STABILITY AND LARGENESS OF THE CORE OF COOPERATIVE GAMES

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Preface

Work for this thesis started in the year 1996 by reading a paper on large core by Prof.W.W.Sharkey published in the year 1982. In this thesis we have focused on the core of a cooperative game and it's coincidence with the other known solution concept defined in the book titled Games and Economic Behaviour by von-Neumann and Morgenstern way back in 1944. In this thesis one of the significant results has been the equivalence of the two concepts namely largeness of the core and stability of it in case of symmetric games. We present several equivalent conditions for largeness of the core both in TU and NTU games. We also provide several examples to highlight the strength and weaknesses of the results wherever necessary.

I was never a student of ISI Delhi, hence I only had an aquaintance with Prof.T.Parthasarathy. I was relieved when he accepted me as a student without much prior knowledge. Prof.Parthasarathy reignited my academic abilities and infused in me the confidence I required after years of my near academic dormance. He brought me from bylanes of life to the highways! With great reverence I record my deepest gratitude to the teacher, supervisor and above all the person that Prof.Parthasarathy is.

In bringing out this dissertation Prof.Parthasarathy's relentless encouragement to me, his very personal understanding and attention to my academic and psychological needs, his constant support and guidance on all associated matters during the last several years is the prime factor. I consider myself extremely fortunate to be his student.

It is Prof.Jos Potters who initiated the introduction of Game Theory to me. Apart from working and coauthoring papers with me, he has given his time and the most valuable comments on nearly all my works. My learning from him will remain invaluable. Most importantly he hardly looked for credentials behind me while agreeing for joint work. I am thankful and indebted to him.

I convey my special regards to Dr.G.Ravindran coauthors of two of my papers, and for all the friendly discussions we had. He has taken pains to travel several times to Chennai for such interactions with the only intention to facilitate my work. I convey my gratitude to the friend that he is.

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During the last few years I have been constantly encouraged and inspired by friends, more than colleagues like Dr.G.S.R. Murthy and Mr.Rajiv Goel. It is Dr.Murthy who had primarily inspired me to research. My sincere thanks to them.

Many improvements in various aspects of this thesis could be brought about because of the extensive suggestions I received from Prof.Kensaku Kikuta and Prof.Shigeo Muto apart from Prof.Jos Potters. My sincere gratitudes to these scientists.

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Amit K.Biswas

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Notations and symbols

The player set	N
Set of all coalitions	2^N
The empty set	ϕ
Cooperative game	(N, v)
Subgame on S	$v\|_S$
Restriction of a vector	$x\|_S$
The core	$C(v)$
Interior of the core	$C^\circ(v)$
Imputation set	$I(v)$
n-dimensional real space	$\mathbf{R}^N$
x dominates y via S	$x \succ_S y$
x dominates y	$x \succ y$
no domination	$\neg(x \succ y)$.
Set of acceptable vectors	$A(v)$
Lower boundary of $A(v)$	$L(v)$
Set of upper vectors	$U(v)$
Totally balanced cover of v	$\bar{v}$
Ordered vector space	$\mathbf{R}^n_<$
All permutations of x	$\pi(x)$
$\cup_{x \in C} \pi(x)$	$\pi(C)$
$\sum_{i=s}^{t} x_i$	$x(s, t)$
$\sum_{i=1}^{t} x_i$	$x(t)$
$\sum_{i \in S} x_i$	$x(S)$
$\{S \ : \ y(S) = v(S)\}$	$\mathcal{S}_y$
k^{th} specified vector	y^k
Indicator vector of coalition T	e_T
Indicator vector for player i	e_i
Minimum of s and t	$s \wedge t$
$a_i > b_i$ for all i	$a \gg b$
equivalence	$\Leftrightarrow$
NTU game	$V(.)$
The boundary of $V(S)$	$\bar{V}(S)$
The interior of $V(S)$	$\dot{V}(S)$

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Chapter 1

Preliminaries

1.1 Introduction

This monograph deals with the area from game theory known as cooperative games. Except the last chapter on NTU games, it deals with transferable utility games. Here we will introduce and discuss the involved game theoretic notions and set a mathematical base for the chapters to come.

In 1944 von Neumann and Morgenstern[45] introduced a theory of solutions for n-person games in characteristic function form in which cooperation and coalition formation is a crucial aspect. The primary mathematical concern regarding this model is the existence of solutions or stable set. In 1968 Lucas[19] described a ten person game which has no solution. However, researchers have gone on to identify properties of such solutions when they exist and their relationship with other known concepts, in particular, the core. Sharkey[43] defined and studied the concept of largeness of the core which arose while he was studying an economic problem involving cost allocation. He showed that largeness of the core is a sufficient condition for the the core to be a stable set.

We present in this thesis some results concerning the coincidence of the stable set and the core. This has been particularly done through the concept of large core. We also present conditions for the core to be large and quite a few examples giving insight into the results proved.

Sharkey[43] proved that largeness of core is sufficient for the core to be a stable set. We identify a subclass namely *the symmetric games*, where largeness of the core turns out to be also necessary and leads to other interesting and easy to check conditions for stability of the core in symmetric games. Subsequently we answer the question if every exact game has a large core. We prove that for games with 5 players or more, every exact game need not posses a large core, however, in the subclass of symmetric games largeness of core and the stability of the core turn out to be equivalent to the concept of the game being exact. For general TU games with 3 or 4 players every exact game has a large core. So for totally balanced *symmetric* games large core, stable core and exactness turn out to be equivalent.

However, for general TU games, largeness of the core always implies core stability and there are examples where the core is stable but not large. It is known that under the extendability condition introduced by Kikuta and Shapley [17] the core is a stable set but the core may not be large. In this thesis, we show that the Kikuta-Shapley condition is sufficient for the core to be stable as well as large for TU games with five or less number of players. We provide a counter example when the number of players is six. We then introduce a stronger extendability condition and show that this condition is necessary and sufficient for the core to be large.

The core of a TU game is perhaps the most intuitive and easiest solution concept in Cooperative Game Theory [33]. The other approach to solution concepts is the stable sets introduced by von Neumann and Morgenstern[45]. Thus far the relation between the two most crucially important solution concepts for cooperative games has been investigated in the context of symmetric transferable utility games and this has been related to the notion of large core. We have further investigated the relation between the von-Neumann-Morgenstern stability of the core and the largeness of it in the case of non-transferable utility (NTU) games. The main findings are basically extensions of existing results obtained in [43], [4], [5] in the case NTU games, which are very similar to the results in TU cases.

During the course of research over the last few years, we have solved a few problems, we have also understood a few problems better, as a result we have a few conjectures and formulations which we present in the last chapter for further attention by researchers. One of which is of a combinatoric nature and can be called as old as the concept of stable set. The problem is precisely to find a necessary and sufficient condition for a stable core. The other appears to be more of an LP problem than of a Game Theory problem. The problem is to find a vector with the largest sum of components in the set of the lower boundary points of the set of acceptable vectors.

Section 1.2 begins with an introduction of the game theoretic model and some basic notions. Much attention is paid to the notion of the core and the stable set.

1.2 Cooperative Games

The Game Model

The game model discussed here consists of two components. Given a finite and non-empty player set N, and a real valued function v on the set 2^N of coalitions of N $[v : 2^N \rightarrow \mathbf{R}]$, the ordered pair (N, v) is called a cooperative Transferable Utility(TU) game, which assigns 0 to the empty coalition ϕ. The function v is called the **characteristic function** of the game. For a coalition $S \subseteq N$ the worth $v(S)$ is interpreted as the savings that can be obtained by players in S in case they decide to cooperate.

To make a few fundamentals explicit; the members of N are called players, it is often convenient to number the players, i.e., to assume that N is equal to $\{1, 2, 3, \ldots, n\}$, where n also denotes the number of players. N itself is also called the grand coalition. Any subset S of $N[S \subseteq N]$ is called a coalition; the players in such a coalition are free to cooperate. $v(N)$ is an important number for the game and is referred to as the worth of the grand coalition. Normally a game is identified with its characteristic function. In case this leads to confusion we will explicitly mention the player set.

The $0-1$ normalization

For a game (N, v) the 0-*normalization* v_0 of v is defined by $v_0(S) = v(S) - \sum_{j \in S} v(\{j\})$ for each $S \subseteq N$. For a game v with $v_0(N) > 0$, i.e., $v(N) > \sum_{j \in N} v(\{j\})$, we call the game v_{01} with $v_{01}(S) = \frac{v_0(S)}{v_0(N)}$, for each $S \subseteq N$ the $0-1$ *normalization* of v. If v is a $0-1$ normalized game then $v(\{i\}) = 0$ for each $i \in N$ and $v(N) = 1$.

Example 1.2.1 *Let $N = \{1, 2, 3\}$ and $v(S) = 1$ for all coalitions with two or three members, $v(\{i\}) = 0$ for $i = 1, 2, 3$.*

The above game is called a simple game as all coalitions take value either 0 or 1.

Subgames

The subgame of a game (N, v) relative to a non-empty coalition S is the game $(S, v_{|S})$ where $v_{|S}$ is defined to be the restriction of v to the subsets of S. So the subgame $v_{|S}$ is defined as follows : $v_{|S}(T) = v(T)$ for each coalition $T \subseteq S$.

Monotonicity in games

A game v is called **monotonic** if $v(S) \leq v(T)$ for all non-empty coalitions $S \subseteq T$. It expresses the property that coalitions possess savings exceeding those of the subcoalitions.

If v satisfies the condition $v(S) + v(T) \leq v(S \cup T)$ for all disjoint coalitions $S, T \subseteq N$ then the game is said to be **superadditive**. Under this situation it is advantagious for already existing disjoint coalitions to join and form larger coalition.

Further a game is called **convex** if the following holds:

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \text{for all} S \text{ and } T \subseteq N \tag{1.1}$$

Balanced games

Finaly, we will mention here the so-called balancedness condition. Let $\mathcal{B}$ be a collection of non-empty coalitions. Such a collection $\mathcal{B}$ with positive weights $(\lambda_S)_{S \in \mathcal{B}}$ is called *balanced* if

$$\sum_{S \in \mathcal{B}, S \ni i} \lambda_S = 1 \quad \text{for each player} \quad i \in N, \tag{1.2}$$

and $\mathcal{B} \subseteq 2^N \setminus \{\phi\}$ is called a *balanced family* of collections if a collection $(\lambda_S)_{S \in \mathcal{B}}$ of positive weight exist for which (*1.2*) holds.

As an example consider the set N to be {1,2,3} and $\mathcal{B}$ to be the set of two person coalitions of N. If the corresponding weights are all equal to $\frac{1}{2}$ then these sets in $\mathcal{B}$ form a balanced collection.

A game v is called *balanced* if it fulfills for every collection $\mathcal{B}$

$$\sum_{S\in\mathcal{B}} \lambda_S v(S) \leq v(N) \quad \text{whenever} \quad \sum_{S\in\mathcal{B}} \lambda_S e_S = e_N \quad \text{with} \quad \lambda_S > 0 \quad (1.3)$$

It is immediately clear that if the worth of the grand coalition is chosen as $\text{Max}\{\sum_{S\in\mathcal{B}} \lambda_S v(S)\}$ over all balanced collections then the game will be balanced. The game v is said to be ***totally balanced*** if each subgame is balanced.

Definition 1.2.1 *Let (N, v) be a TU-game. Let $(S, v_{|S})$ be a subgame. For every subset $S \subsetneq N$ we define $\bar{v}(S) = Max \sum_j x_j v(S_j)$ where the maximum is taken over all balanced collection of subsets $\{S_1, \ldots, S_j \ldots, S_m\}$ of S and x_j's are corresponding balancing coefficients. The game $(N, \bar{v})$ is called the totally balanced cover of (N, v).*

1.3 Solution Concepts

The Core and Solutions

One of the goals of cooperative game theory is to obtain reasonable rules which reflect the strength of a player in a game. In this context the following notions appear.

Allocations

An element $x = (x_i)_{i \in N} \in \mathbf{R}^N$ is called an *allocation*. It can be interpreted to represent a distribution of strength or savings among players. In game v an allocation x is called efficient if,

$$\sum_{j \in N} x_j = v(N).$$

A (set valued) function on a subset of the game space is called a *solution scheme* if it assigns to a game one (a set of) efficient allocation(s) of that game. An allocation x is said to be *individually rational* if for each player i his distribution x_i equals or exceeds the worth he can get by operating on his own, i.e.,

$$x_i \geq v(\{i\}) \quad \text{for each player} \quad i \in N.$$

Let $I(v)$ denote the set of allocations which are both efficient and individually rational in the game v, i.e.,

$$I(v) = \{x \in \mathbf{R}^N : \sum_{j \in N} x_j = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

This set is called the imputation set of the game v.

The imputation set, considered as a set valued function, may serve as an example of a solution scheme. The best known and most widely applied solution scheme, however is the core. In this monograph we deal with certain special properties of the core vis-a-vis coincidence of other solution schemes with the core.

Definition 1.3.1 : *$A(v)$ is called the set of all acceptable vectors of the game v,*

$$A(v) = \{y \in \mathbf{R}^n : y(S) \geq v(S) \quad \textit{for all} \quad S \subseteq N\}, \textit{where}$$

$$y(S) = \sum_{i \in S} y_i \quad \textit{for each} \quad S \subseteq N.$$

We may refer to the sets $I(v)$, $A(v)$ etc. as I, A etc. too, when the referred game is clear from the context. In the literature this is also called the set of aspirations.

The core

If for an efficient allocation x we have $\sum_{j \in S} x_j \geq v(S)$ for each coalition S then x is called a *core allocation* of the game v. The set of all core allocation is denoted by $C(v)$.

$$C(v) = \{x \in \mathbf{R}^N : \sum_{j \in N} x_j = v(N) \text{ and } \sum_{j \in S} x_j \geq v(S) \text{ for each } S \subset N\}. \tag{1.4}$$

The core $C(v)$ of (N, v) is thus the intersection of two sets, that is, $C(v) = \{x \in \mathbf{R}^n : x(N) = v(N), x(S) \geq v(S) \text{ for all } S \subset N\} = I \cap A$. In games with a non-empty core there is an incentive among the players to cooperate and form the grand coalition N. Unfortunately the core of a game need not be non-empty. Non-emptiness of the core is equivalent to the balancedness of the game (see *Theorem 1.3.1*). A game v has a non-empty core if and only if v satisfies (*1.3*). We will see more later.

Let us now look at an aspect which will define preferences of players between two different allocations. In a game v, let x and y be two imputations. Suppose the players in the game are confronted by a choice between x and y. It is clear that unless $x = y$, there will be some players who prefer x to y (those i such that $x_i > y_i$). Because both the vectors are imputations, there will be some players who prefer y to x. Hence, it is not enough to merely say that *some* players prefer x to y. On the other hand, it is not possible that *all* players will prefer x to y [since the sum of the components of x as well as y is $v(N)$]. What is necessary is that the players who prefer x to y be actually strong enough to enforce the choice of x.

Domination

Let x and y be two imputations, and let S be a coalition. We say x ***dominates y via*** S (notation : $x \succ_S y$) if

1. $x_i > y_i$ for all $i \in S$.

2. $\sum_{i \in S} x_i \leq v(S)$.

We say x ***dominates*** y (notation : $x \succ y$) if there is a coalition S such that $x \succ_S y$. Here condition 1 states that the members of S all prefer x to y; condition 2 states that they are capable of obtaining what x offers to them. It is easy to see that the relation $\succ_S$ (for any given S) is a partial order relation. On the other hand, the relation $\succ$, while it is irreflexive, is neither transitive nor antisymmetric (since the coalition

S may be different in different cases). This is a serious difficulty; and we will see the resulting complications in this thesis.

Solutions

A set K, $(K \subseteq I(v))$ is called a solution (stable set) for the game v if

1. If $x, y \in K$, then $\neg(x \succ y)$.
2. If $x \notin K$, then there is a $y \in K$ such that $y \succ x$.

Thus, a stable set satisfies the two conditions of *internal stability*; i.e., no imputation in K dominates another imputation in K, and *external stability*; i.e., any imputation outside K is dominated by some imputation in K. Stable sets were first defined by von Neumann and Morgenstern[45] and they are often called solutions of the game.

Properties of the Core

Theorem 1.3.1 (Bondareva(1963) and Shapley(1967)) *The core of the game v is non-empty if and only if the game v is balanced [satisfying 1.3].*

Let us look at the definition of core from a Linear Programming point of view. A vector x is in the core if $x(S) \geq v(S)$ for all $S \subset N$ and of course the vector must be an imputation i.e., $x(N) = v(N)$. Consider the following Linear Programming Problem[**LPP**] :

Minimize $(x_1 + x_2 + \cdots + x_n)$
subject to
$\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N$.

The core is non-empty if and only if the minimum is at most $v(N)$. The dual of the above programme can be written as follows :

Maximize $\sum_{S \in 2^N} \lambda_S v(S)$
subject to
$\sum_{S \ni i} \lambda_S = 1$ for all $i \in N$ and $\lambda_S \geq 0$ for all $S \subset 2^N$.

By the duality theorem the core is non-empty if and only if $\sum_{S \subset N} \lambda_S v(S) \leq v(N)$ for all feasible points of the dual programme. Now if we take the solution to the dual and consider the coalitions with positive λ_S's, they form a balanced collection; recall (*1.2*). A further reference to (*1.3*) tells us that if the optimum objective value of the dual is less than or equal to $v(N)$ then the core of the game is non-empty.

Domination aspect in the Core : Domination as defined between two vectors ($x \succ_S y$) requires that dominating vector x must have $x(S) > y(S)$ and that $x(S) \leq v(S)$ there by forcing $y(S) < v(S)$. This in turn means that no core vector can be dominated by any other vector; not even another core vector. This gives rise to the following facts :

1. All core vectors are undominated imputations and
2. The core is an internally stable set.

The game in the example *1.2.1* has an empty core, as the balanced collection $\{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ with balancing weights $\frac{1}{2}$ each does not satisfy equation (*1.3*)

$$\sum_{S \in B} \lambda_S v(S) = 1.5 > v(N) = 1.$$

This however does not constraint the game v to have a solution or a stable set. In fact the game v has a continuum stable sets indexed by c apart from a three point stable set.

Stable sets of three person simple majority game :

A finite stable set : $K = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$

Continuum of stable sets : For all c such that $0 \leq c < \frac{1}{2}$ the following set K_c can be checked to be a stable set of v. $K_c = \{(c, t, 1-c-t) : 0 \leq c < 0.5, 0 \leq t \leq 1-c\}$.

In the following we define a few basic concepts which are used throughout this thesis. We may however recall few of these definitions in the chapter when we actually use them.

For a game v, $A(v)$ is non-empty, and is bounded from below.

Definition 1.3.2 : *The lower boundary $L(v)$ of $A(v)$ is defined by*

$$L(v) = \{y \in A(v) : \text{ if } y' \in A(v) \text{ and } y' \leq y \text{ then } y' = y\}.$$

Alternatively call x a lower boundary point of A, if $Q_x \cap A(v) = \{x\}$, where $Q_x = \{ y \in \mathbf{R}^n : y_i \leq x_i, \text{ for all } i = 1, 2, \ldots, n\}$. Then $L(v)$ is precisely the set of all lower boundary points of $A(v)$.

It is a well known fact that $L(v)$ is non-empty since $A(v)$ is a non-empty convex set bounded from below. See [10] page 69, for a proof.
From the definition of $C(v)$ and $L(v)$ it is easy to see that $C(v) \subseteq L(v)$.

Definition 1.3.3 (Sharkey) : *The core of a game* (N, v) *is large if for every* $y \in A(v)$, *there exists* $x \in C$ *such that* $x_i \leq y_i$ *for all* i.

Sharkey proves the following theorem in [43].

Theorem 1.3.2 (Sharkey) : *The core of a game* v *is large if and only if* $C(v) = L(v)$.

As we would see a large core is characterised by the $L(v)$ members, particularly so, because of the above theorem. The following definition will help us to characterise an $L(v)$ element in terms of a collection of '*tight*' coalitions. Following this definition *Lemma 2.2.2* connects it to a lower boundary point and describes an important property which is of strategic importance in quite a few proofs in this thesis.

Definition 1.3.4 : *For* $y \in A(v)$ *define* $\mathcal{S}_y = \{S \subseteq N : y(S) = v(S)\}$.

With the background developed so far, we will study different subclasses of games and attempt to characterise their solutions in the following chapters.

Chapter 2

Symmetric Games

2.1 Introduction

Largeness of the core is sufficient for stability of the core. In general the necessity is not true. In this chapter we answer affirmatively the necessity for symmetric games. We also prove its equivalence to n *specified vectors* being imputations and also to the convexity of the lower boundary of the set of all acceptable pay-off vectors of the game. In this chapter we establish the equivalence of a condition given by Shapley to the newly evolved conditions, thereby we give an alternative proof to Shapley's result. Thus the main results of this chapter are :

1. Core is large $\Leftrightarrow$ $L(A) = C$ $\Leftrightarrow$ $L(A)$ is convex, and

2. Core is large $\Leftrightarrow$ the specified vectors are in the core $\Leftrightarrow$ the core is stable.

In 1944 von Neumann and Morgenstern[45] introduced a theory of solutions for n-person games in characteristic function form in which cooperation and coalition formation is a crucial aspect. The primary mathematical concern regarding this model is the existence of solutions. In 1968 Lucas[19] described a ten person game which has no solution. However researchers have gone on to identify properties of such solutions when they exist and their relationship with other known solution concepts, in particular, the core. Muto[28, 29, 30, 31] and Heijmans[13], studied extensively these aspects of von Neumann and Morgenstern solution concepts for symmetric games and also a special class of symmetric games known as (n, k) games. Sharkey[43] defined and studied the concept of largeness of the core which arose while he was studying an economic problem involving cost allocation. He showed that largeness of the core is a sufficient condition for the stability of the core. The purpose of this chapter is to identify a subclass where largeness of the core turns out to be also necessary and leads to other interesting and easy to check conditions for stability of the core in symmetric games. The convexity of the set of all lower boundary points of the set of all acceptable vectors is shown to play an important role in the largeness and stability of the core in this subclass of games. We need the following definitions in the sequel.

Definition 2.1.1 : *A game v is called symmetric if the characteristic function depends only on the cardinality of the coalitions. Let $f : [0, 1, 2, \ldots, n] \to \mathbf{R}_+$ be a map with the property $f(0) = f(1) = 0$. Such a map defines a symmetric game by $v(S) := f(s)$ whenever $|S| = s$.*

Definition 2.1.2 : *The totally balanced cover of a symmetric game (N, v) is denoted by $(N, \bar{v})$, and is given by $\bar{f}(k) = Max_{0<s\leq k} f(s).(\frac{k}{s})$, for all $k = 1, 2, \ldots, n$.*

Where f and $\bar{f}$ are the corresponding symmetric game defining functions.

2.2 Conditions for Large Core

Since we deal with symmetric games throughout this chapter, we introduce some notations which will be useful for our discussions. First we note that set $W \subseteq \mathbf{R}^n$ is called *symmetric* if $x \in W$ implies that all n-dimensional vectors obtained from x by permuting its coordinates are also contained in W. Let $\mathbf{R}^n_< = \{x \in \mathbf{R}^n : x_1 \leq x_2 \leq \ldots \leq x_n\}$, and for any $x \in \mathbf{R}^n_<$, let $\pi(x)$ be the set of all n dimensional vectors obtained from x by permuting its coordinates. For any $W_< \subseteq \mathbf{R}^n_<$, let $\pi(W_<) = \cup_{x\in W_<}\pi(x)$. For simplicity denote $\sum_{i=s}^{t} x_i$ by $x(s,t)$ for any $x \in \mathbf{R}^n_<$, and use $x(t)$ to denote $x(1,t)$. Let $I_< = \{x \in \mathbf{R}^n_< : x(N) = 1, x_1 \geq 0\}$. Then we have $I = \pi(I_<)$. $I_<$ is called an *ordered imputation set*. For any x, $y \in I_<$ and non-empty $S = \{i(1), \cdots, i(s)\} \subseteq N$ with $i(1) \leq \cdots \leq i(s)$ we say x *dominates* y via S, denoted by $x \succ_< y$ via S if $x_{i(j)} > y_j$ for all $j = 1, \ldots s$ and $\sum_{j=1}^{s} x_{i(j)} \leq f(s)$. The core C is given by $C = \pi(C_<)$ where $C_< = \{x \in I_< : x(s) \geq f(s)$ for all $s = 1, \ldots, n-1\}$. In what follows, our discussions will be proceeded exclusively on ordered imputation set $I_<$, and thus, to simplify notations, we will eliminate $_<$ and use I, A, C, $L(v)$ for $I_<$, $A_<$, $C_<$, $L(v_<)$.

The following is a well known result.

Lemma 2.2.1 : *An n-person $0-1$ normalized symmetric game (N, v) with characteristic function v has a non-empty core iff $\frac{f(s)}{s} \leq \frac{f(n)}{n}$ for all $1 \leq s \leq n-1$. The game is therefore totally balanced iff the function $\frac{f(s)}{s}$ is a weakly monotonic function.*

Lemma 2.2.2 : *Let (N, v) be a TU-game with non-empty core. Then $y \in L(v)$ if and only if $\cup_{S \in \mathcal{S}_y} S = N$. In particular if v is symmetric and $y \in L(v) \setminus C$ then $y_{n-1} = y_n$.*

Proof : If $y \in L(v)$, we have $y' \leq y$ and $y' \in A(v)$ implies $y' = y$. But if there is a player k such that $y(S) > v(S)$ for all coalitions S with $k \in S$, then $y' = y - \varepsilon e_k \in A(v)$ for small positive numbers ε. Recalling the definition of $\mathcal{S}_y = \{S : y(S) = v(S)\}$, conversely, if $\mathcal{S}_y$ covers N, we have: if $y' \leq y$ and $y' \in A$, then $v(S) \leq y'(S) \leq y(S) \leq v(S)$ for all $S \in \mathcal{S}_y$ and therefore $y'_{|S} = y_{|S}$ for all $S \in \mathcal{S}_y$. i.e. $y' = y$ because $\mathcal{S}_y$ covers N.

Now suppose v is symmetric and $y_1 \leq y_2 \leq \cdots \leq y_n$, and $y \in L(v) \setminus C$, so $y(N) > f(n)$ also assume $y_n > y_{n-1}$. Let T be such that $(n-1) \in T$ and $y(T) = v(T)$. Because y is an ordered vector, $|T| \leq n-1$.

But now there is no S with $n \in S$ such that $y(S) = v(S)$. As $S \neq N$ and if $|S| \leq n-1$ and $i(< n) \notin S$ then $y_i < y_n$ and replacement of n with i will yield a contradiction as y_n is largest. ∎

Remark 2.2.1 *In general it is simple to prove that for any* $y \in L(v) \setminus C$*, the coordinates of* y *will be as follows:* $y_1 \leq y_2 \leq \cdots \leq y_l = y_{l+1} = \cdots = y_n$.

Definition 2.2.1 : *For an n-person* $(0-1)$ *normalized symmetric game* (N, v)*, the following vector* y^k *for any given* k*,* $1 \leq k \leq n$ *will be called* **'The Specified Vectors for a Symmetric Game'**.

$$y_i^k = \begin{cases} \frac{\bar{f}(k)}{k} & if\, i \leq k \\ Max\{\bar{f}(i) - y^k(i-1),\ y_{i-1}^k\} & if\, i > k. \end{cases} \tag{2.1}$$

We will see the importance of these vectors in *Lemma 2.2.3*, *Lemma 2.2.4* and *Theorem 2.2.1* etc.

Lemma 2.2.3 : *For an n-person* $(0-1)$ *normalized symmetric game* (N, v)*, the specified vectors are lower boundary points of the set of all acceptable vectors* $A(v)$.

Proof : From the definition of y^k it is clear that $y^k \in A$. So we need to prove that $\cup_{S \in S_{y^k}} S = N$. If $y^k(N) = f(n)$, then we are done. Assume $y^k(N) > f(n)$, and let $l = max\ \{i : y^k(i) = \bar{f}(i)\}$.

Claim 1 : $l > k$

If $l = k$, $y_i^k = \frac{\bar{f}(k)}{k}$ for $i \leq k$ so $y_{k+1}^k > y_k^k$, then by definition $y_{k+1}^k = \bar{f}(k+1) - y^k(k)$. Hence $y^k(k+1) = \bar{f}(k+1)$. This contradicts the maximality of l. Thus the claim holds.

Claim 2 : $y_l^k = y_{l+1}^k = \cdots = y_n^k$

This also follows from the maximality of l. From claim 2, we know $y_1^k + \cdots + y_l^k = \bar{f}(l) = y_1^k + \cdots + y_{l-1}^k + y_{l+1}^k$ etc. This shows that

$\cup_{S \in \mathcal{S}_{y^k}} S = N$ with respect to v. Here we use the fact that $\bar{v}$ is symmetric since v is symmetric. That is, $y^k \in L(v)$ where $A(v) = \{y : y(S) \geq v(S) \text{ for all } S \subseteq N\}$ and $L(v)$ is the lower boundary of $A(v)$. Hence the lemma holds. ∎

Lemma 2.2.4 : *For every $y \in L(v)$ there is an index k such that $y^k(N) \geq y(N)$ where y^k's are the specified vectors.*

Proof : Let $y \in L(v)$ and suppose $y(N) > f(n)$. Let $p \leq n-1$ be the index with $y_1 \leq \ldots \leq y_{p-1} < y_p = \cdots = y_n$. Then $p \geq 2$. Take $z := y^{p-1}$. We have: $y_p \leq \bar{f}(p) - \bar{f}(p-1) \leq z_p$.

The first inequality because of $y(\{1, \ldots, p-1\}) \geq \bar{f}(p-1)$ and $y(\{1, \ldots, p\}) = \bar{f}(p)$;

the second inequality because of $z(\{1, \ldots, p-1\}) = \bar{f}(p-1)$ and $z(\{1, \ldots, p\}) \geq \bar{f}(p)$.

Then

$$\begin{aligned}
y(N) &= (y_1 + \cdots + y_p) + (y_{p+1} + \cdots + y_n) \\
&= \bar{f}(p) + (y_{p+1} + \cdots + y_n) \\
&= \bar{f}(p-1) + (\bar{f}(p) - \bar{f}(p-1)) + (y_{p+1} + \cdots + y_n) \\
&= ((z_1 + \cdots + z_{p-1}) + (\bar{f}(p) - \bar{f}(p-1)) + (y_{p+1} + \cdots + y_n) \\
&\leq (z_1 + \cdots + z_p) + (y_{p+1} + \cdots + y_n) \\
&\leq (z_1 + \cdots + z_p) + (n-p)y_p \\
&\leq (z_1 + \cdots + z_p) + (n-p)z_p \leq z(N)
\end{aligned}$$

so. $y(N) \leq y^{p-1}(N)$. This completes the proof. ∎

Theorem 2.2.1 : *For the symmetric game (N, v), the core of the game (N, v) is large if and only if the specified vectors are imputations.*

Proof : As has been observed already in *Lemma 2.2.3*, the vectors $y^k \in L(v)$ for all k, $1 \leq k \leq n$. So the *only if part* is obvious by Sharkey [43] as $C(v) = L(v)$ when the core is large.

if part : By *Lemma 2.2.4* it is clear that for every $y \in L(v)$, there is an index k, such that $y(N) \leq y^k(N)$. As all the y^k's are imputations $L(v) = C(v)$ and hence the core is large. See *Remark 2.2.2* given below. ∎

Remark 2.2.2 : *From the definition of $A(v)$ and $C(v)$ it is easy to see that $C(v) = L(v)$ if and only if C is large. Refer to Sharkey[43] for a proof.*

Theorem 2.2.2 : *For an n-person $(0-1)$ normalized symmetric game the core is the unique stable set if and only if the core is large.*

Proof : Suppose $y \in L(v)$ and $y_1 \leq y_2 \leq \ldots \leq y_n$. We have to prove that $y(N) = f(n)$. If $y(N) > f(n)$, we can decrease the vector y in the following way: because of $y \in L(v)$ we have $y_p < y_{p+1} = \cdots = y_n$ with $1 \leq p \leq n-2$ We decrease $y_{p+1}, \ldots, y_n$ with the same number ε till the sum $y(N) - (n-p)\,\varepsilon = f(n)$ or $y_{p+1} - \varepsilon = y_p$. In the first case we have an imputation $x \notin C$ and in the second case we have a vector $\bar{y} = (\bar{y}_1(= y_1) \leq \bar{y}_2(= y_2) \leq \cdots \leq \bar{y}_q(= y_q) < \bar{y}_{q+1}(= y_{q+1}) = \cdots = \bar{y}_n(= y_{q+1}))$ with $q < p$. We repeat the same process with $\bar{y}$ and q

instead of p. After finitely many steps we have a vector x and an index k: $x(N) = f(n)$, $x_j = y_j$ for $j \leq k$, $x_{k+1} = \cdots = x_n$ and $x_j < y_j$ for $j \geq k+1$.

The core is stable and therefore there is a coalition T and a core allocation $z \in C$ with $z \succ_T x$.

Let $t := |T|$ and let z^* be the vector obtained from z by ordering the coordinates in a weakly increasing order.

Write $T = \{i(1), \ldots i(t)\}$ with $i(1) < i(2) < \cdots < i(t)$. Note that $i(k) \geq k$ for $k = 1, \ldots, t$. $T^* = \{1, 2, \ldots, t\}$.

Claim : $z^* \succ_{T^*} x$.

Now $t \geq k+1$, since $z_p^* > x_p = y_p$ for $p \leq t \leq k$ and therefore $z_1^* + \cdots + z_t^* > y_1 + \cdots y_t \geq f(t)$ if $t \leq k$.

Then $z^*(N) \geq z_1^* + \cdots + z_k^* + (n-k)z_{k+1}^* > (y_1 + \cdots + y_k) + (n-k)x_{k+1}^* = x(N) = f(n)$. This contradicts that $z \in C$. ∎

Shapley[40], and Menshikova[26] have given an equivalent condition in terms of the characteristic function of the game for stability of the core for symmetric games. We state Shapley's theorem below. Menshikova's conditions are the same, and the authors version of an equivalent condition which is simpler and easy to check is available in *Theorem 2.2.1.*

Theorem 2.2.3 : **[Shapley(1973)]** *Suppose $C(v) \neq \phi$ in an n-person symmetric game. Then $C(v)$ is a stable set if and only if* $\frac{f(n)-\overline{f}(k)}{n-k} \geq \frac{f(t)-\overline{f}(k)}{t-k}$ *for all t, k with $0 \leq k < t < n$, where f and $\overline{f}$ denotes the symmetric game and its* totally balanced cover *of v.*

This theorem is known to many researchers through private communications to Shapley, and we present a slightly different proof of this theorem in Chapter 3.

In the following remark we characterize the largeness of the core of a symmetric game by the convexity of the set of lower boundary vectors.

Remark 2.2.3 : *For an n-person $(0-1)$ normalized symmetric game with non-empty core, $L(v)$ is convex if and only if the core is large.*

The above is easy to prove by taking any $y \in L(v) \setminus C$ so that $y(N) > 1$. Consider y^c the convex combination of all the permutations of y with equal weights. Observe that for all i, $y_i^c = \frac{(n-1)!}{n!} \cdot \sum_{i=1}^{n} y_i = \frac{y(N)}{n} > \frac{1}{n}$. This contradicts the fact that $y^c \in L(v)$ as $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}) \in C$.

Remark 2.2.4 : *The core of a symmetric game when exists is the maximal symmetric convex set within $L(v)$ i.e., if $C(v)$ is the core and if D is another symmetric convex set in $L(v)$ then $D \subseteq C(v)$.*

Example 2.2.1 *Let $C = \{(x_1, x_2, x_3) : x_1 \leq x_2 \leq x_3, \ x_i \geq 0, \ \frac{1}{4} \leq x_1 + x_2 \leq \frac{2}{3}, \sum x_i = 1\}$. Then $\pi(C)$ is a symmetric convex set.*

In fact $\pi(C)$ is the core of a 3-player symmetric game with $f(1) = 0, \ f(2) = 0.25, \ f(3) = 1$.

Remark 2.2.5 : *The core of a symmetric game, when non-empty, either has a non-empty interior or has only one element. Further a single point core can not be large in a symmetric game.*

Theorem 2.2.4 : *In the subclass of balanced games the following statements are equivalent :*

(i) v has a large core.

(ii) $\bar{v}$ has a large core.

Note : This is true for general cooperative games and to prove, observe that the definition of large core depends on two concepts; namely, $C(v)$ and $L(v)$. It is easy to prove that these two are equal for v and $\bar{v}$ using balanced collections and Shapley-Bondareva type conditions.

Example 2.2.2 *Consider the following 6 person symmetric game, where it can be checked that y^1, y^2, y^5, y^6 are imputations and y^3 and y^4 are not. Consequently the game does not have a large core. This example depicts that defining n vectors as has been done is a necessity. Given $k \leq n$, examples can be constructed so that y^k fails to be an imputation, as long as the game does not have a large core.*

s	1	2	3	4	5	6
$f(s)$	0	0.12	0.22	0.28	0.65	1.0
$f(s)$	0	0.12	0.22	0.2933	0.65	1.0

$$y^1 = (0,\ .12,\ .12,\ .12,\ .29,\ .35)$$
$$y^2 = (.06,\ .06,\ .10,\ .10,\ .33,\ .35)$$
$$y^3 = (.0733,\ .0733,\ .0733,\ .0733,\ .3567,\ .3567)$$
$$y^4 = (.0733,\ .0733,\ .0733,\ .0733,\ .3567,\ .3567)$$
$$y^5 = (.13,\ .13,\ .13,\ .13,\ .13,\ .35)$$
$$y^6 = (.1667,\ .1667,\ .1667,\ .1667,\ .1667,\ .1667)$$

Theorem 2.2.5 : *The specified vectors of a symmetric game are extreme points of the ordered core of the game, when the core is large.*

Proof : Take y^k and suppose x^1 and x^2 are two ordered core vectors such that $y^k = \lambda x^1 + (1-\lambda)x^2$, $x^1 \neq y^k \neq x^2$ and $0 < \lambda < 1$.
If $x_1^1 < \frac{\bar{f}(k)}{k}$ then $x_1^2 > \frac{\bar{f}(k)}{k}$. Consequently $x^2(k) > \bar{f}(k)$, and $x^1(k) \geq \bar{f}(k)$ as $x^1 \in C(v)$. Hence a contradiction to the definition of y^k. Similarly we can show that $x_1^1 \not> \frac{\bar{f}(k)}{k}$. So $x_i^1 = x_i^2 = \frac{\bar{f}(k)}{k}$ for all $i \leq k$.
$y_{k+1}^k = Max\{\bar{f}(k+1) - \bar{f}(k), \frac{\bar{f}(k)}{k}\}$, Since x is ordered, $x_{k+1}^j \geq \frac{\bar{f}(k)}{k}$ by the conclusion above. Further $x_{k+1}^j \geq \bar{f}(k+1) - \bar{f}(k)$ as $x^j \in C$. Hence $x_{k+1}^j \geq y_{k+1}^k$ for $j = 1, 2$. Therefore $x_{k+1}^j = y_{k+1}^k$ for $j = 1, 2$.
The equality of other x_i^j s to the corresponding y_i^k follows sequentially in a similar manner. Hence the theorem holds. ∎

Remark 2.2.6 *All extreme points of the (unordered) core are not necessarily 'specified vectors'. Further, all specified vectors are not extreme points of the (unordered) core. In particular one may look at the last specified vector, which is the centre point of the (unordered) core.*

Example 2.2.3 *Consider a four player symmetric game v, with* $f(1) = 0$, $f(2) = 20$, $f(3) = 57$ *and* $f(4) = 100$. *Observe that the core is large. Now consider the vector* $(0, 25, 32, 43)$. *This core vector can not be expressed as a convex combination of the specified vectors, meaning there are extreme points other than the specified vectors.*

Remark 2.2.7 : *Consider a symmetric game* (N, v) *and the corresponding* $(N, \bar{v})$. *The set of specified vectors for* (N, v) *coincides with* *the set of specified vectors for* $(N, \bar{v})$.

This is trivial as the vectors are defined in terms of the totally balanced cover of the game $\bar{v}$ and not v which need not be totally balanced.

Corollary 2.2.1 : *For a symmetric game* (N, v) *if* y^k *'s are as defined in Definition 2.2.1, then* $\frac{f(n)-\bar{f}(s)}{n-s} \geq \frac{f(t)-f(s)}{t-s}$ *for all* t, s *with* $0 \leq s < t < n$, *if and only if* y^k *is an imputation for all* k : $1 \leq k \leq n$, *where* $\bar{f}$ *denotes the* totally balanced cover *corresponding to* v.

Though *Corollary 2.2.1* follows from *Theorems 2.2.2, 2.2.3* and *2.2.1*, a direct proof is interesting. In the following we give the simple direct proof.

Proof of Corollary 2.2.1 : Let us first observe that because of *Remark 2.2.7*, Shapley's condition is true with $\bar{f}(t)$ replacing $f(t)$ in the right hand side of the inequality. In the following we make use of this fact and hence $\bar{f}(t)$ appears in place of $f(t)$.

if part : Let $0 \leq k < t < n$ and consider y^k. Observe that $y^k(t) \geq \bar{f}(t)$ for $k < t < n$.

$\frac{f(n)-\bar{f}(k)}{n-k}$ is the average of $y^k_{k+1}, \cdots, y^k_n$ ... ... (1)

$\frac{y^k(t)-\bar{f}(k)}{t-k}$ is the average of $y^k_{k+1}, \cdots, y^k_t$ ... ... (2)

Because of the ordered nature of the vector y^k it is easy to see that (1) is greater than or equal to (2).

$\frac{f(n)-\bar{f}(k)}{n-k} \geq \frac{y^k(t)-\bar{f}(k)}{t-k} \geq \frac{\bar{f}(t)-\bar{f}(k)}{t-k} \geq \frac{f(t)-\bar{f}(k)}{t-k}$.

This is the end of *if part*.

only if part : $\frac{f(n)-\bar{f}(k)}{n-k} \geq \frac{\bar{f}(t)-\bar{f}(k)}{t-k} \quad 0 \leq k < t < n \quad \cdots \quad \cdots \quad (3)$

Observe that Shapley's condition above is also equivalent to the following two conditions. [Refer Kikuta and Shapley][16]

$$\frac{f(n)-f(t)}{n-t} \geq \frac{f(t)-\bar{f}(k)}{t-k} \quad 0 \leq k < t < n \quad \cdots \quad \cdots \quad (4)$$

$$\frac{f(n)-f(t)}{n-t} \geq \frac{f(n)-\bar{f}(k)}{n-k} \quad 0 \leq k < t < n \quad \cdots \quad \cdots \quad (5)$$

Suppose if possible that a vector y^k defined in *Definition 2.2.1* is such that $y^k(N) > f(n)$. This implies that there exists l such that $y^k_l = y^k_{l+1} = \cdots = y^k_n$ and $\bar{f}(l) = y^k(l)$, i.e. $y^k_l = \bar{f}(l) - y^k(l-1)$.

Now put $t = l$ and $k = l-1$ in (4) above, we get

$\frac{\bar{f}(n)-\bar{f}(l)}{n-l} \geq \frac{\bar{f}(l)-\bar{f}(l-1)}{1} \geq \bar{f}(l) - y^k(l-1) = y^k_l$. Hence $\bar{f}(n) \geq \bar{f}(l) + (n-l)y^k_l = y^k(N)$. This is contrary to the hypothesis that $y^k(N) > f(n)$.

This completes the proof of *only if part*. ∎

Alternate proof to Shapley's Condition : *Theorems 2.2.2, 2.2.1* and *Corollary 2.2.1* put together can be regarded as an alternate proof to *Theorem 2.2.3*. ∎.

The following Theorem sums up the new results of this chapter.

Theorem 2.2.6 : *In a symmetric game (N, v) if the core is non-empty then the following are equivalent.*

1. *The core is large.*
2. *The core is stable.*
3. *The lower boundary of the set of all acceptable vectors is convex.*
4. $\frac{f(n)-\bar{f}(k)}{n-k} \geq \frac{f(t)-\bar{f}(k)}{t-k}$ *for all* t, k *with* $0 \leq k < t < n$.
5. *The specified vectors of the symmetric game (N, v) are all imputations.*

Some more results on symmetric exact games are presented in the chapter on exact games under the section on symmetric games.

Weber Vector : Consider a permutation $\pi = (\pi(1), \cdots, \pi(n))$. The marginal worth vector defined below for each permutation of the player set N is known as a Weber Vector. Refer to Weber [44].

$x^{\pi}_{\pi(j)} = v(\{\pi(1), \pi(2), \ldots, \pi(j)\}) - v(\{\pi(1), \pi(2), \ldots, \pi(j-1)\})$.

Note : The Weber vectors of a symmetric game are all permutations of one another, and if the game is convex then the ordered Weber vector is a core element, i.e. $(0, f(2), f(3)-f(2), f(4)-f(3), \cdots, f(n)-f(n-1))$ is an imputation and the core is large.

2.3 Examples

Example 2.3.1 *: The following game has a non-empty core which however is not a stable set.*

s	1	2	3	4
f(s)	0	0.20	0.65	1.00

Take $x = (0, 0.20, 0.35, 0.45)$, it is easy to see that $x \notin C$ as $x_1+x_2+x_3 = 0.55 < f(3) = 0.65$, However all other core conditions are satisfied. Check that $\nexists y \in C \ni y \succ x$ Domination if possible must occur via coalition $\{1,2,3\} \ni x'_1 = \epsilon;\ x'_2 = 0.20 + \epsilon;\ x'_3 = 0.45 - 2\epsilon > 0.35 \Rightarrow x'_4 = 0.35, x'_3 > 0.35 \Rightarrow x'_1 + x'_2 < 0.30 \Rightarrow x'_1 + x'_2 + x'_4 < 0.65$. Finally we observe that at least one of the specified vectors for this symmetric game has a sum more than $v(N)$

$y^1 = (0, 0.2, 0.45, 0.45); y^1(N) = 1.1$

A **stable** set for this game is explained below :

For $n = 4, f(2)$ and $f(3)$ completely specify the game. When the core is not stable then, $f(3)-f(2) > 1-f(3)$. Consider $f(2) \leq \frac{1}{3}$ and $f(3)-f(2) > 1 - f(3)$. $X = \{x = (x_1, x_2, x_3, x_4) : x_1 = 0.35; x_2 + x_3 + x_4 = 0.65;\ \ x_2 + x_3 \geq 0.20;\ \ x_2 + x_4 \geq 0.20;\ \ x_3 + x_4 \geq 0.20\}$.

Define $K = \pi(X)$. Then K is a symmetric stable set for the game v.

Example 2.3.2 : *Consider the following 6 person symmetric game, which is not super-additive but has a large core.*

s	1	2	3	4	5	6
f(s)	0	0.17	0.20	0.32	0.65	1.0

Consider $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$, $v(S_1) + v(S_2) > v(S_1 \cup S_2)$ but all the specified vectors for this symmetric game are imputations.

$y^1 = (0, .17, .17, .17, .17, .32)$

$y^2 = (.085, .085, .085, .085, .31, .35)$

$y^3 = (.085, .085, .085, .085, .31, .35)$

$y^4 = (.085, .085, .085, .085, .31, .35)$

$y^5 = (.13, .13, .13, .13, .13, .35)$.

Example 2.3.3 : *Consider the following 5 person symmetric game, with a single point core which is naturally not stable :*

s	1	2	3	4	5
f(s)	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1.0

It is easy to see that one of the specified vectors has $y^1(N) = \frac{8}{5}$

Muto[31] gives the following symmetric solution for a game of this type.

$X = \{x \in I \ni x_1 = x_2 = x_3 = x_4 > \frac{1}{5},\ x_5 \leq \frac{1}{5}\}$. Define $K = \pi(X)$ then. K is a symmetric stable set for the game v. However we also present a non-symmetric solution as follows. Define,

$X_1 = \{x \ni \ x = (\frac{1}{5},\ x_2,\ \frac{2}{5} - x_2,\ \frac{1}{5}, \frac{1}{5}) \ : 0 \leq x_2 \leq \frac{1}{5}\}$

$X_2 = \{x \ni \ x = (\frac{1}{5},\ \frac{1}{5},\ \frac{1}{5},\ x_4, \frac{2}{5} - x_4) \ : 0 \leq x_4 \leq \frac{1}{5}\}$

Define $K = X_1 \cup X_2$, then K is a non-symmetric stable set for the game.

Example 2.3.4 : *Consider the following symmetric game whose core is large.*

s	1	2	3	4
f(s)	0	0.20	0.60	1.00

Consider the specified vectors

$y^1 \ = \ (0,\ 0.2,\ 0.4,\ 0.4),$

$y^2 \ = \ (0.1,\ 0.1,\ 0.4,\ 0.4)$

$y^3 \ = \ (0.2,\ 0.2,\ 0.2,\ 0.4)$

observe that $y^1(N) = y^2(N) = y^3(N) = 1$.

Example 2.3.5 : *Consider a game (not necessarily a symmetric one) (N, v) and define a symmetric game v^* based on v as follows:*

$f^*(s) \ = \ Max_{|S|=s} v(S)$ for all s such that $1 \leq s \leq n$

Denote by C^* the core of the game v^* and assume $C^* \neq \phi$. Denote by C the core of the game v and it follows that $C \neq \phi$ as $C^* \subseteq C$. C^* is stable does not necessarily imply C is stable.

An Example: Take $N = \{1,2,3,4\}$ $v(1,2) = 0.20$, $v(3,4) = 0.15$, $v(1,3,4) = 0.60$, $v(2,3,4) = 0.60$. It is easy to check that the game v^* defined as explained above has it's core as the unique stable set. (v^* coincides with the previous example) The core of the game v however is not a stable set. Consider the vector $x^0 = (.45,.45,.10,0)$, $x^0 \notin C$ It is easy to check that this vector can not be dominated by any core element.

Example 2.3.6 : *Under the definitions of v^* above the following examples shows that C could be large but not C^*.*

Let $N = \{1,2,3,4\}$ and $v(1,2) = 0.2, v(1,2,3) = 0.7, v(N) = 1, v(S) = 0$ for all other $S \subseteq N$ Hence v^* is as follows :

s	1	2	3	4
$f^*(s)$	0	0.2	0.7	1.0

C^* is not large as one of the specified vectors $y^1 = (0,\ 0.2,\ 0.5,\ 0.5)$ is not an imputation. However C is large - can be proved as follows : Observe that $C = \{x = (x_1,\ x_2,\ x_3,\ x_4)\ :\ x_1 + x_2 \geq 0.2,\ x_1 + x_2 + x_3 \geq\ 0.7,\ x_1 + x_2 + x_3 + x_4 = 1\}$. We need to show that $L(v) = C$, Let $x = (x_1,\ x_2,\ x_3,\ x_4)\ \in L(v) \Rightarrow\ x_1 + x_2 \geq\ 0.2,\ \ x_1 + x_2 + x_3 \geq\ 0.7$. If possible $x_1 + x_2 + x_3 + x_4 >\ 1$.

Case 1 : $x_1 + x_2\ \geq 1$and $x_1 + x_2 + x_3 + x_4 > 1$. Write $y = (y_1, y_2, 0, 0)$ where $y_1 \leq x_1,\ y_2 \leq x_2 \ni y_1 + y_2 = 1$ contradicting $x \in L(v)$.

Case 2 : $1 > x_1 + x_2 \geq 0.2,\ 0.7 \geq x_1 + x_2 + x_3 \geq 1,\ x_1 + x_2 + x_3 + x_4 > 1$. Write $y\ =\ (y_1,\ y_2,\ 0,\ 0)$ where $y_1\ \leq\ x_1,\ y_2\ \leq\ x_2,\ y_3\ \leq\ x_3\ \ni$

$y_1 + y_2 + y_3 = 1$ contradicting $x \in L(v)$.
Case 3 : $1 > x_1+x_2 \geq 0.2,\ 1 > x_1+x_2+x_3 > 0.7,\ x_1+x_2+x_3+x_4 > 1$. Write $y = (y_1,\ y_2,\ 0,\ 0)$ where $y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3, y_4 = 1 - (x_1 + x_2 + x_3)$ contradicting $x \in L(v)$. Thus $L(v) = C$.

Also recall *Example 2.2.2.*

Concluding remarks

The primary interest in Game Theory has always been the question of existence of a solution or a stable set. Lucas[20] showed that there are games for which a solution may not exist. Search is still on for subclasses where solution may always exist. Do Symmetric games form one such subclass ? –remains the question! However Rabie[35] showed that a solution need not necessarily exhibit the symmetry of the game. In view of these the question assumes greater significance. Do symmetric games always have a solution ? –not necessarily a symmetric one! Sharkey has proved that a convex/subconvex game has a large core, but we had a feeling that these conditions are still too strict as sufficient conditions. One can ask whether exactness is sufficient. Biswas et al[5] have answered this question in the affirmative for symmetric games. For non-symmetric case the reader is referred to Chapters 3 and 4.

Chapter 3

Exact Games

3.1 Introduction

In this chapter we will answer the question if every exact game has a large core. We will prove that, for games with 5 players or more, the answer is no. For TU games with 3 or 4 players every exact game has a large core. For totally balanced *symmetric* games a large core, a stable core and exactness will be proved to be equivalent.

To start with we recall the definitions of both concepts.

Definition 3.1.1 (Sharkey) *[43]A TU-game (N, v) is said to have a* large core, *if for every vector $y \in \mathbf{R}^n$ with $y(S) \geq v(S)$ for all $S \subseteq N$, there is a core element $x \in C(v)$ with $x \leq y$.*

A vector y satisfying the conditions $y(S) \geq v(S)$ for all $S \subset N$ will be called an *upper vector* of the game (N, v). The set of all upper vectors is denoted by $U(v)$. It is a polyhedral set and **not** dependent on the value of the grand coalition $v(N)$ (as we only take the inequalities for

$S \subset N$). The recession cone of the polyhedral set $U(v)$, i.e. the set $\{u \in \mathbf{R}^n : z + tu \in U(v) \text{ for all } z \in U(v),\ t \geq 0\}$ is the cone $\mathbf{R}^n_+$, as can be seen easily and by the structure theorem for polyhedral sets

$U(v) = ch(extrU(v)) + \mathbf{R}^n_+$.

(ch means 'convex hull of' and $extr$ means 'extreme points of').

From this description one can see that the core is large if and only if $v(N) \geq z(N)$ for all extreme points z of $U(v)$.

Definition 3.1.2 (Schmeidler) *[39] A TU-game (N, v) is called exact if for every coalition $T \subset N$, there is a core allocation $x \in C(v)$ with $x(T) = v(T)$.*

Note that an exact game is balanced and that every subgame $(S, v_{|S})$ is also balanced : (N, v) is totally balanced.
Games with large core have been studied, mainly because of the following property:

Proposition 3.1.1 (Sharkey) *[43] If a TU-game has a large core, the core is the unique stable set.*

Proof : For completeness we repeat the proof. If $y \in I(v)$(the imputation set of (N, v)) and y is not a core allocation, there is a coalition S with $y(S) < v(S)$ and $y(T) \geq v(T)$ for $T \subset S$. Take $z|_S > y|_S$ and $z|_S(S) = v(S)$. It is possible to extend $z|_S$ with payoffs for players outside S such that $z = (z|_S, z|_{N \setminus S}) \in U(v)$ and $z(N) \geq v(N)$. There is a core allocation x with $x \leq z$ (largeness of core). Then

$v(S) \leq x(S) \leq z(S) = v(S)$ and $x_S = z_S$. Then x dominates y via S. So, every imputation outside the core is dominated by a core allocation. Hence, the core is a stable set and there can not be another one, as every stable set contains the core and no imputation can be added without violating internal stability. ■

If (N, v) is a balanced game then we define the *totally balanced cover* $(N, \bar{v})$ by $\bar{v}(S) := max\{\sum_{T \subseteq S} y_T v(T) : y_T \geq 0$ and $\sum_{T \subseteq S} y_T e_T = e_S\}$ where, e_T is an n-vector with $(e_T)_i = 1$ for all $i \in T$ and $(e_T)_i = 0$ for all $i \notin T$. As $U(v) = U(\bar{v})$ for every TU-game and $C(v) = C(\bar{v})$ for every balanced game, the game has a large core if and only if the totally balanced cover has a large core. So, we can restrict our attention to totally balanced games. For totally balanced games we have

Proposition 3.1.2 (Sharkey) *[43] If a totally balanced game has a large core, it is an exact game.*

Proof : Take any $T \subset N$ and a core element $z|_T \in C(T, v|_T)$. As before we can extend $z|_S$ to an element $z = (z|_S, z|_{N \setminus S})$ in $U(v)$ and $z(N) \geq v(N)$. There exists a core allocation $x \in C(v)$ with $x \leq z$. Then $v(T) \leq x(T) \leq z(T) = v(T)$. So, x is a core allocation with $x(T) = v(T)$. ■

Remark 3.1.1 *What is really needed in the proof of Propositions 3.1.1 and 3.1.2 is that every core element of a subgame $(S, v|_S)$ can be ex-*

tended to a core element of the whole game. In an unpublished manuscript Kikuta and Shapley[16] proved :

If in a balanced game (N, v) every core element of a subgame can be extended to a core element of the whole game, then the core is stable set. They conjectured that this condition is also necessary for core stability. The question we will answer is the converse of *Proposition 3.1.2*:

Does every exact game have a large core ?

The history of this problem is curious. In Sharkey [43], the author claims to have a counter example in Lucas' famous 10-person game without a stable set. This counter example however is false.

Example 3.1.1 (Lucas[19],cf.Sharkey[43]) *Consider the TU-game (N, v) with $n = 10$ and coalition values $v(S) = 0$ except for the following coalitions :*

$v(12) = v(34) = v(56) = v(78) = v(90) = 1,$
$v(137) = v(139) = v(157) = v(159) = v(357) = v(359) = 2,$
$v(1479) = v(2579) = v(3679) = 2,$
$v(1379) = v(1579) = v(3579) = 3,$
$v(13579) = 4.$

[Numeral 0 in the coalitions stand for player 10]
Sharkey states that the totally balanced cover of the game (N, v) is exact if $v(N) \geq 5$, that the game has a stable core for $v(N) \geq 7$ and has a large core if $v(N) \geq 8$. None of these statements however is true.

For exactness $v(N)$ must be at least 11 and, by *Proposition 3.1.2*, the same must be true for a large core. Finally, the core not is stable for $v(N) < 11$. We show the statement about exactness first.

Take the coalition (1679). This coalition and each of its subcoalitions has value 0. To have exactness there must be a core allocation with $x_i = 0$ for $i = 1, 6, 7, 9$. Then $x_2 \geq 1$ (by coalition (12)). $x_3 \geq 3$ (by coalition (1379)), $x_4 \geq 2$ (by coalition (1479)), $x_5 \geq 3$ (by coalition (1579)). Finally $x_8 \geq 1$ (by coalition (78)), $x_0 \geq 1$ (by coalition (90)). Then $x(N) \geq 11$. The allocation obtained by taking equalities in the above mentioned coalitions is $z = (0, 1, 3, 2, 3, 0, 0, 1, 0, 1)$. It is an extreme point of $U(v)$. It can be proved that the allocation obtained by diminishing the coordinates of z on the places 3, 4 and 5 with same amount $t \in (0, 1]$ gives an imputation in the game with $v(N) = 11 - 3t$ that can not be dominated by any core allocation of that game.

Most recently Biswas et al. [4] proved that the converse of *Proposition 3.1.2* is true for *symmetric* TU- games (coalition values depend on the size of coalition).

3.2 Main Results

In this section we prove that the converse of *Proposition 3.1.2* holds if $|N| = n \leq 4$ and we give a counter example for every number $n \geq 5$.

Theorem 3.2.1 *If (N, v) is a exact game and $n \leq 4$, then (N, v) has a large core (and the core is a stable set). If $n \geq 5$, then there are exact games (N, v) not having a large core.*

Remark 3.2.1 *Kulakovskaja[18] gives, for $n = 4$, the following necessary and sufficient conditions for a zero-normalized balanced game to have a stable core. For every ordering of (i, j, k, l) of the players in N,*

$v(ij) + v(jk) + v(il) \leq v(N)$,

$v(ijk) + v(il) \leq v(N)$,

$v(ijk) + v(ijl) - v(ij) \leq v(N)$.

In fact these conditions check the exactness of the (totally balanced cover of) the game. The first two relations check the exactness conditions for coalitions (i) and the last condition is the exactness condition for (ij). For 3-person coalitions the exactness condition follows from the total balance condition.

Proof of Theorem 3.2.1: The proof of the theorem and the distinction between $n \leq 4$ and $n \geq 5$ mainly rest on the following observation: only if $n \geq 5$, there exists a collection $\mathcal{T}$ of proper coalitions in N, i.e. $\mathcal{T} = \{T_1, \ldots, T_n\}$ with the following properties:

(a) $\{e_T : T \in \mathcal{T}\}$ is a basis of $\mathbf{R}^n$,

(b) the vector equation $\sum_{T \in \mathcal{T}} y_T e_T = e_N + a e_S$ **does not have** a non-negative solution (y_T, a) for any coalition S.

Condition (b) says that $\mathcal{T}$ does not contain a balanced collection and also that it is not possible to combine the vectors e_T, $T \in \mathcal{T}$ in such a way that every player get either weight 1 or $1+a$. For $n \geq 5$, the collection $\mathcal{T}_n := \{(1), (1,2), (1,3), (1,2,4), (1,2,5), \cdots, (1,2,n)\}$ has properties (a) and (b). For $n \leq 4$ such a collection does not exist(see *Lemma 3.2.1*).

If one has a collection $\mathcal{T}$ with properties (a) and (b) one can proceed as follows:

The vector e_N can be written as $\sum_{i=1}^{n} y_{T_i} e_{T_i}$. Let the vectors $\{w_i\}$ be defined by $w_i(T_j) = 0$ if $i \neq j$ and $w_i(T_i) > 0$. If we compute $w_i(N)$, we find that $w_i(N) < 0$ if and only if $y_{T_i} < 0$. We normalize the vector w_j by $w_j(N) = -1$ if $y_{T_j} < 0$ and denote the set of indices j with $y_{T_j} < 0$ by J. By property (b) J contains at least two elements. Next we define a game (N, v) by $v(S) := min_{j \in J} w_j(S)$ for $S \subseteq N$

The game (N, v) is the minimum of $|J|$ additive games. Note that (N, v) has the following properties:

(c) for every coalition $S \subset N$ there is an index J with $w_j(S) = v(S)$,

(d) for every index $j \in J$ and every coalition S we have $w_j(S) \geq v(S)$.

from (c) and (d) we find that (N, v) is an exact game.

(e) $v(N) = -1$ and $v(T) = 0$ for all $T \in \mathcal{T}$(here we use $|J| \geq 2$).

If the game (N, v) ***happens to be non-positive***, the vector $z = 0$ is an element of $U(v)$ (as the collection of 'tight' coalitions T i.e. coalitions with $z(T) = v(T)$, contain the basis $\mathcal{T}$ of $\mathbf{R}^n$). Then $z \in U(v)$ can not be decreased to become a core allocation and the core of (N, v) is not large.

For the collection $\mathcal{T}_n$ we find $e_N = -e_{(1)} - (n-4)e_{(12)} + e_{(13)} + \cdots + e_{(12n)}$. Then $J = \{1, 2\}$ (two coefficients are negative) and

$w_1 = (1, -1, -1, 0, \cdots, 0)$ and $w_2 = \frac{1}{n-4}(0, 1, 0, -1, \cdots, -1)$. We check that $v(S) = min\{w_1(S), w_2(S)\} \leq 0$. $w_1(S) > 0$ iff $1 \in S$ and $S \cap \{2, 3\} = \phi$; $w_2(S) > 0$ iff $2 \in S$ and $S \cap \{4, 5, \cdots, n\} = \phi$. So the game (N, v) is exact and non-positive. $v(N) = -1$ and $z = 0$ is an extreme point of $U(v)$. The game (N, v) does not have a large core. In fact, we think that the game (N, v) defined according to the previous rules is always a non-positive game. If this is true then every collection $\mathcal{T}$ with properties (a) and (b) gives a counter example for the converse of *Proposition 3.1.2*.

To prove the first part of the Theorem we assume that $n < 4$ and that (N, v) is an exact game. We assume that no collection $\mathcal{S}$ with properties (a) and (b) exists. Let z be any extreme point of $U(v)$ and $\mathcal{S}(z) := \{T \subset N \; : \; z(T) = v(T)\}$. The collection $\mathcal{S}(z)$ contains a basis of $\mathbf{R}^n$(as z is an extreme point of $U(v)$). By the absence of property (b) there is, for at least one coalition S, non-negative solution of the vector equation

$$\textstyle\sum_{T \in \mathcal{S}} y_T e_t = e_N + a e_S. \qquad (1)$$

There exists a core element x of (N, v) with $x(S) = v(S)$(exactness of (N, v)). From equation (1) we find,

$\sum_{T \subset T} y_T z(T) = z(N) + az(S)$ and therefore, $\sum_{T \in \mathcal{S}} y_T v(T) \geq z(N) + av(S)$ But also $\sum_{T \subset \mathcal{S}} y_T x(T) = x(N) + ax(S)$ implying, $\sum_{T \in \mathcal{S}} y_T v(T) \leq v(N) + av(S)$. So, $v(N) \geq z(N)$ for every extreme point z of $U(v)$.

If $y \in U(v)$ is an upper vector and $y(N) > v(N)$, there is a vector $w \in ch(extr(U(v))$ and a vector $u \geq 0$ with $y = w + u$. As $y(N) > v(N)$ and $w(N) \leq v(N)$, there is a number $t \in [0, 1)$ with $w(N) + tu(N) = v(N)$. Then $w + tu$ is a core element $\leq y$. So (N, v) has a large core. ■

We are left with the following combinatorial Lemma:

Lemma 3.2.1 *If $n \leq 4$, there is no collection $\mathcal{T}$ satisfying (a) and (b).*

Proof : Suppose That $\mathcal{T}$ is a basis of $\mathbf{R}^n$ with $n = 4$. If $\mathcal{T}$ contains a 3-coalition, say w.l.g. $\{1, 2, 3\}$, there is a coalition $T \ni 4$ in $\mathcal{T}$. Then $e_{(1,2,3)} + e_T = e_N + e_{T \cap (1,2,3)}$. If $\mathcal{T}$ contains only 1-coalitions, it is the balanced collection $\{(1), (2), (3), (4)\}$. So to satisfy (b) $\mathcal{T}$ must contain at least one 2-coalition and no 3-coalitions. Say $(1, 2) \in \mathcal{T}$. There is also a coalition T_3 containing player 3 in $\mathcal{T}$ and a coalition T_4 containing player 4 in $\mathcal{T}$. These coalitions are not the same. Otherwise $(3, 4) \in \mathcal{T}$ and $\mathcal{T}$ contains a balanced collection $\{(1, 2), (3, 4)\}$. At least one of the coalitions T_3 or T_4 is a 2-coalition; otherwise $\mathcal{T} \supset \{(1, 2), (3), (4)\}$, a balanced collection. W.l.g. we may assume $T_3 = (1, 3)$. Then $e_{(12)} + e_{(13)} + e_{T_4} = (2, 1, 1, 1), (2, 1, 2, 1), (3, 1, 1, 1)$ or $(2, 2, 1, 1)$. There exists no collection $\mathcal{T}$ with properties (a) and (b) for $n = 4$. For $n = 3$ the proof is even easier. ■

3.3 Symmetric games

Let $f : [0, 1, 2, \ldots, n] \rightarrow \mathbf{R}_+$ be a map with property $f(0) = f(1) = 0$. Such a map defines a *symmetric game* by $v(S) := f(s)$ whenever $|S| =$

s.

For symmetric games we will investigate the relation between exactness, having a stable core and having a large core.

For any totally balanced TU-game we have:

If the game has a large core, it is exact and it has a stable core. (see Propositions 3.1.1 and *3.1.2).*

Theorem 3.3.1 *For totally balanced, symmetric TU-games (N, v) the following statements are equivalent*

(a) (N, v) has a large core.

(b) (N, v) has a stable core.

(c) (N, v) is exact.

To prove the above Theorem we note that $(a) \Rightarrow (b)$ follows from Sharkey [12], $(b) \Rightarrow (c)$ follows from *Lemma 3.3.1* and *Proposition 3.3.2* and $(c) \Rightarrow (a)$ follows from *Proposition 3.3.1.*

It is well known that symmetric games defined by a function f is balanced iff $\frac{f(s)}{s} \leq \frac{f(n)}{n}$ for all $1 \leq s \leq n-1$. The game is therefore totally balanced iff the function $\frac{f(s)}{s}$ is weakly monotonic function.

Lemma 3.3.1 *A symmetric game defined by a function f is exact if and only if the function $s \rightarrow \frac{f(s)}{s}$ is weakly monotonic and for every pair of integers $s < t < n$, $\frac{f(n)-f(s)}{n-s} \geq \frac{f(t)-f(s)}{t-s}$ (Shapley conditions).*

Proof : If f satisfies the condition in the Lemma and $1 < t \leq n-1$, we define $x' := \frac{f(t)}{t}$ and $x'' := \frac{f(n)-f(t)}{n-t}$. If T is a coalition of size t, we give the players in T the payment x' and the players outside T the payment x''. Call the vector $x := x'e_T + x''e_{N\setminus T}$. Then $x(N) = f(n)$ and $x(T) = f(t)$. We have to prove that x is a core allocation of size s, we have $x(S) \geq (s \wedge t)x' + (s - (s \wedge t))x''$. Where $s \wedge t$ designates the minimum of s and t. If $s \leq t$ we have $sx' \geq f(s)$(by the totally balanced conditions) and, for $s > t$, we have $tx' + (s-t)x'' \geq f(t) + (s-t)\frac{f(s)-f(t)}{s-t} = f(s)$(by Shapley conditions).

The converse can be proved as follows : if f defines an exact game, for every $t : 1 \leq t \leq n-1$ an element x of $\mathbf{R}^n$ can be found such that

$x_1 \leq x_2 \leq \cdots \leq x_n$, $x_1 + x_2 + \cdots + x_s \geq f(s)$ for all s and equality for $s = t$ and $s = n$.

We first replace the first t coordinates by their average $x' = \frac{f(t)}{t}$ and the last $n-t$ coordinates also by their average $x'' = \frac{f(n)-f(t)}{n-t}$, we still have a core allocation. So, we find $s\frac{f(t)}{t} \geq f(s)$ for $s < t$ and $f(t) + (s-t)\frac{f(n)-f(t)}{n-t} \geq f(s)$ for $s > t$.

The last inequality gives the Shapley inequalities. ■

Proposition 3.3.1 *Every symmetric exact game has a large core.*

Proof : We prove that, if the game has no large core, then it is not exact. If the core is not large, then there exists an extreme point y of $U(v)$ with $y(N) > f(n)$ and $y_1 \leq y_2 \leq \cdots \leq y_n$.

There exists an index $k : 2 \leq k \leq n-1$ with $y_{k-1} < y_k = y_{k+1} = \cdots = y_n$. If all the coordinates of y are equal (i.e. $k = 1$), $y_i > \frac{f(n)}{n} \geq \frac{f(n)}{n}$

for all indices i and we can decrease y with a positive vector without leaving $U(v)$. Then y can not be extreme, as one can always increase y with the same positive vector without leaving $U(v)$. If $k = n$, one can decrease y_n without leaving $U(v)$ and again y is not extremal.

We prove that there exists an index $r : k \leq r \leq n-1$ with $y_1 + \cdots + y_r = f(r)$.

If such a vector does not exist and T is a coalition with $y(T) = f(t)$, we have $f(t) \leq y_1 + \cdots + y_t \leq y(T) = f(t)$ and therefore, $T \subset [1, \ldots, k-1]$. If we define $u \subset \mathbf{R}^n$ by $u_i = 0$ for $i < k$ and $u_k + \cdots + u_n = 0$, we have a vector u with $u(T) = 0$ for all T with $y(T) = f(t)$. We can take $u \neq 0$. Then y can not be extreme point of $U(v)$.

Finally, we prove that there is no core allocation x with $x_1 + \cdots + x_{k-1} = f(k-1)$. If such a core element x exists, we may assume that $x_1 \leq x_2 \leq \cdots \leq x_n$. Then $x_k + \cdots x_r \geq f(r) - f(k-1)$ (the index r is as before). For y we have $y_k + \cdots + y_r < f(r) - f(k-1)$.

$$x_1 + \cdots + x_n = (x_1 + \cdots + x_k) + (x_{k+1} + \cdots + x_r) + (x_{r+1} + \cdots + x_n)$$
$$\geq f(k-1) + (f(r) - f(k-1)) + (n-r)\tfrac{f(r)-f(k-1)}{r-k+1}$$
$$\geq f(r) + (n-r)y_k = y(N) > f(n).$$ ■

Summarizing, we have the following:

For totally balanced symmetric games (N, v) the following are equivalent :

(a) the Shapley conditions,

(b) exactness,

(c) having a large core.

From each of these properties follows : (d) the stability of the core. We prove, finally, that (d) stability of the core implies (a) the Shapley conditions.

Proposition 3.3.2 *If the core of a totally balanced symmetric game is stable set, then it satisfies the Shapley conditions.*

Proof : Suppose that f does not satisfy all the Shapley conditions. Let t^* be the index such that $\frac{f(n)-f(t^*)}{n-t^*} < \frac{f(s)-f(t^*)}{s-t^*}$ for some index $t^* < s < n$.

We will give, explicitly, a point $y \in I(v)$ that is not a core allocation and can not be dominated by a core allocation x.

Take $y_i := \frac{f(t^*)}{t^*}$ for $i \leq t^*$ and $y_j := \frac{f(n)-f(t^*)}{n-t^*}$ for $j > t^*$.

If S is a coalition with $|S| = s \leq t^*$, we have $y(S) \geq s\frac{f(t^*)}{t^*} \geq f(s)$ by the totally balancedness condition. If S is a coalition with $|S| = s > t^*$, we find $y(S) \geq f(t^*)+(s-t^*)\frac{f(n)-f(t^*)}{n-t^*} \geq f(s)$, if the Shapley condition for $t^* < s < n$ is satisfied. Hence if $y(S) < v(S)$, then $t^* < s = |S| < n$ and the Shapley condition for $t^* < s < n$ is not satisfied. For $S = \{1, \ldots, s\}$ we have $y(S) < v(S)$. If y can be dominated by a core allocation x, it must be done by a coalition S of size $s > t^*$, not satisfying the Shapley condition $t^* < s < n$.

Suppose that $x \subset U(v)$, $x_S > y_S$ and $x(S) = f(s)$. Let $s' := |S \cup \{1, \ldots, t^*\}|$. Then, $f(s) = x(S) > s'\frac{f(t^*)}{t^*} + (s-s')\frac{f(n)-f(t^*)}{n-t^*} \geq f(t^*) + (s-t^*)\frac{f(n)-f(t^*)}{n-t^*}$.

Furthermore, $x_j \geq \frac{f(n)-f(t^*)}{n-t^*}$ for $j \notin S$ (otherwise $x((S \setminus i) \cup j) < f(s)$), if $x_i > \frac{f(n)-f(t^*)}{n-t^*}$.

$x(N) > f(t^*) + (f(n) - f(t^*)) = f(n)$.

There is no core allocation dominating y and the core is not stable. ■

By *Lemma 3.3.1*, the *Propositions 3.3.1* and *3.3.2* we have *Theorem 3.3.1*.

Finally we prove that *totally exact* TU-games are convex. Totally exact means that the games and all subgames are exact. In Moulin[27] and Kikuta [17] the weaker result is proved that "totally large core"(every subgame has a large core) implies convexity.

Proposition 3.3.3 *Totally exact games are convex.*

Proof : Take two coalition S and T arbitrarily. As the subgame $(S \cup T, v_{|S \cup T})$ is exact, there is a core allocation x of the subgame with $x(S \cap T) = v(S \cap T)$(also trivially if $S \cap T = \emptyset$).Then $v(S) + v(T) \leq x(S) + x(T) = x(S \cup T) + x(S \cap T) = v(S \cup T) + v(S \cap T)$ This is the convexity relation. ■

Conclusion: For balanced symmetric games and balanced games with less than 5 players exactness (of the totally balanced cover), stable core and large core are equivalent. For non-symmetric balanced games with 5 or more players this is no more true. Shapley in his unpublished manuscript [40] also proved the following theorem, before stating which we define the dual of a game.

Definition 3.3.1 *Given a game (N, v) the dual of the game (N, v^*) is defined by $v^*(S) = v(N) - v(S)$ for all $S \subseteq N$.*

Theorem 3.3.2 *The core of a symmetric game is stable if and only if the negative of the dual of the game is totally balanced.*

However, Shapley expressed a doubt that in non-symmetric case the equivalence may not be true. We confirm his doubts to be true through examples in the following section.

3.4 Examples

We recall the generic example of an exact game which does not have a large core, and consider a 5-player version.

Example 3.4.1 *Consider the 5-player exact game defined through two vectors :* $w_1 = (1,\ -1,\ -1,\ 0,\ 0)$ *and* $w_2 = (0,\ 1,\ 0,\ -1,\ -1)$*. Define* $v(S) = Min\{w_1(S), w_2(S)\}$ *for all* $S \subseteq N$*.*

It is easy to check that the negative of the dual of the game $(N, -v^*)$ is given by the following similarly defined game. $-v^*(S) = Min\ \{-w_1(S), -w_2(S)\}$ for all $S \subseteq N$. It is further easy to check that the game $-v^*$ is also exact and hence totally balanced. But the core of the game (N, v) is not stable as $y = (0,\ -1,\ 0,\ 0,\ 0)\ \notin C(v)$ and is undominated by any core element.

Example 3.4.2 *Consider the following 6-player game with a stable core.* $|N| = 6$, $v(N) = 3$, $v(\{1,2\}) = v(\{1,3\}) = v(\{4,5\}) = v(\{4,6\}) = 1$*. All other* $v(S)$*'s can be evaluated from total balance consideration.*

Now consider the negative of the dual of the game and the subgame of $-v^*$ on $\{2,3,5,6\}$. In the following we present the coalitions in flower brackets and the corresponding worth below them.

		$\{2,3,5,6\}$			
		-3			
	$\{2,3,5\}$	$\{2,3,6\}$	$\{2,5,6\}$	$\{3,5,6\}$	
	-2	-2	-2	-2	
$\{2,3\}$	$\{2,5\}$	$\{2,6\}$	$\{3,5\}$	$\{3,6\}$	$\{5,6\}$
-2	-1	1	-1	-1	-2
	$\{2\}$	$\{3\}$	$\{5\}$	$\{6\}$	
	-1	-1	-1	-1	

Now consider the balanced collection $\mathcal{B} = \{\{2,5\}, \{2,6\}, \{3,5\}, \{3,6\}\}$ of $\{2,3,5,6\}$, with the balancing vector $\delta = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$. $\sum_{S \in \mathcal{B}} \delta_S v(S) = -2 > -3 = v(\{2,3,5,6\})$. So the game $-v^*$ is not totally balanced.

These two examples show that Shapley's result in symmetric games is false in both directions in the non-symmetric case.

Chapter 4

Large Core and Extendability

4.1 Introduction

In the case of symmetric TU games, it is known that core stability is equivalent to largeness of the core. For general TU games, largeness of the core always implies core stability and there are examples where the core is stable but not large. It is known that under the extendability condition introduced by Kikuta and Shapley the core is a stable set but the core may not be large. In this chapter, we show that the Kikuta-Shapley condition is sufficient for the core to be large for TU games with five or less number of players. We provide a counter example when the number of players is six. We then introduce a stronger extendability condition and show that this condition is necessary and sufficient for the core to be large. Our proof makes use of a well known result from the theory of convex sets.

The core of a TU game is perhaps the most intuitive and easiest solution concept in Cooperative Game Theory [33]. Another approach to solution concepts is the stable sets introduced by von Neumann and Morgenstern[45]. In this chapter we are concerned with core stability and largeness of core. Sharkey [43] introduced the notion of largeness of the core and showed that largeness of the core implies that the core is a stable set. van Gellekom et al.[11] have given an example of a non-symmetric six person TU game where the core is stable but the core is not large, while Biswas et al.[4] have shown for symmetric TU games, the two concepts coincide. There have been several results which deals with sufficient conditions for the core to be stable [12]. The survey papers by Aumann [1] and Lucas [23], [25] and the references therein give an excellent review of these topics. For more recent results on core and monotonic solutions one can refer to [46], [14].

Kikuta and Shapley[17] have introduced the notion of extendability of the game in the sense that every subgame core element (whenever it exists) can be extended to a core element of the original game. They have shown that extendability implies stability of the core. Recently van Gellekom et al.[11] have given an example showing that stability of the core need not imply extendability property of the game. They have also given a seven person TU game where the game has the extendability property but the core is not large. These results immediately raise two questions : (i) Can we find a TU game with less than 7 players satisfying extendability without having a large core? (ii) Can we strengthen Kikuta-Shapley notion of extendability so that it will imply largeness of core. We answer these two questions in this chapter.

We introduce a different notion of extendability property namely that every lower boundary point of every $(n-1)$ person subgame can be extended to a core element of the original n-person game. This notion will be shown to be equivalent to largeness of the core. Therefore, we will call this concept a bit prematurely, strong extendability. Proof of this result makes use of a well known result from the theory of convex sets. Then we show that when the number of players is at most five, extendability of the game and largeness of the core coincide but for $n = 6$, we give an example to show that the game has the extendability property but the core is not large. Our plan for the chapter is as follows. In section 4.2, we present some preliminaries. In section 4.3, we state and prove our main results. Section 4.4 contains examples and further remarks.

However, the following definitions are necessary for the discussions in this chapter. We now define the notion of extendability due to Kikuta and Shapley[17].

Definition 4.1.1 *Let (N, v) be a TU-game with a non-empty core. We say that (N, v) is extendable if for every non-empty coalition $S \subset N$ for which $C(v|_S) \neq \phi$ and for every core element $y \in C(v|_S)$ there is a core element $x \in C(v)$ with $x_i = y_i$ for all $i \in S$.*

Hence if (N, v) is extendable then every subgame core element can be extended to a core element of (N, v).

Kikuta and Shapley have shown that if the core is large for TU-game (N, v), then (N, v) is extendable. They have also shown that if $C(v) \neq \phi$ and if (N, v) is extendable then $C(v)$ is stable. In a recent paper van Gellekom et al.[11] have given examples to show that in general, stability of $C(v)$ need not imply extendability with a six-person game and extendability need not imply largeness of the core with a seven-person game. Our inspiration for this chapter comes from these two papers. In order to state our main results we need a different notion of extendability.

Definition 4.1.2 *Let (N, v) be TU-game with a non-empty core. We say (N, v) is extendable in the stronger sense if for every $S \subset N$ with $|S| = n - 1$ and every $y \in L(v|_S)$, there is a core element $x \in C(v)$ such that $x_i = y_i$ for all $i \in S$.*

In other words if (N, v) is extendable in the stronger sense, then every $n-1$-player subgame lower boundary point can be extended to a core element of (N, v).

We show in this chapter the following : (i) TU-game (N, v) is extendable in the stronger sense if and only if the core is large and (ii) TU-game (N, v), with $n \leq 5$ is extendable if and only if the game has a large core. Finally, we give an example of a six-person game which is extendable but not large. These results complement the results obtained by van Gellekom et al. [11].

4.2 Main Results

In this section we state and prove some results connecting extendability and largeness of the core. We need the following result from the theory of convex sets [3].

Berge's Theorem : If $D_1, \ldots, D_m \subset \mathbb{R}^m$ are closed convex sets with (a) $\bigcup_i D_i$ is convex and (b) $\bigcap_{j \neq i} D_j \neq \phi$ for every index i then $\bigcap_i D_i \neq \phi$.

Theorem 4.2.1 *Let (N, v) be a TU-game with a non-empty core. Then the core $C(v)$ is large if and only if the game (N, v) is extendable in the stronger sense.*

Proof : If the core is large, it is not difficult to see that the game is extendable in the stronger sense. Take an element $y|_S \in L(v|_S)$. Extend $y|_S$ to $y \in L(v)$. As the core is large $y \in C(v)$.

So we need to prove only the converse.

Suppose there is an element $y \in L(v)$ with $y(N) > v(N)$ and define $D_i := \{x \in C(v) : x_i \leq y_i\}$. We will prove that $\bigcap_i D_i \neq \phi$. Every core element must belong to at least one D_i, since $y(N) > v(N) = x(N)$ for any core element x. Hence we have $\bigcup_i D_i = C(v)$, a convex set. In order to apply Berge's Theorem, we must prove that $\bigcap_{j \neq i} D_j \neq \phi$ for every i.

So take $i \in N$. We will produce a core element belonging to $\bigcap_{j \neq i} D_j \neq \phi$. Reduce $y_{N \setminus i}$ to an element $y^*_{N \setminus i} \leq y_{N \setminus i}$ with $y^*_{N \setminus i} \in L(v_{|N \setminus i})$. This can be done as follows :

As $y \in L(v)$, $y_{N \setminus i} \in A(v|_{N \setminus i})$ in case it is not a $L(v_{|N \setminus i})$ element, there is an index l such that for all $T \subseteq N \setminus i$ containing l, $y(T) > v(T)$. Define $\epsilon := Min_{T \ni l, T \subseteq N \setminus i}\{y(T) - v(T)\}$ and redefine $y_l^* = y_l - \epsilon$ and $y_j^* = y_j$, for $j \neq l$. We repeat this with all remaining indices l such that for all $T \subseteq N \setminus i$ containing l, $y(T) > v(T)$. Thus we produce an an element $y^*_{N\setminus i} \in L(v|_{N\setminus i}$ with $y^*_{N\setminus i} \leq y_{N\setminus i}$. Then $y^*_{N\setminus i}$ can be extended into a core element $x^{(i)}$.

This is an element of $\bigcap_{j\neq i} D_j$. Since i is arbitrary, $\bigcap_{j\neq i} D_j \neq \phi$ for every player i.

Hence $\bigcap_i D_i \neq \phi$. Let x be such an element in $C(v)$, that $x_i \leq y_i$. Since y is a lower boundary point with $y(N) > v(N)$, we arrive at a contradiction. This shows that $C(v)$ is large when (N, v) is extendable in the stronger sence. ■

Theorem 4.2.2 *Let (N, v) be TU-game with non-empty core. Suppose $|N| = n \leq 5$. Then the core $C(v)$ is large if and only if (N, v) is extendable.*

Proof : We will prove the *if* part only, as the *only if* part follows from the proof of *Proposition3.1.1*. Suppose (N, v) satisfies extendability condition and let $y \in L(v)$ with $y(N) > v(N)$. Recall definition of $\mathcal{S}_y$ as the collection of coalitions T with $y(T) = v(T)$.

Then $\mathcal{S}_y$ covers N and y_T can be extended into a core element $(y_T, x_{N\setminus T}) \in C(v)$.

The following situations are not possible:

(a) There are coalitions T_1 and T_2 in $\mathcal{S}_y$ with $T_1 \cup T_2 = N$,

(b) There is a coalition $T^* \in \mathcal{S}_y$ and, for each player $j \notin T^*$ a coalition $T_j \in \mathcal{S}_y$ with $j \in T_j \subsetneq T^* \cup \{j\}$.

If case (a) occurs, we write $A := N \setminus T_2$, $B := T_1 \cap T_2$ and $C := N \setminus T_1$. Then $A \cup B = T_1$ and there is a core allocation $x = (y_A, y_B, x_C)$. Then $B \cup C = T_2$ and $y_B(B) + x_C(C) \geq v(B \cup C) = y_B(B) + y_C(C)$. This means $y(N) < x(N) = v(N)$, a contradiction.

If case (b) occurs, there exists a core allocation $x = (y_{T^*}, \{x_j\}_{j \notin T^*})$. Then $x(T_j) = x_j + y(T^* \cap T_j) \geq v(T_j) = y(T^* \cap T_j) + y_j$. Then $x(N \setminus T^*) \geq y(N \setminus T^*)$ and $x(N) \geq y(N) > v(N)$, a contradiction.

Also impossible is:

(c) There is a coalition $T \subset \mathcal{S}_y$ with $|T| = n - 1$.

In case (c) we have $T_1 := N \setminus i \in \mathcal{S}_y$ for some player $i \in N$. There is also a coalition T_2 containing i. Then T_1 and T_2 are not possible by (a).

Case $n = 4$. There is at least one coalition containing 1, at least one coalition containing 2 up to a coalition containing 4 in $\mathcal{S}_y$. By case (b) these coalitions cannot be all 1-coalitions and there are no 3-coalitions($3 = n - 1$). So, there is a 2-coalition, say w.l.g. $T_1 = (1, 2) \in \mathcal{S}_y$. Then by the impossibility of (a), there is no coalition $T_2 \in \mathcal{S}_y$ containing 3 and 4 and by the impossibility of (b) there are no coalitions T_2' and T_2'' in $\mathcal{S}_y$, one containing 3 and not 4 and one containing 4 and not 3. So, there is no possibility left for $\mathcal{S}_y$.

Case $n = 5$. There is no 4-coalition in $\mathcal{S}_y$. Suppose $T_1 \in \mathcal{S}_y$ with

$|T_1| = 3$. Then, w.l.g. $T_1 = (1,2,3)$. There must be a coalition $T_2 \in \mathcal{S}_y$ containing $(4,5)$, impossible by (a) or two coalitions T_2' and T_2'' in $\mathcal{S}_y$, one containing 4 and not 5 and one containing 5 and not 4. This is not possible by (b). So, every coalition in $\mathcal{S}_y$ has size 1 or 2 and there is at least one 2-coalition, by (b).

Impossible is

(d) $n = 5$ and T_1 and T_2 are two disjoint 2-coalitions in $\mathcal{S}_y$.

In case (d) we have w.l.o.g. $T_1 = (1,2)$ and $T_2 = (3,4)$. There is also a coalition T_3 in $\mathcal{S}_y$ containing 5. This can be $T_3 = (5)$ or $T_3 = (x,5)$ with $x \in \{1,2,3,4\}$. W.l.o.g. we may assume that $x = 1$. Let $x = (y_1, y_2, x_3, x_4, x_5)$ be an extension of (y_1, y_2) into a core allocation of (N, v). In the first case we have $x_3 + x_4 \geq v(34) = y_3 + y_4$ and $x_5 \geq v(5) = y_5$. Then $x(N) \geq y(N) > v(N)$ and a contradiction. In the second case, we also have $x_3 + x_4 \geq y_3 + y_4$ and $y_1 + x_5 \geq y_1 + y_5$. Then again $x(N) \geq y(N) > v(N)$.

Now we try to find a possibility for $\mathcal{S}_y$.

$T_1 = (12)$. By the impossibility of (b) there must be a second 2-coalition T_2 in $\mathcal{S}_y$ and by (d) it is not disjoint from (12). W.l.o.g. $T_2 = (13)$. Again by (b) there must be a third 2-coalition $T_3 \in \mathcal{S}_y$ containing 4 or 5. W.l.g. it contains 4 and T_3 must intersect T_1 and T_2. $T_3 = (14)$ is the only possibility. Finally there must be a coalition T_4 containing 5. This cannot be (5) by the impossibility of (b). It is therefore a 2-coalition intersecting T_1, T_2 and T_3. Then (15) is the only possibility for T_4 but this is after all also impossible by (b) (take $T^* = (12)$). This concludes the proof of the Theorem. ∎

Remark 4.2.1 *For $n = 6$ consider the collection $\mathcal{S}$ consisting of* (12), (13), (23), (124), (135) *and* (236). *This collection is not in contradiction with the impossibility of (a), (b) and (c). This fact will be used in the last section as an example of a 6-person game satisfying extendability without having a large core.*

Biswas and Parthasarathy[6] earlier considered the extension of all lower boundary points of every subgame to the core elements and showed that the core is large. Though this appears to be rather trivial, it is worthwhile to note that the strong extendability, weaker than the above as it may look, eventually implies the original assumption as in [6].

Theorem 4.2.1 is useful in examining whether the core is large. If for some $(n-1)$ player subgame we find a lower boundary point which cannot be extended as a core element of the original game, then the core cannot be large. We illustrate this with an example in section 4.4.

Following van Gellekom, we denote by $U(v)$, the set of upper vectors, that is,

$U(v) = \{x : x \in \mathbf{R}^n,\ x(S) \geq v(S)$ for all proper coalitions $S \subset N\}$.

It is easy to see $U(v) \supseteq A(v)$. It is shown in [11], that the core of a TU game is large if and only if $z(N) \leq v(N)$ for every extreme point z of $U(v)$. This result complements *Theorem 4.2.1* of the present chapter.

4.3 Large core of k-convex games

In the folowing we study largeness of core of a subclass of games known as symmetric k-convex games. We also prove that the subclass is closed under totally balanced games. The k-convex games were defined by Driessen in [9] because of the core structure they possess, which is the same as that of a suitably defined convex game. These games in the general set up are defined through what is known as the gap function which in turn is defined through marginals of the players in a game. However in the symmetric case they can be defined through the vectors defined below.

Definition 4.3.1 *Given a symmetric n-person game v defined by a function $f : \{0, 1, 2, \ldots, n\} \to \mathbf{R}, \Delta_n(v) = f(n) - f(n-1)$, and the corresponding allocation $x(v) = (x_j(v))_{j \in N}$ is as follows:*

$$x_j(v) := \begin{cases} f(j) - f(j-1) & \text{if } j \leq k-1, \\ f(n) - (n-k)\cdot\Delta_n(v) - f(k-1) & \text{if } j = k, \\ f(n) - f(n-1) := \Delta_n(v) & \text{otherwise.} \end{cases} \tag{4.1}$$

The requirement that the allocation $x(v) = (x_j(v))_{j \in N}$ of (4.1) should be nondecreasing is equivalent to the following system of three conditions:

$$f(s) - f(s-1) \leq f(t) - f(t-1) \quad \text{for } 1 \leq s \leq t \leq k-1. \tag{4.2}$$

$$f(n) - f(k-1) \leq (n-k+1)\cdot\Delta_n(v) \tag{4.3}$$

$$f(k-1) - f(k-2) + (n-k)\cdot\Delta_n(v) \leq f(n) - f(k-1) \quad \text{for } k > 1. \tag{4.4}$$

In addition, the requirement that the nondecreasing allocation $x(v) = (x_j(v))_{j\in N}$ of (4.1) should belong to the core of the symmetric game v is equivalent to the next condition:

$$f(n) - f(s) \geq (n-s)\cdot\Delta_n(v) \qquad \text{for all } s\in\{k,k+1,\ldots,n\}. \quad (4.5)$$

Definition 4.3.2 *A symmetric n-person game (N,v) is k-convex if the game v satisfies the four conditions (4.2)–(4.5), or equivalently, the nondecreasing allocation $x(v) = (x_j(v))_{j\in N}$ of (4.1) belongs to the core of v. Alternatively, the symmetric n-person game (N,v) is k-convex if (4.2) holds and the corresponding straight line $\ell_v : \{0,1,2,\ldots,n\}\to\mathbf{R}$ through the points $(n, f(n))$ and $(n-1, f(n-1))$ satisfies the following three conditions:*

$\ell_v(k-1)\leq f(k-1)$ and $\ell_v(s)\geq f(s)$ for all $s\in\{k,k+1,\ldots,n\}$;
$\ell_v(k) - f(k-1)\geq f(k-1) - f(k-2)$ whenever $k>1$.

Evidently, for $k\in\{n-1,n\}$, the k-convexity of a symmetric n-person game v agrees with the convexity notion for v. Since n- and $(n-1)$-convexity are identical notions, we assume, without loss of generality, throughout the section that $k\in\{1,2,\ldots,n-1\}$.

It is well-known that the core of a symmetric n-person game v is non-empty if and only if $\frac{f(s)}{s}\leq\frac{f(n)}{n}$ for all $s\in\{1,2,\ldots,n\}$. Consequently, a symmetric n-person game v is totally balanced if and only if the corresponding sequence of per capita worth $\frac{f(s)}{s}$, $s\in\{1,2,\ldots,n\}$, is nondecreasing. With a symmetric n-person game v, there is associated the totally balanced cover $\bar{v}$, defined in *Definition 2.1.2*

The totally balanced cover $\bar{v}$ may be interpreted as the smallest totally balanced symmetric game that majorizes the game v (that is, if a totally balanced symmetric n-person game w satisfies $w(S) \geq v(S)$ for all $|S| = s \in \{1, 2, \ldots, n\}$, then $w(S) \geq \bar{v}(S)$ for all $|S| = s \in \{1, 2, \ldots, n\}$). Since convex games are known to be totally balanced [41], it holds that $\bar{v} = v$ for every convex symmetric game v. Moreover, it is well-known that both cores coincide, that is $C(\bar{v}) = C(v)$ whenever $C(v) \neq \emptyset$. In particular, $\bar{v}(N) = v(N)$ holds for every symmetric n-person game v with a non-empty core.

Further, both acceptable sets coincide, that is $A(\bar{v}) = A(v)$ for every symmetric n-person game v [4] and therefore, a balanced symmmetric game has a large core if and only if its totally balanced cover has a large core. As a matter of fact, largeness of the core for a symmetric game can be formulated in various equivalent manners as has been seen in *Theorem 2.2.6*.

We aim to investigate, for k-convex symmetric games, the largeness of the core. The next theorem states that the totally balanced cover inherits the k-convexity property. Thus, without loss of generality, in the sequel we study the largeness of the core for k-convex totally balanced symmetric n-person games.

Theorem 4.3.1 *Let $k \in \{1, 2, \ldots, n-1\}$. The totally balanced cover $\bar{v}$ of a k-convex symmetric n-person game v is k-convex.*

Proof : Let the symmetric n person game v be k-convex, that is the four conditions (4.2)–(4.5) applied to v hold. We show that these four conditions (4.2)–(4.5) applied to $\bar{v}$ hold too. Firstly, notice that the convexity condition (4.2) for the game v up to size $k-1$ yields that $\bar{f}(s) = f(s)$ for all $s \in \{1, 2, \ldots, k-1\}$. Thus, the convexity condition (4.2) applied to $\bar{v}$ holds too.

Recall that $\bar{v}(N) = v(N)$ (due to the non-emptiness of the core of k-convex symmetric n-person games). Furthermore, $\bar{f}(n-1) \geq f(n-1)$ by convention of $\bar{v}$ and thus, $\Delta_n(\bar{v}) = \bar{f}(n) - \bar{f}(n-1) \leq f(n) - f(n-1) = \Delta_n(v)$. It follows immediately that condition (4.4) applied to $\bar{v}$ holds too. As a matter of fact, we claim $\bar{v}(n-1) = f(n-1)$. By k-convexity of v, the nondecreasing allocation $x(v) = (x_j(v))_{j \in N}$ of (4.1) belongs to the core of v and besides, $\sum_{j=1}^{n-1} x_j(v) = f(n-1)$ (since $x_n(v) = f(n) - f(n-1)$). As a result, the nondecreasing allocation arising from $x(v)$ restricted to all components, except the very last one, belongs to the core of the subgame $(N \setminus \{n\}, v_{N\setminus\{n\}})$. Next, on its turn, the non-emptiness of the core of the subgame $(N \setminus \{n\}, \bar{v}_{N\setminus\{n\}})$ yields $\bar{f}(n-1) = f(n-1)$ and hence, $\Delta_n(\bar{v}) = \Delta_n(v)$. In summary, the three conditions (4.2)–(4.4) applied to $\bar{v}$ are identical to those applied to v, which are supposed to hold true. It remains to prove condition (4.5) applied to $\bar{v}$. The proof proceeds by induction on s, $s \in \{k, k+1, \ldots, n\}$.

First we prove (4.5) applied to $\bar{v}$ and $s = k$. The case $k = 1$ implies $\bar{f}(k) = \bar{f}(1) = f(1) = f(k)$ and (4.5) is invariant. Thus we may suppose $k > 1$ and $\bar{f}(k) \neq f(k)$. Recall that, by convention of $\bar{f}$, $\bar{f}(k) = \max\left[f(k), k \cdot \frac{\bar{f}(k-1)}{k-1}\right]$. Hence,

$\bar{f}(k) = k \cdot \frac{\bar{f}(k-1)}{k-1} = k \cdot \frac{f(k-1)}{k-1} = f(k-1) + \frac{f(k-1)}{k-1}$
$\leq f(k-1) + [f(k-1) - f(k-2)]$,

where the last inequality follows from the convexity condition (4.2) for the game v up to size $k-1$. From this and (4.4), we deduce $f(n)-\bar{f}(k) \geq f(n)-f(k-1)-[f(k-1) - f(k-2)] \geq (n-k)\cdot\Delta_n(v) = (n-k)\cdot\Delta_n(\bar{v})$. Let $s \in \{k, k+1, \ldots, n-1\}$. By induction hypothesis, suppose that (4.5), applied to $\bar{v}$ and s, holds. Recall that, by convention of $\bar{v}$, $\bar{f}(s+1) = \max\left[f(s+1), (s+1)\cdot\frac{\bar{f}(s)}{s}\right]$. Notice that (4.5) is invariant whenever $\bar{f}(s+1) = f(s+1)$. In case $\bar{f}(s+1) \neq f(s+1)$, then we deduce from the induction hypothesis that $\bar{f}(n)-\bar{f}(s+1) = f(n)-(s+1)\cdot\frac{\bar{f}(s)}{s} = \left[\bar{f}(n) - \bar{f}(s)\right] - \frac{\bar{f}(s)}{s} \geq (n-s)\cdot\Delta_n(\bar{v}) - \frac{\bar{f}(s)}{s}$.

The non-emptiness of the core of the k-convex symmetric n person game v yields $\frac{f(n-1)}{n-1} \leq \frac{f(n)}{n}$ or equivalently, $f(n) \leq n\cdot\Delta_n(v)$. From this and the induction hypothesis, we derive

$\bar{f}(s) \leq \bar{f}(n) - (n-s)\cdot\Delta_n(\bar{v}) = f(n) - (n-s)\cdot\Delta_n(v) \leq s\cdot\Delta_n(v) = s\cdot\Delta_n(\bar{v})$.

From $\bar{f}(s) \leq s\cdot\Delta_n(\bar{v})$ and the former inequality, we conclude that $\bar{f}(n) - \bar{f}(s+1) \geq (n-s)\cdot\Delta_n(\bar{v}) - \frac{\bar{f}(s)}{s} \geq (n-s-1)\cdot\Delta_n(\bar{v})$.

This completes the inductive proof of condition (4.5) applied to $\bar{v}$. □

Example 4.3.1 *Consider the symmetric 8-person game v given by $v(S) := 0, 0, 2, 5, 9, 12, 18, 25, 31$ for coaltions S of size $s = 0, 1, \ldots, 8$ respectively. This game v is totally balanced since the corresponding sequence of per capita worth $\frac{f(s)}{s}$, $s \in \{1, 2, \ldots, 8\}$, is nondecreasing. Moreover, this game is 5-convex since the nondecreasing alloca-*

tion $x(v) = (0,2,3,4,4,6,6,6)$ *of (4.1) belongs to the core of* v. *The core of* v, *however, is not large because the associated specified vector* $y^1 = (0,2,3,4,4,5,7,7)$ *of equation(2.1) does not meet the efficiency principle.*

Theorem 4.3.2 *Let* $k \in \{1,2,\ldots,n-1\}$. *For a k-convex totally balanced symmetric n-person game* v, *the following three statements are equivalent.*

1. *v has a large core.*
2. $f(s) = f(n) - (n-s)\cdot\Delta_n(v)$ *for* $s:\ k \le s \le n$. (4.6)
3. *v is a convex game.*

Proof The implication (iii) $\Longrightarrow$ (i) holds true for every convex game [42]. In order to prove the implication (ii) $\Longrightarrow$ (iii), note that
$f(s+1) - f(s) = \Delta_n(v)$ for all $s \in \{k, k+1, \ldots, n-1\}$, from (4.6).
$f(k) - f(k-1) = f(n) - f(k-1) - (n-k)\cdot\Delta_n(v) \le \Delta_n(v)$, from equations (4.6) and (4.3).
$f(k-1) - f(k-2) \le f(n) - f(k-1) - (n-k)\cdot\Delta_n(v) = f(k) - f(k-1)$ if $k > 1$, from equations (4.6) and (4.4)
Hence, the symmetric game v is convex, provided that (4.6) holds. It remains to prove the implication (i) $\Longrightarrow$ (ii).
Recall that the largeness of the core of the symmetric n-person game v is equivalent to Shapley's conditions.

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(n)-f(s)}{n-s} \quad \text{whenever } 0 \le s < t \le n. \qquad (4.7)$$

For the sake of notation, write $\ell_v(t) := f(n) - (n-t)\cdot\Delta_n(v)$ for all $t \in \{k, k+1, \dots, n\}$. The proof of (4.6) proceeds by backwards induction on s, $s \in \{k, k+1, \dots, n\}$. By convention of $\Delta_n(v)$, (4.6) holds whenever $s \in \{n-1, n\}$. Let $s \in \{k, k+1, \dots, n-2\}$. By induction hypothesis, suppose that $f(s+1) = f(n) - (n-s-1)\cdot\Delta_n(v) := \ell_v(s+1)$. From Shapley's conditions we obtain that

$f(s+1) - \bar{f}(s) \leq \frac{f(n)-\bar{f}(s)}{n-s}$ or equivalently,

$(n-s-1)\cdot \bar{f}(s) \geq (n-s)\cdot f(s+1) - f(n)$.

Further, from the induction hypothesis $f(s+1) = \ell_v(s+1)$ and $\ell_v(s+1) = \ell_v(s) + \Delta_n(v)$, we derive

$$\begin{aligned}
(n-s)\cdot f(s+1) - f(n) &= (n-s)\cdot \ell_v(s+1) - f(n) \\
&= (n-s)\cdot [\ell_v(s) + \Delta_n(v)] - f(n) \\
&= (n-s)\cdot \ell_v(s) + (n-s)\cdot \Delta_n(v) - f(n) \\
&= (n-s-1)\cdot \ell_v(s).
\end{aligned}$$

Hence, $(n-s-1)\cdot \bar{f}(s) \geq (n-s)\cdot f(s+1) - f(n) = (n-s-1)\cdot \ell_v(s)$ and thus, $\bar{f}(s) \geq \ell_v(s)$. By the totally balancedness of v, it holds $\bar{v} = v$. Together with (4.5), it follows that

$f(s) \leq f(n) - (n-s)\cdot\Delta_n(v) = \ell_v(s) \leq \bar{f}(s) = f(s)$.

We conclude that $f(s) = \ell_v(s)$. This completes the inductive proof of (4.6). □

4.4 Examples and remarks

We first give an example to show that *Theorem 4.2.2* may not be true when the number of players is six.

Example 4.4.1 *Let* $N=\{1,2,3,4,5,6\}$ *and* v *is given by* $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 1$, $v(\{1,2,4\}) = v(\{1,3,5\}) = v(\{2,3,6\}) = 2$ *and* $v(N) = 4$ *and* $v(S)$ *is defined suitably for other* S *so that* v *is monotonic, super-additive and totally balanced.*

Though it is not hard to check that this game is extendable, we indicate the proof below. However, that the core is not large is easy to see by verifying that $y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1)$ is a lower boundary point of the game and $y(N) = 4.5 > v(N) = 4$. This example shows the sharpness of *Theorem 4.2.2.*

We now show the extendability of the game given in *Example 4.4.1.* It is enough to prove extendability of one 2-player coalition, w.l.g. $\{1,2\}$ and one 3-player coalition like $\{1,2,4\}$. let $(x_1, x_2) = (1, 0)$ then take $(x_3, x_4, x_5, x_6) = (1, 1, 0, 1)$. Easy to check that $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in C(v)$ let $(x_1, x_2) = (0, 1)$ then take $(x_3, x_4, x_5, x_6) = (1, 1, 1, 0)$. Easy to check that $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in C(v)$. Any other subgame core element of $\{1,2\}$ is now extendable. Similar proofs are possible for $\{1,3\}$, and $\{2,3\}$.

Consider $\{1,2,4\}$ and $(x_1, x_2, x_4) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ Now consider $(x_3, x_5, x_6) \in \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ in the same order, we find $x \in C(v)$. Thus it is easy to see that the $\{1,2,4\}$

subgame core is extendable, hence similar facts are true about $\{1,3,5\}$ and $\{2,3,6\}$. The rest of the extendability property follows from the above.

The following example given in [11] with 7 players also shows that the game is extendable but the core is not large.

Example 4.4.2 *Let $N=\{1,2,3,4,5,6,7\}$ and v is given by $v(1,7) = v(1,7) = 2$, and $v(1,2,7) = v(1,3,7) = v(4,5,7) = v(4,6,7) = 3$, $v(N) = 7$ and $v(S) = 0$ otherwise. In this example, the $(N,\overline{v})$ is extendable but the core of N, $\overline{v})$ is not large. Here $(N,\overline{v})$ stands for the totally balanced cover of the TU-game (N,v).*

Let (N,v) be a TU-game. The restriction of v to $2^N \setminus N$ is denoted by v^o. Then (N,v^o) is called an incomplete TU-game where the value of the grand coalition, namely N is not specified. If the game $(N,v^o,\ v(N) = \beta)$ has a large core, then the game $(N,v^o,\ v(N) \geq \beta)$ is also large. That is if a TU game at $v(N) = \beta$ has large core then the TU-game at $v(N) > \beta$ also has large core. In other words largeness of a core is a prosperity property.

Definition 4.4.1 *A property P on TU-games is called a (strong) prosperity property if for every incomplete game (N,v^o) there exists a number $\alpha_P(v^o) \geq \sum_{i\in N} v(i)$ such that $(N,v^o,v(N))$ has property P if and only if $v(N) \geq \alpha_P(v^o)$*

See [11] for more on prosperity properties. Now the following question arises naturally. Given an incomplete TU -game (N, v^o) what is the least value β so that the TU game $(N, v^o, v(N) = \beta)$ has a large core. In [11] it is shown the least β is given by $\beta = max\{z(N) : z$ is an extreme point of $U(v)\}$. It is not easy to find the least β in practical situations. One can find the least β (easily) in symmetric TU-games using special vectors defined in Biswas et. al. [4]. However the following elementary proposition gives a rough bound for β.

Proposition 4.4.1 *Let (N, v) be a TU-game with $v(\{i\}) = 0$ for all i and $v(S) \geq 0$ for every S. Let $\alpha = max\{v(S) : S \subset N\}$. If $v(N) \geq (n-1)\alpha$, then the core of the TU-game (N, v) is large.*

Before giving the proof of this proposition we would like to make the following remarks.

Remark 4.4.1 *It is well known that every (essential) TU-game is strategically equivalent to a TU-game where the worth of any coalition S is non-negative and worth of every singleton coalition is zero. For a discussion on these, refer to Owen [32].*

Remark 4.4.2 *If two TU-games are strategically equivalent and if one of them has a large core then the other game has also a large core. Combining the above two remarks, it is clear that the non-negativity assumption of v in proposition is not restrictive.*

Remark 4.4.3 *It is not possible to improve α given in Proposition4.4.1 as the following simple example shows.*

Example 4.4.3 *Let* $N = \{1,2,3\}$ *and* v *is given by* $v(1,2) = v(1,3) = 1$, $v(N) = 2$ *and* $v(S) = 0$ *otherwise. In this case* $\alpha = max\{v(1,2), v(1,3)\} = 1$ *and* $v(N) = (n-1)\alpha = 2$. *In this example core is stable and large when* $v(N) = 2$. *In other words, the core is not stable when* $v(N) < 2$ *and the core is stable and large whenever* $v(N) \geq 2$.

We now give the proof of *Proposition 4.4.1* :

Proof: Our assumption is $v(N) \geq (n-1)\alpha$ where $\alpha = max\{v(S) : S \subset N\}$. Clearly $\alpha \geq 0$ since $v(S) \geq 0$. We will simply show that every lower boundary point of the game is a core element. This will imply the core of the game is non-empty and large. Suppose $y \in L(v)$ with $y(N) > v(N)$. Write $y = (y_1, y_2, \ldots y_n)$. Since y is a lower boundary point there exists a non-empty coalition S such that $y(S) = v(S)$. If $y_{i_o} > \alpha$ for some i_o, since $y(N) > v(N)$, we can reduce y_{i_o} a little bit so that the vector $y^* = (y_1, y_2, \ldots, y_{i_o} - \varepsilon, y_{i_o+1} \ldots y_n)$ is an acceptable vector and this contradicts our assumption that $y \in L(v)$ as $y* \leq y$. Thus $y_i \leq \alpha$ for every $i \in N$. Without loss of generality let us suppose $S = \{1, 2, \ldots s\}$ where $y(S) = v(S)$. Suppose $|S| = 1$. Then $y = (0, y_2, \ldots y_n)$ since $y(1) = v(1) = 0$ and consequently $y(N) \leq (n-1)\alpha \leq v(N)$ contradicting our assumption $y(N) > v(N)$. So we shall and do assume $|S| \geq 2$ where $y(S) = v(S)$. That is, $y(N) = v(S) + y(N \backslash S) \leq v(S) + (n-s)\alpha \leq (n-s+1)\alpha$ for $v(S) \leq \alpha$. Hence $v(N) < y(N) \leq (n-s+1)\alpha \leq (n-1)\alpha$, contradicting our hypothesis $v(N) > (n-1)\alpha$. Thus every $y \in L(v)$ is a core element and consequently core of the game is large. This completes the proof of our proposition. ■

Proposition 4.4.1 is of limited use. For instance consider Example 4.4.1. In that example $\alpha = 2$. The Proposition tells us that the core of the game will be large if $v(N) > 10$, but we know from other considerations that this six person game has a large core if $v(N) \geq 4.5$. In fact we can define $\alpha = max\{y(N) \ : \ y \in L(v)\}$. Then any TU-game with $v(N) > \alpha$ will have a large core. We end this section with the following problem. Is it possible to find this α efficiently through linear programming ? We can formulate it as several LP problems and get the value of α but this method requires solving exponential number of LP problems.

In view of the above proposition and examples one may ask whether there are games in which stability, Kikuta-Shapley extendability and largeness occur at different values of $v(N)$ for incomplete TU games. Indeed this can occur as the following example shows :

We consider two games. Let $N_1 = \{1, 2, 3, 4, 5, 6\}$ and v is given by $v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2\}) = 1$, $v(\{1, 2, 4\}) = v(\{1, 3, 5\}) = v(\{2, 3, 6\}) = 2$, $v(N_1) = 4$ and $v(S)$ is defined suitably for other S. For example $v(\{1, 2, 3\}) = 1.5$, $v(\{1, 2, 3, 5, 6\}) = 2$, $v(\{i\}) = 0$ and so on, as we did in *Example 4.4.1*.

Let $N_2 = \{7, 8, 9, 10, 11, 12\}$ with $v(\{7, 9\}) = v(\{7, 8\}) = v(\{10, 11\}) = v(\{10, 12\}) = 1$, $v(N_2) = 3$ and $v(S)$ is suitably defined for other subsets S of N_2, as above.

Example 4.4.4 *Let* $N = N_1 \cup N_2$ *and if* $S \subset N$ *and* $S = S_1 \cup S_2 \neq N$, *where* $S_1 \subset N_1$ *and* $S_2 \subset N_2$. *define* $v(S) = v(S_1) + v(S_2)$.

We can easily check that this game (N, v) has a stable core when $v(N) = 7$ but the game is not extendable and the core is not large. The game is extendable but the core is not large when $v(N) = 8$ and the core is large (and hence stable and the game extendable)when $v(N) = 8.5$.

Chapter 5

NTU Games

5.1 Introduction

In this chapter the realtion between von-Neumann-Morgenstern stability of the core and the largeness of it is investigated in the case of non-transferable utility (NTU) games. The main findings are that, under certain regularity conditions, if the core is large then it is a stable set and for symmetric NTU games the core is a stable set if and only if it is large. Although the stability and largeness of the core has been discussed in the case of an important class of non transferable utility (NTU) games, namely convex NTU games by Sharkey[42], Peleg[33], Ichiishi[15] a direct study of the relation between them has not been made so far. This is an attempt in that direction.

In section 5.2 definitions and preliminary results are mentioned. Section 5.3 presents some of the main results. The results pertaining to the complete characterisation of the stability of the core for symmetric games is obtained in section 5.4.

5.2 Definitions and Preliminary Remarks

Let N be a finite set of players of cardinality n. A coalition is a subset of N. For a set A we shall denote the cardinality of A by $|A|$. For $S \subseteq N$, $\mathbf{R}^S$ is the set of all functions from S to $\mathbf{R}$, the set of real numbers. We would think of elements of $\mathbf{R}^S$ as $|S|-$dimensional vectors whose coordinates are indexed by the members of S. We will begin with a few definitions in the NTU form.

Definition 5.2.1 *An n-person cooperative game without side payment or an NTU game is a pair (N, V) where V is the (characteristic) set function which assigns to every coalition $S \subseteq N$ a set $V(S)$ such that:*
(i) $V(\phi) = \phi$.
(ii) For all non-empty $S \subset N$, $V(S)$ is a proper subset of $\mathbf{R}^S$.
(iii) For all non-empty $S \subset N$, $V(S)$ is closed in $\mathbf{R}^S$.
(iv) For all non-empty $S \subseteq N$, $V(S)$ is comprehensive i.e. if $x \in V(S)$, and $y \in \mathbf{R}^S$ be such that $y \leq x$ then $y \in V(S)$.

For any vector $x \subset \mathbf{R}^N$ we shall denote the i-th component of it by x_i and the S coordinates of it (where $S \subseteq N$) by $x|_S$. For any two vectors $a, b \subset \mathbf{R}^S$ for some $S \subset N$, if $a_i > b_i$ for all $i \in S$ then we shall denote that as $a \gg b$. Often we shall suppress the player set N and denote the game by $V(.)$ itself. Call $\bar{V}(S)$ the boundary of $V(S)$, and $\dot{V}(S)$ the interior of $V(S)$. We assume henceforth that $V(.)$ satisfies the following regularity conditions (see e.g. Aumann[2], Scarf[38]).

C1(No Level Segment): If $x \in V(S)$ then $y \in \mathbf{R}^S$, $[y \leq x; y \neq x]$ implies that $y \in \dot{V}(S)$.

C2(Boundedness of Individually Rational Pay off Vectors): Let $b_j = max\{x | x \in V(\{j\})\}$. For all $S \subseteq N$, the set $\{x \in V(S) | x_j \geq b_j$ for all $j \in S\}$ is bounded.

Definition 5.2.2 *A vector $x \in \mathbf{R}^T$, $(T \subseteq N, T \neq \phi)$ is said to be blocked or dominated by a vector y if there is a coalition $S \subsetneq T$ such that $y_i > x_i$ for all $i \in S$ and $y|_S \in V(S)$. We indicate this domination relation as $y \succ_S x$, i.e. y dominates x via S.*

If vector y dominates a vector x via some coalition S then we shall denote that as $y \succ x$.

Definition 5.2.3 *The core of the game $V(.)$, denoted by $C(V) = \{x \in V(N) |$ there is no y such that $y \succ x\}$.*

Definition 5.2.4 *A set $K \subset V(N)$ is called externally stable if and only if for all $y \in V(N) \setminus K$ there exists $x \in K$ such that $x \succ y$.*

Definition 5.2.5 *A set $K \subseteq V(N)$ is called internally stable if and only if for any two $x, y \in K$ neither $y \succ x$ nor $x \succ y$.*

Definition 5.2.6 *A set $K \subseteq V(N)$ is called stable if and only if it is both externally and internally stable.*

Definition 5.2.7 *The acceptable set of vectors,* $A(V)$, $= \{y \in \mathbf{R}^N | y$ *is not dominated*$\}$.

Definition 5.2.8 *The lower boundary of* $A(V)$, $L(V) = \{x \in A(V) |$ *if* $y \in A(V)$ *and* $y \leq x$ *then* $y = x\}$.

Definition 5.2.9 *The core* $C(V)$ *is large if for all* $x \in A(V)$ *there exists* $y \in C(V)$ *such that* $y \leq x$ *(Cf. Ichiishi[15]).*

Below we make two preliminary observations.

Proposition 5.2.1 *The core,* $C(V)$, *is large if and only if* $L(V) = C(V)$.

Proof: *only if :* Let $C(V)$ be large and let there be $x \in L(V) \setminus C(V)$. Then there exists $y \in C(V)$ such that $y \leq x$, which implies $y = x$. Take $z \in C(V) \setminus L(v)$. Surely $z \in A(V)$. Suppose there is $y \leq z$, $(y \neq z)$ such that $y \in A(V)$. Then $y \in \dot{V}(N)$. This is a contradiction to $y \in A(V)$. So $z \in L(V)$.
if : Let $C(V) = L(V)$. Take $x \in A(V)$. If $x \in L(V)$ then there is nothing to prove. Otherwise consider the set $L_x = \{y \in A(V) | \ y \leq x\}$. It is easily seen that L_x is compact. Consider the map $s : \mathbf{R}^N \mapsto \mathbf{R}$ such that $s(x) = \sum_{i \in N} x_i$. Let $y \in L_x$ minimize $s(.)$ on L_x. Since $s(.)$ is continuous, such a y exists. Surely $y \in L(V)$. Since $C(V) = L(V)$, for all $x \in A(V)$ there is $y \in C(V)$ such that $y \leq x$. ■

Definition 5.2.10 *For any* $y \in \mathbf{R}^N$ *define* $\mathcal{S}_y = \{S \subseteq N | \ y|_S \in \bar{V}(S)\}$.

Proposition 5.2.2 *If $y \in A(V)$ then $y \subset L(V)$ if and only if $\bigcup_{S \in \mathcal{S}_y} S = N$.*

Proof: Let $y \in L(V)$ but $\cup_{S \in \mathcal{S}_y} S \neq N$. Then there is an i such that $y|_S \notin V(S)$ whenever $i \in S$. Then there is an $\epsilon > 0$ such that $y' = (y_1, \ldots, y_{i-1}, y_i - \epsilon, y_{i+1}, \ldots, y_n)$ is in $A(V)$. But then $y \notin L(V)$.
Let $y \in A(V)$, $\cup_{S \in \mathcal{S}_y} S = N$ but $y \notin L(V)$. Then there is a $x \leq y$ ($x \neq y$) such that $x \in A(V)$. This implies that for some $i \subset N$, $x_i < y_i$. Then by C1, there is $S \subseteq N$ such that $i \in S$ and $x|_S \in \mathring{V}(S)$. This contradicts that $x \in A(V)$. ■

5.3 Large Core and Stability

Definition 5.3.1 *For a game (N, V) a subgame of $V(.)$ on $T \subseteq N$ denoted by (T, V_T) is defined by for all $S \subset T$, $V_T(S) = V(S)$.*

It is easily verified that any subgame of a game is itself a game and if the original game satisfies C1 and C2, so does any subgame. For any coalition S we shall denote the core of (S, V_S) by $C(V_S)$.

Theorem 5.3.1 *If the core, $C(V)$, is large then the core is a stable set.*

Proof : Since $C(V)$ is always internally stable we are merely to show that it is externally stable as well. Take $x \in V(N) \setminus C(V)$. Then, without loss of generality (w.l.g. hereafter) there exists $S \subseteq N$ and $u \in C(V_S)$ such that $u \succ x|_S$ (see Ray[36]). By C2, for all $T \subseteq N$ such that $T \cap (N \setminus S) \neq \phi$ there is $M^T \in \mathbf{R}^T$; $M_i^T < \infty$, $\forall i \in T$, such that for all $z \in V(T)$ if $z_i > b_i$ for all $i \in T$ then $z < M^T$. For all $i \in N \setminus S$ define $M_i = Max_T\{M_i^T | i \in T \ \& \ T \cap (N \setminus S) \neq \phi\}$.

Construct $\bar{M} \in \mathbf{R}^{N \setminus S}$ as $(M_i)_{i \in N \setminus S}$. Evidently $(u; \bar{M}) \in A(V)$ and this implies that there exists $y \in C(V)$ such that $y \leq (u; \bar{M})$.

We claim that $u = y|_S$. Suppose not. Then w.l.g. there exists $i \in S$ such that $y_i < u_i$. However, by C1 $(u|_{S \setminus \{i\}}; y_i) \in \dot{V}(S)$.

But then, $y \notin C(V)$. Hence, we have $y \succ_S x$. ■

Definition 5.3.2 *An NTU game (N, V) satisfies Kikuta-Shapley Condition (Cf. Kikuta et al.[16]) if for every $S \subseteq N$, for any $x^S \in C(V_S)$ there exists $x \in C(V)$ such that $x|_S = x^S$.*

We get the following corollary from the proof of *Theorem 5.3.1*

Corollary 5.3.1 *If a game (N, V) has a large core then (N, V) satisfies Kikuta-Shapley Condition, and a game satisfying Kikuta-Shapley condition has a stable core.*

So, Large core $\Rightarrow$ K-S condition $\Rightarrow$ Stable core.

However, *Theorem 5.3.1* is no more valid if we consider general NTU games that do not satisfy C1. Consider, for example the following game:

Example 5.3.1 $N = \{1, 2, 3\}$. $V(N) = \{x \in R^3 | \ x_1 \leq 1, \ x_2 \leq 2, \ x_3 \leq 2\} \cup \{x \in R^3 | \ x_1 \leq 3, \ x_2 \leq 1, \ x_3 \leq 2\}$. $V(\{1, 2\}) =$

$\{x \in R^2 | \ x_1 \leq 3, \ x_2 \leq 2\}$, $V(\{1\}) = V(\{2\}) = \{x \in R | \ x \leq 1\}$, $V(\{3\}) = \{x \in R | \ x \leq 2\}$. *For all other non-empty* $S \subset N$, $V(S)$ $\{x \in R^S | \ x_i \leq 0$ *for all* $i \in S\}$.

This game satisfies C2 but fails to satisfy C1. It is easily seen that $(1,2,2)$ and $(3,1,2)$ are in $C(V)$. Here $C(V)$ is large. To see this consider $x \in A(V)$. Then $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 2$. Now consider the following cases:

a) Let $x_1 \geq 3$ then (i) if $x_2 \geq 2$ then $(1,2,2) \leq x$.

(ii) if $1 \leq x_2 < 2$ then $(3,1,2) \leq x$.

b) Let $1 \leq x_1 < 3$ then if $x_2 \geq 2$ and $(1,2,2) \leq x$.

However, to see that the core is not a stable set, consider the vector $(2,1,2)$ in $V(N)$.

Moreover, for an NTU game satisfying C1 and C2 the core may not be large even when it is a stable set as the following example illustrates. The idea of this example was motivated by a similar example in van Gellekom et al.[11] in the context of TU games;

Example 5.3.2 $N = \{1,\ldots,6\}$. $V(.)$ *is given as follows:*
$V(N) = \{x \in \mathbf{R}^N | \ \sum_{i \in N} x_i \leq 3\}$.
If $\{i,j\}$, $i,j \in N$ *is in* $\{\{1,2\}, \{1,3\}, \{4,5\}, \{4,6\}\}$ *then* $V(\{i,j\}) = \{x \in \mathbf{R}^{\{i,j\}} | \ x_i + x_j^2 = 1$ *if* $0 \leq x_i \leq 1$, $x_i + 2x_j = 2$ *if* $x_i \leq 0$ *and* $x_i + x_j = 1$ *if* $x_i \geq 1\}$ *and* $V(\{i,j\}) = \{x \in \mathbf{R}^{\{i,j\}} | \ x \leq y$ *for some* $y \in V(\{i,j\})\}$. *For all other* $S \subset N \setminus \phi$, $V(S) = \{x \in \mathbf{R}^S | \ \sum_{i \in S} x_i \leq 0\}$.

Notice that this NTU game satisfies C1 and C2. We show that for this game the core is a stable set.

To begin with, notice that $(1,0,0,1,1,0)$ is in $C(V)$ and so the core is non-empty. Consider a vector $y \in V(N) \setminus C(V)$. Then, by Ray (1989) there must exist a coalition $S \subseteq N$ and a vector $x \in C(V_S)$ such that $x \succ_S y$. If $S = N$ then we are done. Let, w.l.g., S be $\{1,2\}$ and let $x = (x_1, x_2)$ be in $C(V_{\{1,2\}})$ such that $x \succ_{\{1,2\}} y$. Then $x_1 \geq 0$, $x_2 \geq 0$. Construct the vector $z \in \mathbf{R}^N$ as $(x_1, x_2, x_3, 1, 0, 0)$ such that $x_1 + x_2 + x_3 = 2$ and $x_3 \geq x_2$. One can easily check that such a vector is obtainable. Then it is easy to see that $z \in C(V)$. If S is any of $\{1,3\}$, $\{4,5\}$, or $\{4,6\}$ then we can proceed in exactly similar manner to show that there is $z \in C(V)$ such that $z \succ y$. If S is any other coalition then for at least one $i \in N$, $y_i < 0$. Since $(0,1,1,1,0,0)$ and $(1,0,0,1,1,0)$ are in $C(V)$, for every $i \in N$ there is a vector z in $C(V)$ such that $z_i = 0$. Hence, $C(V)$ is a stable set.

However, the vector $c = (0,1,1,0,1,1)$ is in $A(V)$ but there cannot exist any vector $x \in C(V)$ such that $x \leq c$. Hence, the core of this game is not large.

However, as shown in the next section if a game is symmetric then the largeness of the core is necessary and sufficient for the core to be a stable set.

5.4 Core Stability in Symmetric Games

Definition 5.4.1 *A game (N, V) is called symmetric if for $S, T \subseteq N$, $x \in V(S)$ and $\pi : S \to T$ is a bijection, then $x^\pi \in \mathbf{R}^T$ defined by $x^\pi_j := x_{\pi^{-1}(j)}$ for $j \in T$ is an element of $V(T)$.*

We can suppress N, the player set and call $V(.)$ symmetric. Since we are concerned with symmetric games throughout this section, we now introduce some concepts and notations which will be useful for our discussions.

First we note that a set $W \subseteq \mathbf{R}^N$ is called *symmetric* if $x \in W$ implies that all n-dimensional vectors obtained from x by permuting its coordinates are also contained in W. Let $W_< = \{x \in W \mid x_1 \leq x_2 \leq \ldots \leq x_n\}$, and for any $x \in \mathbf{R}^N_<$, let $\pi(x)$ be the set of all n dimensional vectors obtained from x by permuting its coordinates. For any $W_< \subseteq \mathbf{R}^N_<$, let $\pi(W_<) = \cup_{x \in W_<} \pi(x)$.

Then we have $V(S) = \pi(V(S)_<)$ for all $S \subseteq N$ if $V(.)$ is symmetric. For a coalition S let us call $V(S)_<$ the *ordered set of attainable pay-offs* for S. Domination by a coalition is defined in the case of ordered sets of attainable pay-offs exactly in an analogous manner to that of the ordinary characteristic functions. A vector $x \in \mathbf{R}^T_<$, $(T \subseteq N, T \neq \phi)$ is said to be dominated by a vector $y \in \mathbf{R}^T_<$ if there is a coalition $S \subseteq T$ such that $y_i > x_i$, for all $i \in S$ and $y|_S \in V(S)_<$. From the above we can define the corresponding notions of $A(V)_<$, $L(V)_<$, $C(V)_<$ etc. In what follows, our discussions will remain confined exclusively to symmetric

games and hence to ordered subsets of $\mathbf{R}^T$ ($T \subseteq N$). So, to simplify notations, we will eliminate $<$ and use $V(S)$, $A(V)$, $C(V)$, $L(V)$ for $V(S)_<$, $A(V)_<$, $C(V)_<$, $L(V)_<$.
Moreover, since we are speaking of large core we implicitly assume that the games we are considering possess a non-empty core.

Lemma 5.4.1 *If $y \in L(V) \setminus C(V)$ then $y_{n-1} = y_n$, and so, in general, for some positive integer $k \leq n-1$, $y_1 \leq y_2 \leq \cdots \leq y_k = y_{k+1} = \cdots = y_n$.*

Proof: Suppose not, i.e. let $y_{n-1} < y_n$. Since $y \notin C(V), y \notin V(N)$. Construct $y' = (y_1, \ldots, y_{n-1}, y_n - \epsilon)$ where $\epsilon > 0$ is small enough so that $y' \notin \dot{V}(N)$ and $y_n - \epsilon > y_{n-1}$(since $V(N)$ is closed such an ϵ is obtainable). If $y' \in L(V)$ then $y \notin L(V)$. On the other hand, if for some $S \subseteq N$, $y'|_S \in \dot{V}(S)$ then, by symmetry, $y \notin A(V)$. ■

Lemma 5.4.2 *For all $y \in L(V) \setminus C(V)$ there exist $y' \in \dot{V}(N)$ and $0 \leq l < k-1$ such that $y' \leq y$, in particular, $y'_j = y_j \ \ \forall j \leq l$, and $y'_j < y_j \ \ \forall j > l$, moreover, $y'_j = y'_{j+1}$ for $j = l+1, \ldots, n-1$ where k is as in Lemma 5.4.1.*

Proof : Since $V(.)$ is symmetric, for all $i \in N$, b_i has a common value which let us call b. Define the continuous function $x : [b, y_n] \to \mathbf{R}^N$ as : for $t \in [b, y_n]$,

$$x(t)_i = \begin{cases} y_j & \text{if } y_j \leq t \\ t & \text{if } y_j > t \end{cases} \tag{5.1}$$

So, $x(b) = b$. Consider $T_x = \{t \in [b, y_n] \mid x(t) \in V(N)\}$. Since $V(N)$ is closed and $x(.)$ is continuous, T_x is compact. Moreover, since $C(V) \neq \phi$ by assumption, $b\ (x(b)) \in V(N)$. So, there is a $\bar{t} \in T_x$ such that $\bar{t} = max\{t \in [b, y_n] \mid x(t) \in V(N)\}$. Since $x(y_n)(= y)$ is not in $V(N)$, $\bar{t} < y_n((= y_{n-1}, y_{n-2}, \ldots, y_k)$, where k is as in *Lemma 5.4.1*. By definition of $\bar{t}$, $x(\bar{t}) \in \bar{V}(N)$. Now take $y' = x(\bar{t})$. ∎

Theorem 5.4.1 *If* (N, V) *is symmetric then* $C(V)$ *is a stable set if and only if* $C(V)$ *is large.*

Proof: *only if :*
Take, if possible, $y \in L(V) \setminus C(V)$. By *Lemma 5.4.1* y is of the form $y_1 \leq y_2 \leq \cdots \leq y_k = \cdots = y_n$ for some positive integer $k \leq n - 1$.
Surely $y \notin V(N)$ as $y \notin C(V)$. Appeal to *Lemma 5.4.2* and consider $y' \leq y$ and $y' \in \bar{V}(N)$ such that
$y'_j = y_j \quad \forall j \leq l,$
$y'_j < y_j \quad \forall j > l,$
$y'_j = y'_{j+1} \quad \forall j \geq l + 1$
where l is as in *Lemma 5.4.2*. Also suppose $l \neq 0$. If $y' \in C(V)$ then $y \notin L(V)$. So, $y' \in V(N) \setminus C(V)$.
Let S be a minimal coalition with respect to cardinality such that there exists $z \in C(V)$ such that $z \succ_S y'$. This is possible because of the stability of the core. Since $y \in A(V)$, $|S|$ must be greater than l.
We claim that $(z_1, \ldots, z_{|S|}) \gg (y'_1, \ldots, y'_{|S|})$.
Suppose not. Then for some $i_1, \ldots i_{|S|}$; where $i_k \in N$ for $k = 1, \ldots, |S|$

and $i_1 < i_2 < \ldots < i_{|S|}$, $\bar{z}(= (z_{i_1}, \ldots z_{i_{|S|}})) \gg (y'_{i_1}, \ldots y'_{i_{|S|}})$ and for all $j \notin \{i_1, i_2, \ldots, i_{|S|}\}$, $j < i_{|S|} \Rightarrow z_j \leq y'_j$. Then for all $k \in \{1, \ldots |S|\}$, $i_k \geq k$ and there is a $\bar{k} \in \{1, \ldots, |S|\}$ such that for all $k \in \{\bar{k}, \ldots |S|\}$, $i_k > k$. Since the vectors y' and z are ordered, for all $k \in \{\bar{k}, \ldots, |S|\}$, $z_{i_k} > y'_{i_k} \geq y_k \geq z_k$. Since $z \in V(S)$ and $(z_1, \ldots, z_{|S|}) \leq z$ with $z_{i_k} > z_k$ for at least one $k \in \{1, \ldots |S|\}$ by C1, $(z_1, \ldots, z_{|S|}) \in \mathring{V}(S)$. But then $z \notin C(V)$. So, $(z_1, \ldots, z_{|S|}) \gg (y'_1, \ldots, y'_{|S|})$.

From *Lemma 5.4.2* we know that for all $i \geq l+1$, $y'_i = y'_{i+1}$. Since z is ordered and $|S|$ is at least $l+1$, $z_j > y'_j$ for all $j \geq |S|+1$.

So, $z_i > y'_i$ for all $i \in N$. As $z \in V(N)$, $y' \notin \mathring{V}(N)$ by C1 and we get a contradiction. If $l = 0$ then $|S| \geq 2$ and similar arguments go through.

if: This is shown in *Theorem 5.3.1*. ■

5.5 Conclusion

Obviously the above is no complete characterisation of the stability of the core for NTU games. One direction in which a complete characterisation may be possible is some modification of the Kikuta-Shapley condition mentioned earlier. To get the relation between large and stable core with assumptions weaker than the assumption C1 in this work can also be pursued.

Chapter 6

Further Remarks and Open Problems

There are few problems that arose in course of the studies which resulted in the previous chapters, which are worth mentioning and pursuing. I present them in the following three sections.

6.1 Concept of 'exstability'

With the Lucas 10-person game without any solution, search for coincidence of other known solution concept with the vN-M solution and necessary and sufficient condition for such coincidence gained importance. In this section we deal with several suficient conditions for core stability and introduce a concept of extended stability of the core and show it to be stronger than stability of the core, we also present a condition in the last section which is far less than any known sufficient conditions and prove its necessity. Finally in that section we conjecture the sufficiency of the same condition.

6.1.1 Introduction

Ever since von Neumann-Morgenstern defined the concept of stable sets (solution), researchers wanted to know whether every TU game has a solution. Contrary to widespread belief Lucas in 1968 [20] came up with a 10 person game which did not have a solution. Attention shifted to find out what other solution concept may coincide with stable set solution when it existed. Core was one of the well known solution concepts which becomes a unique solution when it coincides with vN-M solution. Since then there were attempts to know necessary and sufficient conditions for core solutions. Over the years nubmer of sufficcient conditions could be evolved, some necessary conditions also became known. Unfortunately none of these conditions were both necessary and sufficient. Kikuta and Shapley[17] were possibly the closest, but still far too strong for the necessity. In the mean time in 1982 Sharkey[43] evolved a sufficient condition (called large core) which was very close to the Kikuta-Shapley condition, but unfortunately the K-S condition did not prove to be sufficient for core largeness. Biswas *et al.*[7] generalized the K-S conditions and proved the equivalence of a conditions named '*extendability in a stronger sence*' to Core Largeness. In this chapter we look at restrictions of the K-S conditions and prove the properties of restricted Kikuta Shapley conditions.

The very concept of domination is also extended beyond the set of imputations. Hence postulate that under certain conditions core coincides with the solution to the economy in which the players play the game. The core is also stable with respect to lure of money from outside the economy. That is, as long as an allocation fails to satisfy the acceptability conditions it can always be dominated in the extended sence by a core allocation.

6.1.2 Preliminaries

We extend the concept of domination beyond the set of imputations.

Definition 6.1.1 *Given a game (N, v), and two vectors x, and y both in $\mathbf{R}^n$, we would say that x dominates y in extended sence [written : $x \succ_{ex} y$ and called* **'exdominate'**] *if there is a S : $1 \neq |S| \neq n$, $x_i > y_i \ \forall i \in S$ and $x(S) \leq v(S)$.*

Definition 6.1.2 *The set of all* **Upper Vectors** *$U(v)$ is defined by $U(v) = \{x \subset \mathbf{R}^n : x(S) \geq v(S) \ \forall S \subset N\}$.*

Definition 6.1.3 *A stable set defined through this extended domination [exdomination] would be called an extended stable [exstable] set for the game.*

Remark 6.1.1 *The set of all upper vectors is an exstable set.*

Definition 6.1.4 *A game (N, v) is said to satisfy the Kikuta-Shapley condition if $C(v)|_S = C(v|_S)$ for all $S \subset N$.*

Given any vector $x \notin U(v)$ there is a vector y in $U(v)$ with a suitable value of $y(N)$ which will exdominate x, so the set of all upper vectors is allways a exstable set for the game (N, v). Lower bounds on the value of $y(N)$ can be given through the definition of subconvexity (defined below) and also through the definition of large core. However, the minimum of these lower bounds could be lower than either of the two mentioned above, as is explained through the examples.

Definition 6.1.5 (Sharkey (1982)) *Let* $\mathbf{P} := \{P_1, \ldots, P_k\}$ *be an arbitrary partition of* N *and let* $\mathbf{Q} := \{Q_1, \ldots, Q_k\}$ *be a collection of coalitions such that* $Q_i \subseteq \cup_{j=0}^{i-1} P_j$ *and* $P_k \cup Q_k \neq N$, *where* $P_0 = \phi$. *The TU-game* (N, v) *is said to be subconvex if for all such collections* $\mathbf{P}$ *and* $\mathbf{Q}$ *it is true that* $\sum_{i=1}^{k} [v(P_i \cup Q_i) - v(Q_i)] \leq v(N)$

It is clear that if $v(N)$ is equal or larger than the maximum over all $\mathbf{P}$ and $\mathbf{Q}$ of the left hand side of the last inequality then v would be subconvex. In the following we explain subconveity in a simplified manner.

Definition 6.1.6 *Let* (N, v) *be a TU-game and let* π *be an enumeration of* N. *We define the vector* $y^\pi \subset \mathbf{R}^n$ *by*
$y^\pi := max\{v(Q \cup i) - v(Q) | Q \subset P_i^\pi,\ Q \cup i \neq N\}$.

Where P_i^π is the set of all predecessors of i in π and is defined by $P_i^\pi := \{j | \pi(j) < \pi(i)\}$.

y_i^π is the maximum marginal worth of player i with respect to a subset of his predecessors in any permutation π and the reader is referred to van Gelekom et al[11] for a proof of the fact that '(N, v) is subconvex if and only if $y^\pi(N) \leq v(N)$ for all permutation π of N'.

Example 6.1.1 (Lucas(1969),cf.Sharkey(1982)) *Consider the TU-game*
(N, v) with $n = 10$ and coalition values $v(S) = 0$ except for the following coalitions :

$v(12) = v(34) = v(56) = v(78) = v(90) = 1,$

$v(137) = v(139) = v(157) = v(159) = v(357) = v(359) = 2,$

$v(1479) = v(2579) = v(3679) = 2,$

$v(1379) = v(1579) = v(3579) = 3,$

$v(13579) = 4.$

[Numeral 0 in the coalitions stand for player 10]

If $v(N) = 11$ then the above game has a Large core and hence a stable core, and it can be proved that for any value of $v(N) < 11$ the core of the corresponding TU game is not stable. However, for the game to be subconvex $v(N)$ have to be at least 16.

To get further insight into a few other aspects of the stability, largeness let us define a restriction of the Kikuta Shapley condition.

Definition 6.1.7 *In a TU-game (N, v) for all the subgames $(S, v_{|S})$ which has a core with a non-empty interior, if all such elements $x_{|S} \in C^0(v_{|S})$ is extendable to the core of (N, v) then we call the game v to have Restricted Kikuta Shapley[RKS] condition.*

It is easy to see that even this condition is sufficient for the stability of the core of v.

Example 6.1.2 *$|N| = 7$, $v(N) = 7$, $v\{1,7\} = v\{4,7\} = 2$ $v\{1,2,7\} = v\{1,3,7\} = v\{4,5,7\} = v\{4,6,7\} = 3$. All other $v(S)$'s are defined assuming (N, v) is totally balanced.*

This example above satisfies the Kikuta-Shapley conditions. Yet the vector $y = (2, 1, 1, 2, 1, 1, 0)$ is a lower boundary point with $y(N) = 8 > 7 = v(N)$ so $C(v)$ is not large, however the core is *exstable*. In fact the following statement is easy to prove.

If a game (N, v) satisfies Kikuta-Shapley condition then the core is exstable.

Example 6.1.3 *Consider the following incomplete game (N, v°), which has different characteristics with varrying values of $v(N)$. The game has a non-empty core if $v(N) \geq 2$. The core is stable if $v(N) \geq 4$, the core becomes exact if $v(N) \geq 4.6$ and finally the core is large if $v(N) \geq 4.8$. Also core is exstable at this value of $v(N)$.*

$|N| = 7$, $v\{1,2\} = v\{1,3\} = v\{4,5\} = v\{4,6\} = v\{4,7\} = 1$, $v\{1,4\} = 0.2$ *All other $v(S)$'s are derived from total balance consideration.*

Let $y = (0.2, 0.8, 0.8, 0, 1, 1, 1) \in L(v) \setminus C(v)$. Then $y(N)$ has the maximum sum among elements in $L(v) \setminus C(v)$.

Clearly $y = (0, 1, 1, 0.2, 0.8, 0.8, 0.8) \in C(v)$ when $v(N) = 4.6$ such that $y(\{1,4\}) = 0.2$

When $v(N) = 4$ consider the subgame on $S = \{1, 4\}$ and $x|_S = (0, .2)$ this subgame core element is not extendable to $C(v)$. This subgame core also has a non-empty interior. So the game does not satisfy either the K S conditions or the RKS conditions, yet the core is stable but not exstable.

Example 6.1.4 *Consider the following 6-player game with a stable core.* $|N| = 6$, $v(N) = 3$, $v\{1,2\} = v\{1,3\} = v\{4,5\} = v\{4,6\} = 1$ *All other* $v(S)$*'s can be evaluated from total balance consideration.*

This game does not satisfy Kikuta-Shapley condition but does satisfy restricted Kikuta-Shapley condition and has an exstable core.

6.1.3 Main results

Proposition 6.1.1 $L(v)$ *is exstable for the game* (N, v).

Proposition 6.1.2 *If* (N, v) *is subconvex then* $C(v)$ *is large and then* $C(v)$ *is exstable for the game* (N, v).

Proposition 6.1.3 *If* (N, v) *has Restricted Kikuta-Shapley condition then* $C(v)$ *is exstable for the game* (N, v).

Definition 6.1.8 $C^S = \{x \in I(v) : x(T) \geq v(T), \forall T \neq S\}$.

Observe that $C^S \supseteq C(v)$,

if we define $m(S) = Min_{y \in C^S} y(S)$, then C^S is the core of the game v^S, where $v^S(S) = m(S)$ and $v^S(T) = v(T), \forall T \neq S$.

Also $C^S \setminus C(v) \neq \phi$ if and only if $m(S) < v(S)$.

Remark 6.1.2 *If for every $y|_S \subset C(v|_S)$ which is extendable to the core $C(v)$ there exists $\epsilon_{s \times 1} > 0$ (componentwise), $\sum_{i \in S} \epsilon_i = v(S) - m(S)$ and $z|_{N \setminus S}$ such that $(y|_S + \epsilon : z|_{N \setminus S}) \in C(V)$ then the core $C(v)$ of the TU game (N, v) is stable.*

Let us look at the following statements :

1. $C(v)$ is exstable.

2. For all $S \subset N$ such that $C^o(v|_S) \neq \phi$, $C(v|_S) = C(v)|_S$ [RKS Condition]

3. Given a TU game (N, v) define another game (N, w) as follows :
   $w(S) = v(S) \quad \forall S \subset N$
   $w(N) = v(N) + \epsilon \quad \epsilon > 0$
   Then $C(w)$ is stable $\forall \ \epsilon > 0$

4. $x \in I(v) \setminus C(v)$ there exists $y^S \in C(v)$ for every minimal $S \ni x(S) < v(S)$ and $y^S \succ_S x$

Proof : (1) $\Rightarrow$ (3).
$C(v)$ exstable $\Rightarrow$ $C(w)$ is stable $\forall \ \epsilon > 0$

Take $x \in I \setminus C(w) \Rightarrow x \in \mathbf{R}^n \setminus U(v)$ So $\exists\ y \subset C(v)\ \ni y \succ_{ex} x$ via S

Define z as follows :

$z_i = y_i \quad \forall\ i \in S$ and

$z|_{N \setminus S} \geq y \quad \ni z(N \setminus S) = y(N \setminus S) + \epsilon$.

Then $z \in C(w)$ and $z \succ_S x \Rightarrow C(w)$ is stable.

Proof : (4) $\Rightarrow$ (3).

If $\forall\ x \subset I(v) \setminus C(v)$ there exists $y^S \in C(v)$ for every $S \ \ni x(S) < v(S)$ and $y^S \succ_S x$ then $C(v)$ exstable $\Rightarrow C(w)$ is stable $\forall\ \epsilon > 0$

Take $x \subset \mathbf{R}^n \setminus U(v)$ chose and fix $S \subset N$ such that $x(S) < v(S)$ and $\forall T \subset S \ x(T) \geq v(T)$.

Extend $x|_S$ to an imputation $\bar{x}$ of (N, v), by the given condition $\exists z \in C(v) \ \ni z \succ_S \bar{x} \Rightarrow z \succ_{ex} x$ via S

Proof : (2) $\Rightarrow$ (4).

If $\forall\ S \subset N \ \ni intC(v|_S) \neq \phi\ C(v|_S) = C(v)|_S$ [Restricted S-K Condition] then $\forall\ x \in I(v) \setminus C(v)$ there exists $y^S \in C(v)$ for every minimal $S \ \ni x(S) < v(S)$ and $y^S \succ_S x$

Take $x \in I(v) \setminus C(v)$ and let S be a minimal set such that $x(S) < v(S)$

Define $y_i = x_i + \epsilon \quad \forall\ i \in S$ such that $y(S) = v(S)$

Then $y|_S \in intC(v|_S)$ so by Restricted S K condition there exists an extension of $y|_S$ to $C(v)$, call it y, then $y \succ_S x$

Proof : (4) $\Rightarrow$ (1).

Consider, $y \notin A(v)$, there is a coalition S such that $y(S) < v(S)$ and for all $T \subset S$ $y(T) \geq v(T)$. One can construct an imputation x with $x_i = y_i$ for all $i \in S$. Now if we invoke (4) there exists a $z \in C(v)$, $z \succ_S x$ and hence $z \succ_S y$.

So now we have the following :

(2) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (3)

Conjecture 6.1.1 (3) $\Rightarrow$ (2).

6.2 Specified vectors to check large core

In this section we explore the possibility of finding methods of checking whether the core of a game is large or not through systematic computations, as we have done in the symmetric case in Chapter 2.

First we look at an LP formulation which yields $L(v)$ vectors.

$y \in L(v)$ is characterised by the following properties.

(1) $y(S) \geq v(S), \quad \forall S \subseteq N$

(2) $\cup_{S \in \mathcal{S}_y} S = N$

where $\mathcal{S}_y = \{S : y(S) = v(S)\}$. So for all $y \in L(v)$ there exists at least one $S \subseteq N$ such that $y(S) = v(S)$ and $y(T) \geq v(T) \quad \forall T \subseteq S$ i.e. $y_{|S} \in C(v_{|S})$

Given $x \in C(v_{|S})$ if $N \setminus S \neq \phi$ then consider the following LP.

Note :Minimisation is imposed by the condition (2) above.

Minimise $y(N \setminus S)$

The problem is feasible because A, the set of all acceptable vectors of the game, $A = \{y \ : \ y(S) \geq v(S), \ \ S \subset N\} \neq \phi$

Subject to

$x(T \cap S) + y(T \setminus S) \geq v(S)$

$\forall T \subseteq N, \ \ T \setminus S \neq \phi$

The above constraints are due to the rest of the conditions in (1) which $(x : y_{N \setminus S})$ should satisfy by virtue of being a $L(v)$ element.

Recall definition of $\mathcal{S}_y = \{S : y(S) = v(S)\}$, $\mathcal{S}_y$ defines a $L(v)$ vector whenever $\cup_{S \in \mathcal{S}_y} S = N$

Let (N, v) be a totally balanced TU-game and let $\pi \ : \ N \to \{1, \ldots, n\}$ be a bijective map; π is called an *enumeration* of N. The set of *predecessors* of $i \in N$ is defined by $P_i^\pi := \{j | \pi(j) < \pi(i)\}$. Now define the vector $y^\pi \in \mathbf{R}^N$ by

$$y_i^\pi = Max\{v(Q \cup i) - y^\pi(Q) | Q \subseteq P_i^\pi, \ Q \cup i \neq N\}.$$

Note that $Q \cup i \neq N$ is only restrictive if $\pi(i) = n$.

For every subgame $(S, v|_S)$ such that $|C(v|_S)| = 1$ let $x^S \in \mathbf{R}^S$ be the subgame core element, let $\pi(N \setminus S)$ be an *enumeration* of $N \setminus S$. The set of *predecessors* of $i \in N \setminus S$ is defined similar to above by $P_i^\pi := \{j | \pi(j) < \pi(i)\}$. Now define the vector $x^{N \setminus S} \in \mathbf{R}^{N \setminus S}$ as follows; for convenience of writing we call the vector $(x^S : x^{N \setminus S}) = x^S$. It is easy to see that $x^S \in \mathbf{R}^N$, and for each singleton subgame core there is an x^S. In the following we supress the superscript S in the vector x.

$$x_i^\pi = Max\{v(S \cup Q \cup i) - x^\pi(S \cup Q) | Q \subseteq P_i^\pi, \ S \cup Q \cup i \neq N\}.$$

Note that $Q \cup i \neq N$ is only restrictive if $\pi(i) = n$.

Conjecture 6.2.1 *Core of a TU-game (N, v) is large if and only if*

$$v(N) \geq \begin{cases} y^\pi(N) & for\ all\ \pi \\ x^S(N) & for\ all\ S \subset N, \\ & with\ |C(v|_S)| = 1\, and\ for\ all\ \ \pi. \end{cases}$$

6.3 NASC for Core stability

In this section we present some exploratory understanding to find a necessary and sufficient condition for the core of a game to be a stable set. Given a totally balanced TU game (N, v) let us define the following sets apart from the usual sets like $I(v)$, $C(v)$ etc.:

Definition 6.3.1 $C^S = \{x \in I(v) : x(T) \geq v(T), \forall T \neq S\}$.

Observe that $C^S \supseteq C(v)$,
if we define $m(S) = Min_{y \in C^S} y(S)$, then C^S is the core of the game v^S, where $v^S(S) = m(S)$ and $v^S(T) = v(T), \forall T \neq S$.

Also $C^S \setminus C(v) \neq \phi$ if and only if $m(S) < v(S)$.

Definition 6.3.2 $C_o^S = \{x \in I(v) : x(S) < v(S), \phi \neq S \neq N\}$.

Also $C_o^S \subseteq I(v)\setminus C(v)$ and $\cup_{S\in 2^N} C_o^S = I(v)\setminus C(v)$ and $C^S\setminus C(v) \subseteq C_0^S$.

Definition 6.3.3 $C_{ext}^S = \{y|_S : z|_{N\setminus S} \in I(v) : y \in C^S \setminus C(v)\}$.

This set is defined for all S such that $C^S \setminus C(v) \neq \phi$. Next $C_{ext}^S \subseteq C_o^S$ and there can be cases when this inclusion is strict. If $C(v)$ is stable and $C^S \setminus C(v) \neq \phi$ then for $x \in C^S$ there exists $y \in C(v) : y \succ x$ via S.

The question now is the following :

Can it still be proved that :

$\cup_{S\in 2^N} C_o^S = \cup_{S\in 2^N} C_{ext}^S$

If the answer to the above question is 'YES' then we will have the following result :

Conjecture 6.3.1 *The core $C(v)$ of the TU game (N,v) is stable if and only if for every $y|_S \in C(v^S|_S)$ which is extendable to the core $C(v^S)$ there exists $\epsilon_{s\times 1} > 0$(componentwise),$\epsilon(S) = v(S) - m(S)$ and $z|_{N\setminus S}$ such that $(y|_S + \epsilon : z|_{N\setminus S}) \in C(V)$*

The *if part* is true and easy to prove.

In the example below we depict how the above may actually take place.

Example 6.3.1 *Consider $|N| = 7$ and define a TU game (N,v) as follows :* $v(12) = v(13) = v(45) = v(46) = v(47) = 1$, $v(14) = 0.6$, $v(N) = 3.9$

The above game has a stable core.

If $S = \{1, 4\}$ then $m(S) = 0.55$ and $(0, 0.55)$ is the only subgame core element which is extendable to the core of v^S,

but note that any convex combination x of $(0, 0.5)$ and $(0.1, 0.4)$ is such that $x + \epsilon$ is extendable to $C(v)$ as long as $\epsilon_1 + \epsilon_4 = 0.1$ and $\epsilon > 0$.

In other words the part of $C(v|_S)$ which is extendable to $C(v)$ is :

$CH\{(0, 0.5),\ (0.1, 0.4)\} + \epsilon$.

Observe that there are elements in the above convex hull below which there are no elements $y|_S$ such that $y \in C^S \setminus C(v)$. It is conjectured that such elements must figure in $DomC_{viaT}$, $T \neq S$ and would be covered under C^T_{ext}.

(.1, .90, .90, .4, .6, .5, .5) will occur in C^{46}_{ext}, and C^{47}_{ext}

However question remain : Whether it is possible that $y \in C^S$ and there is no $\epsilon_s > 0, \epsilon_s(S) = v(S) - y(S)$ such that $y|_S + \epsilon$ is extendable to $C(v)$?

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