SOME STUDIES ON UNCERTAINTY MANAGEMENT IN DYNAMICAL SYSTEMS USING FUZZY TECHNIQUES WITH APPLICATIONS

Kausik Kumar Majumdar
Electronics and Communication Sciences Unit
Indian Statistical Institute
203, Barrackpore Trunk Road
Calcutta - 700 108, India

Thesis submitted to the Indian Statistical Institute for partial fulfillment of the requirements for the degree of Doctor of Philosophy

March 2003

To

Late Prof. Jatindra Nath Mitra

A birth centenary gift

And

To

Late Nirmala Mitra

Abstract

Uncertain information processing by fuzzy if-then rules has received a lot of attention. Here we have taken a different path to model a system, about which we do not have precise information namely, modelling the system by fuzzy valued functions without resorting to fuzzy if-then rules. As a result, the phase (state) space of the system becomes a fuzzy set and the underlying fuzzy mapping becomes a fuzzy attainability set mapping.

A fuzzy phase space is a collection of special class of fuzzy subsets (fuzzy points) of R^n for some positive integral n. Let the collection of all fuzzy real numbers be $\Re$. A relationship of fuzzy phase spaces has been established with $\Re^n$. A field like structure for $\Re$ (it has important similarities and differences with the classical fields) has been developed and a vector space like structure (it has important similarities and differences with the classical vector spaces) has been developed for $\Re^n$. A metric has been defined on $\Re^n$. It has been proved that any arbitrary fuzzy subset of R^n can be generated by a suitable collection of members of $\Re^n$ by max-min operations. Some possible role of $\Re^n$ in pattern recognition has been explored.

Uncertain or fuzzy dissipative dynamical systems have been defined in terms of fuzzy attainability set mappings. A criterion for determining dissipativeness of a dynamical system has been formulated. Attractor, stability, robustness, predictability, homoclinicity etc., for fuzzy dynamical systems have been defined. Devaney's definition of chaos has been extended to fuzzy dynamical systems. Fuzzy differentiable dynamical systems have been discussed with a particular emphasis on fuzzy differential inclusion (FDI) relations. An evolutionary algorithm for solving one dimensional FDIs has been developed. Using this algorithm a second order FDI relation has been solved.

Fuzzy fractals have been defined as attractors of contractive iterated fuzzy sets systems, which is a class of discrete fuzzy dissipative dynamical systems. Fuzzy fractals are so defined that, crisp fractals obtained as attractors of Barnsley's chaos games become special cases of fuzzy fractals. Fuzzy fractal based gery level image generation has been discussed. Using this method some fuzzy fractal images have been generated. Model of a simple two dimensional turbulence, as a chaotic occurrence of vortices in a two dimensional dynamic fluid, has been proposed and simulated. Each coordinate of the position of the centre of a vortex in the dynamic fluid plain is given by a chaotic value obtained by an iterated logistic function system for an appropriate parameter value. Each vortex is modelled by a fuzzy valued function, where uncertain parameter and variable values are fuzzy numbers. This is obtained by solving an FDI with the algorithm developed. The simulated turbulence thus obtained has been presented.

The starting stage of an intense tropical storm is a system about which so far very little precise information is available. It is recognized that a sufficiently strong initial disturbance must be present to give rise to a tropical storm of hurricane intensity under favourable climatic and geographical conditions. A model of this initial disturbance in the form of a vortex, created by winds coming from different directions and colliding under certain conditions, has been proposed in terms of FDIs. It has been simulated by the evolutionary algorithm developed earlier in the thesis. Under favourable conditions this initial vortex may become a severe cyclone.

The Lefever and Horsthemke model has been considered for the highly uncertain system of evolution of tumour in a human body. Some dynamical system theoretic properties of the model has been studied by extensive simulation. Comparing these studies with available data the ranges of the uncertain parameters have been estimated. This in turn has helped us to introduce suitable fuzzy numbers to model these uncertain parameters. Thus an FDI relation model of the whole system has been created, where the FDI relation substitutes the original stochastic differential equation to model the uncertain system. This has been justified from the therauptic point of view. The simulation is achieved by the same algorithm mentioned above. Analyzing the simulation results, modification in some of the therauptic planning for a tumour patient has been suggested.

The thesis is concluded with a summary and a brief discussion on the future scope of research.

Acknowledgement

My foremost acknowledgement goes to an unknown little girl of about four or five years of age. She lost her home, her kin and everything else in the devastating cyclone of Orissa on 29th October, 1999. She was spotted by an Indian Air Force pilot sitting among a few strewn cattle corpses on patch of a little raised land surrounded by water. It was her plight which compelled me to undertake some research on cyclone. Though it did not constitute major part of my thesis yet the Chapter 5 was borne out this way.

I am thankful to my supervisor Prof. Dwijesh Dutta Majumder in more ways than one. First, he stood by me at a very critical moment of my career, when I had to change my earlier Ph.D. supervisor, to continue my research under him. Second, it was he who persuaded me to explore the exciting world of cybernetics in general and bio-cybernetics in particular. The warmth of his personality and his vast experience are some of the finest and rarest qualities, I am very much fortunate to have in my supervisor. He has taught me how to orient the research and present it in a professional manner in today's competitive world, which is very vital for success.

I am thankful to Dr. Dipti Prasad Mukherjee for being ever ready to lend me a helping hand where ever and whenever I needed. I am thankful to Mr. Partha Pratim Mohanta, Mr. Narayan Chandra De and Mr. Shyam Shaw for always being nice, cooperative and helpful. I am thankful to Prof. J. Das and Prof. N. R. Pal for encouragement and cooperation. Dr. K. Krishna Kumar, Dr. C. A. Murthy, Prof. Pratul Bandyopadhyay, Prof. B. V. Rao and Dr. Subir Kumar Bhandari were always ready to discuss any point with me, whenever I needed. They meticulously checked some of my manuscripts and made very valuable comments and suggestions despite their tight schedules. I am specially thankful to Prof. H. P. Majumdar for giving me a very crucial support at the time of a dire need, when I had to change my supervisor. I would like to thank Dr. Eyke Hullermeier for some enlightening discussions over e-mails. I am very much grateful to Prof. Didier Dubois for his invaluable advice from time to time. I am also grateful to Prof. Phil Diamond and Prof. Osmo Kaleva, for discussions, suggestions, criticisms and advice. I shall always remember Fer D. Mesman, a former editorial staff of Fuzzy Sets and Systems, for readily giving me important e-mail addresses. I benefited a lot from interactions with Prof. Jyoti Das and Prof. Mihir Kumar Chakrabarty of Department of Pure Mathematics, University of Calcutta. I would like to thank L. V. Fausett for writing [49], which was of great help during simulation works. Dr. Sitabhra Sinha and Mr. Pawan Kumar Singal have helped me by bringing two very important papers from the Indian Institute of Science library at Bangalore. I would like to acknowledge Mr. Arindam Ghosh for giving me two very important papers of Emanuel. I have received many helpful advice and cooperation from Mr. Pabitra Mitra during submission of this thesis.

The Indian Statistical Institute library and the reprography unit under it had been an integral part of my research activity throughout these years. All the staffs of this division have always remained overwhelmingly cooperative to me. The former Chief Librarian Mr. Chitta Bhattacharya was very sensitive to the needs of the library users. He used to instantly attend and solve my problems each time I approached him. All other staffs were also equally responsive.

The invaluable service provided by Mr. Duryadhan Mahankur is deeply acknowledged. His careful, sincere and reliable services lay everywhere, from searching out and bringing down journals during the whole period of my Ph.D. work to standing patiently before the printer and carefully taking the print out page by page. He even worked as an important link between me and my guide.

Generally speaking, I am grateful to all the scientific, non-scientific workers and students of the Indian Statistical Institute for being so nice to me during my stay here for all these years.

Contents

1. Introduction **1**

1.1 General Dynamical Systems and Cybernetics 6

1.2 Uncertain Systems 7

1.3 Dealing with Complexity 8

1.4 Chaos 9

1.5 Scope of the Thesis 11

2. Uncertainty Management in Phase Spaces **20**

2.1 Fuzzy Phase Space Preliminaries 21

2.2 Fuzzy Real Number System $\Re$ 22

2.3 The Space $\Re^n$ 28

2.3.1 The Algebra of $\Re^n$ 28

2.3.2 A Metric on $\Re^n$ 31

2.4 $\Re^n$ as Generator of $F(R^n)$ 32

2.5 Some Applications of $\Re^n$ in Uncertainty Management in Pattern Recognition Systems 35

2.6 Discussion 39

3. Uncertainty Management in Time-Evolution Laws **41**

3.1 Some Extension of Fuzzy Dynamical Systems 42

3.2 Fuzzy Differentiable Dynamical Systems 52

3.3 Discussion 60

4. Application of FDS in Fuzzy Fractal Based Image Generation and Modelling of Turbulence **63**

4.1 Fuzzy Fractals and Image Generation 63

4.2 FDS Modelling of a Turbulence 70

4.3 Discussion 79

5. Modelling and Simulation of Cyclogenesis by FDI **80**

5.1 Modelling of a Climatic Disturbance 80

5.2 Simulation 85

5.3 Discussion 90

6. FDI Approach in Modelling the Evolution of Tumour **91**

6.1 The Model 91

6.2 A Formulation in Terms of FDI 95

6.3 Discussion 102

7. Conclusion **104**

7.1 Summary of Main Points of the Thesis 104

7.2 Future Directions 107

Appendix A **110**

An Interpretation of Time **110**

A.1 Formalization of Time 111

A.2 Main Results 114

Appendix B **116**

An Elaboration of Mandelbrot's Definition for Fractal Sets 116

Appendix C 121

Proof of Chaoticity of Logistic Function 121

Bibliography 126

Papers of the Author 135

List of Figures

2.2.1 Trapezoidal fuzzy numbers M and -M. 25

2.3.1 A two dimensional generalized fuzzy vector (or fuzzy point in $\Re^2$) as a product of two simple trapezoidal fuzzy numbers, where the associated membership function is nothing but a normal, fuzzy convex fuzzy subset of R^2 with compact support. Clearly, 'a typical point of $\Re^2$' is a collection of crisp points (vectors) of R^2. 29

2.5.1 Representation of fuzzy feature vector/pattern vector 38

3.1.1 Transversal vis-a-vis non-transversal intersections. 50

3.2.1 Trapezoidal fuzzy number K 58

3.2.2 Triangular fuzzy number x_0 58

3.2.3 Triangular fuzzy number L 59

3.2.4 Triangular fuzzy number M 59

3.2.5 Fuzzy state space of the system represented by (3.2.12), (3.2.13), (3.2.14) and (3.2.15) is enclosed by curves '1' (extreme left curve of the band) and '2' (extreme right curve of the band). Only the region representing the best system behaviour has been considered. 60

3.2.6 A more generalized solution of (3.2.12), (3.2.13), (3.2.14), (3.2.15) representing the system behaviour for $0 < \alpha \leq 1$, assuming that, $\delta \to 0$ and $\lambda \to 0$. Here the red region is giving the fuzzy flow representing the best possible system behaviour ($\alpha = 1$) and the blue region is giving the fuzzy flow representing the other possible system behaviour ($\alpha < 1$). The scaling of t-axis has been magnified by a factor of 10 compared to Fig 3.2.5. 61

4.1.1 The attrctor of the IFZS $\{ = (w_1, \ldots\ldots, w_N, \mu(w_1), \ldots\ldots, \mu(w_N))$. Each $w_i : \Im \to \Im$ is a contraction mapping. $\mu(w_i(x)) = \varphi_i(\mathbf{u}(w_i^{-1}(x)))$ for all $x \in X$. Note that $x \in \Im$ also. 69

4.1.2 Sierpinski's gasket generated as the fractal attractor of the IFZS described above. The starting point is on the gasket. 70

4.2.1 Feigenbaum diagram of x(s) for very large n and a = 1, irrespective of s, provided $s \in [0, 1]$. $f(s) = \mu s(1 - s)$, $3 \leq \mu \leq 4$. x(s) is plotted along the ordinate and μ along the abscissa. For $\mu = 4$ x(s) can converge on any pixel along the ordinate. Exactly the same diagram will also hold for y(s). 72

4.2.2 Fluid jet 1 and fluid jet 2 are colliding at (r,0). Their resultant radial component of velocity and resultant cross-radial component of velocity after collision have been shown. If the ratio of these two components is constant a spiral shaped vortex is generated. 73

4.2.3 The triangular fuzzy number $<r(0)-\delta, r(0), r(0)+\delta>$. The value r(0) will vary from vortex to vortex. 74

4.2.4 The simulated diagram of a fuzzy vortex centering at (0.5,0.5), where r_0 (in cm) belongs to a fuzzy number, with the best possible value ≤ 0.1. This diagram is the phase (solution) space of (4.2.4) for $\alpha = 1$. The range of θ is a random number $\leq 16\pi$. The enclosed region indicated by the spirals is the fuzzy flow representing the most possible (grade of membership 1) behaviour of the system represented by (4.2.4). We get the spirals by putting $m \in [0.05, 0.1]$ in (4.2.4). If we replace A by $-A$ in (4.2.4) the orientation of the vortex will be reversed. 75

4.2.5 Simulation of a two dimensional fuzzy turbulence in a two dimensional dynamic fluid. Red and blue vortices have different orientation. 76

5.1.1 Graphs showing the evolution with time of the maximum surface wind speed produced by the numerical model of Rotunno and Emanuel. The upper curve starts with a 12 m/s (43.2 km/h) amplitude vortex, while the maximum velocity in an experiment that starts with a 2 m/s (7.2 km/h) amplitude vortex, but is otherwise identical, is shown by the lower curve. The failure of the weak vortex to amplify demonstrates that cyclones in this model can not arise out of weak disturbances; rather a vortex of sufficient amplitude must be provided by independent means, such as a large-scale wave or a thunderstorm complex from a middle -latitude continent. 81

5.1.2 INSAT-1B cloud picture of the vortex of a severe cyclonic storm on the Bay of Bengal taken on 7th May, 1990 at 0600 UTC. The dark spot at the centre of the vortex (the central cloudless region) is known as the *eye* of the cyclone. (Reproduced from *Mausam*, vol - 41, no. 3, 1990). 82

5.1.3 Graph of the fuzzy constant M (not according to the scale). 85

5.2.1 Graph of r_0. Support of r_0 is [500, 1500] (in km). $r_0(1000) = 1$. This scale is compatible with observations [45]. 86

5.2.2 Numerical simulation of (5.2.1), where $\alpha = 1$, that is, the phase space is showing the fuzzy flow (the entire region enclosed between the two curves including the curves also) representing the best system behaviour only. This flow tends to converge to a (fuzzy) point of the (fuzzy) phase space, which is known as the fuzzy attractor of the system. If this vortex ultimately matures into a severe cyclonic storm this fuzzy attractor will become the eye of the cyclone (Fig 5.1.2). 87

6.1.1 The plot of dm/d*t* against m with critical r and v. 93

6.1.2 Plot of dm/d*t* against m for r = 5 (above) and r = 7 (below). 94

6.2.1 An approximation of the solution space of (6.1.2) for r = 5.95, v = 2 and u = 0.1. Resolution of m is taken as 0.01. 97

6.2.2 An approximation of the solution space of (6.1.2) for r = 6.4 and v = 2.7 and u = 0.1. Resolution of m is taken as 0.01. For most of range of m the result is abnormal and unpredictable. 98

6.2.3 An approximation of the solution space of (6.1.2) for r = 5.95, v = 2.5 and u = 0.1. Resolution of m is taken as 0.01. Almost in the whole range of m the result is very abnormal and beyond any meaningful prediction. 99

6.2.4 Region enclosed by the curves (*t*, m(*t*)) for which r = 7 and v = 1 (left), r = 5.95 and v = 1.75 (right) and the *t* axis is the region of the best possible solution (with the degree of possibility $\alpha = 1$) of (6.2.3). 100

List of Tables

4.2.1 Numerical estimation of the constant fuzzy number A for a zero viscosity fluid in a model with moderate shear. 74

5.2.1 Numerical specification of the fuzzy constant M. 87

Chapter 1

Introduction

When using a mathematical model, careful attention must be given to the uncertainties in the model.
- R. P. Feynman on the reliability of the Challenger space shuttle [51]

Management of uncertainty is a challenging problem in computational science in general and modelling problems in particular. This is very much evident in nonlinear dynamical system modelling of various natural and social phenomena, where uncertainty and complexity go hand in hand. Often analytical tools to study such systems are to be complemented by computer simulations.

Advances in computer technology have been steadily extending our capabilities for coping with systems of an increasingly broad range, including the systems those were previously intractable to us by virtue of their nature and complexity. While the level of complexity, we can manage, continues to increase, we begin to realize that there are fundamental limits in this respect. As a consequence, we begin to understand that the necessity for simplification of systems, many of which have become essential for characterizing certain currently relevant problem situations, is often unavoidable. In general, a good simplification should minimize the loss of information relevant to the problem of concern. Information and complexity are thus closely interrelated.

One way of simplifying a very complex system – perhaps the most significant one – is to allow some degree of uncertainty in its description. This entails an appropriate aggregation or summary of the various entities within the system. Statements obtained from this simplified system are less precise (certain), but their relevance to the original system is fully maintained. That is, the information loss that is necessary for reducing the complexity of the system to a manageable level is expressed in uncertainty. The concept of uncertainty is thus connected with both complexity and information.

It is now realized that there are several fundamentally different types of uncertainty and that each of them plays a distinct role in the simplification problem. A mathematical formulation within which these various types of uncertainty can be properly characterized and investigated is now available in terms of the theory of fuzzy sets and fuzzy measures [80].

In this thesis we propose some approaches to modelling and simulation of dynamical systems about which precise information is not available. A dynamical system is a system, which changes with respect to time [60]. By a dynamical system with imprecise or unknown information we shall understand either one or both of the following: (1) a dynamical system whose mathematical formulation, in terms of a set of equations, is not precisely known, and (2) even if this formulation is known, the quantities involved in these equations are not precisely measurable (that is the parameter and/or the input values are uncertain). In all these cases modelling of the inherent uncertainty can effectively be accomplished by fuzzy set theory based techniques, where in many cases the uncertain quantities involved are fuzzy numbers. The relationship between multidimensional fuzzy real number system and uncertain dynamical systems has been explored in [98]. A diverse class of dynamical systems involving uncertain or imprecise quantities from the fields of image generation, turbulence, atmospheric science and medical science have been studied involving extensive use of modern electronic information processing systems [1, 2, 5, 21, 97, 100, 101, 102, 135].

A dynamical system is denoted by the triplet (X, T, π), where X is the phase space, T is the set of time and $\pi: (X \times T) \rightarrow X$ is the time evolution law or the attainability set mapping. Often T is omitted and a dynamical system is denoted by (X, π). When there is no risk of confusion about the phase space a dynamical system is simply expressed by π

(i) A *phase space* X, whose elements or *points* represent possible *states* of the system.

(ii) *Time*, which may be discrete or continuous. It may extend either only into the future (irreversible or noninvertible processes) or into the past as well as the future (reversible or invertible processes). The sequence of time moments for a reversible discrete-time process is in a natural correspondence to the set of all integers; irreversibility corresponds to considering only nonnegative integers. Similarly, for a continuous time process, time is represented by the set of all real numbers in the reversible case and by the set of nonnegative real numbers for the irreversible case (Appendix A) [95].

(iii) The *time-evolution law* or *attainability set mapping*. In the most general setting this is a rule that allows us to determine the state of the system at each moment of time t from its states at all previous times. If our system was initially at a state $x \in X$, it will find itself after time t at a new state, which is uniquely determined by x and t, and thus can be denoted by $\pi(x,t)$. The set $\{\pi(x,t)\}$ is called the *attainability set* or the *reachable set* at time t. Fixing t, we obtain a transformation $\varphi^t: x \rightarrow \pi(x,t)$ of the phase space into itself. These transformations for different t are related to each other. Namely, the evolution of the state x for time s + t can be accomplished by first applying the transformation φ^t to x and then by applying φ^s to the new state $\varphi^t(x)$. Thus we have $\pi(x, t + s) = \pi(\varphi^t(x), s)$ or equivalently, the transformation φ^{t+s} is equal to the composition of φ^t and φ^s [74].

Throughout this thesis we shall denote a dynamical system by (X, π) or when there is no risk of confusion just by π.

In virtually all situations of interest the phase space of a dynamical system possesses a certain structure which the evolution law respects. Different structures give rise to theories dealing with dynamical systems that preserve those structures [74]. For example, when the phase space X is a measure space with a σ-finite measure μ and the time evolution laws are a group or semigroup of measurable transformations on X each of which either preserves μ or transforms it into an equivalent measure then the underlying dynamical system theory is called *ergodic theory*; when X is a topological space, usually a *metrizable, compact* or *locally compact* space and the time evolution laws are group of topological homeomorphisms or semigroup of continuous functions defined on X then the underlying dynamical system is called *topological dynamics*; when X is a *smooth manifold* e.g., a domain or a closed surface in a Euclidean space and the time evolution laws are group of diffeomorhpisms or semigroup of non-invertible differentiable maps on X then the underlying dynamical system is called *differentiable dynamical system*; when X is a collection of *fuzzy sets* (usually nonempty, compact fuzzy subsets of R^n) and the time evolution laws are a semigroup of fuzzy set valued functions defined on X then the underlying dynamical system is called *fuzzy dynamical system* (FDS) [82]. An ergodic theory for fuzzy dynamical systems is also in existence [39].

Any dynamical system theoretic model of a physical phenomenon consists of a number of variables and parameters, each of which is measurable. But unfortunately any measurement is prone to error. The more we want to make a measurement error-free the more time is needed to accomplish that measurement. So achieving accuracy is a time-complex (apart from other complexities) task. We can reduce this complexity by allowing and accommodating uncertainties in our measurements. Fuzzy sets as enunciated by Zadeh [151] are the basis of fuzzy uncertainties. The more uncertainties are involved the more complex is the system. To model these complex systems uncertain systems were born, where complexity is reduced by recognizing and accommodating uncertainty rather than ignoring it [43, 80]. Since fuzzy set theory is a broad based tool to model a wide range of uncertainties [41, 80] FDS (which takes into account uncertainties involved both in the initial input values and in the parameter values) is a very potential means to model a wide range of uncertain systems.

Generally speaking, the approach to model uncertainty in a dynamical system should be two fold namely, (1) phase space approach and (2) attainability set mapping approach. They are often not independent but only complementary to each other. Generally, both of them are simultaneously employed under an FDS modelling. Since crisp phase spaces are special cases of fuzzy phase spaces and crisp attainability set mappings are special cases of fuzzy attainability set mappings, the crisp dynamical systems are only special cases of fuzzy dynamical systems.

Fuzzy set theory has entered the world of uncertain dynamical systems through the study of non-deterministic automata [6]. This approach has given rise to the idea of non-deterministic systems in which the current state does not determine a single unique next state, but rather determines a set of possible next states. In other words $\{\pi(x,t)\}$ is no longer a singleton set. It is a fuzzy set in case of an FDS.

A fuzzy system is described by the evolution of a state (which is a fuzzy subset of the state space) over time. In many important cases (but not in all) this evolution is governed by a differential relation. Fuzziness, so introduced in the representation of a system, reflects our lack of knowledge about it. The introduction of fuzzy set theory into the field of dynamical system theory was motivated mainly by [27]:

(i) The needs for a theory of systems whose structure and/or behaviour involve uncertainties.

(ii) The desire to apply system theoretic methods to systems which have proved impervious to mathematical analysis and computer simulation.

(iii) The design of fuzzy controllers.

Design of fuzzy controllers has perhaps remained the single most important implementation of FDS [72, 86, 87]. Fuzzy control is a nonlinear control based on heuristic control tables, and is actually "another" way of implementing the proportional-integral-derivative (PID) control strategy, although not limited to PID. Any nonlinear system can be approximated as accurately as required with some number of fuzzy control rules. The fuzzy controller is also regarded as a universal approximator, and its use offer many advantages [72]. Traditionally, fuzzy controllers have been designed without an explicit knowledge of the system [66]. The objective function of the controller is approximated to any desired degree of accuracy by a set of fuzzy if-then rules. Stability of such systems have also been extensively studied [72, 128].

(ii) above has so far remained a less explored area. The modelling and simulation capability of the fuzzy set theory based techniques has not been explored much beyond the implementation of if-then rules involving fuzzy linguistic variables, fuzzy relations, fuzzy quantifiers, etc. [135]. Importance of simulation of an uncertain phenomenon, through actual formulation of a fuzzy valued function and not approximating it by fuzzy if-then rules, has been explored in [62, 63, 149]. The necessary tools for this purpose have been developed mainly in [17, 18, 24, 27, 32, 62, 63, 82]. Here we shall concentrate more on this aspect.

Differentiable dynamical systems theory began with Poincare's endeavour to find *qualitative* rather than *quantitative* solution of nonlinear differential equations. As for example, take the equations of the Lorenz system (LS), which came into being as a simple mathematical model for heat convection in atmosphere [93].

$$
\begin{aligned}
dx / dt &= \sigma (y - x), \\
dy / dt &= rx - y - xz, \\
dz / dt &= xy - bz,
\end{aligned}
$$

where σ, r and b are real, positive parameters, which Lorenz fixed at $\sigma = 10$, $b = 8/3$ and $r = 28$ [93]. The corresponding (numerical) solution space generated by computer can be seen in [124]. A very deep exploration of this very versatile yet mysterious system appears in [140]. The divergence of the flow is

$$\partial x' / \partial x + \partial y' / \partial y + \partial z' / \partial z = -(\sigma + b + 1),$$

where $x' = dx / dt$, etc. Since the divergence of the flow is negative, the volume element is a decreasing function of time. All the systems whose 'volume element' is decreasing or the so called phase space is contracting is called *dissipative dynamical system* (DDS). Each DDS must have an *attractor*. Later in this chapter a more general definition of a DDS independent of 'volume element' will be proposed.

Another very important class of dynamical systems is the class of *iterated function systems* or IFS for short, which gives rise to a class of discrete-time dynamical systems. For example, take the *logistic function*

$$f(x) = \mu x(1 - x), \tag{1.0.1}$$

where x is in I = [0, 1] and μ is a positive real number. Like the Lorenz system this system too has generated tremendous amount of interest among the dynamicists. (1.0.1) first appeared in the form $x_{n+1} = \mu x_n(1 - x_n)$ in population studies, where x_n indicates the total population at the nth year. So iterated logistic function system (ILFS) is a dynamical model of yearly population growth. This function was used to model population dynamics. The population dynamics is an uncertain system. May was the first to explore this uncertainty through extensive numerical simulation [106]. He indicated that this uncertainty is chaotic in nature. He also showed that this simple looking system actually has a very complex dynamics. This is also a good example to indicate that uncertainty spawns complexity and vice-versa. Feigenbaum explored the mystery of the quadratic iterators like the ILFS employing computational approach [50]. ILFS has a fuzzy counterpart too, namely the *iterated fuzzy sets systems* (IFZS) [21], which are the most prominent examples of discrete (dissipative) fuzzy dynamical systems. IFZS has been discussed and extended in [97].

Both the above crisp systems are dynamical systems representable completely by a set of mathematical equation(s). When a dynamical system is completely representable by a set of mathematical equations and no noisy or random input or parameter is present, we call the dynamical system *deterministic dynamical system*. Otherwise it will be called *non-deterministic dynamical system*. Though the non-deterministic dynamical systems can not be modelled very precisely like the LS or ILFS, they are often of great significance in real life. For example, the system of creation of a cyclonic storm (*cyclogenesis*). This system is so complex that, for any model of it describable in terms of a precise set of mathematical equations, noise free non-random inputs or parameters would have been a remote possibility. FDS is some time an ideal tool to model this type of imprecise systems [43, 100, 101].

Another recent trend is to replace a stochastic differential equation by a fuzzy differential inclusion relation (which is a special kind of FDS) in the model of an uncertain system. This trend has been introduced in biological system modelling. Like the atmospheric systems biological systems too are often very imprecise. To model the inherent uncertainty in population dynamics, conventionally stochastic differential equations were used. But recently, in population dynamics the stochastic differential equations have been replaced by differential inclusion relations and fuzzy differential inclusion relations [5, 84], with similar qualitative (but not quantitative) results [84]. Likewise in a stochastic differential equation modelling of evolution of tumour in human tissue the stochastic differential equation has been replaced by fuzzy differential inclusion relation, which has yielded some significant results [102, 103].

FDS is an extremely powerful tool for modelling and simulation of very complicated non-deterministic dynamical systems, where the level of uncertainty is very high. By and large the fuzzy system modelling has remained confined to engineering sciences only, particularly to the information science [135]. Attempts have also been made to apply it to economics and other branches of social science [76]. The number of such attempts has increased rapidly in last few years.

1.1 General Dynamical Systems and Cybernetics

In literature there exist many different definitions of general systems [40]. Bertalanfy was one of the first to define general systems [11]. A prominent subclass of general systems is the class of linear systems [160]. Here we present the definition of a general dynamical system. We take a complete, locally compact metric space (X, d) (we call just X) as the state space and give

Definition 1.1.1: A single valued *dynamical system* on a state space X is defined axiomatically in terms of a single valued mapping $\pi : X \times T \rightarrow X$ satisfying the following four axioms, where T is the set of time:

1. π is defined for all $(x, t) \in X \times T$,
2. $\pi(x, 0) = x$ for all $x \in X$,
3. $\pi(x, s + t) = \pi(\pi(x, s), t)$ for all $x \in X$ and $s, t \in T$,
4. π is jointly continuous in (x, t).

Definition 1.1.2: When π is obtained as the solution to a set of differential equations the dynamical system is called *differentiable dynamical system.*

Definition 1.1.3: $\pi(a, t)$ for a given $a \in X$ and t varies over an interval of R is called a *trajectory* or an *orbit* of the dynamical system. Collection of all such orbits of a dynamical system is called the *phase space* of the dynamical system.

We have already seen that the Lorenz system is a dissipative dynamical system. By a dissipative dynamical system we mean one whose phase space is 'shrinking', that is, in general for n-dimensional phase space (having n state variables) the n-dimensional volume element should typically decrease (dissipate) in time so that the volume of the final set will be less than the volume of the initial set. As a consequence of this, dissipative dynamical systems are typically characterized by the presence of attractors [56]. Many important real life systems are dissipative systems. Since attractor of a dissipative system is actually the phase space as $t \rightarrow \infty$ (for all practical purpose, after sufficient lapse of time) the behaviour of the system on the attractor is of ultimate importance. Let us formalize the notions of dissipative dynamical system and attractor in more rigorous terms.

Definition 1.1.4: π is a *dissipative dynamical system* if $\pi(X, t_2) \subseteq \pi(X, t_1)$, where (X, d) is a nonempty compact metric space and $t_1, t_2 \in T$ such that, $t_1 < t_2$. X is the initial phase space of π, i.e., at t_0 (the time at which the system starts) π can start from any point of X.

If $\pi(X, t_2) \subset \pi(X, t_1)$ holds strictly for $t_1 < t_2$ we call the system *strictly dissipative* or *monotonic dissipative*.

Let $0 < t_1 < t_2 < t_3$ and $\pi(X, t_1) = X_1$. If we start from $t = t_1$ instead of $t = 0$ then the definition still holds good or in other words Definition 1.1.4 is well defined. The initial phase space is $X_1 = \pi(X, t_1)$ instead of $X = \pi(X, 0)$. $\pi(X, t_2) = \pi(X, t_1 + t_2 - t_1) = \pi(\pi(X, t_1), t_2 - t_1)$. $\pi(X, t_1) = X_1 \subseteq X$. $X_1 = \pi(X_1, 0)$. So. $\pi(X, t_2) = \pi(\pi(X_1, 0), t_2 - t_1)$. Similarly, $\pi(X, t_3) = \pi(\pi(X_1, 0), t_3 - t_1)$. $t_3 - t_1 > t_2 - t_1$ and π is dissipative. So. $\pi(\pi(X_1, 0), t_2 - t_1) \subseteq \pi(\pi(X_1, 0), t_3 - t_1)$, i.e., $\pi(X, t_2) \subseteq \pi(X, t_3)$.

Definition 1.1.5: In a dissipative system $\lim_{t\to\infty} \pi(X, t) = A \subseteq X$. We define A as *attractor* of π.

This notion of attractor is easy to implement, but can be generalized to a more versatile form.

Progress of sciences and all other components of civilization can be traced to certain types of challenges and responses. The subject of cybernetics emerged through the interaction of traditional sciences when scientists faced a set of problems concerned with communication, control and computation in machines and living tissues. The foundation of cybernetics were laid and its basic principles formulated over centuries by the work of mathematicians, physicists, physicians and engineers. Though the decisive factor in the emergence of cybernetics was the swift development of electronic automation and especially the appearance of the high speed computers which opened up boundless vistas in data processing, simulation and the modelling of control systems [40]. Cybernetics is a special class of dynamical systems, where to achieve the goal of communication, control and computation Weiner envisioned a grand exchange of ideas and methods among various disciplines to enable the scientists to efficiently tackle complex problems of one area with the suitable techniques borrowed from the others [150].

1.2 Uncertain Systems

Uncertainty may enter into a dynamical system (X, π) in two ways. If $\pi(x, 0) = x$ and x can not be 'located' in X accurately, we call x uncertain and this means that the value of $\pi(x, t)$ is also uncertain for any $t \geq 0$. Since the location of x is uncertain in X, each time we want to locate x in X we end up landing upon some neighbourhood N(x) of x in X. The possibility of finding x in N(x) is 1. So the membership value of x in N(x) is 1 [156]. Beside x there are other members of N(x) with membership values in [0, 1]. If $y \neq x$ and $y \in X$ such that, membership value of y is α then possibility of landing at y when searching for x in N(x) is α. Any point outside of N(x) has membership value 0. N(x) is a typical fuzzy subset of X. The same reasoning that holds for an uncertain x holds for $\pi(x, t)$ too for any $t \geq 0$. To model this type of uncertainty we should replace X by the collection of nonempty fuzzy subsets of X, denoted by F(X). Since each crisp point is a singleton fuzzy set. X is a subset of F(X). This will be called the *phase space uncertainty* of a dynamical system.

The other uncertainty in a dynamical system is the uncertainty in determining π. π may be a member of a class of functions C. Each member of C is a possible attainability set mapping of the underlying dynamical system with a given membership value in [0, 1]. C is the fuzzy set of all possible attainability set functions. Following [27] we call C the fuzzy attainability set mapping. The uncertainty will be called *attainability set mapping uncertainty*.

In an uncertain dynamical system both uncertainties may occur simultaneously. In a general FDS both type of uncertainties are treated simultaneously [43].

Conventionally, uncertainty used to be treated by statistical means and they have remained very successful in modelling uncertainty in atmospheric science, engineering, economics, biology, etc. But at least in some cases FDS modelling of uncertainty has been found to be easier to implement and less computation intensive, yet with good results [5, 43, 63, 97, 100, 101, 102]. For example, the starting stage of an intense tropical storm is yet to be understood clearly [47, 48, 113, 133]. An FDI modelling of this uncertain atmospheric system has been able to show the generation of an initial disturbing closed vortex out of a wave under certain conditions [100, 101]. The uncertainty involved in biological systems often used to be modelled by stochastic differential equations [43, 44, 52, 58, 61, 88, 89]. Only recently in some of the biological systems stochastic differential equations have been successfully replaced by differential inclusion relations and FDI relations [5, 84, 102, 103]. On one hand, in some cases, this has given completely new insight into the system and based on this, improved therauptic planing has been suggested [102, 103].

1.3 Dealing with Complexity

Complex systems are much less understood and not even well defined mathematically. Roughly speaking they are in the frontier between simple and chaotic systems: A complex system is a dynamical system depending on many parameters, in constant evolution and distances along trajectories increase (decrease) polynomially and not exponentially. One considers that the brain's neural network is one such system [119]. Some salient feature of complex systems are (a) it is a dynamical system in constant evolution formed by a great number of units, (b) some characteristics of the system are randomly distributed and (c) the system may have several attractors [119].

Our view of complexity is slightly different from that enunciated above. To us a chaotic system is also very much a complex system. Therefore the trajectories may vary exponentially also. Not all uncertainties in a complex system are essentially statistical in nature. There may be other uncertainties too, e.g., fuzzy uncertainty, chaotic uncertainty, Dempster-Shafers' evidence theoretic uncertainty, etc. The possible notions of applications of complex systems are not yet conclusive and perhaps can never be conclusive, like the case of functioning of brain, origin of life, generation of cyclones, evolution of tumour, evolution of social systems, etc. [43].

Historically, the first serious attempt to resolve complexity, which withstood the test of time, was the *probability theory*, which chiefly deals with the uncertainty some time remains involved in determining a value. Then came Zadeh. In his seminal paper of 1965 [152] he further broadened the scope of dealing with the complexity by recognizing and accommodating uncertainty and unspecificity with his revolutionary idea of *fuzzy sets*. In [154] Zadeh showed how complexity of a system can be tackled by the notion of fuzzy sets. After this, innumerable researchers have successfully (to a varied degree) employed fuzzy set theory based techniques to deal with complexities in various systems. Wang and Lengari have attempted to model complex systems via fuzzy logic in [145]. A good example in point is the simulation of the complex decision making tasks of an intelligent human expert by *fuzzy controllers* [86, 87, 120]. Mathematically speaking, fuzzy controllers are nonlinear dynamical systems, where the (complicated) nonlinear objective function is very difficult to determine. That objective function can be approximated by a set of fuzzy if-then rules to any desired degree of accuracy. A relatively new approach is to represent an uncertain system by a fuzzy valued function, which is a fuzzy set of crisp functions [27, 43, 62, 63, 97, 100, 101, 102, 103]. Turbulence is regarded as one of the most complex natural phenomena [16]. Numerous models of turbulence are in existence, each of them highlights some particular aspects of turbulence [125]. A simple model of a two dimensional turbulence has been proposed as a chaotic occurrence of vortices in a two dimensional dynamic fluid [97]. Each such vortex is modelled by a fuzzy valued function taken as a fuzzy set of crisp functions [97]. In [159] Zadeh argues in favour of reducing complexity by replacing conventional numerical computations by computations with words of a natural language. He has even developed a specially meaningful vocabulary for this purpose [155].

1.4 Chaos

The study of dynamical systems started with astronomy. In that sense study of dynamical systems is more than two thousand years old. Theory of differentiable dynamical system was created by Poincare in 1885. Comparing with this, *chaotic dynamical system* (some times also known as *applied nonlinear dynamics* [74]) is a relatively new branch of the classical dynamical systems theory. It is believed to have started with the seminal paper of Li and Yorke titled, "Period Three Implies Chaos" in 1975 [90]. The main result of this paper was only a special case of a much more generalized but lesser known result of Sharkovskii published in 1964 [137]. But a serious attempt to define chaos was first made in [90]. Later Devaney gave a definition of chaos in [30] and this definition is a modification of that of Li and Yorke. Today Devaney's is the most widely accepted definition of chaos, at least theoretically. Here we present that definition in terms of iterated function systems, i.e., in discrete dynamics. Extension of it to the contin-uous dynamical systems is straightforward and will be undertaken wherever required.

Definition 1.4.1: Let $f : X \rightarrow X$. f is said to have *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $x \in X$ and any neighbourhood N of x, there exists a $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$. f^n denotes nth iterate of f.

Definition 1.4.2: f is said to be *topologically transitive* on X if for any pair of nonempty open sets $U, V \subseteq X$ there exists $k > 0$ such that, $f^k(U) \cap V \neq \emptyset$ (null set).

Definition 1.4.3: $x \in X$ is said to be a *periodic point* of f if and only if $f^p(x) = x$ for some integral $p > 0$. p is said to be *period* of x with respect to f if there does not exist any i such that, $0 < i < p$ and $f^i(x) = x$. When $p = 1$ we call p a *fixed point* of f.

Definition 1.4.4: (Devaney [30]) $f : X \rightarrow X$ is said to be *chaotic* over X if and only if

1. f has sensitive dependence on initial conditions,
2. f is topologically transitive, and
3. Periodic points of f are dense in X.

According to Robinson if f satisfies conditions 1 and 2 only from among the above three, then f is said to be *chaotic* [131]. Banks, Brooks, Cairns, Davis and Stacey have shown that, any map which is topologically transitive and also has dense periodic points must have sensitive dependence on initial conditions [8]. So all the three conditions in Devaney's definition of chaotic dynamical system (CDS) are not mutually independent.

The above mathematical definition of chaotic dynamical system is rather restrictive as far as practical applications are concerned. So there are methods compatible with the above definition but which concentrate on determining the satisfiability of only some of the aspects (most often the sensitive dependence on the initial conditions), taken into account by Devaney's definition, by a system to decide whether the system is chaotic. Some of the popular methods to determine chaos in a system are (i) by measuring *topological entropy*, (ii) by measuring *Liapunov exponent*, (iii) by determining whether the attractor (when exists) is *fractal* in nature and also has *homoclinic points*, etc. Detail discussions about some more methods appear in [107, 121].

A recent trend in the contemporary dynamical system theoretic studies is the emphasis on *fractal sets*. To us the most important fact is that, many important dynamical systems have fractal attractors. Usually a dynamical system which has a fractal attractor often (but not always [73]) turns out to be a *chaotic dynamical system*. Studying the fractal attractors is a very standard way to understand such systems [56, 107]. In the last twenty years this notion has found wide spread application in physical, biological and social systems. But *fuzzy chaos* has so far not found any significant application though an attempt has been made in [19]. *Fuzzy fractal* is an even less explored area. Fuzzy fractals come naturally as attractors of the *iterated fuzzy sets systems* (IFZS) [21] exactly the same way as the crisp fractals are generated as attractors of contractive iterated function systems [9]. In [21] Cabrelli et. al., have treated any grey level image as a fuzzy subset of the euclidean plane. A normalized grey level pixel intensity value at each pixel is taken as the fuzzy membership value of that pixel or point. In [21] IFZSs have been used to generate grey level images. The method used here is a bit complicated. Recently some simplification has been introduced in it [97]. Fuzzy chaos and fuzzy fractals have not yet been related as in case of crisp dynamical systems. This is not possible in case of fuzzy differential equation based dynamical systems due to the possibilistic irreversibility [14]. But FDI relations hold a great promise in this direction. Fuzzy fractals and fuzzy chaos hold the promise to find useful applications in FDI modelling of complex natural phenomena. A typical outline of exploration for the future researchers should be as follows. Choose a natural phenomenon. Make a model of it in terms of a set of FDI(s). Solve the system of FDIs. Let the system be dissipative. Then as time $t \to \infty$ the fuzzy trajectories converge to the attractor A. If A is also a fractal subset of X we shall call A a fuzzy fractal [97]. If the dynamical system during evolving on A satisfies the conditions to be a fuzzy chaotic dynamical system then it is a fuzzy chaotic dynamical system with a fuzzy fractal attractor.

Fuzzy chaos has been described by Buckley and Hayashi [19], Teodorescu [142], Kloeden [83] and Diamond [31]. Buckley and Hayashi have shown some applications of fuzzy chaos in [19]. But this is an area which is yet to be fully explored. It turns out that the extension of Definition 1.4.4 to FDSs, which may be regarded as a new definition of fuzzy chaos, is a generalization of the existing notions [43].

1.5 Scope of the Thesis

In this thesis we are developing a methodology for tackling complexity arising due to inherent uncertainty in a general dynamical system. Since the subject is highly multidisciplinary in nature we have taken a special interest in cybernetics. The tools we intend to use here are fuzzy set theory based techniques in general, and iterated fuzzy sets systems (IFZS) [21] and fuzzy differential inclusion (FDI) [7, 62, 63] relations in particular. The thesis is divided into seven chapters. The first or the current one is Introduction.

In Chapter 2 we propose to deal with the phase space uncertainty of an FDS. Phase spaces of crisp dynamical systems in most cases are in one way or the other associated with R^n. In case of FDSs almost no attempt has so far been made to associate fuzzy phase spaces with $\Re^n$, where $\Re$ is the fuzzy real number system. This has been addressed in Chapter 2 [98].

In Section 2.1 we have presented a brief review of fuzzy phase spaces so far used to describe various fuzzy dynamical systems.

In Section 2.2 we have explored fuzzy real number system $\Re$. In subsection A we have highlighted the uncertain quantity (parameter) modelling capability of $\Re$. Precisely for this reason $\Re$ is so important in describing fuzzy phase spaces and fuzzy parameters [95]. Crisp dynamical systems evolve either in R^n or in a subset of R^n. Although $\Re$ does not have as rich a structure as that of R yet $\Re^n$ can play an important role in describing fuzzy phase spaces. We have dealt $\Re^n$ in Section 2.3. Treating $\Re$ in Section 2.2 prepares the necessary background for this purpose. In Section 2.2 we have described a field like algebraic structure over $\Re$ so that we can describe a vector space like structure for $\Re^n$ in the next section. We have defined additive and multiplicative binary operations over $\Re$. We have shown that $\Re$ is closed, commutative and associative under these operations. In subsection C we have defined inverse with respect to both these operations. We have modified the definition of inverse and the notion of the relation between identity and inverse (subsection D) to make the inverse of a fuzzy number unique. This way we have defined an algebraic structure over $\Re$, which we call an F-filed. In an F-field like $\Re$ distributivity does not hold but subdistributivity holds. F-fields have important similarities and differences with crisp fields. In subsection F we have shown that R is embedded in $\Re$ as a crisp field. In subsection G we have defined a metric on $\Re$.

In Section 2.3 we have discussed $\Re^n$, which may be regarded as a tool for modelling multidimensional uncertain quantities [95]. We have constructed a vector space like structure, called generalized fuzzy vector space (GFVS), for $\Re^n$ [94, 98]. Each generalized fuzzy vector may be regarded as a multidimensional fuzzy number. A GFVS is more general than the Katsaras-Liu fuzzy vector space [75] but less general than the Buckley-Aimin topological fuzzy vector space [20]. We have shown that, R^n is contained in $\Re^n$. A metric is defined on $\Re^n$. It has been shown that this metric defined on $\Re^n$ is equivalent to the metric of the metric space $\Re \times \Re \times \Re \times \Re$ .... up to n times.

In Section 2.4 we have shown how an arbitrary fuzzy subset of R^n can be obtained from $\Re^n$ by max-min operations. In Theorem 2.4.1 we have established that, each element of $\Re^n$ is a normal, uppersemicontinuous, convex and compact supported fuzzy subset of R^n. In particular, any fuzzy point of E^n (collection of all normal, uppersemicontinuous, convex and compact supported fuzzy subset of R^n) either belongs to $\Re^n$ or can be generated from elements of $\Re^n$ by max-min operations. But in Section 2.1 we have seen that E^n is the phase space of the FDSs as described by Kloeden [82], De Glas [27], Kaleva [68], Seikkala [136], Hulermeier [62] and many others.

In Section 2.5 we have discussed fuzzy pattern recognition. Our primary objective in this section is to show the relationship of various fuzzy pattern vectors or fuzzy feature vectors with the elements of $\Re^n$. We have shown that every fuzzy pattern vector can be generated from elements of $\Re^n$ by max-min operations. In this section by picking up two examples from applied research [116, 127] we have indicated some possible role of $\Re^n$ in computer vision and pattern recognition.

Fuzzy attainability set mapping (FAM) or fuzzy reachable set mapping is the most important part of an uncertain dynamical system, where the uncertainty is fuzzy uncertainty. All dynamical system theoretic properties of such uncertain dynamical systems or FDSs are described in terms of FAMs. In Chapter 3 we have discussed about attainability set mapping uncertainty of an uncertain or fuzzy dynamical system. In other words in this section we have discussed about fuzzy time-evolution law [43].

In Section 3.1 we have introduced several new notions into FDS as well as extended the scope of some of the existing ones in terms of the fuzzy attainability set mappings (FAM). In subsection A of Section 3.1 we have presented a slightly modified and extended version of Kloeden's original definition of an FDS [82]. In subsection B we have defined fuzzy dissipative dynamical system and presented a criterion to test dissipativeness of an FDS. In subsection C we have defined attractor for an FDS in terms of FAM, where the attractor exists. We have defined nonwandering set for fuzzy attractor and extended Ruelle and Tukens' notion [109] to fuzzy attractor. We have defined Liapunov stability or orbital stability for an FDS. We have defined omega limit set for an FDS and the realm of attraction for a fuzzy attractor. Then we have defined asymptotic stability for an FDS. In subsection D we have defined likely limit set for an FDS and then extended the notion of robustness to the FDS. In subsection E we have discussed fuzzy chaos and extended Deavney's definition of chaotic dynamical system [30] to FDS. Here we have also defined periodic point and homoclinic point for an FDS. We have suggested a criterion for a fuzzy attractor to be a fuzzy chaotic attractor in terms of existence of fuzzy homoclinic points. We have also suggested a practical criterion for a dissipative FDS to be chaotic in terms of existence of fuzzy fractal attractor and fuzzy homoclinic points. In subsection F we have defined Liapunov exponent for an FDS and given a formula to measure it. In subsection G we have defined fuzzy metric entropy and fuzzy Liapunov time (the time up to which the underlying FDS is predictable).

In Section 3.2 we have discussed fuzzy differentiable dynamical systems (FDDS). FDDS can be divided into two parts namely, fuzzy differential equations (FDE) and fuzzy differential inclusions (FDI). In subsection A we have discussed FDEs. In this subsection we have defined autonomous and nonautonomous FDDS. We have stated the notion of Hukuhara derivative [126] and briefly stated various classical developments of FDEs based on this derivative. We have also mentioned some of the recent attempts for their generalization [17]. Fuzzy partial differential equations have recently been defined in [18]. It turns out that the points of the phase spaces of [17] and [18] are elements of $\Re^n$. In subsection B we have discussed about FDI. An existing numerical method for solving a system of FDIs has been discussed [63]. Compared to this, in this subsection we have devised a much less complex algorithm to solve a system of one dimensional FDIs [99]. Since this algorithm only produces the fuzzy solution, which represents the system behaviour with the degree of possibility greater than or equal to a pre-assigned value, this is an example of an evolutionary algorithm [122]. Unfortunately it can not be extended to the multidimensional cases. As an example, we have solved a system of second order FDIs with the help of this algorithm and presented the simulation results.

In Chapter 4 we have discussed fuzzy fractals and fuzzy turbulence. First we have studied a very important class of fuzzy discrete dissipative dynamical systems. Discrete dynamical systems are represented by iterated function systems (IFS). The most interesting IFSs are the dissipative IFSs. Being dissipative dynamical systems they must have attractors. In crisp dissipative IFSs these attractors often turn out to be fractal sets. In particular any fractal can be generated by an IFS. Barnsley has employed this notion to generate various (fractal) images as attractors of randomized, contractive, iterated function systems [9]. This entire notion has been fuzzified by Cabrelli et al., where they have called it iterated fuzzy sets systems or IFZS for short [21]. In Section 4.1 we have defined fuzzy fractals. Attractors of IFZS turn out to be fuzzy fractal images according to this definition. It has been shown that the randomized crisp fractals, such as those described by Barnsley [9], are only special cases of fuzzy fractals. In Section 4.1 fuzzy fractal image generation has been discussed in detail. In subsection A of this section it has been reasoned that any image is a fuzzy subset, where the fuzzy membership value at each point (pixel) of the image is the normalized (i.e., lies in [0, 1]) grey level pixel intensity value. This is a novel interpretation of fuzzy membership value. Here the fuzzy membership value is no longer a measure of vagueness or nonspecificity but a measure of 'lack of uniformity' in the normalized pixel intensity value distribution. In subsection B detail of the pixel intensity value determination has been worked out. In subsection C an image generation operator has been described, which iterates on U^n (the class of normal, uppersemicontinuous and compact supported fuzzy subsets of R^n) to generate the desired images. In subsection D two examples of IFZS have been considered and fuzzy fractal images generated as their attractors have been presented. Barnsley has called the fractal generating dynamical systems as 'chaos games'. Similarly, we can call an IFZS a 'fuzzy chaos game.'

In Section 4.2 we have discussed fuzzy turbulence. As far as we know this is a completely new notion. Turbulence is regarded as one of the most perplexing phenomena of the physical world. Neither it has been properly understood nor there is any universally acceptable definition for it. Our treatment of turbulence in this section is based on a rather simple definition of turbulence given by Brown, which says that, turbulence is the chaotic occurrence of vortices in a dynamic fluid [16]. In this section we have presented a model of turbulence, which ideally follows this definition. For simplicity we have taken this model only two dimensional. However this model can also be extended to three dimension. But that will be much more complicated. This model has two parts namely, the spatial distribution of vortices within the (two dimensional dynamic) fluid and the generation of the vortices. To model the chaotic spatial distribution within a subrectangle of the xy-plane we have used the chaoticity of $f(x) = 4x(1 - x)$ for $x \in [0, 1]$, under iterations (Appendix C). If the rectangle is given by $[a, b] \times [c, d]$ and a vortex is centred at (α,β) then $\alpha \in \{a + (b - a)\lim_{n\to\infty} f^n(x) \mid x \in [0, 1]\}$ and $\beta \in \{c + (d - c)\lim_{n\to\infty} f^n(x) \mid x \in [0, 1]\}$. $\lim_{n\to\infty} f^n(x) \mid x \in [0, 1]$ is chaotic has been proved in Appendix C. We have modelled the vortex at (α,β) by FDI. In this model we have assumed that the vortices are created by collision of fluid jets coming from different directions and colliding. We have taken the projections of the velocity of the resultant fluid jet at a point of the fluid along and perpendicular to the radius vector of that point with respect to certain polar coordinate system. Velocity component along the radius vector is radial velocity and that along the perpendicular to the radius vector is cross-radial velocity (discussion about radial and cross-radial component of a velocity can be found in any standard text of particle dynamics, e.g., Loney [92]). Since both these components are derived from the velocity of irregular fluid jets their values will be very much fluctuating in nature and so also will be their ratio m. We have taken the value of m to lie in some trapezoidal fuzzy real number M. We have used this m in an FDI and simulated the vortex by the Algorithm 3.2.1 developed in Section 3.2. Next we have presented the simulated two dimensional model of a fuzzy turbulence. We have given the complete computer program of this simulation so that any interested reader of this thesis can generate more such turbulence. Here all the parameters have been kept uncontrolled (random quantity), i.e., it is highly unlikely that any two turbulence generated by this program, even with identical inputs, will ever be identical. This a salient feature of a real turbulence, as for example, no two coils of smoke emanating out of a chimney are ever identical or at least the probability of happening so is very low.

In Chapter 5 we have chosen to model and simulate a highly uncertain system of nature namely, the starting stage of a cyclone [100, 101]. Several (crisp dynamical system) models of steady state matured cyclones are available [29, 47, 113]. In these models steady state cyclones are treated as intense cylindrical vortices. A set of differential equations (usually in cylindrical coordinate system) are formulated governing the fluid dynamic (e.g., angular momentum) and thermodynamic (e.g., latent heat transfer from sea to atmosphere) aspects of the cyclonic vortex [29, 47]. But the fact is that, although necessary climatic and geographical conditions for formation of tropical storms prevail over a large part of the globe during the storm seasons, yet only about 80 tropical storms on an average are formed in a year all over the world. About one half to two third of them can ultimately reach hurricane stage [104]. Hurricanes (cyclones) are so rare because they need a strong disturbance to start [47, 48, 133]. Very little is known yet about the starting stage of a storm [48, 133]. In Chapter 5 we have modelled this starting stage, i.e., the strong initial disturbance in terms of a fuzzy differential inclusion relation [100, 101]. In Section 5.1 we have formulated the mathematical model of the disturbance in a manner similar to the generation of fuzzy vortices in the model of a fuzzy turbulence described in Section 4.2 (This also supports the opinion that atmospheric turbulence may some time be responsible for generation of a tropical storm). It is well known that the shape of a cyclonic vortex is spiral (log-spiral to be more precise) [15, 45]. The initial disturbance is in the form of a very strong wind jet or wave [48]. We have resolved the velocity of this wind jet into radial and cross-radial component with respect to certain polar coordinate system. Ratio of these two components is

$$(dr/dt)/\{r(d\theta/dt)\} = m$$

or,

$$dr/d\theta = mr. \qquad (1.5.1)$$

If we take m as a constant the phase (solution) space of the differential equation (1.5.1) will be a log-spiral (both in Chapter 4 and Chapter 5 m denotes the ratio between radial and cross-radial components of velocity and r denotes the radius vector). So the computer generated phase space of (1.5.1) may be taken as a reasonable estimate of a initial disturbing vortex, which under favourable conditions may become a severe cyclonic storm. But m being the ratio of two wind speeds is liable to fluctuate even within a very short time interval. So m can not be a constant in crisp sense. We have taken the values of m to lie in a fixed trapezoidal fuzzy number M (That is, we have fuzzified m). The fuzzy quantity M plays the major trick in uncertainty management of this model. Fluctuation of m with respect to time has been taken care of in M. The initial condition of (1.5.1) is also imprecise. Therefore (1.5.1) should be formulated as an FDI relation. We are interested not in the whole system behaviour of (1.5.1), but only in the best system behaviour of (1.5.1) (that is the solution with the highest degree of possibility). So the α-cuts have been introduced over the solution space of (1.5.1).

In Section 5.2 we have formulated the FDI relation and solved it by Algorithm 3.2.1 and generated the phase space representing the best system behaviour (corresponding to α = 1). This computer simulation is a realistic representation of the initial disturbing vortex. The fuzzy flow of this phase space tends to converge to a fuzzy point. which becomes the central cloudless region of the cyclonic vortex, when it matures. as seen as a small dark spot in satellite images. If this initial disturbing vortex develops into a matured storm this dark spot becomes the eye of the cyclone. In other words, the formative stage of the eye of a cyclone is a very natural example of fuzzy attractor of the fuzzy dynamical system modelling the disturbance. By the Liapunov stability criterion, defined in Chapter 3, this system is a stable one. In fact the system is asymptotically stable. We have also shown that this model is compatible with the model presented in [47]. Since this model represents the very starting stage of a cyclone, it has the potentiality to advance the storm warning.

In Chapter 6, as yet another test of efficacy of the FDSs in modelling uncertainty of complex systems, we have modelled the complex uncertain system of the evolution of tumour in human body as an FDS [102, 103]. To be more precise, here we have substituted a stochastic differential equation by a fuzzy differential inclusion relation, which has given us a new insight into the system that may lead to some modifications in some of the therauptic plans of a tumour patient. Treating biological systems in general and human physiological systems in particular as general dynamical systems is not new. Bertalanffy initiated this approach as a mathematical biologist in 1970 [12]. His equation was used for study of growth in animals and tumours. Tsetlin had also taken a similar approach in modelling some properties of a muscle and a ganglion [143]. Uncertainties inherent in medical diagnostic systems have long been recognized. Adlassnig attempted to tackle this uncertainty by developing a fuzzy logic based medical diagnostic expert system [1]. Many more attempts have been made to apply fuzzy set theory based techniques in medical science. But introduction of FDI in modelling a biological system is only a recent trend [5]. Several fuzzy if-then rule based medical expert systems have been developed for clinical diagnosis [1, 2].

In Section 6.1 we have described a model of the evolution of tumour in human tissues. The original model was due to Garey and Lefever [52, 88]. Later Lefever and Horsthemke improved this model to give it a more complete shape [89]. To model this uncertain system we have replaced the conventional stochastic differential equation by an FDI. The model is based on the coupling of three principal phenomena [89]:

(1) the transformation of normal cells into neoplastic cells (tumour cells) denoted by X,
(2) the replication of transformed cells (that is, X becoming 2X),
(3) the immunological interaction of the host organism with transformed cells, i.e., the immunological propensity of the T-lymphocyte (a kind of white blood corpuscle) and some other cells to fight and destroy the unwanted growth of cells in a tumour.

It has been shown in Chapter 6 that this gives rise to the following equation in normalized form.

$$dm/dt = v + m(1 - um) - r\{m/(1 + m)\}, \qquad (1.5.2)$$

where $m = (k_1/k_2)X$ (m has completely different meaning in Chapter 6 than in Chapters 4 and 5. In Chapter 6 m is proportional to the number of tumour cells in a given volume of tissue), $v = k_1AN/k_2(\lambda - A)$, $r = k_1E_t/(\lambda - A)$ (r has different meaning in Chapter 6 than in Chapter 5. In Chapter 6 r is proportional to the tumour cell destruction rate), $u = k_2\lambda/k_1(\lambda - A)N$ and $t = (\lambda - A)t$. u remains fixed and its value is taken as 0.1 [44]. X, E_0, E_1 are respectively the populations of target cells (tumour cells), of free effector cells and of effector cells having recognized and bound a target cell. A, λ, k_1, k_2 are rate constants. X is the population of target cells. $E_t = E_0 + E_1$. The immunological conditions under which this assumption is reasonable have been discussed in [52, 88]. N is the maximum number of target cells which can be packed in a given volume element. The factor (N - X) expresses simply the existence of an upper limit N for X in the volume element under consideration. This complicated construction is not however our primary concern in this chapter. Our objective here is only to explore the dynamical system theoretic behaviour of (1.5.2) and investigate its possible implication into the therapy administered on a cancer patient by means of induced fluctuations [44, 102].

In subsection B of Section 6.1 we have undertaken some (crisp) dynamical system theoretic analysis of (1.5.2) by means of critical points and Hopf bifurcation. Hopf bifurcation is typically a multidimensional phenomenon, which enters into the solution space of (1.5.2) because it has complex solutions. It has been suggested that the Hopf bifurcation is responsible for the existence of the bistability exhibited by a bimodal probabilistic distribution function in [89].

In Section 6.2 we have taken up the uncertainty management aspect of (1.5.2). u is assumed to remain fixed, but r and v are liable to fluctuate due to inherent uncertainty in determining their values. This fluctuation has been modelled by a stochastic differential equation in [89]. In [89] these fluctuations have been treated as spontaneous physiological activity. But in [44] it has been suggested that these fluctuations may be artificially induced into the body of a tumour patient through suitable (multiple) therapy in order to regress the tumour. Throughout the Section 6.2 we have taken the fluctuations in the values of r and v as entirely induced by an human expert (the cancer specialist administering the therapy). r is proportional to the tumour cell destruction rate and v is proportional to the new tumour cell growth rate. During a therapy these parameters need to be controlled. In the course of treatment the expert often does not resort to statistical analysis to determine the amount of fluctuation needs to be induced. On the contrary, he usually relies on an heuristic subjective estimation based entirely on his skill and past experience. So the uncertainty involved in the fluctuations here is not objective but subjective in nature and hence can not be modelled by probability theory. Fuzzy set theory is the ideal tool to model this type of uncertainty [13, 65]. In this section we have treated the process of evolution of tumour in human body as an evolutionary uncertain system as suggested in [44] and modelled it by fuzzy differential inclusion relations [102]. To convert (1.5.2) into an FDI we must have a reasonable estimate for the entire fluctuating ranges of r and v. In the first part of Section 6.2 we have endeavoured to have reasonable estimation for the ranges of r and v, within which they can fluctuate. We have achieved this goal by extensive computer simulation with the help of the data available from [44, 52, 88, 89]. One remarkable feature of this investigation is that the present model is predictable only for $m \geq 1$ and $v < 2$. This suggests the need for introduction of new models for $m < 1$ or $v \geq 2$. Then in the next part of Section 6.2 we have presented the FDI simulation of the model with the help of Algorithm 3.2.1. The best possible solution (in the sense that the fuzzy membership grade of the solution is 1) region indicates the desired effects of the therapy on the patient. In subsection D we have made therauptic suggestions based on our modelling. We have suggested that, perturbations in the form of increasing arterial pO_2 to 90 mmHg, decreasing venus pO_2 to 20 mmHg, etc. under the heading of "Multistep therapy" and combined perturbation of glucose level, temperature, anti-tumour drugs, radiation, etc. under the heading "Multiplex therapy."

We conclude this thesis in Chapter 7 with a summary and a discussion on future directions.

In Appendix A we have explored the meaning of time from a dynamical system theoretic point of view [95]. In Appendix A our aim is to make a thorough investigation about how the set of real numbers R has been associated with the notion of time. In Section A.1 a formalization of time is presented. To do this first a very broad based definition of event (not in sense of relativity) is proposed. We have proposed that time is an orderer of events. Time itself is a standard event, for example, the event of Sun's diurnal motion with respect to the Earth (as seen by an observer on the Earth). To show that, T is the positive real axis R^+ we have first mapped deffiomorphically the trajectory of the Sun round the Earth onto a circular helix in R^3 whose axis is say, the Z-axis. Each complete circular motion on this helix will give a day's time and total number of units advanced along the Z-axis will give the number of days that have elapsed. Then we have mapped deffiomorphically the portion of the helix above the xy-plane onto the positive Z-axis. Then with the notion of reverse process [129] we have extended T to the whole of R (Theorem A.2.1). So time can be represented by R (as is the standard practice in the dynamical systems theory) and that, symbols like dx/dt are well-posed [85] at least from the point of view of t.

In Appendix B we have discussed about Mandelbrot's definition of fractals [105]. We have stated this definition in Chapter 4 and used it to define fuzzy fractals. Mandelbrot's definition of fractals is rather complicated and demands some knowledge of set theory, and we have made an attempt to explain this important definition term by term as pedagogically as possible. In Lemma B.1 we have proved the existence of Hausdorff dimension. Then we present the definition of Hausdorff dimension. Then we have stated topological dimension. In Lemma B.2 we have established that, if the Hausdorff dimension of a metric space A is s then the topological dimension (small inductive dimension) of A = [s]. [s] is the greatest integer less than or equal to s. From this we have derived an equivalent definition for fractals.

In Appendix C we have discussed chaoticity of logistic function $f(x) = \mu x(1 - x)$, where $x \in [0, 1]$. It is well known that this function is chaotic under iteration on [0, 1] for $\mu \geq 4$. Devaney has given a proof of the same for $\mu > 2 + \sqrt{5}$ and says that the proof for the case $\mu \geq 4$ is difficult [30]. But we need the fact that f is chaotic under iteration for $\mu = 4$ in Section 4.2, where we have constructed a fuzzy dynamical system modelling of a turbulence. So, for the sake of completeness of our model we have given an original proof of this important fact in Appendix C. First we have shown that, for each positive integer n there are 2^n fixed points of f^n ($\mu \geq 4$) in [0, 1]. Each of them is not necessarily of period n. This result and the reasoning to prove it are of great help in subsequently proving that, (1) f is topologically transitive on S for $\mu \geq 4$ and (2) periodic points of f are dense. Then by the result of Banks et al., the sensitive dependence on initial condition follows immediately [8]. So f is chaotic under iteration on [0, 1] for $\mu \geq 4$.

Chapter 2

Uncertainty Management in Phase Spaces

The closer one looks at a real world problem,
the fuzzier becomes its solution.
- L. A. Zadeh [154]

In this chapter we will be concerned with modelling the uncertainty inherent in the phase space of a dynamical system. We have already seen in the last chapter that uncerttain or fuzzy states in the phase space give rise to fuzzy phase spaces. A fuzzy phase space is a collection of fuzzy points. It is already well recognized that, the fuzzy numbers are great tools to measure uncertain quantities [43, 96]. In this chapter we shall explore the role of the fuzzy numbers in describing fuzzy points or fuzzy states [98].

In classical crisp dynamical systems usually the phase space is either R^n or a subset of R^n, where R is the (crisp) real number system and n is a positive integer. Here our aim is to investigate how far this is true in case of fuzzy phase spaces. If $\Re$ is the system of fuzzy real numbers, what is the relation between $\Re^n$ and a fuzzy phase space? This is a very important question needs to be answered. Fuzzy quantities, fuzzy real numbers and fuzzy arithmetic have been discussed in [35, 36, 69, 76, 81, 123]. Multidimensional fuzzy real number system $\Re^n$ and its application in pattern recognition have been discussed in [98]. R^n is algebraically a vector space and topologically a metric space. Fuzzy vector space and fuzzy topological vector space were first defined by Katsaras and Liu [75]. More recently a more generalized fuzzy topological vector space has been defined by Buckley and Yan [20]. We have defined a vector space like structure on $\Re^n$, which is more general than Katsaras and Liu structure but less general than Buckley and Yan structure. $\Re^n$ is the phase space of some important FDSs [17, 18]. For all other FDSs fuzzy states can be derived from $\Re^n$. A metric has also been defined on $\Re^n$. We have shown how arbitrary fuzzy points, fuzzy feature vectors or fuzzy pattern vectors can be obtained from $\Re^n$.

In this chapter we shall try to find connections among these various phase or state spaces defined for different fuzzy dynamical systems. We have already noticed in the beginning of this chapter that the classical crisp dynamical systems mostly evolve on n-tuples of crisp real numbers. That is, in this case the underlying phase spaces are either R^n or some subset of R^n for some positive integral n. The reason behind this is that R^n has a very rich algebraic and topological structure. Algebraically R^n is a vector space and topologically it is a complete metric space. The roots of the structures of R^n lies in R itself. R is an algebraic field and topologically it is a complete metric space, which is also endowed with a total or linear order relation. An immediate question is, "How far all these developments of the phase spaces of crisp dynamical systems can be carried out to their fuzzy counterpart?" In other words this question means, "Do all the fuzzy phase spaces belong to $\Re^n$?" For the systems defined by Buckley and Feuring in [17] and [18] the answer is "Yes." In this chapter we shall try to find an answer to the above question for other fuzzy dynamical systems.

In almost all cases fuzzy dynamical systems evolve in $F(R^n)$, the space of fuzzy subsets of R^n, for some positive integral n. Of particular importance are the nonempty compact subsets of $F(R^n)$ on which a Hausdorff distance metric can be defined. More restricted subsets of $F(R^n)$ have been considered for various special purposes. A lot of mathematical theories have already come up about them. But almost no attempt has so far been made to connect them with $\Re^n$. In this chapter we have made an attempt to develop a theory of fuzzy phase spaces starting right from $\Re$. In Section 2.2 we shall concentrate into developing an algebraic structure over $\Re$. Our ultimate aim in this chapter is to develop an algebraic as well as a topological structure over $\Re^n$. Next we shall show how various fuzzy phase spaces can be obtained from $\Re^n$. During this entire development we shall try to follow the path of similar developments in R^n as closely as possible.

2.1 Fuzzy Phase Space Preliminaries

According to Kloeden [82], "...... the underlying state spaces of the fuzzy dynamical systems are complete, locally compact metric spaces. ..... a fuzzy dynamical system can be considered as a crisp dynamical system on the state space of nonempty compact fuzzy subsets of the underlying state space." In [21] discrete FDSs in the form of IFZSs are evolving in the space $F^*(X)$, where (X, d) is a compact metric space (hence complete) $F^*(X)$ is a collection of all fuzzy subsets of X such that, if $u \in F^*(X)$ then

(1) u is uppersemicontinuous (u.s.c) on (X, d);
(2) u is normal, that is, $u(x_0) = 1$ for some $x_0 \in X$.

These properties yield the following results [21]:

(a) For each $0 < \alpha \leq 1$, the α-level set, defined as $[u]^\alpha = \{x \in X \mid u(x) \geq \alpha\}$ is a nonempty compact subset of X;
(b) The closure of $\{x \in X \mid u(x) > 0\}$, denoted by $[u]^0$ is also a nonempty compact subset of X.

Discrete fuzzy dynamical systems have also been discussed in [10] and the phase space here is taken as the space of all the fuzzy subsets of R^n for some positive integral n. More general fuzzy dynamical systems have been defined in [43] by extending Kloeden's definition of fuzzy dynamical systems [82], where the underlying phase space is taken as $(\Im, \delta)$. $\Im$ is the collection of all the nonempty compact fuzzy subsets of a complete locally compact metric space (X, d) and δ is the metric defined on $\Im$ (δ is a metric). De Glas has considered fuzzy subsets of $X \subset R^n$ as the state space of fuzzy systems [27]. Fuzzy differentiable dynamical systems need a more specialized underlying phase space to be defined. Kaleva has defined fuzzy differential equations on (E^n, D) [67], where $E^n = \{u : R^n \to [0, 1] \mid u$ satisfies (i) - (iv) below$\}$:

(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that, $u(x_0) = 1$;
(ii) u is fuzzy convex;

(iii) u is upper semicontinuous:
(iv) $[u]^0 = \text{closure}\{x \in R^n \mid u(x) > 0\}$ is compact.

The metric D: $E^n \times E^n \to R^+ \cup \{0\}$ is defined by the equation

$$D(u, v) = \sup_{0 \le \alpha \le 1} d([u]^\alpha, [v]^\alpha) \qquad (2.1.1).$$

where $u, v \in E^n$ and d is the Hausdorff metric (Definition 2.1.1) defined on the collection of all nonempty compact convex subsets of R^n. Seikkala has defined fuzzy differential equations on E^1 or just E [136]. Hullermeier has also taken E^n as the state space for modelling uncertain dynamical systems through fuzzy differential inclusions [62]. Buckley and Feuring have defined fuzzy differential equations on a more general state space namely, the space of n-tuples of fuzzy real numbers [17]. They have defined fuzzy partial differential equations also on the space of n-tuples of fuzzy real numbers [18]. Since the phase space of a dynamical system must always be a metric space, we define Hausdorff distance and Hausdorff metric as follows.

Definition 2.1.1: *Hausdorff distance* between two nonempty compact subsets A, B of a metric space (X, d) is defined as $h(A, B) = \max\{D(A, B), D(B, A)\}$, where

$$D(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y) \qquad (2.1.2).$$

h is a metric on the collection of nonempty compact subsets of (X, d) [46] and we call this metric *Hausdorff metric*.

2.2 Fuzzy Real Number System $\mathfrak{R}$

A. Motivation

The concept of fuzzy numbers stems from the fact that each time we want to locate a point (value) on the real line R, irrespective of the degree of sophistication of our technique, we may end up locating some other point within some neighbourhood of that point. So, by fuzzy number we mean certain neighbourhood of a crisp number r as a whole instead of that crisp number r alone. If N(r) is the neighbourhood of r which is taken as the corresponding fuzzy number then for each $x \in N(r)$ we assign a grade of membership $\mu(x)$ to x denoting the certainty or the possibility measure [156] with which we may end up locating x when we actually intend to locate r. According to Dubois and Prade in [36], "A *fuzzy number* M is defined as a bounded support upper semicontinuous fuzzy interval with a unique modal value m: it is a representation of the imprecisely specified quantity "approximately m"." The definitions of fuzzy numbers in [54] and [81] are but slight modification of this. We call $\mathfrak{R}$ the *system of fuzzy real numbers*. In this section we are going to build up an elaborate algebraic structure on $\mathfrak{R}$ [97] and we have also defined a metric on $\mathfrak{R}$. Whenever we want to choose a value in R we end up choosing a fuzzy number in $\Re$ instead. The motivation behind all this is to recognize and accommodate *fuzzy uncertainty* in describing (complex) physical systems (by a collection of measurable attributes i.e., mathematical modelling, where each attribute is quantified by fuzzy numbers). When more than one attributes are involved, say n in number, we replace $\Re$ by $\Re^n$ [94, 98]. (We shall develop an elaborate algebraic structure on $\Re^n$ in the next section and we shall also define a metric on it.) On one hand this justifies Zadeh's contention, "Fuzziness comes from the description of complex systems." [146], and on the other it is in complete harmony with the way in which *fuzzy points* and *fuzzy vectors* are defined by the fuzzy systems community [25, 107, 116, 127]. Elaborate discussion on fuzzy numbers, fuzzy arithmetic and interval analysis appear in [35, 36, 69, 76, 81, 123].

A good way to justify the use of the fuzzy real numbers is the use of fuzzy *linguistic variable*. According to Zadeh, "... the main contribution of fuzzy logic is a methodology for computing with words. No other methodology serves this purpose." [159]. To elaborate, let us take one example of Zadeh from [158]. Let Age be a linguistic variable and its *fuzzy constraints* [159] are young, not young, very young, more or less young, not very young, old, not old, not very young and not very old, etc. Each of these values represents a label of a fuzzy subset of a *universe of discourse* which is associated with Age (in years) - e.g., the interval [0, 100]. For young (According to Zadeh, "The label *young* may be regarded as a *linguistic value* of the variable *age*, with the understanding that it plays the same role as the numerical value 25 but is less precise and hence less informative." [146]) we can choose $\mu_{[20,30]}$ from $\Re$. Similarly, for very young we can choose $\mu_{[15,25]}$ from $\Re$, etc. So we can regard $\Re$ as the collection of all possible fuzzy constraints associated with all possible linguistic variables. $\Re^n$ may be a possible mathematical frame work from where Zadeh's *computing with words* [159] may take off.

The *certainty factor* (CF) of transmission of uncertainty from *premise* to *conclusion* as described by Zadeh in [157] can be explained most naturally in terms of fuzzy real numbers. For example, let us consider the proposition, "If X is small then Y is large with CF = 0.8" from [157]. Let *small* be represented by $\mu_{[a,b]}$ and *large* by $\nu_{[c,d]}$. Let $u \in [a, b]$ such that, $\mu(u) = 1$, where $\mu(u)$ denotes the degree of 'X taking the value u'. One instance of the proposition is "If X is u then Y is v", where $v \in [c, d]$, such that, $\nu(v) = 0.8 \times \mu(u) = 0.8$. Interpretation of any other instance is similar.

B. Basic Definitions and Important Results

Definition 2.2.1: Let F(R) be the collection of fuzzy subsets of R. $A \in F(R)$ is a *fuzzy real number* if and only if (1) A is a normal fuzzy set, (2) $[A]^\alpha$ must be a closed interval of R for every $\alpha \in (0, 1]$ and (3) support of A is a compact interval of R.

In view of Theorem C.2 of [81] it is clear that, if the support of fuzzy number A, denoted by $[A]^0$ is [a, b] and $A(x) = 0$ for $x \in [a, b]$ then x = either a or b or both (for $a \neq b$). There exist $c, d \in R$ such that, $a \le c \le d \le b$ and $A([c, d]) = \{1\}$. A is increasing and continuous from right in [a, c). A is decreasing and continuous from left in (d, b]. Obviously, a fuzzy number A is a convex fuzzy set. To summarize, when $A: R \to [0, 1]$

is a fuzzy number. A is a normal, upper semicontinuous, fuzzy convex fuzzy subset of R. In other words, $A \in E^1$.

We can take the collection of all fuzzy real numbers A or $A_{[a,b]}$, where [a, b] is the support of A, and denote this collection by $\Re$. We call $\Re$ the *system of fuzzy real numbers*.

The binary operations '+' and '*' are called addition and multiplication respectively, are defined on $\Re$ as follows.

Definition 2.2.2: Let $\mu, \nu \in \Re$. Let $\delta = \mu + \nu$. Then δ is defined as follows.

$$\delta(c) = \sup_{c=a+b} \inf[\mu(a), \nu(b)] \ \forall\ a, b \in R \text{ s.t., } c = a + b. \tag{2.2.1}$$

And similarly in case of $\lambda = \mu * \nu$.

$$\lambda(c) = \sup_{c=a.b} \inf[\mu(a), \nu(b)] \ \forall\ a, b \in R \text{ s.t., } c = a \,.\, b. \tag{2.2.2}$$

Lemma 2.2.1: (a) $\Re$ is closed under '+'.
(b) $\Re$ is closed under '*'.

Proof: For the proofs one may consult [81].

Lemma 2.2.2: Associativity holds in $\Re$ both with respect to + and *.

Proof: Refer to [81].

C. Inverses of Fuzzy Numbers

Our next target is to define additive and multiplicative inverses in $\Re$. This is perhaps the biggest challenge of this section, for it is generally accepted that unique inverse for a fuzzy number does not exist [151]. Here in no way we are contradicting the established facts. We only intend to introduce some modifications to obtain a reasonable algebraic structure. To endow a fuzzy number with a 'unique' inverse we shall have to modify the classical definition of inverse. We shall accomplish it here in such a way that the principal spirit of the notion of inverse remains preserved despite allowing other compromises. The spirit of inverse lies in the symmetry property of a system. For example an algebraic system called group is symmetric (in fact groups are standard mathematical tools for modelling symmetry), because every element of a group is invertible i.e., every element of a group has a 'mirror image' with respect to some fixed element, in this case the identity element. If $x \in (G, +)$, where G is a group with respect to +, $x + (-x) = e$. $-x$ is inverse of x and e is the identity element of G. Uniqueness of $-x$ is an obvious consequence of the group axioms. Here inverse is defined with respect to the identity and this automatically preserves the mirror image property of inverse. We shall see shortly that defining inverse with respect to identity will not preserve the mirror image property in case of $\Re$. Therefore in $\Re$ we shall define inverse of an element only as the mirror image (in some sense) of that element without regard to the identity. Of course in $\Re$ identity and inverse will be related but in a less stringent way.

In this section we must keep in mind that, we are dealing with fuzzy sets and not with crisp elements and therefore we may have to relax our penchant for precision. This means, for example, we may have to replace a rather stringent predicate = by a more relaxed one like ≥ or ⊒.

Definition 2.2.3: Let $\mu \in \Re$. *Additive inverse* of μ is defined to be ν, where $\nu(x) = \mu(-x)$ $\forall\, x \in R$.

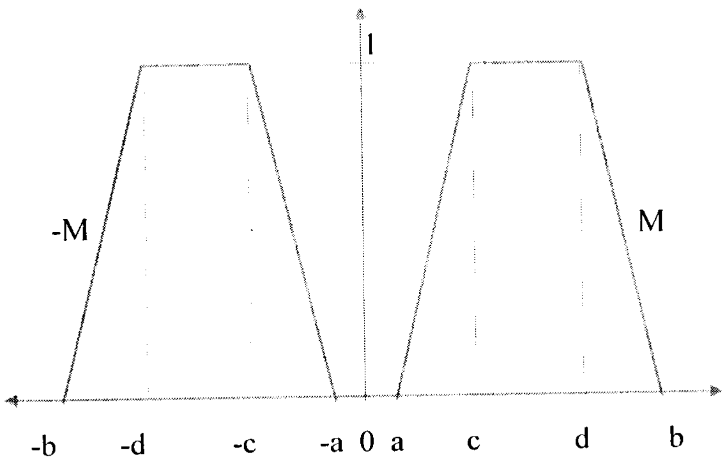


Fig 2.2.1 Trapezoidal fuzzy numbers M and -M.

Here we have taken inverse of fuzzy number μ by considering the graph of μ on the euclidean plane. We have taken the mirror image of μ on the euclidean plane with respect to the y-axis (i.e., as if a mirror is placed along the y-axis to reflect the image of the graph of μ). Automatically we get a mirror image of x on R with respect to the same mirror. The entire scheme becomes very clear at the Fig 2.2.1. Obviously additive inverse so defined for a fuzzy number is unique. We shall subsequently define additive identity of $\Re$ and the relation of additive inverse with it.

Definition 2.2.4: Let $\mu \in \Re$ and $0 \notin [\mu]^0$. Let us define δ such that, $\delta(a) = \mu(1/a)$ $\forall$ a $\notin$ R and a $\neq$ 0. Then we call δ the *multiplicative inverse* of μ.

This time to determine a multiplicative inverse of a fuzzy number we have taken mirror image of μ_x on the euclidean plane with respect to the unit circle. Apart from changing the mirror from the y-axis to the unit circle everything remains same here. It is clear that, multiplicative inverse of a fuzzy number, when exists, is unique. We shall be defining multiplicative identity of $\Re$ and its relation with multiplicative inverse in next few lines.

D. Identity and Its Relation with Inverse

Definition 2.2.5: Let $\underline{0} = \mu_0$, where $\mu_0(p) = 1$ if $p \in [0, 0]$ and $\mu_0(p) = 0$ if $p \notin [0, 0]$. We call $\underline{0}$ *additive identity* of $\Re$. Similarly, we define $\underline{1} = \mu_1$, where $\mu_1(p) = 1$ if $p \in [1, 1]$ and $\mu_1(p) = 0$ if $p \notin [1, 1]$. We call $\underline{1}$ *multiplicative identity* of $\Re$.

From Definitions 2.2.3 and 2.2.4 it is clear that both additive and multiplicative inverses of fuzzy numbers are unique. Both addition and multiplication on the set of real fuzzy numbers $\Re$ are commutative. Clearly $\mu + \underline{0} = \mu \ \forall \ \mu \in \Re$ and $\mu * \underline{1} = \mu \ \forall \ \mu \in \Re$. Also $\mu + \nu \geq \underline{0}$.

E. An Algebraic Structure on $\Re$

Fuzzy subgroups of a (crisp) group were first defined by Rosenfeld in [132]. Subsequently this definition was generalized by Negoita and Ralescu and by Anthony and Sherwood [4]. Here we have just established that, $\Re$ satisfies the following properties.

(1) $\Re$ is closed under addition, which is also commutative.
(2) $\Re$ is closed under multiplication, which is also commutative.
(3) Both addition and multiplication in $\Re$ are associative.
(4) Additive and multiplicative inverses uniquely exist in $\Re$ (up to a new definition of inverse).
(5) Additive and multiplicative identities uniquely exist in $\Re$.

Definition 2.2.6: Let G be a set (of fuzzy sets), closed under a binary operation, say +. For a,b,c $\in$ G, a + (b + c) = (a + b) + c. For each a $\in$ G, $\exists$ e $\in$ G s.t., a + e = a. For each a $\in$ G $\exists$ a unique $a^1 \in$ G, called inverse of a. The relation between identity and inverse is given by $a + a^1 \geq e$ ($\geq$ is some predicate which is a generalization of =). Then G is called an *F-group*. If the F-group is commutative then it is called *abelian F-group*.

By now we have proved the following Theorem 2.2.1, which induces an algebraic F-group structure on $\Re$ both with respect to + and $*$. Note that F-group has important differences with classical crisp groups as well as with classical fuzzy groups.

Theorem 2.2.1: $\Re$ is an abelian F-group with respect to '+' and also the collection of all elements of $\Re$, whose multiplicative inverse exist, form an abelian F-group with respect to '$*$'.

Generally distributivity property does not hold in $\Re$, but subdistributivity does, i.e., for $\mu, \nu, \delta \in \Re$, $\mu * (\nu + \delta) \leq \mu * \nu + \mu * \delta$. However when μ, ν, δ become crisp numbers (i.e., $[\mu]^0 = [a, a]$ for some $a \in R$: $\mu(a) = 1$ and $\mu(b) = 0$ for $b \neq a$. Replace μ by $\{a\}$ etc.) then of course distributivity holds.

Fuzzy field has been defined in [110]. Here we shall define F-field on line of F-group. By fuzzy field we shall understand the classical fuzzy fields only.

Definition 2.2.7: A set Γ will be called an *F-field* if and only if (1) Γ is an abelian F-group with respect to addition, (2) all the elements of Γ whose inverse exist with respect to multiplication, also form an abelian F-group with respect to multiplication, and (3) Γ also satisfies the subdistributivity property.

We have proved the following

Theorem 2.2.2: $\Re$ is an F-field.

F. The Relation Between R and $\Re$

Theorem 2.2.3: The crisp real number system R is embedded in $\Re$. In other words ordinary crisp real numbers are only special cases of fuzzy real numbers.

Proof: We will show that, R is embedded in $\Re$ as a field. We know that, $\mu \in \Re$, where $\mu(y) = 1$ if $y \in [x, x]$ and $\mu(y) = 0$ if $y \neq x$, for some x. Take the collection of all such $\mu \in \Re$ and denote it by F. It is a routine matter to verify that F is a field with respect to the standard binary operations defined on $\Re$. Let ϕ: F → R, such that, $\phi(\mu) = x$. ϕ is clearly one to one and onto. Also $\phi(\mu + \nu) = x + z$, where $\nu(y) = 1$ if $y \in [z, z]$ and $\nu(y) = 0$ if $y \neq z$. The '+' in the argument of ϕ is as defined in Definition 2.2.2. Similarly, $\phi(\mu * \nu) = x \cdot z$. '*' is defined as in Definition 2.2.2 and '.' is the product operation in R. So ϕ is a field isomorphism between F and R. i.e., R as a field is embedded in $\Re$.

G. A Metric on $\Re$

To define a metric on $\Re$, i.e., to induce the notion of distance between two points $\mu^{l}_{[a,b]}$ and $\mu^{m}_{[c,d]}$ of $\Re$ we can follow the notions of Goetschel and Voxman [54]. *l* varies over indices of all membership functions mapping R into [0, 1], satisfying the conditions laid down in (1), (2) and (3) of Definition 2.2.1 and which vanish outside [a, b]. *m* is defined similarly. Normally, we shall omit *l*, *m*. Let $\mu^* = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq \mu(x)\}$ and $\nu^* = \{(x, y) \mid c \leq x \leq d, 0 \leq y \leq \nu(x)\}$. Let H be the Hausdorff metric defined on compact subsets of R^2 and define D: $\Re \times \Re \to [0, \infty)$ by $D(\mu, \nu)) = H(\mu^*, \nu^*)$. Then D is a metric on $\Re$. To see this, it suffices to establish the triangle inequality for metric spaces. $H(\mu^*, \nu^*) = \sup [(x - x')^2 + (y - y')^2]^{1/2}$, $(x,y) \in \mu^*$ and $(x',y') \in \nu^*$. H satisfies triangle inequality [64], so does D. Henceforth we shall denote the metric space of fuzzy numbers by ($\Re$, D). When there is no risk of confusion we shall simply write $\Re$ in place of ($\Re$, D).

2.3 The Space $\Re^n$

R^n is the most important of all spaces in theory as well as in practice, not only in mathematics but in all other branches of science. R^n occupies such a central position in science because of its unique structure. This structure has two main parts: algebraic and topological. Algebraically R^n is a vector space and topologically it is a complete metric space. In this section we shall try to find out an algebraic structure for $\Re^n$ and also we shall define a metric on it.

2.3.1 The Algebra of $\Re^n$

We have defined an algebraic F-field structure over the collection of all fuzzy real numbers $\Re$ in the Section 2.2. The current subsection is a continuation of this trend in case of $\Re^n$. Development of an algebraic structure over $\Re^n$ is motivated by an analogy in the classical Linear Algebra, where an n-dimensional vector space R^n is defined over the field of real numbers R. Similarly, here also we shall try to develop a vector space like structure (though it does have important differences with classical vector spaces) for $\Re^n$ over the F-field $\Re$. Fuzzy vector spaces have already been defined by Katsaras and Liu in [75]. Recently Buckley and Aimin have defined fuzzy topological vector space [20]. Our definition of fuzzy vector space is more general than the first but less general than the second. None of them has been developed from the fuzzy numbers. To utilize the uncertain quantity modelling role of fuzzy real numbers here we have undertaken the development of fuzzy vector space from the system of fuzzy real numbers $\Re$. We would like to call the algebraic structure developed here as generalized fuzzy vector space or GFVS. In fact in this section we will show that $\Re^n$ is a GFVS. The following definition is essentially on line of the classical definition of crisp vector (linear) space [59] with necessary modifications.

Definition 2.3.1: *Generalized (real) fuzzy vector space* **V** is a collection of fuzzy subsets μ of R^n and $\Re$ be the F-field of fuzzy real numbers. **V** is an abelian F-group with respect to additive binary operation '+'. And for every $\alpha \in \Re$ and $\mu \in \mathbf{V}$, there is defined an element $\alpha\mu \in \mathbf{V}$, subject to

1. $\alpha(\mu + \nu) \leq \alpha\mu + \alpha\nu$
2. $(\alpha + \beta)\mu \leq \alpha\mu + \beta\nu$
3. $\alpha(\beta\mu) = (\alpha\beta)\mu$
4. $\underline{1}\mu = \mu$.

where $\alpha\beta = \alpha * \beta$, $\alpha, \beta, \underline{1} \in \Re$ and $\mu, \nu \in \mathbf{V}$. Let $\lambda = \alpha\mu$, then $\lambda(\mathbf{a}) = \sup_{\mathbf{a} = b\mathbf{c}} \inf(\alpha(b), \mu(\mathbf{c}))$.

a is a fixed crisp vector of R^n equal to given value of b**c**, where as usual, b varies over R and **c** varies over R^n.

It is a routine matter to verify that the generalized fuzzy vector space is well defined. From this point onwards by fuzzy vector space we shall mean generalized fuzzy vector space unless otherwise specified. From the above definition it is clear that, fuzzy vector is nothing but a bunch of crisp vectors in R^n for some positive integral n (Fig 2.3.1).

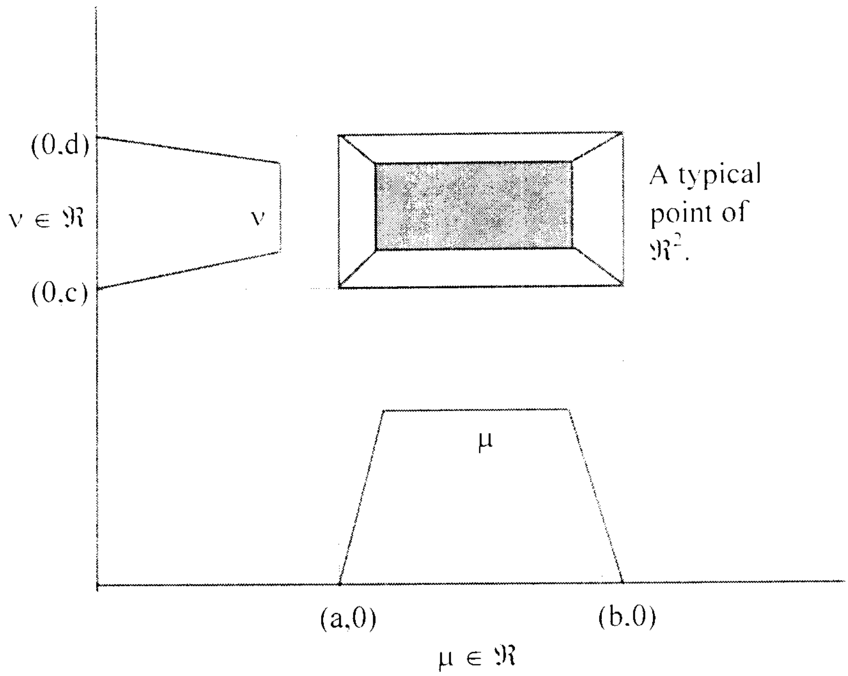


Fig 2.3.1 A two dimensional generalized fuzzy vector (or fuzzy point in $\Re^2$) as a product of two simple trapezoidal fuzzy numbers, where the associated membership function is nothing but a normal, fuzzy convex fuzzy subset of R^2 with compact support. Clearly, 'a typical point of $\Re^2$' is a collection of crisp points (vectors) of R^2.

We have already defined F-group and F-field in the previous section and by now we have completed the definition of generalized fuzzy vector space too. Our next theorem will help us to reformulate the definition of GFVS in an equivalent yet intuitively more appealing form.

Theorem 2.3.1: Let R be the field of crisp real numbers. Let V_n be an n dimensional (crisp) vector space over R. Let B = $\{e_1, e_2, \ldots\ldots\ldots\ldots, e_n\}$ be a basis of V_n, $e_i = (0, 0, \ldots\ldots, 1(\text{ith position}), \ldots.., 0)^T$. Let B′ = $\{\mu_{e1}, \mu_{e2}, \ldots\ldots\ldots\ldots, \mu_{en}\}$, where $\mu_{ei}(e_i) = 1$ and $\mu_{ei}(v) = 0$ for $v \neq e_i$, $v \in R^n$. Take the linear span [59] of B′ over $\Re$ and denote it by χ. Then χ is a GFVS.

Proof: First we will show that $B' \subset \Re^n$ in the sense that each coordinate of an element of B′ belongs to $\Re$. $\mu_{ei} \in B'$ for some $i \in \{1, \ldots\ldots, n\}$. ith coordinate of e_i is 1 and all other coordinates are 0. To get the ith coordinate of μ_{ei}, $\underline{1} \in \Re$ is taken. To get all other coordinates of μ_{ei}, $\underline{0} \in \Re$ is taken. $\mu_{ei}(x) = \min\{\mu_0(x_1), \mu_0(x_2), \ldots\ldots\ldots, \mu_1(x_i), \ldots\ldots\ldots, \mu_0(x_n)\}$ (this notion is very standard and has been used by Katsaras and Liu [75]), where $x = (x_1, \ldots\ldots\ldots x_n)$ and μ_1, μ_0 have usual meaning. Clearly, $\mu_{ei}(x) = 1$, for $x = e_i$ and 0 otherwise.

Now take the linear span of B′ over $\mathfrak{R}$ defining membership of αe_i according to Definition 2.3.1. Let $x \in \chi$. Then $x = \alpha_1 e_1 + \ldots\ldots\ldots\ldots + \alpha_n e_n$ where $\alpha_i \in \mathfrak{R}$ for all $i \in \{1, \ldots\ldots, n\}$. Now it is a routine matter to verify that χ satisfies all the properties of Definition 2.3.1.

Definition 2.3.2: Let V_n be an n-dimensional crisp vector space over the field of crisp real numbers R. B is the basis of V_n as described in Theorem 2.3.1. $\mathfrak{R}$ be the F-field of fuzzy real numbers. B′ be as described in Theorem 2.3.1. Then the *corresponding fuzzy vector space* denoted by $\mathbf{V_n}$ is the linear span of B′ over $\mathfrak{R}$. We also write $\mathfrak{R}^n$ in place of $\mathbf{V_n}$. $\mathfrak{R}^n$ is the collection of all elements of the form $\mu = (\mu_1, \ldots\ldots\ldots \mu_n)$ where $\mu_i \in \mathfrak{R}$ for all $i \in \{1, 2, \ldots\ldots, n\}$ and $\mu((x_1, x_2, \ldots\ldots\ldots x_n)) = \min\{\mu_1(x_1), \mu_2(x_2), \ldots\ldots\ldots \mu_n(x_n)\}$, $x_i \in R$ for all $i \in \{1, 2, \ldots\ldots\ldots, n\}$).

Definition 2.3.3: *Dimension* of the corresponding fuzzy vector space $\mathbf{V_n}$ in Definition 2.3.2 is n. B′ will be called a *basis* of $\mathbf{V_n}$.

Definition 2.3.4: $\mathbf{v_1}, \mathbf{v_2}, \ldots\ldots, \mathbf{v_k} \in \mathbf{V_n}$ are said to be *linearly independent* if and only if $\lambda_1\mathbf{v_1} + \ldots\ldots\ldots\ldots + \lambda_k\mathbf{v_k} = \underline{\mathbf{0}}$ implies that. $\lambda_1 = \ldots\ldots\ldots = \lambda_k = \underline{0}$. Where $\lambda_1, \ldots\ldots\ldots, \lambda_k \in \mathfrak{R}$. $\underline{\mathbf{0}} = \mu_0$, where $\mu_0(0, \ldots\ldots, 0$ (nth term)) = 1 and $\mu_0(v) = 0$, for $v \neq (0, \ldots\ldots, 0$ (nth term)). $\underline{\mathbf{0}}$ is the additive identity of $\mathbf{V_n}$.

Theorem 2.3.2: The crisp n-dimensional real vector space is contained in $\mathbf{V_n}$ (= $\mathfrak{R}^n$).

Proof: Let E be a real vector space of dimension n. Then we take $\mathbf{V_n}$ defined over $\mathfrak{R}$. In the first step we 'extract' R from $\mathfrak{R}$. To do this, for each $x \in R$ take only the closed bounded interval [x, x] with the membership function $\mu : R \to [0, 1]$ s.t., $\mu(x) = 1$ and $\mu(y) = 0 \ \forall \ y \neq x$. It is a routine matter to verify that $[x, x] + [y, y] = [x + y, x + y]$. Where membership value of $x + y$ is 1, for membership value of both x and y are 1. [-x, -x] exists and membership value of -x is 1 etc. So the collection of fuzzy real numbers of the form [x, x] along with the membership function defined above make an F-field, where $x - x = 0$ whose membership value is 1 and $x * 1/x$ (whenever defined) = 1 with membership value 1. $x * (y + z) = x * y + x * z$, membership values on both sides are 1 by Definition 2.2.2. So this F-field is reduced to a field, with each member having a membership value 1. And this is precisely the field of crisp real numbers R. Now, any n-dimensional vector space defined over R (i.e., R^n for some n) is clearly a subset of $\mathbf{V_n}$, for $\mathbf{V_n}$ contains all the fuzzy sets $\boldsymbol{\mu}$, where $(x_1, \ldots\ldots, x_n) \in R^n$ and $\mu((x_1, \ldots\ldots\ldots x_n)) = 1$, $\mu((y_1, \ldots\ldots\ldots\ldots, y_n)) = 0$ if $y_i \neq x_i$ for any $i \in \{1, \ldots\ldots, n\}$.

In Theorem 2.4.1 in the next section we shall prove that the Katsaras-Liu fuzzy vector space [75] over R^n can be obtained by construction from $\mathbf{V_n}$.

By now we have completed, what may be called a reasonable definition of $\mathfrak{R}^n$ as a fuzzy vector space. Since the vector space structure is the principle algebraic structure of R^n, we have been able to develop a somewhat similar algebraic structure for $\mathfrak{R}^n$ also.

2.3.2 A Metric on $\Re^n$

We have already seen that, each real fuzzy vector **u** is a fuzzy subset of R^n for some positive integral n. Each coordinate of **u** being compact interval of R is a compact set. So by Tychonof's theorem [138] **u** is a compact (fuzzy) subset of R^n. Since there is at least one element in each **u** with membership value 1, each **u** is nonempty. $\Re^n$ is collection of fuzzy sets μ. Where μ is a membership function such that, $\mu(\mathbf{v}) = \min(\mu_1(v_1), \mu_2(v_2), \ldots, \mu_n(v_n))$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ for $\mathbf{v} \in \mathbf{u}$ and $\mu(\mathbf{v}) = 0$ for $\mathbf{v} \notin \mathbf{u}$.

Definition 2.3.9: To define a metric D′ on $\Re^n$, let us take $\mu_{\mathbf{u}}, \mu_{\mathbf{v}} \in \Re^n$. $D'(\mu_{\mathbf{u}}, \mu_{\mathbf{v}}) = \sup\{H((u, x), (v, y))$ for $u \in \mathbf{u}$, $v \in \mathbf{v}$ and $0 \le x \le \mu_{\mathbf{u}}(u)$, $0 \le y \le \mu_{\mathbf{v}}(v)\}$.

Since $R^n \times [0, 1]$ is a complete metric space, $(\Re^n, D')$ is a complete metric space (Theorem 4.4 of Edgar [46]).

Lemma 2.3.1: For any $x \in R$ and any membership value m ($0 \le m \le 1$) given to x, $\exists$ a fuzzy number $\mu \in \Re$ such that, $\mu(x) = m$.

Proof: We will choose a suitable [a, b] and a trapezium shaped μ over it. Obviously μ is a trapezoidal fuzzy number with support [a, b]. We take a < x. Take the straight line on Euclidean plane (X, Y), passing through (a, δ) and (x, m), i.e., $(X - a)/(a - x) = (Y - \delta)/(\delta - m)$. Where δ is a fixed nonnegative crisp real number. We can take δ as small as we please. Take the intersection of this line with Y = 1. Denote this point of intersection by B. Denote (a, δ) by A, (b, 1) by C and (b, δ) by D, where b > x. μ is so chosen that the graph of $\mu(X)$ is the trapezium ABCD for $a \le X \le b$ and $\mu(X) = 0$ otherwise and thus we get $\mu(x) = m$.

Theorem 2.3.3: The resulting metric topology $(\Re^n, D')$ is homeomorphic to the product topology of $\Re^n$, where $\Re$ is equipped with the D-metric topology as described at Subsection G of Section 2.2.

Proof: To prove this, all we need to show is that, $(\Re, D_i')$ is topologically homeomorphic to $(\Re, D)$, where D_i' is the induced metric on the ith coordinate set by D′, i.e., in other words $p_i((\Re^n, D')) = (\Re, D_i')$, where p_i is the ith projection map of $(\Re^n, D')$ onto $(\Re, D_i')$.

A typical element μ of $\Re^n$ is, more explicitly, of the form $\mu_{(I_1, I_2, \ldots, I_n)}$, where I_i is compact interval of R. $\mu(x_1, \ldots, x_n) = \min(\mu_1(x_1), \mu_2(x_2), \ldots, \mu(x_n))$, where $x_i \in I_i$ and $\mu_i \in \Re$. We define $p_i(\mu_{(I_1, I_2, \ldots, I_n)}) = \mu_i$. Notice that x_i varies over R. $\mu_i(x_i) > 0$ if $x_i \in I_i$ and $\mu_i(x_i) = 0$ if $x_i \in R - I_i$. Let $x_i = x_i'$ for some $x_i' \in I_i$. Let $\mu_i(x_i') = m$. If $m = \min(\mu_1(x_i'), \ldots, \mu_i(x_i'), \ldots, \mu_n(x_i'))$, where $x_i' \in I_i$ $\forall$ i. But for this $\mu_j(x_j') \ge m$ $\forall$ $j \ne i$. We can always choose values of $\mu_j(x_j') \ge m$ by Lemma 2.3.1 $\forall$ $j \ne i$. So, for any x_i' we can reconstruct $\mu_i(x_i')$. This way we get $p_i((\Re^n, D')) = (\Re, D_i')$. It remains to show that D_i' and D are equivalent metrics on $\Re$. But, $D'(\mu, \nu) = D_i'(\mu_i, \nu_i)$, where for $j \ne i$, $\mu_j = \nu_j = \delta_{[a,a]} = \delta$ such that, $\mu_j(a) = \nu_j(a) = \delta(a) = 1$ and $\mu_j(x) = \nu_j(x) = \delta(x) = 0$ when $x \ne a$ for some fixed

$a \in R$. Now from definition of D in Subsection G of Section 2.2 and definition of D' in Definition 2.3.9 it is clear that, $D_i'(\mu_i, v_i) = D(\mu_i, v_i)$. This completes the proof.

2.4 $\Re^n$ as Generator of $F(R^n)$

Application of fuzzy set theory based techniques often involves the metric space of normal fuzzy convex fuzzy sets over R^n, where the metric is denoted by the supremum of the Hausdorff distances between corresponding level sets. This metric has been found to be very convenient in studying fuzzy dynamical systems, fuzzy differential equations, fuzzy differential inclusions and chaotic iterations of fuzzy sets. So it would be very natural for us to want to find out the relationship of the normal fuzzy convex fuzzy sets over R^n with $\Re^n$. First we shall show that $F(R^n)$ is generated by $\Re^n$. The set of normal fuzzy convex fuzzy sets of R^n is a subset of $F(R^n)$. In the last section we have already described the defining properties of the normal fuzzy convex fuzzy subsets of R^n. Let us reformulate them in the following

Definition 2.4.1: E^n be collection of all the functions u of the form $u : R^n \rightarrow I$ and satisfying the following conditions

(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that, $u(x_0) = 1$;
(ii) u is fuzzy convex, i.e., for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$;
(iii) u is uppersemicontinuous; and
(iv) the closure of $\{x \in R^n \mid u(x) > 0\}$, denoted by $[u]^0$, is compact.

u is called *compact supported normal fuzzy convex fuzzy subset* of R^n. For the sake of brevity we shall call them *basic n-fuzzy set.*

Theorem 2.4.1: For every $\mu \in \Re^n$, μ is a basic n-fuzzy set.

Proof: We have already seen in Section 2.2 that, when μ is a fuzzy number, $\mu : R \rightarrow I$, is a normal, upper semicontinuous, fuzzy convex fuzzy set. So, for each fuzzy number $\mu \in \Re$ μ is a basic 1-fuzzy set. Now we shall show that, for any $\mu \in \Re^n$, $n > 1$, μ is a normal, upper semicontinuous, fuzzy convex fuzzy set.

In each fuzzy number μ_i is normal. So by Definition 2.3.2 $\mu = \min(\mu_1, \ldots\ldots, \mu_n)$ is clearly normal. To prove the fuzzy convexity of μ we must show that for any $(x_1, \ldots\ldots, x_n), (y_1, \ldots\ldots, y_n) \in R^n$ $\mu(\lambda(x_1, \ldots\ldots, x_n) + (1 - \lambda)(y_1, \ldots\ldots, y_n)) \geq \min\{\mu(x_1, \ldots\ldots, x_n), \mu(y_1, \ldots\ldots, y_n)\}$, where $\lambda \in [0, 1]$. We have $\mu(\lambda(x_1, \ldots\ldots, x_n) + (1 - \lambda)(y_1, \ldots\ldots, y_n)) = \mu_i(\lambda x_i + (1 - \lambda)y_i)$ for some i (Definition 2.3.2). $\mu_i(\lambda x_i + (1 - \lambda)y_i) \geq \min\{\mu_i(x_i), \mu_i(y_i)\}$, for μ_i is fuzzy convex for each i. But $\mu((x_1, \ldots\ldots, x_n)) = \mu_j(x_j) \leq \mu_i(x_i)$. Similarly, $\mu(y_1, \ldots\ldots, y_n) \leq \mu_i(y_i)$. So $\min\{\mu(x_1, \ldots\ldots, x_n), \mu(y_1, \ldots\ldots, y_n)\} \leq \min\{\mu_i(x_i), \mu_i(y_i)\} \leq \mu_i(\lambda x_i + (1 - \lambda)y_i) = \mu(\lambda(x_1, \ldots\ldots, x_n) + (1 - \lambda)(y_1, \ldots\ldots, y_n))$. To prove upper semicontinuity of μ we observe that each μ_i is upper semicontinuous. Let we are to verify upper semicontinuity of μ at an arbitrary point $(x_1, \ldots\ldots, x_n)$. Let $B_\epsilon((x_1, \ldots\ldots, x_n))$ is an open ϵ-ball centering $(x_1, \ldots\ldots$

x_n). p_i be the ith projection map on R^n. $p_i(B_{\in}((x_1, \ldots\ldots\ldots x_n))) = N_{\in'}(x_i)$. is a $\in'$ open neighbourhood of x_i in R.

$$\lim_{\in' \to 0} \mu_i(\bigcap_{\in' > 0} N_{\in'}(x_i)) = \mu_i(\lim_{\in' \to 0} \bigcap_{\in' > 0} N_{\in'}(x_i)).$$

Since the above equation is true for any i and $\mu(x_1, \ldots\ldots x_n) = \mu_i(x_i)$ for some i. we have

$$\lim_{\in \to 0} \mu(\bigcap_{\in > 0} B_{\in}((x_1, \ldots\ldots\ldots x_n))) = \mu(\lim_{\in \to 0} \bigcap_{\in > 0} B_{\in}((x_1, \ldots\ldots\ldots x_n))).$$

This implies that, μ is upper semicontinuous. To verify property (iv) of Definition 2.4.1 in case of μ we notice that, cartesian product of compact supports of all μ_i, $i \in \{1, \ldots\ldots\ldots n\}$ gives the support of μ. Hence by Tychonof's theorem $[\mu]^0$ is compact.

Theorem 2.4.1 also establishes that each element of $\Re^n$ is a fuzzy vector as defined in Definition 3.1 of [20]. Of course the fuzzy vectors defined in [20] are more generalized entities compared to the GFVS defined in this chapter. Our next target is to express any arbitrary basic n-fuzzy set of E^n as fuzzy union of fuzzy intersections of μ's, where $\mu \in \Re^n$. Let $\mathbf{u} \in E^n$. We shall show that for each $\mathbf{u}$ there exists a sequence $\{\mu^{k1} \cap \mu^{k2}\}$ such that, μ^{k1}, $\mu^{k2} \in \Re^n$ for each k (k varies over some suitable index set), satisfying the equation

$$\mathbf{u} = \cup_k (\mu^{k1} \cap \mu^{k2}).$$

The fuzzy set intersection and union are in sense of Zadeh [152]. This method has an exact analog in General Topology, where basic open sets are obtained as (finite) intersection of subbasic open sets. Then any arbitrary open set is expressed as union of basic open sets [138].

To prove the above equality we shall show that, for any $x = (x_1, \ldots\ldots\ldots, x_n)$ such that, $\mathbf{u}(\mathbf{x}) = m \in (0, 1]$ there exist μ^{k1} and μ^{k2}, as described above, such that, $\mu^{k1} \cap \mu^{k2}(\mathbf{x}) = m$ and $\mu^{k1} \cap \mu^{k2}(y) = 0$ for any $y \neq x$. In place of the above equality we can also write

$$\mathbf{u}(X) = \sum_{x \in X} (\mu^{x1} \cap \mu^{x2}(x))/x$$

for $X \subseteq R^n$ in standard terminology.

Let $i \in \{1, \ldots\ldots\ldots, n\}$. Let us define μ_i in the following manner

$$\begin{aligned} \mu_i(y_i) &= 0 \quad \text{for } -\infty \le y_i < x_i - \in \ (\in > 0) \\ &= 1 \quad \text{for } x_i - \in \le y_i \le x_i - \in/2 \\ &= m \quad \text{for } x_i - \in/2 < y_i \le x_i \\ &= 0 \quad \text{for } x_i < y_i \le \infty \end{aligned}$$

It is straight forward to verify that, $\mu_i \in \Re$. Let $j \in \{1, \dots\dots\dots n\}$ and $j \neq i$. Take $\mu_j \in \Re$, such that,

$$\mu_j(y_j) = 1 \text{ if } y_j = x_j \text{ and}$$
$$= 0 \text{ if } y_j \neq x_j.$$

Now consider fuzzy numbers μ_1 with support $[x_1, x_1]$, μ_2 with support $[x_2, x_2]$, ... μ_i with support $[x_i - \in, x_i]$, ... and μ_n with support $[x_n, x_n]$. From Definition 2.3.2 we have $\mu \in \Re^n$ with support $([x_1, x_1], [x_2, x_2], \dots\dots [x_i - \in, x_i], \dots\dots [x_n, x_n])$. We take $\mu = \mu^{k1}$. Let $\mu' \in \Re^n$, where $\mu_{j0} \in \Re$ for all $j \in \{1, \dots\dots\dots n\}$ and $\mu' = \min(\mu_{10}, \dots\dots\dots \mu_{n0})$. μ_{j0} for all j is given by

$$\mu_{j0}(y_j) = 1 \text{ for } y_j \in [x_j, x_j] \text{ and}$$
$$= 0 \text{ for } y_j \notin [x_j, x_j].$$

Clearly,

$$\mu'((x_1, \dots\dots, x_n)) = 1 \text{ and}$$
$$= 0 \text{ otherwise.}$$

We take $\mu' = \mu^{k2}$. From the above construction we have $\mu^{k1} \cap \mu^{k2}(x) = m$ and $\mu^{k1} \cap \mu^{k2}(y) = 0$ for any $y \neq x$. Obviously $\mu^{k1} \cap \mu^{k2}$ does not belong to $\Re^n$.

Definition 2.4.2: We shall call the above *fuzzy union-intersection* operation *max-min* operation or just *m-m* operation.

Definition 2.4.3: We call $\mu \in \Re^n$, *n dimensional elementary fuzzy set* or simply *n-set*.

The biggest disadvantage of the m-m operation over the collection of n-sets is that, the collection of n-sets is not closed under this operation. But it is clear from the above calculation that the m-m operation is indeed a very useful method to obtain not only any basic n-fuzzy subset of R^n but also any other member of $F(R^n)$ starting from the n-sets as base sets. So, we have already proved that,

Theorem 2.4.2: We can get any fuzzy subset of R^n (including the Katsaras-Liu fuzzy vector space over R^n) by m-m operation over the collection of all n-sets or in other words $F(R^n)$ can generated by $\Re^n$ by m-m operations.

Definition 2.4.4: Collection of all fuzzy subsets of R^n is denoted by $F(R^n)$. Collection of all n-sets be denoted by L_n. Let L_n be called a *generator* of $F(R^n)$.

Theorem 2.4.2 is significant from an application point of view. Fuzzy vector has immense potentiality for application, particularly in pattern recognition and computer vision. But in any experiment the object under investigation may be of any arbitrary shape and size. In those cases fuzzy vector in the form of fuzzy feature vector, fuzzy pattern vector, etc., may be any arbitrary fuzzy subset of R^n for some n. Compared to this, the Katsaras-Liu fuzzy vector space over R^n or the basic n-fuzzy sets may be too restricted entities to come in any use. In these cases m-m operation is a very useful way (though may not be very simple yet computationally) to get these arbitrary fuzzy feature vectors or fuzzy pattern vectors starting from n-sets, though the collection of n-sets is not closed under m-m operation. In the next section we shall explore some relevance of $\Re^n$ in fuzzy pattern recognition.

2.5 Some Applications of $\Re^n$ in Uncertainty Management in Pattern Recognition Systems

Fuzzy set theory provides an approximate but effective and flexible way of representing, manipulating and utilizing vaguely defined data and information and of describing the behaviours of the systems that are too ill-defined to admit of precise mathematical analysis by classical methods and tools [42, 116, 118].

The most obvious manner in which the fuzzy set theory can be applied to Pattern Recognition (PR), is to treat the classes as fuzzy sets [42, 116]. Hence each pattern does not necessarily belong to one class, rather, there is a certain degree of possibility that it belongs to each of the classes to possibly different degrees, the extent being measured in terms of the membership functions. If the information processing ceases immediately after classification then nothing extra can be gained by using fuzzy set theory, since one has to ultimately decide in favour of just one class, possibly the one for which the unknown pattern has the maximum membership value. The different possibilities acquire added significance only if the import of the decision propagates into a network of other related decisions. In such situations it is possible to use fuzzy set theory to advantage for combining the evidence borne in a collection of membership function values [42].

The concept of fuzzy sets can be used at the feature level for representing input data as an array of membership values denoting the degree of possibility of certain properties, at the classification level, for representing class membership of objects and for providing an estimate (or a representation) of missing information in terms of membership values [13]. In this section we shall be considering two specific examples to elaborate this point with a view to suggest how the GFVS $\Re^n$ can be connected to them.

A. Recognition of Speech Pattern by Fuzzy Techniques

Our first example is from the applications based on the theory of fuzzy sets in the problems of computer recognition of vowels and identifying the person from his/her spoken words, where only three features of the unknown utterance are considered [116].

The concept of pattern classification may be viewed as a partition of feature space or a mapping from the feature space to the decision space [117]. More often than not, situations in the field of natural and social sciences are too complex for precise mathematical analysis. Classes of objects in that field do not have well-defined criteria of membership. To demonstrate such complexly behaved systems, the concepts of fuzzy sets and the subsequent developments in decision process could be applied to a reasonable extent. This concept is approximate but provides an effective and more flexible basis for analysis of systems which are not precisely defined [116].

An unknown pattern has been considered as $X = (x_1, \ldots\ldots, x_n)^T$, where T denotes the transpose, x_n denotes the measured nth feature of the event, represented by a point in the multidimensional vector space R^n. There are m ill-defined pattern classes $C_1, \ldots\ldots, C_m$. Let $R_1, \ldots\ldots, R_m$ be the m reference vectors, where each class C_j contains h_j number of prototype patterns $R_j^{(l)}$, for $l \in \{1, \ldots\ldots, h_j\}$, $R_j \in \{R_j^{(l)}\}$. The pattern X can then be assigned to be a member of that class to which it shows maximum similarity as measured by certain distance based membership values [116]. Here we suggest an alternative but equivalent scheme.

Each x_i be replaced by a triangular fuzzy number $<x_i-\delta, x_i, x_i+\delta>$. Then $(<x_1-\delta, x_1, x_1+\delta>, \ldots\ldots, <x_n-\delta, x_n, x_n+\delta>)^T \in \Re^n$. So by now we have the fuzzy pattern $X_F = (<x_1-\delta, x_1, x_1+\delta>, \ldots\ldots, <x_n-\delta, x_n, x_n+\delta>)^T$ in place of the crisp pattern $X = (x_1, \ldots\ldots, x_n)^T$. Similarly, if $R_j^{(l)} = (y_{j1}^{(l)}, \ldots\ldots, y_{j2}^{(l)})^T$ then we can replace $R_j^{(l)}$ by $R_{Fj}^{(l)} = (<y_{j1}^{(l)}-\delta, y_{j1}^{(l)}, y_{j1}^{(l)}+\delta>, \ldots\ldots, <y_{jn}^{(l)}-\delta, y_{jn}^{(l)}, y_{jn}^{(l)}+\delta>)^T$. Obviously $R_{Fj}^{(l)} \in \Re^n$. δ is a fixed positive crisp real number. We can choose δ as small as we please. We call $\{R_{Fj}^{(l)}\}$, for $l \in \{1, \ldots\ldots, h_j\}$, the class of fuzzy patterns C_{Fj}. Next we present the modified version of the methods for determining the membership function for each class C_{Fj} with respect to the pattern X_F. This is only a slight extension of the original methods devised for determining the membership function for each C_j with respect to the pattern X in [116] to the fuzzy cases.

Method I: We define a membership function $\mu_j(X_F)$ associated with the fuzzy pattern X_F for the jth class of fuzzy patterns C_{Fj} as

$$\mu_j(X_F) = [1 + (h(X_F,R_{Fj})/E)^p]^{-1}. \qquad (2.5.1)$$

where E is an arbitrary positive constant, p is an integer, h is the hausdorff metric (Definition 2.1.1). Clearly $h(X_F,R_{Fj}) = d(X,R_j)$ as $\delta \to 0$, where d is the usual euclidean distance metric in R^n or a weighted distance metric as

$$d(X,R_j) = \min\{\sum_n (W_{jn}^{(l)}(x_n - R_{jn}^{(l)})^2\}^{1/2}, \qquad (2.5.2)$$

where $W_{jn}^{(l)}$ ($|W_{jn}^{(l)}| < 1$) corresponds to the 1st prototype in the class C_j and denotes the magnitude of the weighting coefficient along the nth coordinate. We can keep δ sufficiently small so that the efficiency in classification, matching and selection of patterns as achieved in [116] is not compromised.

Constants E and p in (2.5.1) have the effect of altering fuzziness of the clusters C_{Fj}. The grade of membership of X_F as defined by (2.5.1) is 1 when $h(X_F,R_{Fj}) = 0$, and 0 when $h(X_F,R_{Fj}) = \infty$. The value of $\mu_j(X_F)$ increases as the value of $h(X_F,R_{Fj})$ decreases. Thus an unknown fuzzy pattern X_F is recognized to be a member of the kth class of fuzzy patterns C_{Fk} if

$$\mu_k(X_F) = \max\{\mu_1(X_F), \ldots\ldots\ldots\ldots \mu_m(X_F)\}.$$

The fuzzification of pattern serves at least two purpose. Firstly, it enables us to describe and compute everything in $\Re^n$ instead of R^n (complexity of computation remains the same) and secondly, it allows us to accommodate the uncertainties in measuring each feature x_i within the description of the pattern.

Vectors having imprecise or incomplete specification are usually either ignored or discarded from the design and test sets. Impreciseness or ambiguity in such data may arise from various sources. For example, instrumental error or noise corruption in the experiment may lead to partially reliable information available on a feature measurement. Again, in some cases the expense incurred in extracting a very precise exact value of a feature may be very high or it may be difficult to decide on the most relevant features to be extracted. In such cases it is not appropriate to give an exact numerical representation to uncertain feature data. Rather, it is reasonable to represent uncertain feature information by fuzzy subsets [13]. The fuzzy pattern X_F described above by means of a multidimensional fuzzy real number (the so called generalized fuzzy vector developed in this chapter) may help us realize this goal (as we shall see more clearly in our next example).

Method II: Let $p_1, \ldots\ldots\ldots, p_N$ be N measurable attributes (features) of the unknown pattern X. Each p_i is 'normalized', i.e., have values in [0, 1]. We can write

$$X = (p_1, \ldots\ldots\ldots p_N)^T, \tag{2.5.3}$$

where

$$p_i = \{1 + |(x'_i - x_i)|^p/|E|\}^{-1}. \tag{2.5.4}$$

x'_i is the ith reference constant determined from representative events of all classes. Constants E and p has the same effect as in Method I. If x_i varies over a closed and bounded interval of R (this can be done easily by taking all the ith coordinate values of the patterns belonging to all the pattern classes to lie within a minimum closed bounded interval of R) then it is a routine matter to verify that p_i will be a fuzzy number. That is, $X \in \Re^N$.

If there are h number of prototypes in a class C_j, each reference point may then be represented as

$$R_j^{(l)} = (p_{1j}^{(l)}, \ldots\ldots\ldots\ldots p_{Nj}^{(l)})^T, \tag{2.5.5}$$

where $p_{ij}^{(l)}$ denotes the degree to which the property p_i is possessed by the lst prototype in C_j. Then the similarity vector $S_j(X)$ for the pattern X with respect to the jth class has the form

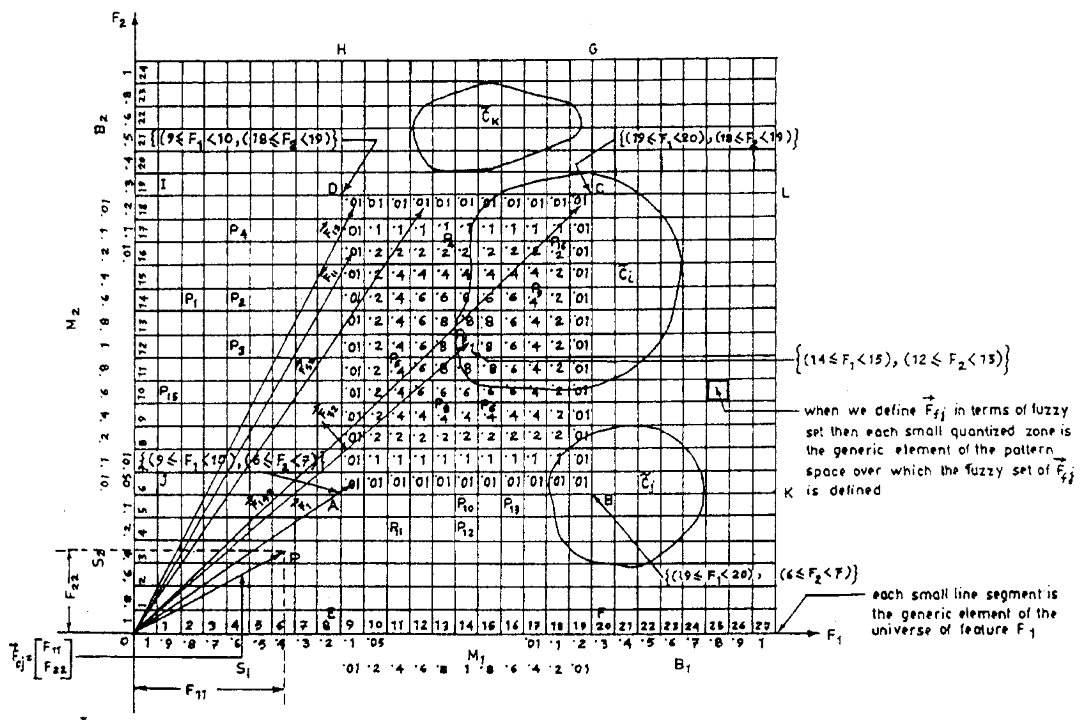
$$S_j(X) = (s_{1j}, \ldots\ldots\ldots\ldots\ldots s_{Nj}). \tag{2.5.6}$$

$$s_{ij} = (1/h)\sum s_{ij}^{(l)}, \tag{2.5.7}$$

$$s_{ij}^{(l)} = (1 + W|1 - (p_i/p_{ij}^{(l)})|)^{-2Z}, \tag{2.5.8}$$

where the numerical value of s_{ij} denotes the grade of similarity of the ith property with that of C_j. W is any positive constant dependent on each of the properties and Z is an arbitrary integer. With the knowledge of all the similarity vectors one can decide $X \in C_k$ if $|S_j(X)| < |S_k(X)|$, $k,j \in \{1, \ldots\ldots, m;\ k \neq j\}$, where $|S_j(X)| = \max(|s_{1j}|, \ldots\ldots\ldots\ldots |s_{Nj}|)$. Generally $S_j(X) \notin \Re^N$. But then $S_j(X)$ can be obtained from $\Re^N$ by max-min operations.

B. Recognition of Two Dimensional Visual Objects by Fuzzy Techniques



KEY: The entries in the small quantized zones of the area ABCD are obtained by considering "min" operator between the corresponding elements of the fuzzy sets M_1 and M_2. For instance, the possibility value 0.01 of $\vec{F}_i$ at $\{(9 \leq F_1 < 10), (6 \leq F_2 < 7)\}$ is obtained by considering min $\{\cdot 01/(9 \leq F_1 < 10), \cdot 01/(6 \leq F_2 < 7)\}$ of the fuzzy sets M_1 and M_2 respectively. Instead of "min" operator we may use algebraic product etc. depending upon the way we want to write the relation formed by the antecedent clauses of an one dimensional fuzzy implication.

Fig 2.5.1 Representation of fuzzy feature vector/pattern vector. Reproduced from [127].

Our second example is taken from fuzzy computer vision [127]. Here pattern classification is achieved in a two dimensional fuzzy feature vector space as shown in Fig 2.5.1. A geometric definition of fuzzy vectors has appeared in [127], which has been stated subsequently in Definition 2.5.1.

Definition 2.5.1: Let F_{fj} be a *fuzzy vector* having n components each of which is a fuzzy set A_i defined over the universe F_i. The fuzzy vector F_{fj} is a fuzzy set in the cartesian product $F_1 \times F_2 \times \ldots \times F_n$. Each element of the fuzzy set is a vector having same initial point, but different terminal points. Each terminal point of each vector in the set carries one membership value indicating its (vector's) possibility to belong to the set F_{fj}. A fuzzy vector F_{fj} is represented as $F_{fj} = \{(\mu_{Ffj}(F), F) \; \forall \; F \in F_1 \times F_2 \times \ldots \times F_n\}$, where $\mu_{Ffj} : F_1 \times F_2 \times \ldots \times F_n \rightarrow [0, 1]$ is the membership function of F_{fj} and $\mu_{Ffj}(F)$ is the grade of membership of $F \in F_1 \times F_2 \times \ldots \times F_n$ in F_{fj} which is a set of vectors. Here F_{fj} is a pattern vector. The suffix f stands for a particular pattern in the class of patterns (Fig 2.5.1).

Each fuzzy pattern vector or fuzzy feature vector F_{ij} can easily be obtained from $\Re$ by just a single min operation between two suitably chosen fuzzy numbers from $\Re$. Once we get F_{ij} for each i, we need only a finite number of max operations to get M_i ($i = 1, 2$). This is true for any fuzzy point in R^2, no matter however complicated the shape is. The concept of fuzzy vectors will be very effective in analyzing shapes of very complicated objects, say for example, analyzing the two dimensional satellite or RADAR images of a cyclone (Fig 5.1.2). Two sets of possible features to be extracted from the satellite or RADAR image of a cyclone are {grey level pixel intensity, curvature} and {grey level pixel intensity, the angle between the radius vector and the tangent}. We know that in the image of a cyclone the cloud bands tend to curve in log-spiral shape (Fig 5.1.2) [45]. So to detect possible cyclone from the satellite or RADAR images by a machine (computer) we must train the machine to recognize log-spiral shapes (of cloud bands). Computationally log-spiral shapes may be identified either by constant curvature or by constant angle between the radius vector and the tangent. In the images of cyclones due to the inherent uncertainty neither of these quantities can be crisp constant. Instead they can be viewed as fuzzy numbers, i.e., members of $\Re$. Also the grey level pixel intensity distribution function satisfies all the properties of a fuzzy membership function, defined on R, to become a fuzzy number (for the detail method see Section 4.1). So the Fig 5.1.2 is actually in $\Re^2$ rather than in R^2.

Here the entire image of the storm may be treated as a fuzzy fractal (Definition 4.1.5), where the grey level pixel intensity will give an estimate for the density of cloud at a point (pixel). Measuring the density of a cloud by grey level pixel intensity is quite standard [45]. This fuzzy fractal is a fuzzy subset of R^2 (which is also a point of $\Re^2$). We may take the relative density of cloud at a point P (with suitable gradation or quantization) = the grey level pixel intensity at P = the membership value assigned to P, where relative density at a point = actual density at that point / maximum possible density (of cloud) at any point at any time. Note that here the fuzzy membership function is no longer a measure of uncertainty but a measure of lack of uniformity.

2.6 Discussion

In this chapter we have remained concerned with the mathematical and structural analysis of a generalized fuzzy vector space and the application of it in the management of the phase space uncertainty in a dynamical system. Here uncertain quantities have been modelled by fuzzy sets and the uncertain dynamical systems have become fuzzy dynamical systems. Since usually the phase spaces of the crisp dynamical systems are subsets of R^n, in this chapter we have established a relationship of the fuzzy phase spaces with $\Re^n$. This gives us the opportunity to utilize the uncertain quantity modelling capability of the fuzzy numbers to model the uncertainty or ambiguity involved in the states (phases) of the dynamical system. Taking two specific examples of fuzzy pattern recognition we have shown how useful this approach can be in tackling real life problems. The fuzzy number system developed in this chapter will be useful in modelling the uncertain quantities in the real life dynamical systems to be studied in Chapters 4, 5 and 6. Modelling uncertain quantities of a dynamical system by fuzzy numbers gives rise to fuzzy phase spaces, which we have studied here.

Unfortunately the fuzzy phase spaces are not as directly related with $\Re^n$ as the crisp phase spaces are related with R^n or C^n (C is the set of crisp complex numbers). The reason behind this lies in the structural differences between R and $\Re$. Unfortunately $\Re$ is not equipped with as rich an algebraic and a topological structure as that of R. Also ordering $\Re$ is not very easy and various fuzzy mathematicians have so far ordered $\Re$ in various different ways. In this chapter we have considered $\Re$ as an orderless structure. Nevertheless we have been able to show here that any arbitrary fuzzy state of an arbitrary fuzzy state space can be generated from $\Re^n$ by max-min operations. Also in the last section we have indicated some possible role of multidimensional fuzzy numbers in the uncertainty management in some of the pattern recognition systems.

In the next chapter we shall deal with the management of attainability set mapping uncertainty. In a general fuzzy dynamical system fuzzy attainability set mappings or fuzzy time-evolution laws evolve within a fuzzy phase space. In such a system the compound uncertainty is due to the fuzzy states and also due to the fuzzy time-evolution laws.

Chapter 3

Uncertainty Management in Time-Evolution Laws

In the previous chapter we have dealt with uncertainty management in phase spaces. In the present chapter we shall concentrate on uncertainty management in time-evolution laws or attainability set mappings or reachable set mappings [43]. These mappings will evolve within uncertain phase or state spaces. Uncertain quantities in an attainability set mapping will always be modelled by fuzzy numbers, which we have studied in detail in the last chapter. We have mentioned that the study of dynamical systems is mainly the study of time-evolution laws. We have mentioned in Chapter 1 that, according to De Glas one of the motivations behind the introduction of fuzzy set theory into the field of dynamical system theory was the needs for a theory of systems whose structure and / or behaviour involve uncertainties, which can not be encompassed within the purview of probability theory [27].

The potential application to systems theory has from the beginning significantly motivated and influenced the direction of development of the theory of fuzzy sets. Fuzzy systems were first discussed in 1965 by Zadeh in his expository paper [153]. The first systematic treatment of abstract fuzzy dynamical systems was in 1973 by Nazaroff [111], who fuzzified Halkin's crisp topological polysystems to obtain fuzzy topological polysystems. These were further investigated by Warren [147]. They, however, suffer the shortcoming of not explicitly exhibiting the time dependence of the systems. Time dependent fuzzy sets were considered by Lientz [91], but are not, strictly speaking, fuzzy dynamical systems as they do not admit variations in initial conditions. In 1982 Kloeden defined and developed the notion of fuzzy dynamical systems in [82] on line of the classical notion of mathematical dynamical systems developed since the time of Poincare in the 1880s and whose essence has been mentioned at the beginning of Chapter 1.

In a parallel development almost at the same time fuzzy dynamical system theory was developed from a more system theoretic and less topological outlook. There are systems, where some deterministic dynamical characteristics are unknown or deliberately ignored as well as uncertainties attached to their mathematical model are such in nature that, the probabilistic approach can not be used. This observation led Chang and Zadeh to the concept of fuzzy systems [24]. In 1983 De Glas considered fuzzy dynamical system (or dynamical fuzzy system as he himself liked to call it) F as the fuzzy set of all possible crisp dynamical systems representing an uncertain dynamical system [27]. He even considered α-cuts of this fuzzy set to signify the system behaviour with the degree of possibility α or more. De Glas was the first to consider stability and attractor of an FDS [28].

Bassanezi et al. have defined discrete FDS in terms of iterated fuzzy valued continuous function of a fuzzy variable [10]. Discrete FDS in terms of iterated fuzzy sets systems (IFZS) have been described by Cabrelli et al. in [21].

In an FDS (X,F) if F is obtained as a solution of a fuzzy differential equation the FDS (X,F) is called *fuzzy differentiable dynamical system* (FDDS). Fuzzy differential equations were first defined and discussed by Kaleva [67] and Seikkala [136] under the name of *fuzzy initial value problem* (FIVP). The fuzzy differentiation involved in [67, 136] is Hukuhara differentiation [126]. Another kind of fuzzy differential equation has been defined by Kandel et al. [70, 71]. Recently Buckley and Feuring have considered a very generalized notion of fuzzy differential equation [17]. Henceforth we shall call it *BF-system*. BF-system subsumes all other first order linear fuzzy differential equations. For various definitions of fuzzy derivatives it gives various classes of fuzzy differential equations.

So far the development of FDDS has progressed in two different directions. One of them is fuzzy differential equation which we have mentioned in the last paragraph. The other course of development is through fuzzy differential inclusion (FDI) introduced by Aubin [7] and Hullermeier [62]. Henceforth we shall call it *AH-system*. Due to possibilistic irreversibility [14] the fuzzy solution sets of a BF-system generally tends to blow up. That is, if x(t) is a solution of a BF-system usually diam $(x(t)) \to \infty$ as $t \to \infty$. On the other hand in an AH-system diam $(x(t)) < \infty$ as $t \to \infty$. Naturally an AH-system is a much better tool for modelling and simulation of an imprecise or uncertain system compared to the BF-systems. Diamond has undertaken dynamical system theoretic studies of the AH-systems [32, 33].

Under the above background of mathematical theory of FDSs in Section 3.1 we introduce the notion of dissipativeness into the FDS and formulate a criterion to determine it. In Section 3.1 we shall also introduce a few other new concepts into the theory of FDS, as for example, chaoticity (as defined by Devaney), predictability, robustness, homoclinicity, etc.

Hullermeier has made elaborate discussions about the direct numerical simulation (DNS) of FDIs [63]. In Section 3.2 we have provided an algorithm (Algorithm 3.2.1) to solve an one dimensional FDI. So far the fuzzy differentiable dynamical systems have been described by first order and first degree fuzzy differential equations or inclusions. In Section 3.2 we have formulated a second order FDI, solved it with the help of Algorithm 3.2.1 and simulated the solution space for the best system behaviour. Throughout this thesis we shall interchangeably use the terms FDI and AH-system.

3.1 Some Extension of Fuzzy Dynamical Systems

We have defined a (crisp) dynamical system on a complete, locally compact metric space (X, d) (we call just X) (Definition 1.1.1). In this section our aim is to present a rigorous definition of FDS in terms of fuzzy time-evolution law (fuzzy attainability or reachable set mapping) and then defining attractor, stability, periodicity, chaoticity, predictability, etc. for an FDS in terms of the fuzzy time-evolution law. As a link between classical crisp dynamical system and the FDS we shall start with a definition of the crisp *generalized dynamical systems* (GDS).

A. Basic Definition of Fuzzy Dynamical Systems

Definition 3.1.1: *GDS* is defined axiomatically in terms of an attainability set mapping F : $X \times T \to C$ ((C, h) be the metric space of all nonempty compact subsets of X with the Hausdorff metric (Definition 2.1.1) h), where for each $(x, t) \in X \times T$ the attainability set F(x, t) is the set of all points in X attainable at time $t \in T$ (unless otherwise mentioned T will always mean the set of time in this thesis) from the initial point x, satisfying following four generalizations of axioms (a) - (d) above:

(A) F is defined for all $(x, t) \in X \times T$;
(B) $F(x, 0) = \{x\}$ for all $x \in X$;
(C) $F(x, s + t) = F(F(x, s), t)) = \cup \{F(y, t): y \in F(x, s)\}$ for all $x \in X$ and $s, t \in T$;
(D) F is jointly continuous in (x, t).

Definition 3.1.2: A *trajectory* of a GDS F is defined to be a single-valued mapping ϕ : $[t_0, t_1] \to X$ for which

$$\phi(t) \in F(\phi(s), t - s) \tag{3.1.1}$$

for all $t_0 \leq s \leq t \leq t_1$.

The existence of trajectories and attainability by trajectories of all points in the attainability sets follows from axioms (A) - (D) [82].

Before we extend Kloeden's definition of fuzzy dynamical system we shall have to do a little ground work. Let D be a metric on $X \times I$ ($I = [0, 1]$) defined by

$$D((x_1, r_1), (x_2, r_2)) = \max\{d(x_1, x_2), |r_1 - r_2|\},\ x_1, x_2 \in X \text{ and } r_1, r_2 \in I.$$

Then $(X \times I, D)$ is also a complete, locally compact metric space [82]. In the last chapter we defined support for fuzzy numbers. Here we are extending that definition to arbitrary fuzzy subset.

Definition 3.1.3: The *support* of an arbitrary fuzzy subset μ_A, denoted by supp μ_A, such that,

$$\text{supp } \mu_A = \text{closure}\{x \in X \mid \mu_A(x) > 0\}.$$

Definition 3.1.4: The *endograph* of a fuzzy subset μ_A, denoted by end μ_A, is the subset

$$\text{end } \mu_A = \{(x, r) \in X \times I \mid \mu_A(x) \geq r\}$$

of $X \times I$ and the *supported endograph*, denoted by send μ_A, the subset

$$\text{send } \mu_A = \text{end } \mu_A \cap \text{supp } \mu_A \times I.$$

μ_A is a compact fuzzy subset of X if and only if μ_A is an upper semi-continuous function on X and supp μ_A is compact subset of X or equivalently, if and only if send μ_A is a compact subset of X × I [82]. Let $\mathfrak{I}$ denote the collection of all nonempty compact fuzzy subsets of X, and let δ be the metric on $\mathfrak{I}$ defined by

$\delta(\mu_A, \mu_B) = H(\text{send } \mu_A, \text{send } \mu_B)$, H is the Hausdorff metric for nonempty compact subsets of X × I.

$(\mathfrak{I}, \delta)$ is not a complete metric space [82]. The metric space (C, h) is embedded in the metric space $(\mathfrak{I}, \delta)$ under the mapping $i : C \to \mathfrak{I}$ defined by $i(A) = \chi_A$, the characteristic function of A. Each individual point of X is compact subset of X. So $\{x\} \subset$ C, for all $x \in X$. So $\{x\} \subset \mathfrak{I}$ for $x \in X$.

And now we are in a position to present an extended version of Kloeden's definition of fuzzy dynamical system.

Definition 3.1.5: *Fuzzy dynamical system* (FDS) $(\mathfrak{I}, \sigma)$ on a state space $\mathfrak{I}$ is defined axiomatically in terms of a fuzzy attainability set mapping (FAM) (Definition 3.1.6) $\sigma : \mathfrak{I} \times T \to \mathfrak{I}$ satisfying the following four axioms:

(A) $\sigma(x, t)$ is defined for all $(x, t) \in \mathfrak{I} \times T$;
(B) $\sigma(x, 0) = \mu_x$ for all $x \in \mathfrak{I}$ (μ_x has support $\{x\}$);
(C) $\sigma(x, s + t) = \sigma(\sigma(x, s), t)$ for all $x \in \mathfrak{I}$ and $s, t \in T$;
(D) σ is jointly continuous in (x, t).

Since each individual point of X is also a member of $\mathfrak{I}$, an FDS σ can start from X as well. In fact in Kloeden's original definition of an FDS σ always starts from X.

Definition 3.1.6: σ(x, t) is called the *fuzzy attainability set* or *fuzzy reachable set* of $(\mathfrak{I}, \sigma)$ at time t starting from the fuzzy point (which is a nonempty compact fuzzy subset of X) $x \in \mathfrak{I}$. The time-evolution law σ is also called *fuzzy attainability set mapping* (FAM).

Of course an attainability set is also a nonempty compact fuzzy subset of X and hence is an element of $\mathfrak{I}$. Definition 3.1.5 is a slightly more generalized version of Kloeden's definition FDSs in terms of FAMs. De Glas has defined an FDS as representable by a fuzzy subset of all the possible crisp time-evolution laws representing an uncertain crisp system [27]. This idea turns out to be very useful in fuzzy dynamical system modelling.

Axioms (*A*) - (*D*) of an FDS are fuzzified version of axioms (A) - (D) of a crisp GDS and because the metric space (C, h) is embedded in the metric space $(\mathfrak{I}, \delta)$ by the identification mapping $i : C \to \mathfrak{I}$ defined by $i(A) = \chi_A$ for each $A \in C$. Thus crisp dynamical systems can be considered as a subclass of fuzzy dynamical systems [82].

Definition 3.1.7: A *fuzzy trajectory* of an FDS σ is a mapping $\psi : [t_0, t_1] \to \Im$ for which $\psi(t)$ is a singleton fuzzy subset of X for each $t_0 \leq t \leq t_1$ and

$$\psi(t) \in \sigma(\psi(s), t - s) \tag{3.1.2}$$

for all $t_0 \leq s \leq t \leq t_1$.

Unlike a crisp trajectory a fuzzy trajectory need not be continuous [82]. Also note that, for $t_0 \leq t \leq t_1$

$$\psi(t) = r(t)\chi_{\phi(t)}, \tag{3.1.3}$$

where $\phi(t) = \text{supp}\ \psi(t)$ and $r(t) = \psi(t)(\phi(t))$. If $r(t) \geq \alpha$ for all t then it is *α-trajectory* of De Glas [27]. Existence of fuzzy trajectories has been proved in Theorem 5.2 of [82].

FDSs have been defined in terms of *fuzzy cocycles* also [32]. Attainabiliy set mappings are special class of cocycles. Another important notion of FDS is that, let S be the set of all possible crisp models of an imprecise dynamical system. By a fuzzy model of the system a suitable fuzzy subset of S is understood [27]. This clearly comes down to the notion of (1) identify and accommodate the initial (fuzzy) imprecision by choosing a suitable fuzzy initial point in the phase space and (2) then identifying and accommodating the propagation of imprecision in determining any subsequent state by a suitable fuzzy attainability set mapping. The description of FDSs in terms of FAMs is perhaps the most powerful notion so far. Also such description is an immediate generalization of the notion of crisp dynamical systems to the FDSs. Therefore it is quite natural to expect that, many important notions of the crisp dynamical systems can be extended to the FDSs described in terms of FAMs. Throughout this thesis we will be concerned about FDSs described in terms of FAMs only.

B. Fuzzy Dissipative Dynamical Systems

We have defined a (crisp) dissipative dynamical system in Definition 1.1.4. Now we want to extend it to the FDSs to define fuzzy dissipative dynamical systems.

Definition 3.1.8: $(\Im, \sigma)$, where $\Im$ is defined over X as above, is a *fuzzy dissipative dynamical system* if $\sigma(X, t_2) \subseteq \sigma(X, t_1)$ ($\mu_{\sigma(X,t_2)} \leq \mu_{\sigma(X,t_1)}$ in fuzzy set terminology), where (X, d, μ) is a nonempty compact metric space, which is also a fuzzy set with membership function μ, $t_1, t_2 \in T$ such that, $t_1 < t_2$. X is the initial phase space of σ, i.e., at t_0 (the time at which the system starts) σ can start from any point of X. If $\sigma(X, t_2) \subset \sigma(X, t_1)$ holds strictly for $t_1 < t_2$ we call the system is *strictly dissipative* or *monotonic dissipative*.

Given the importance of dissipative dynamical systems in real life let us suggest here an easy to implement test for dissipativeness of an FDS. If there exists a non-negative k $\in$ R such that,

$$\lim_{t\to\infty} \delta(X, \sigma(X, t)) \leq k \tag{3.1.4}$$

holds then we call σ a dissipative FDS. Clearly, $X = \sigma(X, 0)$ and dissipativeness implies that $\sigma(X, t) \subseteq \sigma(X, 0)$ for $t \geq 0$, which means $\delta(\sigma(X, 0), \sigma(X, t)) \leq k$ for some non-negative $k \in R$ and all non-negative $t \in T$. On the other hand if (3.1.4) holds and $\sigma(X, t) \not\subset \sigma(X, 0)$ for some $t > 0$ then we can extend X in the following manner

$$\text{closure}(\bigcup_{t\geq 0}\sigma(X, t)) = X'. \tag{3.1.5}$$

Because of (3.1.4) X′ is a bounded metric space. Being union of fuzzy subsets X′ is also a fuzzy set. But X′ is not compact in general. However in most cases of interest X is a compact subset of R^n for some n and in that case X′ being closed bounded subset of R^n is compact (by generalized Heine-Borel theorem). We extend d to X′ and get the space (X′, d, μ). We just replace (X, d, μ) by (X′, d, μ) and define $(\mathfrak{I}', \delta)$ on (X′, d, μ). Because of (3.1.4)

$$\lim_{t\to\infty} \delta(X', \sigma(X', t)) \leq k \tag{3.1.6}$$

holds, which implies σ is dissipative on X′ or rather on $\mathfrak{I}'$.

C. Fuzzy Attractor and Stability of Fuzzy Dynamical Systems

Now we are in a position to define fuzzy attractor. De Glas was the first to define and discuss fuzzy attractor (or rather α-attractor) in [28]. The authors of [21] have used the term 'attractor' only, instead of 'fuzzy attractor'. In a recent paper Bassanezi et al. have dealt attractors and stability of fuzzy dynamical systems [10]. Fuzzy attractor has been defined in terms of fuzzy cocycle by Diamond [32]. Here we have defined a fuzzy attractor in terms of an FAM σ of the FDS $(\mathfrak{I}, \sigma)$.

Definition 3.1.9: In a fuzzy dissipative dynamical system $\lim_{t\to\infty} \sigma(X, t) = A \subseteq X$. We define A as *fuzzy attractor* of σ. Of course A is a fuzzy subset of X. If we introduce the notion of α-cuts into it we will get $\lim_{t\to\infty} [\sigma(X, t)]^\alpha = [A]^\alpha$ for $0 \leq \alpha \leq 1$. $[A]^\alpha$ is the *α-attractor* defined by De Glas in [28].

Now, how to define the membership function μ_A for A? Note that the fuzzy trajectory given by ψ(t) in the expression (3.1.2) converges to A as $t \to \infty$. Also each point of A belongs to a fuzzy trajectory ψ(t) for some t. From equation (3.1.3) it transpires that membership value of ψ(t) is r(t). So for each $x \in A$, there exists a $t \in T$ such that, $x = \psi(t)$, and

$$\mu_A(\psi(t)) = r(t) = \mu_A(x). \tag{3.1.7}$$

But through a given point $x \in A$ more than one trajectories may pass. In that case to determine a unique value for $\mu_A(x)$ we are to take the fuzzy set union of $\mu_A(\psi_c(t))$ over all c, where $\psi_c(t)$ is a (fuzzy) trajectory passing through x.

$$\cup_c \mu_A(\psi_c(t)) = \sup_c\{r_c(t)\} = \mu_A(x). \tag{3.1.8}$$

Attractors have played an increasingly important role in thinking about (classical) dynamical systems. An excellent exposition of (crisp) attractors was presented by Milnor in [109]. In this chapter we shall be extending some properties of attractors presented in [109] to the fuzzy attractors.

Since attractor is actually the phase space of the underlying dynamical system as $t \to \infty$, we may consider attractor as a matured state of the phase space. Naturally, the dynamical system evolving within the attractor is rather matured compared to the initial state. To understand a dynamical system therefore the study of attractor, when one exists, is of considerable importance.

Every attractor has an important superset called nonwandering set. We define $\Omega(\sigma)$ as *fuzzy nonwandering set* of a fuzzy dynamical system σ.

Definition 3.1.10: *Fuzzy nonwandering set* of a fuzzy dynamical system σ be denoted by $\Omega(\sigma)$, which is a fuzzy subset of X. Then $x \in \Omega(\sigma)$ implies that $\sigma(x, t) \subseteq \Omega(\sigma)$ for all $t \in T$.

$\mu_{\Omega(\sigma)} : X \to [0, 1]$ and for any $y \in \Omega(\sigma)$ $\mu_{\Omega(\sigma)}(y) > 0$. Now following Ruelle and Tukens [109] we can give the following

Definition 3.1.11: A subset A of the fuzzy nonwandering set $\Omega(\sigma)$ is a *fuzzy attractor* if it has a neighbourhood U such that.

$$\bigcap_{t>0} \sigma(U, t) = A. \tag{3.1.9}$$

Of course $\sigma(U, t_2) \subseteq \sigma(U, t_1)$ for $t_1 < t_2$. i.e., σ is dissipative in U.

Note that the above definition of fuzzy attractor implies that all fuzzy trajectories sufficiently close to the fuzzy attractor A must converge to A. This is a standard stability condition of a dynamical system [109], and when this stability condition is satisfied we call the underlying fuzzy dynamical system σ, *stable*. This form of stability is known as Liapunov stability. Let us state this more formally in the following

Definition 3.1.12: A is a closed subset (in sense of fuzzy topology [23]) of X with $\sigma(A, t) = A$ ($\mu_{\sigma(A,t)} = \mu_A$ in fuzzy set terminology) for any $t \geq 0$ will be called *Liapunov stable* (also called *orbitally stable*) if A has arbitrarily small neighbourhoods U with $\sigma(U, t) \subset U$ for all $t > 0$.

There is another important form of stability called asymptotical stability. Before we can define it for a fuzzy dynamical system we need to develop some more concepts.

Definition 3.1.13: *Omega limit set* $\omega(x)$ of a point $x \in X$ is the collection of all accumulation points (y will be called an *accumulation point* of the sequence $\{\sigma(x, t)\}_{t \geq 0}$, if and only if there exists a fuzzy trajectory through x which converges to y for some $t \geq 0$) of the sequence $\{\sigma(x, t)\}_{t \geq 0}$.

Definition 3.1.14: The *realm of attraction* of an attractor A, denoted by $\rho(A)$, is the collection of all points $x \in X$ for which $\omega(x) \subseteq A$.

Obviously, for any (fuzzy) attractor A of a fuzzy dynamical system σ, $\sigma(A, t) = A$ for all $t \geq 0$.

Definition 3.1.15: A fuzzy attractor A of a fuzzy dynamical system is called *asymptotically stable* if it is Liapunov stable and its realm of attraction $\rho(A)$ is an open set (in sense of fuzzy topology [23]).

In the asymptotically stable case if we choose U with closure (in sense of fuzzy topology) contained in $\rho(A)$, then it follows that A is equal to the intersection of the sequence of forward images $U \supset \sigma(U) \supset \sigma^2(U) \supset \sigma^3(U)$ .... where $\sigma^i(U) = \sigma(U, i)$. This definition of stability for a fuzzy system is an extension of the notions of stability discussed in [28, 33, 141]. Stability is perhaps the single most important aspect of any fuzzy system and therefore has remained the most widely studied one from different angles, e.g., [72, 77, 128]. Here we have kept our attention confined only within some of the most basic forms of stability of a fuzzy dynamical system.

D. Robustness of Fuzzy Dynamical Systems

Let us next extend the concept of robustness to fuzzy attractors. Again, we shall have to do some ground work.

Definition 3.1.16: The *likely limit set* $\Lambda = \Lambda(\sigma)$ of a fuzzy dynamical system σ is the smallest closed subset (in sense of fuzzy topology [23]) of X with the property that $\omega(x) \subset \Lambda$ for every point $x \in X$ ((X, d) is a compact metric space as mentioned in Definition 3.1.8.) outside of a set of lebesgue measure zero. (Lebesgue measure can be defined on X, for (X, d) is compact and embeddable in R^n for some finite n.)

Clearly, the likely limit set Λ is the largest attractor of σ [109].

Definition 3.1.17: The likely limit set $\Lambda(\sigma_1)$ of a fuzzy dynamical system σ_1 is called *robust* if and only if the likely limit set $\Lambda(\sigma_2)$ of any other fuzzy dynamical system σ_2 has Hausdorff distance (Definition 2.1.1) with $\Lambda(\sigma_1)$ δ then σ_1 converges uniformly to σ_2 on X implies $\delta \to 0$.

E. Fuzzy Chaotic Dynamical Systems

Our next target is to define fuzzy chaotic dynamical system. Crisp chaotic dynamical system has been defined by Devaney [30]. This has been stated in Definition 1.4.4. We shall make an extension of this definition to a fuzzy dynamical system σ. But *fuzzy chaos* has already been defined by Kloeden [83] and Buckley and Hayashi [19]. Kloeden has defined fuzzy chaos in terms of iterative maps on fuzzy sets on line of what Li and Yorke did for certain crisp cases [90]. Li and Yorke's definition of chaos is only slightly different from that of Devaney. Buckley and Hayashi have defined fuzzy chaos in terms of iterated function system. A *chaotic fuzzy set* A is represented by μ_A and this μ_A is then taken as the limit of an iterated function, which is known to be chaotic under iterations, subjected to suitable conditions (e.g., the logistic function for suitable parameter values). Apart from Kloeden and Buckley and Hayashi fuzzy chaos has also been defined by Teodorescu [142] and Diamond [31]. Next we shall extend Devaney's definition of chaos to fuzzy dynamical systems.

Definition 3.1.18: Let σ be a fuzzy dynamical system. A point $p \in X$ is called *periodic* of *period* S (> 0) if and only if $p \in \sigma(p, S)$ and $p \notin \sigma(p, S')$ for any $0 < S' < S$.

Clearly p is a fuzzy singleton of X and hence a member of $\Im$.

Definition 3.1.19: σ will be called a *fuzzy chaotic dynamical system* if and only if the following three conditions are satisfied.

(1) There exists $\in > 0$ such that, for any $x \in \Im$ and any neighbourhood N(x) of x there exist $y \in N(x)$ and $t \geq 0$ such that, $\delta(\sigma(x, t), \sigma(y, t)) > \in$ (δ is Hausdorff distance). We say that, σ has *sensitive dependence on initial conditions*. As usual here x, y are nonempty compact fuzzy subsets of X, i.e., fuzzy points of X and hence individual elements of $\Im$.

(2) For any two nonempty fuzzy open sets A and B in $(\Im, \delta)$ such that, $\mu_A \wedge \mu_B = 0$, there exists $t > 0$, such that $\mu_{\sigma(A, t)} \wedge \mu_B \neq 0$. We call σ is *topologically transitive*.

(3) Let P be the collection of periodic points of σ. $P \subseteq \Im$. If P is dense in $\Im$, i.e., equivalently, in each open subset of $(\Im, \delta)$ there is at least one member of P, *density of periodic points* property is satisfied by σ.

Notice that, if we take t in the set of all non-negative integers instead of all non-negative real numbers, we get the discrete chaotic fuzzy dynamical systems of iterated functions on fuzzy sets. But this time the definition of fuzzy chaos is on line of Devaney rather than Li and Yorke as adopted by Kloeden in [83].

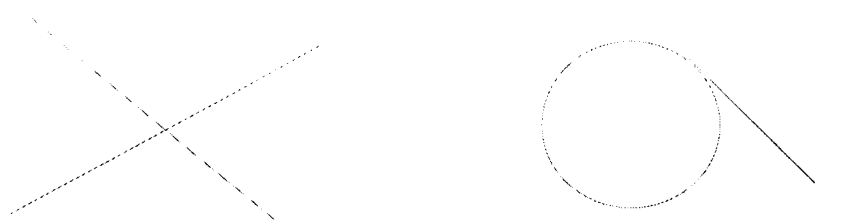
Buckley and Hayashi have iterated the *logistic function* for suitable parameter values to generate chaotic membership functions over the set of real numbers R to get the concomitant chaotic fuzzy numbers [19]. But in reality *chaotic fuzzy sets* may be far more complicated and may not be generated by iterations of any known function.

In a crisp dynamical system an attractor plays a decisive role to determine whether the underlying dynamical system is chaotic. For a crisp dynamical system we know that the presence of a *homoclinic point* in the attractor implies that the underlying dynamical system is behaving *chaotically* [109]. In this section we want to extend this idea to the fuzzy dynamical systems.

To give a formal definition of homoclinic points we need to know what does it mean by *transversality*. A precise mathematical definition of transversality is not very easy to state. We shall however present here a geometric notion of transversality. Rigorous mathematical detail may be found in [57].

If two curves C_1 and C_2 intersect in R^n at a point B. Then C_1 and C_2 will be called *transversal* to each other at B if and only if they can not be pulled apart from each other at B by a small deformation of either of them or both. i.e., in other words, their intersection at B is 'stable' (Fig 3.1.1).

Definition 3.1.20: In a crisp dynamical system p will be a *homoclinic point* if and only if two different trajectories T_1 and T_2 *transversally* intersect at p such that, neighbouring points of p on T_1 converge to p and neighbouring points of p on T_2 diverge from p under the dynamical system.



Transversal intersection   Non-transversal intersection

Fig 3.1.1 Transversal vis-a-vis non-transversal intersections.

Likewise, we define fuzzy homoclinic point of a fuzzy dynamical system in terms of changing membership values.

Definition 3.1.21: In a fuzzy dynamical system let two fuzzy trajectories T_1' and T_2' intersect transversally at p'. If membership values along T_1' increase towards p' and membership values along T_2' decrease towards p', then p' will be called a *fuzzy homoclinic point* of the underlying fuzzy dynamical system.

Horse shoe effect, described by Smale in [139] is another very important aspect of a multidimensional (crisp) chaotic dynamical system. Like fuzzy homoclinic point fuzzy horse shoe effect can also be defined in a similar manner.

This membership interpretation of p′ is equivalent to crisp notions of converging towards (increasing membership values) and diverging from (decreasing membership values) p.

Definition 3.1.22: When there will be a fuzzy homoclinic point in a fuzzy attractor of a fuzzy dynamical system, we call that attractor *fuzzy chaotic attractor*.

Definition 3.1.23: For all practical purposes we shall call a fuzzy dissipative dynamical system evolving in (X, d) chaotic if

- the fuzzy attractor A is a fractal subset (this means A is a fuzzy subset whose Hausdorff-Besicovitch dimension is greater than its topological dimension, the explanation of these terms is appearing in Appendix B) of X, and
- A contains at least one fuzzy homoclinic point.

According to the above criteria of chaotic fuzzy dynamical system it is clear that the shape of the fuzzy attractor will usually be very complicated and the membership function defined on the attractor will take abruptly fluctuating values within any arbitrarily small neighbourhood of almost every point (in sense of measure theory) of A. Ideally, the set of fuzzy homoclinic points will be *dense* in A in the relative metric topology of A as a subspace of (X, d).

F. Fuzzy Liapunov Exponent

Liapunov exponent is perhaps the most important of all the measurable quantities of a dynamical system. Fuzzy system theorists have already defined and measured various forms of Liapunov exponent of a fuzzy system [141]. Here, we would like to define the notion of Liapunov exponent for a fuzzy dynamical system, which will be called fuzzy Liapunov exponent. Note that we have been describing the fuzzy dynamical system in terms of time series. In this type of description of a dynamical system the Liapunov exponents are relatively easy to measure [26]. For a fuzzy dynamical system σ(x, t) we define the Liapunov exponent λ on line of equation 2.19 of [26] as follows.

Definition 3.1.24: For a fuzzy dynamical system (ℑ, σ) evolving on (ℑ, δ) the *fuzzy Liapunov exponent* λ is given by the equation

$$\lambda = \lim_{k\to\infty} (1 / k) \sum_{k=1}^{\infty} \lim_{t_k \to t_{k-1}} \log \delta(\sigma(x, t_k), \sigma(x, t_{k-1})) / |t_k - t_{k-1}|. \quad (3.1.10)$$

Equation (3.1.10) is in complete harmony with [141]. Generally, σ(x, t) is a fuzzy subset of X (attainability set) for any given x and t. So, fuzzy Liapunov exponent λ is in general a fuzzy number (because the Hausdorff distance between two fuzzy sets may not be determined with crisp precision). In a multidimensional fuzzy system if the largest Liapunov exponent tends to zero or a negative value (i.e., membership value at 0 or at the negative value is 1 and membership values are less than 1 outside a progressively smaller compact intervals of R containing 0 or the negative value) the system is stable. Liapunov exponents are very crucial in analyzing local instability and predictability of a dynamical system.

G. Fuzzy Metric Entropy and Fuzzy Liapunov Time

Definition 3.1.25: In a fuzzy dynamical system σ, evolving in $(\mathfrak{I}, \delta)$, the *local metric entropy* (LME) [148] is defined at any point x to be the sum (in the sense of fuzzy arithmetic) of all the fuzzy Liapunov exponents at x, where each fuzzy Liapunov exponent is such a fuzzy number that, no crisp member of that fuzzy number is non-positive or in other words this is a fuzzy number of the form μ whose support is [a, b] and $a \geq 0$.

Definition 3.1.26: Sum of all the Liapunov exponents at x is the *divergence* of the phase space at x.

The greater is the value of LME the more the system is chaotic and the less it is predictable. From [148] we can conclude that, $T_\lambda \approx 1 / \lambda$ (in sense of fuzzy arithmetic), where λ is the average value (calculated according to the arithmetic of fuzzy numbers as described in Section 2.2) of the maximum Liapunov exponent over the elapsed time. Clearly $\lambda \in \mathfrak{R}$. We have seen in the last chapter that $1 / \lambda$ is defined and is in $\mathfrak{R}$ if and only if the support of λ does not contain 0.

Definition 3.1.27: T_λ, when defined, will be called *fuzzy Liapunov time*, i.e., the system is predictable up to a time T_λ.

Note that, the fuzzy Liapunov time determined according to the Definition 3.1.27 is a fuzzy subset of T (an analytical treatment of T has been presented in Appendix A). The time of Definition 3.1.27 may be called *fuzzy time*. In Appendix A we can see that the set of time T = R. A particular instance of fuzzy time is a fuzzy subset of T, more precisely a particular instance of fuzzy time is a fuzzy real number i.e., a member of $\mathfrak{R}$.

3.2 Fuzzy Differentiable Dynamical Systems

In the previous section we extended some important dynamical system theoretic notions to the FDSs. In the current section we shall be concentrating on the special case when the underlying fuzzy attainability set mapping of an FDS is a solution to a set of fuzzy differential equations (FDEs) or fuzzy differential inclusions (FDIs), i.e., fuzzy differentiable dynamical systems (FDDSs).

A. Fuzzy Differential Equations

Definition 3.2.1: When one or more of the FDEs are time dependent we call the system *nonautonomous FDDS*. Time dependent expressions in a nonautonomous system are of the form

$$x'(t) = f(x, t), \quad x(0) = x_0, \tag{3.2.1}$$

where (3.2.1) is a fuzzy differential equation. x' is fuzzy derivative (explained below) of the fuzzy valued function x with respect to t.

Definition 3.2.2: When none of the FDEs is time dependent we call the system *autonomous FDDS*. A fuzzy differential equation of an autonomous system takes the form

$$x'(t) = f(x), \quad x(t_0) = x_0. \tag{3.2.2}$$

FDDS was first defined by De Glas [28]. But his definition of 'fuzzy derivative of a real valued function' [28] is rather restrictive. Puri and Ralescu have presented two definitions of derivative of a fuzzy valued function whose domain of definition is an open subset of some normed (crisp) space [126]. One of them is H-derivative or Hukuhara derivative and the other is a more generalized notion called canonical derivative. Since the H-derivative had great influence on latter works we are presenting the definition below.

Let U be an open subset of R. Let $S_0(R^n)$ be the collection of fuzzy subsets ($u: R^n \to [0, 1]$) of R^n satisfying the following properties:

(i) u is upper semicontinuous;
(ii) u is fuzzy convex, i.e., $u(\lambda x_1 + (1 - \lambda)x_2) \geq \min[u(x_1), u(x_2)]$ for $x_1, x_2 \in R^n$, $\lambda \in [0, 1]$;
(iii) closure($[u]^\alpha$) is compact for $\alpha \in (0, 1]$.

Definition 3.2.3: A fuzzy function $F: U \to S_0(R^n)$, which associates with each point $y \in U$ a fuzzy subset $F(y)$ of R^n with properties (i), (ii) and (iii) described above, is called *H-differentiable* at $y_0 \in U$ if there exists $DF(y_0) \in S_0(R^n)$ such that, the limits

$$\lim_{h \to 0+} \frac{F(y_0 + h) - F(y_0)}{h} \quad \text{and} \quad \lim_{h \to 0+} \frac{F(y_0) - F(y_0 - h)}{h} \tag{3.2.3}$$

both exist and are equal to $DF(y_0)$.

Kaleva was the first to define fuzzy differential equation (FDE) in terms of H-derivative [67]. At the same time Seikkala also defined FDE in terms of a slightly generalized form of H-derivative [136]. Their formulation of fuzzy initial value problem (FIVP) is as follows.

$$x'(t) = f(t, x(t)), \quad x(a) = x_0, \tag{3.2.4}$$

where $x(t)$ is a fuzzy valued function defined on T (whose range set is certain class of fuzzy subsets of R^n). $x'(t)$ is the H-derivative (or some generalized form of H-derivative) of x(t). f is also a fuzzy valued function and x_0 is an initial value of (3.2.4), which is a fuzzy point in the phase (solution) space. In their works and in almost all subsequent works x(t) takes values only in the class of normal subsets of $S_0(R^n)$ denoted by E^n. This E^n is the phase (solution) space of the FDEs. We have already established in Section 2.4 about how a member of E^n can be obtained from $\Re^n$.

Recently Buckley and Feuring have generalized (3.2.4) in terms of various definitions of derivatives of a fuzzy valued function [17].

$$\frac{dX}{dt} = f(t, X, K), \quad X(0) = C, \qquad (3.2.5)$$

assuming that we have adopted some definition for the derivative of the unknown fuzzy function X(t). In (3.2.5) all capital letters denote fuzzy quantity. $X: T \rightarrow S(R^n)$, where $S(R^n)$ is the collection of fuzzy subsets of R^n. $K = (K_1, \ldots, K_n)$ is an n dimensional fuzzy number and C is a fuzzy number. X(0) is of course the initial value of X. (3.2.5) is the Buckley-Feuring form or BF-form of FDEs. In their paper Buckley and Feuring have taken K_i, $i \in \{1, \ldots, n\}$ and C as triangular fuzzy numbers [17], but here for the sake of generality we prefer to keep them as arbitrary fuzzy numbers. To obtain a solution for (3.2.5) the entire range of crisp solution set of (3.2.5) is considered and then α-cuts over various fuzzy quantities responsible for generating that solution set is taken. Then the required solution set is constructed by using Zadeh's extension principle.

Solution of an FDE usually involves summation of two fuzzy numbers [67, 136]. But if we add two fuzzy quantities generally the diameter of the resultant fuzzy quantity is greater than any of the constituent fuzzy quantity. As a result diam(X(t)) = ∞ as t → ∞, where X(t) is the solution of an FDE. This render FDEs unsuitable for modelling. Also due to the same reason the classical dynamical systems theoretic notions are not possible to extend to the FDEs [32, 33].

B. Fuzzy Differential Inclusions with Solutions by an Evolutionary Algorithm

Since the solution or the phase space of (3.2.5) is supposed to represent the behaviour of the system represented by (3.2.5), determining the degree of possibility of the solution set is very crucial to determine the system behaviour. In this situation if we only concentrate on determining the solution set of highest possibility, which obviously represents the best system behaviour, we shall be able to reduce the computational complexity to a great extent. Taking the solution space with highest degree of possibility has been emphasized by Hullermeier [62] and Diamond [32] and their approach is termed as fuzzy differential inclusion (FDI), which was originally proposed by Aubin [7] and Hullermeier [62] and developed by Hullermeier [62, 63] and Diamond [32, 33]. Diamond has extended the classical dynamical systems theoretic studies to the systems represented by fuzzy differential inclusion (FDI) relations [32, 33].

FDI is a reformulation of (3.2.4) in a different form, where an equality (=) is replaced by an inclusion ($\in$) as formulated below:

$$x'(t) \in f(t, x(t)),\ x(a) \in x_0. \qquad (3.2.6)$$

Unlike in equation (3.2.4), in (3.2.6) x(t) is a crisp trajectory and $f(t, x(t))$ is a fuzzy set of crisp functions (actually in (3.2.6) $x'(t) \in f(t, x(t))$ means $x'(t) \in [f(t, x(t))]^0$). The derivative of x(t) is <u>not</u> the Hukuhara derivative but the classical derivative. Equation (3.2.6) in more general form in terms of arbitrary α-cuts becomes

$$x'(t) \in [f(t, x(t))]^\alpha,\ x(a) \in [x_0]^\alpha, \qquad (3.2.7)$$

where $0 \le \alpha \le 1$.

A classical FDE due to Kaleva [67] or Seikkala [136] proceeds from a fuzzification of the differential operator and considers the entire fuzzy flow (by the *fuzzy flow* of (3.2.4) we mean the FAM x(t) as a solution of (3.2.4)) which describes the system behaviour. As opposed to this the FDI proceeds from a generalization of differential inclusion relations and considers a fuzzy set of individual, crisp solutions, much the same way De Glas thought of an FDS in [27]. As long as the crisp initial value problem

$$x'(t) = f_0(t, x(t)),\ x(a) = \lambda \in [x_0]^0,$$

where f_0 is a crisp member of the fuzzy set of crisp functions $f(t, x(t))$ and λ is a crisp member of x_0, has a unique solution, existence of a unique solution of (3.2.7) as α-cuts of fuzzy set of crisp solutions is guaranteed. The α-cut of the fuzzy set of crisp solutions of (3.2.7) is the fuzzy flow representing the system behaviour with possibility α or more.

Be it (3.2.4) or (3.2.7) the solution is more difficult compared to the classical crisp differential equations. So the quest for a qualitative rather than a quantitative solution to (3.2.4) or (3.2.7) is even more natural than the crisp case.

A numerical method for solving the FDIs like (3.2.7) has been developed by Hullermeier [63], which is as follows. The admissible domain of t is γ to δ. We partition $[\gamma, \delta]$ in n equal subintervals, where the ith subinterval is denoted by I_i.

To get a solution of (3.2.7) we shall have to determine the fuzzy reachable set x(t) for any arbitrary value of t. Like equation (17) of [62] we can write

$$\frac{y_{i+1} - y_i}{\Delta t} \in f(t_i, y_i), \qquad (3.2.8)$$

where f is a fuzzy valued function with values in E^1. Note that a fuzzy set of crisp functions, when considered as a single function, maps a crisp or fuzzy set onto a fuzzy set and hence may be called a fuzzy valued function. E^n is the collection of normal fuzzy sets of $S_0(R^n)$. E^n is a metric space (E^n, D), where D is the Hausdorff metric on E^n. Based on (3.2.8) the following generalized difference equation scheme can be defined for fuzzy valued function of a crisp variable

$$Y(t_{i+1}) = \bigcup_{y \in Y(t_i)} \{y + \Delta t. f(t_i, Y(t_i))\}, \quad Y(0) = x_0. \qquad (3.2.9)$$

Since the set $Y(t_i)$ may have very complicated structures, it is generally not possible to represent them exactly [62]. So in addition to discretization of time discretization of a class of subsets of R^n also will have to be considered. Let A be a class of subsets of R^n which can be represented by means of a certain data structure. Denote by $A(Y) \in A$ the approximation of a set $Y \in R^n$. The following approximation of (3.2.9) has been defined in [62]:

$$Z(t_{i+1}) = A\left(\cup z + \Delta t A(f(t_i, z))\right), \quad Z(0) = A(X). \qquad (3.2.10)$$

The implementation of solution of (3.2.7) is based on iteration of (3.2.10). The class A used in (3.2.10) for approximating sets was implemented as different classes of geometrical bodies, such as convex hulls or more general classes including nonconvex sets. All this have been very elaborately described in [63], where some examples have also been presented with simulated results.

However the entire process of finding numerical solutions according to (3.2.9) is very computational intensive, which may be a disadvantage for applying the FDIs for modelling and simulation despite its immense potentiality. Here we have been able to find out an easier method to solve one dimensional FDIs. But unfortunately it can not be extended to solve the multidimensional cases.

In one dimensional formulation of an FDI in the form of (3.2.7) the graph of x(t) is a (crisp) trajectory in R^2 and f(t, x(t)) is a fuzzy valued function (as a fuzzy set of crisp functions) defined on R^2 with values in E^2. $x(a) \in R$ is an initial (crisp) point (that is at t = a). x_0 is a fuzzy point in R. x(a) can take any value from x_0, i.e., x(a) varies over all crisp members of the fuzzy point x_0. x_0 is known as the fuzzy initial point. t is always a crisp quantity. Then f(t, x(t)) is fuzzy valued because x(t) can start (at t = a) from any member of the fuzzy point x_0 and some fuzzy valued parameters may also be present in f(t, x(t)). Since this algorithm chooses the fuzzy solution representing the best system (represented by the FDIs) behaviour with the degree of possibility α or more, this is an evolutionary algorithm [122].

Algorithm 3.2.1 [99, 102]:

START

Step 1: fix $\alpha \in [0, 1]$;

Step 2: calculate $[f(t, x(t))]^{\alpha}$ by Zadeh's extension principle [112] or otherwise as the case may be (the α-level set $[f(t, x(t))]^{\alpha}$ is an n-tuple of α-cuts of fuzzy numbers for some n);

Step 3: solve (directly or numerically) the ordinary crisp DE $x'(t) = f(t, x(t))$, $x(0) = x_0$ only for the boundary values of the α-level set of step 2;

Step 4: the region of R^2 enclosed by the graphs of the solutions obtained in step 3, possibly along with the coordinate axes, is the α-level set of the solution of the FDI (3.2.7) for the fixed α of step 1.

END

This is the most important algorithm of this thesis and we will use it time and again for simulation. In step 2 standard numerical methods for solving ordinary crisp differential equations may be employed when methods for direct solution are not available. Of course in computer simulation all the methods used are numerical methods only. In each implementation we shall present a more case specific version of the above algorithm, where a single step of this original version may be divided into more steps for the convenience of implementation.

Justification: Now, let us justify the Algorithm 3.2.1.

In step 1 we fix α, where $0 \le \alpha \le 1$. The more is the value of α the higher is the possibility of the solution set of (3.2.7) to represent the actual system behaviour. Often only the highest value of α is taken to represent the best system behaviour.

In step 2 we calculate the α-level fuzzy set on the right side of the expression $x'(t) \in [f(t, x(t))]^{\alpha}$ in (3.2.7). This gives the set in which $x'(t)$ can take its values.

In step 3 we choose only those values for $x'(t)$ which fall on the boundary of the fuzzy set $[f(t, x(t))]^{\alpha}$ and solve $x'(t)$ only for those values just like ordinary (crisp) differential equations. If $f_0(t_0, x(t_0))$ is a boundary point and we resort to numerical methods, which is always the case for solving by computer, starting with $i = 0$ at the boundary the following steps are obvious.

$$x(t_{i+1}) = x + \Delta t.\, f_0(t_i, x(t_i)),\quad x(a) = \lambda,$$

from which we get

$$X(t_{i+1}) = \bigcup_{x \in X(t_i)} \{x + \Delta t.\, f_0(t_i, X(t_i))\},\quad X(a) = x_0. \qquad (3.2.11)$$

X(t) is an approximation of the fuzzy attainability set obtained at time t by the fuzzy flow of the solution of (3.2.7) starting at t = a. So by the way of equation (3.2.11) we can reach at an approximation of the fuzzy attainability set given by equation (3.2.9).

In step 4 we demarcate the α-level subset within the fuzzy attainability set by specifying the boundary of the α-level subset. In step 4 we achieve the same goal, which is set to be realized in [62] by iteration of equation (3.2.10). Unfortunately this technique of curving out the α-level subset out of the fuzzy attainability set will not be effective in multidimensional case. This is precisely where the algorithm fails in multidimensional case.

So far the theory of fuzzy differential equations or fuzzy differential inclusions has remained confined to treating first order and first degree equations or inclusions only. Algorithm 3.2.1 is also designed to solve first order first degree FDIs like (3.2.7). But in the following example we shall solve a simple second order fuzzy differential inclusion with the help of this algorithm.

Example 3.2.1 [99, 102]: Find the solution representing the best possible behaviour of the system (in the sense that the degree of possibility of the behaviour by the system is 1) given by

$$x''(t) \in [kx]^{\alpha}, \tag{3.2.12}$$

$$k \in [K]^{0}, \tag{3.2.13}$$

$$x'(0) \in [L]^{\alpha} \text{ for } x(t) \in [M]^{\alpha}, \tag{3.2.14}$$

$$x(0) \in [x_0]^{0}. \tag{3.2.15}$$

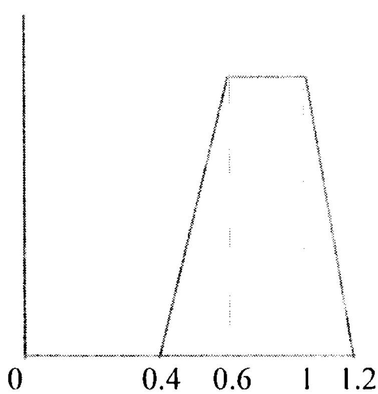


Fig 3.2.1 Trapezoidal fuzzy number K.

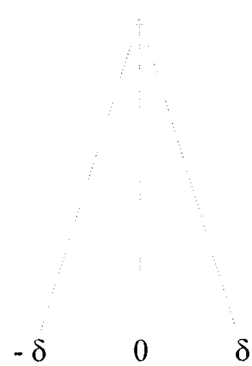


Fig 3.2.2 Triangular fuzzy number x_0.

where $x''(t) = dx'(t)/dt$, $x'(t) = dx(t)/dt$, K is a trapezoidal fuzzy number given by Fig 3.2.1, L is a triangular fuzzy number given by Fig 3.2.2, M is a triangular fuzzy number given by Fig 3.2.3 and x_0 is a triangular fuzzy number given by Fig 3.4. $0 \le \alpha \le 1$. Also show the simulated state space of the system.

Solution: Notice that we are to determine the solution representing the best system behaviour only. In that case $\alpha = 1$. So $k \in [K]^1 = [0.6, 1]$ (Fig 3.2.1). In search of a 'seed' solution we shall have to fix a representative crisp expression in the fuzzy expression $[kx]^\alpha$ or $[kx(t)]^\alpha$ on the right of (3.2.12), surely $k \in [K]^\alpha$. x(t) is a crisp variable but x(0) can take any crisp value from a fuzzy number. This contributes to the 'fuzziness' of x(t) (there may as well be other factors to contribute in this direction), that is x(t) becomes a crisp member of a fuzzy set of crisp functions. So in search of a seed solution it suffices to take

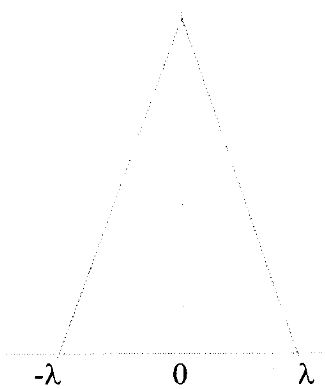


Fig 3.2.3 Triangular fuzzy number L.

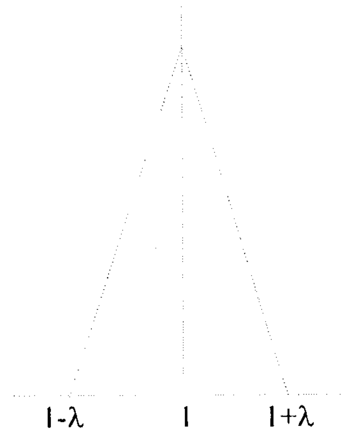


Fig 3.2.4 Triangular fuzzy number M.

$$x''(t) = kx, \qquad (3.2.16)$$

where $k \in [0.6, 1]$ and x is an ordinary crisp variable. We shall proceed by solving (3.2.16) just like an ordinary crisp differential equation. Multiplying both sides of (3.2.16) by $2x'(t)$ and then integrating we get

$$(x')^2 = kx^2 + C, \qquad (3.2.17)$$

where C is an integration constant. From (3.2.14) we get that $x'(0) \in [L]^\alpha$ for $x(t) \in [M]^\alpha$, which means $x'(0) \in [L]^1$ for $x(t) \in [M]^1$. Since L, M are triangular fuzzy numbers given by Fig 3.2.3 and Fig 3.2.4 respectively, we have $x'(0) = 0$ for $x(t) = 1$. Hence from (3.2.17) we get $C = -k$. Or,

$$x'(t) = \sqrt{k}(x^2 - 1)^{1/2}, \qquad (3.2.18)$$

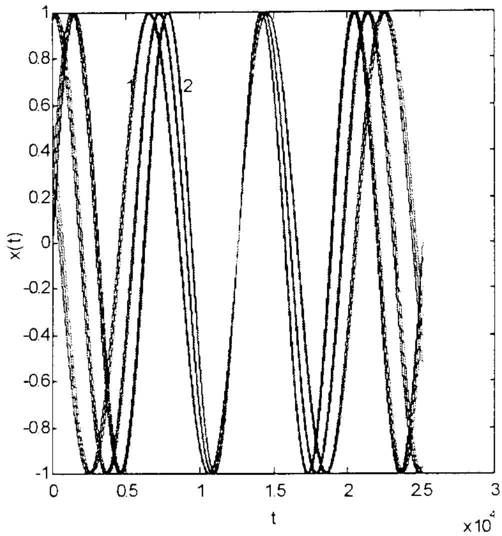


Fig 3.2.5 Fuzzy state space of the system represented by (3.2.12), (3.2.13), (3.2.14) and (3.2.15) is enclosed by curves '1' (extreme left curve of the band) and '2' (extreme right curve of the band). Only the region representing the best system behaviour has been considered.

Solving (3.2.18) we get

$$\sin^{-1}x(t) = \sqrt{k}t + D, \qquad (3.2.19)$$

where D is an integration constant and we are considering the positive square root of k only. From (3.2.15) we get $x(0) \in [x_0]^0$. But we are interested only in $x(0) \in [x_0]^1$ or $x(0) = 0$ (Fig 3.2.2). So

$$x(t) = \sin(\sqrt{k}t), \qquad (3.2.20)$$

where $k \in [0.6, 1]$. (3.2.20) the seed solution of (3.2.12), (3.2.13), (3.2.14) and (3.2.15) for any $k \in [0.6, 1]$. To determine the fuzzy flow corresponding to the best possible behaviour of the system represented by (3.2.12), (3.2.13), (3.2.14) and (3.2.15) in the phase space we shall have to determine the boundary of the fuzzy flow in R^2. The boundary is represented by the graphs of $x(t) = \sin(0.7746t)$ (curve '1' of Fig 3.2.5) and $x(t) = \sin(t)$ (curve '2' of Fig 3.2.5).

3.3 Discussion

In this chapter we have concentrated on uncertainty management in time-evolution laws. Where these uncertainties can not be modelled by probability, they are to be modelled by fuzzy set theory based techniques. Our emphasis was on the latter. Zadeh introduced the notion of fuzzy system more than thirty five years ago. One of the earliest fuzzy systems to be implemented were the fuzzy controllers, first successfully implemented by Pappis and Mamdani in 1977 [120]. Compared to this the theory of fuzzy dynamical systems is a relatively new branch introduced in 1982 and 1983 by Kloeden and De Glas respectively. In the first section of this chapter we have introduced several notions into this theory to extend its scope and applicability, as for example, fuzzy dissipative dynamical systems, chaotic fuzzy dynamical systems, periodicity, robustness, predictability, local metric entropy, fuzzy Liapunov exponent and fuzzy Liapunov time, etc. In the second section fuzzy differential equations and fuzzy differential inclusions have been discussed. An alternative method of solution for one dimensional FDIs has been proposed. With the help of this method solution of a second order FDI has been obtained.

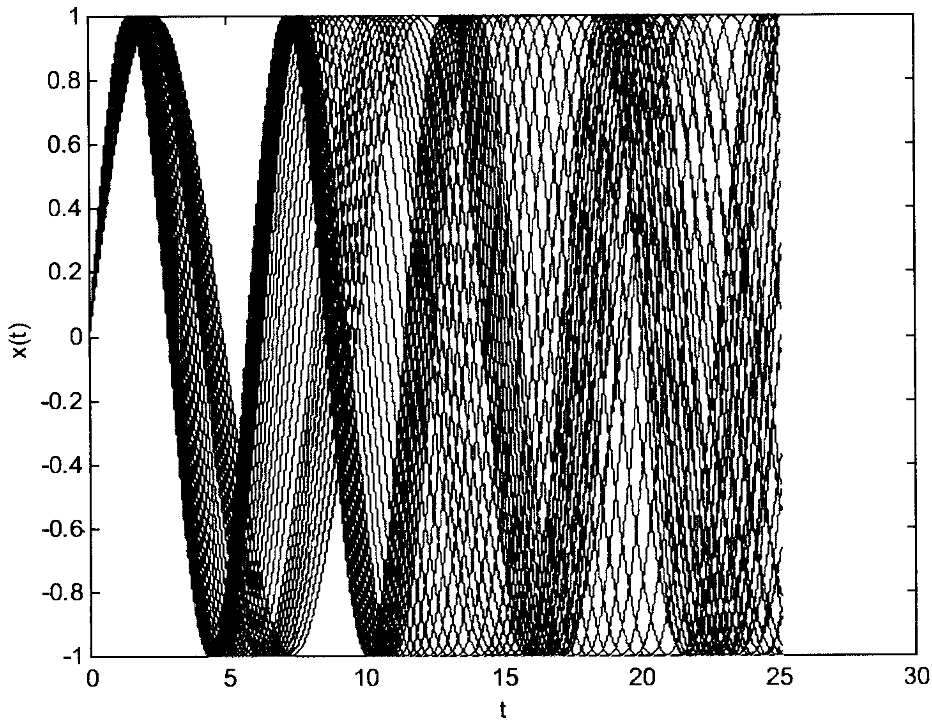


Fig 3.2.6 A more generalized solution of (3.2.12), (3.2.13), (3.2.14), (3.2.15) representing the system behaviour for $0 < \alpha \leq 1$, assuming that, $\delta \to 0$ and $\lambda \to 0$. Here the red region is giving the fuzzy flow representing the best possible system behaviour ($\alpha = 1$) and the blue region is giving the fuzzy flow representing the other possible system behaviour ($\alpha < 1$). The scaling of t-axis has been magnified by a factor of 10 compared to Fig 3.2.5.

In this chapter we have seen that, fuzzy differentiable dynamical systems use fuzzy numbers to model uncertain quantities in a dynamical system. This shows how powerful a tool the system of fuzzy real numbers is to represent uncertain and imprecise information. In Chapter 2 and Chapter 3 we have theoretically dealt with the uncertain dynamical systems, where the uncertainty is fuzzy in nature. In the next three chapters we shall remain preoccupied with the modelling and simulation of some of the very interesting yet uncertain real life dynamical systems, where some of the theoretical tools developed in these two chapters will become useful. We shall begin our study of applications in the next chapter with a very important class of discrete fuzzy dynamical systems.

Chapter 4

Application of FDS in Fuzzy Fractal Based Image Generation and Modelling of Turbulence

In the last two chapters we have discussed management of phase space uncertainty and attainability set mapping uncertainty respectively of a dynamical system. In Section 2.5 we have discussed fuzzy pattern recognition. There we have shown that either fuzzy pattern vectors are multidimensional fuzzy numbers or can be generated from them. In the present chapter we shall discuss *fuzzy fractals* and *fuzzy turbulence* [97]. In Section 4.1 we shall be defining fuzzy fractals and describing their role in image generation. In Section 4.2 we will propose a hypothetical fuzzy dynamical system modelling of a turbulence as a chaotic occurrence of fuzzy vortices (that is, vortices modelled by FDIs) in a (two dimensional) dynamic fluid. All the models will be implemented by simulation, some of them with the help of the evolutionary Algorithm 3.2.1, formulated in the last chapter. All the phase spaces are tacitly assumed to be fuzzy phase spaces, which we have discussed in Chapter 2.

4.1 Fuzzy Fractals and Image Generation

Mandelbrot defined *fractal* as a special class of subsets of a complete metric space for which the Hausdorff-Besicovitch (or just Hausdorff) dimension strictly exceeds the topological dimension [105]. Detail explanation of this definition may be found in Appendix B. Barnsley has shown that, fractal sets can be generated as attractors of a class of randomized, contractive or hyperbolic, iterated function system (IFS) [9]. In other words a given set of randomized, hyperbolic IFS is able to generate a particular image as a fractal set. Such a system is a discrete dynamical system with a fractal attractor. The initial mathematical background for this work was earlier prepared by Hutchinson [64]. Barnsley further extended this background by proposing and proving a number of important mathematical results [9]. A fractal attractor is called a *strange attractor*, where the underlying dynamical system may or may not be chaotic [73]. Of course there is no universal agreement yet over this point [109].

Definition 4.1.1: Let (X, d) be a complete metric space. f: X → X is a *hyperbolic iterated function system* if there exist N contraction mappings $\{w_i\}^{N}_{i=1}$, w_i: X → X for i ∈ {1, ... N}, s.t., $f(D) = \bigcup_{i}^{N} w_i(D)$ for D ⊆ X. When each w_i is an affine transformation then f is called an *Hutchinson operator*.

Definition 4.1.2: Let (X, d) be a complete metric space. f: X → X is a *randomized hyperbolic iterated function system* if $f = (w_1, \ldots\ldots\ldots\ldots, w_N, p_1, \ldots\ldots\ldots\ldots, p_N)$. Each w_i : X → X is a contraction mapping. Each p_i is the probability of occurring w_i. Thus

$\sum_{i=1}^{N} p_i = 1$. And for any $D \subseteq X$, $f(D) = \bigcup_{i=1}^{N} w_i(D)$. f under iterations has been called *chaos game* by Barnsley [9].

Barnsley has shown how a simple randomized, hyperbolic IFS, often consisting of just three to four such simple contractive functions, can generate extremely complicated arbitrary shaped images [9]. This method has been extended, with suitable modifications, by Cabrelli et al. to generate, analyze and / or approximate images as the fractal attractors of iterated fuzzy sets systems (IFZS) [21].

Images with grey or colour levels admit a natural representation in terms of fuzzy sets. In this regard our method incorporates the technique of the discrete dynamical system of IFS. Let f be an IFS as defined in Definition 4.1.2. Since each w_i is contractive (i.e., $d(w_i(x), w_i(y)) \leq sd(x, y)$, $s \in (0, 1)$) f is also contractive or dissipative (take $x, y \in X$ such that, $d(x, y) = k$ is the diameter of the compact set X, then $d(f(x), f(y)) \leq sk$, which satisfies the criterion for dissipativeness (3.1.4), described in the last chapter). So f must have an attractor $A \subseteq X$, i.e., $\lim_{n\to\infty} f^n(X) = A$, where $f^n = f \circ f^{n-1}$. Under each iteration of f, the degree of possibility (by *degree of possibility* we shall mean fuzzy membership measure, for a detailed discussion on *possibility* refer to [156]) of occurrence of w_i is p_i. For a given set of possibilities $\{p_i\}$ there exists a unique invariant fuzzy membership measure μ with support A. The novelty of our approach may be summarized in terms of the following two key points:

(1) The entire setting is provided by a subclass $(\Im, \delta)$ of the class F(X) of fuzzy subsets of X, where $\Im$ is the collection of all the nonempty compact fuzzy subsets of a compact metric space (X, d) and δ is the metric defined on $\Im$ (for the detail of $(\Im, \delta)$ see Section 3.1). All images are considered as fuzzy sets. This leads to two possible interpretations:

   (a) in image representation the value of the fuzzy membership function at a point (pixel) $x \in X$, within the image will be interpreted as the normalized (values lying in [0, 1] only) grey level intensity associated with that point (pixel).

   (b) in pattern recognition, the membership value of x when lies in [0, 1] indicates the possibility that the point x is in the foreground of an image.

(2) Associated with each contraction map w_i, i = 1, ... N, is a grey level map $\varphi_i : I \to I$, where I = [0, 1] is the grey level domain. The collection of maps $\{w_i, \varphi_i\}$ is used to define an operator $T : U^n \to U^n$ (U^n is the collection of normal, uppersemicontinuous members of $\Im$), which is contractive with respect to the metric δ on U^n. This metric is induced by the Hausdorff distance defined on the collection of the nonempty compact subsets of X. Starting with an arbitrary initial fuzzy set $\mathbf{u}_0 \in U^n$, the sequence $\mathbf{u}_n \in U^n$ produced by the iteration $\mathbf{u}_{n+1} = T(\mathbf{u}_n)$ converge in the δ metric to a unique invariant fuzzy set $\mathbf{v} \in U^n$, where $T(\mathbf{v}) = \mathbf{v}$.

Definition 4.1.3: $f: \Im \to \Im$ will be called *iterated fuzzy sets systems* (IFZS) if $f = (w_1, \ldots\ldots\ldots\ldots, w_N, \mu(w_1), \ldots\ldots\ldots\ldots, \mu(w_N))$. Each $w_i : \Im \to \Im$ is a contraction mapping. Each $\mu(w_i)$ is the possibility or fuzzy membership value [156] associated with w_i. For any $\mathbf{u} \in U^n$,

$$f([\mathbf{u}]^{\alpha}) = \bigcup_{i=1}^{N} w_i([\mathbf{u}]^{\alpha}), \qquad (4.1.1)$$

where $[\mathbf{u}]^{\alpha}$ is the α-level set of $\mathbf{u}$ for $0 \leq \alpha \leq 1$.

Definition 4.1.4: The compact set X will be called the *base space* of the IFZS.

If there exists a fuzzy set $\mathbf{v} \in U^n$ such that, $f^n([\mathbf{v}]^{\alpha}) \subseteq [\mathbf{v}]^{\alpha}$ for $0 \leq \alpha \leq 1$ and for all positive integral n, then $\mathbf{v}$ is clearly the fuzzy nonwandering set (Definition 3.1.10) of the discrete fuzzy dynamical system $(\Im, f)$ or just f. When $f^n([\mathbf{v}]^{\alpha}) = [\mathbf{v}]^{\alpha}$ holds for $0 \leq \alpha \leq 1$ and for all positive integral n, $\mathbf{v}$ will be the unique invariant (fuzzy) set of f under iterations [64]. Then clearly $\mathbf{v}$ is an (fuzzy) attractor of the IFZS f. In the last chapter we have defined fuzzy chaotic dynamical systems, where we had to mention about fuzzy fractal sets (Definition 3.1.23). Here we are presenting the definition of fuzzy fractals in a formal way as

Definition 4.1.5: A fuzzy subset A of a complete metric space (X, d) is called a *fuzzy fractal* if A is also a fractal subset of X according to Mandelbrot's definition.

Clearly $\mathbf{v}$ is the fuzzy attractor of the image generating operator T in (2) above. In addition if $\mathbf{v}$ is a fractal subset of X (which is more often than not the case) $\mathbf{v}$ is a grey level fuzzy fractal image generated by T just like Barnsley's chaos games generating crisp fractal images.

In case of image generation $\mu(w_i)$ will depend on a normalized grey level pixel intensity value. We could also take $\mu(w_i) = p_i$, where p_i is a probability measure. That is, $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. When p_i is a probability measure instead of a possibility measure the underlying IFZS reduces to Barnsley's chaos game and the resulting fractal set generated is a crisp fractal. So crisp fractal is only a special case of fuzzy fractal.

A. Images as Fuzzy Sets

A black and white digitized image is a pixel matrix $\{p_{ij}\}$, where p_{ij} is the (i, j) (both are nonnegative integers) coordinate point (pixel) in R^2. Associated with each pixel p_{ij} is a nonnegative grey level or brightness value t_{ij}. We assume a normalized measure for grey levels, i.e., $0 \leq t_{ij} \leq 1$ (0 = black: the background, 1 = white: the foreground) [21].

Definition 4.1.6: The function $h : \{p_{ij}\} \to [0, 1]$ defined by the grey level distribution of the image is called the *image function*.

The digitized image is fully described by its image function h. This is also the situation in the more theoretical case where grey level are distributed continuously on the base space X. At this point one can see that, an image as described by an image function is nothing but a fuzzy set $\mathbf{u} : X \to [0, 1]$.

It is usual to classify the pixels according to their grey levels in the following way. For each $\alpha \in (0, 1]$, we consider the set $\{x \in X \mid \mathbf{u}(x) \geq \alpha\} = [\mathbf{u}]^{\alpha}$, i.e., the set of all pixels whose grey levels exceed the threshold value α is the α-level set (α-cut) of $\mathbf{u}$. For $\alpha \in (0, 1]$, $[\mathbf{u}]^{\alpha}$ represents a thresholding of the grey level distribution at α. Clearly there is a one to one correspondence between the image function h as a fuzzy set and the α-cuts of $\mathbf{u}$.

Since X is a compact metric space $(\mathfrak{I}, \delta)$ is also a compact metric space [46]. In particular it contains all the α-level sets $[\mathbf{u}]^{\alpha}$, $0 \leq \alpha \leq 1$, of all $\mathbf{u} \in U^n$. δ is the Hausdorff metric (Definition 2.1.1). The metric space (U^n, δ) is complete [46]. f is the IFZS of Definition 4.1.3, which being a contraction map (time-evolution law of a dissipative dynamical system) must have an attractor $[\mathbf{v}]^0 = A \subseteq X$, where $\mathbf{v} \in U^n$. In all practical situations X is a compact subset of R^2 and A is a fractal subset of X, which is also a fuzzy subset of X.

Definition 4.1.7: A as described above is called a *fuzzy fractal image* generated by the IFZS f.

One very important thing to notice in this subsection is a novel interpretation of fuzzy membership value. Traditionally fuzzy membership values signify the degree of vagueness or nonspecificity [43]. But in this subsection fuzzy membership value can not be interpreted that way. Here fuzzy membership value means 'lack of uniformity' (in the normalized grey level pixel intensity values). So far we have not gone into the detail of the grey level intensity distribution of the pixels p_{ij} of the fuzzy fractal image A. In the next subsection we are going to take up this issue.

B. Determination of the Grey Level Pixel Intensity

In Barnsley's randomized IFS each component w_i of the hyperbolic map f has a probability of occurrence p_i under each iteration of f (Definition 4.1.2). In the definition of IFZS (Definition 4.1.3) p_i of hyperbolic randomized IFS is to be replaced by $\mu(w_i)$, where $\mu(w_i)$ is the possibility (in sense of Zadeh [156]) of occurrence of w_i under each iteration of f. Since value of possibility is equal to fuzzy membership value, $\mu(w_i)$ is the fuzzy membership value associated with w_i. From image generation and processing point of view we are going to give here an interpretation of $\mu(w_i)$ in terms of the grey level map $\varphi_i : I \to I$, where $I = [0, 1]$ is the grey level domain.

Definition 4.1.8: A function $\varphi : [0, 1] \to [0, 1]$ is said to be nondecreasing right continuous (n.d.r.c) if and only if (i) φ is nondecreasing and (ii) φ is right continuous.

The following lemma justifies this definition.

Lemma 4.1.1: Let $\varphi : [0, 1] \rightarrow [0, 1]$ and X be an infinite compact metric space, then a necessary and sufficient condition for $\varphi \circ \mathbf{u}$ to be uppersemicontinuous for all $\mathbf{u} \in U^n$ is that φ is n.d.r.c.

Proof: [21].

We are now in a position to summarize properties, which should be satisfied by a set of grey level maps $\{\varphi_i\}$, $i = 1, \ldots, N$, so that they can be associated with $\mu(w_i)$ of an IFZS f.

(1) $\varphi_i : [0, 1] \rightarrow [0, 1]$ is nondecreasing.
(2) φ_i is right continuous in $[0, 1)$.
(3) $\varphi_i(0) = 0$.
(4) For at least one $j \in \{1, \ldots, N\}$, $\varphi_j(1) = 1$.

Properties (1), (2) and (4) above and Lemma 4.1.1 together will guarantee that the associated IFZS maps U^n into itself. Property 3 is a natural assumption in the consideration of the grey level functions, which says that, if the grey level of a point (pixel) $x \in X$ is zero, then it should remain zero after being acted upon by the maps φ_i.

C. The image generating fuzzy contraction map

Let us define a contraction map $T_s : U^n \rightarrow U^n$ such that, $\delta(T_s(\mathbf{u}), T_s(\mathbf{v})) \leq s\delta(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in U^n$ and $0 \leq s < 1$. Since T_s is a fuzzy valued contraction map defined over a collection of fuzzy sets, the system of iterated T_s is a discrete dissipative fuzzy dynamical system and hence must have an attractor. This is an IFZS, where T_s consists of N contraction mappings $\{w_i\}_{i=1}^{N}$.

$$T_s(\mathbf{u}(x)) = \sup\{\varphi_1(\mathbf{u}'(w_1^{-1}(x))), \ldots, \varphi_N(\mathbf{u}'(w_N^{-1}(x)))\}. \tag{4.1.2}$$

where

$$\mathbf{u}'(B) = \begin{cases} \sup\{\mathbf{u}(y) \mid y \in B \subseteq X\}, & \text{if } B \neq \varnothing \\ = 0 & \text{if } B = \varnothing. \end{cases} \tag{4.1.3}$$

Obviously, $\mathbf{u}'(\{x\}) = \mathbf{u}(x)$ for any $x \in X$. Since IFZS of iterated T_s must have an attractor

$$\lim_{n\rightarrow\infty} T_s^n(\mathbf{u}) = \mathbf{v}. \tag{4.1.4}$$

$\mathbf{v}$ is the fuzzy attractor of the IFZS, which is the generated fuzzy (fractal) image. In the next subsection we shall be generating two such fuzzy fractal images.

Equivalently, we can take $f: \Im \rightarrow \Im$ of Definition 4.1.3 as the image generating IFZS, where

$$\mu(w_i(x)) = \varphi_i(\mathbf{u}(w_i^{-1}(x))) \qquad (4.1.5)$$

for all $x \in X$ and $\mathbf{u} \in \Im$. When $\mu(w_i(x)) \in [0, 1]$ is a constant function for $i \in \{1, \ldots, N\}$, such that, $\sum_{i=1}^{N} \mu(w_i) = 1$, that is, $\mu(w_i)$ is a random (probability) measure then the IFZS is simply crisp random hyperbolic IFS as described in [9] (Definition 4.1.2) and the fuzzy fractal attractor reduces to crisp fractal attractor. That is the crisp fractals are only special cases of fuzzy fractals.

D. Examples

In the two examples considered in this subsection the base space is $X = [0, 1] \times [0, 1]$. Computer approximation of the attractors of the IFZS has been shown as normalized grey level distribution. The brightness value t_{ij} of a pixel p_{ij} representing a point $x \in X$ obeys $0 \le t_{ij} \le 1$, with $t_{ij} = \mathbf{v}(x)$, where $\mathbf{v}$ is as defined by the equation (4.1.4). $t_{ij} = 0$ if x is in the background.

Example 4.1.1 [21]: N = 4.

$$w_1((x_1, x_2)^T) = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.04 \end{bmatrix}$$

$$w_2((x_1, x_2)^T) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0.4 \end{bmatrix}$$

$$w_3((x_1, x_2)^T) = \begin{bmatrix} 0.355 & -0.355 \\ 0.355 & 0.355 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.266 \\ 0.078 \end{bmatrix}$$

$$w_4((x_1, x_2)^T) = \begin{bmatrix} 0.355 & 0.355 \\ -0.355 & 0.355 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.378 \\ 0.434 \end{bmatrix}$$

To define a base space IFZS $f = (w_1, \ldots, w_N, \mu(w_1), \ldots, \mu(w_N))$, whose attractor $A \subset X$ is shown in Fig 4.1.1, we consider the identity grey level intensity maps given by $\varphi_i(t) = t$ for $i \in \{1, \ldots, 4\}$. Since $\varphi_i(1) = 1$ for $i \in \{1, \ldots, 4\}$, $\mathbf{v} = \chi_A$, that is, $\mathbf{v}(x) = 1$ if $x \in A$ and $\mathbf{v}(x) = 0$ if $x \notin A$. The attractor $\mathbf{v}$ has been shown in Fig 4.1.1.

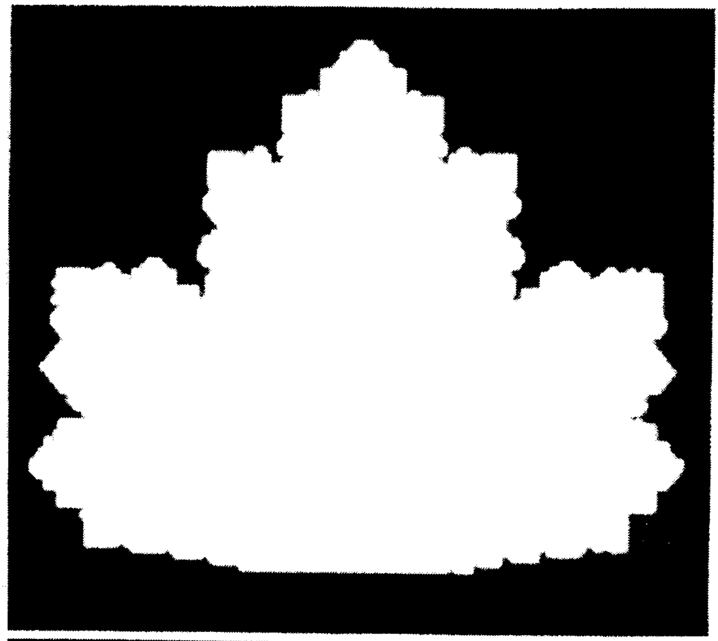


Fig 4.1.1 The attrctor of the IFZS $f = (w_1, \ldots\ldots\ldots\ldots, w_N, \mu(w_1), \ldots\ldots\ldots\ldots, \mu(w_N))$. Each $w_i : \Im \rightarrow \Im$ is a contraction mapping. $\mu(w_i(x)) = \varphi_i(\mathbf{u}(w_i^{-1}(x)))$ for all $x \in X$. Note that $x \in \Im$ also. (Adopted from [21].)

Example 4.1.2: N = 3.

$$w_1((x_1, x_2)^T) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.005 \\ 0.005 \end{bmatrix}$$

$$w_2((x_1, x_2)^T) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.005 \\ 0.255 \end{bmatrix}$$

$$w_3((x_1, x_2)^T) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.125 \\ 0.255 \end{bmatrix}$$

To define a base space IFZS $f = (w_1, \ldots\ldots\ldots\ldots, w_N, \mu(w_1), \ldots\ldots\ldots\ldots, \mu(w_N))$, whose attractor $A \subset X$ is shown in Fig 4.1.2, we consider the constant grey level intensity maps given by $\varphi_i(t) = 1/3$ for $i \in \{1, 2, 3\}$. Since $\varphi_i(1) = 1$ for $i \in \{1, \ldots\ldots, 4\}$, $\mathbf{v} = \chi_A$, that is, $\mathbf{v}(x) = 1$ if $x \in A$ and $\mathbf{v}(x) = 0$ if $x \notin A$. The attractor $\mathbf{v}$ has been shown in Fig 4.1.2. Note that here $\varphi_i(t)$ is a probability distribution function and the generated attractor is the Sierpinski's gasket, which is a crisp fractal. Definition 4.1.3 and Definition 4.1.5 have been formulated in such a manner that the crisp fractal of Fig 4.1.2 becomes only a special case of fuzzy fractal. This is true in general.

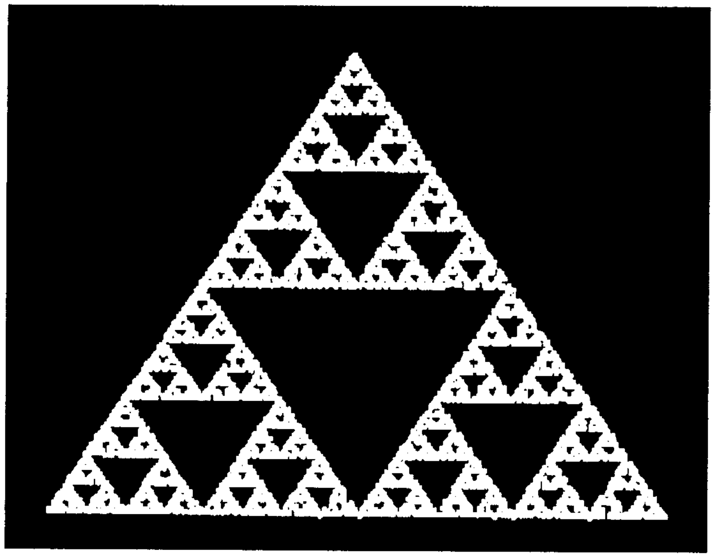


Fig 4.1.2 Sierpinski's gasket generated as the fractal attractor of the IFZS described above. The starting point is on the gasket.

4.2 FDS Modelling of a Turbulence

In this section we will be concerned about another novel application of fuzzy set theory based techniques in general and FDS in particular. Here we have proposed a hypothetical model of a two dimensional turbulence. Turbulence is considered as one of the most complex natural phenomena, which has defied persistent efforts to reveal its mysteries by some of the most brilliant fluid dynamicists. Here we have made only an humble attempt to model an ideal turbulence (according to some definition) in terms of an FDS. Our purpose in this section is not to delve deep into the mystery of turbulence but only to indicate the potentiality for application of fuzzy dynamical systems. This idealistic model at least serves this purpose.

A. Preliminary Discussion

Turbulence occurs in a moving fluid in the form of a spontaneous yet completely irregular appearance and disappearance of vortices. Despite relentless efforts of last more than hundred years, particularly by the fluid dynamicists, turbulence has not yet been completely understood. Naturally there is no mathematical model yet which can completely describe a turbulence. "Historically, many models have been proposed and many are currently in use. It is important to appreciate that there is a broad range of turbulent flows and also a broad range of questions to be addressed. Consequently it is useful and appropriate to have a broad range of models, that vary in complexity, accuracy and other attributes." [125]. An usual description of turbulence starts with the Navier-Stokes equation describing the motion of the underlying body of the fluid [125]. Here we have not considered Navier-Stokes equation. In this section starting from a simple definition of turbulence we shall try to present a mathematical model of an ideal turbulence in terms of an FDS. This may serve as yet another example to emphasize the potentiality of FDS to model various complex real life phenomena. To start with a definition of turbulence let us state,

Definition 4.2.1 (Brown [16]): *Turbulence* is chaotic occurrence of vortices in a dynamic fluid.

Devaney's definition of chaos has already been stated in Definition 1.4.4. For the sake of simplicity we shall keep our model only two dimensional. It is a common practice to use a discrete distribution of point vortices in the analysis of two and three dimensional flows of fluid [144]. In [144] an elliptic function (doubly periodic function in the complex plane) has been used to describe the distribution of vortices in a two dimensional channel flow. It is also a very common practice to describe the distribution of such vortices by probability distribution functions [125]. Here we have described the distribution of vortices by a two dimensional chaotic function. We have outlined an extension of this method to three dimensional case in Section 7.2.

We locate a two dimensional vortex by the centre of its core. In a two dimensional cross-section of a turbulent fluid to model the chaotic distribution of vortices we need some chaotic function. Let the two dimensional cross-section be the xy-plane. According to our idealized model if in the xy-plane the coordinate of the centre of a vortex is given by (x(s),y(s)) for some s then x(s) and y(s) are given by equation (4.2.1) below.

$$x(s) = a.\lim_{n\to\infty} f^n(s) \text{ and } y(s) = b.\lim_{n\to\infty} f^n(s), \qquad (4.2.1)$$

where $f(s) = 4s(1 - s)$, $s \in [0, 1]$ and coordinates of the vertices of the rectangle enclosing the cross-sectional area is given by (0,0), (a,0), (a,b) and (0,b). For practical computational purpose n can never be infinite. Only a sufficiently large value of n can be taken. Here it is important to remember that for every value of s there may not be a vortex with centre of the core (or just centre for short) at (x(s),y(s)) but every centre of the core of a vortex will be given by (x(s),y(s)) for some s. In Appendix C we have mathematically proved that x(s) and y(s) as given by equation (4.2.1) are chaotic. Here we are only invoking these results for our modelling purpose. In actual computer simulation the value of n is finite but large, say $n \geq 20{,}000$. Then we get the famous Feigenbaum diagram (Fig 4.2.1).

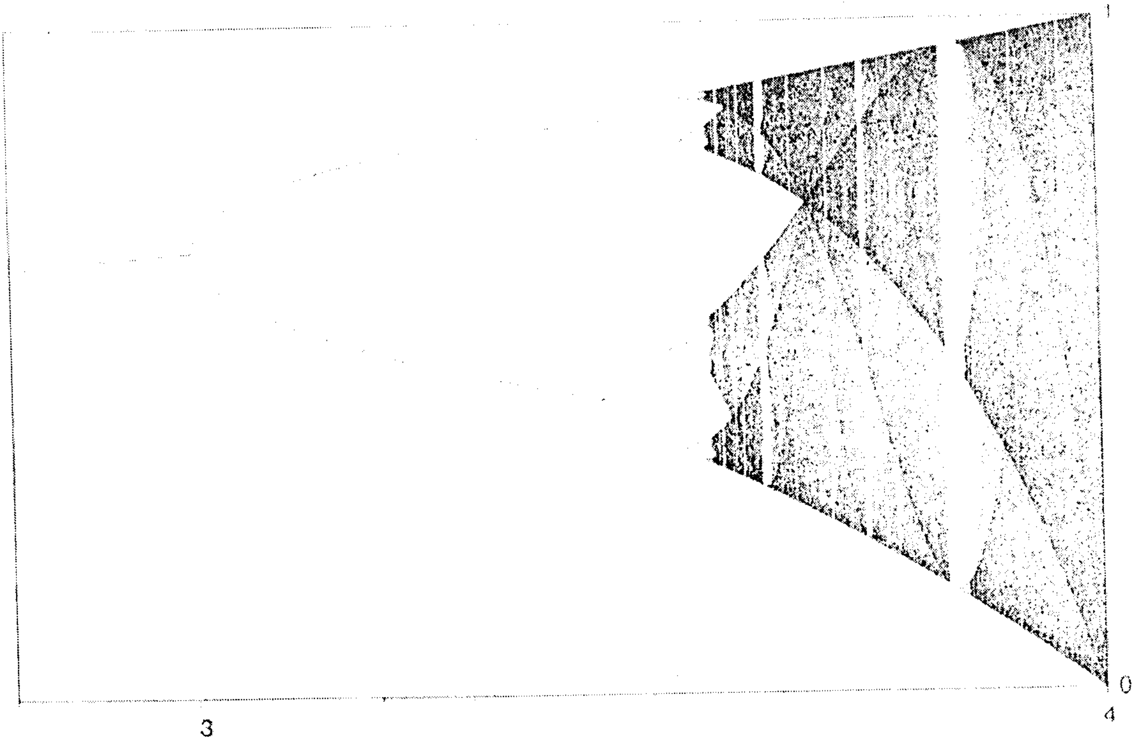


Fig 4.2.1 Feigenbaum diagram of x(s) for very large n and a = 1, irrespective of s, provided $s \in [0, 1]$. $f(s) = \mu s(1 - s)$, $3 \le \mu \le 4$. x(s) is plotted along the ordinate and μ along the abscissa. For $\mu = 4$ x(s) can converge on any pixel along the ordinate. Exactly the same diagram will also hold for y(s). (Reproduced from [124].)

We assume that the vortex centering (x',y') (we shall write (x',y') in place of (x'(s),y'(s)) whenever there is no risk of confusion) is generated by two local tiny fluid jets coming from different directions and colliding. We are not bothered about the origin of the wind jets here. Number of such collisions will be more at higher Reynold's number. After collision the resultant wind jet will create a spiral like vortex under certain conditions. To understand these conditions let us concentrate into the Fig 4.2.2 below. In Fig 4.2.2 for convenience the coordinate system is polar (r,θ), where $r = ((x - x')^2 + (y - y')^2)^{1/2}$ and $\theta = \tan^{-1}((y - y')/(x - x'))$. On the other hand $x = x' + r.\cos\theta$ and $y = y' + r.\sin\theta$.

The Fig 4.2.2 below is self-explanatory. For detail explanation of radial and cross-radial component of velocity any text on particle dynamics, e.g., Loney [92] may be consulted. From elementary particle dynamics we know that the expression for radial component of velocity at a point (r,θ) is dr/dt and that of the cross-radial component is $r(d\theta/dt)$ [92] with respect to some polar coordinate system. If

$$\frac{\text{Resultant radial component of velocity}}{\text{Resultant cross-radial component of velocity}} = \frac{dr/dt}{r(d\theta/dt)} = m, \qquad (4.2.2)$$

where m is a constant holds then

$$dr/d\theta = mr. \tag{4.2.3}$$

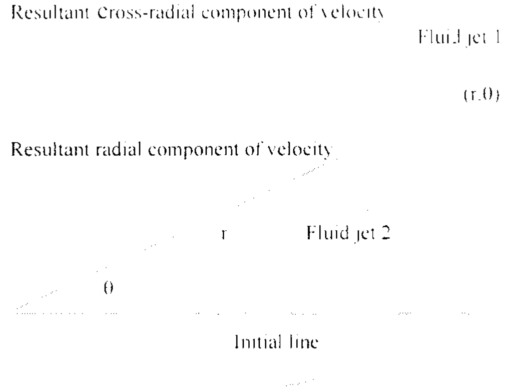


Fig 4.2.2 Fluid jet 1 and fluid jet 2 are colliding at (r,θ). Their resultant radial component of velocity and resultant cross-radial component of velocity after collision have been shown. If the ratio of these two components is constant a spiral shaped vortex is generated.

B. Dealing Nonlinearity with Fuzzy Quantities

In equation (4.2.3) if m is a constant then the solution of (4.2.3) will produce a log-spiral, which describes the shape of a vortex. But in reality m can not be constant. Instead the values of m are liable to fluctuate within some range. Note that (4.2.3) is a simple linear equation modelling the generation of a vortex in a dynamic fluid. But the generation of a vortex is supposed to be a nonlinear phenomenon. Then the immediate question is how is it possible to model a nonlinear phenomenon by a simple linear equation? We assume that the nonlinearity is responsible for giving rise to uncertainty in the model. Had there been no uncertainty involved in the model it would have been perfectly deterministic. In that case we could have described it with a set of linear equations. Here we take the opposite approach. We assume that the model is deterministic in the first place. So we describe it with a set of linear equations. Next, to bring it closer to the reality we try to accommodate the elements of inherent uncertainty within it. In this spirit the generation of vortex in a dynamic fluid has been described by (4.2.3). Our next task is to accommodate the uncertainty, inherent in the underlying nonlinearity of the system, within (4.2.3) (thereby brining it closer to the real life nonlinear model). To accomplish this, we should notice that m being the ratio of two wind speeds, can not be a fixed value even for a short time. m will fluctuate within some closed bounded interval of R. In the most general case this interval is a fuzzy subset of R, the membership distribution function, whose support is the interval. Without much loss of generality we may assume that this membership distribution function satisfies all the three properties of Definition 2.2.1. So this fuzzy subset of R is actually a fuzzy number, say $A \in \Re$. Similarly, the measurement of r(0), the initial value of r, is also subject to uncertainty due to measurement error (uncertainties involved in m and r(0) are different, in m it is due to smoothing out the nonlinearity and in r(0) it is due to measurement error). Classically, the probability distribution of the error in measuring r(0) will follow the normal distribution law. So when we are going to model this uncertainty with a fuzzy quantity we should opt for a triangular fuzzy number, say B, to describe r(0), such that, $r(0) \in B \in \Re$.

A being a fuzzy number must be increasing and continuous from right in [a, c), decreasing and continuous from left in (d, b], where [a, b] is the support of A and $a \leq c \leq d \leq b$. Also $A([c, d]) = \{1\}$, which means that the values of highest possibility within the universe of discourse of A lies in [c, d]. So a trapezoidal fuzzy number <a, c, d, b> (Fig 2.2.1) may be an approximation of A.

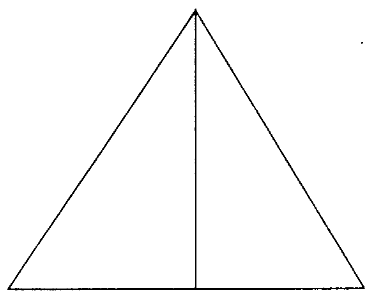


Fig 4.2.3 The triangular fuzzy number <r(0)-δ, r(0), r(0)+δ>. The value r(0) will vary from vortex to vortex.

For the purpose of modelling, next we shall have to make some reasonable estimate for the values of a, c, d and b, which we have given in Table 4.2.1. One very important thing to note that if $m \in [A]^0$ then also $m \in [-A]^0$. So $m \in [A]^0 \cup [-A]^0$. We shall take the triangular fuzzy number B as <r(0)-δ, r(0), r(0)+δ> (Fig 4.2.3), where $\delta > 0$ is a crisp real number, whose value depends upon the employed measurement techniques. The more accurate the technique is the smaller is the value of δ.

a	c	d	b
0.001	0.05	0.1	0.2

Table 4.2.1. Numerical estimation of the constant fuzzy number A for a zero viscosity fluid in a model with moderate shear.

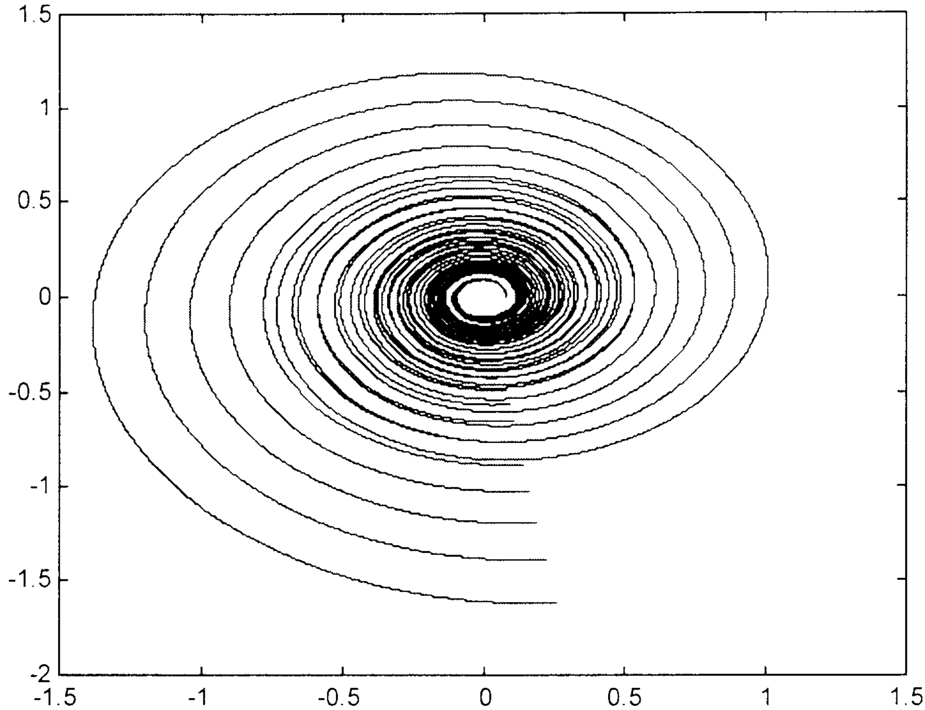


Fig 4.2.4 The simulated diagram of a fuzzy vortex centering at (0.5,0.5), where r_0 (in cm) belongs to a fuzzy number, with the best possible value ≤ 0.1. This diagram is the phase (solution) space of (4.2.4) for $\alpha = 1$, where $x = r(0)\cos\theta\exp(m\theta)$ is taken along x-axis and $y = r(0)\sin\theta\exp(m\theta)$ is taken along y-axis. The range of θ is a random number ≤ 16π. The enclosed region indicated by the spirals is the fuzzy flow representing the most possible (grade of membership 1) behaviour of the system represented by (4.2.4). We get the spirals by putting $m \in [0.05, 0.1]$ in (4.2.4). If we replace A by –A in (4.2.4) the orientation of the vortex will be reversed.

Here we are not too bothered about the actual numerical specifications. These may be determined more accurately through appropriate experiments.

So by now (4.2.3) is no longer an ordinary differential equation (ODE). Instead we shall have to reformulate it as a fuzzy differential inclusion (FDI) relation to make the modelling more realistic. The FDI formulation of (4.2.3) is described below.

$$r'(\theta) \in [mr(\theta)]^{\alpha}, \quad m \in [A]^{\alpha}, \ r(0) \in [B]^{\alpha}, \qquad (4.2.4)$$

where as usual $0 \le \alpha \le 1$. Similar formulation also holds for –A. The solution of (4.2.4) by Algorithm 3.2.1 described in Section 3.2 yields the following phase space or solution space diagram (Fig 4.2.4), which resembles a generated vortex in a dynamic fluid. For convenience we describe step by step the solution of (4.2.4) below.

C. Simulation

(1) fix $\alpha \in [0, 1]$;

(2) calculate the α-level set on the right hand side of $r'(\theta) \in [mr(\theta)]^{\alpha}$ by Zadeh's extension principle [112], which is a 2-tuple of α-cuts of fuzzy numbers;

(3) solve (directly or numerically) the ordinary crisp DEs for $r'(\theta)$ taking only the boundary values of the α-level set;

(4) complete the (crisp) trajectories on R^2 obtained as the graphs of the solutions derived in step 3, the space enclosed by these graphs on R^2 is the α-level set of the solution of the FDI (4.2.4) for the given α.

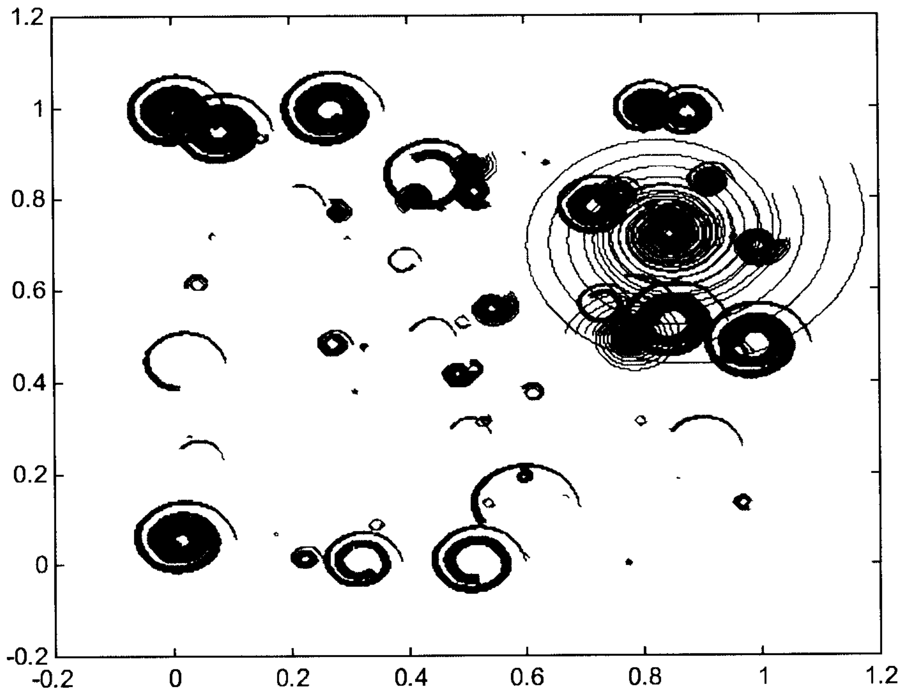


Fig 4.2.5 Simulation of a two dimensional fuzzy turbulence in a two dimensional dynamic fluid. Red and blue vortices have different orientation.

Fuzzy vortex is modelled by (4.2.4) and its chaotic occurrence within a finite rectangular cross-section of area ab is given by (4.2.1). To fill up the enclosed region some other non-boundary solutions have also been included. The simulation diagram is presented in Fig 4.2.5, where a = b = 1 and n = 20,000. r(0) and range of θ varies from vortex to vortex. From equation (4.2.1) and Fig 4.2.1 it appears that (x(s),y(s)) can be any pixel in $[0, 1] \times [0, 1]$. So this model can account for all possible occurrence of vortices in $[0, 1] \times [0, 1]$. In this sense it is the most general representation of fuzzy turbulence.

Definition 4.2.2: Since we get a simulated vortex in a dynamic fluid by solving the FDI (4.2.4), we call the vortex a *fuzzy vortex*. Chaotic occurrence of such fuzzy vortices in a dynamic fluid will be called *fuzzy turbulence*.

The following MATLAB program will generate more two dimensional fuzzy turbulence, with the command gen(argument1, argument2), where the 'argument1' and 'argument2' are positive integers denoting the maximum number of vortices desired and the density of the texture of the FAM respectively. A large value for argument2 (> 10) will slow down the simulation process without any significant enhancement in quality. It will also display the actual number of vortices generated, the number of red or anti-clockwise vortices and the number of blue or clockwise vortices. The program should be stored in a machine, loaded with MATLAB, as an M-file named gen.m in the directory MATLAB\BIN. This program generates turbulence in an uncontrolled manner. The probability of two turbulence generated with the same set of arguments being identical is very low, which is an essential feature of real turbulence.

```
function y = gen(max, flow)

%  max is the maximum number of vortices desired.
%  flow is a positive integer, determining the desired texture of the
%  FAM of (4.2.4). The higher the value of flow the more dense is the
%  texture of the FAM.
%  If max or flow is not an integer it is rounded to the nearest integer

%  smaller than the original input.

num = 1 + floor(max * rand(1));

count1 = 0;
count2 = 0;

% count1 gives the number of red or anti-clockwise vortices.
% count2 gives the number of blue or clockwise vortices.

if max < 0
   disp('Error, the number of vortices can not be negative.');
end

if flow < 1
   disp('Error, the FAM can not have less than 1 trajectories.');
end

for j = 1 : num
   aj = rand(1);
   bj = rand(1);

   a = rand(1);
   b = rand(1);
   c = rand(1);
   d = 0.05 + 0.1*floor(2*rand(1) - 1);
   t = 0:.001:a.*b*16*pi;

   % Generation of fuzzy vortices. Though in this program uncontrolled
```

```
    % parameters have been treated as random, the following for loop
    % generates the phase space of the fuzzy dynamical system represented

    % by the FDI (4.2.4).

    for i = 1 : floor(flow)

    if d >= 0
       xi = 0.02*c.*cos(t).*exp((d + 0.005*i)*t);
       yi = 0.02*c.*sin(t).*exp((d + 0.005*i)*t);
       plot(aj + xi, bj + yi, 'r');
       hold on;
       count1 = count1 + 1;
    end

    if d <= 0
       xi = 0.2*c.*cos(t).*exp((d - 0.005*i)*t);
       yi = 0.2*c.*sin(t).*exp((d - 0.005*i)*t);
       plot(aj + xi, bj + yi, 'b');
       count2 = count2 + 1;
       hold on;
    end

end

% Generation of the fuzzy vortices as the phase space of (4.2.4) ends.

end

hold off;

if max >= 0
   if flow >= 1
      disp('The actual number of vortices generated is')
      disp(floor(num))
      disp('The number of red or anti-clockwise vortices is')
      disp(count1 / floor(flow))
      disp('The number of blue or clockwise vortices is')
      disp(count2 / floor(flow))
   end
end
```

The above program generates a two dimensional turbulence only as a "Chaotic occurrence of fuzzy vortices." Here we have not taken care of the detail of the fluid and its motion in generating the turbulence. In this section our aim is only limited to indicating how FDS can be used to model a turbulence. A vast scope of improvement in this simulation is remaining open. For example the program can be so written to take arguments like 'viscosity' and 'Reynold's number,' which represents the fluid and its flow to a good extent.

4.3 Discussion

In this chapter we have presented iterated fuzzy sets systems (IFZS), which are discrete fuzzy dynamical systems. We have also discussed about the practical significance of the concept of FDI in modelling of complex real life systems. We have shown that fuzzy fractals are generated as attractors of IFZSs. We have defined fuzzy fractal sets. It has been shown that the crisp fractals are only special cases of fuzzy fractals. We have described an arbitrary image as a fuzzy subset of R^2. Then we have described fuzzy fractal image generation by IFZS. Finally we have presented two examples of fuzzy fractal images generated by IFZS.

Next, a turbulence is defined as a chaotic occurrence of vortices in a dynamic fluid. We have considered a finite rectangular cross-section of a dynamic fluid, where vortices are occurring according to some two dimensional chaotic distribution. The vortex is modelled by an FDI. In the next chapter we shall present FDI modelling of an initial disturbing vortex in atmosphere, which under favourable conditions may become an intense storm. In Chapter 6 we shall find more applications of FDIs.

Chapter 5

Modelling and Simulation of Cyclogenesis by FDI

In the last chapter we have modelled a turbulence as a chaotic occurrence of vortices in a dynamic fluid. In the present one we intend to model the occurrence of a strong vortex in the dynamic atmosphere of the earth [99, 100]. Under favourable climatic and geographical conditions such a vortex can become an intense storm. Here we are particularly interested about the starting stage of a cyclone over the tropical seas.

In many atmospheric systems modelling uncertainty management plays a very significant role. Yet the fuzzy set theory based techniques have so far found little or no application in this field. Here we intend to describe a model of a climatic disturbance in terms of FDIs. Under favourable conditions this disturbance may mature into a severe cyclonic storm (known as Hurricane in U.S.A).

5.1 Modelling of a Climatic Disturbance

Tropical cyclones rank with earthquakes as the most destructive natural disasters, and their origins in the normally serene tropics have been shrouded in mystery. Forecasters have long been aware of the existence of the several empirical conditions that are necessary but not sufficient for the formation of cyclones. The first of these requires the sea temperature to be at least 26^0C through a depth of at least 60m. A second requirement is the absence of significant vector changes of the mean wind with height through the troposphere (a layer in which temperature is strongly controlled by moist convection and which in the tropics extends upward to about 15 km, containing 80 to 85% of the mass of the atmosphere) [48]. There are other empirical conditions as well, but even when they are satisfied storm formation usually does not take place. In fact necessary climatic and geographical conditions for the formation of the tropical storms prevail over large areas of the earth during storm seasons, yet actual appearance of a storm is a relatively rare phenomenon. According to statistics, only about 80 tropical storms with maximum sustained wind speed 40-50 knots/hour (1 knot = 1.94 km) are observed per year over the entire globe. Of these between one-half and two-third attain hurricane strength [104]. This indicates that there must be a rare coincidence of circumstances before development of a storm. The formation always occurs in connection with some kind of pre-existing disturbance associated with deep cloud layer. All of these disturbances do not intensify into tropical storm. Only a small percentage of these systems start intensifying. Numerous studies have been made to clarify the process of their formation, but no general mechanism has yet been accepted [104]. In this chapter our research will centre around this point.

In one such study described in [48] the evolution with time of maximum surface wind speed in two of the experiments is shown in Fig 5.1.1. The first experiment starts with a maximum wind of 43.2 km/h. After about three days, the vortex rapidly intensifies to a nearly steady state amplitude of about 162 km/h. The second experiment is identical

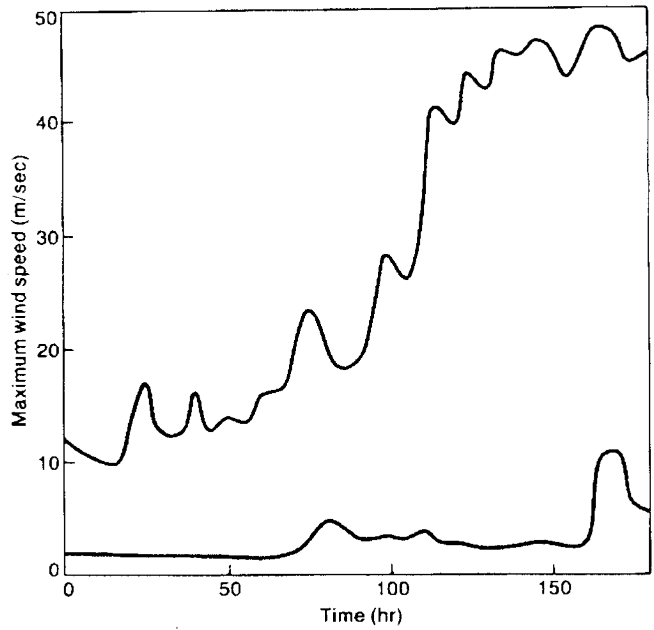


Fig 5.1.1 Graphs showing the evolution with time of the maximum surface wind speed produced by the numerical model of Rotunno and Emanuel. The upper curve starts with a 12 m/s (43.2 km/h) amplitude vortex, while the maximum velocity in an experiment that starts with a 2 m/s (7.2 km/h) amplitude vortex, but is otherwise identical, is shown by the lower curve. The failure of the weak vortex to amplify demonstrates that cyclones in this model can not arise out of weak disturbances; rather a vortex of sufficient amplitude must be provided by independent means. (Reproduced from [48].)

to the first, except that it was started with a maximum wind of only 7.2 km/h. The model storm in this case did not amplify even after 150 hours had elapsed. Apparently, the model needs a sufficient "kick" (disturbance) to get a cyclone started [48].

The need for a strong starting disturbance is consistent with the observation that real cyclones never start spontaneously. Rather, some pre-existing disturbance of independent origin seems to be necessary [48]. Our proposed model of this disturbance is as follows.

To build the model of "a vortex initiated by a sudden disturbance", we take our clues from two dimensional satellite images of cyclonic vortices. Though these features are of a matured cyclone, we accept them as clues towards the development of such shapes. It is natural to assume that the initial disturbing vortex is of log-spiral shaped strong wind vortex that is yet to draw large clouds from the surrounding areas to be prominent enough to be photographed from a satellite. We have taken the following features of such images to infer the initial stage.

- The clouds from the surrounding areas of the eye are curving inwards to the eye in log-spiral shape (as in Fig 5.1.2). It is already recognized that the curved cloud bands surrounding the eye of a disturbance indeed take log-spiral shape [15, 45].
- Here clouds are taking shape along the drag of the wind converging to the eye. So the wind tending to converge to the eye is supposed to traverse a log-spiral path in the vicinity of the eye.

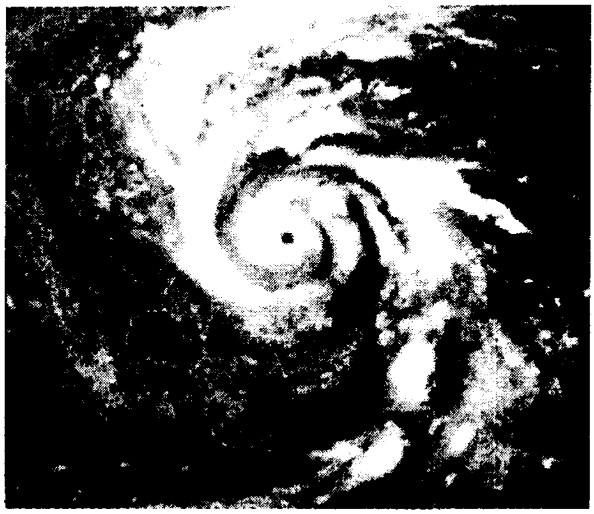


Fig 5.1.2 INSAT-1B cloud picture of the vortex of a severe cyclonic storm on the Bay of Bengal taken on 7th May, 1990 at 0600 UTC. The dark spot at the centre of the vortex (the central cloudless region) is known as the *eye* of the cyclone. (Reproduced from *Mausam*, vol - 41, no. 3, 1990).

In the study of atmospheric phenomena, it is often useful to consider a mathematical model which includes only those processes that are fundamentally important. In such models of cyclones the storm is often treated as a stable cylindrical vortex, where several thermodynamic and fluid dynamic equations are governing the surface fluxes of latent heat, angular momentum etc. To get a model cyclone started, a weak vortex, had to be superimposed on the basic state (of the model) of formation of the storm [48]. The disturbance is considered to be limited within certain layer of the atmosphere. We additionally assume that the wind drag towards the centre of the vortex will be along log-spiral paths similar to that around the eye of a cyclone (Fig 5.1.2). Resolving the velocity of the disturbance or the "kick" (which is nothing but wind jet(s) or wave(s) in reality) along the components of the cylindrical coordinates (r, θ, z) we get,

dr/dt = radial component;
$r(d\theta/dt)$ = cross-radial component;
dz/dt = vertical component.

To get log-spiral shape of the vortex, as seen from the top of the cylinder, i.e., along Z-axis downwards (negative z-direction), the following equation must be satisfied.

$$(dr/dt)/(r(d\theta/dt)) = m$$

$$\text{or, } r^{-1}(dr/d\theta) = m. \qquad (5.1.1)$$

where m is a constant, because the solution to (5.1.1) is

$$r = A \exp(m\theta), \qquad (5.1.2)$$

where A is integration constant to be determined by the initial condition. (5.1.2) is the polar equation of a log-spiral. Note that in Chapter 4 as well as in the present chapter m denotes the ratio between radial and cross-radial components of velocity. Also r denotes the radius vector in both the last and the present chapter.

In reality, the starting disturbance is often provided by large-scale waves that arise from instabilities of the east-to-west flow over sub-Saharan Africa in summer. Most such easterly waves run their course, often as far westwards as Florida, without developing into tropical storms. Only occasionally does a wave develop into a cyclone. Other starting disturbances arise from waves on cold fronts that sometimes manage to penetrate into the tropics, and form thunderstorm complexes that occasionally drift out over the ocean from middle-latitude continents [48]. In view of these facts we may assume that the disturbances propagate parallel to the ground. That is,

$$dz/dt = 0. \qquad (5.1.3)$$

It is important to note that the equations (5.1.1) and (5.1.3) give a simple dynamical model of only a disturbance leading to the development of a tropical cyclone. (5.1.1) and (5.1.3) are valid only for a very short time at the very beginning of the genesis of a storm. Once the storm is formed the model of the storm may be represented by a set of suitable equations like the ones in [29, 47, 113].

The basic problem with equation (5.1.1) is that it can appropriately model only when (r, θ) denotes the precise location of a point. But (5.1.1) is intended to model a vortex created by a certain wind jet, where (r, θ) denotes the position of an average point on the tip of the wind jet creating the vortex (exactly how the jet creates the vortex will be discussed more elaborately later). In reality location of such an average point can not be determined with great precision. A substantial uncertainty will always remain involved which can not be ignored.

Clearly, our model based study of a climatic disturbance leading to the genesis of a storm is a knowledge based system made up of vague or imprecise information. How the vagueness or imprecision or uncertainty arises in equation (5.1.1)?

Well, here value of r depends on value of θ (θ is crisp) and initial value r(0). In practice, (a) the value of m being the ratio of velocity components of wind, is liable to fluctuate even in short time intervals and (b) we can not determine the initial value r(0) precisely. Uncertainties (a) and (b) propagate to the value of r calculated by solving

(5.1.1). So the uncertainty in determining the value of r is the compound uncertainty due to (a) and (b).

To model the system with these uncertainties let us reformulate (5.1.1) as a fuzzy differential inclusion. Our formulation is as follows.

$$r'(\theta) \in [mr(\theta)]^{\alpha},\ r(0) \in [r_0]^{\alpha}. \qquad (5.1.4)$$

Next we are to specify m and r_0. Here r_0 is a fuzzy number containing all possible values of r at $\theta = 0$. For any given value of θ $r(\theta)$ is a fuzzy number in E^1. Fuzziness occurs because we can not determine $r(\theta)$ precisely for any given θ. We take r_0 as a triangular fuzzy number in E^1 (Fig 5.2.1).

To get ideas about m we may concentrate on (5.1.2), the solution of the crisp ODE (5.1.1), where $m = \cot \phi$, ϕ is the angle between the radius-vector and the tangent at (r, θ). In case of log-spiral or equiangular-spiral ϕ is independent of both r and θ. In [45] overlaying of a log-spiral, whose $\phi = 10^0$ on the curved band axis of a tropical storm of moderate intensity (T3), to measure the intensity of the storm from the length of the spiral arc has been suggested (page 11 and Fig 6 at page 12). $\cot 10^0 \approx 5.67$. However at the moment of starting the disturbance ϕ may not be equal to 10^0. In fact the actual numerical simulation of (5.1.1) is very enlightening in this regard. We notice that for values of m close to 5.67 the radius of curvature of the generated log-spiral is very high, i.e., the curve is quite flat in this range of m. The vortex signified by this curve is not likely to survive against the resistance due to wind-shear. To overcome the initial resistance of wind-shear to give rise to an intense stable storm the initial vortex must be more strong and this means that the simulation of (5.1.1) must produce a more 'circle like' log-spiral. In course of time the curved cloud bands i.e., the curved wind patterns of this initial vortex become rather flat due to wind-shear resistance. So we assume that for the sustenance of a stable storm m must take values from a compact interval [a, b], whose interior is nonempty. M is such a fuzzy number that only consists of all feasible values of m. By feasible values of m we mean only those values for which a stable vortex is possible to generate, whose intensity is 'enough' to give rise to a tropical storm (may or may not be of hurricane intensity). We take M as a trapezoidal fuzzy number (Fig 5.1.3).

$M([-\infty, a]) = M([b, \infty]) = \{0\}$, which means for values of m in $[-\infty, a] \cup [b, \infty]$ the generated vortices will not be strong and stable enough to give rise to a tropical storm of a given intensity, say T3, or more in Dvorak scale [45]. For values of m in $[a, c) \cup (d, b]$ (a, c, d and b are to be determined experimentally) the vortices generated may still mature into a tropical storm but only under very strong favourable conditions. The greater is the membership value the more is the possibility of the generated vortex to mature into a tropical storm under relatively less stronger favourable conditions. When $m \in [c, d]$ the possibility of cyclogenesis is the highest. Under favourable conditions for $m \in [c, d]$ a storm of hurricane intensity is very likely to be generated. A relatively small length of [c, d] indicates the lesser possibility of generation of storms of very high intensity. This is of course a very simplified structure for M. Some other structure(s) for the fuzzy number M is (are) also quite possible.

Notice that in the current chapter we have used the phrase *under favourable conditions* in several places. All of these uses imply the same meaning namely, the sea temperature to be at least 26^0C through a depth of at least 60m, an absence of significant vector changes of the mean wind that extends into the troposphere, relative humidity will have to be 85% or more for a long time throughout the region of storm formation and development, etc. These conditions will facilitate the latent and sensible heat transfer from the sea into the atmosphere, which in turn will supply the required rotational kinetic energy to intensify the vortex. Here we are only concerned about the generation of the initial vortex and not about the detail of its maturing into an intense storm.

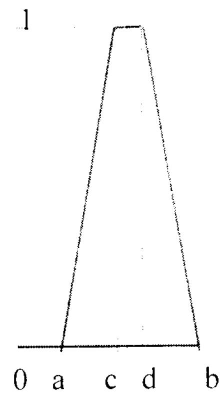


Fig 5.1.3 Graph of the fuzzy constant M (not according to the scale).

5.2 Simulation

By now we have the complete mathematical model of an atmospheric disturbance which has the potential to become a cyclonic storm. The variation of this disturbance along the z-axis is negligible. So FDI relation (5.1.4) is sufficient to represent the model. Let us restate (5.1.4) as

$$r'(\theta) \in [mr(\theta)]^{\alpha},\ r(0) \in [r_0]^{\alpha},\ m \in [M]^{\alpha}. \qquad (5.2.1)$$

where the fixed trapezoidal fuzzy number M is given by Fig 5.1.3. Let us mention here that (5.2.1) should have been written as $r'(\theta) \in [M{*}r_0]^{\alpha}$, which is the mathematically correct form, where $*$ denotes the product of fuzzy numbers. However this form is less convenient to follow and we shall continue with the form appearing in (5.2.1).

Let us recall Algorithm 3.2.1 described in Section 3.2 for solving the one dimensional FDIs like (5.2.1). For a numerical simulation of (5.2.1) we shall follow that method. We have already used it in Example 3.2.1 and in Section 4.2. For convenience let us restate Algorithm 3.2.1 in a more case specific way suitable to solve or simulate (5.2.1).

(1) **fix $\alpha \in [0, 1]$;**

(2) **calculate the α-level set on the right hand side of $r'(\theta) \in [mr(\theta)]^{\alpha}$ by Zadeh's extension principle [112], which is a 2-tuple of α-cuts of fuzzy numbers;**

(3) **solve (directly or numerically) the ordinary crisp DEs for $r'(\theta)$ taking only the boundary values of the α-level set;**

(4) **complete the (crisp) trajectories on R^2 obtained as the graphs of the solutions derived in step 3, the space enclosed by these graphs on R^2 is the α-level set of the solution of the FDI (5.2.1) for the given α.**

The crisp ordinary differential equation $r' = mr$ has solution (5.1.2) for any m and any value of r(0) (A is determined by value of r(0)). $m \in M$ and $r(0) \in r_0$. A membership value will be assigned to each such crisp solution, where the membership value will be determined by the membership values of m and r(0) in the fuzzy sets M and r_0 respectively by Zadeh's extension principle [112]. This way we get the fuzzy set of solutions of the differential equation $r' = mr$, where m and r(0) take values in fuzzy subsets of R. As a solution of (5.2.1) for a given $\alpha \in [0, 1]$, we take only those solutions r whose membership value $\geq \alpha$.

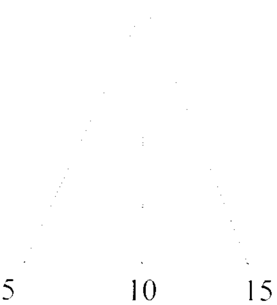


Fig 5.2.1 Graph of fixed triangular fuzzy number r_0. Support of r_0 is [5, 15] in 50 km scale, which is compatible with, $r_0(10) = 1$ [47, 133].

We have already mentioned in the previous section that the values of a, c, d and b are to be determined experimentally. Here we have determined them by computer simulations using synthetic data. It is rather an attempt to make meaningful inference about the values of a, c, d and b, which have been presented in Table 5.2.1. More accurate values of a, c, d and b can be determined only by applying the model in real situations many times over.

a	c	d	b
-0.2	-0.1	-0.05	-0.001

Table 5.2.1. Numerical specification of the fuzzy constant M.

One very important thing to note here is that. m is taking negative values only. Mathematically m could take positive values as well (compare this with the modelling of fuzzy turbulence in Section 4.2, where both M and -M have been considered (Fig 2.2.1)). But in that case the orientation of the vortex would have been just the opposite. The vortex would not then be cyclonic but anticyclonic in northern hemisphere (in southern hemisphere the orientation will just be the opposite). Under the favourable influence of the coriolis force [16], due to the rotation of the earth about its own axis, the cyclonic vortices gather extra angular momentum to intensify. For the same reason the anticyclonic vortices tend to gradually subside rather than intensify. Therefore in this chapter we are concerned about the cyclonic vortices (of northern hemisphere) only.

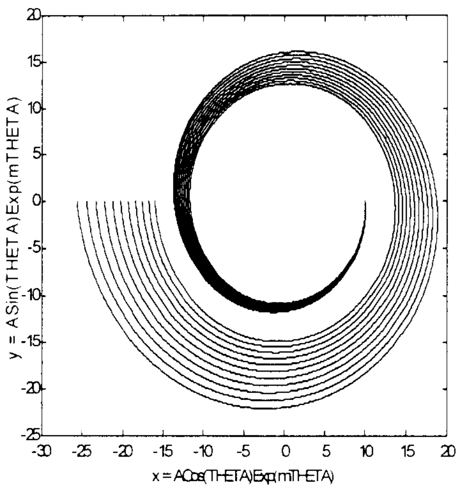


Fig 5.2.2 Numerical simulation of (5.2.1) in 50 km scale, where $\alpha = 1$, that is, the phase space is showing the fuzzy flow representing the best system behaviour only. This flow tends to converge to a (fuzzy) point of the (fuzzy) phase space, which is known as the fuzzy attractor of the autonomous FDDS (5.2.1). If this vortex ultimately matures into a severe cyclonic storm this fuzzy attractor will become the eye of the cyclone (Fig 5.1.2).

Now we are in a position to solve (5.2.1), where M is given by Fig 5.1.3 and Table 5.2.1 and r_0 is given by Fig 5.2.1. For a solution of (5.2.1) we have to evaluate the product of two fuzzy numbers $M * r_0$ by Zadeh's extension principle [112]. Let $M * r_0 = S$, then

$$S(s) = \sup_{s = mr(0)} \inf(M(m).r_0(r(0))).$$

For any given α, let $U = \{s \mid S(s) \geq \alpha\}$. Solve the crisp differential equation $r' = mr$ for such m and r(0) that, $mr(0) = s \in U$. This particular solution is an *α-solution*, i.e., a solution with membership value α or more, of (5.2.1). Collection of all such α-solutions of (5.2.1) will give the *α-flow* of (5.2.1) (in Fig 5.2.2 the 1-flow is shown). This α-flow is actually the fuzzy solution of (5.2.1), which signifies the behaviour of the system given by (5.2.1) with possibility α. The best system behaviour is obviously given by the α-flow, when $\alpha = 1$. Fig 5.2.2 gives the phase space of the best system behaviour of (5.2.1).

The physical significance of the log-spiral shaped fuzzy flow of Fig 5.2.2 is that, it represents the initial disturbance generating the initial vortex (like the ones in [48] (intensity given by the graph in Fig 5.1.1)), which if sufficiently strong, under favourable conditions may develop into a cyclone. As we have seen that the (fuzzy) log-spiral shape of the vortex is due to the (fuzzy) constant ratio between the radial and cross-radial components of the velocity of the disturbance. As the vortex is matured into a severe cyclonic storm the fuzzy flow of Fig 5.2.2 tends to converge to a fuzzy point, which has (0,0) as a crisp member with membership value 1. If this initial disturbing vortex matures into a cyclone, this fuzzy point will ultimately become the *eye* of the cyclone (Fig 5.1.2). So the preliminary stage of formation of the eye of a cyclone is a very natural real life example of a fuzzy attractor (Definition 3.1.9) (of the autonomous fuzzy differentiable dynamical system (5.2.1)). Also from the Definition 3.1.15 (t is replaced by θ in the definitions, because here the FDDS is autonomous) we see that the system represented by (5.2.1) is asymptotically stable for the given values of m and r(0) with a fuzzy limit point as the point of equilibrium (crisp points also are special cases of fuzzy points).

One very interesting aspect to note here is that the inclusion relation (5.2.1) is not explicitly time dependent. That is, it is an autonomous fuzzy differentiable dynamical system. But the development of a storm is surely a time dependent process. The solution of this apparent paradox lies in the facts that, (1) the inclusion relation (5.2.1) is valid only for a very short time during which the change taking place does not depend on time, but rather on the intensity of the disturbance and (2) the fluctuation of intensity of the disturbance is completely accommodated in the fuzzy number M. Once the vortex is created its dynamics with respect to time is taken care of by a set of time dependent differential equations, which we are not bothered about in this thesis.

For a more straight forward interpretation for the generation of the initial vortex we may consider not one, but two simultaneous disturbances, both of them are linear wind jets or waves coming from different directions and colliding and merging to produce the initial vortex (like the one sketched in Fig 4.2.2, but unlike in Fig 4.2.2 here only two wind jets or waves are involved, one of them is very strong and the other is very weak). In this case one of them acts as the radial component of the amplitude of the vortex (the strong one) and the other acts as the cross-radial component. Our mathematical model and the subsequent numerical simulations give a very realistic picture of this *two disturbance* hypothesis. For example, we have seen in Fig 5.1.1 how a 43 km/h initial kick has ultimately generated a 162 km/h storm. If in our model we take this 43 km/h kick as the cross-radial velocity (acts in anti-clockwise direction in the northern hemisphere and hence positive) the radial velocity we can calculate from m. If we take m $\in$ [-0.1, -0.05] so that M(m) = 1, the value of the radial velocity must lie in the range [-4.3, -2.15] km/h (in the southern hemisphere cross-radial velocity will be negative and m will be positive, i.e., the radial velocity will always be negative in both the hemispheres, which means that the wind and along with that also the clouds will always tend to converge towards the centre of the vortex), which at any rate is a minor, almost negligible disturbance. Note that the cross-radial velocity must be obtained from the major kick, i.e., the major kick must act tangentially to the vortex. The minor kick must act along the radius-vector of the vortex. This shows that our choice of M in particular and our model in general is very much realistic.

This model is compatible with the model presented in [47]. Once the sufficiently strong initial cyclonic vortex is created according to our model, the development of the vortex towards maturity (that is, a stable matured intense cyclone) can be simulated as given in [47]. Structure of such a storm in matured stage is given by equation (11) of [47]. The manifold of this structure has been elaborated in Fig 1 of [47] in terms of constant angular momentum surfaces in the r-z plane (r radius, z height), the real life significance of which has been illustrated in Fig 5 of [47]. Fig 1 and Fig 5 of [47] pertain to the vertical cross-section of the vortex of the storm. Fig 5.2.2 on the other hand gives the horizontal cross-section of the vortex of the storm, at the starting, near the surface of the sea. This model however is not compatible with [133], for we have not taken the velocity of the propagation of the disturbance during formation of a weak initial disturbing vortex according to equation (37) of [133]. We have shown instead that the initial disturbing vortex can be generated in a more natural and simple way. We have proposed our model mainly based on the observations made in [15, 45, 47, 48, 133].

The development of the storm is an example of the finite-amplitude air-sea interaction instability described in [47]. In [47] it is argued that when entrainment and other buoyancy reducing effects are taken into account, the mean tropical atmosphere is very close to neutral, rather than conditionally unstable. Because of this, anomalous surface winds are necessary to enhance the surface heat fluxes. This energy is then redistributed in the vertical by convection, which then intensifies the large scale circulation and the surface heat fluxes, and so on. When viewed in this context, the initial vortex must have a large enough amplitude to enhance the surface fluxes enough for the instability to occur [29]. Keeping this in mind we have taken r(0) about 500 km, which is compatible with [47, 133].

This model can also give satisfactory explanation of the phenomena of occurring of the most intense cyclonic storms often either during April-May (just before the onset of the monsoon in the Indian sub-continent) or during October-November (just after withdrawal of the monsoon from the Indian sub-continent) over the Bay-of-Bengal. During these times of the year major changes take place in the wind pattern over the seas surrounding the Indian coasts (Indian Ocean, Bay of Bengal and Arabian Sea). The wind change course from land-to-sea to sea-to-land (April-May) and from sea-to-land to land-to-sea (October-November). Naturally during these periods possibility of simultaneous occurrence of one major and one minor disturbances as described above is very high.

along with the other favourable conditions, and hence the relative high frequency of occurrence of intense cyclones.

5.3 Discussion

An FDS modelling of the incipient stage of the generation of a hurricane or cyclone (cyclogenesis) over the seas in terms of FDIs has been presented. Solution of the FDI has been accomplished by Algorithm 3.2.1. Here computer generated phase space only represents the best system behaviour, i.e., only the solutions with degree of membership 1 have been considered.

We have shown the initial phase of formation of the eye of a cyclone as a fuzzy attractor. It is a very special type of fuzzy attractor namely, a fuzzy limit point to which the fuzzy flow tends to converge. According to Definition 3.1.12 the fuzzy dynamical system of generation of the storm is a stable one. Our model is actually valid for a very short time. It only models the creation of the initial vortex, which under favourable conditions may become a severe cyclonic storm. The most important fact that we have been able to establish through this model is that climatic disturbances in the form of simple linear wind jets or waves are capable of creating log-spiral like cyclonic vortices. The uncertainty managing aspect of an FDS has played a great role in it. The role of such disturbances behind creation of cyclones have long been suspected [48]. Unfortunately we have not been able to test this model in real life. A lot of time and resources are needed for this purpose. However this model has the potentiality to advance the time of cyclone warning by several hours to several days.

Currently the prediction starts with the satellite pictures of cloud bands of a cyclonic vortex once it is already formed. It usually takes five to six days to attain maximum intensity and hit the coastal areas [45], where it becomes a wide spread catastrophe. Therefore presently cyclones can not be predicted more than five to six days before it actually wrecks havoc. According to our model, storm warning is possible by constantly monitoring the wind speeds over high seas of cyclone prone regions during the storm seasons, provided other climatic conditions are also favourable for the generation of a storm. This way a storm warning can be given even before the actual formation of the initial cyclonic vortex.

In the present chapter we have shown how efficiently an atmospheric uncertainty can be modelled by an FDS. In the next chapter we shall test the effectiveness of FDS in modelling uncertainty in a biological system.

Chapter 6

FDI Approach in Modelling the Evolution of Tumour

To demonstrate the efficacy of FDS in modelling uncertain systems we have so far shown application of a class of fuzzy discrete dynamical systems in image generation and application of fuzzy differentiable dynamical systems in modelling a turbulence and the generation of a tropical storm. In this chapter we will show yet another application of fuzzy differentiable dynamical system in modelling uncertainty of the dynamics of evolution of tumour in human tissues. This application has a basic difference with the previous ones, in the sense that, it substitutes a stochastic differential equation by a fuzzy differential inclusion relation in modelling the underlying uncertainty. This substitution turns out to be very fruitful, for it offers us a completely new insight into the model [102, 103].

The evolution of a tumour and turning it into a malignant one (carcinogenesis) is a very complex physiological process, which the scientists have not yet been able to understand fully. Numerous attempts are being made to better understand the process from various points of view. Here we propose to view the entire physiological process from a general system theoretic out look as suggested in [40, 43, 44, 52, 89]. For example, in physics there are a few fundamental equations like Schrodinger's equation, Dirac's equation, Yang-Mill's equation, etc. These equations can not explain everything of the phenomena they are supposed to model. Yet they can explain many things and play very important role in understanding some of the most complicated behaviours of nature. A new approach is suggested by the above mentioned authors so that the medical cybernetics, as studied from a general dynamical system theoretic out look, should adopt this spirit of physics [44, 52, 84, 89]. Systems theoretic pioneer Bertalanffy himself initiated this approach as a mathematical biologist way back in 1970 [12]. His equation is used for study of growth in animals and tumours. Tsetlin had also taken a similar approach in modelling some properties of a muscle and a ganglion [143].

6.1 The Model

A. Motivation and Construction

We present a kinetic model of the onset of tumour which involves the coupling of three principal phenomena [89]:

(1) the transformation of normal cells into neoplastic cells (tumour cells).
(2) the replication of transformed cells.
(3) the immunological interaction of the host organism with transformed cells, i.e., the immunological propensity of the T-lymphocyte (a kind of white blood corpuscle) and some other cells to fight and destroy the unwanted growth of cells in a tumour.

The schematic presentation of (1), (2) and (3) are given below in (1a), (1b) and (1c) respectively. The right side of the arrow of (1a) is deliberately kept empty. This empty place represents normal cells.

$$\xrightarrow{A} X \qquad (1a)$$

$$X \xrightarrow{\lambda} 2X \qquad (1b)$$

$$E_0 + X \xrightarrow{k1} E_1 \xrightarrow{k2} E_0 + P, \qquad (1c)$$

where X denotes the population of the tumour cells. The rate of transformation from normal to neoplastic cells is proportional to a constant A. E_1 is the population of effector cells (T-lymphocytes, macrophages, natural killer cells etc. [89]) having recognized and bound a target cell (tumour cell). E_0 is the population of the free effector cells, i.e., those T-lymphocytes, macrophages, natural killer cells etc., which have not yet identified and attacked a target cell. λ is a replication rate constant. k_1, k_2 are rate constants. The first two steps represent phenomenologically two ways in which the X population can grow: either by the transformations of the normal cells into neoplastic ones (step 1a) or by cellular replication (step 1b). In step 1c E_0 free effector cells bind (recognize and attack) X target cells depending upon the rate constant k_1 to produce E_1 number of bound effector cells. From E_1 number of bound effector cells depending on another rate constant we ultimately get the number $E_0 + P$, where P is the number of tumour cells destroyed in the operation, which will not replicate any more. We assume that $E_t = E_0 + E_1$ is constant. Within this framework the dynamics of the growth in time of the target population X is given by

$$dX/dt = (N - X)\{A + (\lambda/N)X\} - k_1E_tX \ \{1 + (k_1/k_2)X\}, \qquad (6.1.1)$$

[52, 89], where N is the maximum number of target cells which can be packed in a given volume element. Let us notice that equation (6.1.1) does not refer to the *total* population of these cells in the entire tumour. The problem of main interest in this model is the *local* transition mechanism (involving the local interactions of target and effector cells) between tissual states of different nature and not the process of growth of the tumour as a whole. Thus the factor (N - X) expresses simply the existence of an upper limit N for X in the volume element under consideration. For the sake of convenient dynamical system theoretic analysis (6.1.1) can be reformulated in the following simple form:

$$dm/d\iota = v + m(1 - um) - r\{m/(1 + m)\}, \qquad (6.1.2)$$

where $m = (k_1/k_2)X$, $v = k_1AN/k_2(\lambda - A)$, $r = k_1E_t/(\lambda - A)$, $u = k_2\lambda/k_1(\lambda - A)N$ and $\iota = (\lambda - A)t$. u remains fixed and its value is taken as 0.1 [44]. Note that in Chapter 5 we also used m and r. But the meaning of m and r are completely different in this chapter than that in

Chapter 5. Here m is proportional to the number of tumour cells in a given volume of tissue and r is proportional to the rate of destruction of tumour cells.

B. Dynamical System Theoretic Analysis

A *critical point* (that is where dm/d*t* vanishes) of (6.1.2) is given by $v = (1 - u)^3/27u^2$, $r = (1 + 2u)^3/27u^2$ and $m = (1 - u)/3u$. Putting v = 2.7 and r = 6.4 (obtained by putting u = 0.1 in the expressions of v and r respectively) at (6.1.2) we have the following graph of (m, dm/d*t*), which shows the remarkable feature of a cusp catastrophe with an unstable saddle critical point at m = 3. For m > 3 dm/d*t* becomes negative, that is, the density of the malignant cells decreases with time.

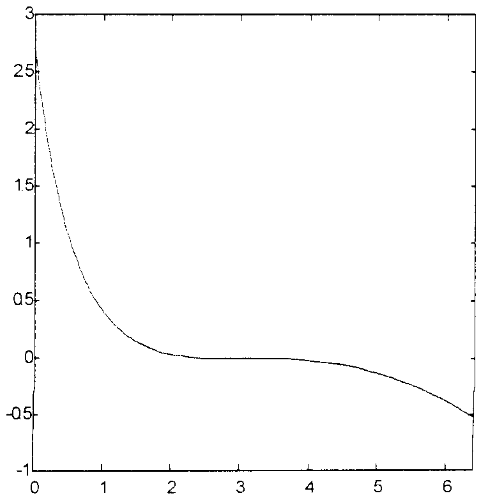


Fig 6.1.1 The plot of dm/d*t* (y-axis) against m (x-axis) with critical r and v, i.e., r = 6.4 and v = 2.7.

In (6.1.2) r is the most important quantity from clinical and therauptic point of view. In fact a therapy of malignancy has been suggested in [44] by an induced statistical fluctuation on r beyond certain threshold limit. Comparing Fig 6.1.1 with Fig 6.1.2 it becomes evident that 6.4 is a threshold value for r (for u = 0.1). If r is much smaller than 6.4 the value of dm/d*t* will not become negative for $0 \leq m \leq 6$ (the admissible range of m [44]) and therefore m, which is proportional to the malignant cell density, will not start to decrease. So to become successful any therapy must strive to fluctuate the value of r beyond 6.4. Only then dm/d*t* will become negative and m will start decreasing, provided of course u remains 0.1. This way the increased fluctuation of r can destabilize a tumour according to the Glansdorff-Prigogine theorem [53], which has been therauptically utilized in [44].

Assuming that at t = 0 m = 0 (when (6.1.2) models the growth of tumour cells then at t = 0 we take m = 0, but latter when from a therauptic point of view the same equation will be used to model the diminishing of m due to the therapy we shall take m = a high positive value at t = 0) (6.1.2) has solutions of the form

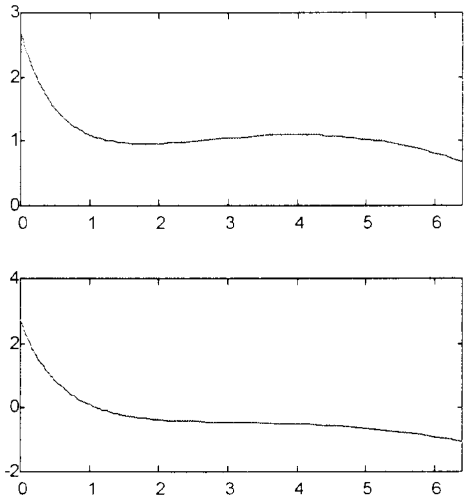


Fig 6.1.2 Plot of dm/d*t* (y-axis) against m (x-axis) for r = 5 (above) and r = 7 (below). Note that for r = 5 dm/d*t* ≠ 0 when m ∈ [0, 6]. But for r = 7 dm/d*t* becomes 0 once for m ∈ [0, 6].

$$\exp(t) = (1 - m/\alpha)^k (1 - m/\beta)^l (1 - m/\nu)^n. \qquad (6.1.3)$$

where α, β, ν are the roots of $v + m(1 - um) - r\{m/(1 + m)\} = -um^3 + (1 - u)m^2 + (v - r + 1)m + v = 0$ and

$$k = \frac{\begin{matrix} 0 & 1 & 1 \\ -1 & \nu+\alpha & \alpha+\beta \\ 1 & \nu\alpha & \alpha\beta \end{matrix}}{\Delta}, \quad \Delta = \begin{matrix} 1 & 1 & 1 \\ \beta+\nu & \nu+\alpha & \alpha+\beta \\ \beta\nu & \nu\alpha & \alpha\beta \end{matrix}$$

$$l = \frac{\begin{matrix} 1 & 0 & 1 \\ \beta+\nu & -1 & \alpha+\beta \\ \beta\nu & 1 & \alpha\beta \end{matrix}}{\Delta}, \quad n = \frac{\begin{matrix} 1 & 1 & 0 \\ \beta+\nu & \nu+\alpha & -1 \\ \beta\nu & \nu\alpha & 1 \end{matrix}}{\Delta}.$$

Each of α, β and ν is a function of u, v and r. To know the exact form of these functions any book of theory of equations, like for example [34], may be consulted.

For u << 1 and v << 1, a *bifurcation* occurs in the phase space of (6.1.2). The values of m lies on two different branches, one corresponds to low values of neoplastic cell population (*micro-cancer focus*) and the other branch corresponds to higher values of the same (*macro-cancer focus*). In [44] and [89] studies have been made about how to shift the focus from macro-cancer to micro-cancer through externally induced statistical fluctuations on r and v. Based on this, multi-pronged therapies of tumour have been reported in [44].

A remarkable feature of (6.1.3) is that for m = α, β or v t becomes minus infinite. This means that either all of α, β and v are negative (so that m can never take these values) or one is negative and rest are complex (again so that m can never take these values). In the latter case since the imaginary parts of the two complex roots have opposite signs a *Hopf bifurcation* [73] occurs in the phase space of (6.1.2). Notice that Hopf bifurcation is a multidimensional phenomenon [73] and can occur only if we allow m to be complex (complex plane is topologically equivalent to R^2). Though for our practical purpose we never allow m to be complex (to be more precise values of m lies in [0, 6] [44]) but for the dynamical system theoretic analysis it is illuminating to allow m to take all possible values (for non-negative t only). In (6.1.3) for any nonnegative t m has three values. For those values of u, v and r for which m has one real and two complex values the Hopf bifurcation occurs. This means that when the *discriminant* [34] of $-um^3 + (1 - u)m^2 + (v - r + 1)m + v = 0$ (which is a function in r and v, assuming that u is fixed) changes sign from negative to positive a Hopf bifurcation occurs. This Hopf bifurcation is responsible for the change in stability of focus from macro-cancer to micro-cancer (or vice versa) in the dynamical system represented by (6.1.2).

6.2 A Formulation in Terms of FDI

Equation (6.1.2) has been treated as a *stochastic differential equation* (SDE) of Ito or Stratonovich type in [44] and [89] and effects of various fluctuations on it have been studied. Fluctuations or, in other words, stochasticity is not intrinsic to this type of fundamental equations. Rather stochasticity arises because we do not follow up all microscopic events but rather try to describe the system macroscopically. The microscopic events are taken care of by statistical approaches. In that way fluctuations mirror a lack of knowledge. This lack of knowledge is introduced, so to speak, at our will in order not to be overloaded with unessential details [58]. Various SDEs are used to model nonlinear stochastic processes [58]. Method to solve an SDE may not always be unique. Depending upon the situation one method is preferred over the others. In the SDE model of [44] and [89] the fluctuation has been introduced as white noise. But white noise is a purely mathematical construct, which though a reasonable model of the noise encountered in physics may not be a suitable description of perturbations in biological systems [5, 84].

Reformulation of stochastic differential equations by FDIs has two advantages [5]:

1) The corresponding FDI is well defined and has a solution.
2) Using FDIs allows us to define "likelihood" and thus develop a non-stochastic analog of systems with uncertain dynamics.

The concept of FDIs generalizes the notions of differential inclusion [5].

On the other hand from a therauptic point of view the induced fluctuations are not objective but subjective in nature, because these are administered by human experts according to their skill. This administration is bound to differ from one expert to another or even for a particular expert at different times. So the fluctuation should not be statistical in nature. Instead the fluctuation or the lack of knowledge involved here is *fuzzy* in nature. Attempts for non-stochastic modelling of uncertainty in a biological system have already been made, where a differential inclusion modelling of population dynamics has been proposed [5, 84]. We have already indicated the need to introduce FDI in place of SDE as the fuzzy set theory has a very long and successful history of dealing with this kind of subjective uncertainty in complex systems [43].

A. Estimating the Uncertainty

In this section we shall subject (6.1.2) to fuzzy fluctuations. We shall induce fuzzy fluctuations on r and v. This means that the entire fluctuating range of r and v will be treated as fuzzy real numbers. Introduction of fuzzy membership functions in place of probability density functions have its own advantages and disadvantages. For example, introduction of fuzzy membership function is much less rigorous and computationally less complex. It may smooth out many finer detail resulting in elimination of unessential information. On the other hand there is no definite method to construct fuzzy membership functions. It depends on the knowledge and skill of the expert who is administering the therapy by induced fluctuations.

For the therauptic purpose fluctuations in r and v have been considered in [44]. So an FDI formulation of (6.1.2) is necessary, where without loss of generality we can take the values of r and v lying in some fuzzy real numbers. This is in complete harmony with the spirit of [84], where the noise is taken to lie within some closed and bounded interval of R. Now the question is "What type of fuzzy real numbers are the most appropriate for this purpose?"

We have already seen that for a therauptic purpose the value of r must be influenced in such a way that it can go beyond some critical value (depending on u, which is fixed). Like in [84] we (arbitrarily) assume some range beyond which the values of r can not go. We take $r \in [1, 9]$. For $r << 1$ the X population and hence also m is close to its maximum value and the tissue is in a cancerous state. On the contrary if $r >> 1$ the m population becomes vanishingly small and the tissue tends to normality [52]. The range [1, 9] is to be estimated from experimental result as typically done by the human experts administering a therapy. Here the range of r is consistent with [44, 52, 89]. The best possible range of r to be attained during a therapy, to reach the micro-cancer focus, should be [5.95, 7] (which is compatible with [44]). Arbitrary large values of r is not desirable, for in that case (4.2) may not represent the dynamics properly. Since we are interested in the best possible solution (i.e., the solution with fuzzy membership grade 1) only (we have discussed about the general case of the other solutions at the end of this sub-section), without loss of generality we can take r to lie in the trapezoidal fuzzy number $R_1 = \langle 1, 5.95, 7, 9 \rangle$ for the following reason. We have seen in Definition 2.2.1 of fuzzy real number, that a fuzzy real number $\mu : R \rightarrow [0, 1]$ is a membership function.

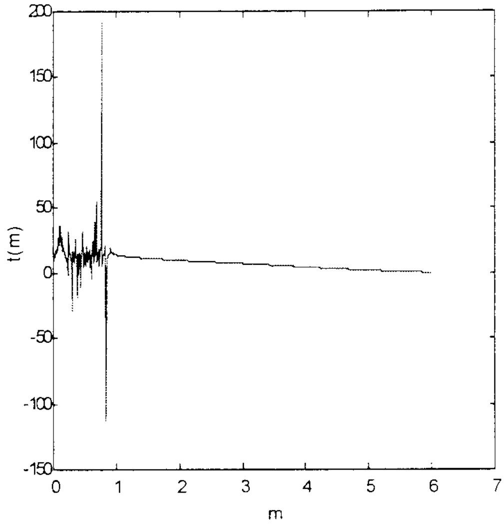


Fig 6.2.1 An approximation of the solution space of (6.1.2) for r = 5.95, v = 2 and u = 0.1. Resolution of m is taken as 0.01.

where R is the set of (crisp) real numbers. μ must satisfy the following properties. (1) The support of μ must be a closed bounded interval of R, say [a, b]. (2) There must be numbers c, d, such that, $a \leq c \leq d \leq b$. μ must be increasing and continuous from right in [a, c]. μ must be decreasing and continuous from left in [d, b]. $\mu(x) = 1$ for any $x \in [c, d]$. Since we are interested in the best possible (fuzzy) solution only, we are interested in the domain of μ, where it takes the value 1, i.e., [c, d]. We are not actually bothered about the shape of μ either in [a, c) or (d, b]. The shape of the fuzzy number on the portion [c, d] are identical in all cases. So as long as only the best possible (fuzzy) solution is required there is absolutely no difference between a trapezoidal fuzzy number and any other arbitrary type of fuzzy number.

Determining the proper range for the values of v is even more challenging. If v, which is proportional to the rate of conversion of the normal cells into the neoplastic ones, remains positive then after stopping the therapy there is a chance of reviving malignancy. On the other hand if v becomes 0 during the therapy (6.1.2) will give abnormal results, which means that it no longer models the dynamics properly. In Fig 6.2.1 an approximation of the solution space of (6.1.2) has been presented for v = 2. The behaviour of the system is very abnormal and chaotic for m < 1. If we increase v this range also increases significantly (Fig 6.2.3), which means that we can not predict the progress of the therapy beyond m is reduced to certain level (e.g., in case of Fig 6.2.1 it is 1). This point has not been taken care of in [44] and [89]. It is to be kept in mind that t(m) is a smooth function and its graph can not exactly be like Fig 6.2.1 or Fig 6.2.3. Here the graphs are generated with a resolution of m as fine as 0.01. Finer resolution will give more realistic figure. But then comes the question of the size of a pixel, no resolution can be finer than that.

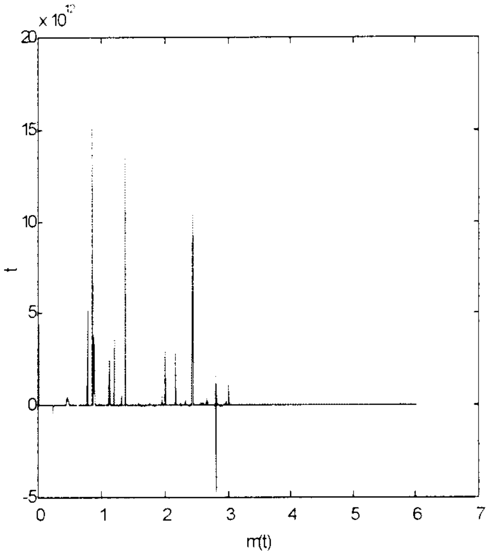


Fig 6.2.2 An approximation of the solution space of (6.1.2) for r = 6.4 and v = 2.7 and u = 0.1. Resolution of m is taken as 0.01. For most of range of m the result is abnormal and unpredictable.

The simulation results of Fig 6.2.1, Fig 6.2.2 and Fig 6.2.3 suggests that for high values of v (>> 2) the system be treated as a nondeterministic system with high degree of unpredictability. Lefever and Horsthemke has modelled this uncertainty by introducing stochastic noise (the so called Gaussian white noise) into the system and then modelling the system by a stochastic differential equation [44, 89]. The introduced Gaussian white noise has a probability distribution which is a function of m and has a standard deviation σ. The value of σ is varied as a result of fluctuation. For certain values of σ this distribution function becomes bimodal, which is interpreted as the cause of bistability in values of m. That is, for the values of m in the range 0 to 6 there exist two points one near 0 and the other near 6, where the system tries to become 'stable'. The point near 0 is called micro-cancer focus and the point near 6 is called the macro-cancer focus [44]. Since the aim of any therapy is to reduce m, in [44] the therapy has been administered to shift the value of m from macro-cancer focus to micro-cancer focus. Right at this point our simulation results suggest that, since for high v (>> 2) the system becomes very unpredictable the success of the therapy can not be guaranteed.

In reality v remains positive and due to the cytotoxicity of the T-lymphocytes tumour cell density X and therefore also m can not grow [52]. In an abnormal situation where m grows unusually, therauptic remedy is needed. In [89] v must satisfy

$$v \ll r/(1 + u). \tag{6.2.1}$$

In view of the limit of r we have taken (or taken in [44, 52]) this means

$$v \leq 1/(1 + 0.1) \approx 0.9. \tag{6.2.2}$$

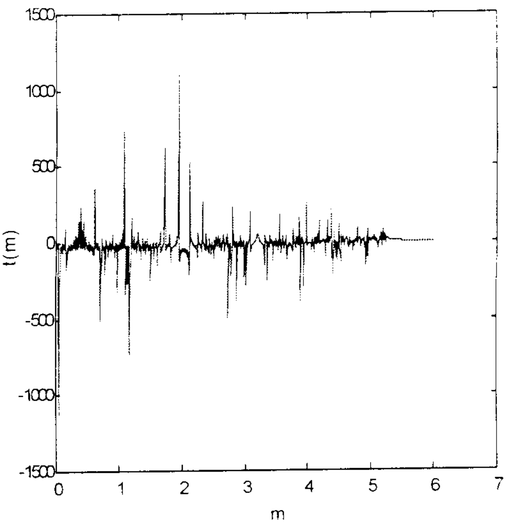


Fig 6.2.3 An approximation of the solution space of (6.1.2) for r = 5.95, v = 2.5 and u = 0.1. Resolution of m is taken as 0.01. Almost in the whole range of m the result is very abnormal and beyond any meaningful prediction.

For the values of v given by (6.2.2) dm/dt is negative for m $\in$ [0, 6] (the admissible range of m [44, 89]). If we take r $\in$ [5.95, 7] then according to (6.2.1) v << 5.4. But since our model is very sensitive to values of v according to this model only for $0 < v \leq 2$ it is possible to devise meaningful therapy. So we can take v $\in$ [0.5, 2]. By introducing fuzzy membership grades in the range we can get a trapezoidal fuzzy number V = <0.5, 1, 1.75, 2> such that, v $\in$ V.

B. Mathematical Formulation

By now we are in a position to give an equivalent FDI formulation of (6.1.2).

$$dm/dt \in [\{-um^3 + (1 - u)m^2 + (v - r + 1)m + v\}/(1 + m)]^\alpha, \quad r \in [R_1]^\alpha,\ v \in [V]^\alpha,\ m(0) \in [<6-\delta, 6, 6+\delta>]^\alpha, \qquad (6.2.3)$$

where $0 \leq \alpha \leq 1$. <6-δ, 6, 6+δ> is a triangular fuzzy number, where δ is a very small positive (crisp) number. We have taken the initial condition to be the worst, i.e., at t = 0 we have taken m(0) $\in$ <6-δ, 6, 6+δ>, which is the highest possible value of m [44]. The region of best possible solution as computed according to Algorithm 3.2.1, described in general terms in Chapter 3, has been shown in Fig 6.2.4. For a better understanding of the simulation results of Fig 6.2.4 we would like to present a more specific version of the algorithm suitable for solving (6.2.3).

C. Simulation

(1) **fix $\alpha \in [0, 1]$;**

(2) **determine all the boundary values of $[R_1]^\alpha$, $[V]^\alpha$ and $[<6-\delta, 6, 6+\delta>]^\alpha$ for the α fixed in step 1;**

(3) **substitute r and v in $dm/dt = \{-um^3 + (1 - u)m^2 + (v - r + 1)m + v\}/(1 + m)$ by boundary values of $[R_1]^\alpha$ and $[V]^\alpha$ respectively determined in step 2 (all possible combination of boundary values to be considered;**

(4) **solve $dm/dt = \{-um^3 + (1 - u)m^2 + (v - r + 1)m + v\}/(1 + m)$ (directly or numerically) for each combination of boundary values of $[R_1]^\alpha$ and $[V]^\alpha$ as substituted for r and v respectively in step 3;**

(5) **in the solution of step 4 m(0) is to be substituted by boundary values of $[<6-\delta, 6, 6+\delta>]^\alpha$;**

(6) **plot the graphs of all solutions obtained by step 4 and step 5 (together) on R^2 and the space enclosed by these graphs, along with the coordinate axes is the α-level set of the solution of the FDI (6.2.3) for the α fixed in step 1.**

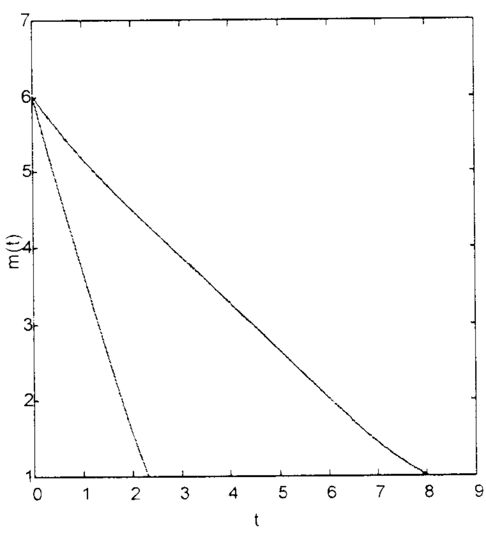


Fig 6.2.4 Region enclosed by the curves (t, m(t)) for which r = 7 and v = 1 (left), r = 5.95 and v = 1.75 (right) and the t axis is the region of the best possible solution (with α = 1) of (6.2.3).

The above algorithm, though consists of six steps, is actually an alternative formulation of Algorithm 3.2.1, where there were only four steps. For convenience of implementation in the particular case of solving (6.2.3) the step 3 of the original version of Algorithm 3.2.1 has been subdivided into three different steps here (steps 3, 4 and 5).

One important thing is to note that in Fig 6.2.4 we have kept all our computations within the range for which $m \geq 1$. We have already seen that for $m < 1$ abnormal results occur. This means that for $m < 1$ (6.2.3) no longer represents the dynamics of the system faithfully. We may have to model the dynamics for this range in a different way. This is also the case for $v > 2$. Designing the dynamical system for $m < 1$ or $v > 2$ is likely to pose great challenge to the future researchers.

The best possible solution region indicates the desired effects of the therapy on the patient. Due to the therapy the possibility of m coming down from 6 to 1 in $t = 2.3$ to 8 is very high (the degree of possibility or the value of fuzzy membership of such an happening is the highest). Remember that m is the normalized (and not actual) tumour cell density and t is the normalized time. To estimate the actual value of the tumour cell density and actual value of time we must know the normalization factor. Let us remember that, $m = (k_1/k_2)X$ and $t = (\lambda - A)t$, where X is the actual tumour cell density and t is the actual time.

For $\alpha < 1$ in (6.2.3) we get a solution which represents the system behaviour with degree of possibility α or more. Obviously the solution representing the best possible system behaviour is included within this solution space (or the fuzzy set of solutions as shown in Fig 3.2.6). For example for $\alpha < 1$ in (6.2.3) the admissible range of t in Fig 6.2.4 will be a larger closed bounded interval containing [2.3, 8] as a proper subset. In general for $\alpha < 1$ the solution of FDI(s) tends to converge less and diverge more. Let us explain this point. The fuzzy trajectory, obtained as the fuzzy set of crisp trajectories (solutions) of the corresponding FDI(s), at any given time will give a fuzzy subset of the solution space. The diameter of the fuzzy trajectory at any given time will generally increase if $\alpha < 1$ (the fuzzy trajectory for $\alpha = 1$ is a proper subset of the fuzzy trajectory for $\alpha < 1$). This makes the fuzzy trajectory obtained as the solution of the system of FDI(s) inconvenient or even completely unsuitable for the dynamical system theoretic analysis. This is exactly the reason why FDIs are preferred over fuzzy differential equations in modelling and simulation, because in fuzzy differential equations the entire solution space is to be considered irrespective of degree of possibility (all solutions for $\alpha \geq 0$ are to be taken). In FDI modelling we have the choice to take the solution representing the best possible system behaviour for the highest value of α, which often turns out to be 1. Once this restriction is imposed the trajectory becomes more 'crisp like' (but not always crisp exactly, particularly when not all the fuzzy quantities are triangular fuzzy numbers). Crisp like trajectories are more convenient for dynamical system theoretic analysis. In the particular case of (6.2.3) it is obvious from the simulation that for $\alpha < 1$ there will not be any qualitative change in the solution (Only [2.3, 8] will be replaced by a bigger interval properly containing this. This has been clearly demonstrated in Fig 3.2.6 for Example 3.2.1.) We practically get the same thing by keeping $\alpha = 1$.

D. Suggested Modification in Therapy Planning

Any therapy devised according to the dynamics given by (6.2.3) must presume that. (1) the value of v will never exceed 2 and (2) after reducing m to approximately 1 the course of the treatment may have to be modified. This has not been taken care of in [44]. Our basic principle of therapy remains the same as [44], where the target was to shift from the macro-cancer focus to the micro-cancer focus. Here however we shall continue this act of shifting only until m comes near to 1. For $m < 1$ we can not predict the effectiveness of the therapies under the present dynamical system modelling. At present a trial and error heuristic method is perhaps the only way out. A fluctuation on r is to be induced in order it to reach the value 6.4 or even surpass it (assuming $u = 0.1$). To fluctuate r the following therapies have been suggested, which are broadly subdivided into two categories.

(1) *Multi-step therapy or multiplicative therapy*: Here we can use perturbation in the form of increasing arterial pO_2 to 90 mmHg, decreasing venus pO_2 to 20 mmHg, using hyperthermia up to 105^0F, hyperglycaemia up to 600 mg% blood glucose level, which also produce pH perturbations (up to 6.5 from the normal tissue level of 7.8). Owing to variation of oxygenation, there is a perturbation in the oxygen index of the blood η as $18.5 < \eta < 42$.

(2) *Multiplex therapy*: Here one uses the combined perturbation of glucose level, temperature, anti-tumour drugs, radiation, thermolabilizer and radiosensitizer. Concentrated glucose solution is administered through superior vena caval catheter and the following perturbations are induced:

   (a) *Hyperglycaemia* - Glucose level is varied between 400 mg/100 ml blood.
   (b) *Hyperthermia* - Temperature variation used is from 98^0F to 103^0F, using high frequency inductive heating.
   (c) *Oxygenation* - Administered through pulmonary ventilation.

Success has been reported under both the above categories of therapies in [44]. If we examine the region of the phase space of (6.2.3) corresponding to the best possible solution (Fig 6.2.4) we observe that under the therapies m reduces from 6 to 1 quite smoothly. This means that the cancer cell population X also reduces accordingly. The corresponding values of t lies in the range of 2.3 to 8 (remember that t is not time but proportional to time). Within this best possible solution region of the FDI relation (6.2.3) the response to the therapies is quite reasonable and predictable just as in case of the conventional SDE modelling.

6.3 Discussion

In our quest to test the effectiveness of the FDSs to model uncertainty of some of the most complex real life systems, which are both scientifically interesting and practically important to the human society. In the current chapter we have substituted an SDE by an FDI in modelling uncertainty of a bio-medical cybernetics. After Krivan and Colombo [84] and Antonelli and Krivan [5], here we have got yet another opportunity to study a bio-medical uncertainty by substituting SDEs with differential inclusion (DI) relations. In

[84] it has been shown that, the solution obtained with DI modelling is qualitatively, but not quantitatively, similar with that obtained from SDE modelling. However our study here has followed a different path. We have fuzzified the classical dynamical system theoretic analysis to model the inherent uncertainty.

Here we have followed the precedence set in [44], [52] and [89], but we have gone much further. We have shown that an FDS (FDI to be more precise) modelling of a biological system is no less effective than a modelling in terms of the stochastic differential equations. Again this is the maiden effort to introduce FDI relations into the realm of medical cybernetics. The model adopted here is due to Lefever and Horsthemke [89]. But we have undertaken the dynamical system theoretic analysis of this model in a new direction and discovered the range within which the model should work. Based on these limitations therauptic suggestions have been made. This shows that a more pragmatic therauptic planning needs to be introduced in treating malignancy as suggested in [44]. Our study also strongly suggests that a new dynamical system needs to be introduced to model the evolution of tumour in human tissues when either (1) the neoplastic cell density X and hence m becomes very low ($m < 1$) or (2) the conversion rate from normal cells to the neoplastic ones becomes very high (for which $v > 2$) or both. These facts were not apparent from the SDE modelling.

Chapter 7

Conclusion

The central theme of this thesis is the management of uncertainty in a general dynamical system by fuzzy set theory based techniques, for fuzzy set theory based techniques are very versatile tools to deal with a broad range of uncertainties [80]. Fuzzy dynamical systems are a class of dynamical systems, which can utilize the fuzzy set theory based techniques for modelling, simulation and analysis of uncertain complex nonlinear systems, which we have shown in Chapters 4 through 6. Like classical crisp dynamical systems fuzzy dynamical systems (FDS) too are defined upon three basic entities namely, fuzzy phase space, time and fuzzy time-evolution law. We have devoted Chapter 2 to the fuzzy phase spaces, Appendix A to time and Chapter 3 to the fuzzy time-evolution laws. In the present thesis we have defined and discussed about quite a few fundamental notions of a general FDS and shown a number of applications of FDS in modelling some of the very complex real life phenomena. In each case of these applications the precise statement of the implementing algorithm is given, followed by extensive computer simulations. In some of these applications we have carefully chosen areas where fuzzy set theory based techniques in general and fuzzy dynamical systems in particular have not been used before, as for example, the modelling of turbulence and cyclogenesis.

Time may have many interpretations in science and philosophy. In Appendix A we have presented an interpretation of time from the dynamical system theoretic point of view. We have shown that representing time by real numbers is mathematically justified and that, writing the symbols like dx/dt are justified, at least from the point of view of t. Since Mandelbrot's definition of fractal requires a considerable preparation to understand, we have presented a detail explanation of it in Appendix B, complete with rigorous mathematical derivations. This definition is essential in defining fuzzy fractal in Section 4.1. In Appendix C we have rigorously proved that the logistic function $f(x) = \mu x(1 - x)$, $x \in [0, 1]$, is chaotic under iterations for $\mu \geq 4$. We have used this result in modelling fuzzy turbulence in Section 4.2.

7.1 Summary of the Main Points of the Thesis

Since in classical crisp dynamical systems phase spaces are often either R^n or some subsets of R^n, in Chapter 2 we have developed an algebraic and metric structure for the multidimensional fuzzy number system. We have tried to make these structures as similar as possible to those of R^n.

In Section 2.1 we have presented a brief discussion on fuzzy phase spaces. Here we have presented the definition of Hausdorff metric over a collection of nonempty compact subset of a complete metric space X. We have extensively used this definition in the rest of the thesis.

In Section 2.2 we have discussed the fuzzy real number system $\mathfrak{R}$. We have defined a field like algebraic structure, christened as F-field, over $\mathfrak{R}$. We have also defined a metric on $\mathfrak{R}$.

In Section 2.3 we have defined a vector space like algebraic structure (christened as generalized fuzzy vector space or GFVS) over a finite number of cartesian product of $\mathfrak{R}$ with itself. We have extended the metric defined on $\mathfrak{R}$ to $\mathfrak{R}^n$.

In Section 2.4 we show that any fuzzy subset of R^n can be obtained from $\mathfrak{R}^n$ by a max-min operation. An FDS either evolves in $\mathfrak{R}^n$ or any point of its phase space, i.e., any fuzzy state can be obtained from $\mathfrak{R}^n$ by max-min operations.

In Section 2.5 we explore the relevance of $\mathfrak{R}^n$ in computer vision and pattern recognition. Obviously, any fuzzy pattern vector or fuzzy feature vector can be generated from a collection of elements of $\mathfrak{R}^n$ by max-min operations. In this section we have considered two examples from applied research, one each from fuzzy speech pattern recognition and fuzzy visual object recognition. We have shown that $\mathfrak{R}^n$ may be a good reference frame to describe these two experiments.

In Section 3.1 a fuzzy dynamical system is defined in terms of fuzzy attainability set mappings. A dynamical system is dissipative when the phase space is 'contracting'. A set theoretic definition of fuzzy dissipative dynamical systems has been given in this section. A mathematical criterion for determining whether a dynamical system is dissipative is formulated. Various classical (crisp) dynamical system theoretic notions like attractor, stability, robustness, chaoticity, homoclinicity, etc., have been extended to fuzzy dynamical systems also. Devaney's definition of chaos has been extended to fuzzy dynamical systems. Predictability and predictability time for fuzzy dynamical systems have been defined.

In Section 3.2 fuzzy differentiable dynamical systems have been discussed. An algorithm to solve one dimensional fuzzy differential inclusions has been described. This algorithm is simpler and easier to apply (when applicable) compared to that given in [63]. We have used the algorithm to solve a second order fuzzy differential inclusion.

In Section 4.1 fuzzy fractals have been introduced. We have defined fuzzy fractal in this section. Then we have shown that fuzzy fractals can be generated by iterated fuzzy sets systems as attractors. In this section we have treated any grey level image as a fuzzy subset of R^2, where the membership value at each point (pixel) is a normalized grey level pixel intensity value at that point (pixel). As implementation of the theory developed in this section we have considered two examples of fuzzy fractal image generation.

In Section 4.2 a simple hypothetical mathematical model of turbulence is proposed. By the term 'turbulence' we have meant chaotic occurrence of vortices in a dynamic fluid. We have kept the model two dimensional for simplicity. We have defined a two dimensional chaotic function, which gives the distribution of the vortices in the dynamic fluid plane. Complete mathematical background for construction of this function has already been prepared in Appendix C. It is assumed that the vortices have been created by collision of fluid jets coming from different directions under certain conditions. To model a vortex, a fuzzy differential inclusion relation has been formulated by considering the ratio of radial and cross-radial components of the resultant velocity of the colliding fluid jets at a point as a constant fuzzy number. The solution space of this FDI, obtained by Algorithm 3.2.1, developed in Section 3.2, gives the simulation of a vortex. We call this a fuzzy vortex. Chaotic occurrence of such fuzzy vortices gives rise to fuzzy turbulence. We have presented a simulated model of a fuzzy turbulence. We also have given a computer program which will generate more such fuzzy turbulence with uncontrolled (random) parameters.

In Section 5.1 a modelling of a climatic disturbance has been provided in terms of fuzzy differential inclusion. It is known that under favourable climatic and geographical conditions a sufficiently strong initial disturbance is needed to give birth to an intense tropical storm, such as a cyclone (known as hurricane in North America). From the fact that curved cloud bands of a cyclonic vortex are of log-spiral shape we have inferred that the ratio of radial and cross-radial components of velocity (with respect to some cylindrical coordinate system) of the initial disturbance is a fuzzy number. The initial condition is also fuzzy and therefore expressed by another fuzzy number. This way we ultimately get a fuzzy differential inclusion modelling of the initial disturbance.

In Section 5.2 we have presented the solution space of the FDI formulated in Section 5.1, with the help of Algorithm 3.2.1. We have only considered the solution space, which represents the best possible system behaviour (in the sense that the degree of possibility of the behaviour by the system is the highest), and this resembles a log-spiral shaped vortex. We have established the compatibility of this model with some of the existing models of the development of a storm. With this model of the initial disturbance, which under favourable conditions may mature into a severe cyclone, we have been able to explain some of the mysteries surrounding the occurrence of Bay-of-Bengal cyclones.

In Section 6.1 we have studied the evolution of tumour in human tissue as a general dynamical system. A mathematical equation has been formulated. In this section we have undertaken some (classical) dynamical system theoretic analysis of this system. We have indicated how Hopf bifurcation can occur in it and what can be its possible implication.

But the parameter values and the initial input values are uncertain quantities. Conventionally these uncertainties used to be treated by formulating the equation as a stochastic differential equation. But in Section 6.2 we have reformulated the equation as a fuzzy differential inclusion relation. In order to accomplish this we had to estimate the range of the uncertain quantities. From the data available from various sources as well as through computer simulations (to determine the range of chaoticity and to exclude it) we have made meaningful inference about the ranges of all the uncertain quantities involved. After completion of the FDI reformulation of the system we have solved it, as usual, by Algorithm 3.2.1. The simulation results have been presented. Based on these results a number of modifications in some of the therapies administered to a tumour patient have been suggested.

7.2 Future Directions

Let us present a chapter wise discussion in the following paragraphs about the future scope of work.

Chapter 2: Our method in Section 2.2 can be extended to define fuzzy complex number system. Exactly the same algebraic development will hold for the fuzzy complex number system as well as for the multidimensional fuzzy complex number system. A metric can also be defined over the space of multidimensional complex numbers just as in Section 2.3. In this chapter we have defined just a barest minimum for the generalized fuzzy vector space. A lot of scope for development is remaining wide open. Also in Section 2.5 we have just treated two examples. Many more example can similarly be treated and more importantly, scope for development of an algorithm to generate an arbitrary fuzzy subset of R^n, starting with a suitable collection of elements of $\Re^n$, with minimum number of max-min operations, is remaining.

Chapter 3: In this chapter we have introduced some new concepts into fuzzy dynamical system in terms of fuzzy time-evolution law or fuzzy attainability set mapping. Many more can still be introduced, for example, we have not defined structural stability for a fuzzy dynamical system and this can be taken up by future researchers. In Section 3.2 we have given Algorithm 3.2.1 for solving one dimensional fuzzy differential inclusions. Unfortunately this can not be extended to the multidimensional cases. The only algorithm to solve the multidimensional fuzzy differential inclusions is due to Hullermeier [63]. But for the proper utilization of fuzzy differential inclusions in modelling and simulation of various natural phenomena finding an easier to implement algorithm to solve them is important.

Chapter 4: Very little work has so far been undertaken in fuzzy fractal based image generation. The pioneering work in this direction is due to Cabrelli et al. [21]. We have introduced some modification and extension into it to make it easier to implement. But the whole method is for the grey level images only, where the values of intensity of red, green and blue (RGB) at each pixel are equal and measured by integral values from 0 to 255. A very challenging work for the future researchers will be to extend it to the colour images also. This means that the intensity values of red, green and blue will be independent of each other at any pixel. Each point (pixel) P on the image will have three membership grades instead of usual one, namely μ_r, μ_g and μ_b, where $\mu_r(P)$ is the membership grade of P to belong to the red region $\mu_g(P)$ is the membership grade of P to belong to the green region and $\mu_b(P)$ is the membership grade of P to belong to the blue region. Values of $\mu_r(P)$, etc. may be determined the way described in Section 4.1. The resultant colour image will be a superposition of the three fuzzy sets as images in red, green and blue.

As far as we know fuzzy turbulence is a completely new concept, introduced only in Section 4.2. Here we have introduced only a simple hypothetical two dimensional model of a turbulence. To be more realistic this model should be three dimensional or if we also want to incorporate time into it, it will be a four dimensional space time model. In this four dimensional space-time reference frame the location of any point on the axis of a vortex will be denoted by (x, y, z) at some instant t. The vortex will be of cylindrical shape, whose axis may make angles α, β and γ respectively with the three dimensional space coordinate axes. Obviously, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. To describe the vortex at (x, y, z) a local cylindrical coordinate system (r, θ, h) is necessary. Also another parameter m as described in Section 4.2 is needed. r will be expressed in terms of m and θ (Section 4.2). So to make a more realistic model of this type of turbulence we will need the following independent parameters: x, y, z, t, any two out of α, β and γ, and m, θ and h, i.e., nine in all. Modelling of turbulence involving all these nine independent parameters (some of them including m are fuzzy numbers) may be the subject of another elaborate piece of research.

Chapter 5: In this chapter we have presented a model of the initial disturbing vortex at the very beginning of a cyclogenesis. Here we have shown how to make fuzzy dynamical system modelling of a complex atmospheric phenomenon. There are scopes of improvement in this model. Here we took m as belonging to a trapezoidal fuzzy number M. In fact a lot depends on the precise nature of M. In the most general case M may not even be a fuzzy number, but a rather more complicated fuzzy subset of R. M may depend on various thermodynamic and fluid dynamic conditions. A lot of scope of research is remaining to determine a more realistic M. Also suitable fuzzification of various existing numerical model of cyclone is possible. In each such case we will get a set of fuzzy differential inclusion relations governing the thermodynamic and fluid dynamic aspects of the cyclone.

In this regard the potentiality of another great work is worthy to mention, although it is not directly very much connected with the mathematical theory of FDS. But it could indeed be a great contribution of the fuzzy set theory based techniques to the atmospheric sciences. Detection and monitoring of cyclones are accomplished through satellite and RADAR imaging. Various feature analysis of such images has been described in great detail by Dvorak [45]. Dvorak's techniques to detect, monitor and predict cyclones are followed all over the world. Implementation of these techniques have so far remained almost entirely manual. The system can at least be partly automated through a fuzzy if-then rule based expert system. This may be a very challenging project for some of the future researchers in fuzzy pattern recognition in particular and fuzzy data mining in general.

Chapter 6: In this chapter we have replaced a stochastic differential equation in a bio-medical dynamical system by a fuzzy differential inclusion relation. This modelling has given us some new insight. We have discovered the range of validity of the model. This implies that new models need to be introduced for certain range of some parameter values, where the existing model fails to represent the system. In this chapter our study opens up the need for research on a new dynamical system to model the evolution of tumour in human tissues when either (1) the neoplastic cell density X and hence m becomes very low ($m < 1$) or (2) the conversion rate from normal cells to the neoplastic ones becomes very high (for which $v > 2$) or both. The author believes that these two are the most important open problems suggested by the present work.

Appendix A

An Interpretation of Time

Time is an illusion.
- A. Einstein

As we have already observed that a dynamical system has three basic concepts namely, phase space, time-evolution law and time. This is true irrespective of whether the dynamical system is deterministic or non-deterministic. In this thesis our emphasis was on non-deterministic dynamical systems, where we have modelled the uncertainty by fuzzy techniques. In Chapter 2 we have discussed fuzzy phase space and in Chapter 3 we have discussed fuzzy time-evolution law. In this appendix we want to discuss about time. The meaning of time has baffled scientists and philosophers alike since the middle age. Time has been analyzed and interpreted in different ways by different schools of thoughts. Here our intention is to present an interpretation of time from a dynamical system theoretic point of view. Though we have borrowed ideas from physics, whenever required, it may not always be in conformity with all the notions of time in physics.

Length, mass and time are the three most fundamental quantities in physics. As quantities in physics they must be measurable [22]. To make them measurable they must somehow be associated with real numbers. The most standard way of associating real numbers to a physical quantity is to define a real valued function whose domain of definition is the set of that physical quantity. In this appendix one of our aims is to make a thorough investigation about how the set of real numbers R has been associated with the notion of time. An outline of the history of some mathematical interpretations of time appears in [60]. But ours is different from any of them. For the physical significance of time we have meticulously followed the relativistic interpretation [130] of time. Time is an orderer of events with respect to an observer. We have chosen the diurnal motion of the Sun round the Earth as the most standard orderer of events. So we have first defined time after this motion. Over this we have built up a physical model to give a better interpretation of time. Finally we have established that this interpretation is equivalent to expressing time as a variable over the whole of R. This enables us to establish that the symbols like dx/dt are well-posed at least from the point of view of time t in the sense of [85]. This enables us to represent a general differentiable dynamical system as

$$dx/dt = f(x, t).$$

Our arguments will hold equally good if the diurnal motion of the Sun round the Earth is replaced by an atomic clock or any other suitable device or any other suitable phenomenon. Here we have taken the space coordinates as real numbers. Now, for example, consider an event space in the special relativity theory. Let each event be expressed as $(x, y, z, t)^T$, x, y, z are space coordinates and t is time coordinate. The suffix T stands for transpose.

A.1 Formalization of Time

Let us start with the following ad hoc definition of time, which we shall refine later in more than one steps.

Definition A.1.1: *Time* is an orderer of events with respect to a given observer. To order we mean a linear order [138] relation.

In special relativity theory an event means a point in the four dimensional Minkowaski's World Space [130], i.e., *event* is a four tuple $(x, y, z, t)^T$ where x, y, z denote three space coordinates and t denotes the time coordinate. Clearly this definition of event can not be taken to define time as *orderer of events*. Of course time can not exist independent of event [129]. So we would like to associate time with event in another manner. For us an event is a simple mathematical description of a physical incident. A mathematical description of a physical incident consists of a number of parameters, usually none of them depends on any other. Each parameter can take some value(s), finite or infinite in number. Set of all such values which a given parameter can take, may be called the *domain* of that parameter. In general the domain of a parameter may be an arbitrary set, as for example, A = {Calcutta, London, Moscow}, B = {Paresh, Sujoy, Mary, Igor}. Now, if you want to say that, Paresh is in Calcutta, you could just write it in mathematical symbol as (Paresh, Calcutta). You may want to say, Paresh is in Calcutta, Sujoy is in Calcutta, Mary is in London, Igor is in Moscow. A mathematical description is like (Paresh, Calcutta) $\cup$ (Sujoy, Calcutta) $\cup$ (Mary, London) $\cup$ (Igor, Moscow). If we have another set C = {Summer, Monsoon, Winter, Spring} and want to describe mathematically that, "Mary was in London during Winter", we can write (Mary, London, Winter). So in general, we can mathematically describe an event by an n tuple. When none of these n parameters depends on any other, n is called *dimension* of the event, i.e., the number of independent parameters involved to describe the event. Collection of all events of a particular context will make an *event space* or for short a *space*. If S denotes the event space as described above, then $S \subseteq A \times B$ or $S \subseteq A \times B \times C$ as the case may be. In a given space dimension of each event is same and this will be called the *dimension* of the event space.

Definition A.1.2: If each parameter domain of an event is some interval of real line R and the event space is equipped with the usual product topology, i.e., the usual Euclidean topology of R^n [138], where n is the dimension of the event space, then the event space or for short the event will be called *continuous event*.

Definition A.1.3: Whenever the event space is equipped with discrete topology [138], the event space or for short the event will be called *discrete event*.

Obviously, there may be events which are neither discrete nor continuous. For the rest of this chapter we shall mostly concentrate on continuous events. Actually we shall be even more specific. We shall concentrate on the event of drawing a continuous curve in a multidimensional Euclidean space R^n. For example, we want to denote the events of drawing the curve of $y = \sin x$ in the Euclidean plane, where $x \in [0, \pi]$. To do this we plot all the points (x, sin x), $x \in [0, \pi]$, on the R^2. Each such point denotes an individual event of plotting that particular point. The event space is $\cup\{(x, \sin x)\}$, where the union is taken over all x in $[0, \pi]$. But, to decide which point comes after what, we need to define an order relation over the event space. When there is no risk of confusion we shall use 'event' and 'event space' (i.e., 'events') interchangeably. Observe that, drawing a continuous curve is a continuous event. However the converse not true.

Here, we shall try to introduce a linear or total order relation [138] '$\leq$' in the event space S, which will enable us to write for any two events E_1 and E_2 in S either $E_1 \leq E_2$ or $E_2 \leq E_1$ (in what sense, we do not bother for the time being). One obvious way to do this is to choose a linearly ordered set $L_{\leq}$ whose cardinality is greater than or equal to that of S. Then there exists a one-to-one map [138] $f : S \to L_{\leq}$ which is invertible on $f(S)$ [138]. $f(S)$ being a subset of $L_{\leq}$ is already linearly ordered [138]. If $f(E_1) = x_1$ and $f(E_2) = x_2$ and $x_1 \leq x_2$ then we define $f^{-1}(x_1) \leq f^{-1}(x_2)$ i.e., $E_1 \leq E_2$. $f(E_1) = x_1$ and $f(E_2) = x_2$ can also be written as (E_1, x_1) and (E_2, x_2) respectively, and since $x_1 \leq x_2$, we define $(E_1, x_1) \leq (E_2, x_2)$. Note that, here we have extended our definition to $S \times L_{\leq}$. This notion had been followed by Einstein when he defined event in a four dimensional space time coordinate system [130], where time t takes the role of elements of $L_{\leq}$.

At this point let us make a closer investigation into the age old orderer of physical events, namely, the Sun rotating round the Earth in a circular path (with respect to an observer on the Earth. For the time being we are not bothered by the eccentricity of orbit). Since this happens all over this planet and perhaps this is the most important of all incidents of our life (though we hardly ever feel it, for it is taken so much for granted), it is quite natural that the entire human being has accepted this incident as an orderer of events as a mere matter of fact. Here the Sun is taken as a 'particle' (physical approximation of mathematical 'point') moving round a circular path. This particle is a mere indicator. 'Time' is represented by each point on the circular path. 'Time' and an indicator to show its 'quantity' are precisely what make a 'clock'. In this sense the Sun rotating in a circular path round the Earth is the most primitive and fundamental of all clocks. Let us denote the points on the circular orbit of the Sun by r. Let r_0 be a fixed point. We observe the Sun starts from r_0 and again comes back to it. This is a one dimensional event according to our definitions (for, the dimension of a curve in R^2 is 1), described by r when the Sun is at r. We define this event as *time*. This definition is more concrete and more down to earth compared to Definition A.1.1. (But still this is not the final one.) So time itself is a standardized event used to order all other events. But, the time we have just defined, can really order other events? In strict mathematical sense it can not. Because, the points on a closed curve (the so called Jordan curve) can not in general be linearly ordered. Of course we have introduced such an order over the points on the orbit of the Sun round the Earth, but not without restriction. And the restriction is, "Our ordering is valid only from event r_0 up to before the next occurrence of r_0". But we shall very often need to order events which can not be accommodated within the two successive occurrences of r_0, namely three occurrences of r_0 itself. So, to build up a consistent mathematical theory of event ordering, we must look forward for some other orderer. But at the same time we may wish to retain as much of this good, age old orderer as possible. To satisfy both the needs we suggest replacement of the closed circular path by an open ended circular helix. In three dimensional Euclidean space the equations of the helix are

$$x = \cos t, \tag{A.1.1}$$
$$y = \sin t, \tag{A.1.2}$$
$$z = t / 2\pi. \tag{A.1.3}$$

We have taken the radius vector of the Sun with respect to the Earth as unit. Sun's diurnal orbit with respect to Earth is given by (A.1.1) and (A.1.2). For any fixed event r_0, to make sure that r_0 is never repeated, we have introduced (A.1.3). (A.1.1), (A.1.2), (A.1.3) together give a circular helix with unit radius in R^3, whose axis is the z-axis. Let $t \in [0, 2\pi)$, the helical path traversed from the point (1, 0, 0) [inclusive] to the point (1, 0, 1) [exclusive] in R^3 has a one to one correspondence with the curve given by $x = \cos t$ and $y = \sin t$, where $t \in [0, 2\pi)$, which also preserves the order of traversing. The curve given by $x = \cos t$ and $y = \sin t$, $t \in [0, 2\pi)$, is the diurnal orbit of the Sun. So each diurnal motion of the Sun is uniquely represented in the helix. During a complete diurnal motion of Sun each point on the circular orbit is uniquely identified by the position of the Sun (indicator). So, in general, each passage of the Sun through each point of its orbit is uniquely recorded on the helix given by (A.1.1), (A.1.2) and (A.1.3). Upon completion of one full diurnal motion, one unit advancement along z-axis is made (to make the advancement along z-axis unit, in (A.1.3) t is divided by 2π). Let h_0 be a fixed point on the helix. Each point h traversed on the helix from h_0 along the positive direction of z-axis uniquely represents a particular passage of Sun (say the nth passage, recorded on the z-axis) through a particular point r_0 on its orbit. So the portion of the helix traversed from h_0 along positive direction of z-axis is a standardized set of events by which we tend to order other events.

Let $h_0 = (\cos t_0, \sin t_0, t_0 / 2\pi)$. We are to order the events given by the portion of the helix from h_0 along the positive z direction, henceforth will be denoted by H. We notice that t is real number. Why then we don't try to construct a one to one and onto mapping $f: T \to H$, T is an interval of R, by which we will eventually be able to order H ? Let

$$f(t) = h = (\cos t, \sin t, t / 2\pi). \tag{A.1.4}$$

Obviously f is onto. If $f(t_1) = f(t_2)$ then $t_1 / 2\pi = t_2 / 2\pi$, i.e., $t_1 = t_2$. So f is also one to one. So f is invertible. $f^{-1}(h) = t$, $f^{-1}(h_0) = t_0$ and $T = [t_0, \infty)$. We write $h_1 \le h_2$ iff $f^{-1}(h_1) \le f^{-1}(h_2)$.

But, then we already have got a much simpler and more straightforward orderer of events, namely T. Since f is a one-to-one correspondence which also preserves order, we may replace H by T. We notice that f is continuous on T and f^{-1} is continuous on H. So f is a topological homeomorphism. At this point two questions are immediate. (a) Can T be extended to the whole of R ? and (b) Even after such an extension will T be sufficient to order all physical events? In the next section we shall look for answers to (a) and (b).

A.2 Main Results

Theorem A.2.1: T can be extended to the whole of R.

Proof: Without loss of generality we can choose $h_0 = (1, 0, 0)$. This makes $t_0 = 0$. So T = $[0, \infty)$. Now let us ponder over the direction of rotation of the Sun round the Earth. In the southern hemisphere it goes from east to west through north. Say, our observer is always on the southern hemisphere. Suppose if the direction of rotation of the Sun round the Earth is suddenly altered, i.e., the Sun goes from west to east through north (this has been described as *reverse process* in [129]), what is the most natural way for our observer to try to order this event? Let on the helix, h_1 be the point which denotes sudden change of direction of the Sun. Then the most natural way to order the events of reverse rotation of the Sun should be by altering the sign of t in the equations (A.1.1), (A.1.2) and (A.1.3) of the helix. Changing sign of t in (A.1.1) and (A.1.2) signifies the change of direction of rotation of the Sun round the Earth with respect to the observer. Change of sign of t in (A.1.3) signifies traversing the helix in the negative z direction. If the helix is considered as threads of a screw inserted from positive z direction, the usual motion of the Sun is given by the unscrewing motion of the screw through the threads (helix). So the reverse motion of the Sun should be given by the screwing motion of the screw through the threads (helix). Since the screw has been inserted from the positive z direction, the screwing motion must direct the screw towards the negative z direction. This gives the justification why the events of reverse motion of the Sun should be denoted on the helix along the negative z direction. Now suppose $h_1 = h_0$. So h_0 is the divider on the helix between the events denoting the usual motion of Sun and the events denoting the reverse motion of Sun. If an event E is denoted by a point $h = (\cos t, \sin t, t / 2\pi)$ on H the reverse event, whenever it exists, is denoted by $h^1 = (\cos(-t), \sin(-t), -t / 2\pi)$ on H^1. Where H^1 is the portion of the helix traversed from h_0 (exclusive) through the negative direction of z-axis. But, then by the reasons given above we have got a better orderer of reversible events, namely the interval $(-\infty, 0)$ of R. And our extension of T to R is complete. Henceforth we shall call the extended set T.

And now we are in a position to give our third and final definition of time.

Definition A.2.1: The elements of the set T described above are called *time*.

Since T = R the notions of calculus are as valid in T as they are in R. Symbols like dx/dt are well-posed [85] at least from the point of view of t. An event may occur at t or at -t. It is not within the scope of this appendix to decide at which of the two times should it occur. A detailed discussion of this can be found in [55] and [129].

Theorem A.2.2: All finite dimensional continuous events are orderable by T.

Proof: A finite dimensional continuous event is by definition a subset of R^n for some integral n. If S be the event space, by definition of continuity of S, cardinality of S is equal to c, where c is the cardinality of continuum [138]. So there exist a one to one map of S into T. By our previous arguments it is straight forward to order S by the order of T.

We can explore even more. The question is, in how many different ways S can be ordered keeping its continuity intact ? The answer is, the number of different space filling curves [138] that can be drawn in S. If dimension of S is 1 then S is a continuous curve (and in this case the only curve filling the space S) in R^n. If dimension of S is greater than 1, S will be topologically homeomorphic to some subset of R^m for some minimum m such that, $1 < m \leq n$. Clearly in this case S will be able to accommodate uncountably infinitely many continuous curves. Among them the space filing curves are given by functions of the form $\varphi : I \rightarrow S$, where I is some interval of T. φ is continuous, one to one and onto. Now ordering of S by I is a straight forward matter. In this case however φ is nowhere differentiable. So the symbols like $d\mathbf{x}/dt$ is no longer well-posed, where $\mathbf{x} = \varphi(t)$, $t \in I$ and $\mathbf{x}$ is a vector in R^m.

Appendix B

An Elaboration of Mandelbrot's Definition for Fractal Sets

The definition of fractal we stated at the beginning of Section 4.1 was first proposed by Mandelbrot in 1975. Towards the end of this appendix in Definition B.4 we will present a modified version of this important definition. Later Mandelbrot himself regretted that his definition of fractals does not include some sets as fractal, which (at least according to him) should have been regarded as fractal. Mandelbrot has called these fractals as *borderline fractals* [46]. Hilbert's space filling curve is a typical example of a borderline fractal. In Definition 4.1.5, where we have defined *fuzzy fractals* we needed Mandelbrot's definition of fractal. In this appendix we have given a complete explanation of the terms involved in Mandelbrot's definition. Let us start with the Lemma B.1 below where, we have presented an easy ingenious proof of existence of the Hausdorff-Besicovitch or just Hausdorff dimension.

Let A be a subset of a complete metric space X. Let $\{c_i\}_i$ be an open cover of A. We choose c_i such that, for all i $\text{diam}(c_i) \leq \in$ for some $\in > 0$. Let $H^s_\in(A) = \inf \sum_{i \in \Lambda} (\text{diam } c_i)^s$. Λ is some countable index set, $s \geq 0$. Let $\lim_{\in \to 0} H^s_\in(A) = H^s(A)$.

Lemma B.1: For any subset $A \subseteq X$, there exists unique $s_0 \in [0,\infty)$ such that,

$$\begin{aligned} H^s(A) &= \infty \quad \text{for } s < s_0 \\ H^s(A) &= 0, \text{ finite or infinite for } s = s_0 \\ H^s(A) &= 0 \quad \text{for } s > s_0. \end{aligned}$$

Proof: A being a subspace of X, itself is a metric space. So by Urysohn's embedding theorem, A can be embedded in R^n for some $n \geq 1$ [138] (n may be infinite too). Without loss of generality we take n such that A can not be embedded in R^{n-1}. In this case we call R^n the minimal subspace in which A can be embedded. We call n the *embedding dimension* of A. When A can not be embedded in R^n for any finite n we call *embedding dimension* of A is infinite. First, let embedding dimension of A n be finite.

We claim that $H^{n-1}(A) = \infty$.

$$H^{n-1}(A) = \lim_{\in \to 0} \left(\inf \sum_i (\text{diam } c_i)^{n-1}\right) \qquad \text{(B.1)}$$

Where $A \subseteq \bigcup_i c_i$ and $\text{diam } c_i \leq \in > 0$. c_i is an open set in R^n for all i. Now let us explain the right hand side of (B.1). $(\text{diam } c_i)^{n-1}$ = Lebesgue measure of an (n - 1) cube in R^{n-1} of side length equal to diam c_i. Let I_n be n-cube of edge length diam c_i. $M(I_n) = (\text{diam } c_i)^n$ and $M(I_{n-1}) = (\text{diam } c_i)^{n-1}$. M is Lebesgue measure. We write $I_n = \bigcup_{l \in \Lambda} I^l_{n-1}$, $I^l_{n-1} \cap I^k_{n-1} = \varphi$ for $l \neq k$. Cardinality of Λ = number of points in a line segment of length diam c_i (> 0).

because $(\text{diam } c_i)^n = (\text{diam } c_i)(\text{diam } c_i)^{n-1}$ i.e., $M(I_n) = (\text{diam } c_i)M(I_{n-1})$. i.e., cardinality of Λ is uncountable. So to cover each c_i we need uncountably many n - 1 cubes. Every metric space is second countable and therefore without loss of generality we can take $\{c_i\}$ as a countable open cover of A as long as each c_i is open in R^n. Let d_{ik} is an open (n - 1) cube in R^{n-1} whose edge length is diam c_i. $M(d_{ik}) = (\text{diam } c_i)^{n-1}$. We shall need uncountably many d_{ik} to cover c_i. Since $\{c_i\}$ is an arbitrary open cover of A we need uncountably many d_{ik} to cover A. So,

$H^{n-1}_{\epsilon}(A) = \inf \sum_{i,k} M(d_{ik})$ (i varies over countable set. k varies over uncountable set.) Edge length of $d_{ik} \leq \epsilon$.

$$H^{n-1}(A) = \lim_{\epsilon \to 0} (\inf \sum_{i,k} M(d_{ik}))$$

$$= \lim_{\epsilon \to 0} (\inf \sum_{p=(i,k)} (\text{diam } c_p)^{n-1})$$

p varies over uncountable set, for k is uncountable and $M(d_{ik}) = (\text{diam } c_p)^{n-1}$. diam c_i = diam $c_p \leq \epsilon > 0$. Hence $H^{n-1}(A) = \infty$.

Now we claim that $H^{n+1}(A) = 0$. To show this let us write the equation

$$H^{n+1}(A) = \lim_{\epsilon \to 0} (\inf \sum_i (\text{diam } c_i)^{n+1}).$$

c_i is an open set in R^n. $(\text{diam } c_i)^n$ = Lebesgue measure of the n cube containing c_i. $A \subseteq \bigcup_i c_i$, where i is countable.

Since diam $c_i \leq \epsilon$ and $\epsilon \to 0$. We can take diam $c_i < (1/i^2)$. Hence

$$H^{n+1}(A) = \lim_{\epsilon \to 0} (\inf \sum_i (\text{diam } c_i)^{n+1})$$

$$\leq \lim_{\epsilon \to 0} (\inf \epsilon \sum_i (\text{diam } c_i)^n)$$

$$< \lim_{\epsilon \to 0} (\inf \epsilon \sum_i (1/i^2)^n)$$

$$= \lim_{\epsilon \to 0} \epsilon \inf \sum_i (1/i^2)^n = 0. \quad [\sum_i (1/i^2)^n = \text{constant for } n \geq 1.]$$

$H^s(A) = \infty$ for $s = n - 1$ and $H^s(A) = 0$ for $s = n + 1$. Let $\varphi(s) = H^s(A)$ when A is fixed and $s \in [0, \infty)$. From the above arguments it is clear that, for $s \leq n - 1$ $\varphi(s) = \infty$ and for $s \geq n + 1$ $\varphi(s) = 0$. So $\varphi(s)$ has at least one infinite jump discontinuity in [n - 1, n + 1]. We will show that, $\varphi(s)$ has only one such infinite jump discontinuity in [n - 1, n + 1]. To prove this let this be not true. i.e., there are at least two such infinite jump discontinuities s_0, s_1 (say) in [n - 1, n + 1], $s_0 < s_1$. But this is impossible, for $H^s(A) = 0$ whenever $s > s_0$. And therefore $H^s(A) = 0$ for $s_0 < s \leq s_1$ and also for $s > s_1$. So s_1 can not be an infinite jump discontinuity of $\varphi(s)$.

On the other hand when embedding dimension of A is infinite, A can only be embedded in R^∞. From the above reasoning it is clear that $s = s_0 = \infty$. This completes our proof.

Definition B.1: We define s_0 as *Hausdorff dimension* of A.

Definition of Hausdorff dimension appears in any standard text book on fractals, for example Edgar [46]. But before presenting the definition lengthy back ground works are needed to be done. Here, by Lemma B.1 we have accomplished the task in a simpler yet shorter way.

Hausdorff dimension is a bit too complicated and hard to compute. So there is an easier version of this notion namely, the box counting dimension or the capacity dimension first enunciated by Kolmogorov. Box counting dimension is another important fractal dimension.

Definition B.2: If $N_\delta(A)$ is the smallest number of sets of diameter at most δ which cover A. Then if

$$D_b(A) = \lim_{\delta \to 0} \frac{\log N_\delta(A)}{\log(1/\delta)},$$

we call $D_b(A)$ the *box counting dimension* or *capacity dimension* of A, provided the limit exists.

Unfortunately, it is not true that the box counting dimension and the Hausdorff dimension are always equal. For example, box counting dimension of the set of rational numbers in [0,1] is 1 (being dimension of a dense subset). But Hausdorff dimension of this set is 0 (since the set is countable). However box counting dimension and Hausdorff dimension do agree for a large class of sets [124].

Next, we define an important topological dimension namely, the small inductive dimension. This is the 'topological dimension' of Mandelbrot's definition of fractals.

Definition B.3: (Edgar [46]) A subset of a metric space is *clopen* iff it is both closed and open. A metric space is called *zero dimensional* iff there is a base for the open sets consisting of clopen sets. In general, the *small inductive dimension* for metric spaces is defined in an inductive manner. Each metric space A will be assigned a dimension, written ind A, chosen from the set {-1,0,1,2,3,....,∞}, consisting of integers ≥ -1 together with an extra symbol ∞, considered to be larger than all of the integers. The empty space ∅ has ind ∅ = -1. If k is a nonnegative integer, then we say that ind A ≤ k iff there is an open base of A consisting of sets U with ind $\partial U \leq k - 1$. ∂U is boundary of U. We say ind $A = k$ iff ind $A \leq k$ but ind A is not less than or equal to $k - 1$. Finally, if ind $A \leq k$ is false for all integers k, then we say that ind $A = \infty$.

Lemma B.2: If the Hausdorff dimension of a metric space A is s then ind $A = [s]$. $[s]$ is the greatest integer less than or equal to s.

Proof: Let embedding dimension of A be m. Two cases may occur. Either $m - 1 < s < m$ or $s = m$.

Case I: $s = m$.

For any open set $U \subseteq A$ there exists open set $U' \subseteq R^m$ such that $U = U' \cap A$. Since U' is open in R^m, U' is union of countable number of infinitesimal m-cubes. Intersection of A with each such m-cube gives a basic open set of A (in relative topology). Since Hausdorff dimension of $A = m$ at least one basic open m-cube must be in A. But boundaries of this basic open m-cube have dimension $m - 1$, which implies small inductive dimension of A is m. But m is integer. So $m = s = [s]$.

Case II: $m - 1 < s < m$.

A has an empty interior in R^m. Otherwise A would have contained at least one open m-cube and in that case $s = m$ would hold true. Since $s > m - 1$, A must contain an open $(m - 1)$-cube. To prove this consider

$$H^k(A) = \lim_{\epsilon \to 0} (\inf \sum_i (\text{diam } c_i)^k), \quad c_i, \epsilon \text{ have usual meaning.}$$

If $k < s$, $H^k(A) = \infty$, if $k > s$ $H^k(A) = 0$. i.e., If $k = m - 1$, $H^k(A) = \infty$, i.e., i becomes uncountable (see the proof of Lemma B.1). To keep i countable we take a subset of A say B, which can be covered by a collection of infinitesimal open $(m - 1)$-cubes of edge length equal to diam c_i as diam $c_i < \epsilon$ and $\epsilon \to 0$. It is clear that,

$$H^k(B) = \infty \text{ for } k < m - 1$$
$$H^k(B) = 0 \text{ for } k > m - 1.$$

i.e., Hausdorff dimension of B is $m - 1$. So B must contain an infinitesimal open $(m - 1)$-cube. But $B \subseteq A$. So A contains an infinitesimal open $(m - 1)$-cube. But this open cube belongs to an open basis of A. Boundary of this open cube has small inductive dimension $m - 2$. i.e., Small inductive dimension of A is $m - 1$. i.e., ind $A = m - 1 = [s]$.

Let us rewrite Mandelbrot's definition of fractals as

Definition B.4: A subset of a complete metric space A will be called *fractal* if $s - [s] > 0$, where s is Hausdorff dimension of A.

We must note that if in case of a subset A of a complete metric space $s - [s] > 0$ holds it is indeed a fractal set. However the converse is not true in general. For example, in case of the graph of Hilbert`s space filling curve. If this graph in R^2 is A and Hausdorff dimension of A is s then $s - [s] = 0$ (for $s = 2$).

Appendix C

Proof of Chaoticity of Logistic Function

First order difference equations arise in many contexts in the biological, economic and social sciences. Iterated logistic function system is one of them, where the logistic function is given by the quadratic map $f(x) = \mu x(1 - x)$. This apparently very simple dynamical system has a very complicated dynamics. This dynamics has been explored for various values of μ, where $x \in I = [0, 1]$. May [106] and Feigenbaum [50] have studied the dynamics of iterated f for $0 < \mu < 4$ by extensive numerical calculations. Devaney has shown by symbolic dynamics that, dynamics of f under iterations is chaotic for $\mu = 4$ [30]. He has also shown that, the dynamics of f under iterations for $\mu > 2 + \sqrt{5}$ is chaotic [30]. Here we have established by classical analytical and geometric reasoning that, the dynamics of f under iterations is chaotic for $\mu \geq 4$.

Let us consider the logistic function $f(x) = \mu x(1 - x)$, $\mu \geq 4$. For $\mu > 4$, f will take values greater than 1 and f^i $(i \geq 2)$ will take negative values. But whenever f^i takes values greater than 1 or less than 0, f^{i-1} always takes value less than 0. We discard those x for which $f^i(x) < 0$ for some i. The set

$$S = \{x \mid 0 \leq \lim_{i \to \infty} f^i(x) \leq 1\}$$

is a Cantor set (To be more precise, S is topologically homeomorphic to the classical middle-third Cantor set). Formally, S is known as the nonwandering or prisoner set of f. S is obtained as $i \to \infty$. S being intersection of nonwandering sets of f^i for each $i > 0$, is the smallest nonwandering set of f under iterations.

For $\mu = 4$ $f^i(x) \in I$ for all $x \in I$ and for any natural i. If $f^i(x) = 0$, then $f^{i+1}(x) = 0$ and if $f^i(y) = 1$, then $f^{i+1}(y) = 0$. Clearly, for $\mu = 4$, $S = I$. We will show that, for $\mu \geq 4$, the dynamics of iterated f is chaotic on S. A (crisp) chaotic dynamical system has been defined in Definition 1.4.4.

We divide S into two mutually disjoint sets, namely the set of end points E and the set of interior points T. $E = \{x \in S \mid f^i(x) = 0, \text{ for some } i\}$. $T = \{x \in S \mid f^i(x) \neq 0 \text{ for any } i\}$.

Clearly, no periodic point other than 0 belongs to E. Also E is countable. But $f^i(x) = y$ will never happen for $x, y \in E$ and for some i unless $y = 0$. Topological transitivity (Definition 1.4.2) does not hold in E.

A. f has points of all periods

Though in Theorem C.1 we prove a more general result, yet mainly for historical reason, we want to start this section with the following lemma. The proof of which is rather geometric in nature.

Lemma C.1: $f(x) = \mu x(1 - x)$, $\mu \geq 4$ has a point with period 3.

Proof: For $\mu > 4$, maximum value of $f(x) > 1$, where $0 \leq x \leq 1$. $f([0, 1]) \supset [0, 1]$, that is, $f^{-1}([0, 1]) \subset [0, 1]$, or more particularly, $f^{-1}([0, 1]) = [0, a]$ or $f([0, a]) = [0, 1]$. Since $\mu > 4$, $a \in (0, \frac{1}{2})$. So $f(f([0, a]) = f([0, 1]) \supset [0, 1]$, or $f^{-2}([0, 1]) \subset [0, a]$, or $f^{-2}([0, 1]) = [b, a]$, where $0 < b < a$, or $f^{2}([b, a]) = [0, 1]$. Likewise, $f^{3}([c, a]) = [0, 1]$ for some $c \in (b, a)$. We can take the maximum c (< a) such that, $f^{3}([c, a]) = [0, 1]$ holds. f^{3} is continuous in $[c, a]$, f^{3} is differentiable in (c, a). Since $0 < c < a < 1$, graph of f^{3} in $[c, a]$ must intersect the line joining the points (c,c) and (a,a) in the euclidean plane. In other words $f^{3}(p) = p$ for some $p \in [c, a]$. $f(x) = x$ holds only for $x = 0$ and $x = (\mu - 1) / \mu$. For $\mu \geq 4$, $(\mu - 1) / \mu \geq \frac{3}{4}$. So $f(p) \neq p$, for $0 < p \leq a < 1/2$. To show that there is a $p \in [c, a]$ such that, $f^{3}(p) = p$ but $f^{2}(p) \neq p$, we note that, $a = [1 - \{1 - (4/\mu)\}^{1/2}] / 2$. Since $f(a) = 1$, $f^{2}(a) = 0$. $f^{2}(b) = 1$. Let us take the maximum b (< a) for which $f^{2}(b) = 1$ holds. $f^{3}(b) = 0$ and $f^{3}(a) = 0$. f^{3} is continuous in $[b, a]$, differentiable in (b, a). Since $f^{2}([b, a]) = [0, 1]$, $f^{3}([b, a]) \supset [0, 1]$. So $\exists\, d \in (b, a)$ such that, $f(d) = 1$. So by previous argument there is a point of period 3 in $(b, d]$ and another point of period 3 in $[d, a)$. Since $f(d) = 1$, d can not be a point of period 3. So there are at least two distinct points of period 3 in (b, a) but only one point of period 2. And we are done. A similar reasoning will show that the same result holds for f for $\mu = 4$ also.

By Sharkovskii's theorem [137], f (for $\mu \geq 4$) has a periodic point of period i in [0, 1] for each positive integer i. All periodic points except 0, belong to T. Next we shall state and prove our main theorem of this section.

Theorem C.1: For each positive integer n there are 2^{n} fixed points of f^{n} ($\mu = 4$) in I. Each of them is not necessarily of period n.

Proof: We shall prove it by induction on n. For n = 1 we have $f(0) = f(1) = 0$ and $f(1/2) = 1$. f is monotonic increasing in [0, 1/2] and monotonic decreasing in [1/2, 1]. f has two fixed points, namely 0 and ¾.

So, for n = 1 our assertion holds. Let it be true for n = k > 1. We shall additionally assume that if a, b are any two successive zeros of $f^{k}(x)$ then there is a unique $c \in (a, b)$ such that,

1) $f^{k}(c) = 1$,
2) f^{k} is monotonic increasing in [a, c] and
3) f^{k} is monotonic decreasing in [c, b].

We first show that f^{k+1} satisfies 1, 2 and 3. Then as a consequence it has 2^{k+1} fixed points.

We have $f^{k+1}(a) = f^{k+1}(b) = f^{k+1}(c) = 0$. It implies that a, c are two successive zeros of f^{k+1}. We also observe that f^{k+1} is continuous in [a, c], differentiable in (a, c). Hence by Roll's theorem $\exists\ \alpha \in (a, c)$ such that $(f^{k+1}(\alpha))' = 0$, or $4(f^k(\alpha))'(1 - 2f^k(\alpha)) = 0$. Since $f^k(x)$ is monotonic increasing in (a, c). Hence $(f^k(\alpha))' > 0$. So $f^k(\alpha) = ½$, that is, $f^{k+1}(\alpha) = 1$. This α is unique in (a, c) follows from the fact that f^k satisfies 2 and 3. Now we shall show that f^{k+1} is monotonic increasing in [a, α] and monotonic decreasing in [α, c]. Since $f^2(1/2) = 0$, $f^k(1/2) = 0$ for $k \geq 2$. Either $b \leq ½$, or $a \geq ½$. So f is monotonic in [a, b]. Now let f^{k+1} is not monotonic in [a, α]. So $\exists$ x, y $\in$ [a, α] such that, $f^{k+1}(x) = f^{k+1}(y)$, where $x \neq y$. But $f(x) \neq f(y)$. Therefore $f^k(f(x)) \neq f^k(f(y))$, for f^k is monotonic in [a, α]. This contradiction implies that f^{k+1} is monotonic in [a, α]. Since $f^{k+1}(a) = 0$ and $f^{k+1}(\alpha) = 1$ it is monotonic increasing in [a, α]. Similarly it is monotonic decreasing in [α, c]. So by induction for any natural n and for any two successive zeros a, b of $f^n(x)$ there is a unique $c \in (a, b)$ such that, 1, 2 and 3 hold.

Now take $g_n(x) = f^n(x) - x$, $x \in [a, c]$. Since $g_n(x)$ is continuous in [a, c] and $0 \leq a < c < b \leq 1$. $g_n(a) \leq 0$ and $g_n(c) > 0$. By intermediate value theorem $\exists\ \gamma \in [a, c)$ such that, $g_n(\gamma) = 0$, that is, $f^n(\gamma) = \gamma$. Leave alone the trivial case when a = 0. We want to show that only one such γ exists in (a, c). But before going to accomplish that, by similar reasoning we find $\delta \in (c, b)$ such that, $g_n(\delta) = 0$. So, for two successive roots a, b of $f^n(x)$ we are getting at least two roots of $g_n(x)$. But $f^n(x)$ and $g_n(x)$ are polynomials of same degree, that is, 2^n. Hence $g_n(x)$ has exactly two roots between any two successive roots of $f^n(x)$ that is, $f^n(x)$ has 2^n fixed points.

A similar argument will establish the same result for $f(x) = \mu x(1 - x)$, $x \in [0, 1]$, where $\mu > 4$.

Example: Take $f^3(x)$. It has $2^3 = 8$ fixed points. 0 and ¾ are included in them. But 0, ¾ are the two fixed points of $f(x)$. In fact 0, ¾ are included in the list of fixed points of $f^n(x)$ for any n. Similarly $f^{15}(x)$ has all the fixed points of $f^5(x)$ and $f^3(x)$ as its own fixed points. But still there are $2^{15} - (2^5 - 2) - (2^3 - 2) - 2$ fixed points of $f^{15}(x)$ each of which has an orbit of length 15.

For $\lim_{n\to\infty} f^n(x)$ will have at least one (actually $\lim_{n\to\infty} \{2^n - \Sigma_{i \in \{\text{factors of } n \text{ excluding } 1\}} (2^i - 2) - 2\}$ in number) orbit of length $\lim_{n\to\infty} n$. So it has the potentiality of becoming a dense orbit. In fact when we shall prove topological transitivity of f we shall see that every such orbit is indeed dense. We have actually proved more. Let us restate the Theorem C.1 below.

Theorem C.1: For $f(x) = \mu x(1 - x)$, $\mu \geq 4$ and $f : I \to I$, $f^n(x)$ has 2^n fixed points in I, not all of them of period n. Let $S_n = \{x \in I \mid 0 \leq f^n(x) \leq 1$, for some natural n$\}$. Then S_n is union of 2^{n-1} closed subintervals of I, successive pairs are overlapping only at end points only when $\mu = 4$. (The last one is easy to establish by induction). Precisely two branches of the graph of f^n is described on each such subinterval of S_n, one is where f^n is monotonic increasing from 0 to 1 and another is where f^n is monotonic decreasing from 1 to 0.

B. Topological Transitivity

Now we are in a position to establish topological transitivity of $f(x) = \mu x(1 - x)$ for $\mu \geq 4$, over the nonwandering set or attractor of f over I, that is, $S = E \cup T$ as defined above. As usual we shall always iterate f within I.

Theorem C.2: f is topologically transitive on S for $\mu \geq 4$.

Proof: Let us note that in the proof of the Theorem C.1 if a, b are two successive zeros of $f^n(x)$ then $\exists\, c \in (a, b)$ such that, $f^{n+1}(c) = 0$ and a, c are two successive zeros of $f^{n+1}(x)$. $[a, c] \subset [a, b]$. After n iterations of f over I, let S_n be the nonwandering or prisoner set. Clearly, $\lim_{n\to\infty} S_n = S$. Let U, V be open sets of S such that $U \cap V = \varnothing$. We will show that, for any such pair U, V there exists a natural q such that, $f^q(U) \cap V \neq \varnothing$.

Since U,V are open in S, let $U = I_\delta \cap S$ and $V = J_\gamma \cap S$, where I_δ and J_γ are open intervals in I of length δ and γ respectively. Let $x \in U$. Then $x \in I_\delta$. For any given positive integral n there exists a closed bounded subinterval $D_n \in S_n$ such that, $x \in D_n$. Clearly $D_n \supset D_{n+1} \supset D_{n+2} \supset \ldots\ldots$ where $D_{n+1} \in S_{n+1}$, $D_{n+2} \in S_{n+2}$, etc. We take n so large that $D_n \subset I_\delta$. But we have $f^n(D_n) = I$. Also f^n is strictly monotonic in D_n. So f^n is invertible and $(f^n)^{-1}(I) = D_n$ or, $f^{-n}(I) = D_n$. Similarly, we have $F_m \subset J_\gamma$ where $F_m \in S_m$. Let $f^{-n}(F_m) = H_m \subset D_n$. Since f^n is continuous and F_m is closed, hence H_m is also closed. $f^n(H_m) = F_m$. f^{-n} is also continuous on F_m. So H_m is closed bounded interval. S is the smallest nonwandering set of f under iterations. So $f^n(S) = S$ for all $n \geq 1$, which implies $f^n(H_m \cap S) \subseteq F_m \cap S$, for $f^n(H_m) = F_m$. But $H_m \cap S \subseteq U$ and $F_m \cap S \subseteq V$. Hence $f^n(U) \cap V \neq \varnothing$. Take n = q and we are done.

C. Density of Periodic Points

Theorem C.3: Periodic points of f are dense in S.

Proof: We already know that for any natural number n there are 2^n periodic points of f in I with period n or less. Since at each of these points values of all iterates of f remain in I, all the periodic points are in S and except 0, all are in T to be precise. For any natural n, S_n contains all the 2^n fixed points of f^n, each subinterval of S_n containing exactly two such fixed points. (We have already seen that S_n has exactly 2^{n-1} compact, i.e., closed and bounded subintervals.) Since $\lim_{n\to\infty} S_n = S$, any open set U of S contains $S \cap$ (interior of a subinterval of S_n for sufficiently large n). So U contains at least two periodic points of period n or less i.e., periodic points are dense in S.

D. Sensitive Dependence on Initial Condition

In subsection C and subsection D we have established topological transitivity and density of periodic points respectively for the iterated logistic function system for $\mu \geq 4$. By the result of Banks et. al. [8], sensitive dependence on initial condition then follows immediately.

Hence by Definition 1.4.4 the iterated logistic function system is chaotic for $\mu \geq 4$.

Bibliography

[1] Klaus-Peter Adlassnig, Fuzzy Set Theory in Medical Diagnosis. *IEEE Transactions on Systems, Man and Cybernetics*, vol – SMC-16, pp. 260 – 265, 1986.

[2] J. Anderson, W. Bandler, L. J. Kohout and C. Trayner, The Design of a Fuzzy Medical Expert System, *Approximate Reasoning in Expert Systems*, edited by Madan M. Gupta, Abraham Kandel, Wyllis Bandler, Jerry B. Kiszka, North-Holland, 1985.

[3] R. A. Anthes, *Tropical Cyclones: Their Evolution, Structures and Effects*, Mono. no. 41, American Meteorological Society, 1982.

[4] J. M. Anthony and H. Sherwood, A Characterization of Fuzzy Subgroups, *Fuzzy Sets and Systems*, vol - 7, pp. 297 - 305, 1982.

[5] P. Antonelli and V. Krivan, Fuzzy Differential Inclusions as Substitutes for Stochastic Differential Equations in Population Biology, in *Volterra-Hamilton Models in the Ecology and Evolution of Colonial Organisms*, P. L. Antonelli and R. H. Bradbury (eds), *Series in Mathematical Biology and Medicine*, vol - 2, World Scientific, 1996.

[6] M. A. Arbib and E. G. Manes, A Category-Theoretic Approach to Systems in a Fuzzy World, *Synthese*, vol - 30, pp. 381 - 406, 1975.

[7] J.-P. Aubin, Fuzzy Differential Inclusions, *Probl. Cont. Inform. Theory*, vol - 19, pp. 55 - 57, 1990.

[8] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, On Devaney's Definition of Chaos, *American Mathematical Monthly*, vol - 99, pp. 332 - 334, 1992.

[9] Michael F. Barnsley, *Fractals Everywhere*, Academic Press, San Diego, CA, 1988.

[10]Rodney C. Bassanezi, Laecio C. de Barros and Pedro A. Tonelli, Attractors and Asymptotic Stability for Fuzzy Dynamical Systems, *Fuzzy Sets and Systems*, vol - 113, pp. 473 - 483, 2000.

[11]L. von Bertalanfy, An Outline of General Systems Theory, *British Journal of Philosophy of Science*, vol – 1, no. 2, 1950.

[12]L. von Bertalanffy, Principles of Growth, *Fundamental Aspects of Normal and Malignant Growth*, Elsevier, Amsterdam, 1970.

[13]James C. Bezdek and Sankar Kumar Pal (edited), *Fuzzy Models for Pattern Recognition: Methods That Search for Structure in Data*, IEEE Press, New Jersey, 1992.

[14]V. N. Bobylev, A Possibilistic Argument for Irreversibility, *Fuzzy Sets and Systems*, vol - 34, pp. 73 - 80, 1990.

[15]P. Bouthemy and A. Benveniste, Modeling of Atmospheric Disturbances in Meteorological Pictures, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol - 6, pp. 587 - 600, 1984.

[16]Robert A. Brown, *Fluid Mechanics of the Atmosphere*, Academic Press, San Diego, 1991.

[17]J. J. Buckley and T. Feuring, Fuzzy Differential Equations, *Fuzzy Sets and Systems*, vol - 110, pp. 43 - 54, 2000.

[18]J. J. Buckley and T. Feuring, Introduction to Fuzzy Partial Differential Equations, *Fuzzy Sets and Systems*, vol - 110, pp. 241 - 248, 2000.

[19]J. J. Buckley and Y. Hayashi, Applications of Fuzzy Chaos to Fuzzy Simulation, Fuzzy Sets and Systems, vol - 99, pp. 151 - 157, 1998.

[20]J. J. Buckley, Aimin Yan, Fuzzy Topological Vector Spaces over R. *Fuzzy Sets and Systems*, vol - 105, pp. 259 - 275, 1999.

[21]Carlos A. Cabrelli, Bruno Forte, Ursula M. Molter and Edward R. Vrscay, Iterated Fuzzy Sets Systems: A New Approach to the Inverse Problem for Fractals and Other Sets, *Journal of Mathematical Analysis and Applications*, vol - 171, pp. 79 - 100, 1992.

[22]Peter Caves, Definition and Measurement in Physics, MEASUREMENT Definitions and Theories, edited by C. West Churchman and Philburn Ratoosh, John Wiley and Sons, Inc. New York, 1959.

[23]C. L. Chang, Fuzzy Topological Spaces, *Journal of Mathematical Analysis and Applications*, vol - 24, pp. 182 - 190, 1968.

[24]S. S. L. Chang and L. A. Zadeh, On Fuzzy Mappings and Control, *IEEE Transactions on Systems Man and Cybernetics*, vol - 2, pp. 30 - 34, 1972.

[25]B. B. Chaudhury, Some Shape Definitions in Fuzzy Geometry of Space, *Pattern Recognition Letters*, vol - 12, pp. 551 - 535, 1991.

[26]Guanrong Chen and Xiaoning Dong, *From Chaos to Order: Methodologies, Perspective and Applications*, World Scientific, Singapore, 1999.

[27]Michel de Glas, Theory of Fuzzy Systems, *Fuzzy Sets and Systems*, vol - 10, pp. 65 - 77, 1983.

[28]Michel de Glas, Invariance and Stability of Fuzzy Systems, *Journal of Mathematical Analysis and Applications*, vol - 99, pp. 299 - 319, 1984.

[29]M. DeMaria and J. D. Pickle, A simplified system of equations for simulation of tropical cyclones, *Journal of Atmospheric Sciences*, vol - 45, pp. 1542 - 1554, 1988.

[30]Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley Publishing Company, Redwood City, California, 1989.

[31]Phil Diamond, Chaos and Fuzzy Representations of Dynamical Systems, *Proceedings of 2nd International Conference on Fuzzy Logic and Neural Networks, Iizuka, Japan*, 17 - 22 July 1992, pp. 51 - 58.

[32]Phil Diamond, Time-Dependent Differential Inclusions, Cocycle Attractors and Fuzzy Differential Equations, *IEEE Transactions on Fuzzy Systems*, vol - 7, pp. 734 - 740, 1999.

[33]Phil Diamond, Stability and Periodicity in Fuzzy Differential Equations, *IEEE Transactions on Fuzzy Systems*, vol - 8, pp. 583 - 590, 2000.

[34]L. E. Dickson, *New First Course in the Theory of Equations*, John Wiley & Sons, New York, 1947.

[35]Didier Dubois and Henri Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980.

[36]Didier Dubois and Henri Prade, Fuzzy Numbers: An Overview, *Readings in Fuzzy Sets for Intelligent Systems*, edited by Didier Dubois, Henri Prade and Ronald R. Yager, Morgan Kauffman Publishers, Inc., California, 1993.

[37]Didier Dubois and Henri Prade, General Introduction, *Fundamentals of Fuzzy Sets*, edited by Didier Dubois and Henri Prade, Kluwer Academic Publishers, Boston, 2000.

[38]Didier Dubois, Etienne Kerre, Radko Mesiar and Henri Prade, Fuzzy Interval Analysis, *Fundamentals of Fuzzy Sets*, edited by Didier Dubois and Henri Prade, Kluwer Academic Publishers, Boston, 2000.

[39]D. Dumitrescu, Corina Haloiu and Adina Dumitrescu, Generators of Fuzzy Dynaimcal Systems, *Fuzzy Sets and Systems*, vol - 113, pp. 447 - 452, 2000.
[40]D. Dutta Majumder, Cybernetics and General Systems - A Unitary Science?, *International Journal of Systems & Cybernetics*, vol - 8, pp. 7-15, 1979.
[41]D. Dutta Majumder, Fuzzy Mathematics and Uncertainty Management for Decision Making in Science and Society, *Journal of Computer Science and Informatics*, Computer Society of India, vol – 23, pp. 1 – 31, 1993.
[42]D. Dutta Majumder, Pattern Recognition (PR) and Machine Learning (ML): Current Trends and Some Applications, *Golden Jubilee Publication of Indian Institute of Technology, Kharagpur*, edited by J. Mishra, Narosa Publishing Hause, New Delhi, 2001.
[43]D. Dutta Majumder and Kausik Kumar Majumdar, Complexity Analysis, Uncertainty Management and Fuzzy Dynamical Systems: A Cybernetic Approach with Some Case Studies, submitted to *International Journal of Systems & Cybernetics* (*Kybernetes*).
[44]D. Dutta Majumder and P. K. Roy, Cancer Self-remission and Tumour Instability — A Cybernetic Analysis: Towards a Fresh Paradigm for Cancer Treatment, *Kybernetes*, vol - 29, pp. 896 - 927, 2000.
[45]Vernon F. Dvorak, Tropical Cyclone Intensity Analysis Using Satellite Data, *NOAA Technical Report NESDIS 11*, Satellite Applications Laboratory, Washington, D. C., September 1984.
[46]Gerald A. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York, 1990.
[47]Kerry A. Emanuel, An Air-Sea Interaction Theory for Tropical Cyclones. Part I: Steady-State Maintenance, *Journal of the Atmospheric Sciences*, vol – 43, pp. 585 – 604, 1986.
[48]Kerry A. Emanuel, Toward a General Theory of Hurricanes, *American Scientist*, vol - 76, pp. 370 - 379, 1988.
[49]Laurene V. Fausett, *Applied Numerical Analysis Using MATLAB*, Prentice Hall International Inc., New Jersey, 1999.
[50]Michel J. Feigenbaum, Quantitative Universality for a Class of Nonlinear Transformations, Journal of Statistical Physics, vol - 19, pp. 25 - 52, 1978.
[51]R. P. Feynman, On the Reliability of the [Challenger] Shuttle, in *What Do You Care What Other People Think?* pp. 224 – 225, Bantam, New York, 1989.
[52]R. P. Garey and R. Lefever, A Kinetic Approach to the Immunology of Cancer: Stationary State Properties of Effector-Target Cell Reactions, *Journal of Theoretical Biology*, vol - 73, pp. 417 - 438, 1978.
[53]P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Stability, Structure and Fluctuation*, Wiley, London, 1971.
[54]R. Goetschel and W. Voxman, Topological Properties of Fuzzy Numbers, *Fuzzy Sets and Systems*, vol - 10, pp. 87 - 99, 1983.
[55]T. Gold, The Arrow of Time, *American Journal of Physics*, vol - 30, pp. 403 - 410, 1962.
[56]Celso Grebogi, Edward Ott and James A.Yorke, Chaos, Strange Attractors, and Fractal Basin Boundaries in Nonlinear Dynamics, *Science*, vol - 238, pp. 632 - 638, 1987.

[57]Victor Guillemin and Alan Pollack, *Differential Topology*, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1974.
[58]H. Haken, Transition Phenomena in Nonlinear Systems, *Stochastic Nonlinear Systems in Physics, Chemistry and Biology*, edited by L. Arnold and R. Lefever, Springer-Verlag, Berlin, 1981.
[59]I. N. Herstein, *Topics in Algebra*, 2nd edition, Wiley Eastern Limited, New Delhi, 1987.
[60]Morris W. Hirsch, The Dynamical Systems Approach to Differential Equations, *Bulletin* (new series) *of the American Mathematical Society*, vol - 11, pp. 1 - 64, 1984.
[61]W. Horsthemke and R. Lefever, *Noise-induced Transitions: Applications to Physics, Chemistry and Biology*, Springer, Berlin, 1984.
[62]Eyke Hullermeier, An Approach to Modeling and Simulation of Uncertain Dynamical Systems, *International Journal of Uncertainty, Fuzziness, Knowledge-Based Systems*, vol - 5, pp. 117 - 137, 1997.
[63]Eyke Hullermeier, Numerical Methods for Fuzzy Initial Value Problems, *International Journal of Uncertainty, Fuzziness, Knowledge-Based Systems*, vol - 7, pp. 1 - 23, 1999.
[64]John E. Hutchinson, Fractals and Self Similarity, *Indiana University Mathematics Journal*, vol - 30, pp. 713 - 747, 1981.
[65]*IEEE Transactions on Fuzzy Systems*, vol – 2, no. 1, special issue on fuzzy set theory vs. probability theory, 1994.
[66]Tor A. Johansen, Fuzzy Model Based Control: Stability, Robustness, and Performence Issues, *IEEE Transactions on Fuzzy Systems*, vol – 2, pp. 221 – 234, 1994.
[67]Osmo Kaleva, Fuzzy Differential Equations, *Fuzzy Sets and Systems*, vol - 24, pp. 301 - 317, 1987.
[68]Osmo Kaleva, The Cauchy Problem for Fuzzy Differential Equations, *Fuzzy Sets and Systems*, vol - 35, pp. 389 - 396, 1990.
[69]Abraham Kandel, *Fuzzy Mathematical Techniques with Applications*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1986.
[70]A. Kandel, M. Friedman and M. Ming, On Fuzzy Dynamical Processes, *Proceedings of FUZZ-IEEE'96*, New Orleans, 8 - 11 September, 1996, pp. 1813 - 1818.
[71]A. Kandel, M. Friedman and M. Ming, On Fuzziness and Duality in Fuzzy Differential Equations, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol - 4, pp. 553 - 560, 1996.
[72]A. Kandel, Y. Lou and Y.-Q. Zhang, Stability Analysis of Fuzzy Control Systems, *Fuzzy Sets and Systems*, vol – 105, pp. 33 – 48, 1999.
[73]Tomasz Kapitaniak and Steven R. Bishop, *The Illustrated Dictionary of Nonlinear Dynamics and Chaos*, John Wiley & Sons Ltd., Chichester, England, 1999.
[74]Anatole Katok and Boris Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1995.
[75]A. K. Katsaras and D. B. Liu, Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces, *Journal of Mathematical Analysis and Applications*, vol - 58, pp. 135 - 146, 1977.
[76]A. Kaufmann and M. M. Gupta, *Fuzzy Mathematical Models in Engineering and Management Science*, North-Holland, 1988.

[77]Jerry B. Kiszka, Madan M. Gupta and Peter M. Nikiforuk, Energetistic Stability of Fuzzy Dynamic Systems, *IEEE Transactions on Systems, Man and Cybernetics*, vol – SMC-15, pp. 783 – 792, 1985.

[78]George J. Klir, Where Do We Stand on Measures of Uncertainty, Ambiguity, Fuzziness, and the Like?, *Fuzzy Sets and Systems*, vol - 24, pp. 141 - 160, 1987.

[79]George J. Klir, Generalized Information Theory, *Fuzzy Sets and Systems*, vol - 40, pp. 127 - 142, 1991.

[80]George J. Klir and Tina A. Folger, *Fuzzy Sets, Uncertainty, and Information*, Prentice-Hall of India, 1988.

[81]George J. Klir and Bo Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall of India Private Limited, New Delhi, 1997.

[82]P. E. Kloeden, Fuzzy Dynamical Systems, *Fuzzy Sets and Systems*, vol - 7, pp. 275 - 296, 1982.

[83]P. E. Kloeden, Chaotic Iterations of Fuzzy Sets, *Fuzzy Sets and Systems*, vol - 42, pp. 37 - 42, 1991.

[84]V. Krivan and G. Colombo, A Non-stochastic Approach for Modelling Uncertainty in Population Dynamics, *Bulletin of Mathematical Biology*, vol – 60, pp. 721 – 751, 1998.

[85]M. M. Lavrent'ev, V. G. Romanov, S. P. Shishatskii, *Ill-posed Problems of Mathematical Physics and Analysis*, Volume 64, Translations of Mathematical Monographs, American Mathematical Society, 1985.

[86]Chuen Chien Lee, Fuzzy Logic in Control Systems: Fuzzy Logic Controller - Part I, *IEEE Transactions on Systems, Man, and Cybernetics*, vol - 20, pp. 404 - 418, 1990.

[87]Chuen Chien Lee, Fuzzy Logic in Control Systems: Fuzzy Logic Controller - Part II, *IEEE Transactions on Systems, Man, and Cybernetics*, vol - 20, pp. 419 - 435, 1990.

[88]R. Lefever and R. P. Garay, A Mathematical Model of the Immune Surveillance Against Cancer, *Theoretical Immunology*, edited by G. I. Bell, A. S. Perelson and G. H. Pimbley, M. Dekker, New York, 1978.

[89]R. Lefever and W. Horsthemke, Bistability in Fluctuating Environments. Implications in Tumour Immunology", *Bulletin Mathematical Biology*, vol - 41, pp. 469 - 490, 1979.

[90]Tien-Yien Li and James A. Yorke, Period Three Implies Chaos, *American Mathematical Monthly*, vol - 82, pp. 985 - 992, 1975.

[91]R. P. Lientz, On Time Dependent Fuzzy Sets, *Information Science*, vol - 4, pp. 367 - 376, 1972.

[92]S. L. Loney, *An Elementary Treaties on the Dynamics of a Particle and of Rigid Bodies*, Macmillan India Limited, Madras, 1972.

[93]E. N. Lorenz, Deterministic Non-periodic Flow, *Journal of Atmospheric Sciences*, vol - 20, pp. 130 - 141, 1963.

[94]Kaushik Majumdar, Generalized Fuzzy Vector Space, *Proceedings of the 4th Asian Fuzzy Systems Symposium*, held at Tsukuba, Japan, from 31st May to 3rd June, 2000, pp. 239 - 241.

[95]Kausik Kumar Majumdar, A Mathematical Analysis of Time, *Science Philosophy Interface*, vol - 5, no. 2, pp. 61 - 69, 2000.

[96]Kausik Kumar Majumdar, Fuzzy - Another Paradigm to Tackle Complexity, accepted for presentation in the 10th *IEEE-FUZZ Conference*, Melbourne, Australia, 2nd to 5th December, 2001.

[97]Kausik Kumar Majumdar. Fuzzy Fractals and Fuzzy Turbulence, accepted for publication in *IEEE Transactions on Systems, Man and Cybernetics, Part – B.*
[98]Kausik Kumar Majumdar, The Multidimensional Fuzzy Number System with Some Applications in Pattern Recognition, under review in *Pattern Recognition.*
[99]Kausik Kumar Majumdar, One Dimensional Fuzzy Differential Inclusions, to appear in *Journal of Intelligent & Fuzzy Systems.*
[100]Kausik Kumar Majumdar, A Mathematical Model of the Nascent Cyclone, accepted for presentation at the 2002 IEEE Int. Geoscience and Remote Sensing Symposium, Toronto, Canada, 24 - 26 June, 2002. An extended version has been accepted for publication in *IEEE Transactions on Geoscience and Remote Sensing.*
[101]Kausik Kumar Majumdar, A Fuzzy Dynamical System Modelling of a Disturbance Leading to Cyclogenesis, to appear in *Journal of Intelligent & Fuzzy Systems.*
[102]Kausik Kumar Majumdar and D. Dutta Majumder, Fuzzy Differential Inclusions in Atmospheric and Medical Cybernetics, revised version submitted to *IEEE Transactions on Systems, Man, and Cybernetics*, Part – B.
[103]Kausik Kumar Majumdar and Dwijesh Dutta Majumder, A Fuzzy System Modelling of the Evolution of Tumour in Human Tissues, accepted for publication in *Journal of Intelligent & Fuzzy Systems.*
[104]J. C. Mandal, A Semi-Implicit Multiple-Nested Grid Model for Simulation of Flow in a Tropical Storm, *Mausam*, vol - 47, pp. 1 - 20, 1996.
[105]Benoit B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman and Co, New York, 1982.
[106]Robert M. May, Simple Mathematical Models with Very Complicated Dynamics, *Nature*, vol - 261, pp. 149 - 157, 1976.
[107]Alistair Mees and Colin Sparrow, Some Tools for Analyzing Chaos, *Proceedings of the IEEE*, vol - 75, pp. 1058 - 1070, 1987.
[108]R. E. Mercer, J. L. Barron and D. Cheng, *Fuzzy Points: Algebra and Application*, Department of Computer Science Technical Report, The University of Western Ontario.
[109]John Milnor, On the Concept of Attractor, *Communications in Mathematical Physics*, vol - 99, pp. 177 - 195, 1985.
[110]Sudarsan Nanda, Fuzzy Fields and Fuzzy Linear Spaces, *Fuzzy Sets and Systems*, vol - 19, pp. 89 - 94, 1986.
[111]G. J. Nazaroff, Fuzzy Topological Polysystems, *Journal of Mathematical Analysis and Applications*, vol - 41, pp. 487 - 485, 1973.
[112]H. T. Nguyen, A Note on the Extension Principle for Fuzzy Sets, *Journal of Mathematical Analysis and Applications*, vol - 64, pp. 369 - 380, 1978.
[113]K. Ooyama, Numerical Simulation of Life Cycle of Tropical Cyclones, *Journal of Atmospheric Science*, vol - 26, pp. 3 - 40, 1969.
[114]E. Ott, C. Grebogi and J. A. York, Controlling Chaos, *Physical Review Letter*, vol – 64, pp. 1196-1199, 1990.
[115]Nikhil R. Pal and James C. Bezdek, Measuring Fuzzy Uncertainty, *IEEE Transactions on Fuzzy Systems*, vol - 2, pp. 107 - 118, 1994.
[116]Sankar K. Pal and Dwijesh Dutta Majumder, Fuzzy Sets and Decision Making Approaches in Vowel and Speaker Recognition, *IEEE Transaction on Systems Man and Cybernetics*, vol - 7, no - 8, pp. 625 - 629, 1977.

[117]Sankar K. Pal and Dwijesh Dutta Majumder, *Fuzzy Mathematical Techniques in Pattern Recognition*, Wiley Eastern, New Delhi, 1986.
[118]Sankar K. Pal and Sushmita Mitra, *Neuro-Fuzzy Pattern Recognition: Methods in Soft Computing*, John Wiley & Sons, Inc. New York, 1999.
[119]J. Palis, Chaotic and Complex Systems, *Current Science*, vol – 82, pp. 403 – 406, 2002.
[120]C. P. Pappis and Ebrahim H. Mamdani, A Fuzzy Logic Controller for a Traffic Junction, *IEEE Transactions on Systems, Man, and Cybernetics*, vol - SMC-7, pp. 707 - 717, 1977.
[121]Thomas S. Parker and Leon O. Chua, Chaos: A Tutorial for Engineers, *Proceedings of the IEEE*, vol - 75, pp. 982 - 1008, 1987.
[122]Witold Pedrycz (editor), *Fuzzy Evolutionary Computation*, Kluwer Academic Publishers, Massachusetts, U.S.A., 1997.
[123]Witold Pedrycz and Fernando Gomide, *An Introduction to Fuzzy Sets Analysis and Design*, A Bradford Book, The MIT Press, Cambridge, Massachusetts, 1998.
[124]Heinz-Otto Peitgen, Hartmut Jurgens and Dietmar Saupe, *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag, New York, 1992.
[125]Stephen B. Pope, *Turbulent Flows*, Cambridge University Press, Cambridge, 2000.
[126]Madan L. Puri and Dan A. Ralescu, Differentials of Fuzzy Functions, *Journal of Mathematical Analysis and Applications*, vol - 91, pp. 552 - 558, 1983.
[127]Kumar S. Ray and Jayati Ghoshal, Neuro-genetic Approach to Multidimensional Fuzzy Reasoning for Pattern Classification, *Fuzzy Sets and Systems*, vol - 112, pp. 449 - 483, 2000.
[128]K. S. Ray and D. Dutta Majumder, Application of Circle Criteria for Stability Analysis of Linear SISO and MIMO Systems Associated with Fuzzy Logic Controllers, *Transactions of IEEE on Systems, Man, and Cybernetics*, vol – SMC-14, pp. 345 - 349, 1984.
[129]Hans Reichenbach, *The Direction of Time*, Edited by Maria Reichenbach, University of California Press, Berkeley and Los Angeles, 1956.
[130]Robert Resnick, *Introduction to Special Relativity*, John Wiley & Sons, Inc., New York, 1968.
[131]Clark Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, second edition, CRC Press, Boca Raton, Florida, 1999.
[132]Azriel Rosenfeld, Fuzzy Groups, *Journal of Mathematical Analysis and Applications*, vol - 35, pp. 512 - 517, 1971.
[133]Richard Rotunno and Kerry A. Emanuel, An Air-Sea Interaction Theory for Tropical Cyclones. Part II: Evolutionary Study Using a Nonhydrostatic Axisymmetric Numerical Model, *Journal of the Atmospheric Sciences*, vol – 44, pp. 542 – 561, 1987.
[134]P. Roy, D. Dutta Majumder and J. Biswas, Spontaneous Cancer Regression: Implications for Fluctuation, *Indian Journal of Physics*, vol - 73-B, pp. 777 - 883, 1999.
[135]Enrique H. Ruspini, Piero P. Bonissone and Witold Pedrycz (editors), *Handbook of Fuzzy Computation*, Institute of Physics Publishing, Bristol and Philadelphia, 1998.
[136]Seppo Seikkala, On the Fuzzy Initial Value Problem, *Fuzzy Sets and Systems*, vol - 24, pp. 319 - 330, 1987.
[137]A. N. Sharkovskii, Coexistence of Cycles of Continuous Map of a Line onto Itself, *Ukrainian Mathematics Journal*, vol - 16, pp. 61 - 71, 1964.

[138]George F. Simmons. *Introduction to Topology and Modern Analysis*. McGraw-Hill International Edition, 1963.

[139]Stefen Smale. Differentiable Dynamical Systems. *Bulletin of American Mathematical Society*, vol - 13, pp. 747 - 817, 1967.

[140]Colin Sparrow. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. Springer-Verlag, New York, 1982.

[141]M. A. L. Tathachar and Pramod Viswanath. On the Stability of Fuzzy Systems. *IEEE Transactions on Fuzzy Systems*, vol - 5, pp. 1997.

[142]H.-N. Teodorescu, Chaos in Fuzzy Systems and Signals. Proceedings of 2nd International Conference on Fuzzy Logic and Neural Networks. Iizuka, Japan, 17 - 22 July 1992, pp. 21 - 50.

[143]M. L. Tsetlin. *Automaton Theory and Modeling of Biological Systems*. Academic Press, New York, 1973.

[144]S. K. Venkatesan. Some Vortex Algorithms for 2D Euler Flows in *Some Applied Problems in Fluid Mechanics*, edited by H. P. Majumdar, Indian Statistical Institute. Calcutta, 1993.

[145]Liang Wang and Reza Langari. Complex Systems Modeling via Fuzzy Logic, *IEEE Transactions on Systems, Man, and Cybernetics - Part B: Cybernetics*, vol - 26, pp. 100 - 106, 1996.

[146]Pei Wang. The Interpretation of Fuzziness. *IEEE Transactions on Systems, Man, and Cybernetics - Part B*, vol - 26, no. 2, pp. 321 - 326, 1996.

[147]R. H. Warren. Optimality in Fuzzy Topological Polysystems. *Journal of Mathematical Analysis and Applications*, vol - 54, pp. 309 - 315, 1976.

[148]Mozheng Wei. Quantifying Local Instability and Predictability of Chaotic Dynamical Systems by Means of Local Metric Entropy. *International Journal of Bifurcation and Chaos*, vol - 10, pp. 135 - 154, 2000.

[149]F. Wenstop. Fuzzy Sets Simulation Models in a Systems Dynamics Perspective, in D. Dubois, H. Prade and R. R. Yager, editors. *Readings in Fuzzy Sets for Intelligent Systems*, Morgan Kauffman, 1994.

[150]Norbert Wiener, *Cybernetics: or Control and Communication in the Animal and the Machine*, John Wiley & Sons, Inc., New York, 1949.

[151]Ronald R. Yager, On the Lack of Inverses in Fuzzy Arithmetic. *Fuzzy Sets and Systems*, vol - 4, pp. 73 - 82, 1980.

[152]L. A. Zadeh, Fuzzy Sets, *Information and Control*, vol - 8, pp. 338 - 353, 1965.

[153]L. A. Zadeh, Fuzzy Sets and Systems, *Proceedings of Symposium on Systems Theory*, Polytechnic Institute Press, Brooklyn, NY, 1965.

[154]L. A. Zadeh. Outline of a New Approach to the Analysis of Complex Systems and Decision Processes. *IEEE Transactions on Systems, Man, and Cybernetics*, vol - 3, pp. 28 - 44, 1973.

[155]L. A. Zadeh, PRUF - A Meaning Representation Language for Natural Languages. *International Journal of Man-Machine Studies*, vol - 10, pp. 395 - 460, 1978.

[156]L. A. Zadeh. Fuzzy Sets as a Basis for Theory of Possibility. *Fuzzy Sets and Systems*, vol - 1, pp. 3 - 28, 1978.

[157]L. A. Zadeh. The Role of Fuzzy Logic in the Management of Uncertainty in Expert Systems. *Fuzzy Sets and Systems*, vol - 11, pp. 199 - 227, 1983.

[158]L. A. Zadeh, Fuzzy Sets. *Readings in Fuzzy Sets for Intelligent Systems*. Edited by Didier Dubois, Henri Prade and Ronald R. Yager. Morgan Kaufmann Publishers, Inc., San Mateo, California, 1993.

[159]L. A. Zadeh, Fuzzy Logic = Computing with Words. *IEEE Transactions on Fuzzy Systems*, vol - 4, pp. 103 - 111, 1996.

[160]L. A. Zadeh and C. A. Desoer. *Linear System Theory: The State Space Approach*, McGraw Hill Inc., New York, 1963.

Papers of the Author

1. Fuzzy Dynamical Systems Modeling of a Disturbance Leading to Cyclogenesis, K. K. Majumdar, to appear in *Journal of Intelligent & Fuzzy Systems*.
2. Fuzzy - A New Paradigm to Tackle Complexity, K. K. Majumdar, accepted for presentation at the IEEE Fuzzy System Society's conference held in Melbourne, Australia from 2nd to 5th December, 2000.
3. A Mathematical Analysis of Time, K. K. Majumdar, *Science Philosophy Interface*, vol - 5, no. 2, pp. 61 - 69, 2000.
4. A Mathematical Model of the Nascent Cyclone, K. K. Majumdar, accepted for presentation at the International Geoscience and Remote Sensing Symposium 2002 (IGARSS'02), hosted in Toronto, Canada, from 24 to 28 June, 2002. An extended version has been accepted for publication in *IEEE Transactions on Geoscience and Remote Sensing*.
5. Complexity Analysis, Uncertainty Management and Fuzzy Dynamical Systems: A Cybernetic Approach with Some Case Studies, D. Dutta Majumder and K. K. Majumdar, submitted to *Kybernetes* (International Journal of General Systems and Cybernetics).
6. Fuzzy Differential Inclusions in Atmospheric and Medical Cybernetics, K. K. Majumdar and D. Dutta Majumder, revised version submitted in *IEEE Trans. on Systems, Man and Cybernetics, Part B: Cybernetics*.
7. One Dimensional Fuzzy Differential Inclusions, K. K. Majumdar, to appear in *Journal of Intelligent & Fuzzy Systems*.
8. Generalized Fuzzy Vector Space, *Proceedings of Asian Fuzzy Systems Symposium*, Tsukuba, Japan, pp. 239 – 243, 2000.
9. A Fuzzy System Modeling of Evolution of Tumour in Human Tissue, K. K. Majumdar and D. Dutta Majumder, accepted for publication in *Journal of Intelligent & Fuzzy Systems*.
10. The Multidimensional Fuzzy Number System with Some Applications in Pattern Recognition, K. K. Majumdar, under review in *Pattern Recognition*.
11. Fuzzy Fractals and Fuzzy Turbulence, K. K. Majumdar, accepted for publication in *IEEE Trans. on Systems, Man, and Cybernetics, Part – B: Cybernetics*.