group. We introduce two equivalence relations $\sim$ and $\approx$ on $L(G)$ as follows : For v, w in $L(G)$ we define

1) $v \sim w$ if $\mathrm{ad}(v)$ and $\mathrm{ad}(w)$ are linearly conjugate.

2) $v \approx w$ if $w = w_0 + d\tau(v)$ for some $w_0 \in L(Z)$, $\tau \in \mathrm{Aut}(G)$.

Definition : Let G be a compact connected Lie group. Then G is said to be *rigid for automorphism flows* if for $v, w \in L(G)$, $v \sim w$ implies $v \approx w$.

Theorem 4.2.2 : *Let G be a compact connected Lie group. Then G is rigid for automorphism flows if and only if for any two one-parameter automorphism flows ϕ_1 and ϕ_2 on G the following statements are equivalent.*

1) ϕ_1 and ϕ_2 are algebraically conjugate.
2) ϕ_1 and ϕ_2 are topologically conjugate.
3) ϕ_1 and ϕ_2 are topologically conjugate at the identity.

Proof : Suppose G is rigid for automorphism flows. Let ϕ_1 and ϕ_2 be any two one-parameter automorphism flows on G which are topologically conjugate at the identity. By Proposition 4.1.1 there exist v and w such that $\phi_1 = \phi^v$ and $\phi_2 = \phi^w$. Applying Proposition 4.1.2 we see that $v \sim w$. Since G is rigid for automorphism flows from Lemma 4.2.1 it follows that ϕ_1 and ϕ_2 are algebraically conjugate. Therefore 3) implies 1). Now applying Theorem 4.1.4 we see that 1), 2) and 3) are equivalent.

Now assume that 1), 2) and 3) are equivalent for any two one-parameter automorphism flows on G. Let v, w be two elements of $L(G)$ such that $v \sim w$. Then from Proposition 4.1.2 it follows that ϕ^v and ϕ^w are topologically conjugate at the identity. From our assumption it follows that ϕ^v and ϕ^w are algebraically conjugate. Applying Lemma 4.2.1 we see that $v \approx w$. □

In the next Section we give several examples of compact Lie groups which are rigid for automorphism flows. Here we make the following two simple observations about compact Lie groups satisfying this property.

Proposition 4.2.3 : *Suppose G is a compact connected Lie group, $T \subset G$ is a maximal torus, $C \subset L(T)$ is a Weyl chamber and $R_1, \dots, R_k : L(T) \to \mathbb{R}$ are the real roots of G with respect to T. Furthermore assume that for any two points*

$v, w \in \overline{C}$, *the numbers* $R_1(v), \ldots, R_k(v)$ *and* $R_1(w), \ldots, R_k(w)$ *are same up to a permutation if and only if* $v = w$. *Then* G *is rigid for automorphism flows.*

Proof : Let v, w be two elements of $L(G)$ such that $v \sim w$. Then applying Proposition 1.3.2 we see that there exist $v_1, w_1 \in \overline{C}$ and inner automorphisms τ_1, τ_2 such that $v = d\tau_1(v_1)$ and $w = d\tau_2(w_1)$. Since for x, y in $L(G)$, $x \approx y$ implies $x \sim y$ and $\sim$ is an equivalence relation, we deduce that $v_1 \sim w_1$. Now from our hypothesis it follows that $v_1 = w_1$. Since $v \approx v_1$ and $w \approx w_1$ we conclude that $v \approx w$. □

Proposition 4.2.4 : *Let* G *be a compact connected Lie group and let* $K \subset G$ *be a finite central subgroup which is invariant under* $\mathrm{Aut}(G)$. *Then* G *is rigid for automorphism flows if and only if* G/K *is rigid for automorphism flows.*

Proof : If $\pi : G \to G/K$ is the projection map then the map $d\pi$ is a Lie algebra isomorphism from $L(G)$ to $L(G/K)$. It follows that $d\pi$ preserves the relation $\sim$. Since K is a finite central subgroup which is invariant under $\mathrm{Aut}(G)$, there is an isomorphism $i : \mathrm{Aut}(G) \mapsto \mathrm{Aut}(G/K)$ satisfying $i(\theta) \circ \pi = \pi \circ \theta$ for all θ in $\mathrm{Aut}(G)$. Since $d\pi$ is a Lie algebra isomorphism, this implies that for $v, w \in L(G)$, $v \approx w$ if and only if $d\pi(v) \approx d\pi(w)$. Hence G is rigid for automorphism flows if and only if G/K is rigid for automorphism flows. □

4.3 Some examples

Example 4.3.1 : $\mathrm{Sp}(n), n \geq 1$.

Let $\mathbb{H}$ denote the standard quaternion algebra and $< , >$ denote the standard symplectic scalar product on $\mathbb{H}^n$. Then $\mathrm{Sp}(n)$ is defined as the set of all elements in $GL(n, \mathbb{H})$ which preserves $< , >$. We choose a maximal torus T and a fundamental Weyl chamber C of $L(T)$. Then after suitable identifications we have $L(T) = \mathbb{R}^n$ and $\overline{C} = \{x \in \mathbb{R}^n \mid x_n \geq x_{n-1} \geq \cdots \geq x_1 \geq 0\}$. Furthermore For any x in $L(T)$, $R_1(x), \ldots, R_k(x)$ are given by $\pm 2x_i$, $1 \leq i \leq n$ and $\pm x_i \pm x_j$, $1 \leq i < j \leq n$.

We shall show that these groups are rigid for automorphism flows. Take x and y in $\overline{C}$ such that $R_1(x), \ldots, R_k(x)$ is a permutation of $R_1(y), \ldots, R_k(y)$. We claim that $x = y$. Suppose not. Note that $2x_n$ = maximum of $R_1(x), \ldots, R_k(x)$ and $2y_n$ = maximum of $R_1(y), \ldots, R_k(y)$. Hence $x_n = y_n$. Choose l such that $x_l \neq y_l$ but $x_i = y_i$ for $i > l$. Remove the numbers $x_i + x_j$, $l < i < j \leq n$ and $2x_i$, $l < i \leq n$ from $R_1(x), \ldots, R_k(x)$ and denote the remaining numbers by $x_1', \ldots, x_t'$. Similarly remove the numbers $y_i + y_j$, $l < i < j \leq n$ and $2y_i$, $l < i \leq n$ from $R_1(y), \ldots, R_k(y)$ and denote the remaining numbers by $y_1', \ldots, y_t'$. Note that maximum of $x_1', \ldots, x_t'$ is $x_n + x_l$ and maximum of $y_1', \ldots, y_t'$ is $y_n + y_l$. On the other hand $y_1', \ldots, y_t'$ is a permutation of $x_1', \ldots, x_t'$. Hence $x_n + x_l = y_n + y_l$ i.e. $x_l = y_l$, which is a contradiction. Now from the above claim and Proposition 4.2.3 it follows that $\mathrm{Sp}(n)$ is rigid for automorphism flows.

Example 4.3.2 : $SO(2n+1), n \geq 1$.

We recall that $SO(2n+1)$ is the group of all orthogonal linear transformations of $\mathbb{R}^{2n+1}$ with unit determinant. After suitable identifications we have $L(T) = \mathbb{R}^n$ and $\overline{C} = \{x \in \mathbb{R}^n \mid x_n \geq x_{n-1} \geq \cdots \geq x_1 \geq 0\}$. Also for any x in $L(T)$, $R_1(x), \ldots, R_k(x)$ are given by $\pm x_i$, $1 \leq i \leq n$ and $\pm x_i \pm x_j$, $1 \leq i < j \leq n$.

Lemma 4.3.3 : *Let $x_1 \leq x_2 \leq \ldots \leq x_m$ and $y_1 \leq y_2 \leq \ldots \leq y_m$ be two finite sequences in $\mathbb{R}$ such that the numbers $x_i + x_j$, $1 \leq i < j \leq m$ are same as $y_i + y_j$, $1 \leq i < j \leq m$ up to a permutation. Then either $m = 2^l$ for some l or $x_i = y_i \ \forall i = 1, \ldots, m$.*

Proof : Define two functions $f_1, f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f_1(t) = \sum_1^m t^{x_i}, \quad f_2(t) = \sum_1^m t^{y_i}.$$

Under the hypothesis as in the lemma, we have

$$\sum_{i<j} t^{x_i + x_j} = \sum_{i<j} t^{y_i + y_j}.$$

This implies that for all t in $\mathbb{R}^-$. $f_1^2(t) - f_2^2(t) = f_1(t^2) - f_2(t^2)$. Define $h, g : \mathbb{R}^+ \to \mathbb{R}$ by $h = f_1 - f_2$, $g = f_1 + f_2$. Then $h(t)g(t) = h(t^2)$ for all t in $\mathbb{R}$. Let $h^{(k)}$ denote the k'th derivative of h. We consider the following two cases separately.

1. $h^{(k)}(1) \neq 0$ for some k.

Choose $l \geq 0$ such that $h^{(i)}(1) = 0$ for $i \leq l$ and $h^{(l+1)}(1) \neq 0$. Now differentiating $l+1$ times the identity $h(t)g(t) = h(t^2)$, and using the fact $h^{(i)}(1) = 0$ for $i \leq l$ we get

$$h^{(l+1)}(1)g(1) = 2^{l+1}h^{(l+1)}(1).$$

Hence $g(1) = 2^{l+1}$. But $g(1) = 2m$. So $m = 2^l$.

2. $h^{(k)}(1) = 0 \;\; \forall k$.

In this case we have

$$\sum x_i(x_i - 1)\cdots(x_i - k) \;\; = \;\; \sum y_i(y_i - 1)\cdots(y_i - k) \;\; \forall k.$$

This implies that

$$\sum x_i^k = \sum y_i^k \;\; \forall k.$$

We shall deduce from this fact that $x_i = y_i \;\; \forall i$. Suppose this is not the case. Interchanging x and y if necessary we can find j such that $x_j > y_j$ and $x_i = y_i$ for $i > j$. We choose a $c \in \mathbb{R}$ such that

$$x_i + c, \; y_i + c \geq 0 \quad \forall i = 1, \ldots, m.$$

Now for large k,

$$(x_j + c)^k > j(y_j + c)^k \geq \sum_{i \leq j}(y_i + c)^k.$$

This implies that

$$\sum_1^m (x_i + c)^k > \sum_1^m (y_i + c)^k,$$

which contradicts the fact that $\sum x_i^k = \sum y_i^k \;\; \forall k$. This proves the lemma. □

Now we will prove that $SO(2n+1)$ is rigid for automorphism flows. Take v, w in the closure of C such that $v \sim w$. Put $m = 2n+1$ and define $x_1 \leq x_2 \leq \ldots \leq x_m$ and $y_1 \leq y_2 \leq \ldots \leq y_m$ by

$$\begin{aligned} (x_1, \ldots, x_m) &= (-v_n, \ldots, -v_1, 0, v_1, \ldots, v_n), \\ (y_1, \ldots, y_m) &= (-w_n, \ldots, -w_1, 0, w_1, \ldots, w_n) . \end{aligned}$$

Since $v \sim w$, the numbers $x_i + x_j$, $1 \leq i < j \leq m$ and $y_i + y_j$, $1 \leq i < j \leq m$ are same up to a permutation. Since m is odd, Lemma 4.3.3 implies $x_i = y_i$ for all $i = 1, .., m$ i.e. $v = w$. Now applying Proposition 4.2.3 we see that $SO(2n+1)$ is rigid for automorphism flows.

Example 4.3.4 : Spin$(2n+1)$, $n \geq 1$.

For any k , Spin(k) is defined as the simply connected covering group of $SO(k)$. Let p_k denote the covering map of Spin(k) onto $SO(k)$ and Z_k denote the center of Spin(k). Then $SO(k)$ can be identified with Spin$(k)/Z_k$ with p_k as the projection map of Spin(k) onto Spin$(k)/Z_k$. Now from Proposition 4.2.4 it follows that Spin$(2n+1)$ is rigid for automorphism flows for all n.

We will conclude with a few examples of compact Lie groups which are not rigid for automorphism flows. We will use the following fact.

Proposition 4.3.5 : *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two elements of $\mathbb{R}^n$ such that*

$$\sum_1^n t^{x_i} = \sum_1^n t^{y_i} \quad \forall t \in \mathbb{R}^+.$$

Then $x_1, \ldots, x_n$ are same as $y_1, \ldots, y_n$ up to a permutation.

Proof : Clearly it is enough to consider the case when $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$. Suppose $x_i \neq y_i$ for some i. Then interchanging x and y if necessary we may assume that there exists a j such that $x_j > y_j$ and $x_i = y_i$ for

$i \geq j+1$. Now from our hypothesis it follows that

$$\sum_1^j t^{x_i} = \sum_1^j t^{y_i} \quad \forall t \in \mathbb{R}^+.$$

We choose t_0 large enough so that $t_0^{x_j} > n t_0^{y_j}$. Then putting $t = t_0$ in the previous identity we see that

$$\sum_1^j t_0^{x_i} \geq t_0^{x_j} > n t_0^{y_j} \geq \sum_1^j t_0^{y_i},$$

which gives a contradiction. □

Example 4.3.6 :

Now we will take G to be $SO(2n)$, $n \geq 1$. After suitable identifications we have $L(T) = \mathbb{R}^n$, $\overline{C} = \{ x \mid x_n \geq \cdots \geq x_2 \text{ and } x_2 \geq |x_1| \}$. Also for any x in $L(T)$, $R_1(x), \ldots, R_k(x)$ are given by $\pm x_i \pm x_j$, $1 \leq i < j \leq n$.

For any $m \geq 2$ define two polynomial P, Q by

$$\begin{aligned} P(t) &= \tfrac{1}{2}t^{-m}(t+1)^{2m} + \tfrac{1}{2}t^{-m}(t-1)^{2m} \text{ and} \\ Q(t) &= \tfrac{1}{2}t^{-m}(t+1)^{2m} - \tfrac{1}{2}t^{-m}(t-1)^{2m}. \end{aligned}$$

It is easy to check that for all t in $\mathbb{R}$

$$\begin{aligned} (P-Q)(t)(P+Q)(t) &= t^{-2m}(t-1)^{2m}(t+1)^{2m} \\ &= t^{-2m}(t^2-1)^{2m} \\ &= (P-Q)(t^2). \end{aligned}$$

This implies that $P^2(t) - P(t^2) = Q^2(t) - Q(t^2)$ for all t in $\mathbb{R}$. Note that since $P(t) = P(1/t)$ and $Q(t) = Q(1/t)$ there exist $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$ in $\overline{C}$ such that

$$P(t) = \sum_1^m t^{x_i} + \sum_1^m t^{-x_i} \quad Q(t) = \sum_1^m t^{y_i} + \sum_1^m t^{-y_i}.$$

Hence for all t in $\mathbb{R}$,

$$\begin{aligned} \sum_1^k t^{R_i(x)} &= P^2(t) - P(t^2) - 2m \\ &= Q^2(t) - Q(t^2) - 2m \\ &= \sum_1^k t^{R_i(y)}. \end{aligned}$$

Applying Proposition 4.3.5 we see that $R_1(x), \ldots, R_k(x)$ and $R_1(y), \ldots, R_k(y)$ are same up to a permutation i.e. $x \sim y$.

Now if $G = SO(2n)$, $n \geq 1$, $L(Z) = 0$ and $\mathrm{Aut}(G) = \mathrm{Int}(G)$. So in this case $x \approx y$ if and only if $y = d\tau(x)$ for some inner automorphism τ of G. Since $y, x \in \overline{C}$, $y = d\tau(x)$ for some inner automorphism τ of G if and only if $x = y$. Hence x and y do not belong to the same equivalence class of $\approx$.

Example 4.3.7 :

Let us take G to be $U(n)$, the set of all unitary transformations of $\mathbb{C}^n$. Then after suitable identifications we have $L(T) = \mathbb{R}^n$ and $L(Z) = \{(y, ..., y) \mid y \in \mathbb{R}\}$. Also for any x in $L(T)$ the numbers $R_1(x), \ldots, R_k(x)$ are given by $x_i - x_j$, $i \neq j$, $1 \leq i, j \leq n$.

We choose two distinct positive integers k, $l > 0$ and define two functions p, $q : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$p(t) = (t+k)(t+l), \quad q(t) = t^{-1}(1+kt)(t+l).$$

If $m = (k+1)(l+1)$ then we choose integers $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ such that

$$p(t) = \sum_1^m t^{v_i} \text{ and } q(t) = \sum_1^m t^{w_i}.$$

Let T be a maximal torus of $U(m)$. Then we define $v, w \in L(T) \cong \mathbb{R}^m$ by $v = (v_1, \ldots, v_m)$ and $w = (w_1, \ldots, w_m)$. Since $p(t)p(1/t) = q(t)q(1/t)$ for all t in $\mathbb{R}$, from Proposition 4.3.5 it follows that $R_1(v)), \ldots, R_k(v)$ and $R_1(w)), \ldots, R_k(w)$ are same up to a permutation i.e. $v \sim w$.

Now we will prove that v and w do not belong to the same equivalence class of $\approx$. Since $\mathrm{Aut}(U(m))$ is generated by $\mathrm{Int}(U(m))$ and the automorphism $A \mapsto \overline{A}$ it is easy to see that for any two elements y, z in the Lie algebra of $U(m)$, $y \approx z$ if and only if y can be obtained from z by applying operations of the following type.

a) $(x_1, \ldots, x_m) \to (-x_1, \ldots, -x_m)$.

b) $(x_1, \ldots, x_m) \to (x_1 + t, \ldots, x_m + t)$ for some t in $\mathbb{R}$.

c) $(x_1, \ldots, x_m) \to (x_{\pi(1)}, \ldots, x_{\pi(m)})$ for some permutation π of 1,...,m.

Since the coefficient of t^2 in p is 1, it is easy to see that there exists a j such that $v_j \neq v_i$ for all $i \neq j$. Observe that this property remains invariant under the operations a), b) and c). Since no coefficient in q is equal to 1, it follows that w does not satisfy this property. Hence v and w do not belong to the same equivalence class of $\approx$.

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