Connections on Small Vertex Models

R. Srinivasan

A thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the degree of Doctor of Philosophy
1998

Preface

This thesis is devoted to the classification of a special class of commuting squares called vertex models.

The first chapter is indroductory in nature and is included for the sake of completeness and convenience of reference. It starts with the description of the basic construction and the invariants called the principal and dual graph for an inclusion of II_1 factors. After defining a commuting square we describe the special class of commuting squares called vertex model given by an $M_n \otimes M_k$ biunitary matrix . Finally we state some results, without proof, about vertex models from [KSC].

The second chapter is the core of the thesis. It is devoted to the classification of vertex models given by an $M_n \otimes M_2$ biunitary matrix. If $B(k,n)$ denotes the collection of $M_n \otimes M_k$ biunitary matrices, then $B(2,3)$ is classified up to the natural equivalence relation on it. Further, a simple model form for a representative from each equivalence class in $B(2,n)$ and also necessary conditions for two such 'model connections' to be themselves equivalent are obtained. Then we go on to show that $B(2,n)$ contains a $(3n-6)$ parameter family of pairwise inequivalent connections and show that the number $(3n-6)$ is sharp. Finally, it is deduced that every graph that can arise as the principal graph of a finite depth subfactor of index 4 actually arises for one arising from a vertex model corresponding to $B(2,n)$ for some n.

In the appendix, we give elementary and direct proofs of two known results in the literature (due to Kosaki-Yamagami and Bisch, respectively) using the techniques of bimodules and elementary matrix manipulations. The first result is the computation of principal and dual graphs for the inclusion of II_1 factors $N = P \rtimes H \subseteq P \rtimes G = M$, where G is a discrete group acting as outer automorphisms of a II_1 factor P, and H is a subgroup of G such that $[G : H] < \infty$. It is this proof that has basically been reproduced in [JS]. The second result (due to Bisch) is that if $N \subseteq M \subseteq P$ is an inclusion of II_1 factors such that $N \subseteq P$ has finite depth, then $N \subseteq M$ and $M \subseteq P$ have finite depth.

Acknowledgements

Basically, I have to thank my advisor Prof. V. S. Sunder for everything. Nobody has been so kind to me and patient with me like him in my life. I am deeply indebted to him for devoting his time and efforts which helped me a lot to do this work. I also thank him for his encouragement which helped me to continue my research career during a difficult phase of my life, and for bearing with my insincerity and mistakes at many points of our interaction. I dedicate this thesis and my future works to him.

I thank Dr. Vijay Kodiyalam for helping in the combinatorial argument used to sort out the proof of the statement about the principal graphs in the last section of second chapter.

I would like to thank my friends Mihir and Balaji of Banglore centre of the ISI, for being with me during a depressing period of my life. I thank Mihir for also being with me when the work presented in the appendix of this thesis was done in ISI, Banglore.

Finally, I thank the Indian Statistical Institute and the Institute of Mathematical Sciences for providing the financial support and research facilities for this work.

Contents

Chapter 1

Preliminaries

In this chapter we recall some facts about subfactors. For a proof of these facts presented here one can refer to [GHJ] or [JS], from where most of the meterial here is reproduced. We start with the description of the basic construction for $N \subseteq M$, an inclusion of II_1 factors of finite index. Then we describe two invariants for $N \subseteq M$ called the principal and the dual graph. After defining a commuting square, we describe a special class of commuting squares called the vertex models, a smaller class of which will be analysed in Chapter 2. We also reproduce certain facts about vertex models (proved in [KSC]) without proof. We end this chapter with a proposition, which removes the ambiguity in a fact proved in [KSC].

1.1 The basic construction

If M is a finite von Neumann algebra with a fixed faithful, nomal, tracial state tr, by an M *−module*, we mean a Hilbert space $\mathcal{H}$ equipped with an action of M into $\mathcal{L}(\mathcal{H})$. One such example is the standard module $L^2(M)$, with its distinguished cyclic trace vector Ω, obtained through the GNS construction applied to the the tracial state tr (ref [JS]).

Now suppose $N \subseteq M$ is an inclusion of finite von Neumann algebras. Consider the module $L^2(M)$ (which is actually an $M - M$ *− bimodule* -see the appendix). Since $\mathcal{H}$ is the completion of $M\Omega$, it follows that the subspace $\mathcal{H}_1 = [N\Omega]$ can be naturally identified with $L^2(N, tr)$. Let e_N denote the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathcal{H}_1$, then it is a fact (see [JS], for instance) that $e_N(M\Omega) \subseteq N\Omega$. Hence the projection induces, by restriction, a map $E_N : M \rightarrow N$, which is called the conditional expectation of M onto N. This map satisfies the following properties:

(i) $e_N x e_N = E_N(x)e_N \;\; \forall\, x \in M$, and consequently E_N defines a Banach space projection of M onto N. (ii) E_N is an $N-N$ bilinear map i.e. $E_N(n_1 m n_2) = n_1 E_N(m) n_2 \;\; \forall\, n_1, n_2 \in N$. (iii) $tr(E_N(x)) = tr(x) \;\; \forall\, x \subset M$.

DEFINITION 1.1.1 *The passage from the intial inclusion $N \subseteq M$ to the von Neumann algebra $M_1 = < M, e_N > = (M \cup \{e_N\})''$, consequently to the tower $N \subseteq M \subseteq M_1$, is called the basic construction. The projection e_N is called the Jones projection for the basic construction of $N \subseteq M$.*

Suppose $N \subseteq M$ is an inclusion of II_1 factors of finite index. If J_M denote the extention of the adjoint map in M to $L^2(M)$, then $M_1 = J_M N' J_M$. Hence M_1 is also a II_1 factor.

The trace tr defines a Markov trace for $N \subseteq M$, i.e. it extends to a trace on M_1 so that $tr(xe_N) = \tau tr(x) \;\; \forall\, x \in M$, where $\tau = [M : N]^{-1}$, and its extention to M_1 is a Morkov trace for $M \subseteq M_1$. It is further true that $[M_1 : M] = [M : N]$.

Using this, we can iterate the basic contruction and obtain Jones' tower of the basic contruction for $N \subseteq M$:

$$N \subseteq M \subseteq M_1 \subseteq M_2 \cdots \subseteq M_n \subseteq \cdots$$

We end this section by stating a fact - see [J] or [GHJ] - regarding the basic construction for finite dimensional C^*-algebras.

PROPOSITION 1.1.2 *Let $A \subseteq B$ be an inclusion of finite dimensional C^*-algebras. If τ is a Markov trace for the inclusion $A \subseteq B$, then it extends to a state tr on $< B, e_A >$ which is a Markov trace for the inclusion $B \subseteq < B, e_A >$.*

1.2 The Principal and the dual graphs

We start with a brief description of inclusions of finite dimensional C^*-algebras and the associated Bratelli diagrams. If $A \subseteq B$ is an inclusion of finite dimensional C^*-algebras, then (by the Wedderburn-Artin theorem) A and B are isomorphic to direct sums of finitely many matrix algebras over $\mathbb{C}$. Let $A \cong \oplus_{i=1}^{n} M_{n_i}(\mathbb{C})$ and $B \cong \oplus_{i=1}^{m} M_{m_i}(\mathbb{C})$.

If $\pi(X)$ denote the set of minimal central projections of X, then the Bratteli diagram for the inclusion $A \subseteq B$ is the graph Λ whose set of vertices is the disjoint union of two sets labelled by $\pi(A)$ and $\pi(B)$ and $p_i \in \pi(A)$ is joined to $q_j \in \pi(B)$ by m_{ij} bonds, where m_{ij} is 'the multiplicity of $M_{n_i}(\mathbb{C})$ in $M_{m_i}(\mathbb{C})$' under the inclusion. The $m \times n$ matrix with its entries as m_{ij} is called the inclusion matrix for $A \subseteq B$. It is a fact (ref [JS]) that if $N \subseteq M$ is an inclusion of finite dimensional C^*- algebras with Bratteli diagram Λ, then the Bratteli diagram for the inclusion $M \subseteq M_1$, where M_1 is the finite dimensional C^*- algebra obtained by the basic contruction, is given by a 'reflection' of Λ, i.e. the inclusion matrix is given by the transpose.

Suppose $N \subsetneq M$ is an inclusion of II_1 factors with finite index and

$$N = M_{-1} \subseteq M = M_0 \subseteq M_1 \subseteq M_2 \cdots \subseteq M_n \subseteq \cdots$$

is the tower of basic contruction, then $\{M_i' \cap M_j : -1 \leq i \leq j\}$ is a grid of finite dimensional C^*- algebras, which is canonically associated with the inclusion $N \subsetneq M$, and is consequently an 'invariant' of the intial inclusion.

Because of a periodicity of order two, we need to only consider $i = -1$ and $i = 0$. It is a fact that (ref [JS]) $N' \cap M_{n+1}$ contains a copy of the basic construction for the inclusion $N' \cap M_{n-1} \subseteq N' \cap M_n$, and consequently that the Bratteli diagram for the inclusion $N' \cap M_n \subseteq N' \cap M_{n+1}$ contains a 'reflection' of the Bratteli diagram for the inclusion $N' \cap M_{n-1} \subseteq N' \cap M_n$. The graph obtained by starting with the Bratteli diagram for the tower $\{N' \cap M_n : n \geq -1\}$ of relative commutants, and removing all those parts which are obtained by reflecting the previous stage, is called the Principal graph invariant for the inclusion $N \subseteq M$. A similar reasoning applies for $i = 0$ also, and the resulting graph is called the dual graph invariant for the inclusion $N \subseteq M$.

1.3 Commuting squares

First we define the notion of a commuting square.

DEFINITION 1.3.1 *Suppose D is a finite von Neumann algebra with a finite, faithful, normal tracial state tr and A, B and C are subalgebras of D such that $A \subseteq B \cap C$; then the following diagram*

$$\begin{array}{ccc} C & \subseteq & D \\ \cup & & \cup \\ A & \subseteq & B \end{array} \tag{1.3.1}$$

is said to be a commuting square if $E_B(C) = E_C(B) = A$, where the conditional expectation is defined on D with respect to the trace tr.

Suppose

$$\begin{array}{ccc} B_0 & \subseteq & B_1 \\ \cup & & \cup \\ A_0 & \subseteq & A_1 \end{array} \tag{1.3.2}$$

is a commuting square of finite dimensional C^* -algebras with respect to a trace on B_1 which is a Markov trace for the inclusion $B_0 \subseteq B_1$. Assume that this commuting square is 'symmetric' or 'non-degenerate' (see [HS] or [P1]).

Let $B_2 =< B_1, e >$ denote the basic construction for the inclusion $B_0 \subseteq B_1$, where e denotes the projection in B_2 which implements the conditional expectation of B_1 onto B_0. Define $A_2 =< A_1, e > \subseteq B_2$. Then the following is also a commuting square (with respect to the unique trace on B_2 which extends the given trace on B_1 and is a Markov trace for the inclusion $B_1 \subseteq B_2$) :

$$\begin{array}{ccc} B_1 & \subseteq & B_2 \\ \cup & & \cup \\ A_1 & \subseteq & A_2 \end{array} \tag{1.3.3}$$

Suppose that $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots B_n \subseteq \cdots$ is the tower of basic construction with the Jones projections e_n, and $A_n = < A_{n-1}, e_{n-1} >$ for $n \geq 2$; then the following is also a commuting square, for $n \geq 2$:

$$\begin{array}{ccc} B_{n-1} & \subseteq & B_n \\ \cup & & \cup \\ A_{n-1} & \subseteq & A_n \end{array} \tag{1.3.4}$$

Now if we define $R = (\cup_n B_n)''$ and $R_0 = (\cup_n A_n)''$, then $R_0 \subset R$ is a subfactor and its relative commutants are described by a result due to Ocneanu which we state here for the first relative commutant. (See [O1] or [JS].)

THEOREM 1.3.2 **(Ocneanu Compactness)** *Let A_n, B_n, R_0, R be as above then*

$$R_0' \cap R = A_1' \cap B_0.$$

The above construction of the subfactor is canonical and we call the subfactor the horizontal subfactor associated with the commuting square 1.3. By applying the basic construction to the pair $A_1 \subseteq B_1$, in a similar way we construct another subfactor which we call the vertical subfactor.

1.4 The vertex models

Consider the following commuting square:

$$\begin{array}{ccc} A_0^1 & \overset{L}{\subseteq} & A_1^1 \\ K \cup & & \cup H \\ A_0^0 & \overset{G}{\subseteq} & A_1^0 \end{array} \qquad (1.4.5)$$

where G, H, K and L are the corresponding inclusion matrices; then the following are equivalent:

(i) $G = L = [n]$ and $H = K = [k]$.

(ii) the square (1.4.5) is isomorphic to a commuting square of the form

$$\begin{array}{ccc} W(1 \otimes M_k(\mathbb{C}))W^* & \subseteq & M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subseteq & M_n(\mathbb{C}) \otimes 1 \end{array}, \qquad (1.4.6)$$

where $W = ((W_{\beta b}^{\alpha a})) \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ is unitary. (We use the convention that $1 \leq \alpha, \beta \leq n$, $1 \leq a, b \leq k$.)

If $W = ((W_{\beta b}^{\alpha a})) \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ is any unitary matrix, then the square (1.4.6) is a commuting square iff W is **biunitary** - i.e., both W and $\tilde{W}$ given by $\tilde{W}_{\beta b}^{\alpha a} = W_{\alpha b}^{\beta a}$ are unitary. We call such a commuting square a vertex model. We shall use the symbol $B(k, n)$ to denote the set of such biunitary matrices.

Two biunitary matrices W and W' are said to be **equivalent** if the corresponding commuting squares are isomorphic. It is true that if $W, W' \in B(k,n)$, then W and W' are equivalent if and only if there exists unitary matrices $U, U' \in M_n$, $A, A' \in M_k$ such that $(U \otimes A)W = W'(U' \otimes A')$.

Given $W \in B(k,n)$, the basic construction yields a grid of commuting squares and consequently, a *horizontal* (respectively *vertical*) subfactor $A^0_\infty \subset A^1_\infty$ (respectively $A^\infty_0 \subseteq A^\infty_1$) with index k^2 (respectively n^2). As this construction is canonical, isomorphic commuting squares (i.e., equivalent biunitary matrices) yield isomorphic horizontal (respectively vertical) subfactors. The following theorem, which decribes the relative commutants of the horizontal subfactor when the biunitary matrix is in a particular form, is proved in [KSC]. (See also [BHJ].) Before stating that result we describe what is called the Cayley graph of a group.

For a group $G \subseteq U(N)$, let π denotes the standard (or identity) representation of G in $U(N)$, and let $C(\hat{G}, \pi)$ denote the bipartite graph obtained as follows: let $\mathcal{G}$ denote the bipartite graph with the set of even (respectively odd) vertices being given by $\mathcal{G}^{(0)} = \hat{G} \times \{0\}$ (respectively $\mathcal{G}^{(1)} = \hat{G} \times \{1\}$), where $\hat{G}$ denotes the (unitary) dual of G, and the number of bonds joining $(\rho, 0)$ and $(\sigma, 1)$ is given by $(\rho \otimes \pi, \sigma)$; here and everywhere we write $< \rho \otimes \pi, \sigma >$ to denote the multiplicity of σ in $\rho \otimes \pi$. Finally, let $C(\hat{G}, \pi)$ denote the connected component in $\mathcal{G}$ containing $(tr, 0)$, where tr denotes the trivial representation of G. Now we state the theorem.

THEOREM 1.4.1 *Let $\{\gamma_1, \gamma_2, ..., \gamma_n\}$ be any collection of $k \times k$ unitary matrices, with $\gamma_1 = 1$ and define $W^{\alpha a}_{\beta b} = \delta^\alpha_\beta (\gamma_\alpha)^a_b$; then W is a biunitary and the principal graph of the horizontal subfactor given by the vertex model corresponding to W is $C(\hat{G}, \pi)$, where G is the group generated by $\{\gamma_1, \gamma_2, ..., \gamma_n\}$.*

Also it is proved in [KSC] that when $n = k = 2$, any $W \in B(2,2)$ is equivalent to a biunitary matrix of the form

$$W(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

where $\omega \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Further neither the vertical nor the horizontal subfactor is irreducible. We explicitly point out the ambiguity in such a representation.

PROPOSITION 1.4.2 *$W(\omega)$ is equivalent to $W(\omega')$ if and only if $Re(\omega) = Re(\omega')$.*

Proof: Let

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \text{ and } A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then it can be easily verified that $(U \otimes A)W(\omega) = W(\bar{\omega})(U \otimes A')$.

Conversely suppose

$$(U \otimes A)\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D' \end{pmatrix}(U' \otimes A')$$

where $D = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$, $D' = \begin{pmatrix} 1 & 0 \\ 0 & \omega' \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

Then the following equations hold:

$$aU = a'U' \tag{1.4.7}$$

$$bUD = b'U' \tag{1.4.8}$$

$$cU = c'D'U' \tag{1.4.9}$$

$$dUD = d'D'U' \tag{1.4.10}$$

Suppose $a \neq 0$. Then $a' \neq 0$ by equation 1.4.7, and so also $d' \neq 0$ (as D, D', U, U' are unitary matrices). From equations 1.4.7 and 1.4.10, we see that $d'^{-1}da^{-1}a'U'DU'^{*} = D'$. Now by comparing the eigenvalues, (as U' is an unitary matrix) we conclude that $\{d'^{-1}da^{-1}a', d'^{-1}da^{-1}a'\omega\} = \{1, \omega'\}$. Hence either $\omega = \omega'$ or $\omega = \bar{\omega}'$. Suppose that $a = 0$, then as A is an unitary matrix, it is the case that $b \neq 0$. Exactly in a similar way, using equations 1.4.8 and 1.4.9, we can again conclude that either $\omega = \omega'$ or $\omega = \bar{\omega}'$. □

Chapter 2

Connections in Small Vertex Models

This chapter is the core of the thesis. It is devoted to the study of small vertex models - i.e. the vertex models given by a biunitary matrix in $B(2,n)$. We attempt to classify the connections in small vertex models. We are able to classify $B(2,3)$ completely and for $B(2,n)$ we specify the maximum number of independent parameters with pairwise inequivalent connections.

First we get a simple model for a representative from each equivalence class in $B(2,n)$. In the second section we classify $B(2,3)$ completely, i.e. we obtain necessary and sufficient conditions for two such representatives in the model form to be themselves equivalent. In the third section we obtain necessary conditions (and some sufficient conditions also) for two matrices from $B(2,n)$ in the model form to be equivalent. We also prove that the vertical subfactor associated with the commuting square (corresponding to a connection in $B(2,n)$) is always reducible and we provide some conditions for the the horizontal subfactor to be irreducible or reducible.

Using these facts we go on to prove that $B(2,n)$ contains a $(3n-6)$-parameter family of pairwise inequivalent connections and show that the number $(3n-6)$ is sharp - i.e. there does not exist a subset of $B(2,n)$ homeomorphic to an open subset of Euclidean space of dimension $(3n-5)$ and consisiting of pairwise inequivalent connections. Finally, using Theorem 1.4.1, we deduce that every graph that can arise as the principal graph of a finite depth subfactor of index 4 actually arises as the principal graph of the horizontal subfactor for a vertex model corresponding to a biunitary matrix in $B(2,n)$ for some n.

2.1 A model form for a matrix in $B(2,n)$

In this section we prove that every biunitary matrix in $B(2,n)$ is equivalent to a biunitary matrix in a model form - similar to the one for $n = 2$ described at the end of Chapter 1 - with $(3n-5)$ independent parameters. First we prove that any biunitary matrix in $B(2,n)$ is equivalent to a block diagonal matrix of the form prescribed in the following Proposition.

PROPOSITION 2.1.1 *Any biunitary matrix $W \in B(2,n)$ is equivalent to a matrix of the form*

$$\begin{pmatrix} C & US \\ VS & -UVC \end{pmatrix},$$

where U, V are diagonal unitary matrices and C, S are positive diagonal matrices such that $C^2 + S^2 = 1$.

Proof: Let $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $\check{W} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where $a, b, c, d \in M_n$. Then W is a biunitary matrix if and only if both W and $\check{W}$ are unitary matrices. The unitarity of W and $\check{W}$ (i.e. the relation $WW^* = 1 = \check{W}\check{W}^*$) implies the following equations:

$$\begin{aligned} aa^* + bb^* &= 1 \\ cc^* + dd^* &= 1 \\ a^*a + c^*c &= 1 \\ b^*b + d^*d &= 1 \\ aa^* + cc^* &= 1 \\ bb^* + dd^* &= 1 \\ a^*a + b^*b &= 1 \\ c^*c + d^*d &= 1 \end{aligned}$$

By premultiplying by $1 \otimes u$, where $u \in M_n$ is a suitable unitary matrix (i.e. by working with an equivalent biunitary matrix), we may assume, without loss of generality, that a is positive. Then it follows from the above equations that $0 \leq a \leq 1$. Let $C = a$. So there exist a unique positive matrix $S \in M_n$ such that, $0 \leq S \leq 1$ and $C^2 + S^2 = 1$. Then from the above equations we can conclude that b, c, d are normal and also that $bb^* = cc^* = S^2$ and $dd^* = C^2$. So there exist unitary matrices $U, V, T \in M_n$ such that $b = US, c = VS$ and $d = TC$, and such that U and V commute with S (hence, also with C) and T commutes with C (hence, also with S).

So we find that W is equivalent to the biunitary matrix

$$\begin{pmatrix} C & US \\ VS & TC \end{pmatrix},$$

where C, S, U, V and T are as above.

The biunitarity of W (i.e. the relation $WW^* = \tilde{W}\tilde{W}^* = 1$) also implies the following equations:

$$\begin{aligned} SC(V + TU^*) &= 0. \\ SC(U + V^*T) &= 0. \\ SC(U + TV^*) &= 0. \\ SC(V + U^*T) &= 0. \end{aligned} \qquad (2.1.1)$$

Since U, V and T leave the eigenspaces $\{H_i\}_{i\in I}$ of C invariant, we may, by conjugating W by a unitary matrix of the form $\Gamma \otimes 1$ (where Γ is a unitary matrix which diagonalises C), assume that

$$C = \oplus_{i\in I} c_i 1_{H_i} \ , \ S = \oplus_{i\in I} s_i 1_{H_i}$$

$$U = \oplus_{i\in I} U_i \ , \ V = \oplus_{i\in I} V_i \ , \ T = \oplus_{i\in I} T_i,$$

where 1_{H_i} denotes the identity in $\mathcal{L}(H_i)$, $0 \leq c_i, s_i \leq 1$ and $U_i, V_i, T_i \in \mathcal{L}(H_i)$.

Thus we see that $W = \oplus_{i\in I} W_i$, where W_i is a biunitary matrix in $M_{n_i} \otimes M_2$ and n_i is the dimension of H_i, and that

$$W_i = \begin{pmatrix} c_i I_{n_i} & U_i s_i \\ s_i V_i & c_i T_i \end{pmatrix}.$$

Note that in order to complete the proof of this proposition we need to show that each of this W_i is equivalent to a biunitary matrix of the form presented in the proposition. To prove this we now consider two cases depending on whether c_i is zero or non-zero.

Suppose $c_i = 0$, then, by pre-multiplying W_i by the unitary matrix $U_i^* \otimes 1$, we may assume that $U_i = 1_{H_i}$. Now, by conjugating W_i by a unitary matrix $p \otimes 1$, where p is a unitary matrix which diagonalises V_i, we can conclude that W_i is equivalent to a matrix of the desired form.

Suppose that $c_i \neq 0$; If $s_i = 0$ we can assume, by conjugating by a unitary matrix $p \otimes 1$ (where p is a unitary matrix which diagonalises T_i), that the matrix W_i is in the required form. If s_i is also non-zero, then first conclude from the set of equations 2.1.1 that $T_i = -U_iV_i = -V_iU_i$. Now, by conjugatng W_i by a unitary matrix of the form $p \otimes 1$, where p is a unitary matrix which simultaneously diagonalises the commuting unitaries U_i, V_i, T_i, we may conclude that W_i is equivalent to a matrix of the desired form.

Hence, in any case, we find that W is equivalent to a matrix of the form

$$\begin{pmatrix} C & US \\ VS & -UVC \end{pmatrix},$$

where U, V are diagonal unitary matrices and C, S are positive diagonal matrices such that $C^2 + S^2 = 1$. □

The next proposition gives a subclass of $B(2,n)$, in the model form promised in the begining of this chapter, which has at least one element equivalent to any given biunitary matrix in $B(2,n)$.

PROPOSITION 2.1.2 *Any $W \in B(2,n)$ is equivalent to a biunitary matrix of the form*

$$W(\omega,\theta,\phi,C) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & \theta S \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & \phi S & 0 & 0 & -\theta\phi C \end{pmatrix},$$

where $\theta = diag(\theta_1, \theta_2, ..\theta_{n-2})$, $\phi = diag(\phi_1, \phi_2, ...\phi_{n-2})$, $C = diag(C_1, C_2, ...C_{n-2})$, $S = diag(S_1, S_2, ...S_{n-2})$, $\theta_i, \phi_i, \omega \in \{z \in \mathbb{C} : |z| = 1\}$, $Im(\omega) > 0$, $0 < C_i, S_i \leq 1$, and $C_i^2 + S_i^2 = 1$.

Proof: From Proposition 2.1.1, we may assume that $W = \sum_{i=1}^{n} E_{ii} \otimes W_i$ (where W_i is a 2×2 unitary matrix and $\{E_{ij} : 1 \leq i, j \leq n\}$ denotes - here and elsewhere- the usual system of matrix units in M_n), and that W_i has the form

$$W_i = \lambda_i \begin{pmatrix} C_i & \theta_i S_i \\ \phi_i S_i & -\theta_i \phi_i C_i \end{pmatrix} ,$$

where $\lambda_i = 1$, θ_i, ϕ_i are complex numbers of unit modulus, and $0 \leq C_i, S_i \leq 1$ and $C_i^2 + S_i^2 = 1$.

Note next that if $D = diag(d_1, \cdots, d_n) \in M_n$ is a diagonal unitary matrix, and if W, W_i are as above, and if $V_1, V_2 \in M_2$ are unitary, then

$$(D \otimes V_1)\ W\ (1 \otimes V_2) \ = \ \sum_{i=1}^{n} d_i(E_{ii} \otimes V_1 W_i V_2). \qquad (*)$$

Set $V_1 = 1, V_2 = W_1^*$; if the (1,1) entry of $W_i W_1^*$ is $\omega_i \tilde{C}_i$, with $\tilde{C}_i \geq 0$ and $|\omega_i| = 1$, define $d_i = \overline{\omega}_i$. We may now deduce from equation (*) that we may reduce to the case where W_1 is the identity matrix, and W_i are as above, with $\lambda_i = 1\ \forall i$.

Next, let U be the unitary matrix which diagonalises (the new) W_2. Then, by setting $d_i = \overline{\omega_i}$ if $\omega_i \tilde{C}_i'$ is the $(1,1)$ entry of $U^* W_i U$, with $\tilde{C}_i' \geq 0$ and $|\omega_i| = 1$, and by setting $U = V_1^* = V_2$, we find that we may reduce to the case where W is as above, and in addition, $W_1 = 1$ and $W_2 = diag(1, \omega)$, where ω is a complex number of unit modulus.

If $Im(\omega) \geq 0$, the proof of the Proposition is complete. If $Im(\omega) < 0$, then set

$$V_1 = V_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

and $d_1 = 1$, $d_2 = \overline{\omega}$, $d_i = -\overline{\theta_i \phi_i}\ \forall\ i = 3, \cdots, n$, to conclude that W is indeed equivalent to a biunitary matrix of the prescribed form. □

2.2 Classification of $B(2,3)$

In this section we classify $B(2,3)$ completely up to the natural equivalence relation on it. We shall use the notation $\Omega(2,n) = \mathbf{T}^+ \times \mathbf{T}^{n-2} \times \mathbf{T}^{n-2} \times [0,1]^{n-2}$, where

$\mathbb{T}$ is the unit circle in the complex plane and $\mathbb{T}^+ = \{\omega \in \mathbb{T} : Im(\omega) \geq 0\}$; we shall denote a typical pair of points of $\Omega(2,n)$ by $P = (\omega, \theta, \phi, C)$ and $P' = (\omega', \theta', \phi', C')$ and the corresponding biunitary matrices by W and W'.

We shall find it convenient to isolate a few simple assertions as lemmas, since we will need to repeatedly use them.

LEMMA 2.2.1 *Suppose $a, b, c, d \neq 0$. Consider the following equations.*

$$a = m_1 b \ , \ c = m_2 d \ , \ a = m_1' c \ , \ b = m_2' d;$$

For these equations to be consistent, it is necessary that the following condition is satisfied:

$$\frac{m_1}{m_2} = \frac{m_1'}{m_2'}.$$

Proof: Both ratios are equal to $\frac{ad}{bc}$. □

LEMMA 2.2.2 *Suppose* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a unitary matrix, and suppose $\theta, \phi, \omega_j, j = 0, 1, 2$ are complex numbers of unit modulus, and suppose C and S are non-negative real numbers satisfying $C^2 + S^2 = 1$.*

Assume that $\omega_2 \neq 1$ and that the following equations are satisfied:

$$a(C - \omega_0) + b\phi S = 0. \tag{2.2.2}$$

$$a\theta S - b(\theta\phi C + \omega_0\omega_1) = 0. \tag{2.2.3}$$

$$c(C - \omega_0\omega_2) + d\phi S = 0. \tag{2.2.4}$$

$$c\theta S - d(\theta\phi C + \omega_0\omega_1\omega_2) = 0 \tag{2.2.5}$$

Consider the following distinct possibilities:

Case (i): $a = d = 0$.

In this case, $S = 0$, $\theta\phi = -\omega_0\omega_1$, $\omega_0\omega_2 = 1$.

Case (ii): $b = c = 0$.

In this case, $S = 0$, $\theta\phi = -\omega_1\omega_2$, $\omega_0 = 1$.

Case (iii): $a, b, c, d \neq 0$.

In this case, $S \neq 0, C \neq 1$ and

$$Re(\omega_2) \; = \; -[S^2 + Re(\bar{\omega}_1 \theta\phi)\; C^2] \; ; \tag{2.2.6}$$

further, if $\omega_2 \neq \pm 1$, then,

$$\omega_0 \; = \; \frac{(\omega_1 - \theta\phi)C}{\omega_1(1+\omega_2)} \; . \tag{2.2.7}$$

In particular, equation 2.2.6 is satisfied in all three cases.

Proof: If $a = 0$, then necessarily $|b| = |c| = 1$ and $d = 0$; so equation 2.2.2 implies that $S = 0$, whence $C = 1$; then equations 2.2.3 and 2.2.4 imply that $\theta\phi = -\omega_0\omega_1$ and that $\omega_0\omega_2 = 1$, respectively.

If $b = 0$, then necessarily $|a| = |d| = 1$ and $c = 0$, and equations 2.2.3 implies that $S = 0$ so $C = 1$; and equations 2.2.2 and 2.2.5 imply that $\omega_0 = 1$ and that $\theta\phi = -\omega_1\omega_2$, respectively.

So assume $a, b, c, d \neq 0$. If $S = 0$, then the equations 2.2.2 and 2.2.4 would imply that $\omega_0 = \omega_0\omega_2 = 1$. Hence, as we have assumed that $\omega_2 \neq 1$, conclude that $S \neq 0$. Now, deduce from 2.2.2 and 2.2.3 that

$$(\omega_0 - C)\; (\theta\phi C + \omega_0\omega_1) \; = \; \theta\phi S^2 \; ; \tag{2.2.8}$$

similarly, deduce from equations 2.2.4 and 2.2.5 that

$$(\omega_0\omega_2 - C)\; (\theta\phi C + \omega_0\omega_1\omega_2) \; = \; \theta\phi S^2 \; .$$

These equations may be re-written as

$$\begin{aligned} \omega_0^2\omega_1 + \omega_0(\theta\phi - \omega_1)C \; &= \; \theta\phi S^2 \\ \omega_0^2\omega_2^2\omega_1 + \omega_0\omega_2(\theta\phi - \omega_1)C \; &= \; \theta\phi S^2 \; , \end{aligned}$$

from which we may deduce that

$$\omega_0^2(1-\omega_2^2)\omega_1 + \omega_0(1-\omega_2)(\theta\phi - \omega_1)C \; = \; 0 \; . \tag{2.2.9}$$

If $\omega_2 = -1$, then the first term in equation 2.2.9 vanishes, and we find that we must have $\theta\phi = \omega_1$ or $C = 0$; it is clear that equation 2.2.6 is satisfied in either case.

If $\omega_2 \neq -1$, then since $\omega_2 \neq 1$, by hypothesis, we may infer from equation 2.2.9 that

$$\omega_0 = \frac{(\omega_1 - \theta\phi)C}{\omega_1(1+\omega_2)} .$$

Substituting this expression for ω_0 into equation 2.2.8, we find that

$$\begin{aligned}
&(\omega_0 - C)\ (\theta\phi C + \omega_0\omega_1) \\
&= \frac{C}{\omega_1(1+\omega_2)}(\omega_1 - \theta\phi - \omega_1(1+\omega_2)) \cdot \frac{C}{\omega_1(1+\omega_2)}(\theta\phi\omega_1(1+\omega_2) + \omega_1(\omega_1 - \theta\phi)) \\
&= -\frac{C^2}{(1+\omega_2)^2}(\bar{\omega}_1\theta\phi + \omega_2)(\omega_1 + \theta\phi\omega_2) ;
\end{aligned}$$

and hence,

$$\begin{aligned}
&\theta\phi S^2 - (\omega_0 - C)\ (\theta\phi C + \omega_0\omega_1) \\
&= \theta\phi \left[S^2 + \frac{C^2}{(1+\omega_2)^2}(\bar{\omega}_1\theta\phi + \omega_2)(\omega_1\overline{\theta\phi} + \omega_2)\right] .
\end{aligned}$$

Thus we find that the equation $(\omega_0 - C)\ (\theta\phi C + \omega_0\omega_1) = \theta\phi S^2$ will be satisfied precisely when

$$\begin{aligned}
0 &= (1+\omega_2)^2 S^2 + C^2(\bar{\omega}_1\theta\phi + \omega_2)(\omega_1\overline{\theta\phi} + \omega_2) \\
&= \omega_2^2(S^2 + C^2) + \omega_2(2S^2 + 2C^2 Re(\bar{\omega}_1\theta\phi)) + (S^2 + C^2) \\
&= \omega_2^2 - 2\alpha\omega_2 + 1 \text{ , (say),}
\end{aligned}$$

where $\alpha = -(S^2 + C^2 Re(\bar{\omega}_1\theta\phi))$.

On the other hand, it is clear that if a complex number ω satisfies the equation $\omega^2 - 2\alpha\omega + 1 = 0$, where α is real and $|\alpha| \leq 1$, then $\omega = \alpha \pm i\sqrt{1-\alpha^2}$, so that $Re(\omega) = \alpha$, and hence equation 2.2.6 is satisfied. □

LEMMA 2.2.3 *Suppose $\theta, \phi, \omega_j, j = 1,2$ are complex numbers of unit modulus, and C, S are non-negative real numbers satisfying $C^2 + S^2 = 1$.*

Assume that $\omega_2 \neq \pm 1$. Define

$$\omega_0 = \frac{(\omega_1 - \theta\phi)C}{\omega_1(1+\omega_2)} .$$

(a) The following conditions are equivalent:

(i) $Re(\omega_2) = -(S^2 + Re(\bar{\omega}_1\theta\phi)\ C^2)$;

(ii) $|\omega_0| = 1$.

Further, if these equivalent conditions are satisfied, then,

$$(\omega_0 - C)\ (\theta\phi C + \omega_0\omega_1) \;=\; (\omega_0\omega_2 - C)\ (\theta\phi C + \omega_0\omega_1\omega_2) \;=\; \theta\phi S^2 .$$

(b) Assume, further, that $S \neq 0$ (i.e., $C \neq 1$), and that the equivalent conditions of (a) are satisfied. Define

$$m_1 \;=\; -\frac{\phi S}{C - \omega_0} \;=\; \frac{\theta\phi C + \omega_1\omega_0}{\theta S}\ ;\ m_2 \;=\; -\frac{\phi S}{C - \omega_0\omega_2} \;=\; \frac{\theta\phi C + \omega_0\omega_1\omega_2}{\theta S} .$$

Then, m_1 and m_2 are well defined and non-zero, and

(i)

$$\frac{m_1}{m_2} \;=\; \frac{\omega_1 + \omega_2\theta\phi}{\omega_1\omega_2 + \theta\phi}\ ;$$

and

(ii) $m_1\overline{m_2} \;=\; -1$.

Proof: (a) We have $|\omega_0|^2 = \frac{(1-Re(\bar{\omega}_1\theta\phi))C^2}{1+Re(\omega_2)}$, and hence,

$$\begin{aligned} |\omega_0| = 1 \quad &\Leftrightarrow \quad 1 + Re(\omega_2) \;=\; (1 - Re(\bar{\omega}_1\theta\phi))C^2 \\ &\Leftrightarrow \quad Re(\omega_2) \;=\; (1 - Re(\bar{\omega}_1\theta\phi))C^2 - S^2 - C^2 \\ &\Leftrightarrow \quad Re(\omega_2) \;=\; -\left(S^2 + Re(\bar{\omega}_1\theta\phi)\ C^2\right) , \end{aligned}$$

as desired.

If conditions (i) and (ii) are satisfied, then, the validity of the desired equations is verified by the reasoning given at the end of the proof of the last lemma.

(b) (i)

$$\begin{aligned} \frac{m_1}{m_2} \;&=\; \frac{C - \omega_0\omega_2}{C - \omega_0} \\ &=\; \frac{\frac{C}{\omega_1(1+\omega_2)}\,(\omega_1(1+\omega_2) - \omega_2(\omega_1 - \theta\phi))}{\frac{C}{\omega_1(1+\omega_2)}\,(\omega_1(1+\omega_2) - (\omega_1 - \theta\phi))} \\ &=\; \frac{\omega_1 + \omega_2\theta\phi}{\omega_1\omega_2 + \theta\phi} . \end{aligned}$$

(ii) Condition (a)(i) may be re-written as follows:

$$\begin{aligned} 0 &= \omega_2 + \bar{\omega}_2 + 2\left(S^2 + Re(\bar{\omega}_1\theta\phi)C^2\right) \\ &= \omega_2 + \bar{\omega}_2 + 2 - 2C^2(1 - Re(\omega_1\theta\phi)) \\ &= (1 + \omega_2) - 2C^2(1 - Re(\omega_1\theta\phi)) + (1 + \bar{\omega}_2) \ . \end{aligned}$$

Notice now that if $|z| = 1$, then $\frac{1+\bar{\omega}_2}{1+\omega_2} = \bar{\omega}_2$; divide the above equation by $(1+\omega_2)$, and use this observation (twice), as well as the fact that $2Re(z) = z + \bar{z}$, to find that:

$$\begin{aligned} 0 &= 1 - C^2\left(\frac{1 - \bar{\omega}_1\theta\phi}{1+\omega_2} + \frac{1 - \omega_1\overline{\theta\phi}}{1+\omega_2}\right) + \bar{\omega}_2 \\ &= 1 - C\left(\frac{(\omega_1 - \theta\phi)C}{\omega_1(1+\omega_2)} + \frac{(\bar{\omega}_1 - \overline{\theta\phi})C}{\bar{\omega}_1(1+\bar{\omega}_2)}\bar{\omega}_2\right) + \bar{\omega}_2 \\ &= 1 - C(\omega_0 + \bar{\omega}_0\bar{\omega}_2) + \bar{\omega}_2 \\ &= S^2 + C^2 - C(\omega_0 + \omega_0\bar{\omega}_2) + \bar{\omega}_2 \\ &= S^2 + (C - \omega_0)(C - \overline{\omega_0\omega_2}) \ , \end{aligned}$$

and consequently,

$$m_1\overline{m_2} = \frac{S^2}{(C - \omega_0)(C - \overline{\omega_0\omega_2})} = -1 \ .$$

□

LEMMA 2.2.4 *Suppose $a, b, c, d \neq 0$, and suppose $\theta, \phi, \omega_j, j = 0, 1, 2$ are complex numbers of unit modulus, and C, S are non-negative numbers satisfying $C^2 + S^2 = 1$ and $S \neq 0$. Assume that $\omega_2 = -1$ and that the equations 2.2.2-2.2.5 are satisfied. Then,*

(a) either (i) $C = 0$, or (ii) $\theta\phi = \omega_1$, and $\omega_0 = \pm 1$.

(b) Define

$$m_1 = -\frac{\phi S}{C - \omega_0} = \frac{\theta\phi C + \omega_1\omega_0}{\theta S} \ ; \ m_2 = -\frac{\phi S}{C - \omega_0\omega_2} = \frac{\theta\phi C + \omega_0\omega_1\omega_2}{\theta S} \ .$$

Then m_1 and m_2 are well defined, non-zero, and

$$\frac{m_1}{m_2} = \begin{cases} -1 & \text{if } C = 0 \\ \frac{C\pm 1}{C\mp 1} & \text{if } \theta\phi = \omega_1 \ . \end{cases}$$

Proof: Applying Lemma 2.2.2 we conclude that $-1 = -(S^2 + C^2) = -(S^2 + Re(\theta\phi)C^2)$ and consequently conclude that either $C = 0$ or $\theta\phi = \omega_1$. If $\theta\phi = \omega_1$, then equations 2.2.2 and 2.2.3 become

$$a(C - \omega_0) + b\phi S \;=\; 0 \;=\; a\theta S - b\omega_1(C + \omega_0)\ ,$$

from which we may deduce (since $a, b \neq 0$) that

$$\omega_1(C^2 - \omega_0^2) \;=\; -\theta\phi S^2 = -\omega_1 S^2\ ,$$

which implies that $\omega_0^2 \;=\; 1$, thereby proving (a).

The proof of (b) is an easy verification. □

In the next two Propositions we give the exact description of when P and P' (as in the first paragraph of this section) afford equivalent connections W and W' when $n = 3$. In the first Proposition we assume that either $1 \in \{\omega, \omega'\}$ or $(P, P') \in \Omega_0$ and present necessary and sufficient conditions for W and W' to be equivalent - where Ω_0 denotes the following set:

$$\Omega_0 \;=\; \Big\{(P, P') : Im(\omega), Im(\omega') > 0,\ 1 \neq 1_{\{1\}}(C) + 1_{\{1\}}(C')\Big\}\ .$$

We exhaust the remaining cases in the second Proposition.

PROPOSITION 2.2.5 *In order for W and W' to denote equivalent connections, it is necessary that one of the following ten relations is satisfied; if it is further the case that either $1 \in \{\omega, \omega'\}$ or $(P, P') \in \Omega_0$, then each of the ten conditions (i)-(x) is also sufficient for W and W' to be equivalent.*

(i) $\omega = \omega'$ *and* $C = C' = 0$.

(ii) $\omega = \omega', S = S', C = C'$ *and* $(\theta\phi)(\theta'\phi') = \omega$.

(iii) $\omega = \omega', S = S', C = C'$, *and* $(\theta\phi)(\bar{\theta}'\bar{\phi}') = 1$.

(iv) (α) $Re(\omega') \;=\; -(S^2 + Re(\theta\phi)C^2),\ Re(\omega) \;=\; -(S'^2 + Re(\theta'\phi')C'^2)$; *and*

(β) *if it is the case that* $C = C' = 1$, *then*

$$\omega = -(\overline{\theta'\phi'})\ , \textit{and}\ \omega' = -(\overline{\theta\phi})\ ,$$

or

$$\bar{\omega} = -(\overline{\theta'\phi'}) \text{ , and } \bar{\omega}' = -\overline{(\theta\phi)} \text{ ,}$$

(γ) *if it is the case that* $C \neq 1 \neq C'$ *and* $Im(\omega), Im(\omega') > 0$, *then*

$$\frac{1 + \omega'\theta\phi}{\omega' + \theta\phi} = \frac{1 + \omega\theta'\phi'}{\omega + \theta'\phi'} .$$

(*v*) (α) $Re(\omega') = -(S^2 + Re(\bar{\omega}\theta\phi)C^2)$, $Re(\omega) = -(S'^2 + Re(\theta'\phi')C'^2)$; *and*

(β) *if it is the case that* $C = C' = 1$, *then*

$$\omega = -(\theta'\phi') \text{ , and } \omega' = -(\omega\overline{\theta\phi}) \text{ ,}$$

or

$$\bar{\omega} = -(\theta'\phi') \text{ , and } \bar{\omega}' = -(\omega\overline{\theta\phi}) \text{ ,}$$

(γ) *if it is the case that* $C \neq 1 \neq C'$ *and* $Im(\omega), Im(\omega') > 0$, *then*

$$\frac{1 + \bar{\omega}\omega'\theta\phi}{\omega' + \bar{\omega}\theta\phi} = \frac{1 + \bar{\omega}\theta'\phi'}{\bar{\omega} + \theta'\phi'} .$$

(*vi*) (α) $Re(\omega') = -(S^2 + Re(\theta\phi)C^2)$, $Re(\omega) = -(S'^2 + Re(\bar{\omega}'\theta'\phi')C'^2)$; *and*

(β) *if it is the case that* $C = C' = 1$, *then*

$$\omega = -(\omega'\overline{\theta'\phi'}) \text{ , and } \omega' = -(\theta\phi) \text{ ,}$$

or

$$\bar{\omega} = -(\omega'\overline{\theta'\phi'}) \text{ , and } \bar{\omega}' = -(\theta\phi) \text{ .}$$

(γ) *if it is the case that* $C \neq 1 \neq C'$ *and* $Im(\omega), Im(\omega') > 0$, *then*

$$\frac{1 + \bar{\omega}'\theta\phi}{\bar{\omega}' + \theta\phi} = \frac{1 + \bar{\omega}'\omega\theta'\phi'}{\omega + \bar{\omega}'\theta'\phi'} .$$

(*vii*) (α) $Re(\omega') = -(S^2 + Re(\omega\theta\phi)C^2)$, $Re(\omega) = -(S'^2 + Re(\omega'\theta'\phi')C'^2)$; *and*

(β) *if it is the case that* $C = C' = 1$, *then*

$$\omega = -(\bar{\omega}'\theta'\phi') \text{ , and } \omega' = -(\omega\theta\phi) \text{ ,}$$

or

$$\bar{\omega} = -(\bar{\omega}'\theta'\phi') \text{ , and } \bar{\omega}' = -(\omega\theta\phi) \text{ ; and}$$

(γ) if it is the case that $C \neq 1 \neq C'$ and $Im(\omega), Im(\omega') > 0$, then

$$\frac{1 + \bar{\omega}\bar{\omega}'\theta\phi}{\bar{\omega}' + \bar{\omega}\theta\phi} = \frac{1 + \omega'\omega\theta'\phi'}{\omega + \bar{\omega}'\theta'\phi'}.$$

(viii) $\omega = \omega' = 1$, *and the matrix* $\begin{pmatrix} C & \theta S \\ \phi S & -\theta\phi C \end{pmatrix}$ *is unitarily equivalent to a constant multiple of* $\begin{pmatrix} C' & \theta' S' \\ \phi' S' & -\theta'\phi' C' \end{pmatrix}$.

(ix) $\omega = \omega' = -1, S = S', C = C'$ *and* $(\theta\phi)(\theta'\phi') = 1$.

(x) $\omega = \omega' = -1, S = S', C = C'$, *and* $(\theta\phi)(\theta'\phi') = -1$.

Proof: First we present the condition for W and W' to be equivalent as biunitary matrices in the form of a set of equations. By definition W is equivalent to W' if and only if there exist unitary matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ , } A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in M_2$$

and $U = (u_{i,j}), U' = (u'_{i,j}) \in M_3$ such that $(U \otimes A)W = W'(U' \otimes A')$; *i.e.*, if and only if the following equations hold:

$$au_{1,1} = a'u'_{1,1} \quad (2.2.10)$$

$$bu_{1,1} = b'u'_{1,1} \quad (2.2.11)$$

$$cu_{1,1} = c'u'_{1,1} \quad (2.2.12)$$

$$du_{1,1} = d'u'_{1,1} \quad (2.2.13)$$

$$au_{1,2} = a'u'_{1,2} \quad (2.2.14)$$

$$b\omega u_{1,2} = b'u'_{1,2} \quad (2.2.15)$$

$$cu_{1,2} = c'u'_{1,2} \quad (2.2.16)$$

$$d\omega u_{1,2} = d'u'_{1,2} \quad (2.2.17)$$

$$(aC + b\phi S)u_{1,3} = a'u'_{1,3} \quad (2.2.18)$$

$$(a\theta S - b\theta\phi C)u_{1,3} = b'u'_{1,3} \quad (2.2.19)$$

$$
\begin{aligned}
(cC + d\phi S)u_{1,3} &= c'u'_{1,3} && (2.2.20)\\
(c\theta S - d\theta\phi C)u_{1,3} &= d'u'_{1,3} && (2.2.21)\\
au_{2,1} &= a'u'_{2,1} && (2.2.22)\\
bu_{2,1} &= b'u'_{2,1} && (2.2.23)\\
cu_{2,1} &= c'\omega'u'_{2,1} && (2.2.24)\\
du_{2,1} &= d'\omega'u'_{2,1} && (2.2.25)\\
au_{2,2} &= a'u'_{2,2} && (2.2.26)\\
b\omega u_{2,2} &= b'u'_{2,2} && (2.2.27)\\
cu_{2,2} &= c'\omega'u'_{2,2} && (2.2.28)\\
d\omega u_{2,2} &= d'\omega'u'_{2,2} && (2.2.29)\\
(aC + b\phi S)u_{2,3} &= a'u'_{2,3} && (2.2.30)\\
(a\theta S - b\theta\phi C)u_{2,3} &= b'u'_{2,3} && (2.2.31)\\
(cC + d\phi S)u_{2,3} &= c'\omega'u'_{2,3} && (2.2.32)\\
(c\theta S - d\theta\phi C)u_{2,3} &= d'\omega'u'_{2,3} && (2.2.33)\\
au_{3,1} &= (a'C' + c'\theta'S')u'_{3,1} && (2.2.34)\\
bu_{3,1} &= (b'C' + d'\theta'S')u'_{3,1} && (2.2.35)\\
cu_{3,1} &= (a'\phi'S' - c'\theta'\phi'C')u'_{3,1} && (2.2.36)\\
du_{3,1} &= (b'\phi'S' - d'\theta'\phi'C')u'_{3,1} && (2.2.37)\\
au_{3,2} &= (a'C' + c'\theta'S')u'_{3,2} && (2.2.38)\\
b\omega u_{3,2} &= (b'C' + d'\theta'S')u'_{3,2} && (2.2.39)\\
cu_{3,2} &= (a'\phi'S' - c'\theta'\phi'C')u'_{3,2} && (2.2.40)\\
d\omega u_{3,2} &= (b'\phi'S' - d'\theta'\phi'C')u'_{3,2} && (2.2.41)\\
(aC + b\phi S)u_{3,3} &= (a'C' + c'\theta'S')u'_{3,3} && (2.2.42)\\
(a\theta S - b\theta\phi C)u_{3,3} &= (b'C' + d'\theta'S')u'_{3,3} && (2.2.43)\\
(cC + d\phi S)u_{3,3} &= (a'\phi'S' - c'\theta'\phi'C')u'_{3,3} && (2.2.44)\\
(c\theta S - d\theta\phi C)u_{3,3} &= (b'\phi'S' - d'\theta'\phi'C')u'_{3,3} && (2.2.45)
\end{aligned}
$$

First we will prove that in order for W to be equivalent to W' it is necessary that one of the relations (i)-(x) should be satisfied. The proof will be broken up into the consideration of numerous cases, depending on whether certain entries of U are not, or are, equal to 0, and on the values of ω, ω', and in each case we will end up by proving that one of the ten conditions (i)-(x) holds.

First assume $\omega \neq 1 \neq \omega'$.

Case(1): $u_{1,1} \neq 0$

The unitarity of A and A', together with equations 2.2.10-2.2.13 (which say that $u_{1,1}A = u'_{1,1}A'$) imply that $|u_{1,1}| = |u'_{1,1}|$. Let $u_{1,1} = zu'_{1,1}$ where $|z| = 1$; it follows that $A' = zA$. So (by replacing the pair (A, U) by $(zA, z^{-1}U)$, in case $z \neq 1$) we may assume, without loss of generality, that $A = A'$ and $u_{1,1} = u'_{1,1}$.

Since (a, c) is the first column of a unitary matrix and hence not zero, deduce from 2.2.14 and 2.2.16 that $u_{1,2} = u'_{1,2}$; similarly, deduce from equations 2.2.15 and 2.2.17 that $\omega u_{1,2} = u'_{1,2}$. The assumption $\omega \neq 1$ now implies that $u_{1,2} = u'_{1,2} = 0$.

Similarly equations 2.2.22 and 2.2.23 imply that $u_{2,1} = u'_{2,1}$, while equations 2.2.24 and 2.2.25 imply that $u_{2,1} = \omega' u'_{2,1}$; so, as before, $u_{2,1} = u'_{2,1} = 0$.

We now consider two sub-cases depending upon whether the entry $u_{2,2}$ is non-zero or zero.

Case (1.1): $u_{2,2} \neq 0$.

Equations 2.2.26 and 2.2.27 now read: $au_{2,2} = au'_{2,2}$ and $b\omega u_{2,2} = bu'_{2,2}$. Since $\omega \neq 1$, it cannot be the case that both a and b are non-zero. So either $a = 0$ or $b = 0$. Since A is unitary, this means that either $a = d = 0$ or $b = c = 0$. Hence there are two possibilities. We consider them separately in two sub-sub-cases

Case (1.1.1) : $b = c = 0$ (and so $a \neq 0 \neq d$).

We will show in this case that either relation (i) or (iii) is satisfied. It follows from (the assumptions of this case and) equation 2.2.26 and 2.2.29 that $u_{2,2} = u'_{2,2}$ and $\omega = \omega'$.

Now we will consider each of the possibilities of $u_{1,3}$ and $u_{2,3}$ being either non-zero or zero.

Suppose now that $u_{1,3} \neq 0$. Since equation 2.2.19 implies that $a\theta S u_{1,3} = 0$, we first conclude that $S = 0$. Deduce next from equation 2.2.18 and also from equation 2.2.21 that $u'_{1,3} = u_{1,3} = -\theta\phi u_{1,3}$, and that hence, $\theta\phi = -1$. Since we already know that $u_{1,2} = u_{2,1} = 0$, we find from the orthogonality of the first two rows of U (and the assumption $u_{1,3} \neq 0$) that $u_{2,3} = 0$. The orthogonality of the second and third rows of U now implies that $u_{3,2} = 0$. Thus the second entry of the first and third columns of U are 0; the orthogonality of the first and third columns then shows that necessarily $u_{3,1} \neq 0 (\neq u_{3,3})$. In a similar way, we can prove that $S' = 0, u_{3,1} = u'_{3,1}$ and $\theta'\phi' = -1$. So, in this (sub-sub-)case, we do have $\omega = \omega', S = (0 =)S', C = (1 =)C', (\theta\phi)(\theta'\phi') = 1$ - i.e., the relation (iii) of the proposition holds.

If $u_{1,3} = 0$, then 2.2.18 implies $u'_{1,3} = 0$. Suppose for a moment that $u_{2,3} \neq 0$; then 2.2.31 would imply that $S = 0$. We could then infer from equations 2.2.33 and 2.2.30 that $-\theta\phi u_{2,3} = \omega' u'_{2,3}$ and $u_{2,3} = u'_{2,3}$, and hence that $\theta\phi = -\omega'$. The orthogonality of the first and third rows of U would show (as before) that $u_{3,1} = 0$. Then the orthogonality of the second and third rows of U would show that $u_{3,2} \neq 0$. In a similar way, the fact that $u_{3,2} \neq 0$ will imply that $S' = 0$ and $\theta'\phi' = -\omega$ and the relation (iii) is satisfied in this case also. So, we are through in this case also.

Finally, if $u_{1,3} = u_{2,3} = 0$, then equations 2.2.18 and 2.2.30 imply that $u'_{1,3} = 0$ and $u'_{2,3} = 0$. By orthogonality, $u_{3,1} = u'_{3,1} = u'_{3,2} = u_{3,2} = 0$. From 2.2.43 we get $a\theta S u_{3,3} = d\theta' S' u'_{3,3}$. Taking absolute values, we get $S = S'$. If $C = (C') \neq 0$ - i.e. the relation (i) is not satisfied - then from equations 2.2.45 and 2.2.42, we get $\theta\phi u_{3,3} = \theta'\phi' u'_{3,3}$ and $u_{3,3} = u'_{3,3}$. So it follows $(\theta\phi)(\theta'\phi') = 1$. Hence either the relation (i) or the relation (iii) is satisfied in this case. We have completed the proof in the case (1.1.1).

Case (1.1.2) : $a = d = 0$ (and so $b \neq 0 \neq c$).

We will show in this case that either relation (i) or (ix) is satisfied. It follows (under the standing assumptions in this case) from equations 2.2.27 and 2.2.28 that $\omega = \omega'$ and that $u_{2,2} = \bar{\omega} u'_{2,2} \neq 0$. Since ω and ω' are both, by definition, in the upper half plane, and since ω is assumed to be not equal to 1, it must be the case that $\omega = \omega' = -1$.

Suppose now that $u_{1,3} \neq 0$. Then equation 2.2.18 shows that $S = 0$ (and so $C = 1$). Deduce now from equation 2.2.19 and 2.2.20 that $-\theta\phi u_{1,3} = u'_{1,3} = u_{1,3}$, and hence that $\theta\phi = -1$. Arguing exactly as in the third paragraph of the proof of Case (1.1.1), we may conclude that in this case, we have: $u_{2,3} = u_{3,2} = 0, u_{3,1} \neq 0, S' = 0$ and $\theta'\phi' = -1$. Thus, we do have $\omega = \omega'(= -1), S = (0 =)S', C = (1 =)C'$ and $(\theta\phi)(\theta'\phi') = 1$ - i.e. the relation (ix) holds.

If $u_{1,3} = 0$, then it follows from equation 2.2.21 that $u'_{1,3} = 0$. Suppose for a moment that $u_{2,3} \neq 0$; then, we infer from equation 2.2.30 that $S = 0$. Now, equations 2.2.31 and 2.2.32 show that $-\theta\phi u_{2,3} = u'_{2,3}$ and $u_{2,3} = \omega' u'_{2,3}$; and hence, $\theta\phi = -\omega$. Arguing exactly as in the fourth paragraph of the Case (1.1.1), we may conclude that $u_{3,1} = 0, u_{3,2} \neq 0, S' = 0$ and $\theta'\phi' = -\bar{\omega}$, and so $(\theta\phi)(\theta'\phi') = 1$; and we again find that relation (ix) holds.

Finally, if $u_{1,3} = u_{2,3} = 0$, we find from equations 2.2.21 and 2.2.31 that $u'_{1,3} = 0$ and $u'_{2,3} = 0$. By orthogonality, we get $u_{3,1} = u'_{3,1} = u'_{3,2} = u_{3,2} = 0$. If $C \neq 0$, - i.e., if relation (i) does not hold - then equations 2.2.43 and 2.2.44 imply that $-\theta\phi u_{3,3} = u'_{3,3}$ and $u_{3,3} = -\theta'\phi' u'_{3,3}$. Hence $(\theta\phi)(\theta'\phi') = 1$, and again, relation (ix) is seen to hold.

We now come to the remaining sub-case of Case (1).

Case (1.2) : $u_{2,2} = 0$.

We will prove that in this case the relation (iv) is satisfied.

Equations 2.2.26 and 2.2.27 imply that $u'_{2,2} = 0$ (since a or b must be non-zero). Thus U and U' are both 3×3 unitary matrices which have 0 entries in the (1,2), (2,1) and (2,2) places; a moment's thought shows that such a unitary matrix has non-zero entries precisely at the (1,1), (2,3) and (3,2) places; i.e., $u_{3,1} = u'_{3,1} = u'_{1,3} = u_{1,3} = u_{3,3} = u'_{3,3} = 0$. Let ω_0 and ω'_0 be the complex numbers of unit modulus such that $u'_{2,3} = \omega_0 u_{2,3}$ and $u_{3,2} = \omega'_0 u'_{3,2}$.

Now, equations 2.2.30-2.2.33 and equations 2.2.38-2.2.41 may be re-written thus:

$$\begin{aligned} a(C - \omega_0) + b\phi S &= 0. \\ a\theta S - b(\theta\phi C + \omega_0) &= 0. \\ c(C - \omega_0\omega') + d\phi S &= 0. \\ c\theta S - d(\theta\phi C + \omega_0\omega') &= 0 \end{aligned} \tag{2.2.46}$$

$$\begin{aligned} a(C' - \omega'_0) + c\theta' S' &= 0 \\ a\phi' S' - c(\theta'\phi' C' + \omega'_0) &= 0 \\ b(C' - \omega'_0\omega) + d\theta' S' &= 0 \end{aligned}$$

$$b\phi'S' - d(\theta'\phi'C' + \omega_0'\omega) = 0 \tag{2.2.47}$$

There are three possibilities now, which we consider separately.

Case (1.2.1) : $a = d = 0$.

It follows from two applications of Case (i) of Lemma 2.2.2 that $S = 0$, $\omega_0 = -\theta\phi$, $\omega_0\omega' = 1$, (so that $\omega' = -\overline{\theta\phi}$), and that $S' = 0$, $\omega_0' = -\theta'\phi'$, $\omega_0'\omega = 1$ (so that $\omega = -\overline{\theta'\phi'}$). The relation (iv) is easily seen to be satisfied.

Case (1.2.2) : $b = c = 0$.

It follows from two applications of Case (ii) of Lemma 2.2.2 that $S = 0$, $\omega' = -\theta\phi$, $\omega_0 = 1$ (so that $\omega' = -\theta\phi$), and that $S' = 0$, $\omega = -\theta'\phi'$, $\omega_0' = 1$ (so that $\omega = -\theta'\phi'$). Thus the relation (iv) is satisfied.

Case (1.2.3) : $a, b, c, d \neq 0$.

It follows from two applications of Case (iii) of Lemma 2.2.2 that $S \neq 0$ and $Re(\omega') = -(S^2 + Re(\theta\phi)C^2)$, and that $S' \neq 0$ and $Re(\omega) = -(S'^2 + Re(\theta'\phi')C'^2)$.

If it is also assumed that $\omega, \omega' \neq \pm 1$, we may deduce from Case (iii) of Lemma 2.2.2 that

$$\omega_0 = \frac{(1 - \theta\phi)C}{1 + \omega'}, \quad \omega_0' = \frac{(1 - \theta'\phi')C'}{1 + \omega}.$$

Now, first by applying (b) (i) of Lemma 2.2.3 separately to ω_0 and ω_0' and then using Lemma 2.2.1 we may verify that the condition (γ) is satisfied and hence the relation (iv) is indeed satisfied.

Now the proof for the Case (1) is complete.

Case (2) : $u_{1,1} = 0$.

The proof is again divided into many sub (and sub-sub-) cases.

Case (2.1) : $u_{1,2} \neq 0$.

Using the unitarity of A and A', we see from equations 2.2.11-2.2.17 that $|u_{1,2}| = |u'_{1,2}|$. By resoning as in Case(1), we can assume without loss of generality that $a = a'$, $b\omega = b'$, $c = c'$, $d\omega = d'$ and $u_{1,2} = u'_{1,2}$. Then, equations 2.2.26 and 2.2.27 imply that $u_{2,2} = u'_{2,2}$, while equations 2.2.28 and 2.2.29 imply that $u_{2,2} = \omega' u'_{2,2}$. By our standing assumption that $\omega' \neq 1$, we may conclude that $u_{2,2} = u'_{2,2} = 0$.

Now, equations 2.2.22-2.2.25 may be re written as:

$$au_{2,1} = au'_{2,1},\ bu_{2,1} = b\omega u'_{2,1},\ cu_{2,1} = c\omega' u'_{2,1},\ du_{2,1} = d\omega\omega' u'_{2,1} \qquad (\dagger)$$

We consider two cases now, according as whether or not $u_{2,1} = 0$.

Case (2.1.1) : $u_{2,1} \neq 0$.

In this case, we immediately see from the first two equations of ($\dagger$) that either a or b must vanish. Since A is unitary, we thus have two possibilities: (i) $a = d = 0$, and (ii) $b = c = 0$. Similar to case (1) we consider these two possibilities separately.

(i) $a = d = 0$. This and ($\dagger$) immediately yield $u_{2,1} = \omega u'_{2,1} \neq 0$, and $\omega = \omega'$. We will show that either relation (i) or (ii) is satisfied in this case.

Suppose $u_{1,3} \neq 0$. Then deduce from equation 2.2.18 that $S = 0$. Equations 2.2.20 and 2.2.19 then imply that $u_{1,3} = u'_{1,3}$ and $\theta\phi = -\omega$. The orthogonality of the first two rows of U now shows that $u_{2,3} = 0$. The orthogonality of the last two rows of U then shows that $u_{3,1} = 0$. The orthogonality of the first and last rows of U then shows that $u_{3,2} \neq 0$. Deduce then from equation 2.2.39 that $u'_{3,2} \neq 0$; then appeal to equation 2.2.38 to conclude that $S' = 0$, and then (again) from equation 2.2.39 that in fact $u'_{3,2} = u_{3,2}$. Finally conclude from equation 2.2.40 that $\theta'\phi' = -1$. So we have $(\theta\phi)(\theta'\phi') = \omega$, and the relation (ii) is satisfied in this instance.

Suppose now that $u_{1,3} = 0$. Then 2.2.19 implies that $u'_{1,3} = 0$. Suppose in addition that $u_{2,3} \neq 0$; then 2.2.30 implies that $S = 0$. Equations 2.2.32 then implies that $u_{2,3} = \omega' u'_{2,3}$ and consequently 2.2.31 implies that $\theta\phi = -1$. The orthogonality of the last two rows of U show that necessarily $u_{3,1} \neq 0$. The non-vanishing of $u_{3,1}$ will, together with equations 2.2.34-2.2.36, show (by arguing as in the last paragraph) that $S' = 0$ and $\theta'\phi' = -\omega$. Thus, $(\theta\phi)(\theta'\phi') = \omega$, and the relation (ii) is again satisfied.

Finally, if $u_{1,3} = u_{2,3} = 0$, then 2.2.19 and 2.2.31 imply that $u'_{1,3} = u'_{2,3} = 0$.

Again using orthogonality conclude that $u_{3,1} = u'_{3,1} = u'_{3,2} = u_{3,2} = 0$. Now using 2.2.42 we get the relation $b\phi S u_{3,3} = c\theta' S' u'_{3,3}$ which in turn (by taking absolute value) implies that $S = S'$. If $C(= C') \neq 0$ (i.e.the relation (i) is not satisfied), then equations 2.2.43 and 2.2.44 imply that $-b\theta\phi u_{3,3} = b\omega u'_{3,3}$ and $c u_{3,3} = -c\theta'\phi' u'_{3,3}$. Hence, $(\theta\phi)(\theta'\phi') = \omega$, and the relation (ii) is again satisfied.

We now come to the second possibility in this sub-case:

(ii) $b = c = 0$.

In this case we will show that either the relation (i) or (x) is satisfied.

First, equation (†) implies that $\omega = \omega'$ and that $u_{2,1} = u'_{2,1}$. Our assumptions that ω and ω' are in the upper half-plane and not equal to 1 then force $\omega = \omega' = -1$.

Suppose $u_{1,3} \neq 0$. Deduce from equations 2.2.18-2.2.21 that $S = 0, u_{1,3} = u'_{1,3}$ and $\theta\phi = -\omega$. Then use orthogonality to get $u_{2,3} = u'_{2,3} = u_{3,1} = u'_{3,1} = 0$. and $u_{3,2} \neq 0$. Then conclude from 2.2.38-2.2.41 that $S' = 0, u_{3,2} = u'_{3,2}$ and $\theta'\phi' = -1$, so that $(\theta\phi)(\bar{\theta}'\bar{\phi}') = -1$ and the relation (x) is satisfied.

Next, if $u_{1,3} = 0 \neq u_{2,3}$, then deduce from 2.2.30-2.2.33, that $S = 0$. $u_{2,3} = \omega\omega' u'_{2,3}$ and $\theta\phi = -1$. Deduce from orthogonality that $u_{3,2} = u'_{3,2} = 0$ and that $u_{3,1} \neq 0$; and then deduce from 2.2.34-2.2.37 that $S' = 0$ and $\theta'\phi' = -\bar{\omega}$, so that $(\theta\phi)(\theta'\bar{\phi}') = -1$. Hence the relation (x) holds.

And finally, if $u_{1,3} = u_{2,3} = 0$, then deduce 2.2.18 and 2.2.30 that $u'_{1,3} = u'_{2,3} = 0$. Also from orthogonality deduce that $u_{3,1} = u'_{3,1} = u'_{3,2} = u_{3,2} = 0$, and hence that $u_{3,3} \neq 0$. Now use this and equations 2.2.42-2.2.45, in an analogous way to the previous case, to conclude that $C = C'$, and that if $C(= C') \neq 0$ (i.e. the relation (i) is not satisfied), then $(\theta\phi)(\bar{\theta}'\bar{\phi}') = -1$. Thus the relation (x) does hold in any of these situations.

Case (2.1.2) : $u_{2,1} = 0$.

It follows from equation (†) that also $u'_{2,1} = 0$. We will show that the relation (v) is satisfied in this case.

First, we deduce, from unitarity of U and U' (and by arguing exactly as in the second paragraph of the proof of Case (1.2)), that both U and U' have non-zero entries at precisely the (1,2), (2,3) and (3,1) places. Let $u'_{2,3} = \omega_0 u_{2,3}$ and $u_{3,1} = \omega'_0 u'_{3,1}$ where $|\omega_0| = |\omega'_0| = 1$.

First re-write equations 2.2.30-2.2.33 as

$$\begin{aligned} a(C - \omega_0) + b\phi S &= 0 \\ a\theta S - b(\theta\phi C + \omega_0\omega) &= 0 \\ c(C - \omega_0\omega') + d\phi S &= 0 \\ c\theta S - d(\theta\phi C + \omega_0\omega'\omega) &= 0 \end{aligned} \tag{2.2.48}$$

and equations 2.2.34-2.2.37 as

$$\begin{aligned} a(C' - \omega'_0) + c\theta' S' &= 0 \\ a\phi' S' - c(\theta'\phi' C' + \omega'_0) &= 0 \\ b(C' - \omega'_0\bar{\omega}) + d\theta' S' &= 0 \\ b\phi' S' - d(\theta'\phi' C' + \omega'_0\bar{\omega}) &= 0\ . \end{aligned} \tag{2.2.49}$$

As in the proof of Case (1.2), we need to separately consider three possibilities:

Case (i): $a = d = 0$.

Deduce from Case (i) of Lemma 2.2.2 that $S = 0$, $\theta\phi = -\omega_0\omega$, $\omega' = \bar{\omega}_0$ and that $S' = 0$, $\theta'\phi' = -\omega'_0$, $\omega = \omega'_0$; deduce that $C = 1 = C'$ and that $\omega = -\theta'\phi'$, $\omega' = -\omega\overline{\theta\phi}$; i.e., condition $(v)(\beta)$ is satisfied.

Case (ii): $b = c = 0$.

Deduce from Case (ii) of Lemma 2.2.2 that $S = 0$, $\theta\phi = -\omega\omega'$, $\omega_0 = 1$ and that $S' = 0$, $\theta'\phi' = -\bar{\omega}$, $\omega'_0 = 1$; deduce that $C = 1 = C'$ and that $\omega = -\theta'\phi'$, $\bar{\omega}' = -\omega\overline{\theta\phi}$; i.e., condition $(v)(\beta)$ is satisfied.

Case (iii): $a, b, c, d \neq 0$.

It follows from two applications of Case (iii) of Lemma 2.2.2 that $S \neq 0$, $Re(\omega') = -(S^2 + Re(\bar{\omega}\theta\phi)C^2)$, and that $S' \neq 0$, $Re(\omega) = -(S'^2 + Re(\theta'\phi')C'^2)$.

If $\omega, \omega' \neq \pm 1$, we may deduce from Case (iii) of Lemma 2.2.2 that

$$\omega_0 = \frac{(\omega - \theta\phi)C}{\omega(1+\omega')}, \; \omega_0' = \frac{(1-\theta'\phi')C'}{1+\bar{\omega}}.$$

Now apply (b) (ii) of Lemma 2.2.3 and Lemma 2.2.1 to verify that the part (γ) of the relation (v) is also satisfied.

Case (2.2): $u_{1,1} = u_{1,2} = 0$.

Then by orthogonality of the first two rows of U) we find that $u_{1,3} \neq 0$ and $u_{2,3} = 0$, and that at least one of $u_{2,1}$ or $u_{2,2}$ must be non-zero. We consider these cases separately.

Case (2.2.1): $u_{2,1} \neq 0$

We will show that the relation (vi) is satisfied in this case. As before using equations 2.2.22-2.2.25, we can assuume without loss of generality that $a = a'$, $b = b'$, $c\bar{\omega}' = c'$, $d\bar{\omega}' = d'$ and $u_{2,1} = u'_{2,1}$. We may next deduce from equations 2.2.26-2.2.29 that $u_{2,2} = u'_{2,2} = \omega u_{2,2}$. Since $\omega \neq 1$, we get $u_{2,2} = u'_{2,2} = 0$. Then, using orthogonality, we find that both U and U' have non-zero entries precisely in the (1,3), (2,1) and (3,2) places. Let $u'_{1,3} = \omega_0 u_{1,3}$ and $u_{3,2} = \omega_0' u'_{3,2}$, where $|\omega_0| = |\omega_0'| = 1$. Re-write equations 2.2.18 to 2.2.21 thus:

$$\begin{aligned} a(C - \omega_0) + b\phi S &= 0 \\ a\theta S - b(\theta\phi C + \omega_0) &= 0 \\ c(C - \omega_0\bar{\omega}') + d\phi S &= 0 \\ c\theta S - d(\theta\phi C + \omega_0\bar{\omega}') &= 0. \end{aligned} \tag{2.2.50}$$

Next, re-write equations 2.2.38 to 2.2.41 as

$$\begin{aligned} a(C' - \omega_0') + c\bar{\omega}'\theta' S' &= 0 \\ a\phi' S' - c(\bar{\omega}'\theta'\phi' C' + \omega_0') &= 0 \\ b(C' - \omega_0'\omega) + d\bar{\omega}'\theta' S' &= 0 \\ b\phi' S' - d(\bar{\omega}'\theta'\phi' C' + \omega_0'\omega) &= 0. \end{aligned} \tag{2.2.51}$$

As in the proof of earlier cases (1.2) and (2.1.2), we need to separately consider three possibilities:

Case (i): $a = d = 0$.

Deduce from Case (i) of Lemma 2.2.2 that $S = 0$, $\theta\phi = -\omega_0$, $\omega' = \omega_0$ and that $S' = 0$, $\bar{\omega}'\theta'\phi' = -\omega_0'$, $\omega = \bar{\omega}_0'$; deduce that $C = 1 = C'$ and that $\omega = -\omega'\overline{\theta'\phi'}$, $\omega' = -\theta\phi$; i.e., condition $(vi)(\beta)$ is satisfied.

Case (ii): $b = c = 0$.

Deduce from Case (ii) of Lemma 2.2.2 that $S = 0$, $\theta\phi = -\bar{\omega}'$, $\omega_0 = 1$ and that $S' = 0$, $\bar{\omega}'\theta'\phi' = -\omega$, $\omega_0' = 1$; deduce that $C = 1 = C'$ and that $\bar{\omega} = -\omega'\overline{\theta'\phi'}$, $\bar{\omega}' = -\theta\phi$; i.e., condition $(vi)(\beta)$ is satisfied.

Case (iii): $a, b, c, d \neq 0$.

It follows from two applications of Case (iii) of Lemma 2.2.2 that $S \neq 0$, $Re(\omega') = -(S^2 + Re(\theta\phi)C^2)$, and that $S' \neq 0$, $Re(\omega) = -(S'^2 + Re(\bar{\omega}'\theta'\phi')C'^2)$.

If it is the case that $\omega, \omega' \neq \pm 1$, we may deduce from Case (iii) of Lemma 2.2.2 that

$$\omega_0 = \frac{(1 - \theta\phi)C}{(1 + \bar{\omega}')}, \quad \omega_0' = \frac{(\omega' - \theta'\phi')C'}{\omega'(1 + \omega)}.$$

As before verify the part (γ) of the relation (vi) by using (b) (ii) of Lemma 2.2.3 and Lemma 2.2.1

Case(2.2.2): $u_{2,1} = 0$

As already noted - see the paragraph at the start of the discussion of Case (2.2) - this forces $u_{2,2} \neq 0$. We will show that in this case, the relation (vii) will be satisfied.

As before, using equations 2.2.26-2.2.29, we may assume, without loss of generality, that $a = a'$, $b\omega = b'$, $c\bar{\omega}' = c'$, $d\omega\bar{\omega}' = d'$ and that $u_{2,2} = u'_{2,2}$. Using orthogonality, we see that both U and U' have non-zero entries precisely in the (1,3), (2,2) and (3,1) places. Let $u'_{1,3} = \omega_0 u_{1,3}$ and $u_{3,1} = \omega_0' u'_{3,1}$where $|\omega_0| = |\omega_0'| = 1$. Re-write equations 2.2.18-2.2.21 thus:

$$\begin{aligned} a(C - \omega_0) + b\phi S &= 0 \\ a\theta S - b(\theta\phi C + \omega_0\omega) &= 0 \\ c(C - \omega_0\bar{\omega}') + d\phi S &= 0 \end{aligned}$$

$$c\theta S - d(\theta\phi C + \omega_0\bar{\omega}'\omega) = 0. \tag{2.2.52}$$

Also, deduce from equations 2.2.34-2.2.37 that

$$\begin{aligned} a(C' - \omega_0') + c\bar{\omega}'\theta' S' &= 0 \\ a\phi' S' - c(\bar{\omega}'\theta'\phi' C' + \omega_0') &= 0 \\ b(C' - \omega_0'\bar{\omega}) + d\bar{\omega}'\theta' S' &= 0 \\ b\phi' S' - d(\bar{\omega}'\theta'\phi' C' + \omega_0'\bar{\omega}) &= 0. \end{aligned} \tag{2.2.53}$$

As in the proof of earlier cases, we need to separately consider three possibilities:

Case (i): $a = d = 0$.

Deduce from Case (i) of Lemma 2.2.2 that $S = 0$, $\theta\phi = -\omega_0\omega$, $\omega' = \omega_0$ and that $S' = 0$, $\bar{\omega}'\theta'\phi' = -\omega_0'$, $\omega = \omega_0'$; deduce that $C = 1 = C'$ and that $\omega = -\bar{\omega}'\theta'\phi'$, $\omega' = -\bar{\omega}\theta\phi$; i.e., condition $(vii)(\beta)$ is satisfied.

Case (ii): $b = c = 0$.

Deduce from Case (ii) of Lemma 2.2.2 that $S = 0$, $\theta\phi = -\omega\bar{\omega}'$, $\omega_0 = 1$ and that $S' = 0$, $\bar{\omega}'\theta'\phi' = -\bar{\omega}$, $\omega_0' = 1$; deduce that $C = 1 = C'$ and that $\bar{\omega} = -\bar{\omega}'\theta'\phi'$, $\bar{\omega}' = -\bar{\omega}\theta\phi$; i.e., condition $(vii)(\beta)$ is satisfied.

Case (iii): $a, b, c, d \neq 0$.

It follows from two applications of Case (iii) of Lemma 2.2.2 that $S \neq 0$, $Re(\omega') = -(S^2 + Re(\bar{\omega}\theta\phi)C^2)$, and that $S' \neq 0$, $Re(\omega) = -(S'^2 + Re(\bar{\omega}'\theta'\phi')C'^2)$.

Similar to the earlier cases if $\omega, \omega' \neq \pm 1$, we may deduce from Case (iii) of Lemma 2.2.2 that

$$\omega_0 = \frac{(\omega - \theta\phi)C}{\omega(1 + \bar{\omega}')}, \quad \omega_0' = \frac{(\omega' - \theta'\phi')C'}{\omega'(1 + \bar{\omega})}$$

and then, by using (b) (ii) of Lemma 2.2.3 and Lemma 2.2.1, that the part (γ) of the relation (vii) is also satisfied.

Finally, in order to complete the proof of 'if' part of the proposition, we need to drop the assumption that $\omega \neq 1 \neq \omega'$. We separately consider three possibilities : (i) $\omega = 1 \neq \omega'$, (ii) $\omega' = 1 \neq \omega$ (iii) $\omega = \omega' = 1$.

(i) $\omega = 1 \neq \omega'$.

Since $\omega = 1$, equations 2.2.10-2.2.17 and 2.2.22-2.2.29 may be re-written as:

$$u_{1,1}A = u'_{1,1}A' \ , \quad u_{1,2}A = u'_{1,2}A' \ ,$$
$$u_{2,1}A = u'_{2,1}D'A' \ , \quad u_{2,2}A = u'_{2,2}D'A' \ ,$$

where we write D' for the diagonal matrix $diag(1, \omega')$. (Notice, by the way, from the above equations and the unitarity of the 3×3 matrices U and U', that $|u_{i,j}| = |u'_{i,j}|$, $\forall 1 \leq i, j \leq 3$.)

Since $\omega' \neq 1$, the matrix D' is linearly independent from the identity matrix. At the same time, as U is a 3×3 unitary, it cannot have a 2×2 block being identically zero. Thus, we find that exactly one vector in the set $\{(u_{1,1}, u_{1,2}), (u_{2,1}, u_{2,2})\}$ is non-zero.

Suppose $(u_{1,1}, u_{1,2}) \neq 0$. Then, using the unitarity of A, A' and the equations 2.2.10-2.2.17, we may (and do) assume without loss of generality that $A = A'$, and $(u_{1,1}, u_{1,2}) = (u'_{1,1}, u'_{1,2})$. It follows then, from the untarity of U and U', that $u_{2,3} \neq 0 \neq u'_{2,3}$.

Now, from the equations 2.2.30-2.2.33, we can deduce the validity of the equations 2.2.46, and hence we may conclude, using Lemma 2.2.2 as before, that $Re(\omega') = -(S^2 + Re(\theta\phi)C^2)$.

Note that the unitarity of U, coupled with the fact that $u_{2,1} = u_{2,2} = 0$, enables us to conclude that $u_{3,3} = 0$, and hence that the vector $(u_{3,1}, u_{3,2}) \neq (0, 0)$. By comparing the two sets of equations 2.2.34-2.2.37 and 2.2.38- 2.2.41, and using the fact that the matrix $\begin{pmatrix} C' & \theta'S' \\ \phi'S' & -\theta'\phi\prime C' \end{pmatrix}$ is unitary, we may derive the following relations:

$$\begin{pmatrix} C' & \theta'S' \\ \phi'S' & -\theta'\phi'C' \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \omega'_0 \begin{pmatrix} a \\ c \end{pmatrix}$$

and

$$\begin{pmatrix} C' & \theta'S' \\ \phi'S' & -\theta'\phi'C' \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = \omega'_0 \begin{pmatrix} b \\ d \end{pmatrix}$$

where ω_0' is a complex number of unit modulus.

The fact that the vectors (a,c) and (b,d) are orthogonal forces the above unitary matrix to be a scalar matrix and as the $(1,1)$ entry is positive we conclude that it is the identity matrix. Hence we deduce that $S'=0,\ C'=1,\ \theta'\phi'=-1$ and that $Re(\omega)=1=-(S'^2+Re(\theta'\phi')C'^2)$. Also note that when $S=0$, the (β) condition in the relation (iv) is satisfied. Hence the relation (iv) is satisfied.

If it turns out, instead, that $(u_{2,1},u_{2,2})\neq 0$, we would, by proceeding as in the case discussed above, first assume without loss of generality that $A=D'A'$ and that $(u_{2,1},u_{2,2})=(u'_{2,1},u'_{2,2})$; we would then deduce from the uniarity of U and U' that $u_{1,3}\neq 0\neq u'_{1,3}$, and eventually, from the equations 2.2.18-2.2.21, find that the equations 2.2.50 are valid. Now conclude, using Lemma 2.2.2 as before, that $Re(\omega')=-(S^2+Re(\theta\phi)C^2)$.

Again, using the unitarity of U and the fact that $(u_{1,1},u_{1,2})=0$, conclude that $(u_{3,1},u_{3,2})\neq 0$. Now, by comparing the two sets of equations 2.2.34-2.2.37 and 2.2.38- 2.2.41, and using the fact that the matrix $\begin{pmatrix} C' & \theta'S' \\ \phi'S' & -\theta'\phi\prime C' \end{pmatrix}$ is unitary, we may derive the following relations:

$$\begin{pmatrix} C' & \theta'S' \\ \phi'S' & -\theta'\phi'C' \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \omega_0'D'\begin{pmatrix} a \\ c \end{pmatrix}$$

and

$$\begin{pmatrix} C' & \theta'S' \\ \phi'S' & -\theta'\phi'C' \end{pmatrix}\begin{pmatrix} b \\ d \end{pmatrix} = \omega_0'D'\begin{pmatrix} b \\ d \end{pmatrix}$$

where ω_0' is a complex number of unit modulus.

The fact that the vectors (a,c) and (b,d) are orthogonal forces the above unitary matrix to be a scalar multiple of D', but by again comparing the $(1,1)$ entry we would find that $\omega_0'=1$. Hence we deduce that $S'=0,\ C'=1,\ \theta'\phi'=-\omega'$ and that $Re(\omega)=1=-(S'^2+Re(\bar{\omega}'\theta'\phi')C'^2)$. Also when $S=0$, (vi)(β) is clearly satisfied. So we find that the relation (vi) is satisfied in this case.

(ii) $\omega'=1\neq\omega$.

Since $\omega'=1$, equations 2.2.10-2.2.17 and 2.2.22-2.2.29 may be re-written as:

$$u_{1,1}A=u'_{1,1}A'\ ,\quad u_{1,2}AD=u'_{1,2}A'\ ,$$

$$u_{2,1}A = u'_{2,1}A' \ , \quad u_{2,2}AD = u'_{2,2}A' \ ,$$

where we write D for the diagonal matrix $diag(1,\omega)$. (Notice, as before, from the above equations and the unitarity of the matrices U and U', that $|u_{i,j}| = |u'_{i,j}|$, $\forall 1 \leq i,j \leq 3$.)

Since $\omega' \neq 1$, by resoning exactly as in the Case (i), we find that exactly one vector in the set $\{(u_{1,1},u_{2,1}),(u_{1,2},u_{2,2})\}$ is non-zero.

Suppose $(u_{1,1},u_{2,1}) \neq 0$. Then, using the unitarity of A, A' and the equations 2.2.10-2.2.17, we may assume without loss of generality that $A = A'$, and $(u_{1,1},u_{2,1}) = (u'_{1,1},u'_{2,1})$. Now, deduce using the unitarity U and U', that $u_{3,2} \neq 0 \neq u'_{3,2}$. Now, from equations 2.2.38-2.2.41, we would find that the equations 2.2.47 hold. Hence, using Lemma 2.2.2, conclude that $Re(\omega) = -(S'^2 + Re(\theta'\phi')C'^2)$.

Also note that the unitarity of U, coupled with the fact that $u_{1,2} = u_{2,2} = 0$, enables us to conclude that $u_{3,3} = 0$, and hence that the vector $(u_{1,3},u_{2,3}) \neq (0,0)$. Proceeding in an analogous way to case (i), using equations 2.2.34-2.2.41, we would find that the relation (iv) is satisfied in this case.

If it turns out, instead, that $(u_{1,2},u_{2,2}) \neq 0$, we would, by proceeding as in the case discussed above, first assume without loss of generality that $AD = A'$ and that $(u_{1,2},u_{2,2}) = (u'_{1,2},u'_{2,2})$; we would then deduce that $u_{3,1} \neq 0 \neq u'_{3,1}$. Now, from equations 2.2.34 -2.2.47, conclude that the set of equations 2.2.48 are valid. Hence, using the lemma 2.2.2 as before, we again find that $Re(\omega) = -(S'^2 + Re(\theta'\phi')C'^2)$.

Again, using the unitarity of U, we find that $(u_{1,3},u_{2,3}) \neq 0$. The equations 2.2.34-2.2.41 and arguments anlogous to the previous cases imply that the unitary matrix $\begin{pmatrix} C & \theta S \\ \phi S & -\theta\phi C \end{pmatrix}$ is equal to D. Hence we find that relation (v) is satisfied in this case.

(iii) $\omega = 1 = \omega'$. First note, from the unitarity of U, that at least one entry in the set $\{u_{1,1},u_{1,2},u_{2,1},u_{2,2}\}$ is non-zero and hence - from equations 2.2.10-2.2.17 and 2.2.22-2.2.29 - we may assume (without loss of generality, as before) that $A = A'$. Also note that the vectors $(u_{1,3},u_{2,3})$ and $(u_{3,1},u_{3,2})$ have the same norm. So they have to be both zero or non-zero. If they are both non-zero, then arguing as in cases (i) and (ii), we can conclude that the relation (iv) is satisfied.

Suppose both the vectors $(u_{1,3}, u_{2,3})$ and $(u_{3,1}, u_{3,2})$ are zero. Then, conclude from equations 2.2.18-2.2.21 and 2.2.30-2.2.41 that also $(u'_{1,3}, u'_{2,3})$ and $(u'_{3,1}, u'_{3,2})$ are zero. The unitarity of U, U' then implies that $u_{3,3}$ and $u'_{3,3}$ are complex numbers of unit modulus Then, equations 2.2.42-2.2.45 may be simplified thus:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C & \theta S \\ \phi S & -\theta\phi C \end{pmatrix} = \omega_0 \begin{pmatrix} C' & \theta' S' \\ \phi' S' & -\theta'\phi' C' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

where ω_0 is a complex number of unit modulus.

Hence the matrix $\begin{pmatrix} C & \theta S \\ \phi S & -\theta\phi C \end{pmatrix}$ is unitarily equivalent to a constant multiple of $\begin{pmatrix} C' & \theta' S' \\ \phi' S' & -\theta'\phi' C' \end{pmatrix}$, and the relation (viii) is satisfied in this case.

Finally, the proof of the 'if' part is complete.

Now, we will first prove the sufficiency of the relations (i)-(iii) and (viii)-(x) and then dispose the case $S = 0 = S'$ for the remaining cases. Finally, with the assumption that $I_{\{0\}}(S) + I_{\{0\}}(S') \neq 2$ (i.e. atleast one among S and S' is non-zero), we will prove the sufficiency of the relations (iv)-(vii) separately for the cases $1 \in \{\omega, \omega'\}$ and $(P, P') \in \Omega_0$. The proof basically consists of an explicit presentation of unitary matrices A, A', U and U', which would satisfy the set of equations 2.2.10-2.2.45.

(i) Suppose $\omega = \omega', C = C' = 0$;

Let $z = (\theta\phi)(\bar{\theta}'\bar{\phi}')$ and $z^{\frac{1}{2}}$ denote any fixed square root of z. Now choose

$$A = A' = \begin{pmatrix} \phi z^{\frac{1}{2}} & 0 \\ 0 & \theta\phi\bar{\theta}' \end{pmatrix}, \quad U = I, \quad U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{\frac{1}{2}} \end{pmatrix}.$$

It can be easily checked that the equations 2.2.10-2.2.45 are satisfied and hence that $(U \otimes A)W = W'(U' \otimes A')$. So $W(\omega, \theta, \phi, 0)$ is equivalent to $W(\omega, \theta', \phi', 0)$ for all values of θ, ϕ, θ', and ϕ'.

(ii) Suppose $\omega = \omega', S = S', C = C'$, and $(\theta\phi)(\theta'\phi') = \omega$.

Choose

$$A = \begin{pmatrix} 0 & \bar{\phi}\bar{\phi}' \\ -1 & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & \omega\bar{\phi}\bar{\phi}' \\ -1 & 0 \end{pmatrix} \quad \text{and}$$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad U' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\theta\phi \end{pmatrix};$$

verify that equations 2.2.10-2.2.45 are satisfied, and conclude that $(U \otimes A)W = W(U' \otimes A')$.

(iii) Suppose $\omega = \omega', S = S', C = C'$, and $(\theta\phi)(\bar{\theta}'\bar{\phi}') = 1$;

Now choose

$$A = A' = \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}, \quad U = U' = I.$$

Again, equations 2.2.10-2.2.45 are satisfied, and so, $(U \otimes A)W = W'(U' \otimes A')$.

(viii) Suppose now that the relation (viii) is satisfied; this means that $\omega = 1 = \omega'$, and that

$$A_0 \begin{pmatrix} C & \theta S \\ \phi S & -\theta\phi C \end{pmatrix} = \omega_0 \begin{pmatrix} C' & \theta' S' \\ \phi' S' & -\theta'\phi' C' \end{pmatrix} A_0$$

where A_0 is a 2×2 unitary matrix and ω_0 is a complex number of unit modulus. Define

$$A = A' = A_0, \quad U = I, \quad U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega_0 \end{pmatrix},$$

and verify that equations 2.2.10-2.2.45 are satisfied to conclude that $(U \otimes A)W = W'(U' \otimes A)$.

(ix) Suppose $\omega = \omega' = -1, S = S', C = C'$ and $(\theta\phi)(\theta'\phi') = 1$;

Define

$$A = A' = \begin{pmatrix} 0 & -\theta'\theta \\ 1 & 0 \end{pmatrix}, \quad U = I, \quad U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\theta\phi \end{pmatrix},$$

and verify that equations 2.2.10-2.2.45 are satisfied to conclude that $(U \otimes A)W = W(U' \otimes A')$.

(x) Suppose $\omega = \omega' = -1, S = S', C = C'$ and $(\theta\phi)(\overline{\theta'\phi'}) = -1$;

Define

$$A = \begin{pmatrix} \phi\bar{\phi}' & 0 \\ 0 & 1 \end{pmatrix}, \quad A' = \begin{pmatrix} \phi\bar{\phi}' & 0 \\ 0 & -1 \end{pmatrix}, \quad U = U' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

verify that equations 2.2.10-2.2.45 are satisfied, and conclude that $(U \otimes A)W = \tilde{W}(U' \otimes A')$.

Assume that $S = S' = 0$.

Note that, under the above assumption, the relations (iv), (v), (vi) and (vii) are basically the (β) part of the corresponding relations. We prove the sufficiency (i.e. provide the desired unitaries A , A', U and U') separately for each relation.

(iv) Suppose the relation (iv) (i.e.(β) of (iv)) is satisfied. We consider the two cases in the relation (β).

If $\omega = -\overline{\theta'\phi'}$ and $\omega' = -\overline{\theta\phi}$, then define

$$A = A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\theta'\phi' & 0 \end{pmatrix}, \quad U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\theta\phi \\ 0 & 1 & 0 \end{pmatrix},$$

and verify that the equations 2.2.10-2.2.45 are satisfied.

If $\omega = -\theta'\phi'$ and $\omega' = -\theta\phi$, then define

$$A = A' = I, \quad U = U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and verify that the equations 2.2.10-2.2.45 are satisfied.

(v) Suppose the relation (v) (i.e.(β) of (v)) is satisfied.

If $\omega = -\theta'\phi'$ and $\omega' = -\omega\overline{\theta\phi}$, then define

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \theta'\phi' & 0 & 0 \end{pmatrix}, \quad U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\bar{\omega}\theta\phi \\ 1 & 0 & 0 \end{pmatrix},$$

and verify that the equations 2.2.10-2.2.45 are satisfied.

If $\omega = -\theta'\phi'$ and $\omega' = -\bar{\omega}\theta\phi$, then define

$$A = I \ , \ A' = diag\{1, \omega\} \ , \ U = U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ,$$

and verify that the equations 2.2.10 2.2.45 are satisfied.

(vi) Suppose the relation (vi) is satisfied.

If $\omega = -\omega'\overline{\theta'\phi'}$ and $\omega' = -\theta\phi$, then define

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ A' = \begin{pmatrix} 0 & 1 \\ \bar{\omega}' & 0 \end{pmatrix} , \ U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -\omega'\theta'\phi' & 0 \end{pmatrix} , \ U' = \begin{pmatrix} 0 & 0 & \theta\phi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ,$$

and verify that the equations 2.2.10-2.2.45 are satisfied.

If $\omega = -\omega'\theta'\phi'$ and $\omega' = -\overline{\theta\phi}$, then define

$$A = I \ , \ A' = diag\{1, \omega'\} \ , \ U = U' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ,$$

and verify that the equations 2.2.10-2.2.45 are satisfied.

(vii) Suppose the relation (vii) is satisfied.

If $\omega = -\bar{\omega}'\theta'\phi'$ and $\omega' = -\omega\theta\phi$, then define

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ A' = \begin{pmatrix} 0 & \omega \\ \bar{\omega}' & 0 \end{pmatrix} , \ U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -\bar{\omega}'\theta'\phi' & 0 & 0 \end{pmatrix} , \ U' = \begin{pmatrix} 0 & 0 & \omega\theta\phi \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} ,$$

and verify that the equations 2.2.10-2.2.45 are satisfied

If $\omega = -\omega'\overline{\theta'\phi'}$ and $\omega' = \omega\overline{\theta\phi}$, then define

$$A = I \ , \ A' = diag\{1, \omega'\omega\} \ , \ U = U' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} ,$$

and verify that the equations 2.2.10-2.2.45 are satisfied.

Assume that $1 \in \{\omega, \omega'\}$ and that $2 \neq I_{\{0\}}(S) + I_{\{0\}}(S')$.

First note that, if $\omega = \omega' = 1$ and the pair (P, P') satisfies part (α) of any one of the relations (iv), (v) or (vi), then it follows that $S = S' = 0$, which is a case we have already disposed. So we only consider the case when exactly one of ω, ω' is equal to 1.

(iv) Suppose that the relation (iv) is satisfied and that exactly one of ω, ω' is 1. To be specific, suppose $\omega = 1 \neq \omega'$; (the other case is similarly treated by replacing every thing by a corresponding 'primed' expression;)

As $\omega = 1$, the pair (P, P') satisfying the relation (iv) implies that $S' = 0$ and that $\theta'\phi' = -1$. Note, by our standing assumption, that $S \neq 0$.

Suppose $\omega' = -1$, then -again as the relation (iv) is satisfied- it follows that either $C = 0$ or $\theta\phi = 1$. If it is the case that $C = 0$, then define

$$A = A' = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi & \omega_0 \\ \phi & -\omega_0 \end{pmatrix}, \quad U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega_0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω_0 is any fixed square root of $\theta\phi$.

If it is the case that $\theta\phi = 1$, then define

$$A = A' = \begin{pmatrix} \frac{1}{\sqrt{2}}(1-C)^{-1/2}\phi S & \frac{1}{\sqrt{2}}(1-C)^{1/2} \\ -\frac{1}{\sqrt{2}}(1+C)^{-1/2}\phi S & \frac{1}{\sqrt{2}}(1+C)^{1/2} \end{pmatrix}, \quad U = U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Verify in both the cases that the equations 2.2.10-2.2.45 are satisfied and hence conclude that $(U \otimes A)W = W'(U' \otimes A')$.

Now assume that $\omega' \neq -1$. Define $\omega_0 = \frac{(1-\theta\phi)C}{(1+\omega')}$. Using Lemma 2.2.3 conclude that $|\omega_0| = 1$ and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C+\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\omega'} = \frac{(\theta\phi C+\omega_0\omega')}{\theta S} = m_2$ (say); Note that, by Lemma 2.2.3, m_1, m_2 are well defined and that $m_1 \neq 0 \neq m_2$.

Now choose complex numbers b, d such that $|b|^2 = \frac{1}{1+|m_1^2|}$, and $|d|^2 = \frac{1}{1+|m_2^2|}$. If we define $c = m_2 d$, and $a = m_1 b$, then it clearly follows that $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$. Again using (b) (ii) of Lemma 2.2.3 we conclude that $a\bar{c} + b\bar{d} = b\bar{d}(1 + m_1\bar{m}_2) = 0$.

Thus, if we define

$$A = A' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega_0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

(all of which are clearly unitary), then all of the equations 2.2.10-2.2.45 - with the possible exception of 2.2.30-2.2.33 - are readily seen to be satisfied; equations 2.2.30-2.2.33, however, are exactly equivalent to the system 2.2.46 of equations; these latter equations are satisfied by the very definition of a, b, c, d. Thus, finally, $(U \otimes A)W = W'(U' \otimes A')$.

(v) Suppose the relation (v) is satisfied. Note that when $\omega = 1$, the relation (v) is same as the relation (iv), whose sufficiency has already been established. So suppose that $\omega \neq 1 = \omega'$. This immediately implies -as the condition (v) is satisfied- that $S = 0$ and that $\bar{\omega}\theta\phi = -1$. Also it follows from our assumption that $S' \neq 0$.

Suppose $\omega = -1$, then - as the relation (v) is satisfied - either $C' = 0$ or $\theta'\phi' = 1$. If it is the case that $C' = 0$, then define

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta' & \theta' \\ \omega_0' & -\omega_0' \end{pmatrix}, \quad A' = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta' & -\theta' \\ \omega_0' & \omega_0' \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega_0' & 0 & 0 \end{pmatrix}, \quad U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where ω_0' is a fixed square root of $\theta'\phi'$.

If it is the case that $\theta'\phi' = 1$, then define

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}}(1 - C')^{-1/2}\theta' S' & -\frac{1}{\sqrt{2}}(1 + C')^{-1/2}\theta' S' \\ \frac{1}{\sqrt{2}}(1 - C')^{1/2} & \frac{1}{\sqrt{2}}(1 + C')^{1/2} \end{pmatrix},$$

$$A' = \begin{pmatrix} \frac{1}{\sqrt{2}}(1 - C')^{-1/2}\theta' S' & \frac{1}{\sqrt{2}}(1 + C')^{-1/2}\theta' S' \\ \frac{1}{\sqrt{2}}(1 - C')^{1/2} & -\frac{1}{\sqrt{2}}(1 + C')^{1/2} \end{pmatrix},$$

$$U = U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Verify in both the cases that the equations 2.2.10-2.2.45 are satisfied and hence conclude that $(U \otimes A)W = W'(U' \otimes A')$.

Assume, now, that $\omega \neq -1$ and define $\omega_0' = \frac{(1-\theta'\phi')C}{(1+\bar{\omega})}$. Using Lemma 2.2.3 conclude that $|\omega_0'| = 1$ and that $-\frac{\theta' S'}{C' - \omega_0'} = \frac{(\theta'\phi' C' + \omega_0')}{\phi' S'} = m_1$ (say); $-\frac{\theta' S'}{C' - \omega_0' \bar{\omega}} = \frac{(\theta'\phi' C' + \omega_0' \bar{\omega})}{\phi' S'} = m_2$ (say). Note m_1, m_2 are well defined, non-zero.

Now choose c, d such that $|c|^2 = \frac{1}{1+|m_1^2|}$, and $|d|^2 = \frac{1}{1+|m_2^2|}$. Define $b = m_2 d$, and $a = m_1 c$. Clearly $|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1$.

Now using (b) (ii) of Lemma 2.2.3 conclude that $a\bar{b} + c\bar{d} = 0$. Hence clearly the matrices A, A', U, U', defined below, are all unitary:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ A' = \begin{pmatrix} a & \omega b \\ c & \omega d \end{pmatrix}, \ U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega_0' & 0 & 0 \end{pmatrix}, \ U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now verify as before that the equations 2.2.10-2.2.45 are satisfied and hence conclude that $(U \otimes A)W = W'(U' \otimes A)$.

(vi) Suppose the relation (vi) is satisfied. Note that when $\omega' = 1$, the relation (vi) is same as the relation (iv), whose sufficiency has already been established.

Suppose that $\omega = 1 \neq \omega'$. This immediately implies that $S' = 0$ and $\bar{\omega}'\theta'\phi' = -1$.

When $\omega' = -1$, it is the case that either $C = 0$ or $\theta\phi = 1$. If $C = 0$, then define

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi & \omega_0 \\ \phi & -\omega_0 \end{pmatrix}, \ A' = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi & \omega_0 \\ -\phi & \omega_0 \end{pmatrix},$$

$$U' = \begin{pmatrix} 0 & 0 & \omega_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω_0 is a fixed square root of $\theta\phi$.

If $\theta\phi = 1$, then define

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}}(1-C)^{-1/2}\phi S & \frac{1}{\sqrt{2}}(1-C)^{1/2} \\ -\frac{1}{\sqrt{2}}(1+C)^{-1/2}\phi S & \frac{1}{\sqrt{2}}(1+C)^{1/2} \end{pmatrix},$$

$$A' = \begin{pmatrix} \frac{1}{\sqrt{2}}(1-C)^{-1/2}\phi S & \frac{1}{\sqrt{2}}(1-C)^{1/2} \\ \frac{1}{\sqrt{2}}(1+C)^{-1/2}\phi S & -\frac{1}{\sqrt{2}}(1+C)^{1/2} \end{pmatrix},$$

$$U = U' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Verify in both the cases that the equations 2.2.10-2.2.45 are satisfied and hence conclude that $(U \otimes A)W = W'(U' \otimes A')$.

Now assume that $\omega' \neq -1$. Define $\omega_0 = \frac{(1-\theta\phi)C}{(1+\bar{\omega}')}$. Using Lemma 2.2.3 conclude that $|\omega_0| = 1$ and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C+\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\omega'} = \frac{(\theta\phi C+\omega_0\bar{\omega}')}{\theta S} = m_2$ (say); By Lemma 2.2.3, m_1, m_2 are well defined, non-zero complex numbers.

Now choose complex numbers b, d such that $|b|^2 = \frac{1}{1+|m_1^2|}$, and $|d|^2 = \frac{1}{1+|m_2^2|}$. If we define $c = m_2 d$, and $a = m_1 b$, then it clearly follows that $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$. As before using (b) (ii) of Lemma 2.2.3 we conclude that $a\bar{c} + b\bar{d} = b\bar{d}(1 + m_1\bar{m}_2) = 0$. Hence the matrices A, A', U, U' defined below are all unitary:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ A' = \begin{pmatrix} a & b \\ \bar{\omega}'c & \bar{\omega}'d \end{pmatrix}, \ U' = \begin{pmatrix} 0 & 0 & \omega_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now, by verifying the equations 2.2.10-2.2.45, we would find that the relation (vi) is also a sufficient condition.

(vii) Suppose that $\omega = 1 \neq \omega'$ (respectively $\omega \neq 1 = \omega'$), then the relation (vii) is same as the relation (vi) (respectively the relation (v)). The proof of the sufficiency of the relations (iv)-(vii), when the condition $1 \in \{\omega, \omega'\}$ is satisfied, is complete now.

Assume that $(P, P') \in \Omega_0$ and that $2 \neq I_{\{0\}}(S) + I_{\{0\}}(S')$.

Note, from the above assumptions, that it actually follows that $S \neq 0 \neq S'$.

(iv) Suppose that the relation (iv) is satisfied. Define $\omega_0 = \frac{(1-\theta\phi)C}{(1+\omega')}$, $\omega_0' = \frac{(1-\theta'\phi')C'}{(1+\omega)}$. As before, by applying Lemma 2.2.3, we conclude that ω_0 is of unit modulus and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C+\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\omega'} = \frac{(\theta\phi C+\omega_0\omega')}{\theta S} = m_2$ (say). Note that m_1 and m_2 are well defined and non-zero. Again applying Lemma 2.2.3 to

ω_0', we conclude that ω_0' is of unit modulus and that $-\frac{\theta' S'}{C'-\omega_0'} = \frac{(\theta'\phi' C' + \omega_0')}{\phi' S'} = m_1'$ (say), and $-\frac{\theta' S'}{C'-\omega_0'\omega} = \frac{(\theta'\phi' C' + \omega_0'\omega)}{\theta' S'} = m_2'$ (say). Notice that again m_1', m_2' are well defined and non-zero. Now from the condition (γ), using (b) (i) of Lemma 2.2.3 derive that $m_1 m_2' = m_1' m_2$.

Choose a complex number d such that $|d| = \frac{1}{1+|m_2^2|}$. Define $c = m_2 d$, $b = m_2' d$, and $a = m_1 b$. Using the relation $m_1 m_2' = m_1' m_2$, we find that $a = m_1' c$ also. Also, by (b) (ii) of Lemma 2.2.3, it is the case that $m_1 \bar{m}_2 = -1$ and consequently $a\bar{c} + b\bar{d} = 0$. Again, by (b) (ii) of Lemma 2.2.3, $m_1' \bar{m}_2' = -1$, and hence $|m_1 m_2| = |m_1' m_2'| = 1$. But the relation $m_1 m_2' = m_1' m_2$ then implies that $|m_1| = |m_1'|$ and $|m_2| = |m_2'|$. Now note that the definition of c, d implies that $|c|^2 + |d|^2 = 1$ and also that $|a|^2 + |b|^2 = (1+|m_1|^2)|m_2'|^2|d|^2 = (1+|m_1|^2)|m_2|^2|d|^2 = (|m_2|^2 + |m_1 m_2|^2)|d|^2 = (|m_2|^2 + 1)|d|^2 = 1$. Thus if we define

$$A\ (= A') = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then A is a unitary matrix.

Now define

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega_0' & 0 \end{pmatrix}, \text{ and } U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega_0 \\ 0 & 1 & 0 \end{pmatrix}$$

and verify that the equations 2.2.10-2.2.45 are satisfied to conclude that $(U \otimes A)W = W'(U' \otimes A')$.

(v) Suppose the relation (v) is satisfied. Define $\omega_0 = \frac{(\omega - \theta\phi)C}{\omega(1+\omega')}$, $\omega_0' = \frac{(1-\theta'\phi')C'}{(1+\bar{\omega})}$. Use lemma 2.2.3 to conclude that ω_0 and ω_0' are of unit modulus and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C + \omega\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\omega'} = \frac{(\theta\phi C + \omega\omega_0\omega')}{\theta S} = m_2$ (say); $-\frac{\theta' S'}{C'-\omega_0'} = \frac{(\theta'\phi' C' + \omega_0')}{\phi' S'} = m_1'$ (say); and $-\frac{\theta' S'}{C'-\omega_0'\bar{\omega}} = \frac{(\theta'\phi' C' + \omega_0'\bar{\omega})}{\theta' S'} = m_2'$ (say).

Now, choose d such that $|d| = \frac{1}{1+|m_2^2|}$, and define $c = m_2 d$, $b = m_2' d$, and $a = m_1 b$. As in the proof of (iv) using the condition (γ), and (b) (ii) of Lemma 2.2.3, conclude that $m_1 m_2' = m_1' m_2$ and consequently that $a = m_1' c$ also. Similar to the previous case, by applying Lemma 2.2.3, conclude that $a\bar{c} + b\bar{d} = 0$, and that $|c|^2 + |d|^2 = |a|^2 + |b|^2 = 1$, and finally that each of the following matrices is unitary:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ A' = \begin{pmatrix} a & \omega b \\ c & \omega d \end{pmatrix},\ U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega_0' & 0 & 0 \end{pmatrix},\ U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega_0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now verify that the equations 2.2.10-2.2.45 are satisfied and conclude that $(U \otimes A)W = W'(U' \otimes A')$.

(vi) Suppose relation (vi) is satisfied. Define $\omega_0 = \frac{(1-\theta\phi)C}{(1+\bar{\omega}')}$, $\omega_0' = \frac{(1-\bar{\omega}'\theta'\phi')C'}{(1+\omega)}$. As in the previous cases use lemma 2.2.3 to conclude that ω_0, ω_0' are of unit modulus and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C+\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\bar{\omega}'} = \frac{(\theta\phi C+\omega_0\bar{\omega}')}{\theta S} = m_2$ (say); $-\frac{\bar{\omega}'\theta' S'}{C'-\omega_0'} = \frac{(\bar{\omega}'\theta'\phi' C'+\omega_0')}{\phi' S'} = m_1'$ (say), and $-\frac{\bar{\omega}'\theta' S'}{C'-\omega_0'\omega} = \frac{(\bar{\omega}'\theta'\phi' C'+\omega_0'\omega)}{\theta' S'} = m_2'$ (say). Exactly in a similar way to the previous cases, define a, b, c, d and A, and conclude - using Lemma 2.2.3- that A is a unitary matrix, and finally define

$$A' = \begin{pmatrix} a & b \\ \bar{\omega}' c & \bar{\omega}' d \end{pmatrix},\ U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \omega_0' & 0 \end{pmatrix},\ U' = \begin{pmatrix} 0 & 0 & \omega_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

to verify that the equations 2.2.10-2.2.45 are satisfied.

(vii) Suppose (vii) is satisfied. Define $\omega_0 = \frac{(\omega-\theta\phi)C}{\omega(1+\bar{\omega}')}$, $\omega_0' = \frac{(1-\bar{\omega}'\theta'\phi')C'}{(1+\bar{\omega})}$. Use Lemma 2.2.3 to cnclude that ω_0, ω_0' are of unit modulus and that: $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C+\omega_0\omega)}{\theta S} = m_1$, $-\frac{\phi S}{C-\omega_0\bar{\omega}'} = \frac{(\theta\phi C+\omega_0\bar{\omega}'\omega)}{\theta S} = m_2$ (say); $-\frac{\bar{\omega}'\theta' S'}{C'-\omega_0'} = \frac{(\bar{\omega}'\theta'\phi' C'+\omega_0')}{\phi' S'} = m_1'$ (say), and $-\frac{\bar{\omega}'\theta' S'}{C'-\omega_0'\bar{\omega}} = \frac{(\bar{\omega}'\theta'\phi' C'+\omega_0'\bar{\omega})}{\theta' S'} = m_2'$ (say). Again $m_1, m_2, m_1', m_2' \neq 0$. Define a, b, c, d and A exactly as in the proof of the sufficiency of the relations (iv), (v) and (vi) and finally define

$$A' = \begin{pmatrix} a & \omega b \\ \bar{\omega}' c & \omega\bar{\omega}' d \end{pmatrix},\ U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \omega_0' & 0 & 0 \end{pmatrix},\ U' = \begin{pmatrix} 0 & 0 & \omega_0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and verify that the equations 2.2.10-2.2.45 are satisfied.

The proof of the Proposition is finally complete. □

In the following proposition we consider the remaining cases of the preceding proposition. We have to consider the case when exactly one among C, C' is equal to 1 and the case when $-1 \in \{\omega, \omega'\}$. Note that this exhausts all the possibilities.

PROPOSITION 2.2.6 *(a) Assume $\omega \neq 1 \neq \omega'$. Then, $W(\omega, \theta, \phi, 1)$ is not equivalent to $W(\omega', \theta', \phi', C')$ for all $C' < 1$ and for all values of $\theta, \phi, \theta', \phi'$.*

(b) Assume $C, C' < 1$. In order for $W(-1, \theta, \phi, C)$ and $W(-1, \theta', \phi', C')$ to be equivalent, it is necessary and sufficient that one of the relations (i)-(iii), (ix), (x) of Proposition 2.2.5 holds.

(c) Let $f : \mathbf{T} \to \mathbf{C}$ be the map given by $f(z) = \frac{1+z}{1-z}$. Assume $C, C' < 1$ and $1 \neq \omega' \neq -1$. Then, $W(-1, \theta, \phi, C)$ is equivalent to $W(\omega', \theta', \phi', C')$ if and only if one of the following five relations is satisfied.

(1) $C' = 0, Re(\omega') = C^2 - S^2$ and $\theta\phi = \pm 1$.

(2) $\theta'\phi' = 1, Re(\omega') = -(S^2 + Re(\theta\phi)C^2)$ and $C' = \pm f(\omega')f(\theta\phi)$.

(3) $\theta'\phi' = 1, Re(\omega') = -(S^2 + Re(-\theta\phi)C^2)$ and $C' = \pm f(\omega')f(-\theta\phi)$.

(4) $\theta'\phi' = \omega', Re(\omega') = -(S^2 + Re(\theta\phi)C^2)$ and $C' = \pm f(\bar{\omega}')f(\theta\phi)$.

(5) $\theta'\phi' = \omega', Re(\omega') = -(S^2 + Re(-\theta\phi)C^2)$ and $C' = \pm f(\omega')f(-\theta\phi)$.

Proof: (a) We shall suppose that $W(\omega, \theta, \phi, 1)$ is equivalent to $W(\omega', \theta', \phi', C')$, and arrive at a contradiction. Under the assumed equivalence, it follows from Proposition 2.2.5, because of the reason $C' \neq 1$, that one among the relations (iv)-(vii) of Proposition 2.2.5 is satisfied. In other words, as $\omega \neq 1 \neq \omega'$, in the proof of Proposition 2.2.5 the only cases possible are Case (1.2), Case (2.1.2), Case (2.2.1) and Case(2.2.2) (since the other cases lead to the conclusion that $C = C'$).

Suppose we are in the Case (1.2); by looking at the equation 2.2.46, under the assumption that $\omega' \neq 1$ and $S = 0$, we conclude that either $a = d = 0$ or $b = c = 0$. But both these possibilities imply that $S' = 0$ (see the proof of the Case (1.2.1) and the Case (1.2.2)). The desired contradiction has been reached.

When the relations (v), (vi) and (vii) are satisfied, by reasoning in an exactly similar way, - by looking at the proof of Case (2.1.2), (2.2.1) and (2.2.2) respectively - we can deduce that $S' = 0$. The proof of (a) is complete now.

(b) As we have proved the sufficiency of the relations (i)-(iii), note that it is enough to prove that when $\omega = -1 = \omega'$ (as $\omega, \omega' \neq 1$) the Cases (1.2), (2.1.2), (2.2.1) and (2.2.2) in the proof of necessity in Proposition 2.2.5 lead to one among the relations (i)-(iii) itself (because all the other cases anyway lead to one among the relations (i)-(iii), (ix), (x)). We consider the four cases separately.

Case (1.2): As $S \neq 0 \neq S'$ we are in the Case (1.2.3). As the relation (iv) (α) holds, it is the case that either $C = 0$ or $\theta\phi = 1$ and either $C' = 0$ or $\theta'\phi' = 1$.

Suppose it is the case that $C = 0$ and $C' = 0$, note that the relation (i) is satisfied.

Suppose $C = 0$ and $\theta'\phi' = 1$, As the sets of equations 2.2.46 and 2.2.47 are satisfied, first by using Lemma 2.2.4 and then by applying Lemma 2.2.1, conclude that $\frac{C'+\omega_0'}{C'-\omega_0'} = -1$. Hence it is the case that $C' = 0$ and the relation (i) is again satisfied.

In an exactly similar way we may prove that the relation (i) is satisfied when $C' = 0$ and $\theta\phi = 1$.

Suppose that $\theta\phi = 1 = \theta'\phi'$ and $C \neq 0 \neq C'$; Again applying Lemma 2.2.4 and Lemma 2.2.1 to the sets of equations 2.2.46 and 2.2.47 implies that $\frac{\omega_0+C}{\omega_0-C} = \frac{\omega_0'+C'}{\omega_0'-C'}$, and hence it follows that $C' = \omega_0'\bar{\omega}_0 C$. But, because of our assumption that $C \neq 0 \neq C'$, it is the case that $\omega_0 = \omega_0'$ and $C = C'$, and the relation (iii) is satisfied.

Case (2.1.2): As $S \neq 0 \neq S'$ it is the case that $a, b, c, d \neq 0$. Also, as the relation (v) (α) is satisfied, conclude that either $C = 0$ or $\theta\phi = -1$ and either $C' = 0$ or $\theta'\phi' = 1$.

Suppose that $C = 0$ and $\theta'\phi' = 1$. Exactly as in the previous case applying Lemma 2.2.4 and then applying the Lemma 2.2.1 to the sets of equations 2.2.48 and 2.2.49 conclude that $C' = 0$ and the relation (i) is satisfied. The proof of the relation (i) is being satisfied when $C' = 0$ and $\theta\phi = -1$ is exactly similar.

Suppose that $\theta\phi = -1$ and $\theta'\phi' = 1$; In an exactly similar way application of Lemma 2.2.4 and Lemma 2.2.1 straightaway implies that $C = C'$, and the relation (iii) is satisfied.

In an exactly similar way - by using the corresponding set of equations - 2.2.50 and 2.2.51 (2.2.52 and 2.2.53 respectively) - and the Lemmas 2.2.4 and 2.2.1 we may prove that either the relation (i) or (iii) ((i) or (ii) respectively) is satisfied in the Case (2.2.1) (Case (2.2.2) respectively). This completes the proof (b).

(c) First we prove that it is necessary that one among the relations (1)-(5) should hold for $W(-1,\theta,\phi,C)$ to be equivalent to $W(\omega',\theta',\phi',C')$, where $1 \neq \omega' \neq -1$ and $C \neq 1 \neq C'$.

Suppose $W(-1,\theta,\phi,C)$ is equivalent to $W(\omega',\theta',\phi',C')$, for $Im(\omega') > 0$ and $C, C' \neq 1$; then as $\omega' \neq -1$, it follows from Proposition 2.2.5 that that one among the relations (iv)-(vii) is satisfied. These relations are arrived (as $\omega' \neq 1$) in the cases (1.2), (2.1.2), (2.2.1) and (2.2.2) respectively in the proof of necessity in Proposition 2.2.5. We will consider each of these cases separately below, assuming that the corresponding relation holds. Due to our assumption that $S \neq 0 \neq S'$, it is the case that $a, b, c, d \neq 0$ in all the following cases we are going to consider.

Case (1.2): We will prove that either the relation (1) or the relation (2) is satisfied in this case. First, from the condition (iv) (α), note that it is the case that either $C' = 0$ or $\theta'\phi' = 1$.

Suppose that $C' = 0$; Let m_1 and m_2 be as defined in Lemma 2.2.3. Then as $\omega_0 = \frac{(1-\theta\phi)C}{(1+\omega')}$ (see the Case (1.2) in the proof of Proposition 2.2.5), by applying (b) (i) of Lemma 2.2.3 (note that $\omega' \neq \pm 1$) to ω_0, conclude that $\frac{m_1}{m_2} = \frac{1+\omega'\theta\phi}{\omega'+\theta\phi}$ Also let m_1' and m_2' be as defined in Lemma 2.2.4 with the set of equations 2.2.2-2.2.5 replaced by the set of equations 2.2.47. By applying Lemma 2.2.4 to the set of equations 2.2.47 conclude that $\frac{m_1'}{m_2'} = -1$. Now, by applying Lemma 2.2.1, conclude that

$$\frac{1+\omega'\theta\phi}{\omega'+\theta\phi} = -1.$$

The above equation may be re-written as $(\omega'+1)(\theta\phi+1) = 0$. Since $\omega' \neq -1$ it follows that $\theta\phi = -1$. As the relation (iv) is satisfied, it also follows that $Re(\omega') = C^2 - S^2$. Thus the relation (1) is satisfied.

Suppose that $\theta'\phi' = 1$; Let m_1, m_2, m_1', m_2' be exactly as defined in the previous paragraph. Again as $\omega_0 = \frac{(1-\theta\phi)C}{(1+\omega')}$, by applying (b) (i) of Lemma 2.2.3 to ω_0 first conclude that $\frac{m_1}{m_2} = \frac{1+\omega'\theta\phi}{\omega'+\theta\phi}$ and then by applying Lemma 2.2.4 to the set of equations 2.2.47 conclude that $\frac{m_1'}{m_2'} = \frac{C\pm 1}{C\mp 1}$. Hence, by applying Lemma 2.2.1, conclude that

$$\frac{1+\omega'\theta\phi}{\omega'+\theta\phi} = \frac{C'\pm 1}{C'\mp 1},$$

and hence that $C' = \pm f(\omega')f(\theta\phi)$. Hence the relation (2) is satisfied in this case.

Case (2.1.2): In this case we will prove that either the relation (1) or (3) is satisfied. First note that (v) (α) again implies that either $C' = 0$ or $\theta'\phi' = 1$.

Suppose that $C' = 0$; Now, first by applying (b) (i) of Lemma 2.2.3 (note $\omega' \neq \pm 1$) to ω_0 ($= \frac{(1+\theta\phi)C}{(1+\omega')}$) and Lemma 2.2.4 to the set of equations 2.2.49, and then by applying the Lemma 2.2.1, conclude that

$$\frac{1-\omega'\theta\phi}{\omega'-\theta\phi} = -1.$$

The above equation may be re-written as $(\omega'+1)(-\theta\phi+1) = 0$ to conclude, as $\omega' \neq -1$, that $\theta\phi = 1$. As the relation (v) is satisfied it also follows that $Re(\omega') = C^2 - S^2$. Thus the relation (1) is satisfied.

Suppose that $\theta'\phi' = 1$; First by applying (b) (i) of Lemma 2.2.3 to ω_0 and Lemma 2.2.4 to the set of equations 2.2.49, and then, using Lemma 2.2.1, conclude that

$$\frac{1-\omega'\theta\phi}{\omega'-\theta\phi} = \frac{C'\pm 1}{C'\mp 1},$$

and hence that $C' = \pm f(\omega')f(-\theta\phi)$. Hence the relation (3) is satisfied in this case.

Case (2.2.1): In this case we will prove that either the relation (1) or (4) is satisfied in this case. As before first note that (vi) (α) implies that either $C' = 0$ or $\theta'\phi' = \omega'$.

Suppose that $C' = 0$; As before, first by applying (b) (i) of Lemma 2.2.3 to ω_0 ($= \frac{(1-\theta\phi)C}{(1+\bar{\omega}')}$) and Lemma 2.2.4 to the set of equations 2.2.51, and then by applying the Lemma 2.2.1, conclude that

$$\frac{1+\bar{\omega}'\theta\phi}{\bar{\omega}'+\theta\phi} = -1.$$

The above equation can be re-written as $(\bar{\omega}'+1)(\theta\phi+1) = 0$. Since $\omega' \neq -1$ it follows that $\theta\phi = -1$. As the relation (vi) is satisfied, it also follows that $Re(\omega') = C^2 - S^2$. Thus the relation (1) is satisfied.

Suppose that $\theta'\phi' = \omega'$; Again apply (b) (i) of Lemma 2.2.3 to ω_0 and Lemma 2.2.4 to the set of equations 2.2.51, to conclude that

$$\frac{1+\bar{\omega}'\theta\phi}{\bar{\omega}'+\theta\phi} = \frac{C'\pm 1}{C'\mp 1},$$

and hence that $C' = \pm f(\bar{\omega}')f(\theta\phi)$. Hence the relation (4) is satisfied in this case.

Case (2.2.2): In this case we will prove that either the relation (1) or (4) is satisfied . As before first note that (vii) (α) implies that either $C' = 0$ or $\theta'\phi' = \omega'$.

Suppose that $C' = 0$; as before, first by applying (b) (i) of Lemma 2.2.3 ω_0 ($= \frac{(1+\theta\phi)C}{(1+\bar{\omega}')}$) and Lemma 2.2.4 to the set of equations 2.2.53, and then by applying the Lemma 2.2.1, conclude that

$$\frac{1-\bar{\omega}'\theta\phi}{\bar{\omega}'-\theta\phi} = -1.$$

Re-writte the above equation as $(\bar{\omega}'+1)(\theta\phi-1) = 0$, to conclude, again as $\omega' \neq -1$, that $\theta\phi = 1$. As the relation (vii) is satisfied it also follows that $Re(\omega') = C^2 - S^2$. Thus the relation (1) is satisfied.

Suppose that $\theta'\phi' = \omega'$; again apply (b) (i) of Lemma 2.2.3 to ω_0 and Lemma 2.2.4 to the set of equations 2.2.53, to conclude that

$$\frac{1-\bar{\omega}'\theta\phi}{\bar{\omega}'-\theta\phi} = \frac{C'\pm 1}{C'\mp 1},$$

and hence that $C' = \pm f(\bar{\omega}')f(-\theta\phi)$. Hence the relation (5) is satisfied in this case. The proof of the necessity of one of the relations (1)-(5) to hold is complete now.

Now we will prove that the relations (1)-(5) are all sufficient.

(1) Suppose the relation (1) is satisfied and $\theta\phi = -1$. Define $\omega_0 = \frac{(1-\theta\phi)C}{(1+\omega')}$. Conclude, using Lemma 2.2.3, that ω_0 is of unit modulus and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C+\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\omega'} = \frac{(\theta\phi C+\omega_0\omega')}{\theta S} = m_2$ (say). Let ω_0' be a fixed square root of $\theta'\phi'$, and notice that $\theta'\bar{\omega}_0' = \omega_0'\bar{\phi}' = m_1'$ (say), and set $m_2' = -m_1'$. Using the relation $\theta\phi = -1$ and (b) (i) of Lemma 2.2.3, we may derive that $\frac{m_1}{m_2} = -1$ and consequently that the relation $m_1m_2' = m_1'm_2$ is satisfied. Now choose a complex number d such that $|d| = \frac{1}{1+|m_2^2|}$. Define $c = m_2d$, $b = m_2'd$, and $a = m_1b$. Using the relation $m_1m_2' = m_1'm_2$, we find that $a = m_1'c$ also. Also, by (b) (ii)

of Lemma 2.2.3, it is the case that $m_1\bar{m}_2 = -1$ and consequently $a\bar{c} + b\bar{d} = 0$. Also clearly $m'_1\bar{m}'_2 = -1$, and hence $|m_1 m_2| = |m'_1 m'_2| = 1$. But the relation $m_1 m'_2 = m'_1 m_2$ implies that $|m_1| = |m'_1|$ and $|m_2| = |m'_2|$. Now the very definition of c, d implies that $|c|^2 + |d|^2 = 1$, and also $|a|^2 + |b|^2 = (1 + |m_1|^2)|m'_2|^2|d|^2 = (1 + |m_1|^2)|m_2|^2|d|^2 = (|m_2|^2 + |m_1 m_2|^2)|d|^2 = (|m_2|^2 + 1)|d|^2 = 1$. Thus if we define $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then A is a unitary matrix. Also, clearly, all the following matrices are unitary:

$$A' = A\ ,\ U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega'_0 & 0 \end{pmatrix}\ ,\ U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega_0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now verify that the equations 2.2.10-2.2.45 are satisfied to conclude that $(U \otimes A)W = W'(U' \otimes A')$.

If $\theta\phi = 1$, then define $\omega_0 = \frac{(1+\theta\phi)C}{(1+\omega')}$, and conclude, using Lemma 2.2.3, that ω_0 is of unit modulus and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C-\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\omega'} = \frac{(\theta\phi C-\omega_0\omega')}{\theta S} = m_2$ (say). Define $\omega'_0, m'_1, m'_2, a, b, c, d$ and A exactly as in the last paragraph and use exactly similar arguments to find that A is a unitary matrix. Finally define

$$A' = AD\ ,\ U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega'_0 & 0 & 0 \end{pmatrix}\ ,\ U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega_0 \\ 1 & 0 & 0 \end{pmatrix},$$

and verify that the equations 2.2.10-2.2.45 are satisfied.

(2) Suppose the relation (2) is satisfied;

First we consider the case $C' = f(\omega')(f(\theta\phi)$. Define $\omega_0 = \frac{(1-\theta\phi)C}{(1+\omega')}$. As before, use Lemma 2.2.3 to conclude that $|\omega_0| = 1$ and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C+\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\omega'} = \frac{(\theta\phi C+\omega_0\omega')}{\theta S} = m_2$ (say). Let $m'_1 = \frac{\theta' S'}{1-C'}$ ($= \frac{1+C'}{\phi' S'}$ using the relation $\theta'\phi' = 1$) and $m'_2 = -\frac{\theta' S'}{1+C'}$ ($= \frac{C'-1}{\phi' S'}$ again using $\theta'\phi' = 1$). By an applcation of Lemma 2.2.3(b)(i) to this ω_0, we find that

$$\frac{m_1}{m_2} = \frac{1 + \omega'\theta\phi}{\omega' + \theta\phi}\ ,\ \text{while}\ \frac{m'_1}{m'_2} = \frac{C' + 1}{C' - 1}.$$

Now, using the relation $C' = f(\omega')f(\theta\phi)$, we find that $m_1 m'_2 = m'_1 m_2$. Choose a, b, c, d and define A exactly as before, and use exactly the same arguments to assert the unitarity of A, and finally define

$$A' = A \ , \ U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \ U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega_0 \\ 0 & 1 & 0 \end{pmatrix} \ ,$$

and verify that the equations 2.2.10-2.2.45 are satisfied to conclude that $(U \otimes A)W = W'(U' \otimes A')$.

Suppose next that $C' = -f(\omega')f(\theta\phi)$; define ω_0, m_1, m_2 exactly as in the preceding paragraph and define $m_1' = -\frac{\theta' S'}{1+C'}$ $(= \frac{C'-1}{\phi' S'}$ using the relation $\theta'\phi' = 1)$, $m_2' = -\frac{\theta' S'}{C'-1}$ $(= \frac{C'+1}{\phi' S'}$ again using $\theta'\phi' = 1)$. Now by applying Lemma 2.2.3(b)(i) to ω_0, we find that

$$\frac{m_1}{m_2} = \frac{1+\omega'\theta\phi}{\omega' + \theta\phi} \ , \ \text{while} \ \frac{m_1'}{m_2'} = \frac{C'-1}{C'+1}.$$

Hence, using the relation $C' = -f(\omega')f(\theta\phi)$, we find that $m_1 m_2' = m_1' m_2$.

Define a, b, c, d, A and U' exactly in the preceding paragraph and define

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Procede exactly in a similar way to conclude that W is equivalent to W'.

(3) Suppose relation (3) is satisfied and $C' = f(\omega')f(-\theta\phi)$; Define $\omega_0 = \frac{(1+\theta\phi)C'}{(1+\omega')}$. As before, using Lemma 2.2.3 conclude that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C - \omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\omega'} = \frac{(\theta\phi C - \omega_0\omega')}{\theta S} = m_2$ (say). Let $m_1' = \frac{\theta' S'}{1-C'}$ $(= \frac{1+C'}{\phi' S'})$ and $m_2' = -\frac{\theta' S'}{1+C'}$ $(= \frac{C'-1}{\phi' S'})$. Using the relation $C' = f(\omega')f(-\theta\phi)$, we may derive that $m_1 m_2' = m_1' m_2$. Now choose a, b, c, d and define A exactly as before and finally define

$$A' = AD \ , \ U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ , \ U' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega_0 \\ 1 & 0 & 0 \end{pmatrix} .$$

and verify that the equations 2.2.10-2.2.45 are satisfied to conclude that $(U \otimes A)W = W'(U' \otimes A')$.

Suppose $C' = -f(\omega')f(-\theta\phi)$; The proof is same as for the case considered in the preceding paragraph exept the definition of m_1', m_2' and U. Define

$$m_1' = -\frac{\theta' S'}{1+C'} \ (= \frac{C'-1}{\phi' S'}), \ m_2' = -\frac{\theta' S'}{C'-1} \ (= \frac{C'+1}{\phi' S'}), \ U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Now precede exactly in a similar way as in the preceding paragraph to verify the equivalence.

(4) Suppose that the relation (4) is satisfied and that $C' = f(\bar{\omega}')f(\theta\phi)$; define $\omega_0 = \frac{(1-\theta\phi)C}{(1+\bar{\omega}')}$. Using Lemma 2.2.3 $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C+\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\bar{\omega}'} = \frac{(\theta\phi C+\omega_0\bar{\omega}')}{\theta S} = m_2$ (say). Let $m_1' = \frac{\bar{\omega}'\theta' S'}{1-C'}$ $(= \frac{1+\bar{\omega}'\theta'\phi' C'}{\phi' S'}$ using $\theta'\phi' = \omega'$) and $m_2' = -\frac{\bar{\omega}'\theta' S'}{1+C'}$ $(= \frac{\bar{\omega}'\theta'\phi' C'-1}{\phi' S'}$). Using the relation $C' = f(\bar{\omega}')f(\theta\phi)$, we may derive that $m_1 m_2' = m_1' m_2$. Now choose a, b, c, d and define A exactly as before and finally define

$$A' = D'^{*} A \ , \ U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ , \ U' = \begin{pmatrix} 0 & 0 & \omega_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

and verify that the equations 2.2.10-2.2.45 are satisfied to conclude that $(U \otimes A)W = W'(U' \otimes A')$.

Suppose that $C' = -f(\bar{\omega}')f(\theta\phi)$; define $m_1' = -\frac{\bar{\omega}'\theta' S'}{1+C'}$ $(= \frac{\bar{\omega}'\theta'\phi' C'-1}{\phi' S'}$ using $\theta'\phi' = \omega' 1)$, $m_2' = -\frac{\omega'\theta' S'}{C'-1}$ $(= \frac{\bar{\omega}'\theta'\phi' C'+1}{\phi' S'})$, and $\omega_0, m_1, m_2, A, A', U'$ exactly as in the previous paragraph, and finally define

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} .$$

Now procede exactly in a similar way to conclude that W and W' are equivalent.

(5) Suppose that the relation (5) is satisfied and that $C' = f(\bar{\omega}' f(-\theta\phi)$; define $\omega_0 = \frac{(1+\theta\phi)C}{(1+\omega')}$. Use Lemma 2.2.3 to conclude that $|\omega_0| = 1$ and that $-\frac{\phi S}{C-\omega_0} = \frac{(\theta\phi C-\omega_0)}{\theta S} = m_1$ (say); $-\frac{\phi S}{C-\omega_0\bar{\omega}'} = \frac{(\theta\phi C-\omega_0\omega')}{\theta S} = m_2$ (say). Let $m_1' = \frac{\bar{\omega}'\theta' S'}{1-C'}$ $(= \frac{1+\omega'\theta'\phi' C'}{\phi' S'}$ using $\theta'\phi' = \omega'$) and $m_2' = -\frac{\omega'\theta' S'}{1+C'}$ $(= \frac{\bar{\omega}'\theta'\phi' C'-1}{\phi' S'})$. Using the relation $C' = f(\omega')f(-\theta\phi)$, we may derive that $m_1 m_2' = m_1' m_2$. Now choose a, b, c, d and define A exactly as before and finally define

$$A' = D'^{*} A D \ , \ U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \ U' = \begin{pmatrix} 0 & 0 & \omega_0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

and verify that the equations 2.2.10-2.2.45 are satisfied to conclude that $(U \otimes A)W = W'(U' \otimes A')$.

Suppose that $C' = -f(\bar{\omega}')f(-\theta\phi)$; define $m'_1 = -\frac{\bar{\omega}'\theta' S'}{1+C'}(= \frac{\bar{\omega}'\theta'\phi' C'-1}{\phi' S'}$ using $\theta'\phi' = \omega'$), $m'_2 = -\frac{\bar{\omega}'\theta' S'}{C'-1}(= \frac{\bar{\omega}'\theta'\phi' C'+1}{\phi' S'})$, and $\omega_0, m_1, m_2, A, A', U'$ exactly as in the previous paragraph, and finally define

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Now procede exactly in a similar way to conclude that W and W' are equivalent.

This completes the proof of the Proposition 2.2.6. □

2.3 Classification of $B(2, n)$

Here we try to classify $B(2, n)$ up to the natural equivalence relation on it. The method of analysis is more or less similar to the case $n = 3$. As points in $\Omega(2, n)$ give at least one connection from each equivalence class, similar to the case $n = 3$ here also we attempt to describe when a pair of points $P, P' \in \Omega(2, n)$ afford equivalent connections. In the following Proposition we state four conditions and prove that it is necessary that a pair of points should satisfy at least one of them in order to afford equivalent connections. We also prove that two of those conditions are even sufficient also. Finally using Ocneanu's Compactness we prove the reducebility of the vertical subfactor and provide some conditions for the horizontal subfactor to be irreducible, and also for being reducible.

PROPOSITION 2.3.1 *Let n be arbitrary. Assume that $Im(\omega), Im(\omega') > 0$, and $C_i, S_i, C'_i, S'_i \neq 0$ for all i.*

(a) If $W(\omega, \theta, \phi, C)$ is equivalent to $W(\omega', \theta', \phi', C')$, then one of the following relations holds:

(0) $\omega = \omega'$, and there exists a permutation $\sigma \in S_{n-2}$ such that

$$Ad(P_\sigma)(C) = C', \ Ad(P_\sigma)(\theta) = \zeta\theta', \text{ and } Ad(P_\sigma)(\phi) = \zeta\phi' ,$$

where ζ is some complex number of unit modulus, P_σ denotes the permutation matrix correspoding to σ, and we write $Ad(P_\sigma) = P_\sigma(\cdot)P_\sigma^{-1}$

(1) $\omega = \omega'$, *and there exists a permutation* $\sigma \in S_{n-2}$ *such that*

$$Ad(P_\sigma)(C) = C', \; Ad(P_\sigma)(\theta) = \zeta\theta'^{*}, \; \text{and } Ad(P_\sigma)(\phi) = \omega\bar{\zeta}\phi'^{*} \;,$$

where ζ *is some complex number of unit modulus.*

(2) There exist i, i' *such that* $(Re(\omega'), Re(\omega)) \in \Lambda_i \times \Lambda'_{i'}$, *where* $\Lambda_i = \{-(S_i^2 + Re(\theta_i\phi_i)C_i^2), -(S_i^2 + Re(\bar{\omega}\theta_i\phi_i)C_i^2)\}$ *and* $\Lambda'_i = \{-(S_i'^2 + Re(\theta'_i\phi'_i)C_i'^2), \; (S_i'^2 + Re(\omega'\theta'_i\phi'_i)C_i'^2)\}$.

(3) There exist i, j, i', j' *such that* $(Re(\omega'), Re(\omega)) = (-m_{i,j}, -m'_{i',j'})$ *where* $m_{i,j} = 1 - (1 + Re(\theta_i\phi_i\bar{\theta}_j\bar{\phi}_j))C_i^2C_j^2 - (1 + Re(\theta_i\phi_j\bar{\theta}_j\phi_i))S_i^2S_j^2 - 2(Re(\theta_i\theta_j) + Re(\phi_i\phi_j))C_iC_jS_iS_j$, *and* $m'_{i,j}$ *is the corresponding 'primed' expression.*

(b) In (a), conditions (0) and (1) are also sufficient conditions for $W(\omega, \theta, \phi, C)$ *to be equivalent to* $W(\omega, \theta', \phi', C')$.

(c) (1) The vertical subfactor associated with $W(\omega, \theta, \phi, C)$ *is always reducible.*

(2) (i) The horizontal subfactor associated with $W(\omega, \theta, \phi, C)$ *is irreducible if either of the following conditions holds:*

(α) $\omega \neq 1$ *and* $S \neq 0$;

(β) θS *and* ϕS *are not scalar multiples of one another.*

(ii) The horizontal subfactor is reducible if $S = 0$.

Proof: Similar to the case $n = 3$ we first write the condition for $P, P' \in \Omega(2, n)$ to afford equivalent connections, as a set of equations. Thus, in order for W to be equivalent to W', i.e. $(U \otimes A)W(\omega, \theta, \phi, C) = W(\omega', \theta', \phi', C')(U' \otimes A')$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;, \; A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U(2)$$

$$U = \begin{pmatrix} u_{1,1} & u_{1,2} & P^t \\ u_{2,1} & u_{2,2} & Q^t \\ X & Y & Z \end{pmatrix}, \quad U' = \begin{pmatrix} u'_{1,1} & u'_{1,2} & P'^t \\ u'_{2,1} & u'_{2,2} & Q'^t \\ X' & Y' & Z' \end{pmatrix} \in U(n),$$

$P, Q, X, Y, P', Q', X', Y' \in M_{(n-2)\times 1}$ and $Z, Z' \in M_{n-2}$, where X^t denotes the matrix transpose of X, it is necessary and sufficient that the following set of equations holds.

$$\begin{aligned}
u_{1,1}A &= u'_{1,1}A' && (2.3.54)\\
u_{1,2}AD &= u'_{1,2}A' && (2.3.55)\\
P^t(aC + b\phi S) &= a'P'^t && (2.3.56)\\
P^t(a\theta S - b\theta\phi C) &= b'P'^t && (2.3.57)\\
P^t(cC + d\phi S) &= c'P'^t && (2.3.58)\\
P^t(c\theta S - d\theta\phi C) &= d'P'^t && (2.3.59)\\
u_{2,1}A &= u'_{2,1}D'A' && (2.3.60)\\
u_{2,2}AD &= u'_{2,2}D'A' && (2.3.61)\\
Q^t(aC + b\phi S) &= a'Q'^t && (2.3.62)\\
Q'(a\theta S - b\theta\phi C) &= b'Q'^t && (2.3.63)\\
Q'(cC + d\phi S) &= c'\omega'Q'^t && (2.3.64)\\
Q^t(c\theta S - d\theta\phi C) &= d'\omega'Q'^t && (2.3.65)\\
aX &= (a'C' + c'\theta'S')X' && (2.3.66)\\
bX &= (b'C' + d'\theta'S')X' && (2.3.67)\\
cX &= (a'\phi'S' - c'\theta'\phi'C')X' && (2.3.68)\\
dX &= (b'\phi'S' - d'\theta'\phi'C')X' && (2.3.69)\\
aY &= (a'C' + c'\theta'S')Y' && (2.3.70)\\
b\omega Y &= (b'C' + d'\theta'S')Y' && (2.3.71)\\
cY &= (a'\phi'S' - c'\theta'\phi'C')Y' && (2.3.72)\\
d\omega Y &= (b'\phi'S' - d'\theta'\phi'C')Y' && (2.3.73)\\
Z(aC + b\phi S) &= (a'C' + c'\theta'S')Z' && (2.3.74)\\
Z(a\theta S - b\theta\phi C) &= (b'C' + d'\theta'S')Z' && (2.3.75)\\
Z(cC + d\phi S) &= (a'\phi'S' - c'\theta'\phi'C')Z' && (2.3.76)\\
Z(c\theta S - d\theta\phi C) &= (b'\phi'S' - d'\theta'\phi'C')Z' && (2.3.77)
\end{aligned}$$

As before, we consider cases depending on whether various entries of U are zero or non-zero.

Case(1) : $u_{1,1} \neq 0$.

Using the unitarity of A and A' and from equation 2.3.54, we can assume without loss of generality, as for the case $n = 3$, that $A = A'$ and $u_{1,1} = u'_{1,1}$. Also by the assumption $\omega, \omega' \neq 1$, using equations 2.3.55 and 2.3.60 and arguments exactly similar to the one used in the previous section, we find that $u_{1,2} = u'_{1,2} = u_{2,1} = u'_{2,1} = 0$. We consider two sub-cases depending upon whether the entry $u_{2,2}$ is non-zero or zero.

Case (1.1): $u_{2,2} \neq 0$.

As $Im(\omega), Im(\omega') > 0$, First we deduce from equation 2.3.61 that either $a = 0$ or $b = 0$. But $a = d = 0$ implies that $\omega = \bar{\omega}'$. But as we have assumed that $Im(\omega), Im(\omega') > 0$, It is the case that that $b = c = 0$, and that $\omega = \omega'$, and $u_{2,2} = u'_{2,2}$. We will show that the relation (0) is satisfied in this case.

As $S_i \neq 0$ for all i (by the assumption in the statement of the Proposition), we find - from equations 2.3.57 and 2.3.63 that $P' = Q' = 0$. At the same time the equations 2.3.56 and 2.3.62, imply that $P'' = Q'' = 0$. Also by the assumption $S'_i \neq 0$ for all i, we find - from equations 2.3.67 and 2.3.71 - that $X' = Y' = 0$, while the equations 2.3.66 and 2.3.70 imply that $X = Y = 0$. The unitarity of U and U' is now seen to imply that also Z and Z' are unitary.

The equations 2.3.74 and 2.3.77 may be rewritten as:

$$ZC = C'Z' \tag{2.3.78}$$

$$ZC\theta\phi = \theta'\phi'C'Z' \tag{2.3.79}$$

Since C and C' are invertible positive operator, (as follows from the assumption in the statement of the Proposition that $C_i \neq 0$ for all i) and since equation 2.3.78 may be re-written as

$$(ZZ'^*)(Z'CZ'^*) = C' \; ,$$

we may deduce from the uniqueness of polar decomposition that $Z = Z'$.

Next, we may deduce from equations 2.3.78 and 2.3.79 - using the invertibility of the matrix C' - that $Z\theta\phi Z^* = \theta'\phi'$.

Thus,

$$ZC = C'Z \; , \text{ (hence also } ZS = S'Z) \text{ and } Z\theta\phi = \theta'\phi'Z \; .$$

Notice now that

$$\begin{aligned} Z\theta S &= Z(\theta\phi)\phi^* S \\ &= (\theta'\phi')Z\phi^* S \\ &= (\theta'\phi')(Z\phi^* Z^*)ZS \\ &= (\theta'\phi')(Z\phi^* Z^*)S'Z \ . \end{aligned}$$

Hence, we may deduce from equation 2.3.75 that

$$a(\theta'\phi')(Z\phi^* Z^*)S'Z = d\theta' S'Z \ ;$$

deduce from the invertibility of $S'Z$ that

$$Z\phi^* Z^* = \zeta\phi'^* \ ,$$

where $\zeta = d/a$. Since $Z\theta\phi Z^* = (\theta'\phi')$, we thus find that also

$$Z\theta Z^* = \zeta\theta' \ .$$

Let $\mathcal{A}$ (resp., $\mathcal{A}'$) denote the *-subalgebra of M_{n-2} generated by $\{\theta, \phi, C\}$ (resp., $\{\theta', \phi', C'\}$). The preceding analysis shows that the map $Ad(Z) = Z(\cdot)Z^*$ maps $\mathcal{A}$ onto $\mathcal{A}'$ (since it carries the generators to non-zero multiples of the generators).

Note now that $\mathcal{A}$ and $\mathcal{A}'$ are contained in the algebra of diagonal matrices. If $\{e_\alpha : \alpha \in \Lambda\}$ denotes the set of minimal projections in the abelian C^*-algebra $\mathcal{A}$, and if $Ze_\alpha Z^* = e'_\alpha$, then clearly $\{e'_\alpha : \alpha \in \Lambda\}$ is the set of minimal projections in $\mathcal{A}'$. The fact that some unitary matrix - i.e., Z - simultaneously conjugates each e_α into e'_α, clearly implies now that we can find some permutation $\sigma \in S_{n-2}$ such that $Ad(P_\sigma)$ maps each e_α into e'_α.

It follows easily now from the construction that

$$Ad(P_\sigma)(\theta) = \zeta\theta',\ Ad(P_\sigma)(\phi) = \bar{\zeta}\phi',\ \text{and}\ Ad(P_\sigma)(C) = C' \ .$$

Case (1.2): $u_{2,2} = 0$.

We will prove that the relation (2) is satisfied in this case. From the equation 2.3.61 we conclude that $u'_{2,2} = 0$. Suppose $Q^t = (q_1, q_2, \cdots q_{n-2})$. As $u_{2,1} = u_{2,2} =$

0, we find that Q is a unit vector; hence there exists an index i such that $q_i \neq 0$. Then, we find from equations 2.3.62-2.3.65 that

$$\begin{aligned} q_i(aC_i + b\phi_i S_i) &= aq'_i & (2.3.80)\\ q_i(a\theta_i S_i - b\theta_i\phi_i C_i) &= bq'_i & (2.3.81)\\ q_i(cC_i + d\phi_i S_i) &= c\omega' q'_i & (2.3.82)\\ q_i(c\theta_i S_i - d\theta_i\phi_i C_i) &= d\omega' q'_i & (2.3.83) \end{aligned}$$

The unitarity of the matrix $\begin{pmatrix} C_i & \phi_i S_i \\ \theta_i S_i & -\theta_i\phi_i C_i \end{pmatrix}$ would imply that necessarily that $|q'_i| = |q_i| \neq 0$. Also, as $S_i \neq 0$, we may infer from equations 2.3.80 and 2.3.81 that $a, b, c, d \neq 0$. Let $Y^t = (y_1, y_2, \cdots y_{n-2})$. Since $u_{1,2} = u_{2,2} = 0$, we find that Y is a unit vector; hence there exists an index i' such that $y_{i'} \neq 0$. Then, we find from equations 2.3.70-2.3.73, that the following equations hold:

$$\begin{aligned} ay_{i'} &= (aC'_{i'} + c\theta'_{i'}S'_{i'})y'_{i'}\\ b\omega y_{i'} &= (bC'_{i'} + d\theta'_{i'}S'_{i'})y'_{i'}\\ cy_{i'} &= (a\phi'_{i'}S'_{i'} - c\theta'_{i'}\phi'_{i'}C'_{i'})y'_{i'}\\ d\omega y_{i'} &= (b\phi'_{i'}S'_{i'} - d\theta'_{i'}\phi'_{i'}C'_{i'})y'_{i'} & (2.3.84) \end{aligned}$$

Again, using the unitarity of the matrix $\begin{pmatrix} C'_{i'} & \phi'_{i'}S'_{i'} \\ \theta'_{i'}S'_{i'} & -\theta'_{i'}\phi'_{i'}C'_{i'} \end{pmatrix}$, deduce that $y_{i'}$ and $y'_{i'}$ have the same absolute value.

Let $\omega_0 = q'_i q_i^{-1}$ and $\omega'_0 = y'_{i'}y_{i'}^{-1}$. Now, first by re-writing the above two equations in the form as in equations 2.2.2-2.2.5, and then by applying Lemma 2.2.2 separately to the two sets of equations above (exactly as in the previous section), conclude that

$$(Re(\omega'), Re(\omega)) = (-(S_i^2 + Re(\theta_i\phi_i)C_i^2), -({S'_{i'}}^2 + Re(\theta'_{i'}\phi'_{i'}){C'_{i'}}^2)) .$$

Hence the relation (2) is satisfied in this case.

Case (2) $u_{1,1} = 0$

Case (2.1) $u_{1,2} \neq 0$

Using 2.3.55 and the unitarity of A and A', we can assume without loss of generality that $AD = A'$ and $u_{1,2} = u'_{1,2}$. Also as $\omega' \neq 1$, we find from 2.3.61

that $u_{2,2} = u'_{2,2} = 0$. There are two cases now, depending on whether $u_{2,1}$ is not or is 0, which we consider separately.

Case (2.1.1) $u_{2,1} \neq 0$

As $Im(\omega), Im(\omega') > 0$ we may deduce from equation 2.3.60 that $a = d = 0$, $\omega = \omega'$, and $u_{2,1} = \omega u'_{2,1}$. We will show that the relation (1) is satisfied in this case.

As S is invertible, (i.e.$S_i \neq 0$ for all i) we find from 2.3.56 and 2.3.62 that $P^t = Q^t = 0$. From 2.3.57 and 2.3.63, we get $P'^t = Q'^t = 0$. As S' is invertible, we find from 2.3.66 and 2.3.70 that $X' = Y' = 0$, and then from 2.3.67 and 2.3.71 we get $X = Y = 0$. Now it follows that Z and Z' are unitary.

From 2.3.75 and 2.3.76 we have

$$\begin{aligned} -Z\theta\phi C &= \omega C'Z' \\ ZC &= -\theta'\phi'C'Z' \end{aligned}$$

It follows (as before, from the uniqueness of polar decomposition and the invertibility of the positive operators C, C') that $Z\theta\phi = -\omega Z'$ and $Z = -\theta'\phi'Z'$. These equations together with the equation $bZ\phi S = c\theta' S' Z'$ (which is a consequence of equation 2.3.74) are seen to imply (after some minor manipulations) that

$$Ad(Z')(C) = C',\ Ad(Z')(\theta) = \zeta\theta'^*, \text{ and } Ad(Z')(\phi) = \omega\bar{\zeta}\phi'^*\ ,$$

where ζ is some scalar of unit modulus.

Arguing exactly as in the proof of Case (1.1), we may deduce the existence of a permutation $\sigma \in S_{n-2}$ such that

$$Ad(P_\sigma)(C) = C',\ Ad(P_\sigma)(\theta) = \zeta\theta'^*, \text{ and } Ad(P_\sigma)(\phi) = \omega\bar{\zeta}\phi'^*\ .$$

Case (2.1.2) $u_{2,1} = 0$

It follows from equation 2.3.60 that also $u'_{2,1} = 0$. Using the unitarity of U and the fact that $(u_{2,1}, u_{2,2}) = 0$, deduce that $Q^t (= (q_1, \cdots, q_{n-2}))$ is a unit vector and hence that there exists an index i such that $q_i \neq 0$. Similarly the unitarity of U and the fact that $(u_{1,1}, u_{1,2}) = 0$, implies that the vector $X (= (x_1, \cdots, x_{n-2}))$ is a unit vector and hence that there exists an index i' such that $x_{i'} \neq 0$.

Then, we find from equations 2.3.62-2.3.65 that

$$\begin{aligned} q_i(aC_i + b\phi_i S_i) &= aq'_i \\ q_i(a\theta_i S_i - b\theta_i\phi_i C_i) &= b\omega q'_i \\ q_i(cC_i + d\phi_i S_i) &= c\omega' q'_i \\ q_i(c\theta_i S_i - d\theta_i\phi_i C_i) &= d\omega\omega' q'_i \end{aligned} \qquad (2.3.85)$$

Also we find from the equations 2.3.66-2.3.69 that the following equations hold:

$$\begin{aligned} ax_{i'} &= (aC'_{i'} + c\theta'_{i'}S'_{i'})x'_{i'} \\ b\bar{\omega}x_{i'} &= (bC'_{i'} + d\theta'_{i'}S'_{i'})x'_{i'} \\ cx_{i'} &= (a\phi'_{i'}S'_{i'} - c\theta'_{i'}\phi'_{i'}C'_{i'})x'_{i'} \\ d\omega x_{i'} &= (b\phi'_{i'}S'_{i'} - d\theta'_{i'}\phi'_{i'}C'_{i'})x'_{i'} \end{aligned}$$

Now using the unitarity of the matrix $\begin{pmatrix} C_i & \phi_i S_i \\ \theta_i S_i & -\theta_i\phi_i C_i \end{pmatrix}$ (resp. the matrix $\begin{pmatrix} C'_{i'} & \phi'_{i'}S'_{i'} \\ \theta'_{i'}S'_{i'} & -\theta'_{i'}\phi'_{i'}C'_{i'} \end{pmatrix}$) deduce that $|q'_i| = |q_i| \neq 0$ (resp. $|x_i| = |x'_i|$). Also, as $S_i \neq 0$, we may infer from the set of equations 2.3.85 that $a, b, c, d \neq 0$.

Let $\omega_0 = q'_i q_i^{-1}$ and $\omega'_0 = x'_{i'}x_{i'}^{-1}$. Now, by applying Lemma 2.2.2 twice to the two sets of equations above (exactly as before), conclude that

$$(Re(\omega'), Re(\omega)) = (-(S_i^2 + Re(\bar{\omega}\theta_i\phi_i)C_i^2), -({S'_{i'}}^2 + Re(\theta'_{i'}\phi'_{i'}){C'_{i'}}^2) .$$

Hence the relation (2) is satisfied in this case.

Case (2.2) $u_{1,2} = 0$

We break this into cases depending on whether $u_{2,1}$ vanishes or not.

Case (2.2.1) $u_{2,1} \neq 0$

As before using the unitarity of A, A' and equation 2.3.60 we may assume that $u_{2,1} = u'_{2,1}$ and $A = D'A'$. Using the unitarity of U and the fact that $(u_{1,1}, u_{1,2}) = 0$, deduce that $P''(= (p_1, \cdots, p_{n-2}))$ is a unit vector and hence that there exists an index i such that $p_i \neq 0$. As $\omega \neq 1$, the matrix D is linearly independent from the identity matrix. Hence using equation 2.3.61 conclude that $u_{2,2} = u_{2,2} = 0$. Now the unitarity of U and the fact that $(u_{1,2}, u_{2,2}) = 0$, implies that the vector $Y(=(y_1,\cdots,y_{n-2}))$ is a unit vector and hence that there exists an index i' such that $y_{i'}\neq 0$.

Then, we find from equations 2.3.56-2.3.59 that

$$\begin{aligned} p_i(aC_i + b\phi_i S_i) &= ap'_i \\ p_i(a\theta_i S_i - b\theta_i\phi_i C_i) &= bp'_i \\ p_i(cC_i + d\phi_i S_i) &= c\omega' p'_i \\ p_i(c\theta_i S_i - d\theta_i\phi_i C_i) &= d\omega' p'_i \end{aligned}$$

Also we find from the equations 2.3.70-2.3.73 that the following equations hold:

$$\begin{aligned} ay_{i'} &= (aC'_{i'} + c\omega'\theta'_{i'}S'_{i'})y'_{i'} \\ b\omega y_{i'} &= (bC'_{i'} + d\omega'\theta'_{i'}S'_{i'})y'_{i'} \\ cy_{i'} &= (a\phi'_{i'}S'_{i'} - c\omega'\theta'_{i'}\phi'_{i'}C'_{i'})y'_{i'} \\ d\omega y_{i'} &= (b\phi'_{i'}S'_{i'} - d\omega'\theta'_{i'}\phi'_{i'}C'_{i'})y'_{i'} \end{aligned}$$

Again the unitarity of the matrix $\begin{pmatrix} C_i & \phi_i S_i \\ \theta_i S_i & \theta_i\phi_i C_i \end{pmatrix}$ (resp. the matrix $\begin{pmatrix} C'_{i'} & \phi'_{i'}S'_{i'} \\ \theta'_{i'}S'_{i'} & -\theta'_{i'}\phi'_{i'}C'_{i'} \end{pmatrix}$) implies that $|p'_i| = |p_i| \neq 0$ (resp. $|y_{i'}| = |y'_{i'}|$). Also, as $S_i \neq 0$, we may infer from the above set of equations that $a,b,c,d\neq 0$.

Let $\omega_0 = p'_i p_i^{-1}$ and $\omega'_0 = y'_{i'}y_{i'}^{-1}$. Now, by applying Lemma 2.2.2 twice to the two sets of equations above (exactly as before), conclude that

$$(Re(\omega'), Re(\omega)) = \left((S_i^2 + Re(\theta_i\phi_i)C_i^2), -({S'_{i'}}^2 + Re(\omega'\theta'_{i'}\phi'_{i'}){C'_{i'}}^2) \right).$$

Hence the relation (2) is satisfied in this case.

Case (2.2.2) $u_{2,1}=0$

First, suppose $u_{2,2}\neq 0$. Then, using the unitarity of A and equation 2.3.61, we may assume without loss of generality that $u_{2,2}=u'_{2,2}$ and $AD = D'A'$. As before using the unitarity of U and the fact that $(u_{1,1},u_{1,2})=0$, deduce that $P^t(=(p_1,\cdots,p_{n-2}))$ is a unit vector and hence that there exists an index i such that $p_i\neq 0$. Also the unitarity of U and the fact that $(u_{1,1},u_{2,1})=0$, implies that the vector $X(=(x_1,\cdots,x_{n-2}))$ is a unit vector and hence that there exists an index i' such that $x_{i'}\neq 0$.

Then, we find from equations 2.3.56-2.3.59 that

$$\begin{aligned} p_i(aC_i + b\phi_i S_i) &= ap'_i \\ p_i(a\theta_i S_i - b\theta_i \phi_i C_i) &= b\omega p'_i \\ p_i(cC_i + d\phi_i S_i) &= c\omega' p'_i \\ p_i(c\theta_i S_i - d\theta_i \phi_i C_i) &= d\omega\omega' p'_i \end{aligned}$$

Also we find from the equations 2.3.66-2.3.69 that the following equations hold:

$$\begin{aligned} ax_{i'} &= (aC'_{i'} + c\omega'\theta'_{i'} S'_{i'})x'_{i'} \\ b\omega x_{i'} &= (bC'_{i'} + d\omega'\theta'_{i'} S'_{i'})x'_{i'} \\ cx_{i'} &= (a\phi'_{i'} S'_{i'} - c\omega'\theta'_{i'}\phi'_{i'} C'_{i'})x'_{i'} \\ d\omega x_{i'} &= (b\phi'_{i'} S'_{i'} - d\omega'\theta'_{i'}\phi'_{i'} C'_{i'})x'_{i'} \end{aligned}$$

Again the unitarity of the matrix $\begin{pmatrix} C_i & \phi_i S_i \\ \theta_i S_i & -\theta_i \phi_i C_i \end{pmatrix}$ (resp. the matrix $\begin{pmatrix} C'_{i'} & \phi'_{i'} S'_{i'} \\ \theta'_{i'} S'_{i'} & -\theta'_{i'}\phi'_{i'} C'_{i'} \end{pmatrix}$) implies that $|p'_i| = |p_i| \neq 0$ (resp. $|x_{i'}| = |x'_{i'}|$). Also, as $S_i \neq 0$, we may infer from the above set of equations that $a, b, c, d \neq 0$.

Let $\omega_0 = p'_i p_i^{-1}$ and $\omega'_0 = x'_{i'} x_{i'}^{-1}$. Now, by applying Lemma 2.2.2 twice to the two sets of equations above, conclude exactly as before that

$$(Re(\omega'), Re(\omega)) = (-(S_i^2 + Re(\omega\theta_i\phi_i)C_i^2), -(S'^{\,2}_{i'} + Re(\omega'\theta'_{i'}\phi'_{i'})C'^{\,2}_{i'})).$$

Hence the relation (2) is satisfied in this case also.

Next we consider the final case when $u_{2,2}$ is also zero.

Case 2.2.3: $\begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} = 0$

First note that P, Q, X and Y are all unit vectors. So, there exist indices i, j such that $p_i \neq 0 \neq q_j$. We see from equations 2.3.56-2.3.59 and 2.3.62-2.3.65 that

$$\begin{aligned} p_i(aC_i + b\phi_i S_i) &= a'p'_i \\ p_i(a\theta_i S_i - b\theta_i \phi_i C_i) &= bp'_i \\ p_i(cC_i + d\phi_i S_i) &= c'p'_i \\ p_i(c\theta_i S_i - d\theta_i \phi_i C_i) &= d'p'_i \end{aligned} \qquad (2.3.86)$$

and

$$\begin{aligned} q_j(aC_j + b\phi_j S_j) &= a'q'_j \\ q_j(a\theta_j S_j - b\theta_j\phi_j C_j) &= b'q'_j \\ q_j(cC_j + d\phi_j S_j) &= c'\omega' q'_j \\ q_j(c\theta_j S_j - d\theta_j\phi_j C_j) &= d'\omega' q'_j \end{aligned} \tag{2.3.87}$$

Arguing exactly as in the proof of Case (1.2) (of this proposition), we find that $|p'_i| = |p_i|$ and $|q'_j| = |q_j|$. Further, the fact that $S_i, S_j \neq 0$ implies (as before) that $a, b, c, d \neq 0$. Setting $\omega_0 = p_i p_i'^{-1} q_j^{-1} q'_j$, we see that equations 2.3.86 and 2.3.87 imply the following identities:

$$\begin{aligned} a(\omega_0 C_i - C_j) + b(\omega_0\phi_i S_i - \phi_j S_j) &= 0 \\ a(\omega_0\theta_i S_i - \theta_j S_j) - b(\omega_0\theta_i\phi_i C_i - \theta_j\phi_j C_j) &= 0 \\ c(\omega_0\omega' C_i - C_j) + d(\omega_0\omega'\phi_i S_i - \phi_j S_j) &= 0 \\ c(\omega_0\omega'\theta_i S_i - \theta_j S_j) - d(\omega_0\omega'\theta_i\phi_i C_i - \theta_j\phi_j C_j) &= 0 \end{aligned}$$

The consistency of the above equations demands that

$$\begin{aligned} (\omega_0 C_i - C_j)(\omega_0\theta_i\phi_i C_i - \theta_j\phi_j C_j) + (\omega_0\phi_i S_i - \phi_j S_j)(\omega_0\theta_i S_i - \theta_j S_j) &= 0 \\ (\omega_0\omega' C_i - C_j)(\omega_0\omega'\theta_i\phi_i C_i - \theta_j\phi_j C_j) + (\omega_0\omega'\phi_i S_i - \phi_j S_j)(\omega_0\omega'\theta_i S_i - \theta_j S_j) &= 0 \end{aligned} \tag{2.3.88}$$

The fact that $\omega' \neq \pm 1$ enables us to derive the following consequence of the two equations above:

$$\omega_0 = \frac{(\theta_i\phi_i + \theta_j\phi_j)C_iC_j + (\theta_i\phi_j + \theta_j\phi_i)S_iS_j}{\theta_i\phi_i(1 + \omega')} .$$

Substituting this value for ω_0 in equation 2.3.88, we get

$$\omega'^2 + 2m_{i,j}\omega' + 1 = 0.$$

where, of course, $m_{i,j}$ is as in the statement of relation (3) in the proposition. It follows that $Re(\omega') = -m_{i,j}$.

As $X, Y \neq 0$, in a similar way to the previous cases, it follows (from equations 2.3.66-2.3.73) that there exist indices i', j' such that $|x_{i'}| = |x'_{i'}| \neq 0 \neq |y'_{i'}| = |y_{i'}|$

and

$$\begin{aligned}
a x_{i'} &= (a' C'_{i'} + c' \theta'_{i'} S'_{i'}) x'_{i'} \\
b x_{i'} &= (b' C'_{i'} + d' \theta'_{i'} S'_{i'}) x'_{i'} \\
c x_{i'} &= (a' \phi'_{i'} S'_{i'} - c' \theta'_{i'} \phi'_{i'} C'_{i'}) x'_{i'} \\
d x_{i'} &= (b'_{i'} \phi'_{i'} S'_{i'} - d' \theta'_{i'} \phi'_{i'} C'_{i'}) x'_{i'}
\end{aligned} \tag{2.3.89}$$

and

$$\begin{aligned}
a y_{j'} &= (a' C'_{j'} + c' \theta'_{j'} S'_{j'}) y'_{j'} \\
b\omega y_{j'} &= (b' C'_{j'} + d' \theta'_{j'} S'_{j'}) y'_{j'} \\
c y_{j'} &= (a' \phi'_{j'} S'_{j'} - c' \theta'_{j'} \phi'_{j'} C'_{j'}) y'_{j'} \\
d\omega y_{j'} &= (b' \phi'_{j'} S'_{j'} - d' \theta'_{j'} \phi'_{j'} C'_{j'}) y'_{j'}
\end{aligned} \tag{2.3.90}$$

Again, setting $\omega'_0 = x_{i'} x'^{-1}_{i'} y^{-1}_{j'} y'_{j'}$, we find the following consequence of the above sets of equations:

$$\begin{aligned}
a'(\omega'_0 C'_{i'} - C'_{j'}) + c'(\omega'_0 \theta'_{i'} S'_{i'} - \theta'_{j'} S'_{j'}) &= 0 \\
a'(\omega'_0 \phi'_{i'} S'_{i'} - \phi'_{j'} S'_{j'}) - c'(\omega'_0 \theta'_{i'} \phi'_{i'} C'_{i'} - \theta'_{j'} \phi'_{j'} C'_{j'}) &= 0
\end{aligned} \tag{2.3.91}$$

and

$$\begin{aligned}
b'(\omega'_0 \omega C'_{i'} - C'_{j'}) + d'(\omega'_0 \omega \theta'_{i'} S'_{i'} - \theta'_{j'} S'_{j'}) &= 0 \\
b'(\omega'_0 \omega \phi'_{i'} S'_{i'} - \phi'_{j'} S'_{j'}) - d'(\omega'_0 \omega \theta'_{i'} \phi'_{i'} C'_{i'} - \theta'_{j'} \phi'_{j'} C'_{j'}) &= 0
\end{aligned} \tag{2.3.92}$$

The consistency of these two sets of equations implies that

$$\begin{aligned}
&(\omega'_0 C'_{i'} - C'_{j'})(\omega'_0 \theta'_{i'} \phi'_{i'} C'_{i'} - \theta'_{j'} \phi'_{j'} C'_{j'}) + \\
&\quad (\omega'_0 \phi'_{i'} S'_{i'} - \phi'_{j'} S'_{j'})(\omega'_0 \theta'_{i'} S'_{i'} - \theta'_{j'} S'_{j'}) = 0 ,
\end{aligned}$$

and

$$\begin{aligned}
&(\omega'_0 \omega C'_{i'} - C'_{j'})(\omega'_0 \omega \theta'_{i'} \phi'_{i'} C'_{i'} - \theta'_{j'} \phi'_{j'} C'_{j'}) + \\
&\quad (\omega'_0 \omega \phi'_{i'} S'_{i'} - \phi'_{j'} S'_{j'})(\omega'_0 \omega \theta'_{i'} S'_{i'} - \theta'_{j'} S'_{j'}) = 0
\end{aligned}$$

and we may deduce as before that $Re(\omega) = -m'_{i',j'}$; i.e., the relation (3) is satisfied. Finally the proof of (a) is complete.

(b) If condition (0) is satisfied, we may define

$$A = A' = \begin{pmatrix} \theta'_1 & 0 \\ 0 & \theta_{\sigma^{-1}(1)} \end{pmatrix}$$

and

$$U = U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & P_n \end{pmatrix}$$

and verify that equations 2.3.54 to 2.3.77 are satisfied; and thus, it is indeed true that $(U \otimes A)W(\omega, \theta, \phi, C) = W(\omega, \theta', \phi', C'')(U' \otimes A')$.

If condition (1) is satisfied, we may define

$$A = \begin{pmatrix} 0 & 1 \\ -\omega\zeta & 0 \end{pmatrix} , \; A' = \begin{pmatrix} 0 & \omega \\ -\omega\zeta & 0 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 0 & 1 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & -\theta'\phi' P_n \end{pmatrix} , \; U' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & P_n \end{pmatrix} .$$

and verify that equations 2.3.54 to 2.3.77 are satisfied; and thus, it is indeed true that $(U \otimes A)W(\omega, \theta, \phi, C) = W(\omega, \theta', \phi', C'')(U' \otimes A')$.

(c) (1) By Ocneanu's Compactness result (Theorem 1.3.2), we know that

$$A_0^{\infty\prime} \cap A_1^{\infty} = (M_n \otimes 1) \cap W(M_n \otimes 1)W^* .$$

It is easily seen that if $X = E_{11} \otimes 1$, then $WXW^* = X$, and so we see that $A_0^{\infty\prime} \cap A_1^{\infty}$ contains a non-trivial projection, thus establishing reducibility of the vertical subfactor.

(2) In this case, Ocneanu's Compactness result says that

$$A_{\infty}^{0} \cap A_{\infty}^{1} = (1 \otimes M_2) \cap W(1 \otimes M_2)W^* .$$

The above algebra does not reduce to the scalars - i.e., the horizontal subfactor is reducible - precisely when it is possible to find non-scalar matrices

$$X = \begin{pmatrix} x_1 I_n & x_2 I_n \\ x_3 I_n & x_4 I_n \end{pmatrix} , \; Y = \begin{pmatrix} y_1 I_n & y_2 I_n \\ y_3 I_n & y_4 I_n \end{pmatrix} \in 1 \otimes M_2$$

- where $x_i, y_i \in \mathbb{C}$ - such that $WX = YW$.

Easy calculation shows that this matrix equation is satisfied if and only if the following relations hold:

$$
\begin{aligned}
x_1 = y_1 \quad &, \quad x_4 = y_4 \; , \\
x_2 = y_2 = \omega y_2 \quad &, \quad x_3 = y_3 = \omega x_3 \; , \\
x_3 \theta S \quad &= \quad x_2 \phi S \; , \qquad\qquad (**) \\
x_2(1 + \theta\phi)C \quad &= \quad (x_1 - x_4)\theta S \; , \\
x_3(1 + \theta\phi)C \quad &= \quad (x_1 - x_4)\phi S \; .
\end{aligned}
$$

(i) (α) If $\omega \neq 1$ and $S \neq 0$, it follows at once from the second and fourth lines of (**) that the equations above are satisfied if and only if $x_2 = y_2 = x_3 = y_3 = 0$, and $x_1 = y_1 = x_4 = y_4$ - i.e., if and only if $X = Y = \zeta I_{2n}$ for some $\zeta \in \mathbb{C}$. Hence the horizontal subfactor is irreducible in this case.

(β) Suppose θS and ϕS are not scalar multiples of one another. Then, in particular, $S \neq 0$, and we may also deduce from the third line of (**) that $x_3 = x_2 = 0$; since $S \neq 0$, either of the last two lines then forces $x_1 = x_4$.

(ii) If $S = 0$, it is readily seen that a non-scalar solution to the above system of equations is provided by

$$
x_i = y_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}
$$

□

We end this section with the following Proposition, which asserts the existence of a continous $(3n - 6)$-parameter family of pairwise inequevalent connections in $B(2, n)$. It also asserts that the number $(3n - 6)$ is sharp. What we mean by the sharpeness of the number $(3n - 6)$ is that there does not exist a subset $\mathcal{B} \subset B(2, n)$ with the following two properties: (i) no two distinct elements of $\mathcal{B}$ are equivalent (as connections); and (ii) $\mathcal{B}$ is homeomorphic to an open subset of Euclidean space of dimension $(3n - 5)$.

PROPOSITION 2.3.2 *There exist non-empty open sets* $\Omega \subset \mathbb{T}$, $\Theta \subset \mathbb{T}^{n-2}$, $\Phi_0 \subset \mathbb{T}^{n-3}$, $\Gamma \subset (0, 1)^{n-2}$ *such that if* $(\omega, \theta, \phi, C), (\omega', \theta', \phi' C') \in \Omega \times \Theta \times \Phi \times \Gamma$, *where* $\Phi = \{1\} \times \Phi_0$ *and* $(\omega, \theta, \phi, C) \neq (\omega', \theta', \phi' C')$ *then* $W(\omega, \theta, \phi, C)$ *is not equivalent to* $W(\omega', \theta', \phi', C')$. *Thus, there exist a* $(3n - 6)$ *parameter family of pairwise inequivalent connections and that is the best possible number.*

Further, we may assume that $1 \notin \Omega \cup \Gamma$; hence all these connections have the property that the associated vertical subfactor is reducible and has index n^2, and the horizontal subfactor is irreducible and has index 1.

Proof: Fix $0 < x_1 < x_2 < \frac{\pi}{4}$, and define $\Omega_0 = \{e^{ix} \in \mathbb{T} : x_1 \leq x \leq x_2\}$.

Fix $\pi/2 < y_1 < y_2 < 3\pi/4$, such that $0 < y_2 - y_1 < x_1$, and let $\Theta_0 = \{e^{ix} \in \mathbb{T} : y_1 \leq x \leq y_2\}$.

The definitions have the following (easily verified) consequences.

Suppose $\omega, \omega' \in \Omega_0$ and $\zeta, \zeta' \in \Theta_0$ are arbitrary. Then,

- $\zeta\zeta' \notin \Omega_0$;
- $Re(\omega) + Re(\zeta) \neq 0$;
- $Re(\omega) + Re(\omega'\zeta) \neq 0$;
- $Re(\omega) - Re(\zeta\bar{\zeta}') \neq 0$.

Define $f : [0,1] \times \mathbb{T} \times \mathbb{T} \rightarrow I\!R$ by $f(C, \zeta, \omega) = Re(\omega) + (1 - C^2) + C^2 Re(\zeta)$. Then $0 \notin f(\{1\} \times \Theta_0 \times \Omega_0)$. The compactness of $(\{1\} \times \Theta_0 \times \Omega_0)$ and continuity of f imply the existence of an $\epsilon > 0$ such that for all $C \in (1 - \epsilon, 1), \omega \in \Omega_0, \zeta \in \Theta_0$, we have $Re(\omega) + (1 - C^2) + C^2 Re(\zeta) \neq 0$. In a similar way, by considering suitable continuous functions, we can see that if ϵ is chosen sufficiently small, then the following relations are also valid:

Suppose $\omega \in \Omega_0$, and $\theta, \phi, \theta', \phi' \in \mathbb{T}$ are such that $\theta\phi, \theta'\phi' \in \Theta_0$, and suppose $C, C' \in (1 - \epsilon, 1)$. Then we simultaneously have:

$$Re(\omega) + (1 - C'^2) + C'^2 Re(\bar{\omega}'\theta\phi) \neq 0 \ ,$$

and

$$Re(\omega) + m \neq 0 \ ,$$

where $m = 1 - (1 + Re(\theta\phi\bar{\theta}'\bar{\phi}'))C^2C'^2 - (1 + Re(\theta\phi'\bar{\theta}'\bar{\phi}))S^2S'^2 - 2(Re(\theta\bar{\theta}') + Re(\phi\bar{\phi}'))CC'SS'$.

Let Ω denote the interior of Ω_0, and $\Gamma_0 = (1-\epsilon, 1)$. Let $\{\Theta'_i : 1 \leq i \leq n-2\}$ be a collection of pairwise disjoint open subsets of Θ_0. Define $\Gamma = \{diag(C_1, \cdots, C_{n-2}) : C_i \in \Gamma_0 \forall i\}$, and $\Theta' = \{diag(\zeta_1, \cdots, \zeta_{n-2}) : \zeta_i \in \Theta'_i \forall i\}$.

Define $\Theta_1 = \Theta'_1, \Phi_1 = \{1\}$ and for $1 < i \leq n-2$, choose non-empty open subsets $\Theta_i, \Phi_i \subset \mathbb{T}$ such that $\Theta_i\Phi_i \subset \Theta'_i$. Let $\Theta = \{diag(\theta_1, \cdots, \theta_{n-2}) \in M_{n-2} : \theta_i \in \Theta_i \forall i\}$ and $\Phi = \{diag(\phi_1, \cdots, \phi_{n-2}) \in M_{n-2} : \phi_i \in \Phi_i \forall i\}$.

Suppose now that $W(\omega, \theta, \phi, C)$ is equivalent to $W(\omega', \theta', \phi', C')$, where $(\omega, \theta, \phi, C), (\omega', \theta', \phi', C') \in \Omega \times \Theta \times \Phi \times \Gamma$.

First notice that if $\zeta, \zeta' \in \Theta'$ and if $\sigma \in S_{n-2}$ are such that $(Ad(P_\sigma))(\zeta) = \zeta'$, then necessarily $\zeta = \zeta'$ and σ is the identity permutation.

Our choice of (ϵ and consequently of) Γ ensures that neither of the relations (2) or (3) of Proposition 2.3.1 (a) can occur. Suppose the relation (1) were to hold; this would imply that (in the notation of the proposition) $(Ad(P_\sigma))(\theta\phi) = \omega(\theta'\phi')^*$; in particular, looking at any one diagonal entry of this matrix equation, we would be able to produce elements $\zeta_1, \zeta_2 \in \Theta_0$ such that $\omega = \zeta_1\zeta_2$, which we have already observed to be impossible. Thus the relation (1) can also not hold.

Thus, by Proposition 2.3.1 (a), the relation (0) must necessarily hold. Then the permutation σ (whose existence is the content of (0)) must satisfy the condition $(Ad(P_\sigma))(\theta\phi) = (\theta'\phi')$, which can only happen when σ is the identity permutation (by the discussion in the paragraph preceding the last one). Hence $\phi = \zeta\phi'$ - where ζ is as in the statement of Proposition 2.3.1 (a) (0); since $\phi_1 = \phi'_1 = 1$, we see that necessarily $\zeta = 1$; but relation (0), when $\sigma = id$ and $\zeta = 1$, then just says that $(\omega, \theta, \phi, C) = (\omega', \theta', \phi', C')$.

Now we will prove that the number $(3n-6)$ is sharp.

For this, let $F : \mathbb{T} \times \mathbb{T}^{n-2} \times \mathbb{T}^{n-2} \times [0,1]^{n-2} \longrightarrow B(2,n)$ denote the (obviously continuous) mapping given by $F(\omega, \theta, \phi, C) = W(\omega, \theta, \phi, C)$. Suppose now that there exists a subset $\mathcal{B}$ with the following two properties: (i) no two distinct elements of $\mathcal{B}$ are equivalent (as connections); and (ii) $\mathcal{B}$ is homoemorphic to an open subset of Euclidean space of dimension $(3n-5)$. Then $F^{-1}(\mathcal{B})$ is a subset of $\mathbb{T}^{2n-3} \times [0,1]^{n-2}$ which is homeomorphic to an open subset of $\mathbb{T}^{2n-3} \times [0,1]^{n-2}$, and is consequently itself open - see, for instance [Spa], Th. 4.8.16.

So it suffices to show that any open subset of $\mathbb{T}^{2n-3} \times (0,1)^{n-2}$ contains two distinct points whose images under F are equivalent, as connections. It clearly suffices to establish this assertion when the open subset is a product $\Omega \times \Theta \times \Phi \times \Gamma$ with open factors.

So, suppose $\Omega \subset \mathbb{T}$, $\Theta, \Phi \subset \mathbb{T}^{n-2}$, $\Gamma \subset (0,1)^{n-2}$ are open subsets. Let $\phi \in \Phi$, $\theta \in \Theta$ be arbitrary. As Φ is is assumed to be open, for all $\epsilon > 0$ there exists a $\phi'' \in \Phi$ such that, $\phi_1 \neq \phi_1''$ and $Arg(\phi_1 \phi_1'') < \epsilon$, where ϕ_1 and ϕ_1'' are the (1,1) th entry of ϕ and ϕ'' respectively. Let $\zeta = \phi_1 \phi_1''$. Now define $\theta_i' = \theta_i \zeta$ for $i = 1, 2, \ldots, n-2$ and $\phi_i' = \phi_i \zeta$ for $1 = 2, 3, \ldots, n-2$. Now choose ϵ - as Θ and ϕ are open - so that $\theta' \in \Theta$ and $\phi' \in \Phi$ Now it is easily seen that the pair $(\omega, \theta, \phi, C)$ and $(\omega, \theta', \phi', C)$ satisfies the relation (0) in Proposition 2.3.1, and hence - using (b) of the same proposition - we conclude that $W(\omega, \theta, \phi, C)$ is equivalent to $W(\omega, \theta', \phi', C)$. Finally, the proof is complete. □

2.4 The Principal Graph of the Horizontal subfactor

In [P] it is shown that for finite-depth subfactors of index 4, the principal graph has to be one of the extended Dynkin diagrams. We will show that all those diagrams can be obtained from vetex models coming from $B(2,n)$ for some n.

THEOREM 2.4.1 *(Popa) Let $N \subset M$ be an inclusion of II_1 factors, with finite depth and $[M : N] = 4$. Then the principal graph for the inclusion $N \subset M$ is one of the following diagrams: $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$.*

Suppose H is a finite subgroup of $SO(3)$. Let $\phi : SU(2) \to SO(3)$ be the 2-fold covering map (i.e., surjective homomorphism such that $ker\phi = \{+I, -I\}$); let π_n be the (unique, up to isomorphism) irreducible representation of $SU(2)$ of dimension $n+1$. Let $G = \phi^{-1}(H)$, and let $\pi = \pi_1|_G$. The following lemma can be easily seen to be true.

LEMMA 2.4.2 *Let $\rho \in \hat{G}$. Then* *(i)* $(\rho, 0) \in C(\hat{G}, \pi)^{(0)}$ *if and only if* $\pi(-1) = 1$ *if and only if* $\pi = \pi_0 \circ \phi$ *for some* $\pi_0 \in \hat{H}$.

(ii) $(\rho, 1) \in C(\hat{G}, \pi)^{(1)}$ *if and only if* $\pi(-1) \neq 1$ *if and only if* π *does not factor through* H.

PROPOSITION 2.4.3 *For all* $\mathcal{G} \in \{A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}\}$ *there exists an* $n \in \mathbb{N}$ *and* $W \in B(2, n)$ *such that the principal graph of the horizontal subfactor given by the vertex model corresponding to* W *is* $\mathcal{G}$.

Proof: It is enough to show that there exists $G \subset SU(2)$ with the property that $C(\hat{G}, \pi) = \mathcal{G}$. Note that π is self-contragredient and faithful. Using the lemma and some combinatorial arguments one can see, without too much difficulty, that if we let H be the group Z_n, D_n, A_4, S_4 or A_5, then the corresponding Cayley graph $C(\hat{G}, \pi)$ turns out to be the extended Coxeter graph $A_{2n}^{(1)}, D_{n+2}^{(1)}, E_6^{(1)}, E_7^{(1)}$ or $E_8^{(1)}$ respectively. □

Appendix A

Appendix

Here we provide an elementary and direct proof of two known results in the literature (due to Kosaki-Yamagami and Bisch, respectively) using the techniques of bimodules and elementary matrix manipulations. In the first section we recall certain basic facts about II_1 factors and their bimodules, and set up the basic machinery. Then in the second section the principal and dual graphs for the inclusion of II_1 factors $N = P \rtimes H \subseteq P \rtimes G = M$, where G is a discrete group acting as outer automorphisms of a II_1 factor P, and H is a subgroup of G such that $[G : H] < \infty$ are computed. Finally in the third section it is proved that if $N \subseteq M \subseteq P$ is an inclusion of II_1 factors such that $N \subseteq P$ has finite depth, then $N \subseteq M$ and $M \subseteq P$ have finite depth.

A.1 Preliminaries

In this section we recall various elementary facts about II_1 factors and their bimodules. These facts can be found in [GHJ], [JS], [PP] and [S]. We assume throughout that the symbols N and M denote II_1 factors.

A.1.1 A bifinite $N - M$ bimodule is a Hilbert space $\mathcal{H}$ equipped with unital normal *-homomorphisms π and π^{op} of N and M^{op} respectively into $\mathcal{L}(\mathcal{H})$ such that $\pi(N) \subseteq \pi^{op}(M^{op})'$ and both $\pi(N)'$ and $\pi^{op}(M^{op})'$ are finite von Neumann algebras, where M^{op} denotes the opposite algebra of M. The Hilbert space $\mathcal{H}$, thought of as an $N - M$ bimodule, is denoted by ${}_N\mathcal{H}_M$.

A.1.2 Let $M_{n,m}(A)$ denote the set of matrices whose entries come from A. For a Hilbert space $\mathcal{H}$, consider $M_{m,n}(\mathcal{H})$ as a Hilbert space with $\| \zeta \|^2 = \sum \| \zeta_{i,j} \|^2$. We denote $M_{n,n}(\cdot)$ by $M_n(\cdot)$. If N is a II_1 factor, then $L^2(N)$ is an $N - N$ bimodule, and more generally, for any positive integers m and n, $M_{m,n}(L^2(N))$ has a natural structure of an $M_m(N) - M_n(N)$ bimodule.

Let A denote either N or $L^2(N)$. For real numbers r, s, define

$$M_{r,s}(A) = \{x \in M_n(A) : pxq = x\},$$

where n is any integer greater than both r and s, and p, q are projections in $M_n(N)$ such that $Tr(p) = r$ and $Tr(q) = s$, where Tr denotes the non-normalised trace defined by $Tr(a) = \sum_i tr\ a_{i,i}$. As before, when $r = s$ we abbreviate $M_{r,r}(A)$ to $M_r(A)$.

A.1.3 If $\alpha : M \to M_r(N)$ is a unital normal *-homomorphism such that $[M_r(N) : \alpha(M)] < \infty$, then such an α is called a cofinite morphism of M. Let $\mathcal{H} = M_{1,r}(L^2(N))$; then $\mathcal{H}$ is a natural $N - M_r(N)$ bimodule. Also $\mathcal{H}$ is an $N - M$ bimodule (where the right action of M is given by $\zeta.a = \zeta\alpha(a)$) which is bifinite as $[M_r(N) : \alpha(M)] < \infty$. Write $\mathcal{H}_\alpha$ for $M_{1,r}(L^2(N))$ viewed as an $N - M$ bimodule via α and write $d_\alpha = r$.

A.1.4 Let ${}_N L_M(\mathcal{H}_\alpha)$ denote the space of bounded operators on $\mathcal{H}_\alpha$ which are right M-linear and left N-linear. Then, $T \in {}_N L_M(\mathcal{H}_\alpha)$ iff there exists a matrix $\hat{T} \in M_{d_\alpha}(N) \cap \alpha(M)'$ such that $T\zeta = \zeta\hat{T}$ for all $\zeta \in \mathcal{H}_\alpha$.

A.1.5 Every $N - M$ bimodule is equivalent to a bimodule $\mathcal{H}_\alpha$ given by a cofinite morphism α as in A.1.3.

A.1.6 For $N \subseteq M$ such that $[M : N] = r < \infty$, any $\lambda = (\lambda_i)_{i \in I}$ - where $\lambda_i \in M \forall\, i \in I$ and the cardinality of I is no smaller than r - is called a (right) basis for M/N, if $E_N(\lambda\lambda^*) = Q_\lambda$ is a projection in $M_I(N)$ with $Tr(Q_\lambda) = [M : N]$. (We always regard λ as a column vector, and we have used the notation $E_N(\lambda\lambda^*)$ for the $I \times I$ matrix with (i, j)-th entry $E_N(\lambda_i\lambda_j^*)$.)

The tuple λ is called an ONB for M/N if it satisfies the following conditions:

(i) $I = \{1, 2, \cdots, n+1\}$ where $n < [M : N] \leq n+1$;

(ii) $E_N(\lambda_i \lambda_j^*) = 0$ if $i \neq j$;

(iii) $E_N(\lambda_i \lambda_i^*) = 1$ for $i = 1, 2, 3, \ldots n$ and $E_N(\lambda_{n+1} \lambda_{n+1}^*)$ is a projection of trace equal to $([M : N] - n)$.

A.1.7 λ is a (right) basis for M/N iff $\lambda^* e \lambda = 1$ where e is the projection which implements the conditional expectation of M onto N. (Here we write $\lambda^* e \lambda$ for the matrix product $\sum_{i \in I} \lambda_i^* e \lambda_i$. Similarly, in the sequel we shall juxtapose symbols to mean matrix products. Also we shall only use (right) bases here and we shall henceforth simply call them bases.)

A.1.8 If $N \subseteq M$, $[M : N] = r < \infty$, and if λ is a basis for M/N, then $\theta : M \to M_r(N)$, defined by $\theta(m) = E_N(\lambda m \lambda^*)$ for all $m \in M$, is a cofinite morphism. (Here, $E_N(\lambda m \lambda^*)$ denotes the matrix obtained by applying the map E_N entrywise to the matrix $\lambda m \lambda^*$.) Also ${}_N L^2(M)_M$ is equivalent to $M_{1,r}(N)_\theta$. (The cofinite morphism θ was introduced by Ocneanu. See [O])

A.1.9 Let $\mathcal{H}_\alpha, \mathcal{H}_\beta$ be $N - M$, $M - P$ bifinite bimodules given by cofinite morphisms $\alpha : M \to M_{d_\alpha}(N), \beta : P \to M_{d_\beta}(M)$, respectively. Define $\alpha^r : M_r(M) \to M_{rd_\alpha}(N)$ in the obvious way. Then the $N - P$ bimodule $\mathcal{H}_{\alpha^{d_\beta} \circ \beta}$ given by the cofinite morphism $\alpha^{d_\beta} \circ \beta : P \to M_{d_\beta d_\alpha}(N)$, is a model for ${}_N(\mathcal{H}_\alpha)_M \otimes_M {}_M(\mathcal{H}_\beta)_P$.

A.1.10 Let $\mathcal{H}$ be an $N - M$ bimodule; (a model of) the contragredient of $\mathcal{H}$ is an $M - N$ bimodule $\overline{\mathcal{H}}$ for which there exists an anti-unitary operator $J : \mathcal{H} \to \overline{\mathcal{H}}$ satisfying $J(a.\zeta.b) = b^*.J(\zeta).a^*$ for all $a \in N$, $b \in M$ and $\zeta \in \mathcal{H}$.

A.1.11 Let $\mathcal{H} = {}_N L^2(M)_M \cong M_{1,r}(L^2(N))_\theta$ be as in A.1.3 above. Then the contragredient

$$\overline{\mathcal{H}} = {}_M L^2(M)_N \cong L^2(M)_\alpha,$$

where α is the inclusion of N into M. Hence

$$\mathcal{H} \otimes_M \overline{\mathcal{H}} \cong M_{1,r}(L^2(N))_{\theta_N} \cong {}_N L^2(M)_N,$$

where θ_N is the restriction of θ to N. Also,

$$(\mathcal{H} \otimes_M \overline{\mathcal{H}}) \otimes_N \mathcal{H} \cong M_{1,r^2}(L^2(N))_{\theta_N^r \circ \theta} \cong_N L^2(M_1)_M.$$

Iterating the same yields

$$_N L^2(M_n)_M \otimes_M {}_M L^2(M)_N \cong {}_N L^2(M_n)_N \cong M_{1,r^{n+1}}(L^2(N))_{\theta_N^{(n+1)}};$$

$$_N L^2(M_n)_N \otimes_N {}_N L^2(M)_M \cong {}_N L^2(M_{n+1})_M \cong M_{1,r^{n+2}}(L^2(N))_{\theta^{(n+2)}}.$$

In the above, $\theta^{(n)} : M \longrightarrow M_{I^n}(N)$ is defined by

$$\theta^{(n)}_{(i_1,i_2,\cdots,i_n),(j_1,j_2\cdots,j_n)} = \theta_{i_1,j_1} \circ \theta_{i_2,j_2} \circ \cdots, \circ\theta_{i_n,j_n}.$$

A.1.12 Clearly,

$$_N L_M(L^2(M_n)) = N' \cap J_{M_n} M' J_{M_n} = N' \cap M_{2n}.$$

Similarly,

$$_N L_N(L^2(M_n)) = N' \cap J_{M_n} N' J_{M_n} = N' \cap M_{2n+1}$$

. It follows from (A.1..11) and (A.1.4) that

$$N' \cap M_{2n} \cong {}_N L_M(M_{1,r^{n+1}}(L^2(N))_{\theta^{(n+1)}}) \cong M_{r^{n+1}}(N) \cap \theta^{(n+1)}(M)';$$

$$N' \cap M_{2n+1} \cong {}_N L_N(M_{1,r^{n+1}}(L^2(N))_{\theta_N^{(n+1)}}) \cong M_{r^{n+1}}(N) \cap \theta^{(n+1)}(N)'.$$

Similarly, we get

$$M' \cap M_{2n} \cong M_{r^n}(M) \cap \theta^{(n)}(M)'$$

$$M' \cap M_{2n+1} \cong M_{r^n}(M) \cap \theta^{(n)}(N)'. \qquad \text{(A.1.1)}$$

A.2 The subfactor corresponding to a subgroup

We assume throughout this section that $N = P \rtimes H \subseteq P \rtimes G = M$, where G is a discrete group acting as outer automorphisms of a II_1 factor P denoted by α, and H is a subgroup of G with $[G : H] < \infty$. Let $\mathcal{G}$ (resp., $\mathcal{H}$) denote the principal (resp., dual) graph for the subfactor $N \subseteq M$. The bipartite graphs $\mathcal{G}$ and $\mathcal{H}$ admit the following descriptions: (in the sequel, we use the notation $\bigsqcup$ to denote disjoint unions.)

Proposition A.2.1 *Let $H\backslash G/H = \bigsqcup Hg^iH$ be the decomposition of G into H-double cosets with $g^1 = 1$, the identity of G. Let $H_i = (g^i)^{-1}Hg^i \cap H$. Define $\dot{\mathcal{G}}^{(0)} = \bigsqcup \hat{H}_i \times \{i\}$, and $\dot{\mathcal{G}}^{(1)} = \hat{H}(= \hat{H}_1)$. Let $\dot{\mathcal{G}}$ be the bi-partite graph with the set of even vertices being given by $\dot{\mathcal{G}}^{(0)}$, and the set of odd vetices being given by $\dot{\mathcal{G}}^{(1)}$. If $\sigma \in \hat{H}_i$ and $\rho \in \hat{H}$, join the vertex $(\sigma, i) \in \dot{\mathcal{G}}^{(0)}$ to $\rho \in \dot{\mathcal{G}}^{(1)}$ by $< \rho_{|H_i}, \sigma >$ bonds. Then $\mathcal{G}$ is the connected component in $\dot{\mathcal{G}}$ of $(triv, 1)$, where triv denotes the trivial representation of H_1.*

Proposition A.2.2 *Let $\mathcal{H}^{(0)} = \hat{G} \times \{0\}$, $\mathcal{H}^{(1)} = \hat{H} \times \{1\}$, and let $\dot{\mathcal{H}}$ be the bipartite graph with $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ as the sets of even and odd vertices, with the even vertex $(\sigma, 0)$ connected to the odd vertex $(\rho, 1)$ by $< \sigma_{|H}, \rho >$ bonds. Then $\mathcal{H}$ is the connected component of $\dot{\mathcal{H}}$ containing $(triv, 0)$, where triv denotes the trivial representation of H.*

The results described above have been proved by Kosaki and Yamagami - see [KY] - using bimodules arising from bundles. Here an elementary and more direct proof of the same is presented using the theory of tensor products of bimodules (via composition of morphisms).

Let $g \mapsto \lambda_g$ denote the unitary representation of G in M and α_g be the corresponding outer automorphism on P.

Let $[G : H] = n$ and let Hg_i, $i = 1, 2 \cdots, n$ be the distinct right cosets of H in G. If we set $\lambda = (\lambda_{g_i})_{i=1}^n \in M_{1,n}(M)$, then $E_N(\lambda_{g_ig_j^{-1}}) = 1_H(g_ig_j^{-1})\lambda_{g_ig_j^{-1}} = \delta_{i,j}$; and so, $E_N(\lambda\lambda^*) = I$ and hence λ forms an ONB for M/N (see A.1.6).

The group G has an action β on the set $\{g_1, g_2, \cdots, g_n\}$ thus: for $g \in G$, β_g sends g_i to the coset representative of Hg_ig^{-1}; i.e.,

$$\beta_g(g_i) = g_j \text{ if there exists } h' \in H \text{ such that } g_j = h'g_ig^{-1}. \qquad (A.2.1)$$

Note that for $h \in H$, $\beta_h(g_i) = g_j$ only if g_i, g_j lie in the same double coset of H.

Note that $N' \cap M \subseteq P' \cap M = \mathbb{C}$.

Proposition A.2.3 *$N' \cap M_1 \cong \oplus_{i=1}^d \mathbb{C}$, where d is the number of double cosets of H in G.*

Proof: By 1.11, $N' \cap M_1 \cong M_n(N) \cap \theta(N)'$, where $\theta : M \to M_n(N)$ is given by $\theta(m) = E_N(\lambda m \lambda^{-1})$. Observe that for $r \in P$ and $1 \leq i, j \leq n$, we have $\theta_{g_i,g_j}(r) = E_N(\lambda_{g_i} r \lambda^*_{g_j}) = E_N(\alpha_{g_i}(r)\lambda_{g_i g_j^{-1}}) = \alpha_{g_i}(r)\delta_{i,j}$. If $X = (X_{g_i,g_j})_{1 \leq i,j \leq n} \in M_n(N)$, then clearly

$$X \in \theta(N)' \Leftrightarrow X \in \theta(P)' \cap \{\theta(\lambda_h) : h \in H\}'.$$

Now,

$$\begin{aligned} X \in \theta(P)' &\Leftrightarrow (X\theta(r))_{g_i,g_j} = (\theta(r)X)_{g_i,g_j} \ \forall\, 1 \leq i,j \leq n,\ r \in P \\ &\Leftrightarrow X_{g_i,g_j}\alpha_{g_j}(r) = \alpha_{g_i}(r)X_{g_i,g_j} \ \forall\, 1 \leq i,j \leq n,\ r \in P \\ &\Leftrightarrow \lambda_{g_i}^{-1}X_{g_i,g_j}\lambda_{g_j}r = r\lambda_{g_i}^{-1}X_{g_i,g_j}\lambda_{g_j} \ \forall\, 1 \leq i,j \leq n,\ r \in P \\ &\Leftrightarrow \lambda_{g_i}^{-1}X_{g_i,g_j}\lambda_{g_j} \in P' \cap M = \mathbb{C} \ \forall\, 1 \leq i,j \leq n \\ &\Leftrightarrow X_{g_i,g_j} = C'_{g_i,g_j}\lambda_{g_i g_j^{-1}} \text{ for some } C'_{g_i,g_j} \in \mathbb{C},\ \forall\, i,j; \end{aligned}$$

but $X_{g_i,g_j} \in N$ implies $C'_{g_i,g_j} = \delta_{i,j}C'_{g_i}$ for all $1 \leq i,j \leq n$, where $C'_{g_i} \in \mathbb{C}$.

Thus,

$$X \in M_n(N) \cap \theta(P)' \Leftrightarrow X_{g_i,g_j} = \delta_{i,j}C'_{g_i}\ \forall i,j \text{ , for some } C'_{g_i} \in \mathbb{C}. \qquad \text{(A.2.2)}$$

Now $\theta_{g_i,g_j}(\lambda_h) = E_N(\lambda_{g_i h g_j^{-1}}) = 1_H(g_i h g_j^{-1})\lambda_{g_i h g_j^{-1}} = \delta_{g_i,\beta_h(g_j)}\lambda_{g_i h g_j^{-1}}$. So, if $h \in H$, then

$$X\theta(\lambda_h) = \theta(\lambda_h)X \Leftrightarrow X_{g_i,\beta_h(g_j)}\lambda_{\beta_h(g_j)hg_j^{-1}} = \lambda_{g_i h(\beta_{h^{-1}}(g_i))^{-1}}X_{(\beta_{h^{-1}}(g_i))^{-1},g_j}\ \forall\, i,j.$$

It follows that if X satisfies the equivalent conditions of A.2.2, if further, $X \in \{\theta(\lambda_h) : h \in H\}'$, and if $g_i = \beta_h(g_j)$ for some $h \in H$ (so that $g_j = \beta_{h^{-1}}(g_i)$), then $C'_{g_i}\lambda_{g_i h g_j^{-1}} = C_{g_j}\lambda_{g_i h g_j^{-1}}$; so $C_{g_i} = C_{g_j}$ if g_i, g_j lie in the same double coset. Thus $N' \cap M_1 \cong \oplus_{i=1}^{d}\mathbb{C}$, where d is the number of double cosets of H in G. $\square$

The group G has an action β^p on the p-fold Cartesian power $\{g_1, g_2, \cdots, g_n\}^p$ as follows:

$$\beta^p_g((g_{i_1}, g_{i_2}, \cdots, g_{i_p})) = (g_{i'_1}, g_{i'_2}, \cdots, g_{i'_p}) \qquad \text{(A.2.3)}$$

if and only if

$$Hg_{i'_s}g_{i'_{s+1}}\cdots g_{i'_p} = Hg_{i_s}g_{i_{s+1}}\cdots g_{i_p}g^{-1} \ ,\ \forall\, 1 \leq s \leq p\ .$$

(This is easily seen to define an action of G.) If $g^1, g^2, \cdots, g^d$ are distinct double coset representatives, with g^1 being the identity and $H_i = H \cap (g^i)^{-1}Hg^i$

for $i = 1, 2, \cdots, d$, let π^p denote the permutation representation of G on $\mathbb{C}^{n^p}$ (given by the action β^p), and let π_k^p denote the permutation representation of H_k on $\mathbb{C}^{n^p}$ (given by the action β_p restricted H_k).

We shall find it convenient to use the following notation in the sequel: if $p \in \mathbb{N}$ and if $\mathbf{i} = (i_1, \cdots, i_p) \in \{1, 2, \cdots, n\}^p$, then we shall write $g_{\mathbf{i}} = (g_{i_1}, \cdots, g_{i_p})$ and $!g_{\mathbf{i}}! = g_{i_1} g_{i_2} \cdots g_{i_p}$.

PROPOSITION A.2.4 *(i)* $N' \cap M_{2p-1} \cong \oplus_{i=1}^{d} \pi_i^{p-1}(H_i)' \ \forall\ p \geq 2$.

(ii) $N' \cap M_{2p-2} \cong \pi_1^{p-1}(H)' \ \forall\ p > 2$.

Proof: (i) From A.1.12, we have

$$N' \cap M_{2p-1} \cong M_{n^p}(N) \cap \theta^{(p)}(N)',$$

where $\theta^{(p)} : M \rightarrow M_{n^p}(N)$ is given by

$$\theta^{(p)}_{g_{\mathbf{i}}, g_{\mathbf{j}}}(m) = E_N(\lambda_{g_{i_1}} E_N(\lambda_{g_{i_2}} \cdots E_N(\lambda_{g_{i_p}} m \lambda_{g_{j_p}^{-1}}) \cdots \lambda_{g_{j_2}^{-1}}) \lambda_{g_{j_1}^{-1}}),$$

for $g_{\mathbf{i}}, g_{\mathbf{j}} \in \{g_1, g_2, \cdots, g_n\}^p$.

For $r \in P$, note that

$$\begin{aligned} \theta^{(p)}_{g_{\mathbf{i}}, g_{\mathbf{j}}}(r) &= E_N(\lambda_{g_{i_1}} E_N(\lambda_{g_{i_2}} \cdots E_N(\lambda_{g_{i_p}} r \lambda_{g_{j_p}^{-1}}) \cdots \lambda_{g_{j_2}^{-1}}) \lambda_{g_{j_1}^{-1}}) \\ &= E_N(\lambda_{g_{i_1}} E_N(\lambda_{g_{i_2}} \cdots \alpha_{g_{i_p}}(r) E_N(\lambda_{g_{i_p}} \lambda_{g_{j_p}^{-1}}) \cdots \lambda_{g_{j_2}^{-1}}) \lambda_{g_{j_1}^{-1}}) \\ &= \delta_{\mathbf{i}, \mathbf{j}} \alpha_{!g_{\mathbf{i}}!}(r). \end{aligned}$$

As before, observe that $\theta^{(p)}(N)' = \theta^{(p)}(P)' \cap \{\theta^{(p)}(\lambda_h) : h \in H\}'$, and compute as follows:

For a fixed $r \in P$, $\mathbf{i}, \mathbf{j} \in \{1, 2, \cdots, n\}^p$, notice that

$$(X\theta^{(p)}(r))_{g_{\mathbf{i}}, g_{\mathbf{j}}} = (\theta^{(p)}(r)X)_{g_{\mathbf{i}}, g_{\mathbf{j}}} \ \Leftrightarrow\ X_{g_{\mathbf{i}}, g_{\mathbf{j}}} \alpha_{!g_{\mathbf{j}}!}(r) = \alpha_{!g_{\mathbf{i}}!}(r) X_{g_{\mathbf{i}}, g_{\mathbf{j}}}$$

and hence,

$$X \in M_{n^p}(N) \cap \theta^{(p)}(N)' \ \Leftrightarrow\ \lambda_{(!g_{\mathbf{i}}!)^{-1}} X_{g_i, g_j} \lambda_{!g_{\mathbf{j}}!} \in P' \cap M = \mathbb{C} \ \forall\ \mathbf{i}, \mathbf{j};$$

(Notice however that, in order for the matrix X to have entries from N, the last condition means that $X_{g_{\mathbf{i}}, g_{\mathbf{j}}} = 0$ unless $(!g_{\mathbf{i}}!)(!g_{\mathbf{j}}!)^{-1} \in H$.)

Thus we find that $X \in \theta^{(p)}(P)'$ if and only if there exist scalars $C_{g_{\mathbf{i}}, g_{\mathbf{j}}} \in \mathbb{C}$ such that

$$X_{g_{\mathbf{i}}, g_{\mathbf{j}}} = \begin{cases} C_{g_{\mathbf{i}}, g_{\mathbf{j}}} \lambda_{(!g_{\mathbf{i}}!)(!g_{\mathbf{j}}!)^{-1}}, & \text{if } (!g_{\mathbf{i}}!)(!g_{\mathbf{j}}!)^{-1} \in H \\ 0, & \text{otherwise} \end{cases} \tag{A.2.4}$$

Temporarily fix $p \in \mathbb{N}$: we write $\mathbf{i}, \mathbf{j}$ for elements of $\{1, 2, \cdots, n\}^p$, and $\mathbf{i}_-, \mathbf{j}_-$ for the elements of $\{1, 2, \cdots, n\}^{p-1}$ defined by $\mathbf{i}_- = (i_2, i_3, i_4, \cdots, i_p)$, etc.

Given an element

$$((C_{g_{\mathbf{i}_-}, g_{\mathbf{j}_-}, g_k}))_{g_{\mathbf{i}_-}, g_{\mathbf{j}_-} \in \{g_1, g_2, \cdots g_n\}^{p-1}, g_k \in \{g_1, g_2, \cdots g_n\}} \in \oplus_{k=1}^{n} M_{n^{p-1}}(\mathbb{C})$$

define $C_{g_{\mathbf{i}}, g_{\mathbf{j}}}$ as in equation A.2.4 by

$$C_{g_{\mathbf{i}}, g_{\mathbf{j}}} = \begin{cases} C_{g_{\mathbf{i}_-}, g_{\mathbf{j}_-}, g_k}, & \text{if } !g_{\mathbf{i}}!, !g_{\mathbf{j}}! \in Hg_k \\ 0, & \text{if } (!g_{\mathbf{i}}!)(!g_{\mathbf{j}}!)^{-1} \notin H \end{cases} \tag{A.2.5}$$

Notice now that if $(g_{\mathbf{i}_-}, g_{\mathbf{j}_-}, g_k)$ is arbitrary, then there is a unique pair $g_{\mathbf{i}}, g_{\mathbf{j}}$ such that $!g_{\mathbf{i}}!, !g_{\mathbf{j}}! \in Hg_k$, since the first co-ordinates are determined as follows: g_{i_1} is the coset representative of $Hg_k(g_{i_2}g_{i_3} \cdots g_{i_p})^{-1}$.) So there is a one-to-one correspondence between matrices $(C_{g_{\mathbf{i}}, g_{\mathbf{j}}})$ as described in A.2.4 and $\oplus_{i=1}^{n} M_{n^{p-1}}(\mathbb{C})$ via A.2.5.

Define

$$X_{g_{\mathbf{i}}, g_{\mathbf{j}}} = \begin{cases} C_{g_{\mathbf{i}_-}, g_{\mathbf{j}_-}, g_k} \lambda_{(!g_{\mathbf{i}}!)(!g_{\mathbf{j}}!)^{-1}}, & \text{if } !g_{\mathbf{i}}!, \ !g_{\mathbf{j}}! \in Hg_k \\ 0, & \text{if } (!g_{\mathbf{i}}!)(!g_{\mathbf{j}}!)^{-1} \notin H \end{cases} \tag{A.2.6}$$

where $g_{\mathbf{i}_-} = (g_{i_2}, g_{i_3}, g_{i_4} \cdots, g_{i_p}), g_{\mathbf{j}_-} = (g_{j_2}, g_{j_3}, \cdots, g_{j_p})$. It must be clear from the preceding that the assignment $X \mapsto ((C_{g_{\mathbf{i}_-}, g_{\mathbf{j}_-}, g_k}))$, defined by equation A.2.6, defines a *-algebra isomorphism $M_{n^p}(N) \cap \theta^{(p)}(P)' \cong \oplus_{i=1}^{n} M_{n^{p-1}}(\mathbb{C})$.

Now notice that, if $h \in H$, $\mathbf{i}, \mathbf{j} \in \{1, 2, \cdots, n\}^p$, then

$$\begin{aligned}
&\theta^{(p)}_{g_{\mathbf{i}}, g_{\mathbf{j}}}(\lambda_h) \\
&= E_N(\lambda_{g_{i_1}} E_N(\lambda_{g_{i_2}} \cdots E_N(\lambda_{g_{i_p}} \lambda_h \lambda_{g_{j_p}^{-1}}) \cdots \lambda_{g_{j_2}^{-1}}) \lambda_{g_{j_1}^{-1}}) \\
&= 1_H(g_{i_p} h g_{j_p}^{-1}) 1_H(g_{i_{p-1}} g_{i_p} h (g_{j_{p-1}} g_{j_p})^{-1}) \cdots 1_H((!g_{\mathbf{i}}!) h (!g_{\mathbf{j}}!)^{-1}) \lambda_{(!g_{\mathbf{i}}!) h (!g_{\mathbf{j}}!)^{-1}} \\
&= \delta_{g_{\mathbf{i}}, \beta_h^p(g_{\mathbf{j}})} \lambda_{(!(\beta_h^p(g_{\mathbf{j}})!) h (!(g_{\mathbf{j}})!)^{-1}}.
\end{aligned}$$

It follows easily that $X \in M_{n^p}(N) \cap \{\theta^{(p)}(\lambda_h) : h \in H\}'$ iff

$$X_{\beta_h^p(g_{\mathbf{i}}), \beta_h^p(g_{\mathbf{j}})} \lambda_{!(\beta_h^p(g_{\mathbf{j}}))! h (!(g_{\mathbf{j}})!)^{-1}} = \lambda_{!(\beta_h^p(g_{\mathbf{i}}))! h (!(g_{\mathbf{i}})!)^{-1}} X_{g_{\mathbf{i}}, g_{\mathbf{j}}} \tag{A.2.7}$$

$\forall\ h \in H,\ \mathbf{i},\mathbf{j} \in \{1,2,\cdots,n\}^p$.

Notice now that if both $!(g_{\mathbf{i}})!$ and $!(g_{\mathbf{j}})!$ lie in the same right coset Hg_k, then both $(!\beta_h^p(g_{\mathbf{i}}))!$ and $(!\beta_h^p(g_{\mathbf{j}}))!$ lie in the same right-coset $H\beta_h(g_k)$. Hence if X and $((C_{g_{\mathbf{i}\_},g_{\mathbf{j}\_},g_k}))$ are related as in equation A.2.6, then X satisfies equation A.2.7 iff

$$C_{\beta_h^{p-1}(g_{\mathbf{i}\_}),\beta_h^{p-1}(g_{\mathbf{j}\_}),\beta_h(g_k)} \;=\; C_{g_{\mathbf{i}\_},g_{\mathbf{j}\_},g_k}\ \forall\ k,\ \mathbf{i}\_,\ \mathbf{j}\_ \ . \qquad \text{(A.2.8)}$$

Thus, $X \in M_{n^p}(N) \cap \theta^{(p)}(N)'$ iff X is given by equation A.2.6, where the scalars $C_{g_{\mathbf{i}\_},g_{\mathbf{j}\_},g_k}$ satisfy equation A.2.8; this says that for each $k,l \in \{1,2,\cdots,n\}$, such that $Hg_kH \;=\; Hg_lH$, the matrix $((C_{\cdot,\cdot,g_k}))$ completely determines the matrix $((C_{\cdot,\cdot,g_l}))$ by equation A.2.8. Assertion (i) of the Proposition now follows easily.

(ii) Begin by observing that

$$M_{n^p}(N) \cap \theta^{(p)}(M)' = M_{n^p}(N) \cap \theta^{(p)}(N)' \cap \{\theta^{(p)}(\lambda_{g_l}) : 1 \leq l \leq n\}';$$

and that for $\mathbf{i},\mathbf{j} \in \{1,2,\cdots,n\}^p,\ l \in \{1,2,\cdots,n\}$,

$$\begin{aligned} &\theta^{(p)}_{g_{\mathbf{i}},g_{\mathbf{j}}}(\lambda_{g_l}) \\ &= E_N(\lambda_{g_{i_1}} E_N(\lambda_{g_{i_2}} \cdots E_N(\lambda_{g_{i_p}} \lambda_{g_l} \lambda_{g_{j_p}^{-1}}) \cdots \lambda_{g_{j_2}^{-1}})\lambda_{g_{j_1}^{-1}}) \\ &= 1_H(g_{i_p} g_l g_{j_p}^{-1}) 1_H(g_{i_{p-1}} g_{i_p} g_l (g_{j_{p-1}} g_{j_p})^{-1}) \cdots 1_H((!g_{\mathbf{i}}!) g_l (!g_{\mathbf{j}}!)^{-1}) \lambda_{(!g_{\mathbf{i}}!)g_l(!g_{\mathbf{j}}!)^{-1}} \\ &= \delta_{g_{\mathbf{i}},\beta_{g_l}^p(g_{\mathbf{j}})} \lambda_{!(\beta_{g_l}^p(g_{\mathbf{j}}))!g_l(!(g_{\mathbf{j}})!)^{-1}} \ . \end{aligned}$$

Hence if $X \in M_{n^p}(N)$, an easy computation shows that X commutes with $\theta^{(p)}(\lambda_{g_l})$ iff

$$X_{\beta_{g_l}^p(g_{\mathbf{i}}),\beta_{g_l}^p(g_{\mathbf{j}})} \lambda_{!(\beta_{g_l}^p(g_{\mathbf{j}}))!g_l(!(g_{\mathbf{j}})!)^{-1}} \;=\; \lambda_{!(\beta_{g_l}^p(g_{\mathbf{i}}))!g_l(!(g_{\mathbf{i}})!)^{-1}} X_{g_{\mathbf{i}},g_{\mathbf{j}}}\ \forall\ \mathbf{i},\mathbf{j}. \qquad \text{(A.2.9)}$$

As before, we find that $X \in M_{n^p}(N) \cap \theta^{(p)}(M)'$ iff X is given by equation A.2.6, where the scalars $((C_{g_{\mathbf{i}\_},g_{\mathbf{j}\_},g_k}))$ satisfy equation A.2.8 as well as:

$$C_{\beta_{g_l}^{p-1}(g_{\mathbf{i}\_}),\beta_{g_l}^{p-1}(g_{\mathbf{j}\_}),\beta_{g_l}(g_k)} \;=\; C_{g_{\mathbf{i}\_},g_{\mathbf{j}\_},g_k}\ \forall\ l,k,\ \mathbf{i}\_,\mathbf{j}\_ \ . \qquad \text{(A.2.10)}$$

Assertion (ii) follows quite easily from this. □

Proof of Proposition A.2.1: It is seen from the proof of Proposition A.2.4 that the isomorphisms established in that proposition are such that the following composite maps

$$N' \cap M_{2p-2} \;\cong\; \pi_h^{p-1}(H)' \;\subseteq\; M_{n^{p-1}}(\mathbf{C})$$

and

$$N' \cap M_{2p-1} \;\cong\; \oplus_{i=1}^{d} \pi_i^{p-1}(H_i)' \;\subseteq\; \oplus_{i=1}^{d} M_{n^{p-1}}(\mathbb{C})$$

are given by

$$X \;\mapsto\; ((C_{g,g'}))$$

and

$$X \;\mapsto\; \oplus_{i=1}^{d}((C_{g,g'}))$$

respectively, where, of course, X and $((C_{g,g_i}))$ are related as in equation A.2.6.

Since $\pi_i^{p-1} = \pi_h^{p-1}|_{H_i}$, it follows that the inclusion $N' \cap M_{2p-2} \subseteq N' \cap M_{2p-1}$ is described by the matrix $W = ((W_{\rho,(i,\sigma)}))$, with rows indexed by the set $\{\rho \in \hat{H} : < \pi_h^{p-1}, \rho > \neq 0\}$ and columns indexed by the set $\{(\sigma, i) : \sigma \in \hat{H}_i, < \pi_i^{p-1}, \sigma > \neq 0,\ i = 1, 2 \cdots, d\}$, and $W_{\rho,(i,\sigma)} = < \rho_{|H_i}, \sigma >$.

As the principal graph is always connected, it follows from the above that the Bratteli diagram for the inclusion $N' \cap M_{2p-2} \subseteq N' \cap M_{2p-1}$ is the connected component of $(1, triv)$, where $triv$ is the trivial representation, in the bipartite graph with odd vertices indexed by $\hat{H}$, even vertices by $\coprod_{i=1}^{d} \hat{H}_i$, and with adjacency relations as described above; and the proof of the Proposition is complete.

$\Box$

Proof of Proposition A.2.2: The starting point for the computation is the equations A.1.1, and the fact that

$$M' \cap M_1 \;\cong\; N' \cap M \;\subseteq\; P' \cap M \;=\; \mathbb{C}$$

Let $X = (X_{g_{\mathbf{i}},g_{\mathbf{j}}}) \in M_{n^p}(M)$; then for a fixed $r \in P$, $\mathbf{i}, \mathbf{j} \in \{1, 2, \cdots, n\}^p$, notice that

$$(X\theta^{(p)}(r))_{g_{\mathbf{i}},g_{\mathbf{j}}} = (\theta^{(p)}(r)X)_{g_{\mathbf{i}},g_{\mathbf{j}}} \;\Leftrightarrow\; X_{g_{\mathbf{i}},g_{\mathbf{j}}}\alpha_{!g_{\mathbf{j}}!}(r) = \alpha_{!g_{\mathbf{i}}!}(r)X_{g_{\mathbf{i}},g_{\mathbf{j}}};$$

and hence,

$$X \in M_{n^p}(M) \cap \theta^{(p)}(P)' \;\Leftrightarrow\; \lambda_{(!g_{\mathbf{i}}!)^{-1}} X_{g_{\mathbf{i}},g_{\mathbf{j}}} \lambda_{!g_{\mathbf{j}}!} \in P' \cap M = \mathbb{C}\ \forall\ \mathbf{i}, \mathbf{j};$$

Thus we find that $X \in M_{n^p}(M) \cap \theta^{(p)}(P)'$ if and only if there exist scalars $C_{g_{\mathbf{i}},g_{\mathbf{j}}} \in \mathbb{C}$ such that

$$X_{g_{\mathbf{i}},g_{\mathbf{j}}} \;=\; C_{g_{\mathbf{i}},g_{\mathbf{j}}}\lambda_{(!g_{\mathbf{i}}!)(!g_{\mathbf{j}}!)^{-1}}. \tag{A.2.11}$$

Arguing as before, we see that, for X as above, it is further true that $X \in M_{np}(N) \cap \{\theta^{(p)}(\lambda_h) : h \in H\}'$ iff

$$C_{\beta_h^p(g_{\mathbf{i}}),\beta_h^p(g_{\mathbf{j}})} = C_{g_{\mathbf{i}},g_{\mathbf{j}}} \ \forall\ \mathbf{i},\mathbf{j},h; \tag{A.2.12}$$

and consequently

$$M' \cap M_{2p+1} \cong \pi^p(H)'. \tag{A.2.13}$$

Equations A.1.1 clearly imply that

$$M' \cap M_{2p} = (M' \cap M_{2p+1}) \cap \{\theta^{(p)}(\lambda_{g_k}) : 1 \leq k \leq n\}'.$$

If X is given by equation A.2.11, an easy computation shows that $X \in (M' \cap M_{2p+1}) \cap \{\theta^{(p)}(\lambda_{g_k}) : 1 \leq k \leq n\}'$ if and only if the scalars $C_{\cdot,\cdot}$, in addition to A.2.12, satisfy

$$C_{\beta_g^p(g_{\mathbf{i}}),\beta_g^p(g_{\mathbf{j}})} = C_{g_{\mathbf{i}},g_{\mathbf{j}}} \ \forall\ \mathbf{i},\mathbf{j},g \in G, \tag{A.2.14}$$

and we find, consequently, that

$$M' \cap M_{2p} \cong \pi^p(G)'. \tag{A.2.15}$$

The argument for the rest of the proof of Proposition A.2.2 is exactly like the proof of Proposition A.2.1. □

A.3 Intermediate subfactors

In this section it is proved that if $N \subseteq M \subseteq P$ an inclusion of II_1 factors such that $N \subseteq P$ has finite depth, then $N \subseteq M$ and $M \subseteq P$ have finite depth. This result, which was proved by Bisch - see [B] - follows easily from another fact which we shall prove - see Proposition A.3.2.

We begin by stating an elementary lemma which is proved in[JS].

LEMMA A.3.1 *Let $N \subseteq M \subseteq P$ be an inclusion of II_1 factors. If $\lambda = \{\lambda_i\}_{i\in I}$ is a basis for M/N, and if $\eta = \{\eta_j\}_{j\in J}$ is a basis for P/M, then $\lambda\eta = \{\lambda_i\eta_j\}_{i\in I,j\in J}$ is a basis for P/N.*

Proposition A.3.2 *Let $N \subseteq M \subseteq P$ be inclusion of II_1 factors such that $[P : N] < \infty$. Let $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \cdots$ be the tower obtained by basic construction for $N \subseteq M$ and $N \subseteq P \subseteq P_1 \subseteq P_2 \subseteq \cdots$ be the tower obtained by basic construction for $N \subseteq P$. Then for each $n \geq 1$, there exist a projection $p_n \in N' \cap P_{2n-1}$ such that $p_n(N' \cap P_{2n-1})p_n \cong N' \cap M_{2n-1}$.*

Proof: Let $\{\lambda_i\}_{i \in I}$ be a basis for M/N and $\{\eta_j\}_{j \in J}$ be an ONB for P/M such that $\eta_1 = 1$. (Then in particular, $E_M(\eta_j) = 0$ for all $j \neq 1$). Then, by Lemma A.3.1, $\{\lambda_i \eta_j\}_{i \in I, j \in J}$ forms a basis for P/N. Hence, by (1.5), we have co-finite morphisms

$$\theta : P \rightarrow M_{I \times J}(N), \quad \psi : M \rightarrow M_I(N)$$

defined by

$$\theta_{(i,j),(i',j')}(p) = E_N(\lambda_i \eta_j p \eta_{j'}^* \lambda_{i'}^*), \quad \psi_{i,i'}(m) = E_N(\lambda_i m \lambda_{i'}^*)$$

respectively.

As before we consider $\theta^{(n)} : P \rightarrow M_{(I \times J)^n}(N)$ and $\psi^{(n)} : M \rightarrow M_{I^n}(N)$ defined by

$$(\theta^{(n)}(p))_{(\mathbf{i},\mathbf{j}),(\mathbf{i}'\mathbf{j}')} = E_N(\lambda_{i_1}\eta_{j_1} E_N(\lambda_{i_2}\eta_{j_2} \cdots E_N(\lambda_{i_n}\eta_{j_n} p \eta_{j_n'}^* \lambda_{i_n'}^*) \cdots)\eta_{j_2'}^* \lambda_{i_2'}^*)\eta_{j_1'}^* \lambda_{i_1'}^*)$$

for $p \in P, (\mathbf{i},\mathbf{j}), (\mathbf{i}',\mathbf{j}') \in (I \times J)^n$, and

$$(\psi^{(n)}(m))_{\mathbf{i},\mathbf{i}'} = E_N(\lambda_{i_1} E_N(\lambda_{i_2} \cdots E_N(\lambda_{i_n} m \lambda_{i_n'}^*) \cdots)\lambda_{i_2'}^*)\lambda_{i_1'}^*)$$

for $m \in M, \mathbf{i}, \mathbf{i}' \in I^n$.

By (A.1.12), we have

$$N' \cap P_{2n+1} \cong \theta^{(n)}(N)' \cap \theta^{(n)}(1)(M_{(I \times J)^n}(N))\theta^{(n)}(1) \qquad (A.3.16)$$

and

$$N' \cap M_{2n+1} \cong \psi^{(n)}(N)' \cap \psi^{(n)}(1)(M_{I^n}(N))\psi^{(n)}(1). \qquad (A.3.17)$$

In the sequel, we shall consistently identify $I^n \times J^n$ with $(I \times J)^n$; we shall also find it convenient to use the notation $\mathbf{1} = (1, 1, \cdots 1)$. Thus, our assumption about η_1 implies that

$$(\theta^{(n)}(x))_{(\mathbf{i},\mathbf{1}),(\mathbf{i}',\mathbf{1})} = (\psi^{(n)}(x))_{\mathbf{i},\mathbf{i}'} \ \forall \ x \in M, \ \forall \ \mathbf{i}, \mathbf{j}. \qquad (A.3.18)$$

Define $\tilde{p}_n{}' \in M_{(I\times J)^n}(N)$ by

$$(\tilde{p}_n{}')_{(\mathbf{i},\mathbf{j}),(\mathbf{i}',\mathbf{j}')} = \delta_{\mathbf{i},\mathbf{i}'}\delta_{(\mathbf{j},\mathbf{j}'),(\mathbf{1},\mathbf{1})}. \tag{A.3.19}$$

We first claim that

$$\tilde{p}_n{}' \in \theta^{(n)}(N)' \cap M_{(I\times J)^n}(N). \tag{A.3.20}$$

As $\tilde{p}_n{}'$ is diagonal we have to verify that

$$(\tilde{p}_n{}')_{(\mathbf{i},\mathbf{j}),(\mathbf{i},\mathbf{j})}(\theta^{(n)}(x))_{(\mathbf{i},\mathbf{j}),(\mathbf{i}',\mathbf{j}')} = (\theta^{(n)}(x))_{(\mathbf{i},\mathbf{j}),(\mathbf{i}',\mathbf{j}')}(\tilde{p}_n{}')_{(\mathbf{i}',\mathbf{j}'),(\mathbf{i}',\mathbf{j}')} \tag{A.3.21}$$

for all $x \in N$, $(\mathbf{i},\mathbf{j}),(\mathbf{i}',\mathbf{j}') \in (I\times J)^n$. If neither $\mathbf{j}$ nor $\mathbf{j}'$ is equal to $\mathbf{1}$, then both sides of equation A.3.21 vanish; while if both $\mathbf{j}$ and $\mathbf{j}'$ are equal to $\mathbf{1}$, then both sides of the equation are seen to be equal to $(\theta^{(n)}(x))_{(\mathbf{i},\mathbf{1}),(\mathbf{i}',\mathbf{1})}$. In case exactly one of $\mathbf{j}$ and $\mathbf{j}'$ is equal to $\mathbf{1}$, the validity of equation A.3.21 is a consequence of the following claim.

Claim: If $x \in N$, $(\mathbf{i},\mathbf{j}),(\mathbf{i}',\mathbf{j}') \in (I\times J)^n$, and if exactly one of $\mathbf{j}$ and $\mathbf{j}'$ is equal to $\mathbf{1}$, then

$$(\theta^{(n)}(x))_{(\mathbf{i},\mathbf{j}),(\mathbf{i}',\mathbf{j}')} = 0.$$

We prove the claim in case $\mathbf{j}' \neq \mathbf{1}$, the other case being similar. Let r be the smallest co-ordinate such that $j'_r \neq 1$; it follows that there exists an element $x' \in N$ such that

$$\begin{aligned}(\theta^{(n)}(x))_{(\mathbf{i},\mathbf{1}),(\mathbf{i}',\mathbf{j}')} &= E_N(\lambda_{i_1}E_N(\lambda_{i_2}\cdots E_N(\lambda_{i_r}x'\eta^*_{j'_r}\lambda^*_{i'_r})\cdots)\eta^*_{j'_2}\lambda^*_{i'_2})\eta^*_{j'_1}\lambda^*_{i'_1}) \\ &= E_N(\lambda_{i_1}E_N(\lambda_{i_2}\cdots E_N(\lambda_{i_r}x'E_M(\eta^*_{j'_r})\lambda^*_{i'_r})\cdots)\eta^*_{j'_2}\lambda^*_{i'_2})\eta^*_{j'_1}\lambda^*_{i'_1}) \\ &= 0\end{aligned}$$

and the claim is proved.

Now consider the projection $\tilde{p}_n = \tilde{p}_n{}'\theta^{(n)}(1)$, which is clearly dominated by $\theta^{(n)}(1)$. Notice that if $X \in \theta^{(n)}(N)' \cap \theta^{(n)}(1)(M_{(I\times J)^n}(N))\theta^{(n)}(1)$, then

$$(\tilde{p}_n X\tilde{p}_n)_{(\mathbf{i},\mathbf{j}),(\mathbf{i}',\mathbf{j}')} = \delta_{(\mathbf{j},\mathbf{j}'),(\mathbf{1},\mathbf{1})}X_{(\mathbf{i},\mathbf{1}),(\mathbf{i}',\mathbf{1})}.$$

It is easy to see that the map

$$\alpha : \tilde{p}_n(\theta^{(n)}(N)' \cap (\theta^{(n)}(1)(M_{(I\times J)^n}(N))\theta^{(n)}(1)))\tilde{p}_n \rightarrow M_{I^n}(N) \tag{A.3.22}$$

defined by

$$(\alpha\ (\tilde{p}_n X \tilde{p}_n)\ )_{\mathbf{i},\mathbf{i}'} = X_{(\mathbf{i},\mathbf{1}),(\mathbf{i}',\mathbf{1})} \tag{A.3.23}$$

is an injective homomorphism.

We claim now that the map α defines an isomorphism, i.e.

$$\begin{aligned} &\tilde{p}_n(\theta^{(n)}(N)' \cap (\theta^{(n)}(1)(M_{(I\times J)^n}(N))\theta^{(n)}(1)))\tilde{p}_n \\ &\cong (\psi^{(n)}(1)(M_{I^n}(N))\psi^{(n)}(1)) \cap \psi^{(n)}(N)'. \end{aligned} \tag{A.3.24}$$

Since $\alpha(\tilde{p}_n) = \psi^{(n)}(1)$ (see A.3.18), it follows that

$$\alpha(\tilde{p}_n(\theta^{(n)}(N)' \cap (\theta^{(n)}(1)(M_{(I\times J)^n}(N))\theta^{(n)}(1)))\tilde{p}_n) \subseteq \psi^{(n)}(1)(M_{I^n}(N))\psi^{(n)}(1).$$

Further, if $X \in \theta^{(n)}(N)' \cap (\theta^{(n)}(1)(M_{(I\times J)^n}(N))\theta^{(n)}(1)))$, $x \in N$, we have, for all $\mathbf{i}, \mathbf{i}' \in I^n, \mathbf{j}, \mathbf{j}' \in J^n$:

$$\begin{aligned}
(\alpha(\tilde{p}_n X \tilde{p}_n)\psi^{(n)}(x))_{\mathbf{i},\mathbf{i}'} &= \sum_{\mathbf{k}\in I^n} X_{(\mathbf{i},\mathbf{1}),(\mathbf{k},\mathbf{1})}\psi^{(n)}(x)_{(\mathbf{k},\mathbf{i}')} \\
&= \sum_{\mathbf{k}\in I^n} X_{(\mathbf{i},\mathbf{1}),(\mathbf{k},\mathbf{1})}\theta^{(n)}(x)_{(\mathbf{k},\mathbf{1}),(\mathbf{i}',\mathbf{1})} \quad \text{(by equation A.3.18)} \\
&= \sum_{(\mathbf{k},\mathbf{j})\in(I\times J)^n} X_{(\mathbf{i},\mathbf{1}),(\mathbf{k},\mathbf{j})}\ \theta^{(n)}(x)_{(\mathbf{k},\mathbf{j}),(\mathbf{i}',\mathbf{1})} \quad \text{(by claim)} \\
&= (X\theta^{(n)}(x))_{(\mathbf{i},\mathbf{1}),(\mathbf{i}',\mathbf{1})} \\
&= (\theta^{(n)}(x)X)_{(\mathbf{i},\mathbf{1}),(\mathbf{i}',\mathbf{1})} \\
&= \sum_{(\mathbf{k},\mathbf{j})\in(I\times J)^n} \theta^{(n)}(x)_{(\mathbf{i},\mathbf{1}),(\mathbf{k},\mathbf{j})}X_{(\mathbf{k},\mathbf{j}),(\mathbf{i}',\mathbf{1})} \\
&= \sum_{\mathbf{k}\in I^n} \theta^{(n)}(x)_{(\mathbf{i},\mathbf{1}),(\mathbf{k},\mathbf{1})}X_{(\mathbf{k},\mathbf{1}),(\mathbf{i}',\mathbf{1})} \quad \text{(by claim)} \\
&= \sum_{\mathbf{k}\in I^n} \psi^{(n)}(x)_{(\mathbf{i},\mathbf{k})}X_{(\mathbf{k},\mathbf{1}),(\mathbf{i}',\mathbf{1})} \quad \text{(by equation A.3.18)} \\
&= (\psi^{(n)}(x)\alpha(\tilde{p}_n X\tilde{p}_n))_{\mathbf{i},\mathbf{i}'};
\end{aligned}$$

and we see that indeed, α maps $\tilde{p}_n(\theta^{(n)}(N)' \cap (\theta^{(n)}(1)(M_{(I\times J)^n}(N))\theta^{(n)}(1)))\tilde{p}_n$ into $(\psi^{(n)}(1)(M_{I^n}(N))\psi^{n}(1)) \cap \psi^{(n)}(N)'$.

On the other hand, it is clear, (by using similar arguments), that if $Y \in \psi^{(n)}(1)(M_{I^n}(N))\psi^{(n)}(1) \cap \psi^{(n)}(N)'$, and if we define $X_{(\mathbf{i},\mathbf{j}),(\mathbf{i}',\mathbf{j}')} = \delta_{(\mathbf{j},\mathbf{j}'),(\mathbf{1},\mathbf{1})}Y_{\mathbf{i},\mathbf{i}'}$, then $X \in \tilde{p}_n(\theta^{(n)}(N)' \cap \theta^{(n)}(1)(M_{(I\times J)^n}(N))\theta^{(n)}(1))\tilde{p}_n$ and $\alpha(X) = Y$.

Thus we have proved A.3.24, and hence - in view of A.3.16 and A.3.17 - the proof of the proposition is complete. □

REMARK A.3.3 *Continuing with the notation of Proposition A.3.2, if $M \subseteq P \subset Q_1 \subset Q_2 \subset \cdots$ is the tower obtained by basic construction for $M \subseteq P$, then for each $n \geq 1$, there exist a projection $q_n \in P' \cap P_{2n}$ such that*

$$q_n(P' \cap P_{2n})q_n \simeq P' \cap Q_{2n}.$$

(Reason: Apply Proposition A.3.2 to the inclusion of II_1 factors $P \subseteq Q_1 \subseteq P_1$.)

COROLLARY A.3.4 *Let $N \subseteq M \subseteq P$ be an inclusion of II_1 factors. If $N \subseteq P$ has finite depth then, also $N \subseteq M$ and $M \subseteq P$ have finite depth.*

Proof: The inclusion $N \subseteq M$ has finite depth, since if p_n is as in Proposition A.3.2, then

$$\begin{aligned} sup_{n\geq 1}(dim(Z(N' \cap M_{2n-1}))) &= sup_{n\geq 1}(dim(Z(p_n(N' \cap P_{2n-1})p_n))) \\ &\leq sup_{n\geq 1}(dim(Z(N' \cap P_{2n-1}))) \\ &< \infty. \end{aligned}$$

Similarly, for $M \subseteq P$, apply Remark A.3.3. □

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