Hypergroups, Graphs and Subfactors

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PREFACE

The main theme of this thesis is hypergroups. In this thesis the theory of hypergroups is applied to study the relation between certain graphs and subfactors of II_1 factors in the context of principal graphs associated with the inclusions of II_1 factors. More general classes of hypergroups are introduced, new examples of hypergroups associated to certain graphs are constructed and classification of small order hypergroups is discussed.

The text of the thesis is arranged in four chapters. The first chapter is on preliminaries of the theory of hypergroups, the second on the application of the theory of hypergroups in the relation between certain graphs and subfactors of II_1 factors, the third on a more general class of hypergroups and the fourth chapter is on some new examples of hypergroups and classification of small order hypergroups.

The first chapter on the preliminaries of the theory of hypergroups collects together the basic known facts about hypergroups which also serves the purpose of fixing notation and terminology for the following chapters. In this chapter the bimodule interpretation as against the relative commutant interpretation of a principal graph associated with the inclusion of a pair of II_1 factors is worked out in detail.

The second chapter is on the notion of an action of a hypergroup on a set. After deriving some consequences of the definition of action, the notion is used here to show that certain bipartite graphs cannot arise as principal graphs for inclusions of II_1 factors.

The third chapter is on the notion of an M_2-graded hypergroup. This notion extends the notion of a hypergroup and captures the algebraic structure possessed by the collection of irreducible bifinite bimodules over a pair of II_1 factors with respect to taking tensor products and contragredients. The notion of a dimension function of a hypergroup is extended to M_2-graded hypergroups and it is proved that every irreducible finite M_2-graded hypergroup possesses a unique dimension function. The results in this chapter also rule out some graphs from arising as principal graphs for inclusions of II_1 factors.

The fourth chapter is on some new examples of hypergroups and classification of hypergroups of small order. Sequences of hypergroups associated to the graphs β_{2n} for all positive integers n and the Coxeter graph E_n for all positive integers except 7 and 10 are described here. More examples given by connected sum of certain graphs are also described here. This chapter concludes with classification of hypergroups of small order which shows that the smallest non-abelian hypergroup is the smallest non-abelian group.

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Contents

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we collect the basic known facts about hypergroups, the prime notion for this thesis. Along with introducing the basics of hypergroups we also use this chapter to fix notation and terminology for the rest of this thesis, the basic reference for this chapter being [$S1$] where the notion of a hypergroup (as the notion is used in this thesis) was first introduced and used to answer some questions in the index theory of II_1 factors which started with the work of V.F.R Jones in [J]. (Various other notions of hypergroups have been discussed in the literature - e.g., [Je] or [Sp] - but throughout this thesis, we shall only discuss the notion as defined and studied in [S1] and later in [S2], [S3] and [BS].) The facts in [$S1$] are included for the sake of completeness and immediate reference.

In the first section we define a hypergroup and discuss some examples; we also define the hypergroup algebra associated with a hypergroup and discuss its left regular representation which gives an association of finite hypergroups with certain kinds of sets of non-negative integral matrices.

In the third section we discuss the dimension function of a hypergroup, the most important notion for a finite hypergroup from our point of view as this notion relates the theory of finite hypergroups to the index theory of II_1 factors.

In the last section of this chapter we consider principal graphs for the inclusions of algebras and give the bimodule description of principal graphs for inclusions of

II_1 factors.

1.2 Hypergroups

The notion of a hypergroup which was motivated by and introduced to study the index theory of II_1 factors captures the natural algebraic structure possessed by the set of equivalence (isomorphism) classes of irreducible bifinite bimodules over a II_1 factor with respect to tensor products and contragredients.

Even though there are varying definitions of hypergroups used in different contexts in the literature (cf. for instsnce [Je] and [Sp]), we stick to the definition given in [S1], introduced in the context of bifinite bimodules over II_1 factors.

Let N be a II_1 factor and $\mathcal{G}(N)$ denote the collection of isomorphism classes of irreducible $N-N$ bifinite bimodules (i.e., Hilbert spaces which admit left and right actions of N which have finite left and right N-dimensions as N modules). We can form the tensor product over N of such bimodules, (ref. [S1]), the result of which may not be an irreducible N-bimodule, but will decompose into a finite direct sum of irreducible N-bimodules. We can find out the multiplicity of a third bimodule from $\mathcal{G}(N)$ in the above direct sum, which is going to be a non-negative integer. Thus we can associate a non-negative integer with any triple from $\mathcal{G}(N)$. Then we can define a function from $\mathcal{G}(N) \times \mathcal{G}(N) \times \mathcal{G}(N)$ to the set of non-negative integers. This function acquires some nice properties due to the existence of an identity for tensor product in $\mathcal{G}(N)$, existence of a contragredient map from $\mathcal{G}(N)$ to $\mathcal{G}(N)$ and due to the associativity of the tensor product over N.

Now we list the properties of the above mentioned function as axioms and define an abstract hypergroup.

Definition 1 *By a (discrete) hypergroup we mean a set $\mathcal{G}$ equipped with a mapping $\mathcal{G} \times \mathcal{G} \times \mathcal{G} \mapsto Z^+$ $(= \{0, 1, 2, ...\})$ - denoted by $(\alpha, \beta, \gamma) \mapsto < \alpha \otimes \beta, \gamma >$ - that satisfies the following conditions:*

0. *(local finitness) : for all α, β in $\mathcal{G}$, $< \alpha \otimes \beta, \gamma > \neq 0$ for at most finitely many γ;*

(1) *(associativity) : for all α, β, γ and κ in $\mathcal{G}$, we have*

$$\sum_{\lambda \in \mathcal{G}} < \alpha \otimes \beta, \lambda >< \lambda \otimes \gamma, \kappa > \; = \sum_{\lambda \in \mathcal{G}} < \alpha \otimes \lambda, \kappa >< \beta \otimes \gamma, \lambda >;$$

(2) *(identity) : there exists an element of $\mathcal{G}$, denoted by 1, such that*

$$< \alpha \otimes 1, \beta > = < 1 \otimes \alpha, \beta > = \; \delta_{\alpha\beta},$$

where the δ on the right side is the Kronecker symbol;

(3) *(contragredient) : there is a self map $\alpha \mapsto \overline{\alpha}$ of $\mathcal{G}$ such that,*

$$< \alpha \otimes \beta, \gamma > = < \overline{\alpha} \otimes \gamma, \beta >,$$

for all α, β, and γ in $\mathcal{G}$.

Remark 2 **(i)** The associativity axiom states the equality of two different ways of computing '$< \alpha \otimes \beta \otimes \gamma, \kappa >$'.

(ii) The contragredient axiom is just an adjoint condition if we consider the right side of the axiom as $< \beta, \overline{\alpha} \otimes \gamma >$.

(iii) As in the identity axiom we will use the symbol δ in the rest of this thesis to denote the Kronecker delta.

Example 3 **(i)** Groups: For a group $\mathcal{G}$, define $< \alpha \otimes \beta, \gamma >= \delta_{\alpha \cdot \beta, \gamma}$, where $\alpha \cdot \beta$ denotes the group product of α and β. It is easy to verify that, with $\overline{\alpha} = \alpha^{-1}$, $\mathcal{G}$ acquires a hypergroup structure with the group identity as the identity of the hypergroup.

(ii) Duals of compact groups:

The set of equivalence (isomorphism) classes of irreducible representations of a compact group (i.e., the unitary dual of a compact group) acquires a hypergroup structure as follows: for irreducible representations π, π' and ρ, define $< \pi \otimes \pi', \rho >$ to be equal to the multiplicity with which ρ features in the tensor product $\pi \otimes \pi'$; let 1 denote the trivial 1-dimensional representation and $\overline{\pi}$ the contragredient representation of π.

(iii) Hypergroup of a II_1 factor:
For a II_1 factor N the collection $\mathcal{G}(N)$ discussed in the begining of this section is a hypergroup with respect to taking tensor products over N and contragredients, with the trivial N-bimodule ${}_NL^2(N)_N$ being the identity.

Definition 4 **(i)** *A subset $\mathcal{H}$ of a hypergroup $\mathcal{G}$ is said to be a subhypergroup of $\mathcal{G}$ if $\mathcal{H}$ is closed under 'products' and the contragredient map in the sense that if α, β belongs to $\mathcal{H}$ then γ belongs to $\mathcal{H}$ for all γ in $\mathcal{G}$ such that $< \alpha \otimes \beta, \gamma > \neq 0$, and $\overline{\alpha}$ belongs to $\mathcal{H}$ for all α in $\mathcal{H}$.*

(ii) *A map $\pi : \mathcal{G} \mapsto \mathcal{G}'$ between hypergroups is said to be a homomorphism if $\pi(1) = 1$, $< \pi(\alpha) \otimes \pi(\beta), \pi(\gamma) > \; = \; < \alpha \otimes \beta, \gamma >$ for all α, β, γ in $\mathcal{G}$ and if $\pi(\mathcal{G})$ is a subhypergroup of $\mathcal{G}'$.*

(iii) *For a hypergroup $\mathcal{G}$ the opposite hypergroup of $\mathcal{G}$ which we shall donote by $\mathcal{G}^{op}$ is a hypergroup for which there exists an anti-isomomorphism $\alpha \mapsto \alpha^0 : \mathcal{G} \mapsto \mathcal{G}^{op}$ such that $< \alpha^0 \otimes \beta^0, \gamma^0 > \; = \; < \beta \otimes \alpha, \gamma >$ for all α, β, γ in $\mathcal{G}$.*

(iv) *A hypergroup $\mathcal{G}$ is said to be abelian if $< \alpha \otimes \beta, \gamma > \; = \; < \beta \otimes \alpha, \gamma >$ for all α, β, γ in $\mathcal{G}$.*

(v) *A hypergroup $\mathcal{G}$ is said to be* ***Hermitian*** *if $\alpha = \overline{\alpha}$, for all α in $\mathcal{G}$.*

Remark 5 It is clear that Hermitian hypergroups are abelian, a fact which is analogous to the fact that a group in which every element has order 2 is necessarily abelian. For finite hypergroups this is same as the fact that the product of two real symmetric matrices is symmetric if and only if the two matrices commute; this is in view of the way in which every finite hypergroup corresponds to a set of non-negative integral matrices, which is discussed below.

The following is a result from [S1] which translates the problem of constructing finite hypergroups into one of constructing certain kinds of sets of non-negative integral matrices.

Proposition 6 *There is an essentially one-to-one correspondence between finite hypergroups with cardinality n and sets $\{A_1, A_2, ..., A_n\} \subseteq M_n(Z^+)$ satisfying the following conditions:*

(a) $A_1 = 1$, *the* $n \times n$ *identity matrix;*

(b) *The collection* $\{A_i\}$ *is linearly independent and selfadjoint, i.e., closed under the formation of transposes;*

(c) $A_i A_j = \sum_{1 \leq k \leq n} A_i(k, j) A_k$; $1 \leq i,\ j \leq n$.

Remark 7 The matrices above arise from the left regular representation of the hypergroup algebra discussed below.

Hypergroup Algebra:
Given an abstract hypergroup $\mathcal{G}$, we can consider $\mathbb{C}\mathcal{G}$, the class of all finitely supported complex-valued functions on $\mathcal{G}$. The space $\mathbb{C}\mathcal{G}$ has a canonical basis $\{f_\alpha : \alpha \in \mathcal{G}\}$, given by $f_\alpha(\beta) = \delta_{\alpha\beta}$. Then, for any f in $\mathbb{C}\mathcal{G}$, $f = \sum_{\alpha \in \mathcal{G}} f(\alpha) f_\alpha$. We can make $\mathbb{C}\mathcal{G}$ an algebra by introducing a product in $\mathbb{C}\mathcal{G}$, defined by $(f_\alpha * f_\beta)(\gamma) = < \alpha \otimes \beta, \gamma >$; which extends to the whole of $\mathbb{C}\mathcal{G}$ linearly; i.e., for $f = \sum_{\alpha \in \mathcal{G}} f(\alpha) f_\alpha$ and $g = \sum_{\beta \in \mathcal{G}} g(\beta) f_\beta$, $(f * g)(\gamma) = \sum_{\alpha, \beta \in \mathcal{G}} f(\alpha) g(\beta) < \alpha \otimes \beta, \gamma >$. The space $\mathbb{C}\mathcal{G}$ becomes an associative algebra with respect to the above defined 'convolution product' by the associative axiom of $\mathcal{G}$. Finally we make $\mathbb{C}\mathcal{G}$ into a pre-Hilbert space by demanding that $\{f_\alpha : \alpha \in \mathcal{G}\}$ is a (necessarily maximal) orthonormal set of vectors in $\mathbb{C}\mathcal{G}$. We will use the symbol V to denote the inner-product so obtained.

Now we can consider the left-regular representation of $\mathbb{C}\mathcal{G}$. For $f \in \mathbb{C}\mathcal{G}$, we define the associated left-multiplication operator L_f on V by $L_f\ g = f * g$. (For a finite hypergroup $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ the A_i's in Proposition 6 correspond to L_{α_i}.) The map $f \mapsto L_f$ from $\mathbb{C}\mathcal{G}$ into $L(V)$ is clearly an algebra homomorphism which is unital (as $L_{f_1} = id_V$) and faithful (as f_1 is an identity for $\mathcal{G}$). If we define $f^*(\alpha) = \overline{(f(\overline{\alpha}))}$ - the complex conjugate of $f(\overline{\alpha})$ - then the contragredient axiom of $\mathcal{G}$ ensures that $< L_f g, h > = < g, L_{f^*} h >$ for $f,\ g,\ h$ in $\mathbb{C}\mathcal{G}$. The injectivity of the regular representation and the fact that identities and adjoints are unique in operator algebras imply that the identity element of a hypergroup is unique, $\overline{1} = 1$ and $\overline{\overline{\alpha}} = \alpha$ for every α in $\mathcal{G}$.

It is easy to verify that the equation $\tau(f) = f(1)$ $(= < L_f f_1, f_1 >)$ defines a faithful positive trace on the involutive algebra $\mathbb{C}\mathcal{G}$.

We quote from [S1] certain consequences of the defining axioms of a hypergroup.

Proposition 8 *Let $\alpha, \beta, \gamma, \kappa, \alpha_1, ..., \alpha_n$ denote elements of an arbitrary hypergroup $\mathcal{G}$. we have.*

(1) $<\alpha \otimes \beta, 1> = \delta_{\overline{\alpha}\beta}$;

(2) $<\alpha \otimes \beta, \gamma> = <\gamma \otimes \overline{\beta}, \alpha>$;

(3) $<\alpha \otimes \beta, \gamma> = <\overline{\beta} \otimes \overline{\alpha}, \overline{\gamma}>$;

(4) *if $m \leq n$ then.* $<\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_n, \kappa>$
$= \sum_{\beta,\gamma \in \mathcal{G}} <\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_m, \beta> <\alpha_{m+1} \otimes \alpha_{m+1} \otimes ... \otimes \alpha_n, \gamma><\beta \otimes \gamma, \kappa>$,
where $<\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_n, \kappa> = (f_{\alpha_1} * f_{\alpha_2} * ... * f_{\alpha_n})(\kappa)$.

1.3 Dimension function

The dimension function is one of the most important notions for a hypergroup introduced in [S1]. This notion relates finite hypergroups to the indices of subfactors of II_1 factors. Here we give the definition of the dimension function and state an existence and uniqueness theorem on the dimension function from [S1].

Definition 9 *Let $\mathcal{G}$ be a hypergroup. A function $\alpha \mapsto d_\alpha$ from $\mathcal{G}$ to $(0, \infty)$ is called a dimension function for $\mathcal{G}$ if for all α, β in $\mathcal{G}$ the function satisfies*

$$d_\alpha d_\beta = \sum_{\gamma \in \mathcal{G}} <\alpha \otimes \beta, \gamma> d_\gamma.$$

Theorem 10 *Every finite hypergroup admits a unique dimension function.*

Remark 11 **(i)** We will prove a more general result in chapter 3;

(ii) If $\alpha \in \mathcal{G}$ then d_α is given by the Perron-Frobenius eigenvalue of the associated operator L_α in the left regular representation of the hypergroup algebra $\mathbb{C}\mathcal{G}$. The vector with its α^{th} coordinate equal to d_α is the common Perron-Frobenius eigenvector for the L_α's, and the uniqueness of the dimension function follows basically from the fact that the Perron-Frobenius eigenvector of a strictly positive matrix is unique upto scaling.

(iii) The dimension function for a finite hypergroup is symmetric in the sense that $d_\alpha = d_{\bar{\alpha}}$; which follows from the fact that $\alpha \mapsto d_{\bar{\alpha}}$ is also a dimension function for $\mathcal{G}$ and from the uniqueness of the dimension function for finite hypergroups.

(iv) If $\mathcal{G}$ is a group, then a dimension function on $\mathcal{G}$ is easily seen to be nothing but a homomorphism of $\mathcal{G}$ into the multiplicative group $\boldsymbol{R}_+^\times$ of positive real numbers; and for groups the above theorem is a consequence of the fact that the only finite subgroup of $\boldsymbol{R}_+^\times$ is $\{1\}$. The group case also shows that the above theorem is false for infinite $\mathcal{G}$. If $\mathcal{G} = Z$ such homomorphisms are determined by the image of 1 which can be any $\lambda > 0$. Also it is easy to see that a finite hypergroup $\mathcal{G}$ is a group if and only if $d_\alpha = 1$ for all α in $\mathcal{G}$.

1.4 Bratteli diagrams and Principal graphs

In this section we explain the notion of a Bratteli diagram and principal graph associated with an inclusion of algebras. In the case when the algebras are II_1 factors we give the 'bimodule' interpretation of the principal graph as opposed to the equivalent 'relative commutant' interpretation of this invariant for the inclusion. A good source for the details of these notions associated with the theory of II_1 factors is [GHJ].

Consider a pair of finite dimensional C^* algebras $N \subseteq M$. Let $\sigma_i, 1 \leq i \leq \nu$ and $\rho_j, 1 \leq j \leq \mu$ represent the irreducible representations of N and M on $\mathcal{H}_{\sigma_i}$ and $\mathcal{H}_{\rho_j}$ respectively. Let dim $\mathcal{H}_{\sigma_i} = n_i$ and dim $\mathcal{H}_{\rho_j} = m_j$. Then we have

$$M \simeq \oplus_{j=1}^{\mu} M_{m_j}(\mathbb{C}) \text{ and } N \simeq \oplus_{j=1}^{\nu} N_{n_i}(\mathbb{C})$$

In order to describe the manner in which N sits inside M, some extra information is needed, which can be encoded in more than one way. One involves the 'inclusion matrix' $\Lambda = ((\lambda_{ij}))$ of the pair $N \subseteq M$ where λ_{ij} denotes the multiplicity with which the irreducible representation σ_i of N features in the restriction to N of the irreducible representation ρ_j of M. The inclusion is determined upto isomorphism by Λ. Another way to represent the inclusion is graph theoretic.

Consider a bipartite graph G with $(\mu + \nu)$ vertices, with one set V_0 of μ vertices being adjacent only to vertices in its complementary set V_1 of ν vertices with edges of the graph defined by the following description.

After the vertices in the two sets V_0 and V_1 are indexed in some definite order the $i-th$ vertex of V_0 is connected to the $j-th$ vertex of V_1 by λ_{ij} nodes. For the graph to represent the pair $N \subseteq M$ fully we also label each vertex of G with appropriate integers n_i(resp., m_j) at the $i-th$ (resp., $j-th$) vertex of V_0 (resp., V_1). The resulting graph is called the 'Bratteli diagram' of the inclusion $N \subseteq M$.

Thus each pair $N \subseteq M$ of finite dimensinal C^* algebras yields the data of the triple $(\vec{n}, \vec{m}, \Lambda)$ where $\vec{n} = (n_1, ..., n_\nu)$, $\vec{m} = (m_1, ..., m_\mu)$ and $\Lambda = ((\lambda_{ij}))$. The three ingredients of the above data $(\vec{n}, \vec{m}, \Lambda)$ are related by the equation $m_j = \sum_{1 \leq i \leq \nu} n_i \lambda_{ij}$, or equivalently $\vec{m} = \vec{n}\Lambda$, where $\vec{m}$ and $\vec{n}$ are viewed as row vectors.

Now we will describe the Bratteli diagram for a tower of algebras.

Suppose $M_1 \subseteq M_2 \subseteq ... \subseteq M_n ...$ is a tower of finite dimensional C^* algebras. Fix an ordering $\left\{\pi_j^{(n)} : 1 \leq j \leq \mu^{(n)}\right\}$ of irreducible representations (i.e., *the spectrum*) of M_n for each n. Then for each n and for $1 \leq i \leq \mu^{(n)}$ and $1 \leq j \leq \mu^{(n+1)}$, define $\lambda_{ij}^{(n)}$ to be the multiplicity with which the irreducible representation $\pi_i^{(n)}$ of M_n features in the restriction to M_n of the irreducible representation $\pi_j^{(n+1)}$ of M_{n+1}. Then it is clear that, if $\dim \mathcal{H}_{\pi_i^{(n)}} = m_i^{(n)}$, then the row vectors $\vec{m}^{(n)} = ((m_i^{(n)}))$, are related by $\vec{m}^{(n+1)} = \vec{m}^{(n)}\Lambda^{(n)}$. Also the above equation shows clearly that the inclusion matrix of the pair $M_n \subseteq M_{n+k}$ is given by the product $\Lambda^{(n)}.....\Lambda^{(n+k-1)}$. Just as the inclusion matrices can be 'stacked up one over the other' as above the Bratteli diagrams of the successive inclusions may be stacked up one over the other so as to get in a natural fashion the Bratteli diagram of the tower.

Now consider a pair of II_1 factors $1 \in N \subseteq M$ with $[M : N] < \infty$. Let $N = M_{-1} \subseteq M = M_0 \subseteq M_1 \subseteq ...$ be the tower of II_1 factors obtained from applying the basic construction to the pair $N \subseteq M$. Since $[M : N] < \infty$ the relative commutants $\{N' \cap M_k : k \geq 0\}$ of N in the members of the tower are going to be finite dimensional C^* algebras. The tower of algebras $\mathbb{C} = (N' \cap N) \subseteq (N' \cap M_0) \subseteq (N' \cap M_1) \subseteq ...$ is called the 'derived tower' associated with the pair $N \subseteq M$. Now as explained before we can construct the Bratteli diagram for the derived tower.

A loose way of describing the principal graph of the pair is as follows: on the Bratteli diagram of the derived tower delete on each level the vertices corresponding to the old vertices (which occur due to the symmetry of the Bratteli diagram) and the edges emanating from them . The result is a connected bipartite graph with a distinguished vertex $*$, the unique vertex on the level zero. The Bratteli diagram for the derived tower can be reconstructed from the principal graph for $N \subseteq M$, given together with the distinguished vertex $*$.

Remark 12 (i) For a finite index inclusion of II_1 factors $N \subseteq M$ there is another principal graph which is 'dual' to the one described above in the sense that this one corresponds to the tower $\{M' \cap M_i : \quad i \geq 0\}$ of the relative commutants of M in the members of the tower of the basic construction for $N \subseteq M$.

(ii) If there exists a positive integer k such that $Z(N' \cap M_{k-2}) \simeq Z(N' \cap M_k)$, where $Z(A)$ denotes the centre of the algebra A, then $N \subset M$ is said to be of finite depth or N is said to be of finite depth in M, and the least such k is called the 'depth' of N in M. In terms of the principal graph, $N \subset M$ is of finite depth if the graph is finite, and the depth is the maximum distance from any vertex to $*$.

(iii) The Bratteli diagram of the derived tower for a finite depth pair $N \subseteq M$ will always exhibit a growth in complexity upto a certain level, after which the remaining structure is obtained by reflecting. If $N \subseteq M$ has depth k, then the Bractteli diagram for $(N' \cap M_l) \subset (N' \cap M_{l+1})$ is identical to the principal graph for $N \subseteq M$, where l is any odd integer such that $l \geq k$.

(iv) For a pair of II_1 factors the principal graph is a conjugacy invariant for the inclusion which is finer than the index as the graph determines the index, but not the other way around.

There is a restriction on the graphs that can occur as principal graphs in view of (a) Kronecker's theorem (cf., [GHJ]) on the classification of integer matrices with small norm: i.e., for a finite matrix X over Z, either $\|X\| = 2cos\frac{\pi}{q}$ for some $q \geq 2$ or $\|X\| \geq 2$; and (b) the fact that for a pair of II_1 factors $N \subseteq M$ with finite depth the index $[M : N]$ is equal to the square of the norm of the principal graph for the pair where the norm of a finite graph is defined as $\max \{ | \lambda | : \lambda$ is an eigenvalue of $\Lambda\}$, where Λ is the adjacency matrix of the graph. So from the above facts it can be deduced that if $[M : N] < 4$, then only A_n, D_n, E_6, E_7, or E_8 can, if at all, occur as principal graphs.

It turns out that the Bratteli diagram of the tower of relative commutants $(N' \cap M_n)$ has an alternate description in terms of the $N - N$ and $N - M$ bimodules arising from the basic construction for $N \subseteq M$. For the sake of completeness we furnish a proof of this fact below. The bimodule approach is due to Ocneanu - see [O1] and [O2]. We begin with a few preliminaries, from [$S1$] on 'bi-finite bimodules' over II_1 factors. Recall that for arbitrary II_1 factors N and M, an $N-M$ bimodule $\mathcal{H}$ is called bi-finite if it has finite left dimension as a left N-module and finite right M-dimension as a right M-module. A vector ψ in such a module $\mathcal{H}$ is said to be a bounded vector if there exists a constant $K>0$ such that $\|a.\psi\| \leq K[tr(a^*a)]^{\frac{1}{2}}$ for all a in N, or equivalently $\|\psi.a\| \leq K[tr(a^*a)]^{\frac{1}{2}}$ for all a in M. The collection of such bounded vectors, which will henceforth be denoted by $\mathcal{H}_0$, is known to satisfy various good properties, such as : (i) $\mathcal{H}_0$ is a dense linear subspace of $\mathcal{H}$ which is *invariant* under left-, (resp. right-) action by N, (resp. M) and (ii) the association $T \mapsto T_{|\mathcal{H}_0}$ sets up a bijection between ${}_N\mathcal{L}_M(\mathcal{H},\mathcal{K})$ (the space of bounded $N-M$ linear operators between the bifinite bimodules $\mathcal{H}$ and $\mathcal{K}$) and ${}_NL_M(\mathcal{H}_0,\mathcal{K}_0)$ (the space of $N-M$ linear maps between the vector spaces $\mathcal{H}_0$ and $\mathcal{K}_0$).

Given $\mathcal{H}$ as above, there exists a unique map: $\mathcal{H}_0 \times \mathcal{H}_0 \mapsto N$, referred to as the N-valued inner-product on $\mathcal{H}$ and denoted by $(\xi,\eta) \mapsto {}_N(\xi,\eta)$, satisfying the following conditions, for all ξ,η,ζ in $\mathcal{H}_0$, a in N and b in M:

$${}_N<\xi,\xi>\in N_+, \quad {}_N<\xi,\eta>=({}_N<\eta,\xi>)^*,$$
$${}_N<\xi+\eta,\zeta>= {}_N<\xi,\zeta>+{}_N<\eta,\zeta>,$$
$${}_N<a\cdot\xi,\eta>= a\cdot{}_N<\eta,\zeta>,$$
$${}_N<\xi.b,\eta>= {}_N<\xi,\eta.b^*> \text{ and } <\xi,\eta>=tr({}_N<\xi,\eta>).$$

For example, if $N,P \subset M$ are subfactors of finite index of the II_1 factor M, then $L^2(M)$ can be viewed as an $N-P$ bimodule, which is bifinite in view of the assumption that both subfactors have finite index; in this case the set of bounded vectors is precisely the dense subspace M of $L^2(M)$, and the 'N-valued inner-product' on $L^2(M)$ is given by ${}_N<x,y>=E_N(xy^*)$.

If N,M and P are II_1 factors, if $\mathcal{H}$ is a bi-finite $N-M$ bimodule and $\mathcal{K}$ is a bi-finite $M-P$ bimodule, then their 'tensor product' is the unique (up to isomorphim) bi-finite $N-P$ bimodule, denoted by $\mathcal{H}\otimes_M\mathcal{K}$, with the following properties: there exists a linear map from (the algebraic tensor product) $\mathcal{H}_0\otimes\mathcal{K}_0$ onto $(\mathcal{H}\otimes_M\mathcal{K})_0$, denoted by $\xi\otimes\eta\mapsto\xi\otimes_M\eta$ which is (i)'balanced over M', meaning $\xi\cdot b\otimes_M\eta=\xi\otimes_M b\cdot\eta$ for all ξ in $\mathcal{H}_0$, η in $\mathcal{K}_0$, b in M, (ii)$N-P$bilinear, meaning $a\cdot\xi\otimes_M\eta\cdot c=a\cdot(\xi\otimes_M\eta)\cdot c$ for all ξ in $\mathcal{H}_0$, η in $\mathcal{K}_0$, a in N and c in P, and (iii) ${}_N<(\xi\otimes_M\eta,\xi'\otimes_M\eta'>=$

$< \xi \cdot {}_M < \eta, \eta' > . \xi' >$ for all ξ, ξ' in $\mathcal{H}_0$ and η, η' in $\mathcal{K}_0$.

Assume henceforth that $N \subset M$ are II_1 factors with $[M : N] < \infty$ and that $N = M_{-1} \subset M = M_0 \subset M_1 \subset ... \subset M_n \subset M_{n+1} \subset ...$ is the tower of the basic construction for $N \subset M$.

Proposition 13 (i) $({}_{M_n}L^2(M_n)_M) \otimes_M ({}_ML^2(M)_M) \simeq ({}_{M_n}L^2(M_n)_M)$;

(ii) $({}_{M_n}L^2(M_n)_N) \otimes_N ({}_NL^2(M)_M) \simeq ({}_{M_n}L^2(M_{n+1})_M)$.

Proof:

Recall that if $N, P \subset M$ are subfactors of finite index of the II_1 factor M, then the set of bounded vectors of $L^2(M)$ is precisely the dense subspace M. For (i), define $x_n \otimes_M y = x_n y$, note that $(x_n y)(x_n' y')^* = (x_n(yy'^*))x_n'^*$, for x_n in M_n and y in M and appeal to the foregoing characterisation of the tensor product.

For (ii), define $x_n \otimes_M y = \tau^{\frac{-(n+1)}{2}} x_n e_n e_{n-1} ... e_0 y$, where e_k is the projection in M_{k+1} which implements the conditional expectation of M_k onto M_{k-1} and $\tau = [M : N]^{-1}$. The fact that M_{n+1} is linearly spanned by products of the form $x_n e_n e_{n-1} ... e_0 y$, with x_n in M_n and y in M implies that the above equation does indeed define a surjective linear map from the algebraic tensor product $M_n \otimes M$ onto M_{n+1}, which is clearly balanced and $M_n - M$ bilinear. (The above 'spanning' fact may be proved using the notion of orthonormal basis introduced in [PP1], explicit details of such arguments being available in [S4].) To complete the verification, note that if $x, x' \in M_n$ and $y, y' \in M$, then

$$
\begin{aligned}
E_{M_n}(\tau^{-(n+1)} x e_n ... e_0 y y'^* e_0 ... e_n x'^*) &= \tau^{-(n+1)} E_{M_n}(x e_n ... e_0 E_N(yy'^*) e_1 ... e_n x'^*) \\
&= \tau^{-(n+1)} E_{M_n}(x E_N(yy'^*) e_n ... e_0 e_1 ... e_n x'^*) \\
&= \tau^{-1} E_{M_n}(x E_N(yy'^*) e_n x'^*) \\
&= \tau^{-1} x E_N(yy'^*) E_{M_n}(e_n) x'^* \\
&= (x . E_N(yy'^*)) x'^*
\end{aligned}
$$

[illegible] once again the foregoing characterisation of tensor products completes the [illegible]

Q.E.D.

Lemma 14 (i) ${}_N\mathcal{L}_N(L^2(M_n)) \simeq N' \cap M_{2n+1}$;

[illegible] $L^2(M_n)) \simeq N' \cap M_{2n}$.

Proof:
Recall - cf. [PP2] - that $N \subset M_n \subset M_{2n+1}$ is also an instance of the basic construction, with the conditional expectation being implemented by the projection

$$e_{[-1,n]} = \tau^{\frac{-n(n+1)}{2}}(e_n e_{n-1}...e_0)(e_{n+1}e_n...e_1)...(e_{2n}e_{2n-1}...e_n),$$

and we may, and do, consider M_{2n+1} as an algebra of operators on $L^2(M_n)$ in such a way that the action of the subalgebra M_n is by left-multiplication. (In fact we have $M_{2n+1} = J_n N' J_n$, where J_n is the canonical modular conjugation on $L^2(M_n)$.) For the same reasons, we have $M_{2n} = J_n M' J_n$. Now simply note that ${}_N\mathcal{L}_N(L^2(M_n)) = N' \cap (J_n N J_n)'$ whereas ${}_N\mathcal{L}_M(L^2(M_n)) = N' \cap (J_n M J_n)'$.

Before proceeding to the main proposition of this section, we state, as a lemma, some elementary facts concerning the basic construction. Some of these are standard, and appear in [PP1] for instance; they are all collected together for convenience of reference.

Lemma 15 **(i)** *The action of M_1 on $L^2(M)$ is given by $x_1 \cdot x_0 = \tau^{-1}E_M(x_1 x_0 e_0)$, for all x_1 in M_1 and x_0 in M;*

(ii) $\tau^{-1}E_M(x_1 e_0)e_0 = x_1 e_0$ *for all x_1 in M_1;*

(iii) *If $x_0, y_0 \in M$, then $x_0 = y_0$ if and only if $x_0 e_0 = y_0 e_0$;*

(iv) *$M_n \subset M_{n+k+1} \subset M_{n+2k+2}$ is an instance of Jones' basic construction, with the projection in M_{n+2k+2} which implements the conditional expectation of M_{n+k+1} onto M_n being given by*

$$e_{[n,n+k+1]} = \tau^{\frac{-k(k+1)}{2}}(e_{n+k+1}...e_{n+1})(e_{n+k+2}...e_{n+2})...(e_{n+2k+1}...e_{n+k+1});$$

in particular, the following relations hold:

$$e_{[n,n+k+1]}(e_{n+2k+2}...e_{n+k+2}) = (e_{n+k+1}...e_{n+1})e_{[n+1,n+k+2]} \quad (1.4.1)$$

and

$$e_{[n,n+k+1]} = e_{[n+1,n+k+1]}\left(\tau^{-k}e_{n+1}e_{n+2}...e_{n+k}(e_{n+2k+1}...e_{n+k+1})\right) \quad (1.4.2)$$

Proof:

(i) It suffices to consider the case $x_1 = ae_0b$, where $a, b \in M$, since elements of that form linearly span M_1; for such x_1, an elementary verification shows that both sides of the desired equation reduce to $aE_N(bx_0)$.

The statements (ii) and (iii) are explicitly proved in [PP1] and the first equation in (iv) is explicitly derived, in the case $n = -1$, in [PP2] and the case of general n follows from the case $n = 0$. Finally, it is elementary to see that both sides of the equation 1.4.1 are equal to

$$\tau^{\frac{-k(k+1)}{2}}(e_{n+k+1}\dots e_{n+1})(e_{n+k+2}\dots e_{n+2})\dots(e_{n+2k+1}\dots e_{n+k+1})(e_{n+2k+2}\dots e_{n+k+2})$$

while 1.4.2 follows easily from the formula for $e_{[n,n+k+1]}$ and the fact that e_p commutes with e_q for $q > p+1$. Q.E.D

Proposition 16 **(i)** *Let ϕ be the composite mapping defined by the following commutative diagram*

$$\begin{array}{ccc} N' \cap M_{2k-1} \subset N' \cap M_{2k+2} & \overset{\phi_2}{\underset{\simeq}{\rightarrow}} & {}_N\mathcal{L}_M(L^2(M_{k+1})) \\ (\simeq) \uparrow \phi_1 & & (\simeq) \downarrow \phi_3 \\ {}_N\mathcal{L}_N(L^2(M_k)) & \overset{\phi}{\longrightarrow} & {}_N\mathcal{L}_M(L^2(M_k) \otimes_N L^2(M)) \end{array}$$

where ϕ_1 and ϕ_2 are given by Lemma 14 and ϕ_3 is given by Proposition 13 (ii). Then $\phi(T) = T \otimes_N id$, meaning $(\phi(T))(x \otimes_N y) = (Tx) \otimes_N y$, for all x in M_k and y in M.

(ii) *Let ψ be the composite mapping defined by the following commutative diagram*

$$\begin{array}{ccc} N' \cap M_{2k} \subset N' \cap M_{2k+1} & \overset{\psi_2}{\underset{\simeq}{\rightarrow}} & {}_N\mathcal{L}_N(L^2(M_k)) \\ (\simeq) \uparrow \psi_1 & & (\simeq) \downarrow \psi_3 \\ {}_N\mathcal{L}_M(L^2(M_k)) & \overset{\psi}{\longrightarrow} & {}_N\mathcal{L}_N(L^2(M_k) \otimes_M L^2(M)) \end{array}$$

where ψ_1 and ψ_2 are given by Lemma 14 and ψ_3 is given by Proposition 13 (i). Thus $\psi(S) = S \otimes_N id$, meaning $(\psi(S))(x \otimes_N y) = (Sx) \otimes_N y$, for all x in M_k and y in M.

Proof:

(i) In view of Lemma 15 (i), and since $N \subset M_k \subset M_{2k+1}$ and $M \subset M_{k+1} \subset M_{2k+2}$ are instances of the basic construction, we see that

$$\phi_1^{-1}(x_{2k+1}).x_k = \tau^{-(k+1)} E_{M_k}(x_{2k+1}x_k e_{[-1,k]}),$$

and

$$\phi_2(x_{2k+2}).x_{k+1} = \tau^{-(k+1)} E_{M_{k+1}}(x_{2k+2}x_{k+1}e_{[0,k+1]}).$$

Also, we have $\phi_3^{-1}(T) = U^* T U$, where U^* is the unique $N - M$ linear unitary operator from $L^2(M_k) \otimes_N L^2(M)$ onto $L^2(M_{k+1})$ such that

$$U^*(x_k \otimes_N x_0) = \tau^{\frac{-(k+1)}{2}} x_k e_k \dots e_0 x_0.$$

In order to prove the assertion, it is enough to assume that ϕ is defined by $\phi(T) = T \otimes_N id_{L^2(M)}$ and show that the diagram of maps in (i) commutes.

So let $T \in {}_N\mathcal{L}_N(L^2(M_k))$ and suppose that $\phi_1(T) = x_{2k+1}$. It is clear that we need to show that, for all x_k in M_k and x_0 in M_0,

$$U^*((\phi_1^{-1}(x_{2k+1})) \otimes_N id)(x_k \otimes_N x_0) = \phi_2(x_{2k+1})U^*(x_k \otimes_N x_0).$$

It follows from the foregoing equations that

$$U^*((\phi_1^{-1}(x_{2k+1}) \otimes_N id)(x_k \otimes_N x_0) = \tau^{\frac{-3}{2}(k+1)} E_{M_k}(x_{2k+1}x_k e_{[-1,k]})e_k \dots e_0 x_0,$$

while

$$\begin{aligned} \phi_2(x_{2k+1})U^*(x_k \otimes_N x_0) &= \tau^{\frac{-3}{2}(k+1)} E_{M_{k+1}}(x_{2k+1}x_k e_k \dots e_0 x_0 e_{[0,k+1]}) \\ &= \tau^{\frac{-3}{2}(k+1)} E_{M_{k+1}}(x_{2k+1}x_k e_k \dots e_0 e_{[0,k+1]})x_0, \end{aligned}$$

since elements of M commmute with $e_{[0,k+1]}$.

Hence it suffices to prove that

$$E_{M_k}(x_{2k+1}x_k e_{[-1,k]})e_k \dots e_0 = E_{M_{k+1}}(x_{2k+1}x_k e_k \dots e_0 e_{[0,k+1]}),$$

for which it suffices, since $M \subset M_{k+1} \subset M_{2k+2}$ is a basic construction, to verify that

$$LHS \times e_{[0,k+1]} = RHS \times e_{[0,k+1]}.$$

Appeal now to Lemma 15 (ii) to deduce that the right side of the above equation is equal to $\left(\tau^{k-1}x_{2k-1}x_k e_k \ldots e_0 e_{[0,k+1]}\right)$; on the other hand, from equation 1.4.1 with $n = -1$, we have $e_k \ldots e_0 e_{[0,k+1]} = e_{[-1,k]}e_{2k+1}\ldots e_{k+1}$, and again by Lemma 15 (iii), we deduce that the left side of the above equation is equal to

$$\begin{aligned} E_{M_k}(x_{2k-1}x_k e_{[-1,k]})e_{[-1,k]}e_{2k+1}\ldots e_{k+1} &= \left(\tau^{k+1}x_{2k+1}x_k e_{[-1,k]}\right)e_{2k+1}\ldots e_{k+1} \\ &= \tau^{k+1}x_{2k+1}x_k e_k \ldots e_0 e_{[0,k+1]}, \end{aligned}$$

and the proof of (i) is complete.

(ii) As before, we find, using Lemma 15 that

$$\begin{aligned} \psi_1^{-1}(x_{2k}) \cdot x_k &= \tau^{-k}E_{M_k}(x_{2k}x_k e_{[0,k]}), \\ \psi_2(x_{2k-1}).x_k &= \tau^{-(k+1)}E_{M_k}(x_{2k+1}x_k e_{[-1,k]}) \end{aligned}$$

and $\psi_3^{-1}(\tilde{T}) = W^*\tilde{T}W$, where W^* is the unique $N-N$ linear unitary operator from $(L^2(M_k) \otimes_M L^2(M))$ onto $L^2(M_k)$ such that $W^*(x_k \otimes_M x_0) = x_k x_0$.

Again, in order to prove the assertion, it is enough if we assume that ψ is defined by $\psi(T) = T \otimes_M id_{L^2(M)}$ and show that the diagram of maps in (ii) commutes.

So let $T \in {}_N\mathcal{L}_N(L^2(M_k))$ and suppose $\psi_1(T) = x_{2k}$. Again it is clear that what we actually have to show is that, for all x_k in M_k and x_0 in M_0,

$$W^*((\psi_1^{-1}(x_{2k})) \otimes_M id)(x_k \otimes_M x_0) = \psi_2(x_{2k})W^*(x_k \otimes_M x_0).$$

It follows from the foregoing equations that

$$W^*((\psi_1^{-1}(x_{2k})) \otimes_M id)(x_k \otimes_M x_0) = \tau^{-k}E_{M_k}(x_{2k}x_k e_{[0,k]})x_0,$$

while

$$\psi_2(x_{2k})W^*(x_k \otimes_M x_0) = \tau^{-(k+1)}E_{M_k}(x_{2k}x_k x_0 e_{[-1,k]}).$$

Hence it suffices to prove that

$$\begin{aligned}\tau^{-1}E_{M_k}(x_{2k}x_kx_0e_{[-1,k]}) &= E_{M_k}(x_{2k}x_ke_{[0,k]})x_0\\ &= E_{M_k}(x_{2k}x_kx_0e_{[0,k]}),\end{aligned}$$

since the elements of M commute with $e_{[0,k]}$. In other words, we need to show that,

$$E_{M_k}(x_{2k}y_ke_{[0,k]}) = \tau^{-1}E_{M_k}(x_{2k}y_ke_{[-1,k]})$$

for all y_k in M_k.

In view of Lemma 15 (iii) and since $N \subset M_k \subseteq M_{2k+1}$ is an instance of the basic construction, it suffices to prove that

$$E_{M_k}(x_{2k}y_ke_{[0,k]})e_{[-1,k]} = \tau^{-1}E_{M_k}(x_{2k}y_ke_{[-1,k]})e_{[-1,k]}.$$

Appeal to Lemma 15 (ii) to deduce that the right side of the above equation is equal to $\left(\tau^kx_{2k}y_ke_{[-1,k]}\right)$; on the other hand, from equation 1.4.2 with $n=-1$ we have $e_{[-1,k]} = e_{[0,k]}\left(\tau^{-k}e_0e_1...e_{k-1}(e_{2k}...e_k)\right)$, and again by Lemma 15 (iii), we deduce that the left side of the above equation is equal to

$$\begin{aligned}&E_{M_k}(x_{2k}y_ke_{[0,k]})e_{[0,k]}\left(\tau^{-k}e_0e_1...e_{k-1}(e_{2k}...e_k)\right)\\ &= \left(x_{2k}y_ke_{[0,k]}\right)(e_0e_1...e_{k-1}(e_{2k}...e_k))\\ &= \tau^kx_{2k}y_ke_{[-1,k]}\end{aligned}$$

and the proof of (ii) is complete. Q.E.D

Remark 17 From the above discussed correspondence between the relative commutants of N in M and the bimodules ($N-N$ and $N-M$) arising from the basic construction for $N \subset M$, the Bratteli diagram of the tower $\{N' \cap M_n : n = -1, 0, 1, ...\}$ admits the following alternate description:

Let $\mathcal{G}_0^{2k}$ denote the set of equivalence classes of irreducible $N-N$ subbimodules of $L^2(M_{k-1})$, $k = 0, 1, 2, ...$ and let $\mathcal{G}_1^{2k+1}$ denotes the set of eqivalence classes of irreducible $N-M$ subbimodules of $L^2(M_k)$, $k = 0, 1, 2, ...$. Connect a vertex β of $\mathcal{G}_0^{2k}$ to a vertex γ of $\mathcal{G}_1^{2k\pm1}$ by $< \beta \otimes_N L^2(M), \gamma >$ bonds. Then for $k \geq 0$, the Bratteli diagram of $(N' \cap M_{2k-1}) \subset (N' \cap M_{2k}) \subset (N' \cap M_{2k+1})$ is given by the nodes of $\mathcal{G}_0^{2k}$, $\mathcal{G}_1^{2k-1}$ and $\mathcal{G}_0^{2k-2}$ with adjacency of nodes as described above. Take $\mathcal{G}_0 = \cup_k \mathcal{G}_0^{2k}$ and $\mathcal{G}_1 = \cup_k \mathcal{G}_1^{2k+1}$. The principal graph of the pair $N \subset M$ (corresponding to the relative commutants of N is the tower $\{M_n\}$ of the basic construction.) is the bipartite graph with even vertices indexed by $\mathcal{G}_0$ and odd vertices indexed by $\mathcal{G}_1$ with $< \beta \otimes_N L^2(M), \gamma >$ bonds between a vertex β in $\mathcal{G}_0$ and a vertex $\gamma \in \mathcal{G}_1$.

Chapter 2

Actions of Hypergroups

2.1 Introduction

In this chapter, we define the notion of an action of a hypergroup on a set. This notion, first introduced in [SV], was motivated by the set up of the inclusion of a II_1 factor pair and one of the principal graphs associated to the pair, and the notion is used here to prove the non-occurence of certain Coxeter graphs as principal graph invariants for the inclusion of II_1 factors.

In the first section we define action, discuss some examples and introduce the notion of irreducibility for an action and then derive some consequences of these notions.

In the next section we discuss the question: 'when is a graph a principal graph?' We describe how principal graphs give rise to actions. Then we use the notion of action to give a proof for the non-occurrence of certain graphs as the principal graph invariants for inclusion of II_1 factors. These graphs include the Coxeter graphs D_{2n+1} and E_7 which were observed to not occur as principal graphs by Ocneanu in [O1] without giving a proof. We prove that the graph β_{2n+1}, (which is similar to the Coxeter graph B_{2n+1}) which we desribe later, also cannot occur as a principal graph for an inclusion of II_1 factors.

Remark 2.1.1 Apart from some nice consequences of the notion of an action, the above mentioned proofs show the use of hypergroups as convenient book keeping devices.

2.2 Actions of Hypergroups

We look at the following picture which motivated the notion of an action of a hypergroup on a set.

Let $N \subseteq M$ be a pair of II_1 factors with $[M : N] < \infty$. Consider the principal graph which corresponds to the tower $\{N' \cap M_n \ : n \geq -1\}$.

Recall from Chapter 1 that the even and odd vertices of the principal graph correspond to the equivalence (isomorphism) classes of irreducible $N-N$ and $N-M$ bifinite bimodules respectively arising in the tower obtained from the basic construction for $N \subseteq M$. The set $\mathcal{G}_0$ of even vertices forms a hypergroup and 'acts' on the set $\mathcal{G}_1$ of odd vertices in the sense that for $\alpha \in \mathcal{G}_0$ and $\beta \in \mathcal{G}_1$, $\alpha \otimes \beta \in Z^+\mathcal{G}_1$ (finite linear combinations of elements of $\mathcal{G}_1$ with coefficients from Z^+). So if we set for α in $\mathcal{G}_0$, $\pi_1(\alpha)$ in $End(Z^+\mathcal{G}_1)$ defined as $\pi_1(\alpha)(\gamma, \kappa) \ = \ <\alpha \otimes_N \kappa, \gamma> =$ the multiplicity of γ in $\alpha \otimes_N \kappa$ for all γ, κ in $\mathcal{G}_1$, we get a map $\pi_1 : Z^+\mathcal{G}_o \to End(Z^+\mathcal{G}_1)$ which behaves like a homomorphism.

Now we will give the formal definition of an action of a hypergroup on a set.

Definition 18 An action of a hypergroup $\mathcal{G}_0$ on a set $\mathcal{G}_1$ is a mapping $\pi_1 : Z^+\mathcal{G}_0 \mapsto End(Z^+\mathcal{G}_1)$, which is a homomorphism in the sense that it satisfies the following conditions:

(i) $\pi_1(1_{\mathcal{G}_o}) = id_{Z^+\mathcal{G}_1}$;

(ii) $\pi_1(\alpha)\pi_1(\beta) \ = \sum_{\gamma \in \mathcal{G}_0} <\alpha \otimes \beta, \gamma> \pi_1(\gamma)$ for all $\alpha,\ \beta \in \mathcal{G}_0$;

(iii) $\pi_1(\overline{\alpha}) \ = \ \pi_1(\alpha)'$ for all $\alpha \in \mathcal{G}_0$; where '$'$' denotes the matrix transpose (the endomorphisms of $Z^+\mathcal{G}_1$ being represented by non-negative integral matrices with respect to the basis given by $\mathcal{G}_1$).

Remark 19 Given an action π_1 of a hypergroup $\mathcal{G}_0$ on a set $\mathcal{G}_1$, the opposite action of π_1, an action of $\mathcal{G}^{op}$ on $\mathcal{G}_1$ denoted by π_1^{op}, defined as $\pi_1^{op}(\alpha^0) \ = \ \pi_1(\overline{\alpha})$ $(= \pi_1(\alpha)')$.

Example 20 (i) *Left-regular action :*

Every hypergroup acts on itself naturally, the action of a hypergroup $\mathcal{G}_o$ on $\mathcal{G}_o$, $\pi_o : Z^+\mathcal{G}_o \to End(Z^+\mathcal{G}_0)$ being given by $\pi_0(\alpha)(\gamma, \kappa) \ = \ <\alpha \otimes \kappa, \gamma>$, for all α, γ, κ in $\mathcal{G}_o$.

(ii) Let $N \subseteq M$ be a pair of II_1 factors with $[M : N] < \infty$. The set $\mathcal{G}(N)$ of equivalence classes of irreducible $N-N$ bifinite bimodules forms a hypergroup which acts on the set $\mathcal{G}(N,M)$ of equivalence classes of irreducible $N-M$ bifinite bimodules via tensor multiplication over N from the left.

(iii) Let $N = M_{-1} \subseteq M = M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$ be the tower of the basic construction applied to the finite index inclusion $N \subseteq M$ of II_1 factors. Let

$$\mathcal{G}_0 = \{\beta \in \mathcal{G}(N) : \beta \subseteq {}_N L^2(M_n)_N \text{ for some } n \geq -1\ \};$$
$$\mathcal{G}_1 = \{\gamma \in \mathcal{G}(N,M) : \gamma \subseteq {}_N L^2(M_n)_M \text{ for some } n \geq 0\ \}.$$

Here also $\mathcal{G}_0$ acts on $\mathcal{G}_1$ via tensor multiplication over N.

Remark 21 In the above example the inclusion $N \subseteq M$ has finite depth precisely when $\mathcal{G}_o$ and $\mathcal{G}_1$ are finite and then the principal graph is finite.

Definition 22 A hypergroup $\mathcal{G}$ is said to be a Z_2 - graded hypergroup if $\mathcal{G}$ is of the form $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$, where $\mathcal{G}_0$ is a subhypergroup of $\mathcal{G}$, the elements of $\mathcal{G}_0$ are thought of as having degree zero and the elements of $\mathcal{G}_1$ are thought of as having degree one and it is further the case that $< \alpha \otimes \beta, \gamma > = 0$ unless $deg(\alpha) + deg(\beta) = deg(\gamma) (mod\ 2)$.

Proposition 23 *Suppose that π_1 is an action of a hypergroup $\mathcal{G}_0$ on a set $\mathcal{G}_1$. Let $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$; define*

(1) $\alpha = \overline{\alpha}\ \ \forall \alpha \in \mathcal{G}_1$;

(2) $deg(\alpha) = i$ *for α in $\mathcal{G}_i, i = 1,2$;*

(3) :

(a) $< \alpha \otimes \beta, \gamma > = 0$ *unless* $deg(\alpha) + deg(\beta) = deg(\gamma)(mod\ 2)$, *for arbitrary α, β, γ in $\mathcal{G}$; and*

(b) $< \alpha_0 \otimes \kappa_1, \gamma_1 > = < \gamma_1 \otimes \alpha_0, \kappa_1 > = < \gamma_1 \otimes \kappa_1, \alpha_0 > = \pi_1(\alpha_0)(\gamma_1, \kappa_1)$, *for all α_0 in $\mathcal{G}_0$ and γ_1, κ_1 in $\mathcal{G}_1$.*

Then $\mathcal{G}$ becomes a Z_2-graded hypergroup if and only if the following conditions are satisfied:

(i) $\pi_1(\mathcal{G}_0)$ *is abelian: and*

(ii) $\sum_{\delta_0\in\mathcal{G}_0} \pi_1(\delta_0)(\alpha_1,\beta_1)\pi_1(\delta_0)(\gamma_1,\kappa_1)$

$= \sum_{\delta_0\in\mathcal{G}_0} \pi_1(\delta_0)(\alpha_1,\gamma_1)\pi_1(\delta_0)(\beta_1,\kappa_1)$, for all $\alpha_1,\beta_1,\gamma_1,\kappa_1 \in \mathcal{G}_1$.

Proof:
Assume that $\mathcal{G}$ is a Z_2- graded hypergroup according to the given prescriptions. Let α_0, $\beta_0 \in \mathcal{G}_0$ and γ_1, $\kappa_1 \in \mathcal{G}_1$; then,

$$\begin{aligned}
\pi_1(\alpha_0)\pi_1(\beta_0)(\gamma_1,\kappa_1) &= \sum_{\delta_1\in\mathcal{G}_1} \pi_1(\alpha_0)(\gamma_1,\delta_1)\pi_1(\beta_0)(\delta_1,\kappa_1) \\
&= \sum_{\delta_1\in\mathcal{G}_1} <\alpha_0\otimes\delta_1,\gamma_1><\beta_0\otimes\kappa_1,\delta_1> \\
&= \sum_{\delta_1\in\mathcal{G}_1} <\gamma_1\otimes\alpha_0,\delta_1><\overline{\beta_0}\otimes\delta_1,\kappa_1> \\
&= \sum_{\delta_1\in\mathcal{G}_1} <\overline{\beta_0}\otimes\gamma_1,\delta_1><\delta_1\otimes\alpha_0,\kappa_1> \\
&\quad \text{by the asssociativity of } \mathcal{G} \\
&= \sum_{\delta_1\in\mathcal{G}_1} <\overline{\beta_0}\otimes\gamma_1,\delta_1><\overline{\alpha_0}\otimes\delta_1,\kappa_1> \\
&= \sum_{\delta_1\in\mathcal{G}_1} \pi_1(\overline{\beta_0})(\delta_1,\gamma_1)\pi_1(\overline{\alpha_0})(\kappa_1,\delta_1) \\
&= \pi_1(\overline{\alpha_0})\pi_1(\overline{\beta_0})(\kappa_1,\gamma_1) \\
&= \pi_1(\alpha_0)'\pi_1(\beta_0)'(\kappa_1,\gamma_1) \\
&= (\pi_1(\beta_0)\pi_1(\alpha_0))'(\kappa_1,\gamma_1) \\
&= \pi_1(\beta_0)\pi_1(\alpha_0)(\gamma_1,\kappa_1).
\end{aligned}$$

Thus $\pi_1(\alpha_0)\pi_1(\beta_0) = \pi_1(\beta_0)\pi_1(\alpha_0)$ for all $\alpha_0,\beta_0 \in \mathcal{G}_0$, which proves (i).

To prove (ii),

$$\begin{aligned}
&\sum_{\delta_0\in\mathcal{G}_0} \pi_1(\delta_0)(\alpha_1,\beta_1)\pi_1(\delta_0)(\gamma_1,\kappa_1) \\
&\qquad = \sum_{\delta_0\in\mathcal{G}_0} <\alpha_1\otimes\delta_0,\beta_1><\gamma_1\otimes\kappa_1,\delta_0> \\
&\qquad = \sum_{\delta_0\in\mathcal{G}_0} <\alpha_1\otimes\gamma_1,\delta_0><\delta_0\otimes\kappa_1,\beta_1>
\end{aligned}$$

$$
\begin{aligned}
& \text{(by the associativity of } \mathcal{G}\text{)} \\
= & \sum_{\delta_0 \in \mathcal{G}_0} \pi_1(\delta_0)(\alpha_1, \gamma_1)\pi_1(\delta_0)(\beta_1, \kappa_1) \\
& \text{which is condition } (ii).
\end{aligned}
$$

Conversely assume that the triple $(\mathcal{G}_0, \mathcal{G}_1, \pi_1)$ with the given prescription satisfies the conditions (i) and (ii). We show that $\mathcal{G}$ is a Z_2-graded hypergroup.

Clearly $1 \in \mathcal{G}_0$ is the identity of $\mathcal{G}$ as $\pi_1(1) = id_{Z^+\mathcal{G}_1}$. The facts that the elements of $\mathcal{G}_1$ are self contragredient and $\pi_1(\overline{\alpha_0}) = \pi_1(\alpha_0)'$ imply that the contragredient axiom also holds in $\mathcal{G}$. So now we need to show that associative axiom holds in $\mathcal{G}$. We will show that in $Z^+\mathcal{G}$, $(\alpha_i \otimes \alpha_j) \otimes \alpha_k = \alpha_i \otimes (\alpha_j \otimes \alpha_k)$ for all $\alpha_i \in \mathcal{G}_i, \alpha_j \in \mathcal{G}_j, \alpha_k \in \mathcal{G}_k$; $i, j, k \in \{0, 1\}$. It is enough to show that $< (\alpha_i \otimes \alpha_j) \otimes \alpha_k, \kappa > = < \alpha_i \otimes (\alpha_j \otimes \alpha_k), \kappa >$ for all $\kappa \in \mathcal{G}$.
This is true if $i = j = k = 0$ as $\mathcal{G}_0$ is a hypergroup.

The associativity of the products $\alpha_0 \otimes \beta_1 \otimes \gamma_0$ and $\alpha_1 \otimes \beta_0 \otimes \gamma_1$ can be verified easily using the fact that $\pi_1(\mathcal{G}_0)$ is abelian (condition (i)).

The associativity of the products $\alpha_0 \otimes \beta_1 \otimes \gamma_1, \alpha_0 \otimes \beta_0 \otimes \gamma_1, \alpha_1 \otimes \beta_0 \otimes \gamma_0$ and $\alpha_1 \otimes \beta_1 \otimes \gamma_0$ follows from the fact that π_1 is an action.

The associativity of the product $\alpha_1 \otimes \beta_1 \otimes \gamma_1$ is a consequence of the condition (ii).
Just to give the flavour of the computation involved, we give below the computation for the products $\alpha_0 \otimes \beta_1 \otimes \gamma_0$ and $\alpha_1 \otimes \beta_1 \otimes \gamma_1$. Consider $< \alpha_0 \otimes \beta_1 \otimes \gamma_0, \kappa >$;
$< (\alpha_0 \otimes \beta_1) \otimes \gamma_0, \kappa > = < \alpha_0 \otimes (\beta_1 \otimes \gamma_0), \kappa > = 0$ for $\kappa \in \mathcal{G}_0$ as $\alpha_0 \otimes \beta_1 \otimes \gamma_0 \in Z^+\mathcal{G}_1$.
So let $\kappa = \kappa_1 \in \mathcal{G}_1$.

$$
\begin{aligned}
< (\alpha_0 \otimes \beta_1) \otimes \gamma_0, \kappa_1 > &= \sum_{\delta_1 \in \mathcal{G}_1} < \alpha_0 \otimes \beta_1, \delta_1 >< \delta_1 \otimes \gamma_0, \kappa_1 > \\
&= \sum_{\delta_1 \in \mathcal{G}_1} \pi_1(\alpha_0)(\delta_1, \beta_1)\pi_1(\gamma_0)(\delta_1, \kappa_1); \\
&= \pi_1(\overline{\alpha_0} \otimes \gamma_0)(\beta_1, \kappa_1).
\end{aligned}
$$

$$
\begin{aligned}
< \alpha_0 \otimes (\beta_1 \otimes \gamma_0), \kappa_1 > &= \sum_{\delta_1 \in \mathcal{G}_1} < \beta_1 \otimes \gamma_0, \delta_1 >< \alpha_0 \otimes \delta_1, \kappa_1 > \\
&= \sum_{\delta_1 \in \mathcal{G}_1} \pi_1(\gamma_0)(\beta_1, \delta_1)\pi_1(\alpha_0)(\kappa_1, \delta_1)
\end{aligned}
$$

$$= \pi_1(\gamma_0 \otimes \overline{\alpha_0})(\beta_1, \kappa_1)$$
$$= \pi_1(\overline{\alpha_0} \otimes \gamma_0)(\beta_1, \kappa_1)$$
since $\pi_1(\mathcal{G}_0)$ is abelian by (i).

Now consider, $< \alpha_1 \otimes \beta_1 \otimes \gamma_1, \kappa >$; since $\alpha_1 \otimes \beta_1 \otimes \gamma_1 \in Z^+\mathcal{G}_1$, $< (\alpha_1 \otimes \beta_1) \otimes \gamma_1, \kappa > = < \alpha_1 \otimes (\beta_1 \otimes \gamma_1), \kappa > = 0$ for $\kappa \in \mathcal{G}_0$.

So let $\kappa = \kappa_1 \in \mathcal{G}_1$;

$$< (\alpha_1 \otimes \beta_1) \otimes \gamma_1, \kappa_1 > = \sum_{\delta_0 \in \mathcal{G}_0} < \alpha_1 \otimes \beta_1, \delta_0 >< \delta_0 \otimes \gamma_1, \kappa_1 >$$
$$= \sum_{\delta_0 \in \mathcal{G}_0} \pi_1(\delta_0)(\alpha_1, \beta_1)\pi_1(\delta_0)(\kappa_1, \gamma_1).$$

$$< \alpha_1 \otimes (\beta_1 \otimes \gamma_1), \kappa_1 > = \sum_{\delta_0 \in \mathcal{G}_0} < \alpha_1 \otimes \delta_0, \kappa_1 >< \beta_1 \otimes \gamma_1, \delta_0 >$$
$$= \sum_{\delta_0 \in \mathcal{G}_0} \pi_1(\delta_0)(\alpha_1, \kappa_1)\pi_1(\delta_0)(\beta_1, \gamma_1).$$

Thus $< (\alpha_1 \otimes \beta_1) \otimes \gamma_1, \kappa_1 > = < \alpha_1 \otimes (\beta_1 \otimes \gamma_1), \kappa_1 >$ for all $\alpha_1, \beta_1, \gamma_1, \kappa_1 \in \mathcal{G}_1$ by condition (ii). Q.E.D.

Definition 24 An action π_1 of a hypergroup $\mathcal{G}_0$ on a set $\mathcal{G}_1$ is said to be irreducible if for all α_1, κ_1 in $\mathcal{G}_1$, there exists α_0 in $\mathcal{G}_0$ such that $\pi_1(\alpha_0)(\gamma_1, \kappa_1) > 0$.

Remark 25 (i) The left-regular action of a finite hypergroup $\mathcal{G}_0$ is irreducible, as the matrix corresponding to no element of $\mathcal{G}_0$ in the regular representation of $\mathbb{C}\mathcal{G}_0$ can have zero rows or columns (which follows from the fact that if $\alpha \to d_\alpha$ denotes the dimension function for $\mathcal{G}_0$, then for γ, κ in $\mathcal{G}_0$, $d_\gamma d_\kappa = \sum_{\alpha_0 \in \mathcal{G}_0} < \gamma \otimes \kappa, \alpha > d_\alpha$, and $d_\alpha > 0$ for all α in $\mathcal{G}_0$ implies that $< \gamma \otimes \kappa, \alpha > \neq 0$ for at least one α.)

(ii) It is easy to verify that the action in Example 20 (iii) is also irreducible; for, if $\gamma_1, \kappa_1 \in \mathcal{G}_1$, $\gamma_1 \subseteq {}_N L^2(M_i)_M$ and $\kappa_1 \subseteq {}_N L^2(M_j)_M$ for some $i, j \geq 0$. Asssume that $j > i$ and that $j = i + k + 1$; notice that ${}_N L^2(M_k)_N \in Z^+\mathcal{G}_0$, and appeal to the easy consequence of Lemma 13(ii) that ${}_N L^2(M_k)_N \otimes_N {}_N L^2(M_i)_M \simeq_N L^2(M_{k+i+1})_M$.

Proposition 26 *Let $\mathcal{G}_0$ be a finite hypergroup. Then,*

(1) $e = \frac{1}{\sum_{\gamma\in\mathcal{G}_0} d_\gamma^2} \sum_{\alpha\in\mathcal{G}_0} d_\alpha \alpha$ *is a self-adjoint projection in the centre of $\mathbb{C}\mathcal{G}_0$, where $\alpha \mapsto d_\alpha$ denotes the unique dimension function of $\mathcal{G}_0$.*

(2) *if π is an action of $\mathcal{G}_0$ on $\mathcal{G}_1$, then the following are equivalent:*

(i) *for all $\gamma, \kappa \in \mathcal{G}_1$ there exists $\alpha \in \mathcal{G}_0$ such that $\pi(\alpha)(\gamma,\kappa) > 0$;*

(ii) *$\pi(e)(\gamma,\kappa) > 0$, for all γ, κ in $\mathcal{G}_1$;*

(iii) *$\pi(e)$ is a rank one projection.*

Proof:

(1) Since $\alpha \mapsto \overline{\alpha}$ is an involution in $\mathcal{G}_0$ and $d_\alpha = d_{\overline{\alpha}}$, for all α in $\mathcal{G}_0$, it follows that $e^* = e$. Also, for all α in $\mathbb{C}\mathcal{G}_0$, the relation $\alpha * e = e * \alpha = d_\alpha e$ follows from the defining property of the dimension function. Hence e belongs to the centre of $\mathbb{C}\mathcal{G}$.

Also,

$$\begin{aligned} e^2 &= e * \frac{1}{\sum_{\gamma\in\mathcal{G}_0} d_\gamma^2} \sum_{\alpha\in\mathcal{G}_0} d_\alpha \alpha \\ &= \frac{1}{\sum_{\gamma\in\mathcal{G}_0} d_\gamma^2} \sum_{\alpha\in\mathcal{G}_0} d_\alpha e * \alpha \\ &= \frac{1}{\sum_{\gamma\in\mathcal{G}_0} d_\gamma^2} \sum_{\alpha\in\mathcal{G}_0} d_\alpha d_\alpha e \\ &= e. \end{aligned}$$

Thus e is a projection.

(2) Since $\pi(e)$ is a (strictly) positive combination of the $\pi(\alpha)$'s , it is clear that the conditions (i) and (ii) are equivalent.

If (ii) holds, then $\pi(e)$ is a matrix with strictly positive entries. Therefore by the Perron-Frobenius theorem there exists a positive eigenvalue of $\pi(e)$ with both geometric and algebraic multiplicity equal to one which is also the maximum of all eigenvalues of $\pi(e)$. But $\pi(e)$ is a projection so that its eigenvalues are either zero or one . Thus the maximum eigenvalue one is of geometric multiplicity one. Hence the range of $\pi(e)$ is one dimensional; i.e., $\pi(e)$ is a rank one projection.

Suppose (iii) holds. Then, since $\pi(1_{\mathcal{G}_0})$ is the identity matrix in $End(Z^+\mathcal{G}_1)$ (as π is a homomorphism) $\pi(e) = \frac{1}{\sum_{\gamma\in\mathcal{G}_0} d_\gamma^2} \sum_{\alpha\in\mathcal{G}_0} d_\alpha \pi(\alpha)$ has strictly positive diagonal entries. Hence the rank one projection $\pi(e)$ has nonzero entries everywhere.

Q.E.D.

Remark 27 Any action of a hypergroup on a set can be written as a direct sum of irreducible actions; as for any action π, the projection $\pi(e)$, up to conjugation by a permutation matrix, is a block diagonal matrix with strictly positive blocks.

2.3 Obstructions for Principal graphs

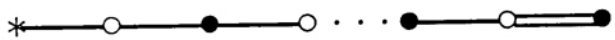
We have observed in Chapter 1 that some Coxeter graphs arise as principal graph invariants of finite index inclusions of II_1 factors, and that the graphs are a conjugacy invariants for the inclusions.

Let $1 \in N \subseteq M$ be a pair of II_1 factors such that $1 \leq [M : N] < 4$ (which forces N to have finite depth in M). Then the fact that the norm of the inclusion matrix for the principal graph of the inclusion $N \subseteq M$ is less than 2, forces the principal graph to be a Coxeter graph of the type A_n, D_n, E_6, E_7, or E_8. An interesting fact here is that whereas A_n, D_n, E_6, and E_8 are found to actually occur as principal graph invariants, D_{2n-1} and E_7 are not principal graphs. Certain other graphs also cannot occur as principal graphs.

It was shown in [J] that the Jones subfactor (the subfactor generated by the Jones projections) of the hyperfinite II_1 factor R with index equal to $4cos^2\frac{\pi}{n+1}$ has principal graph equal to A_n (see also [K]).
It has also been shown that - see [BN] - the Coxeter graph E_6 arises as the principal graph of a suitable subfactor of R. The graphs D_{2n} and E_8 were also found to occur as principal graphs ([K]).

It was observed by Ocneanu in [O1] without giving a proof that D_{2n+1} and E_7 cannot occur as principal graphs for inclusions of II_1 factors.

We can use the notion of action of hypergroups on sets to prove the non-occurence of the Coxeter graphs D_{2n+1} and E_7 and the graph β_{2n+1} as the principal graph of an inclusion of II_1 factors. Here the symbol β_{2n+1} is used to denote the graph below:

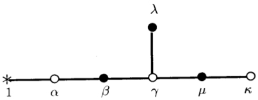


(After the proof presented here was submitted for publication, we found that this fact has been independently proved by [I].) If a graph arises as the principal graph of an inclusion $N \subseteq M$ of II_1 factors, then the set of even vertices forms a hypergroup and acts on the set of odd vertices. Thus a principal graph gives an action of a hypergroup on a set (see Example 20 (iii)).
The idea of our proof is that for a graph to arise as the principal graph of an inclusion of II_1 factors, the adjacency relations of the graph and the relations between the matrices corresponding to the vertices of the graph given by the action, should be consistent. This inconsistency implies non-occurence.

Remark 28 It is a fact that if a finite graph arises as the principal graph of a finite index inclusion of II_1 factors, then the smallest coordinate of the Perron-Frobenius eigenvector of the adjacency matrix of the graph must occur at the distinguished vertex (which Ocneanu labels $*$ in [O1]). For a proof of this see Remark 46(a).

Theorem 29 *(cf., [SV]) The Coxeter graphs E_7, D_{2n+1}, and β_{2n+1} cannot occur as principal graphs for inclusions of II_1 factors.*

Proof:
The case of E_7:
Suppose there is an inclusion $N \subseteq M$ of II_1 factors with principal graph equal to E_7. We label the graph as indicated.



It follows from the Remark 28 and an inspection of the Perron-Frobenius eigenvector of the E_7 graph that if the diagram E_7 occurs as a principal graph, then the distiguished vertex - which corresponds to the identity element of the hypergroup $\mathcal{G}_0$ - must occur as above.
Then in the language of Example 20 (iii), we have $\mathcal{G}_0 = \{1, \beta, \lambda, \mu\}$ and $\mathcal{G}_1 = \{\alpha, \gamma, \kappa\}$. Let π denotes the action of $\mathcal{G}_0$ on $\mathcal{G}_1$. From the bimodule description of the principal graph given in the first chapter, it is easily seen that α is nothing but the isomorphism class of the irreducible $N-M$ bimodule $L^2(M)$. Furthermore, the adjacency relations of the graph show that

$$\left.\begin{array}{l} 1\alpha = \alpha,\ \beta\alpha = \alpha + \gamma,\ \lambda\alpha = \gamma,\ \mu\alpha = \gamma + \kappa \text{ and} \\ \alpha\overline{\alpha} = 1 + \beta,\ \gamma\overline{\alpha} = \beta + \lambda + \mu,\ \kappa\overline{\alpha} = \mu \end{array}\right\} \qquad (2.3.1)$$

where we have used natural abbreviations: thus the second equations in the two sets of equations are short-hand for $\beta \otimes_N \alpha \simeq \alpha \oplus \gamma$ and $\gamma \otimes_M \overline{\alpha} \simeq \beta \oplus \lambda \oplus \mu$ respectively. If we write R_α for the map from $Z^+\mathcal{G}_0$ to $Z^+\mathcal{G}_1$ defined by (tensor-) multiplication on the right by α, and if we similarly write $R_{\overline{\alpha}}$ for the map from $Z^+\mathcal{G}_1$ to $Z^+\mathcal{G}_0$ defined by (tensor-) multiplication on the right by $\overline{\alpha}$, and if we represent these two linear maps by matrices with respect to the ordered bases given by $\mathcal{G}_0 = \{1, \beta, \lambda, \mu\}$ and $\mathcal{G}_1 = \{\alpha, \gamma, \kappa\}$, we then find from the equations 2.3.1 that

$$R_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and so } R_{\overline{\alpha}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

We shall also write R_ρ for the matrix corresponding to the the self map of $Z^+\mathcal{G}_0$ defined by (tensor-) multiplication on the right by ρ in $\mathcal{G}_0$. Deduce from $\alpha\overline{\alpha} = 1 + \beta$ that

$$R_1 + R_\beta = R_{\overline{\alpha}} R_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \text{ so that } R_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

The fact that R_β is a symmetric matrix means that β is self-contragredient. A look at the second column shows that $\beta^2 = 1 + \beta + \lambda + \mu$, which implies that

$$R_\lambda + R_\mu = (R_\beta)^2 - R_1 - R_\beta = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

Since R_1 and R_β are symmetric matrices, the matrices R_λ and R_μ must either be both symmetric or must be transposes of one another. These matrices have integral entries and their sum is seen to have an odd diagonal entry; hence they cannot be transposes of one another, and so, they must be symmetric. Hence $\mathcal{G}_0$ is a Hermitian hypergroup and so the matrices $\{R_\rho : \rho \in \mathcal{G}_0\}$ commute pairwise.

Since $\mathcal{G}_0$ is a commutative hypergroup we see, from the last two columns of the matrix R_β that $\beta\lambda = \lambda\beta = \beta + \mu$ and $\beta\mu = \mu\beta = \beta + \lambda + \mu$. These equations determine the first two columns of the matrices R_λ and R_μ. Also, since $\mathcal{G}_0$ is a commutative hypergroup, we find that the fourth column of R_λ must equal the third column of R_μ. Since R_λ and R_μ are both symmetric matrices, it follows that

$$R_\lambda = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & x & y \\ 0 & 1 & y & z \end{bmatrix} \text{ and } R_\mu = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & y & z \\ 1 & 1 & z & w \end{bmatrix}$$

for some non-negative integers x, y, z and w which must satisfy - in view of the equation we have already obtained for $(R_\lambda + R_\mu)$ -the equations $x + y = y + z = z + w = 1$; *i.e.*, $x = z$, $y = w$ and $x + y = 1$.

The third column of R_λ shows that $\lambda^2 = 1 + x\lambda + y\mu$, so $R_\lambda^2 = R_1 + xR_\lambda + yR_\mu$; comparing the $(\beta, \mu)^{th}$ $(= (2,4)^{th})$ entry of the two sides of this matrix equation , it is seen that we must have $x = 0$, and $y = 1$.

We have thus determined the multiplication table for the hypergroup $\mathcal{G}_0$.

$$\left.\begin{array}{l} \beta^2 = 1 + \beta + \lambda + \mu,\ \beta\lambda = \beta + \mu,\ \beta\mu = \beta + \lambda + \mu, \\ \lambda^2 = 1 + \mu,\ \lambda\mu = \beta + \lambda,\ \mu^2 = 1 + \beta + \mu. \end{array}\right\} \quad (2.3.2)$$

Note next that, in view of the equations 2.3.1, we have

$$\begin{aligned} \beta\alpha = \alpha + \gamma \Rightarrow \beta\gamma &= \beta^2\alpha - \beta\alpha = (1 + \lambda + \mu)\alpha \\ &= \alpha + 2\gamma + \kappa \\ \text{and } \mu\alpha = \gamma + \kappa \Rightarrow \beta\kappa = \beta\mu\alpha - \beta\gamma &= (\beta + \lambda + \mu)\alpha - (\alpha + 2\gamma + \kappa) = \gamma. \end{aligned}$$

This shows that

$$\pi(\beta) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which implies, since $\beta^2 = 1 + \beta + \lambda + \mu$, that

$$\pi(\lambda) + \pi(\mu) = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Note next that

$$\lambda\gamma = \lambda^2\alpha = \alpha + \mu\alpha = \alpha + \gamma + \kappa;$$

$$\mu\gamma = \mu\lambda\alpha = (\beta + \lambda)\alpha = \alpha + 2\gamma.$$

Since the fact that the elements λ and μ are self contragredient elements of $\mathcal{G}_0$ implies that the matrices $\pi(\lambda)$ and $\pi(\mu)$ are symmetric, conclude that

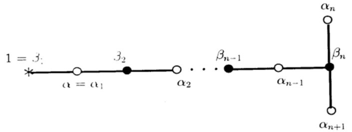
$$\pi(\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \pi(\mu) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Finally, the equation $\beta\lambda = \beta + \mu$ should imply that $\pi(\beta)\pi(\lambda) = \pi(\beta) + \pi(\mu)$, but it is seen that the matrices on the two sides of the alleged equality differ in the (2, 3) as well as the (3, 3) places. This contradiction finally completes the proof that E- cannot arise as the principal graph of any inclusion of II_1 factors.

The case of D_{2n-1}:

Remark 30 We consider only half of the case here, i.e., D_{2n-1} when n is even. There is a subtle difference between the cases $D_{4k-1}(n = 2k)$ and $D_{4k-3}(n = 2k - 1)$ which will become clear in the proofs of these cases. While the proof in the case of D_{4k-1} requires only 'one sided actions' of hypergroup, we require facts about 'dual principal graphs' and 'M_2-graded hypergroups' for the proof in the case of D_{4k-3}. We take up the case of D_{4k-3} in Chapter 3 where we consider M_2-graded hypergroups. Anyhow in the following discussion we treat n as any positive integer till we actually need to distiguish between the even and the odd cases.

Suppose that there is an inclusion $N \subseteq M$ of II_1 factors with principal graph D_{2n-1} and that the vertices of the graph as shown.



For the same reasons as in the case of E_7, the identity of the hypergroup $\mathcal{G}_0$ must occur at the indicated vertex. Thus the even vertices of D_{2n+1} are represented by $1 = \beta_1, \beta_2, \ldots, \beta_n$ and the odd vertices are represented by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}$; i.e.,

$$\mathcal{G}_0 = \{1 = \beta_1, \beta_2, \ldots, \beta_n\} \text{ and } \mathcal{G}_1 = \{\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}\}.$$

The graph implies the following relations:

$$\beta_i \alpha = \begin{cases} \alpha_{i-1} + \alpha_i & \text{for } 1 \leq i < n \\ \alpha_{n-1} + \alpha_n + \alpha_{n+1} & \text{for } i = n \end{cases} \tag{2.3.3}$$

(with the convention that $\alpha_0 = 0$),

$$\alpha_i \overline{\alpha} = \begin{cases} \beta_i + \beta_{i+1} & \text{for } i < n \\ \beta_n & \text{for } i = n \text{ and } n+1 . \end{cases} \tag{2.3.4}$$

Arguing as in the case of E_7, we see that the self map R_{β_2} of $Z\mathcal{G}_0$ of right multiplication by β_2 is given by the $n \times n$ matrix

$$R_{\beta_2} = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 & 0 \\ 1 & 1 & 1 & 0 & . & . & 0 \\ 0 & 1 & 1 & 1 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & 0 & 1 & 1 & 1 \\ 0 & 0 & . & . & 0 & 1 & 2 \end{bmatrix} . \tag{2.3.5}$$

It is known - (cf. [BS]) - that there exists a unique hypergroup $\mathcal{G}_0 = \{1 = \beta_1, \beta_2, \ldots, \beta_n\}$ satisfying 2.3.5, that every element of this hypergroup is self-contragredient and (consequently) that this hypergroup is abelian.

We know that $\mathcal{G}_0$ acts on $\mathcal{G}_1$; if this action is given by $\pi_1 : Z^+\mathcal{G}_0 \to End(Z^+\mathcal{G}_1)$, let us write A_i for the matrix representing $\pi_1(\beta_i)$. Let A_i have the block decomposition

$$A_i = \begin{bmatrix} P_i & Q_i \\ R_i & S_i \end{bmatrix}$$

corresponding to the partition: $n + 1 = (n - 1) + 2$.

It can be proved by induction and straight forward (though laborious and painful) computation, using the fact that each β_i is an appropriate polynomial in β_2, that these matrices have the following descriptions:

$$P_i(l, m) = \begin{cases} 0 & \text{if } l + m \leq i - 1 \text{ or } \mid l - m \mid \geq i \\ 2 & \text{if } l + m \geq 2n - i + 1 \\ 1 & \text{otherwise;} \end{cases}$$

$R_i = Q_i'$ has identical rows, both rows being equal to $[0, ..., 0, 1, 1, ..., 1]$, the first 1 occuring in the $(n - i + 1)^{st}$ column; i.e.,

$$Q_i(l, m) = R_i(m, l) = \begin{cases} 1 & \text{if } l \geq n - i - 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} (S_{4k+1}, S_{4k+2}, S_{4k+3}, S_{4k+4}) &= (I, S_2, S_3, J); \\ \{S_2, S_3\} &= \{I, J\}; \end{aligned}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Most of the computation is trivial; but we do wish to point out what causes the ambiguity as well as the periodicity in the S_j's which is crucial to our argument. The point is that the relations (2.3.3) and (2.3.4) are seen fairly easily to imply that

$$\beta_2\alpha_j = \begin{cases} \alpha_{j-1} + \alpha_j + \alpha_{j+1} & \text{for } 1 \leq j < n - 1 \\ \alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1} & \text{for } j = n - 1 \end{cases}$$

thereby establishing that the first $(n-1)$ columns of A_2 are indeed as asserted. Since P_2 and R_2 have been determined, so also is $Q_2(=R_2')$, since the fact that β_2 is self-contragredient implies that A_2 is a symmetric matrix.

The further information - about $\beta_2\alpha_n$ and $\beta_2\alpha_{n+1}$ - that can be gleaned from the relations (2.3.3) and (2.3.4) is only that

$$\beta_2(\alpha_n+\alpha_{n+1}) = 2\alpha_{n-1}+\alpha_n+\alpha_{n+1}.$$

This means that S_2 is a symmetric 2×2 matrix (with non- negative integral entries) whose two columns have sum equal to $[1,1]'$. This means that necessarily $S_2 = I$ or J.

Also since the two rows (respectively columns) of R_2 (resp., Q_2) are identical, this means that $R_2I = R_2J$ (resp., $IQ_2 = JQ_2$), and consequently, the ambiguity in the (2.2) entry in the block decomposition of A_2 only results in the ambiguity of the (2.2) entry in the block decomposition of any polynomial of A_2.

The fact that the S_i's exhibit the periodic behaviour ascribed to them is an easy consequence of the forms of the Q_i's and R_i's and is established by induction. Begin by noting that - since the hypergroup $\mathcal{G}_0$ is abelian - we have, in view of the form of the matrix for R_{β_2}, $A_{k+1} = A_2A_k - A_k - A_{k-1}$ for $1\le k < n-1$ (with the convention that $A_0 = 0$). Comparing (2.2) entries, we find that

$$S_{k+1} = R_2Q_k + S_2S_k - S_k - S_{k-1}, \text{ for } 1\le k< n-1. \qquad (2.3.6)$$

If the assertions about the matrices P_i, Q_i, R_i, S_i have been verified for $i\le k$, it is seen that R_2Q_k is the 2×2 matrix with 1's everywhere - *i.e.*, $R_2Q_k = I+J$. On the other hand, if $S_2 = I$, it follows from (2.3.6) that $S_3 = (I+J)+I\times I-I-I = J$, that $S_4 = (I+J)+I\times J-J-I = J$, and that $S_5 = (I+J)+I\times J-J-J = I$; while if $S_2 = J$, (2.3.6) implies that $S_3 = (I+J)+J\times J-J-I = I$, $S_4 = (I+J)+J\times I-I-J = J$, and that $S_5 = (I+J)+J\times J-J-I = I$. Hence, in either case, we see that $S_4 = J$ and $S_5 = I$. Then (2.3.6) implies that $S_6 = (I+J)+S_2\times I-I-J = S_2$, and it is clear from the recursion relations (2.3.6) that the S_i's behave in the periodic fashion indicated.

To proceed further we need to discuss two cases depending on the parity of n, where we defer the case of odd n to the next chapter.

The case of D_{4k+1}:
Suppose k is odd; then $n = 2(mod\ 4)$, so that $S_n = S_2$ and $S_{n-1} = I$. The last

column of R_{β_2} shows that $\beta_2\beta_n = \beta_{n-1} + 2\beta_n$. Since π_1 is an action, we should have $A_2A_n = A_{n-1} + 2A_n$, and in particular, $R_2Q_n + S_2S_n = S_{n-1} + 2S_n$; *i.e.*, $(I + J) + S_2.S_2 = I + 2S_2$, or $I + J = 2S_2$ which is not true.

Suppose k is even ; then $n = 0 (mod\ 4)$, and we have $S_n = J$ and $S_{n-1} = S_3$. As before, we should have $R_2Q_n + S_2S_n = S_{n-1} + 2S_n$; *i.e.*, $(I + J) + S_2J = S_3 + 2J$, or $I + J = 2J$ which is also not true.

Hence we have completed the proof that D_{4k+1} cannot arise as a principal graph for an inclusion of II_1 factors.

The case of β_{2n+1}:

Assume there exists a finite-index inclusion $N \subset M$ of II_1 factors with β_{2n+1} as the principal graph corresponding to the tower $\{N' \cup M\}$ of relative commutants of N in the basic construction.

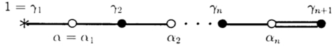
We first argue that the smallest entry of the Perron-Frobenius eigenvector of β_{n+1} occurs at the unique vertex of valency 1 in β_{n+1} for all $n > 1$. For this, set $d = \|A\|$, where A denotes the adjacency matrix of the above graph. It is clear that $d > 2$. Let us assume, only for this paragraph, that the vertices are numbered linearly, in increasing order from the unique vertex with valency 1 (which is asssigned label 1). It is fairly easy to see that, since d is the Perron-Frobenius eigevector of A, then the k^{th} coordinate of v is $P_{k-1}(d)$ if $k \leq n$, and $(\frac{2}{d})P_{n-1}(d)$ if $k = n + 1$, where $\{P_k : k = 0, 1, 2...\}$ are (variants of) the Chebyshev polynomials defined by $P_0(t) = 1, P_1(t) = t$ and $P_{k+1}(t) = tP_k(t) - P_{k-1}(t)$ for $k > 1$. Thus, in order to prove our assertion about the smallest co-ordinate of v occuring at vertex 1, we need to prove that $1 = P_0(d) < P_1(d) < ... < P_{n-1}(d) > \frac{d}{2}$. On the other hand only, we know that $P_k(2cos(z)) = \frac{sin(k+1)z}{sinz}$ for all complex numbers z. Since $d > 2$, we may pick a positive real number y such that $2cosh(y) = d$. Now put $z = iy$, and note that for any k, we have

$$\begin{aligned} P_k(d) - P_{k-1}(d) &= \frac{sin(k+1)z - sin\ kz}{sinz} \\ &= \frac{cos(k+\frac{1}{2})z}{cos\frac{1}{2}z} \\ &= \frac{cosh(k+\frac{1}{2})y}{cosh\ \frac{y}{2}} > 0 \end{aligned}$$

while the inequalities $n > 1$ and $y > 0$ imply that

$$\begin{aligned} P_{n-1}(d) &= P_{n-1}(2cos(iy)) \\ &= \frac{(sinh\ ny)}{(sinh\ y)} \\ &\geq \frac{(sinh\ 2y)}{(sinh\ y)} \\ &= 2coshy \\ &> coshy \\ &= \frac{d}{2}. \end{aligned}$$

Now we argue, as we did in the case of E_7 and D_{2n+1}, that the identity of the hypergroup $\mathcal{G}_0$ must occur where indicated. Assume that the other vertices are as labelled below.



As before, if we let R_α denote the matrix of the operator on right-multiplication by $\alpha = \alpha_1$ from $Z^+\mathcal{G}_0$ to $Z^+\mathcal{G}_1$ with respect to the ordered bases $\{\gamma_1, ..., \gamma_{n+1}\}$ and $\{\alpha_1, ..., \alpha_n\}$ respectively, we see that R_α is given by the $n \times (n+1)$ matrix

$$R_\alpha = \begin{bmatrix} 1 & 1 & 0 & . & . & 0 & 0 \\ 0 & 1 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & 0 & 0 & 1 & 0 \\ 0 & 0 & . & . & 0 & 1 & 2 \end{bmatrix}.$$

from which it may be deduced via the equation $\alpha\overline{\alpha} = 1 + \gamma_2$ that the matrix of the operator R_{γ_2} of $Z^+\mathcal{G}_0$ of right multiplication by γ_2 is given by the $(n+1) \times (n+1)$

matrix

$$R_{\gamma_2} = (R_\alpha)' R_\alpha - I = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 & 0 \\ 1 & 1 & 1 & 0 & . & . & 0 \\ 0 & 1 & 1 & 1 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & 0 & 1 & 1 & 2 \\ 0 & 0 & . & . & 0 & 2 & 3 \end{bmatrix}.$$

The second column of the above equation, for instance, says that $\gamma_2^2 = \gamma_1 + \gamma_2 + \gamma_3$ and hence that $\gamma_3 = \gamma_2^2 - \gamma_1 - \gamma_2$, whence $R_{\gamma_3} = R_{\gamma_2}^2 - R_{\gamma_1} - R_{\gamma_2}$. A similar analysis of columns 2 through n, yields the formulae

$$R_{\gamma_{i+1}} = \begin{cases} R_{\gamma_2} R_{\gamma_i} - R_{\gamma_{i-1}} - R_{\gamma_i} & \text{for } 1 < i < n \\ \frac{1}{2}(R_{\gamma_2} R_{\gamma_n} - R_{\gamma_{n-1}} - R_{\gamma_n}) & \text{for } i = n. \end{cases}$$

It is clear that the R_{γ_i}'s can now be recursively computed from the above equations. The desired contradiction stems from the fact that the matrix $R_{\gamma_{n+1}}$ turns out to have a non-integral entry. The computations are as follows:

For $k = 1, \ldots, n+1$, let v_k denote the $n \times 1$ column-vector defined by

$$v_k' = \left(0, 0, \ldots, 0, 2, 6, 18, 54, \ldots, (2 \cdot 3^m), \ldots, (2 \cdot 3^{k-2})\right),$$

where the first $(n - k + 1)$ entries are equal to 0 and $'$ denotes the transpose.

For $k = 1, \ldots, n+1$, let P_k denote the $n \times n$ matrix defined by

$$P_k(i,j) = \begin{cases} 0, & \text{if } i + j \leq k \text{ or } \mid i - j \mid \geq k \\ 1, & \text{if } k < i + j < 2n - k + 3 \text{ and } \mid i - j \mid < k \\ 4 \cdot 3^m, & \text{if } i + j = 2n - k + 3 + m \quad \text{with } m = 0, 1, 2, \ldots \end{cases}$$

(Thus for instance, if $n = 7$ and $k = 5$, P_5 would be the following matrix:

$$P_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 & 1 & 4 & 12 \\ 0 & 0 & 1 & 1 & 4 & 12 & 36 \end{bmatrix}.$$

Finally define the $(n-1) \times (n+1)$ matrices $A_1, ..., A_{n+1}$ by the block decompositions given by

$$A_k = \begin{bmatrix} P_k & v_k \\ v'_k & 3^{k-1} \end{bmatrix}.$$

It can be verified by a straightforward, if somewhat laborious, induction argument that

$$R_{\gamma_k} = \begin{cases} A_k, & \text{for } 1 \leq k \leq n \\ \frac{1}{2}A_{n+1} & \text{for } k = n+1 \ . \end{cases}$$

This would imply that $< \gamma_{n-1} \otimes \gamma_{n+1}, \gamma_{n+1} >= \frac{3^n}{2}$, which contradicts the requirement that these numbers should be non-negative integers.

We conclude finally that the graph β_{2n+1} could not have arisen as the principal graph of a finite index inclusion of II_1 factors. Q.E.D.

Remark 31 An almost identical argument also shows that the graph β^k_{2n+1} cannot arise as the principal graph of a finite-index inclusion of II_1 factors, where the only distinction between β^k_{2n-1} and β_{2n+1} $(=\beta^2_{2n+1})$ is that the unique double bond in the latter is substituted by k bonds in the former. Also these arguments fail in the case of β_{2n} - or β^k_{2n} for that matter - since it turns out that in this case the adjacency matrix arises as the matrix R_{γ_2} of right-multiplication by the second element of a unique hypergroup $\{1 = \gamma_1, \gamma_2, ..., \gamma_{2n}\}$ with $2n$ elements.

Chapter 3

M_2-graded Hypergroups

3.1 Introduction

In this chapter we define the notion of an M_2-graded hypergroup. This notion extends the notion of hypergroup and is motivated by the example of the 'paragroup' introduced by Ocneanu in the context of a pair of II_1 factors. In fact our axioms in the definition of M_2-graded hypergroup describe the algebraic structure possessed by the collection of all isomorphism classes of irreducible $N-N, N-M, M-N$ and $M-M$ bifinite bimodules with respect to taking tensor products and contragredients.

In this chapter, after defining M_2-graded hypergroups, we derive certain consequences of the definition. We extend the notion of a dimension function to M_2-graded hypergroups and prove that every finite irreducible M_2-graded hypergroup admits a unique dimension function; an easy consequence of this fact rules out several graphs from arising as principal graphs for inclusions of II_1 factors.

We also prove the non-occurence of the Coxeter graph D_{4k+3} as principal graph for an inclusion of II_1 factors in this chapter.

3.2 M_2-graded hypergroups

The notion of an M_2-graded hypergroup is a generalization of the notion of a hypergroup. Whereas the definition of an abstract hypergroup was motivated by the hypergroup of II_1 factors defined in the first chapter, the notion of M_2-graded hypergroup defined first in [V] was motivated by the paragroup (cf., [O1]) of the inclusion of a pair of II_1 factors.

Definition 32 A discrete M_2-graded hypergroup is a set $\mathcal{G} = \bigsqcup_{1\le i,j\le 2} \mathcal{G}_{ij}$ (where $\bigsqcup$ denotes disjoint union) with a map from $\mathcal{G}\times\mathcal{G}$ to $Z^+\mathcal{G}$ denoted by

$$(\alpha,\beta)\mapsto \alpha\otimes\beta = \sum_{\gamma\in\mathcal{G}} <\alpha\otimes\beta,\gamma>\gamma,$$

satisfying the following conditions:

(i) $<\alpha\otimes\beta,\gamma>\neq 0$ for only finitely many γ, for all $\alpha,\beta\in\mathcal{G}$;

(ii) $\mathcal{G}_{ij}\otimes\mathcal{G}_{kl}\subseteq\delta_{jk}Z^+\mathcal{G}_{il}$ and $Z\mathcal{G}$ is an (associative) ring;

(iii) there exist $1_{\mathcal{G}_{ii}}\in\mathcal{G}_{ii}$, $i=1,2$, such that,

$$\begin{aligned} 1_{\mathcal{G}_{ii}}\otimes\alpha &= \alpha \text{ for all } \alpha\in\mathcal{G}_{ij},\ 1\le i,j\le 2,\\ \alpha\otimes 1_{\mathcal{G}_{ii}} &= \alpha \text{ for all } \alpha\in\mathcal{G}_{ji},\ 1\le i,j\le 2; \end{aligned}$$

(iv) there exists a self map of $\mathcal{G}$ denoted by $\alpha\mapsto\overline{\alpha}$ such that,

$$\alpha\in\mathcal{G}_{ij} \text{ implies } \overline{\alpha}\in\mathcal{G}_{ji} (\text{ i.e., } \overline{\mathcal{G}_{ij}}\subseteq\mathcal{G}_{ji})$$

and

$$<\alpha\otimes\beta,\gamma> = <\overline{\alpha}\otimes\gamma,\beta>, \text{ for all } \alpha,\ \beta,\gamma\in\mathcal{G}.$$

Remark 33 If $\mathcal{G}_{12}=\mathcal{G}_{21}=\mathcal{G}_{22}=\emptyset$, an M_2-graded hypergroup can be thought of as a hypergroup as in [S1].

Example 34 **(1)** The above definition is motivated by the following example: let $\mathcal{G}_{11}=\mathcal{G}(N)$, $\mathcal{G}_{12}=\mathcal{G}(N,M)$, $\mathcal{G}_{21}=\mathcal{G}(M,N)$ and $\mathcal{G}_{22}=\mathcal{G}(M)$, where N and M are II_1 factors and $\mathcal{G}(P,Q)$ represents the collection of isomorphism classes of irreducible left P finite, right Q finite $P-Q$ bimodules with the convention that $\mathcal{G}(P,P) = \mathcal{G}(P)$. Then $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$ are hypergroups, $\overline{\mathcal{G}_{12}} = \mathcal{G}_{21}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{22}^{op}$ act on $\mathcal{G}_{12}$ (equivalently $\mathcal{G}_{22}$ and $\mathcal{G}_{11}^{op}$ act on $\mathcal{G}_{21}$) in the sense of Definition 18.

(2) Clearly Z_2-graded hypergroups yield M_2-graded hypergroups. If $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ is a Z_2-graded hypergroup, then $\mathcal{G}$ is an M_2-graded hypergroup with $\mathcal{G}_{11} = \mathcal{G}_{22} = \mathcal{G}_0$ and $\mathcal{G}_{12} = \mathcal{G}_{21} = \mathcal{G}_1$. In particular if $\mathcal{G}$ is a hypergroup, then $\mathcal{G} \times Z_2$ is a Z_2-graded hypergroup, hence an M_2-graded hypergroup.

Remark 35 **(a)** (i) In Example 20(ii) $\mathcal{G}(N)$ and $\mathcal{G}(M)$ are hypergroups and they act on $\mathcal{G}(N,M)$ and $\mathcal{G}(M,N)$, respectively. (ii)It is not hard to see that being given an M_2-graded hypergroup is equivalent to being given hypergroups $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$ with commuting actions π and ρ of $\mathcal{G}_{11}$ and $\mathcal{G}_{22}^{op}$ respectively on $\mathcal{G}_{12}$ satisfying, for all α_{12}, β_{12}, γ_{12}, δ_{12} in $\mathcal{G}_{12}$,

$$\sum_{\kappa_{11}\in\mathcal{G}_{11}} \pi(\kappa_{11})(\alpha_{12},\beta_{12})\pi(\kappa_{11})(\gamma_{12},\delta_{12}) = \sum_{\kappa_{22}\in\mathcal{G}_{22}} \rho(\kappa_{22}^0)(\alpha_{12},\gamma_{12})\rho(\kappa_{22}^0)(\beta_{12},\delta_{12}).$$

The above condition stems from the associative axiom of M_2-graded hypergroup and can be easily proved as in Proposition 23.

(b) Objects with similar structure, with the name "fusion algebra", were discussed in [O3] and in [EK]. The definition that we give seems to be the most natural one in the context of bifinite bimodules over II_1 factor - subfactor pairs.

The hypergroup algebra:

Given an M_2-graded hypergroup $\mathcal{G}$ we can consider $\mathbb{C}\mathcal{G}$, the class of all finitely supported complex valued functions on $\mathcal{G}$. The space $\mathbb{C}\mathcal{G}$ has a canonical basis $\{f_\alpha, \alpha \in \mathcal{G}\}$, given by $f_\alpha(\beta) = \delta_{\alpha\beta}$; and then $f = \sum_{\alpha\in\mathcal{G}} f(\alpha)f_\alpha$ for all $f \in \mathbb{C}\mathcal{G}$. We can make $\mathbb{C}\mathcal{G}$ an algebra by defining a 'convolution product' on $\mathbb{C}\mathcal{G}$, given by $f_\alpha * f_\beta = \sum_{\gamma\in\mathcal{G}} <\alpha\otimes\beta,\gamma> f_\gamma$ (i.e., $f_\alpha * f_\beta(\gamma) = <\alpha\otimes\beta,\gamma>$). More generally, for f, g in $\mathbb{C}\mathcal{G}$,

$$f * g = \sum_{\alpha,\beta,\gamma\in\mathcal{G}} f(\alpha)g(\beta) <\alpha\otimes\beta,\gamma> f_\gamma,$$

where $f = \sum_{\alpha\in\mathcal{G}} f(\alpha)f_\alpha$ and $g = \sum_{\beta\in\mathcal{G}} g(\beta)f_\beta$. The associativity of $Z\mathcal{G}$ ensures that $\mathbb{C}\mathcal{G}$ is an associative algebra with respect to the above product. We can make $\mathbb{C}\mathcal{G}$ an inner-product space, which we denote by V, by demanding that $\{f_\alpha; \alpha \in \mathcal{G}\}$ is a (necessarily maximal) orthonormal set of vectors.

Now we can consider the left regular representation of $\mathbb{C}\mathcal{G}$. For f in $\mathbb{C}\mathcal{G}$, define the left multiplication operator L_f on V by $L_f\, g = f * g$. Clearly $f \mapsto L_f$ is an algebra homomorphism of $\mathbb{C}\mathcal{G}$ into $L(V)$ which is unital - $f_{1_{\mathcal{G}_{11}}} + f_{1_{\mathcal{G}_{22}}}$ is the identity of $\mathbb{C}\mathcal{G}$ and $L_{f_{1_{\mathcal{G}_{11}}}} + L_{f_{1_{\mathcal{G}_{22}}}}$ is the identity of $L(V)$ - and faithful. Further if we define $f^*(\alpha) = \overline{f(\overline{\alpha})}$ - the complex conjugate of $f(\overline{\alpha})$ - then the contragredient axiom ensures that $< L_f\, g, h > = < g, L_{f^*} h >$ for f, g, h in $\mathbb{C}\mathcal{G}$.

Definition 36 An M_2-graded hypergroup is said to be irreducible if the (commuting) actions of $\mathcal{G}_{11}$ and $\mathcal{G}_{22}^{op}$ on $\mathcal{G}_{12}$ are both irreducible.

Remark 37 It follows from Remark 25(ii) that the M_2-graded hypergroup obtained from the paragroup of a factor - subfactor pair is irreducible.

Proposition 38 *The equation* $\tau(f) = f(1_{\mathcal{G}_{11}}) + f(1_{\mathcal{G}_{22}})$ *defines a faithful positive trace on* $\mathbb{C}\mathcal{G}$ *for any* M_2*-graded hypergroup* $\mathcal{G}$.

Proof:

Clearly,

$$\begin{aligned}
\tau(f) &= < L_f f_{1_{\mathcal{G}_{11}}}, f_{1_{\mathcal{G}_{11}}} > + < L_f f_{1_{\mathcal{G}_{22}}}, f_{1_{\mathcal{G}_{22}}} >, \\
\tau(f^* * f) &= < L_{f^* * f} f_{1_{\mathcal{G}_{11}}}, f_{1_{\mathcal{G}_{11}}} > + < L_{f^* * f} f_{1_{\mathcal{G}_{22}}}, f_{1_{\mathcal{G}_{22}}} > \\
&= < L_{f^*} L_f f_{1_{\mathcal{G}_{11}}}, f_{1_{\mathcal{G}_{11}}} > + < L_{f^*} L_f f_{1_{\mathcal{G}_{22}}}, f_{1_{\mathcal{G}_{22}}} > \\
&= < L_f f_{1_{\mathcal{G}_{11}}}, L_f f_{1_{\mathcal{G}_{11}}} > + < L_f f_{1_{\mathcal{G}_{22}}}, L_f f_{1_{\mathcal{G}_{22}}} > \\
&= < f * f_{1_{\mathcal{G}_{11}}}, f * f_{1_{\mathcal{G}_{11}}} > + < f * f_{1_{\mathcal{G}_{22}}}, f * f_{1_{\mathcal{G}_{22}}} > \\
&= < f * (f_{1_{\mathcal{G}_{11}}} + f_{1_{\mathcal{G}_{22}}}), f * (f_{1_{\mathcal{G}_{11}}} + f_{1_{\mathcal{G}_{22}}}) > \\
&= < f, f >.
\end{aligned}$$

(since $f * f_{1_{\mathcal{G}_{ii}}} \in \bigoplus_{j=1}^{2} Z^{\cdot}\mathcal{G}_{ji}$ implies $< f * f_{1_{\mathcal{G}_{11}}}, f * f_{1_{\mathcal{G}_{22}}} > = 0$).
Therefore τ is faithful and positive.

$$\tau(f * g) = < f * g, f_{1_{\mathcal{G}_{11}}} > + < f * g, f_{1_{\mathcal{G}_{22}}} >$$

$$
\begin{aligned}
&= < g, f^* * f_{1_{\mathcal{G}_{11}}} > + < g, f^* * f_{1_{\mathcal{G}_{22}}} > \\
&= < g, f^* * (f_{1_{\mathcal{G}_{11}}} + f_{1_{\mathcal{G}_{22}}}) > \\
&= < g, f^* > \\
&= \sum_{\alpha \in \mathcal{G}} g(\alpha) f(\overline{\alpha}) \\
&= \sum_{\beta \in \mathcal{G}} f(\beta) g(\overline{\beta}), \\
&\quad (\alpha \mapsto \overline{\alpha} \text{ being an involution of } \mathcal{G}, \text{ as shown below}) \\
&= \tau(g * f).
\end{aligned}
$$

Hence τ is a trace. Q.E.D.

Now we will discuss some consequences of the definition of an M_2-graded hypergroup.

Proposition 39 *Let $\mathcal{G} = \bigsqcup_{1 \le i,j \le 2} \mathcal{G}_{ij}$ be an M_2-graded hypergroup. Then,*

(i) *$\mathcal{G}_{ii}$ are hypergroups, $i = 1, 2$;*

(ii) *$1_{\mathcal{G}_{ii}}$ are unique, $i = 1, 2$;*

(iii) *$\alpha \mapsto \overline{\alpha}$ is an involution in $\mathcal{G}$;*

(iv) *$< \alpha \otimes \beta, \gamma > = < \gamma \otimes \overline{\beta}, \alpha > = < \overline{\beta} \otimes \overline{\alpha}, \overline{\gamma} >$ (i.e., $\overline{\alpha \otimes \beta} = \overline{\beta} \otimes \overline{\alpha}$).*

Proof:

(i) By axiom (ii) in the definition, we have, $\mathcal{G}_{ii} \otimes \mathcal{G}_{ii} \subseteq Z^+\mathcal{G}_{ii}$.
Hence we have a map : $\mathcal{G}_{ii} \otimes \mathcal{G}_{ii} \otimes \mathcal{G}_{ii} \mapsto Z^+$ given by $(\alpha, \beta, \gamma) \mapsto < \alpha \otimes \beta, \gamma >$.
The identity of $\mathcal{G}_{ii}$ is $1_{\mathcal{G}_{ii}}$ by axiom (iii) and $\mathcal{G}_{ii}$ is closed under contragredients by axiom (iv). The associativity of $\mathcal{G}_{ii}$ follows from axiom (ii).

(ii) The uniqueness of $1_{\mathcal{G}_{ii}}$ follows from the fact that the identity is unique in a hypergroup.

(iii) We have

$$
\begin{aligned}
1 &= < \alpha_{ij} \otimes 1_{\mathcal{G}_{jj}}, \alpha_{ij} > \\
&= < \overline{\alpha_{ij}} \otimes \alpha_{ij}, 1_{\mathcal{G}_{jj}} >, \text{by the contragredient axiom}
\end{aligned}
$$

$$
\begin{aligned}
&= \; < \overline{\overline{\alpha_{ij}}} \otimes 1_{\mathcal{G}_{jj}}, \alpha_{ij} >, \\
&= \; \delta_{\overline{\overline{\alpha_{ij}}}, \alpha_{ij}}, \text{ since } \overline{\overline{\alpha_{ij}}} \in \mathcal{G}_{ij}. \\
\text{Hence } \overline{\overline{\alpha_{ij}}} &= \; \alpha_{ij}.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
< \alpha \otimes \beta, \gamma > &= < f_\alpha * f_\beta, f_\gamma > \\
&= \tau(f_{\overline{\gamma}} * f_\alpha * f_\beta) \\
&= \tau(f_\beta * f_{\overline{\gamma}} * f_\alpha) \\
&= < f_{\overline{\gamma}} * f_\alpha, f_{\overline{\beta}} > \\
&= < \overline{\gamma} \otimes \alpha, \overline{\beta} > \\
&= < \gamma \otimes \overline{\beta}, \alpha > .
\end{aligned}
$$

Note that $< \alpha \otimes \beta, \gamma > = \tau(f_\alpha * f_\beta * f_{\overline{\gamma}})$.
If g is such that $g(1_{\mathcal{G}_{11}}) + g(1_{\mathcal{G}_{22}}) \in \boldsymbol{R}$, then $\tau(g) = \tau(g^*)$, hence

$$
\begin{aligned}
< \alpha \otimes \beta, \gamma > &= \tau(f_\alpha * f_\beta * f_{\overline{\gamma}}) \\
&= \tau(f_\gamma * f_{\overline{\beta}} * f_{\overline{\alpha}}). \\
&= < \overline{\beta} \otimes \overline{\alpha}, \overline{\gamma} > .
\end{aligned}
$$

Q.E.D.

Definition 40 A dimension function for an M_2-graded hypergroup $\mathcal{G}$ is a map $d : \mathcal{G} \mapsto (0, \infty)$, satisfying the following conditions:

(i) $d(\alpha) = d(\overline{\alpha})$, for all α in $\mathcal{G}$;

(ii) $d(\alpha_{ij})d(\beta_{jk}) = \sum_{\gamma_{ik} \in \mathcal{G}_{ik}} < \alpha_{ij} \otimes \beta_{jk}, \gamma_{ik} > d(\gamma_{ik})$, for all $i, j, k = 1, 2$.

Remark 41 From the above definition it follows that, restricted to the hypergroups $\mathcal{G}_{ii}$, $i = 1, 2$, the function d is a dimension function for $\mathcal{G}_{ii}$.

Theorem 42 *Every finite irreducible M_2-graded hypergroup admits a unique dimension function.*

Proof:

Let $\mathcal{G} = \bigsqcup_{1 \leq i,j \leq 2} \mathcal{G}_{ij}$ denote a finite irreducible M_2-graded hypergroup.

A linear operator T on V - the innerproduct space $\mathbb{C}\mathcal{G}$ with orthonormal basis $\{ f_\alpha : \alpha \in \mathcal{G} \}$ - is identified with the matrix $(t_{\alpha\beta})$, where $t_{\alpha\beta} = < Tf_\beta, f_\alpha >$. Thus L_α (resp., R_α) denotes the matrix whose rows and columns are indexed by $\mathcal{G}$ and whose entry in the position (β, γ) is $< \alpha \otimes \gamma, \beta >$ (resp., $< \gamma \otimes \alpha, \beta >$).

Let π^i_j denote the action of $\mathcal{G}_{ii}$ on $\mathcal{G}_{ij}$, for $i, j = 1, 2$.

Existence:

On $\mathcal{G}_{ii}$ define d to be equal to the dimension function of $\mathcal{G}_{ii}$, for $i = 1, 2$.

Recall, from Proposition 26, that if

$$e_i = \frac{1}{\sum_{\beta_{ii} \in \mathcal{G}_{ii}} d(\beta_{ii})^2} \sum_{\alpha_{ii} \in \mathcal{G}_{ii}} d(\alpha_{ii})\alpha_{ii}, \ i = 1, 2,$$

then $e_i \in \mathbb{R}^+\mathcal{G}_{ii}$, $e_i^2 = e_i = e_i^*$ (the adjoint of e_i in $\mathbb{C}\mathcal{G}_{ii}$), $d(e_i) = 1$ (d on $\mathcal{G}_{ii}$ extends by linearity to a function from $\mathbb{C}\mathcal{G}_{ii}$ to $\mathbb{C}$ which is clearly multiplicative), $\alpha_{ii}e_i = e_i\alpha_{ii} = d(\alpha_{ii})e_i$, and $\pi^i_j(e_i)$ is a projection in $End(\mathbb{C}\mathcal{G}_{ij})$.

Since $\mathcal{G}$ is irreducible $\pi^i_j, i, j = 1, 2$, are irreducible so that the matrices corresponding to $\pi^i_j(e_i)$ have strictly positive entries. Let V_{ij} be the Perron-Frobenius eigenvector of $\pi^i_j(e_i)$. Thus V_{ij} is a (column) vector, in the range of $\pi^i_j(e_i)$, with strictly positive entries, which is unique upto scaling by a positive constant. We assume that V_{ii} is normalised such that $V_{ii}(1_{\mathcal{G}_{ii}}) = 1$, so that it is the dimension function vector for $\mathcal{G}_{ii}$. For $i \neq j$, we shall choose a suitable normalization later. Thus $\pi^i_j(e_i)V_{ij} = V_{ij}, \ i, j = 1, 2$ (the Perron-Frobenius eigenvalue of $\pi^i_j(e_i)$ is 1, since $\pi^i_j(e_i)$ is a projection).

Since V_{ij} belongs to $\mathbb{R}^+\mathcal{G}_{ij}$ it is of the form

$$V_{ij} = \sum_{\alpha_{ij} \in \mathcal{G}_{ij}} C(\alpha_{ij})\alpha_{ij}, \ \text{for } i, j = 1, 2,$$

where $C(\alpha_{ij})$ are nonzero positive constants.

Consider $\rho^i_j(\alpha_{ii})$ belonging to $End(\mathbb{C}\mathcal{G}_{ji})$ given by

$$\rho^i_j(\alpha_{ii})(\gamma_{ji}, \kappa_{ji}) = < \kappa_{ji} \otimes \alpha_{ii}, \gamma_{ji} >, \ i, j = 1, 2.$$

Since in $\mathcal{G}$, $< \kappa_{ji} \otimes \alpha_{ii}, \gamma_{ji} > = < \overline{\alpha_{ii}} \otimes \overline{\kappa_{ji}}, \overline{\gamma_{ji}} >$.

Proof:

Let $\mathcal{G} = \bigsqcup_{1 \le i,j \le 2} \mathcal{G}_{ij}$ denote a finite irreducible M_2-graded hypergroup.

A linear operator T on V – the innerproduct space $\mathbb{C}\mathcal{G}$ with orthonormal basis $\{ f_\alpha : \alpha \in \mathcal{G} \}$ - is identified with the matrix $(t_{\alpha\beta})$, where $t_{\alpha\beta} = < Tf_\beta, f_\alpha >$. Thus L_α (resp., R_α) denotes the matrix whose rows and columns are indexed by $\mathcal{G}$ and whose entry in the position (β, γ) is $< \alpha \otimes \gamma, \beta >$ (resp., $< \gamma \otimes \alpha, \beta >$).

Let π^i_j denote the action of $\mathcal{G}_{ii}$ on $\mathcal{G}_{ij}$, for $i, j = 1, 2$.

Existence:

On $\mathcal{G}_{ii}$ define d to be equal to the dimension function of $\mathcal{G}_{ii}$, for $i = 1, 2$.

Recall, from Proposition 26, that if

$$e_i = \frac{1}{\sum_{\beta_{ii} \in \mathcal{G}_{ii}} d(\beta_{ii})^2} \sum_{\alpha_{ii} \in \mathcal{G}_{ii}} d(\alpha_{ii})\alpha_{ii}, \ i = 1, 2,$$

then $e_i \in \boldsymbol{R}^+\mathcal{G}_{ii}$, $e_i^2 = e_i = e_i^*$ (the adjoint of e_i in $\mathbb{C}\mathcal{G}_{ii}$), $d(e_i) = 1$ (d on $\mathcal{G}_{ii}$ extends by linearity to a function from $\mathbb{C}\mathcal{G}_{ii}$ to $\mathbb{C}$ which is clearly multiplicative), $\alpha_{ii}e_i = e_i\alpha_{ii} = d(\alpha_{ii})e_i$, and $\pi^i_j(e_i)$ is a projection in $End(\mathbb{C}\mathcal{G}_{ij})$.

Since $\mathcal{G}$ is irreducible $\pi^i_j, i, j = 1, 2$, are irreducible so that the matrices corresponding to $\pi^i_j(e_i)$ have strictly positive entries. Let V_{ij} be the Perron-Frobenius eigenvector of $\pi^i_j(e_i)$. Thus V_{ij} is a (column) vector, in the range of $\pi^i_j(e_i)$, with strictly positive entries, which is unique upto scaling by a positive constant. We assume that V_{ii} is normalised such that $V_{ii}(1_{\mathcal{G}_{ii}}) = 1$, so that it is the dimension function vector for $\mathcal{G}_{ii}$. For $i \neq j$, we shall choose a suitable normalization later. Thus $\pi^i_j(e_i)V_{ij} = V_{ij}, \ i, j = 1, 2$ (the Perron-Frobenius eigenvalue of $\pi^i_j(e_i)$ is 1, since $\pi^i_j(e_i)$ is a projection).

Since V_{ij} belongs to $\boldsymbol{R}^+\mathcal{G}_{ij}$ it is of the form

$$V_{ij} = \sum_{\alpha_{ij} \in \mathcal{G}_{ij}} C(\alpha_{ij})\alpha_{ij}, \text{ for } i, j = 1, 2,$$

where $C(\alpha_{ij})$ are nonzero positive constants.

Consider $\rho^i_j(\alpha_{ii})$ belonging to $End(\mathbb{C}\mathcal{G}_{ji})$ given by

$$\rho^i_j(\alpha_{ii})(\gamma_{ji}, \kappa_{ji}) = < \kappa_{ji} \otimes \alpha_{ii}, \gamma_{ji} >, i, j = 1, 2.$$

Since in $\mathcal{G}$, $< \kappa_{ji} \otimes \alpha_{ii}, \gamma_{ji} > = < \overline{\alpha_{ii}} \otimes \overline{\kappa_{ji}}, \overline{\gamma_{ji}} >$, we have,

$$\rho_j^i(\alpha_{ii})(\gamma_{ji}, \kappa_{ji}) = \pi_j^i(\overline{\alpha_{ii}})(\overline{\gamma_{ji}}, \overline{\kappa_{ji}}),$$

so that the matrix corresponding to $\rho_j^i(e_i)$ has strictly positive entries.

Clearly, $\pi_j^i(e_i)$ commutes with $\rho_i^j(e_j)$, for i, j = 1, 2. Therefore V_{ij} is an eigenvector, hence the Perron-Frobenius eigenvector of $\rho_i^j(e_j)$. But $\overline{V_{ji}}$ is the Perron-Frobenius eigenvector of $\rho_i^j(e_j)$. Hence we may assume that $\overline{V_{ji}} = V_{ij}$, and therefore $C(\overline{\alpha_{ij}}) = C(\alpha_{ij})$.

Assertion:

(i) $V_{ij} \odot \alpha_{ji} = \varphi(\alpha_{ji})V_{ii}$, where $\varphi(\alpha_{ji})$ is a positive constant depending on α_{ji};

(ii) $V_{ii} \odot \alpha_{ij} = \upsilon(\alpha_{ij})V_{ij}$, where $\upsilon(\alpha_{ij})$ is a positive constant depending on α_{ij};

(iii) $\varphi(\alpha_{ij}) = C(\overline{\alpha_{ij}})$;

(iv) $\frac{\upsilon(\alpha_{ij})}{C(\alpha_{ij})} = k_{ij}$, a constant depending on (i, j), for $i, j = 1, 2$.

Proof of the Assertion:

Note that for $i = j$ the above assertions are about the hypergroups $\mathcal{G}_{ii}$ and hence $\varphi = \upsilon = C = d$ and consequently $k_{ii} = 1, i = 1, 2$.

Therefore in the remaining part of the proof of the assertion we assume that $i \neq j$.

(i) In $\mathbb{C}\mathcal{G}$ for $\alpha_{ij} \in \mathcal{G}_{ij}$, for $i, j = 1, 2$,

$$(\alpha_{ii} \odot V_{ij}) \odot \alpha_{ji} = \alpha_{ii} \odot (V_{ij} \odot \alpha_{ji})$$

i.e.,

$$R_{\alpha_{ji}}(\pi_j^i(\alpha_{ii})V_{ij}) = \pi_i^i(\alpha_{ii})(R_{\alpha_{ji}}V_{ij}),$$

Multiplying both sides by $d(\alpha_{ii})$ and summing over α_{ii} in $\mathcal{G}_{ii}$, we get

$$R_{\alpha_{ji}}(\pi_j^i(e_i)V_{ij}) = \pi_i^i(e_i)(R_{\alpha_{ji}}V_{ij})$$
$$\text{i.e., } R_{\alpha_{ji}}V_{ij} = \pi_i^i(e_i)(R_{\alpha_{ji}}V_{ij})$$

since $\sum_{\alpha_{ii} \in \mathcal{G}_{ii}} d(\alpha_{ii})\alpha_{ii} = (\sum_{\alpha_{ii} \in \mathcal{G}_{ii}} d(\alpha_{ii})^2)e_i$, and $\pi_i^i(e_i)V_{ii} = V_{ii}$. Therefore $R_{\alpha_{ji}}V_{ij}$ is in the range of $\pi_i^i(e_i)$. Further, $R_{\alpha_{ji}}$ is a matrix with non-negative entries, with non-zero rows and columns (since, π^i_j is irreducible) and V_{ij} has strictly positive entries. Hence $R_{\alpha_{ji}}V_{ij}$ is a positive scalar multiple of the Perron-Frobenius eigenvector of $\pi^i_i(e_i)$ which is V_{ii}.

(ii) The proof of (ii) is exactly like the proof of (i), except that one starts with the equation

$$(\alpha_{ii} \otimes V_{ii}) \otimes \alpha_{ij} = \alpha_{ii} \otimes (V_{ii} \otimes \alpha_{ij}).$$

(iii) We have

$$\begin{aligned} V_{ij} &= \sum_{\alpha_{ij} \in \mathcal{G}_{ij}} C(\alpha_{ij})\alpha_{ij} \\ V_{ij} \otimes \alpha_{ji} &= \varphi(\alpha_{ji})V_{ii} \end{aligned}$$

Since $C(1_{\mathcal{G}_{ii}}) = 1$, we see that

$$\begin{aligned} \varphi(\alpha_{ji}) &= < V_{ij} \otimes \alpha_{ji}, 1_{\mathcal{G}_{ii}} > \\ &= < V_{ij}, 1_{\mathcal{G}_{ii}} \otimes \overline{\alpha_{ji}} > \\ &= < V_{ij}, \overline{\alpha_{ji}} > \\ &= C(\overline{\alpha_{ji}}). \end{aligned}$$

(iv) We have

$$\begin{aligned} (\alpha_{ij} \otimes V_{ij}) \otimes \beta_{ij} &= \alpha_{ji} \otimes (V_{ji} \otimes \beta_{ij}) \\ i.e., \overline{V_{ji} \otimes \overline{\alpha_{ij}}} \otimes \beta_{ij} &= \alpha_{ij} \otimes \phi(\beta_{ij})V_{jj} \\ i.e., \phi(\overline{\alpha_{ij}})V_{ii} \otimes \beta_{ij} &= \phi(\beta_{ij})\overline{V_{jj} \otimes \overline{\alpha_{ij}}} \\ i.e., C(\alpha_{ij})\psi(\beta_{ij})V_{ij} &= C(\overline{\beta_{ij}})\psi(\overline{\alpha_{ij}})V_{ij} \\ i.e., \frac{\psi(\beta_{ij})}{C(\beta_{ij})} &= \frac{\psi(\overline{\alpha_{ij}})}{C(\overline{\alpha_{ij}})}. \end{aligned}$$

Hence,

$$\frac{\psi(\alpha_{ij})}{C(\alpha_{ij})} = k_{ij}, \text{ a strictly positive constant for all } i, j = 1, 2,$$

where $k_{11} = k_{22} = 1$ and $k_{12} = k_{21}$.

Define

$$d(\alpha_{ij}) = \sqrt{k_{ij}}C(\alpha_{ij}).$$

We will verify that the d defined above is a dimension for $\mathcal{G}$. To begin with, note that since $\overline{V_{12}} = V_{21}$ and $k_{12} = k_{21}$, we have $d(\alpha) = d(\overline{\alpha})$ for all α in $\mathcal{G}_{12}$.

We need to check that

$$d(\alpha_{ij})d(\beta_{jk}) = \sum_{\gamma_{ik}\in\mathcal{G}_{ik}} < \alpha_{ij} \otimes \beta_{jk}, \gamma_{ik} > d(\gamma_{ik}), \text{ for all } i,j,k = 1,2.$$

Since $d|\mathcal{G}_{ii}$ is defined to be the dimension function for the hypergroup $\mathcal{G}_{ii}$, the above equation is satisfied for $i = j = k$. The remaining cases are $i = j \neq k$ and $i = k \neq j$ as i, j, k belong to $\{1, 2\}$.

Case(i): $i = j \neq k$.

$$\begin{aligned}
V_{ik} &= \sum_{\gamma_{ik}\in\mathcal{G}_{ik}} C(\gamma_{ik})\gamma_{ik} \\
\text{therefore } V_{ik} \otimes \beta_{ki} &= \sum_{\gamma_{ik}\in\mathcal{G}_{ik}} C(\gamma_{ik})\gamma_{ik} \otimes \beta_{ki} \\
i.e., \varphi(\beta_{ki})V_{ii} &= \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} \sum_{\gamma_{ik}\in\mathcal{G}_{ik}} C(\gamma_{ik}) < \gamma_{ik} \otimes \beta_{ki}, \alpha_{ii} > \alpha_{ii} \\
C(\overline{\beta_{ki}}) \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})\alpha_{ii} &= \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} \sum_{\gamma_{ik}\in\mathcal{G}_{ik}} C(\gamma_{ik}) < \gamma_{ik} \otimes \beta_{ki}, \alpha_{ii} > \alpha_{ii} \\
\text{therefore } d(\alpha_{ii})C(\overline{\beta_{ki}}) &= \sum_{\gamma_{ik}\in\mathcal{G}_{ik}} < \gamma_{ik} \otimes \beta_{ki}, \alpha_{ii} > C(\gamma_{ik}) \\
i.e., d(\alpha_{ii})\frac{1}{\sqrt{k_{ik}}}d(\overline{\beta_{ki}}) &= \sum_{\gamma_{ik}\in\mathcal{G}_{ik}} < \alpha_{ii} \otimes \overline{\beta_{ki}}, \gamma_{ik} > \frac{1}{\sqrt{k_{ik}}}d(\gamma_{ik}) \\
i.e., d(\alpha_{ii})d(\overline{\beta_{ki}}) &= \sum_{\gamma_{ik}\in\mathcal{G}_{ik}} < \alpha_{ii} \otimes \overline{\beta_{ki}}, \gamma_{ik} > d(\gamma_{ik}).
\end{aligned}$$

Case(ii): $i = k \neq j$.

$$\begin{aligned}
V_{ii} &= \sum_{\gamma_{ii}\in\mathcal{G}_{ii}} d(\gamma_{ii})\gamma_{ii} \\
V_{ii} \otimes \beta_{ij} &= \sum_{\gamma_{ii}\in\mathcal{G}_{ii}} d(\gamma_{ii})\gamma_{ii} \otimes \beta_{ij} \\
i.e., \psi(\beta_{ij})V_{ij} &= \sum_{\gamma_{ii}\in\mathcal{G}_{ii}} d(\gamma_{ii})\gamma_{ii} \otimes \beta_{ij} \text{ by the assertion(ii)} \\
i.e., k_{ij}C(\beta_{ij}) \sum_{\alpha_{ij}\in\mathcal{G}_{ij}} C(\alpha_{ij})\alpha_{ij} &= \sum_{\alpha_{ij}\in\mathcal{G}_{ij}} \sum_{\gamma_{ii}\in\mathcal{G}_{ii}} d(\gamma_{ii}) < \gamma_{ii} \otimes \beta_{ij}, \alpha_{ij} > \alpha_{ij}
\end{aligned}$$

$$\begin{aligned} i.e., k_{ij}C(\alpha_{ij})C(\beta_{ij}) &= \sum_{\gamma_{ii}\in\mathcal{G}_{ii}} <\alpha_{ij}\otimes\overline{\beta_{ij}},\gamma_{ii}> d(\gamma_{ii}) \\ \text{So } d(\alpha_{ij})d(\overline{\beta_{ij}}) = d(\alpha_{ij})d(\beta_{ij}) &= \sum_{\gamma_{ii}\in\mathcal{G}_{ii}} <\alpha_{ij}\otimes\overline{\beta_{ij}},\gamma_{ii}> d(\gamma_{ii}). \end{aligned}$$

Uniqueness:

Let $\tilde{d}$ be a dimension function for $\mathcal{G}$. By the definition of a dimension function, restricted to the hypergroups $\mathcal{G}_{ii}$, $\tilde{d}$ is a dimension function for $\mathcal{G}_{ii}$ and hence equal to the unique dimension function for $\mathcal{G}_{ii}$ which is equal to d, so that d and $\tilde{d}$ agree on $\mathcal{G}_{ii}, i = 1, 2$.

So to prove the uniqueness of the dimension function it is enough to show that d and $\tilde{d}$ agree on $\mathcal{G}_{ij}, i, j \in \{1, 2\}$ and $i \neq j$.

Let $\tilde{V}_{ij}$ be the element of $\mathbb{C}\mathcal{G}_{ij}$ whose α^{th} coordinate is $\tilde{d}(\alpha_{ij})$, $i \neq j$. Thus $\tilde{V}_{ij}$ is a vector with strictly positive coordinates.

Assertion:

$\tilde{V}_{ij}$ is an eigenvector of $\pi^i_j(e_i)$.

Proof of the Assertion:

$$\begin{aligned} (\pi^i_j(e_i)\tilde{V}_{ij})(\beta_{ij}) &= \sum_{\gamma_{ij}\in\mathcal{G}_{ij}} (\pi^i_j(e_i))(\beta_{ij},\gamma_{ij})\tilde{V}_{ij}(\gamma_{ij}) \\ &= \sum_{\gamma_{ij}\in\mathcal{G}_{ij}} \frac{1}{\sum_{\kappa_{ii}\in\mathcal{G}_{ii}} d(\kappa_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})\pi^i_j(\alpha_{ii})(\beta_{ij},\gamma_{ij})\tilde{d}(\gamma_{ij}) \\ &= \frac{1}{\sum_{\kappa_{ii}\in\mathcal{G}_{ii}} d(\kappa_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii}) \sum_{\gamma_{ij}\in\mathcal{G}_{ij}} <\alpha_{ii}\otimes\gamma_{ij},\beta_{ij}> \tilde{d}(\gamma_{ij}) \\ &= \frac{1}{\sum_{\kappa_{ii}\in\mathcal{G}_{ii}} d(\kappa_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii}) \sum_{\gamma_{ij}\in\mathcal{G}_{ij}} <\overline{\alpha_{ii}}\otimes\beta_{ij},\gamma_{ij}> \tilde{d}(\gamma_{ij}) \\ &= \frac{1}{\sum_{\kappa_{ii}\in\mathcal{G}_{ii}} d(\kappa_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})\tilde{d}(\overline{\alpha_{ii}})\tilde{d}(\beta_{ij}) \end{aligned}$$

Since $\tilde{d}(\alpha_{ii}) = d(\alpha_{ii}) = d(\overline{\alpha_{ii}})$ for all i, the right hand side of the above becomes

$$= \frac{1}{\sum_{\kappa_{ii}\in\mathcal{G}_{ii}} d(\kappa_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2\tilde{d}(\beta_{ij})$$

$$
\begin{aligned}
&= \tilde{d}(\beta_{ij}) \\
&= \tilde{V}_{ij}(\beta_{ij}).
\end{aligned}
$$

Hence $\tilde{V}_{ij}$ is the Perron-Frobenius eigenvector of $\pi^i_j(e_i)$, so that $\tilde{V}_{ij}$ is a positive multiple of V_{ij}. Thus we get $\tilde{d}(\alpha_{ij}) = Cd(\alpha_{ij})$ and $\tilde{d}(\alpha_{ji}) = C'd(\alpha_{ij})$ for some positive constants C and C'. By the dimension function equation, we get $CC' = 1$.
Also $\tilde{d}(\alpha) = \tilde{d}(\overline{\alpha}) \Rightarrow C = C'$.
Hence, $C = C' = 1$.
Thus, $\tilde{d} = d$.
Hence there exists at most one dimension function for $\mathcal{G}$. Q.E.D.

Corollary 43 *The dimension function of a finite M_2-graded hypergroup satisfies the following equations:*

$$
\sum_{\alpha_{11}\in\mathcal{G}_{11}} d(\alpha_{11})^2 = \sum_{\alpha_{12}\in\mathcal{G}_{12}} d(\alpha_{12})^2 = \sum_{\alpha_{22}\in\mathcal{G}_{22}} d(\alpha_{22})^2.
$$

Proof:

We continue to use the notation of the proof of Theorem 42.

$$
\begin{aligned}
V_{11} \odot V_{12} &= \sum_{\alpha_{11}\in\mathcal{G}_{11}} d(\alpha_{11})\alpha_{11} \odot V_{12} \\
&= \sum_{\alpha_{11}\in\mathcal{G}_{11}} d(\alpha_{11})(d(\alpha_{11})V_{12}), \text{ since } \alpha_{11} \odot V_{12} = d(\alpha_{11})V_{12}, \\
&= \sum_{\alpha_{11}\in\mathcal{G}_{11}} d(\alpha_{11})^2 V_{12}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
V_{11} \odot V_{12} &= V_{11} \odot \sum_{\alpha_{12}\in\mathcal{G}_{12}} C(\alpha_{12})\alpha_{12} \\
&= \sum_{\alpha_{12}\in\mathcal{G}_{12}} C(\alpha_{12}) V_{11} \odot \alpha_{12} \\
&= \sum_{\alpha_{12}\in\mathcal{G}_{12}} C(\alpha_{12})\psi(\alpha_{12}) V_{12} \\
&= \sum_{\alpha_{12}\in\mathcal{G}_{12}} d(\alpha_{12})^2 V_{12}.
\end{aligned}
$$

Thus,

$$\sum_{\alpha_{11}\in\mathcal{G}_{11}} d(\alpha_{11})^2 = \sum_{\alpha_{12}\in\mathcal{G}_{12}} d(\alpha_{12})^2 \tag{3.2.1}$$

Similarly,

$$\sum_{\alpha_{22}\in\mathcal{G}_{22}} d(\alpha_{22})^2 = \sum_{\alpha_{21}\in\mathcal{G}_{21}} d(\alpha_{21})^2 \tag{3.2.2}$$

But,

$$\sum_{\alpha_{21}\in\mathcal{G}_{21}} d(\alpha_{21})^2 = \sum_{\alpha_{12}\in\mathcal{G}_{12}} d(\alpha_{12})^2,$$

since $d(\alpha) = d(\overline{\alpha})$ and $\alpha \mapsto \overline{\alpha}$ is a bijection between $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.

Thus from equations 3.2.1 and 3.2.2,

$$\sum_{\alpha_{11}\in\mathcal{G}_{11}} d(\alpha_{11})^2 = \sum_{\alpha_{12}\in\mathcal{G}_{12}} d(\alpha_{12})^2 = \sum_{\alpha_{22}\in\mathcal{G}_{22}} d(\alpha_{22})^2.$$

Q.E.D.

Remark 44 In [O3], Ocneanu calls the number $\sum_{\alpha_{ij}\in\mathcal{G}_{ij}} d(\alpha_{ij})^2$ "global index" in the context of fusion algebras.

Corollary 45 *For all α in $\mathcal{G}$, $d(\alpha) < 2$ implies $d(\alpha) = 2cos\frac{\pi}{n}$ for some integer $n > 2$.*

Proof:

We have

$$R_{\alpha_{ji}} V_{ij} = \varphi(\alpha_{ji}) V_{ii} \tag{3.2.3}$$

and

$$R_{\alpha_{ij}} V_{ii} = \psi(\alpha_{ij}) V_{ij} \tag{3.2.4}$$

From 3.2.3,

$$R_{\overline{\alpha_{ij}}} V_{ij} \;\; = \;\; \varphi(\overline{\alpha_{ij}}) V_{ii}$$

Thus,

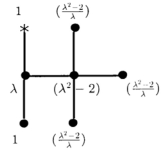
$$\begin{aligned} R_{\overline{\alpha_{ij}}} R_{\alpha_{ij}} V_{ii} &= R_{\overline{\alpha_{ij}}}(\psi(\alpha_{ij})V_{ij}) \\ &= \psi(\alpha_{ij})\varphi(\overline{\alpha_{ij}})V_{ii} \\ &= k_{ij}C(\alpha_{ij})^2 V_{ii} \\ &= d(\alpha_{ij})^2 V_{ii}. \end{aligned}$$

Hence V_{ii} is a Perron-Frobenius eigenvector of the matrix $R_{\overline{\alpha_{ij}}}R_{\alpha_{ij}}$ with the eigenvalue $d(\alpha_{ij})^2$, so that $d(\alpha_{ij})$ is the norm of the non-negative integer matrix $R_{\alpha_{ij}}$. Hence by Kronecker's theorem (i.e., for a finite matrix X over Z, either $\|X\| = 2cos\frac{\pi}{n}$ for some integer $n \geq 2$ or $\|X\| \geq 2$), $d(\alpha_{ij}) < 2 \Rightarrow d(\alpha_{ij}) = 2cos\frac{\pi}{n}$, for some $n > 2$ ($n \neq 2$ as $d(\alpha_{ij}) \neq 0$ for any α_{ij}). Q.E.D.

Remark 46 **(a)** If a pair of finite graphs arise as principal graphs of a finite index inclusion of II_1 factors, then the vertices of the graphs label an M_2-graded hypergroup and hence it follows from the above Corrollory that the smallest co-ordinate of the Perron-Frobenius eigenvector must occurs at the distinguished vertex $*$.

(b) The above corollary rules out several graphs arising as 'principal graphs' for the inclusion of II_1 factors.

Just to cite an example, consider the following graph with the Perron-Frobenius eigenvalues of the matrices corresponding to the vertices marked.



Observe that, $\lambda = \sqrt{3+\sqrt{3}}$, $\frac{\lambda^2-2}{\lambda}$ is between 1 and $\sqrt{2}$ and is not of the form $2cos\frac{\pi}{n}$ for any integer n.

(c) The restriction in the above corollory that we obtained just from the abstract settings of an M_2-graded hypergroup was proved in the case of principal graphs of subfactors in [P, theorem 3.8] and [I,theorem 6.1].

Now we prove that the Coxeter graph D_{4k+3} cannot occur as the principal graph for any inclusion of II_1 factors.

The case of D_{4k+3}:

We will follow the notation we used in the case of D_{4k+1}. So we are considering D_{2n+1} when $n = 2k+1$. In this case also the possibility $S_2 = J$ will leads to a contradiction to the equation $A_2A_n = A_{n-1} + 2A_n$. However, setting $S_2 = I$ does lead to an action of $\mathcal{G}_0$ and the contradiction is not yet reached. What we have shown however in Chapter 1 is that there is a unique hypergroup $\mathcal{G}_0$ and a unique action of this hypergroup on $\mathcal{G}_1$ that is consistent with the equations 2.3.3 and 2.3.4.

Only this much can be proved by considering only 'one-sided' actions, or equivalently only one of the principal graphs. To proceed further, we must note that, corresponding to the tower $\{M' \cap M_n : n \geq 0\}$ of relative commutants of M in the members of the tower of the basic construction, there exists another principal graph $\mathcal{H}_0$ whose odd vertices are in bijection with the $\mathcal{G}_1$ of the original principal graph, in such a way that $\mathcal{H}_0$ admits a right action π_1^0 on $\mathcal{G}_1$ which commutes with the left-action of $\mathcal{G}_0$, as in Remark 35(a)(ii).

First note that the principal graph corresponding to $\mathcal{H}_0$ and $\mathcal{G}_1$ must be a Coxeter diagram the norm of whose associated adjacency matrix is the same as that of D_{2n+1}. This can only be D_{2n+1} or a suitable A_m. Since the set of odd vertices of the graph must have the same cardinality as $\mathcal{G}_1$, we find that the other principal graph must also be D_{2n+1}. Then, we deduce from the earlier analysis that we must have $\mathcal{G}_0 = \mathcal{H}_0$ and that $\pi_1 = \pi_1^0$.

We may now deduce from the Propositon 23 that $\mathcal{G}_0 \sqcup \mathcal{G}_1$ must have the structure of a Z_2-graded hypergroup $\mathcal{G}$ with every element of $\mathcal{G}_1$ self-conjugate. In this hypergroup, we would have : $L_\beta = \pi_0(\beta) \oplus \pi_1(\beta)$ for all β in $\mathcal{G}_0$, where π_0 denotes the action of $\mathcal{G}_0$ on itself given by left-multiplication, π_1 denotes the action of $\mathcal{G}_0$ on $\mathcal{G}_1$, and L_β denotes the matrix of left-multiplication by β on $\mathcal{G}$ with respect to to the ordered basis $\{\beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_{n+1}\}$. Since every element of $\mathcal{G}_0$, as well as of $\mathcal{G}_1$, is self contragredient, we deduce that the hypergroup $\mathcal{G}$ is abelian. Note now that the equation $\alpha_1\alpha_n = \alpha_n\alpha_1 = \beta_n = \alpha_{n+1}\alpha_1 = \alpha_1\alpha_{n+1}$ imply that the last two columns of the matrix L_{α_1} are equal; since this matrix is symmetric, the last two rows are also equal. Then, since $L_{\alpha_1}L_{\alpha_n} = L_{\beta_n}$, it must be the case that the last two rows of L_{β_n} must be equal. However the bottom 2×2 principal submatrix is S_n which is equal to I or J and the desired contradiction has been reached, thus finally completing the proof of the fact that whether n is odd or even, the Coxeter diagram D_{2n+1} cannot arise as the principal graph of any inclusion $N \subseteq M$ of II_1 factors.

Chapter 4

Examples and Classification

4.1 Introduction

In this chapter we discuss some isolated and certain sequences of examples of finite hypergroups. Classification of small order hypergroups is also discussed in this chapter.

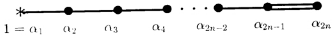
In the first section we discuss some examples of finite hypergroups associated with certain finite graphs in the sense that they are generated by the adjacency matrices of the graphs.

First we describe a sequence of $2n$ element hypergroups generated by the adjacency matrices of the graphs $\beta_{2n,k}$ for all positive integers n and k. Another sequence of n element hypergroups is generated by the adjacency matrices of the Coxeter graphs E_n for $n \geq 6$ where $n \neq 7, 10$. We also describe certain examples given by the 'connected sum' $A_{2n-1}\#G$ of the Coxeter graph A_{2n-1} and graphs associated with certain finite hypergroups which are either groups or group duals.

In the second section we classify finite hypergroups of order ≤ 6 where by the 'order' of a finite hypergroup $\mathcal{G}$ we mean $\sum_{\alpha\in\mathcal{G}} d_\alpha^2$, where $\alpha \mapsto d_\alpha$ denotes the unique dimension function for $\mathcal{G}$.

4.2 Examples

The β_{2n} hypergroups:
The adjacency matrix of the graph β_{2n} generates a $2n$ element Hermitian hypergroup $\mathcal{G} = \{\alpha_1, \alpha_2, \ldots, \alpha_{2n}\}$.
First we label the β_{2n} graph as shown below.



If we write $A_i = L_{\alpha_i}$, then A_1 is the $2n \times 2n$ identity matrix, A_2 is the adjacency matrix of the graph β_{2n} and the remaining A_i's are as described below. Then the adjacency matrix of β_{2n} which we denote by A_2 is given by,

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & 0 & 1 & . & . & . & 0 & 0 & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 1 & 0 \\ 0 & 0 & 0 & . & . & . & 1 & 0 & 2 \\ 0 & 0 & 0 & . & . & . & 0 & 2 & 0 \end{bmatrix}$$

$$\text{i.e., } A_2(l, m) = \begin{cases} 1 \text{ if } \mid l - m \mid = 1 \text{ and } l, m \neq 2n, \\ 2 \text{ if } l \text{ or } m = 2n \text{ and } \mid l - m \mid = 1, \\ 0 \text{ otherwise .} \end{cases}$$

The A_i's have the following block decomposition corresponding to the partition $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_{2n-1}\} \sqcup \{\alpha_{2n}\}$ of $\mathcal{G}$.

$$A_i = \begin{bmatrix} P_i & Q_i \\ Q_i' & R_i \end{bmatrix},$$

where the blocks have the following description:

Case(i) $2 < i < 2n$:

P_i's are the $(2n-1) \times (2n-1)$ symmetric matrices given by

$$P_i(l,m) = \begin{cases} 0 \text{ if } l+m \leq i \text{ or } \mid l-m \mid \geq i \text{ or } l+m = i+2, i+4, \ldots, \\ 1 \text{ if } \mid l-m \mid < i \text{ and } l+m = i+1, i+3, \ldots, 4n-i-1, \\ 4 \cdot 3^r \text{ if } l+m = 4n-i+(2r+1), r = 0,1,2\ldots; \end{cases}$$

Q_i's are the $(2n-1) \times 1$ (column) matrices with the first $2n-i$ entries zero and given by

$$Q_i(l) = \begin{cases} 0 \text{ if } l \leq 2n-i \text{ or } l = 2n-i+2, 2n-i+4, \ldots, \\ 2 \cdot 3^r \text{ if } l = 2n-i+(2r+1), r = 0,1,2,\ldots; \end{cases}$$

and

$$R_i = \begin{cases} [0] \text{ if } i \text{ is even}, \\ [3^{\frac{i-1}{2}}] \text{ if } i \text{ is odd}. \end{cases}$$

Case (ii) $i = 2n$:

$$P_{2n}(l,m) = \begin{cases} 0 \text{ if } l+m \leq 2n \text{ or } l+m = 2n+2, 2n+4, \ldots, \\ 2 \cdot 3^r; \text{ if } l+m = 2n+(2r+1), r = 0,1,2,\ldots; \end{cases}$$

$$Q_{2n}(l) = \begin{cases} 0 \text{ if } l \text{ is even}, \\ 3^r \text{ if } l = 2r+1, r = 0,1,2,\ldots; \end{cases}$$

and

$$R_{2n} = [0].$$

The collection $\{A_1, A_2, \ldots, A_{2n}\}$ satisfies the conditions (a) and (b) of Proposition 6 as A_1 is the identity matrix, the A_i's are symmetric and the first columns of A_i's are mutually orthogonal vectors. The condition $A_iA_j = \sum_{1 \leq s \leq 2n} A_i(s,j)A_s$ for all i and j is easily verified as we need to check it only for $i = 2$ as the A_i's are polynomials in A_2 and the above equation for $i = 2$ is just the adjacency relations of the graph $\mathcal{B}_{2n}$.

Thus the adjacency matrix of the $\mathcal{B}_{2n}$ graph generates a $2n$ element Hermitian hypergroup. For example, when $n = 4$ the matrices of the hypergroup are as follows:

$$A_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, A_2 = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&2\\ 0&0&0&0&0&0&2&0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 1&0&1&0&1&0&0&0\\ 0&1&0&1&0&1&0&0\\ 0&0&1&0&1&0&1&0\\ 0&0&0&1&0&1&0&2\\ 0&0&0&0&1&0&4&0\\ 0&0&0&0&0&2&0&3 \end{bmatrix}, A_4 = \begin{bmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&1&0&1&0&1&0&0\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&2\\ 0&0&1&0&1&0&4&0\\ 0&0&0&1&0&4&0&6\\ 0&0&0&0&2&0&6&0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&2\\ 1&0&1&0&1&0&4&0\\ 0&1&0&1&0&4&0&6\\ 0&0&1&0&4&0&12&0\\ 0&0&0&2&0&6&0&9 \end{bmatrix}, A_6 = \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&1&0&1&0&2\\ 0&0&1&0&1&0&4&0\\ 0&1&0&1&0&4&0&6\\ 1&0&1&0&4&0&12&0\\ 0&1&0&4&0&12&0&18\\ 0&0&2&0&6&0&18&0 \end{bmatrix},$$

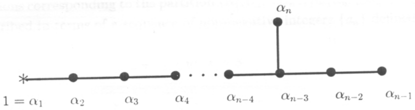
$$A_7 = \begin{bmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&2\\ 0&0&0&0&1&0&4&0\\ 0&0&0&1&0&4&0&6\\ 0&0&1&0&4&0&12&0\\ 0&1&0&4&0&12&0&18\\ 1&0&4&0&12&0&36&0\\ 0&2&0&6&0&18&0&27 \end{bmatrix}, A_8 = \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&2&0\\ 0&0&0&0&0&2&0&3\\ 0&0&0&0&2&0&6&0\\ 0&0&0&2&0&6&0&9\\ 0&0&2&0&6&0&18&0\\ 0&2&0&6&0&18&0&27\\ 1&0&3&0&9&0&27&0 \end{bmatrix}.$$

Remark 47 Arguing exactly as above we can show that there exist Hermitian hypergroups corresponding to $\beta_{2n,k}$ for all positive integers n and k, where the only difference between $\beta_{2n,k}$ and β_{2n} ($=\beta_{2n,2}$) is that the 2 bonds between the last two vertices in the latter is replaced in the former by k bonds and the modification in the A_i's is that the 3's occuring in the description of latter one is replaced in the former one by (k^2-1)'s .

The E_n hypergroups:

There exist Hermitian hypergroups of cardinality n corresponding to the Coxeter graph E_n except when $n = 7$ or 10. The hypergroups corresponding to the Coxeter graphs E_6 and E_8 are described in [S2]. In the following discussion we assume that $n \geq 11$.

We label the E_n graph as follows:



We denote the hypergroup by $\mathcal{G} = \{\alpha_i : 1 \leq i \leq n\}$ such that if we write $A_i = L_{\alpha_i}$, then A_1 is the $n \times n$ identity matrix and A_2 is the adjacency matrix of E_n given by

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & . & . & . & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\text{i.e.,} A_2(l,m) = \begin{cases} 1 \text{ if } \mid l-m \mid = 1; 1 \le l,m \le n-1, \\ \quad \text{or } \ l \ne m \text{ and } l,m \in \{n, n-3\}, \\ 0 \text{ otherwise }; \end{cases}$$

The adjacency relations of the graph imply that,

$$\begin{aligned} A_i &= A_2A_{i-1} - A_{i-2} \text{ for } 2 \le i \le n-3; \\ A_{n-2} + A_n &= A_2A_{n-3} - A_{n-4}; \\ A_2A_{n-2} &= A_{n-3} + A_{n-1}; \\ A_2A_{n-1} &= A_{n-2}; \text{ and} \\ A_2A_n &= A_{n-3}, \end{aligned}$$

with the convention that $A_0 = 0$. The A_i's are symmetric matrices which admit block decompositions corresponding to the partition $\{\alpha_1, \alpha_2, ...\alpha_{n-3}\} \sqcup \{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$ of $\mathcal{G}$ and are described in terms of a sequence of non-negative integers $\{a_n\}$ defined as follows:

$a_1 = 1, a_2 = 0, a_3 = 1$ and $a_k = a_{k-3} + a_{k-2}$ for $k > 3$.

We have

$$A_i = \begin{bmatrix} P_i & Q_i \\ Q_i^{'} & R_i \end{bmatrix},$$

where the blocks have the folowing description:

Case(i) $2 < i \le n-3$:

$$P_i(l,m) = \begin{cases} 0 \text{ if } l+m \le i \text{ or } \mid l-m \mid \ge i \text{ or } l+m = i+2, i+4, ..., \\ 1 \text{ if } \mid l-m \mid < i \text{ and } l+m = i+1, i+3, ..., i+2(n-i)-5, \\ a_{7-t} \text{ if } l+m = i+2(n-i-2)+(2t+1), t = 0,1,2....; \end{cases}$$

$$Q_i(l,1) = \begin{cases} 0 \text{ if } l \le (n-i-2) \text{ or } l = (n-i-2)+2, (n-i-2)+4, ..., \\ a_{5+t} \text{ if } l = (n-i-2)+(2t+1), t = 0,1,2,...; \end{cases}$$

$$Q_i(l,2) = \begin{cases} 0 \text{ if } l \le (n-i-1) \text{ or } l = (n-i-1)+2, (n-i-1)+4, ..., \\ a_{3-t} \text{ if } l = (n-i-1)+(2t+1), t = 0,1,2,...; \end{cases}$$

$$Q_i(l,3) = \begin{cases} 0 \text{ if } l \leq (n-i-2) \text{ or } l = (n-i-2)+2, (n-i-2)+4, ..., \\ a_{4-t} \text{ if } l = (n-i-2)+(2t+1), t = 0,1,2,...; \end{cases}$$

The first five R_i's are given by,

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, R_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The R_i's for $6 \leq i \leq n-3$ are determined by the first five R_i's together with the sequence $\{a_i\}$ in the following fashion:

$$\begin{aligned} R_{2i} &= a_i R_2 + a_{i-1} R_4; \\ R_{2i+1} &= a_i R_1 + a_{i-2} R_3 + a_{i-1} R_5. \end{aligned}$$

Case(ii) $i = n-2$:

From the adjacency relations we have: $A_{n-2} + A_n = A_2 A_{n-3} - A_{n-4}$. The matrix A_{n-2} is given by the following description:

$$\begin{aligned} P_{n-2}(l,m) &= \begin{cases} 0 \text{ if } l+m \leq (n-2) \text{ or } l+m = n, n+2, n+4, ..., \\ a_{5-t} \text{ if } l+m = (n-2)+(2t+1), t = 0,1,2....; \end{cases} \\ Q_{n-2}(l,1) &= \begin{cases} a_{2+t} \text{ if } l = (2t-1), t = 1,2,3,..., \\ 0 \text{ otherwise ;} \end{cases} \\ Q_{n-2}(l,2) &= \begin{cases} a_t \text{ if } l = 2t, t \geq 1, \\ 0 \text{ otherwise ;} \end{cases} \\ Q_{n-2}(l,3) &= \begin{cases} a_t \text{ if } l = (2t-3), t \geq 2; \\ 0 \text{ otherwise ;} \end{cases} \end{aligned}$$

and $R_{(n-2)} = R_{(n-6)}$.

Case(iii) $i = n$:

The matrix A_n is determined in terms of the preceeding A_i 's by the equation $A_n = A_2 A_{n-3} - A_{n-4} - A_{n-2}$ given by the adjacency relations and A_n admits the following description:

$$P_n(l,m) = \begin{cases} 0 \text{ if } l+m \leq (n-2) \text{ or } l+m = n, n+2, ..., \\ a_{4-t} \text{ if } l+m = (n-2)+(2t+1), t = 0,1,2....; \end{cases}$$

$$Q_n(l,1) = \begin{cases} a_{t+1} \text{ if } l = (2t-1), t \geq 1, \\ 0 \text{ otherwise ;} \end{cases}$$

$$Q_n(l,2) = \begin{cases} a_t \text{ if } l = 2(t+1), t \geq 0, \text{with } a_0 = 0, \\ 0 \text{ otherwise ;} \end{cases}$$

$$Q_n(l,3) = \begin{cases} a_t \text{ if } l = (2t-1), t \geq 1, \\ 0 \text{ otherwise ;} \end{cases}$$

and $R_{(n)} = R_{(n-8)}$.

Case(iv) $i = n-1$:

The matrix A_{n-1} is determined in terms of the preceding A_i's by the relation $A_{n-1} = A_2 A_{n-2} - A_{n-3}$ given by the adjacency relations and A_{n-1} admits the following description:

$$P_{n-1}(l,m) = \begin{cases} 0 \text{ if } l+m \leq (n-1) \text{ or } l+m = n+1, n+3, \ldots, \\ a_{3-t} \text{ if } l+m = n+2t, t = 0,1,2,\ldots; \end{cases}$$

$$Q_{n-1}(l,1) = \begin{cases} a_t \text{ if } l = 2t, t \geq 1, \\ 0 \text{ otherwise ;} \end{cases}$$

$$Q_{n-1}(l,2) = \begin{cases} 1 \text{ if } l = 1, \\ a_t \text{ if } l = 2t+3, t \geq 1, \\ 0 \text{ otherwise ;} \end{cases}$$

$$Q_{n-1}(l,3) = \begin{cases} a_t \text{ if } l = 2(t+1), t \geq 0, \text{ with } a_0 = 0, \\ 0 \text{ otherwise ;} \end{cases}$$

and $R_{(n-1)} = R_{(n-11)}$ for $n \geq 11$, with the convention that $R_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that when $n = 11$, $R_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The collection $\{A_1, A_2, \ldots, A_n\}$ satisfies the conditions (a) and (b) of Proposition 6 as A_1 is the identity matrix, the A_i's are symmetric and the first columns of A_i's are mutually orthogonal vectors. Now consider the condition $A_i A_j = \sum_{1 \leq s \leq n} A_i(s,j) A_s$. The above equation for $i = 2$ and $1 \leq j \leq n-3$ follows from the adjacency relations of the graph E_n. The relation is true for $1 \le i \le n-3$ and $1 \le j \le n-3$ as the A_i's, $1 \le i \le n-3$ are polynomials in A_2 and the relation is true for $i = 2$. For $n-2 \le i \le n$ and $1 \le j \le n$, the equation can be verified using the above structure of the matrices A_i.

The above equation can be verified for $j = n-2, n-1$ and n using the prescribed structure of the A_i's. For example when $n = 11$ the matrices of the hypergroup E_n are as follows: $A_1 = 11 \times 11$ identity matrix,

$$A_2 = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&1&0&2&0 \\ 0&0&0&0&0&0&1&0&1&0&1 \\ 0&0&0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \end{bmatrix}, A_3 = \begin{bmatrix} 0&0&1&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&0&0&0 \\ 0&1&0&1&0&1&0&0&0&0&0 \\ 0&0&1&0&1&0&1&0&0&0&0 \\ 0&0&0&1&0&1&0&1&0&0&0 \\ 0&0&0&0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&1&0&2&0&1&0 \\ 0&0&0&0&0&0&1&0&1&0&1 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&1&0&0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0&0 \\ 0&1&0&1&0&1&0&0&0&0&0 \\ 1&0&1&0&1&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0&0&0 \\ 0&0&1&0&1&0&1&0&1&0&1 \\ 0&0&0&1&0&1&0&2&0&1&0 \\ 0&0&0&0&1&0&2&0&2&0&1 \\ 0&0&0&0&0&1&0&2&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&1&0&1&0&1&0 \end{bmatrix}, A_5 = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&1&0&1&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0&0&0 \\ 1&0&1&0&1&0&1&0&1&0&1 \\ 0&1&0&1&0&1&0&2&0&1&0 \\ 0&0&1&0&1&0&2&0&2&0&1 \\ 0&0&0&1&0&2&0&3&0&1&0 \\ 0&0&0&0&1&0&2&0&1&0&1 \\ 0&0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&1&0&1&0&1&0&1 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}, A_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 & 3 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$A_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 & 3 & 0 & 4 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 & 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}, A_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 2 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

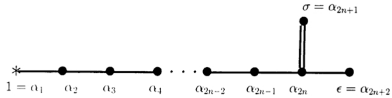
Connected sum of hypergroups:

We construct some finite hypergroups by taking the 'connected sum' of certain finite hypergroups in the following sense.

First, by the graph $G(\mathcal{G})$ associated with a finte hypergroup $\mathcal{G} = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ which is a group or a group dual we mean, a bipartite graph with $k+1$ vertices having k even vertices and a lone odd vertex such that the i^{th} even vertex α_i is connected to the odd vertex by d_{α_i} bonds, where $\alpha_i \mapsto d_{\alpha_i}$ denotes the dimension function of the hypergroup.

By the connected sum $A_{2n-1}\#G(\mathcal{G})$ of the Coxeter graph A_{2n-1} and the graph $G(\mathcal{G})$ associated with the hypergroup $\mathcal{G}$ which is a group or a group dual we mean the graph obtained by identifying the $(2n-1)^{th}$ vertex of the graph A_{2n-1} with the vertex corresponding to the identity element of the hypergroup in the graph $G(\mathcal{G})$. Note that the number of vertices of the graph $A_{2n-1}\#G(\mathcal{G})$ is equal to $2n-1+k$, where k is the cardinality of $\mathcal{G}$ and $A_1\#G(\mathcal{G}) = G(\mathcal{G})$.
First we consider a sequence of finite hypergroups given by the graph $A_{2n-1}\#G(S_3^\wedge)$, where $S_3^\wedge$ denotes the unitary dual of the group S_3.

We label the resulting graph as follows :



There exists a Hermitian hypergroup of cardinality $2n+2$ associated to the above graph for all positive integers n. If we denote the hypergroup by $\mathcal{G} = \{\alpha_i : 1 \leq i \leq 2n+2\}$, then $\{\alpha_i : 1 \leq i \leq 2n-1\}$ represents the vertices of the Coxeter graph A_{2n-1}, α_{2n} represents the odd vertex in $G(S_3^\wedge)$. The vertices α_{2n+1} and α_{2n+2} represent respectively the 2-dimensional and the non-trivial 1-dimensional irreducible unitary representations of the group S_3. With the above labelling of the graph, we have L_{α_1} is equal to the $(2n+2) \times (2n+2)$ identity matrix and L_{α_2} is equal to the adjacency matrix of the graph $A_{2n} := G(S_3^{*})$ given by

$$L_{\alpha_2} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & . & . & . & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\text{i.e., } L_{\alpha_2}(l, m) = \begin{cases} 1 \text{ if } \mid l - m \mid = 1 \text{ and } l, m \leq 2n, \\ \quad \text{or } l \neq m, l, m \in \{2n, 2n+2\}, \\ 2 \text{ if } l \neq m \text{ and } l, m \in \{2n, 2n+1\}, \\ 0 \text{ otherwise }. \end{cases}$$

The adjacency relations of the graph imply that,

$$\begin{aligned} L_{\alpha_i} &= L_{\alpha_2} L_{\alpha_{i-1}} - L_{\alpha_{i-2}}, \quad 3 \leq i \leq 2n, \\ 2L_{\alpha_{2n+1}} + L_{\alpha_{2n+2}} &= L_{\alpha_2} L_{\alpha_{2n}} - L_{\alpha_{2n-1}}; \end{aligned}$$

so that L_{α_i}'s for $3 \leq i \leq 2n$ are determined as polynomials in L_{α_2} and the sum $2L_{\alpha_{2n+1}} + L_{\alpha_{2n+2}}$ is also determined by L_{α_2}. We make choices for $L_{\alpha_{2n+1}}$ and $L_{\alpha_{2n+2}}$ such that the collection $\{L_{\alpha_i} : 1 \leq i \leq 2n+2\}$ corresponds to a hypergroup.

The L_{α_i}'s admit block decomposition corresponding to the partition of $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}\} \sqcup \{\alpha_{2n-1}, \alpha_{2n+2}\}$ of $\mathcal{G}$ as follows:

$$L_{\alpha_i} = \begin{bmatrix} P_i & Q_i \\ Q_i^{'} & R_i \end{bmatrix}$$

The blocks have the following description:

Case (i) $3 \leq i \leq 2n$:

The P_is are $n \times n$ symmetric matrices given by,

$$P_i(l, m) = \begin{cases} 0 \text{ if } l + m \leq i \text{ or } \mid l - m \mid \geq i \text{ or } l + m = i+2, i+4, \ldots, \\ 1 \text{ if } \mid l - m \mid \leq i \text{ and } l + m = i+1, i+3, \ldots, 4n - i + 1 \\ 5 \cdot 4^k \text{ if } l + m = i + 2(2n - i + 1) + (2k+1), k \geq 0; \end{cases}$$

The Q_i's are $n \times 2$ matrices given as below:

$$Q_i(l,m) = \begin{cases} 0 \text{ if } l \leq 2n-i+1 \text{ or } l = 2n-i+3, 2n-i+5, \ldots \\ 2 \times 4^{k-1} \text{ if } \text{ if } l = 2n-i+2k, k \geq 1 \text{ and } m = 1, \\ 4^{k-1} \text{ if } l = 2n-i+2k, k \geq 1 \text{ and } m = 2; \end{cases}$$

The R_i's are 2×2 symmetric matrices given as follows:
$R_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ if i is even $R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R_3 = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$. The R_i's for $3 < i < 2n$ and i odd are given by the following recursion relations:

$$R_i = \begin{bmatrix} x_i & y_i \\ y_i & z_i \end{bmatrix}$$

where $x_i = 4x_{i-2} + (-1)^{\frac{i-1}{2}}$, $y_i = 2x_{i-2}$ and $z_i = 2y_{i-2}$.

Case(ii) $i > 2n$:

Our choice of A_{2n+1} and A_{2n+2} is completely in terms of the preceding A_i's in the following fashion:

$$P_{2n+1} = [(Q_1)_1(Q_2)_1, \ldots, (Q_{2n})_1] \text{ and } P_{2n+2} = [(Q_1)_2(Q_2)_2, \ldots, (Q_{2n})_2],$$

where $(Q_i)_s$ denotes the s^{th} column (vector) of Q_i.

Equivalently,

$$P_{2n+1}(l,m) = \begin{cases} 0 \text{ if } l+m \leq 2n+1 \text{ or } \text{ or } l+m = 2n+3, 2n+5, \ldots, \\ 2 \cdot 4^{k-1} \text{ if } l+m = 2n+2k, k \geq 1; \end{cases} \quad \text{and}$$

$$P_{2n+2}(l,m) = \begin{cases} 0 \text{ if } l+m \leq 2n+1 \text{ or } \text{ or } l+m = 2n+3, 2n+5, \ldots, \\ 4^{k-1} \text{ if } l+m = 2n+2k, k \geq 1; \end{cases}$$

$$Q_{2n+1} = \begin{bmatrix} (R_1)^1 \\ (R_2)^1 \\ \cdot \\ \cdot \\ \cdot \\ (R_{2n})^1 \end{bmatrix} \text{ and } Q_{2n+2} = \begin{bmatrix} (R_1)^2 \\ (R_2)^2 \\ \cdot \\ \cdot \\ \cdot \\ (R_{2n})^2 \end{bmatrix}$$

where $(R_i)^r$ denotes the r^{th} row (vector) of R_i. Finally,

$$R_{2n+1} = \begin{bmatrix} x_{2n-1} + y_{2n-1} & x_{2n-1} \\ x_{2n-1} & 0 \end{bmatrix} \text{ and } R_{2n+2} = \begin{bmatrix} x_{2n-1} & 0 \\ 0 & 2y_{2n-1} \end{bmatrix}.$$

Using the above structure of the matrices, it can be verified that the collection $\{L_{\alpha_i} : 1 \leq i \leq 2n-2\}$ satisfies the conditions of the Proposition 6 and hence $\mathcal{G} = \{\alpha_1, \alpha_2, \ldots, \alpha_{2n-3}, \alpha_{2n-2}\}$ is a Hermitian hypergroup.

For example we list the the matrices for $A_5\#G(S_3^\wedge)$ below.

$$L_{\alpha_1} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, L_{\alpha_2} = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&2&1\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix},$$

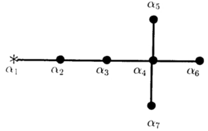
$$L_{\alpha_3} = \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 1&0&1&0&1&0&0&0\\ 0&1&0&1&0&1&0&0\\ 0&0&1&0&1&0&2&1\\ 0&0&0&1&0&5&0&0\\ 0&0&0&0&2&0&3&2\\ 0&0&0&0&1&0&2&0 \end{bmatrix}, L_{\alpha_4} = \begin{bmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&1&0&1&0&1&0&0\\ 1&0&1&0&1&0&2&1\\ 0&1&0&1&0&5&0&0\\ 0&0&1&0&5&0&8&4\\ 0&0&0&2&0&8&0&0\\ 0&0&0&1&0&4&0&0 \end{bmatrix},$$

$$L_{\alpha_5} = \begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&1.&0&0\\ 0&0&1&0&1&0&2&1\\ 0&1&0&1&0&5&0&0\\ 1&0&1&0&5&0&8&4\\ 0&1&0&5&0&20&0&0\\ 0&0&2&0&8&0&13&6\\ 0&0&1&0&4&0&6&4 \end{bmatrix}, L_{\alpha_6} = \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&2&1\\ 0&0&0&1&0&5&0&0\\ 0&0&1&0&5&0&8&4\\ 0&1&0&5&0&20&0&0\\ 1&0&5&0&20&0&32&16\\ 0&2&0&8&0&32&0&0\\ 0&1&0&4&0&16&0&0 \end{bmatrix},$$

$$L_{\alpha_7} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 0 & 8 & 0 & 0 \\ 0 & 0 & 2 & 0 & 8 & 0 & 13 & 6 \\ 0 & 2 & 0 & 8 & 0 & 32 & 0 & 0 \\ 1 & 0 & 3 & 0 & 13 & 0 & 19 & 13 \\ 0 & 0 & 2 & 0 & 6 & 0 & 13 & 0 \end{bmatrix}, L_{\alpha_8} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 & 6 & 4 \\ 0 & 1 & 0 & 4 & 0 & 16 & 0 & 0 \\ 0 & 0 & 2 & 0 & 6 & 0 & 13 & 0 \\ 1 & 0 & 0 & 0 & 4 & 0 & 0 & 12 \end{bmatrix}.$$

As above there exist hypergroups associated with the connected sums $A_3\#G(Z_2 \oplus Z_2)$ and $A_5\#G(Z_4)$.

We label the graph $A_3\#G(Z_2 \oplus Z_2)$ as follows:

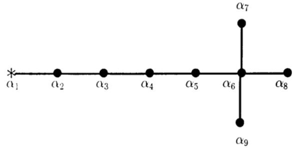


Let the hypergroup be denoted by $\{\alpha_i : 1 \leq i \leq 7\}$ so that L_{α_1} is the 7×7 identity matrix and L_{α_2} is the adjacency matrix of the graph $A_3\#G(Z_2 \oplus Z_2)$. The matrices $L_{\alpha_i}, 2 \leq i \leq 7$ are as follows:

$$L_{\alpha_2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, L_{\alpha_3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, L_{\alpha_4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{\alpha_5} = \begin{bmatrix} 0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&1&0&0&1&1 \\ 0&1&0&2&0&0&0 \\ 1&0&0&0&1&0&1 \\ 0&0&1&0&0&1&0 \\ 0&0&1&0&1&0&0 \end{bmatrix}, L_{\alpha_6} = \begin{bmatrix} 0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0 \\ 0&0&1&0&1&0&1 \\ 0&1&0&2&0&0&0 \\ 0&0&1&0&0&1&0 \\ 1&0&0&0&1&1&0 \\ 0&0&1&0&0&0&1 \end{bmatrix}, L_{\alpha_7} = \begin{bmatrix} 0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0 \\ 0&0&1&0&1&1&0 \\ 0&1&0&2&0&0&0 \\ 0&0&1&0&1&0&0 \\ 0&0&1&0&0&0&1 \\ 1&0&0&0&0&1&1 \end{bmatrix}$$

We label the graph $A_5\#G(Z_4)$ as follows:



Let the hypergroup be denoted by $\{\alpha_i : 1 \leq i \leq 9\}$ so that L_{α_1} is the 9×9 identity matrix and L_{α_2} is the adjacency matrix of the graph $A_5\#G(Z_4)$. The matrices $L_{\alpha_i}, 2 \leq i \leq 9$ are as follows:

$$L_{\alpha_2} = \begin{bmatrix} 0&1&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&1&0&1&1&1 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \end{bmatrix}, L_{\alpha_3} = \begin{bmatrix} 0&0&1&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&0&0 \\ 0&0&1&0&1&0&1&1&1 \\ 0&0&0&1&0&3&0&0&0 \\ 0&0&0&0&1&0&0&1&1 \\ 0&0&0&0&1&0&1&0&1 \\ 0&0&0&0&1&0&1&1&0 \end{bmatrix},$$

$$L_{\alpha_4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}, L_{\alpha_5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 3 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 3 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 2 \end{bmatrix},$$

$$L_{\alpha_6} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 3 & 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 6 & 0 & 4 & 4 & 4 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 \end{bmatrix}, L_{\alpha_7} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 1 \end{bmatrix},$$

$$L_{\alpha_8} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}, L_{\alpha_9} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

4.3 Classification of small order hypergroups

Let $\mathcal{G}$ be a finite hypergroup of cardinality n. Let $N = \sum_{\alpha \in \mathcal{G}} d_\alpha^2$, where $\alpha \mapsto d_\alpha$ denotes the unique dimension function for $\mathcal{G}$. We call N the 'order' of the finite hypergroup $\mathcal{G}$. Since $d_\alpha \geq 1$ for all $\alpha \in \mathcal{G}$, it follows that $N \geq n$. Define $\mathcal{G}_0 := \{\alpha \in \mathcal{G}$ such that $d_\alpha = 1\}$ and $\mathcal{G}_1 = \mathcal{G} \setminus \mathcal{G}_0$. Then $\mathcal{G}_0 \subseteq \mathcal{G}$ is a sub-hypergroup of $\mathcal{G}$, which is actually a group. The group $\mathcal{G}_0$ acts on the set $\mathcal{G}_1$ naturally. Let n_0 and n_1 denote the cardinality of $\mathcal{G}_0$ and $\mathcal{G}_1$ respectively so that $n = n_0 + n_1$. Since $d_1 = 1$ where 1 is the identity element of the hypergroup $\mathcal{G}$, $\mathcal{G}_0$ is non-empty and $n_0 \geq 1$. For an $\alpha \in \mathcal{G}_1$, it follows from Corollary 45 that $d_\alpha \geq \sqrt{2}$.

In the following discussion we assume that $N \leq 6$. Then, as $d_\alpha \geq 1$ for all $\alpha \in \mathcal{G}$, it follows that $n \in \{1, 2, 3, 4, 5, 6\}$. For different values of n we find out all the possible hypergroups of order at most 6.

We have the following result:

Theorem 48 *Let $\mathcal{G}$ be a finite hypergroup of cardinality n and order N with $\mathcal{G}_0$, $\mathcal{G}_1$, n_0 and n_1 as defined above. If $N \leq 6$, then the only possible hypergroups are:*

(1) *$n = 1$:*

the hypergroup is the trivial group $\mathcal{G} = \{1\}$ and $N = 1$;

(2) *$n = 2$:*

(i) *$\mathcal{G} = Z_2$ (and $N = 2$), or*

(ii) *$\mathcal{G}$ is given by the multiplication table,*

	1	α
1	1	α
α	α	$\alpha + 1$.

and $N = \frac{5+\sqrt{5}}{2}$.

(3) *$n = 3$:*

(i) *$\mathcal{G} = Z_3$ (and $N = 3$), or*

(ii) *$\mathcal{G}$ is given by the multiplication table,*

	1	α	β
1	1	α	β
α	α	1	β
β	β	β	$1 + \alpha + k\beta$.

where $k = 0$, or 1.
When $k = 0$, $N = 4$ and when $k = 1$, $N = 6$. The case $k = 1$ corresponds to S_3^*.

(4) $n = 4$:

(i) $\mathcal{G} = Z_4$ *or* $Z_2 \times Z_2$ *(with $N = 4$ in either case), or*

(ii) *$\mathcal{G}$ is given by the multiplication table,*

	1	α	$\overline{\alpha}$	β
1	1	α	$\overline{\alpha}$	β
α	α	$\overline{\alpha}$	1	β
$\overline{\alpha}$	$\overline{\alpha}$	1	α	β
β	β	β	β	$1 + \alpha + \overline{\alpha}$

and $N = 6$;

(5) $n = 5$:
$\mathcal{G} = Z_5$ *and* $N = 5$;

(6) Z_6, *or* $Z_2 \times Z_3$, *or* S_3 *(with $N = 6$ in each case).*

Proof:

(1) $n = 1$:
Here $n_0 = 1$ and $n_1 = 0$.
Then $\mathcal{G} = \mathcal{G}_0$ is the trivial singleton group and clearly $N = 1$.

(2) $n = 2$:
Since $n_0 \geq 1$, it follows that $n_1 \leq 1$.

(i) $n_1 = 0$: Then $\mathcal{G} = \mathcal{G}_0 \simeq Z_2$ and $N = 2$.

(ii) $n_1 = 1$:
Then $\mathcal{G}_0$ is the trivial group.

Let $\mathcal{G}_1 = \{\alpha\}$.
We have,

$$L_\alpha = \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix}; 0 < k \in Z,$$

so that $d_\alpha = \frac{k+\sqrt{k^2+4}}{2} > 1$. However, $N \leq 6 \Rightarrow d_\alpha^2 \leq 5$ and hence $k \leq 1$. Thus $k = 1$ and $d_\alpha = \frac{1+\sqrt{5}}{2}$. Here $N = 1 + (\frac{1+\sqrt{5}}{2})^2 = \frac{5+\sqrt{5}}{2}$.
Thus, when $n = 2$, the possible hypergroups are Z_2 and $\{1, \alpha\}$ where $L_\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

(3) $n = 3$:
Since $n_0 \geq 1$, it follows that $n_1 \leq 2$.

(i) $n_1 = 0$:
Then $\mathcal{G} = \mathcal{G}_0 \simeq Z_3$ and $N = 3$.

(ii) Let $n_1 = 1$:
Then $\mathcal{G}_0 \simeq Z_2$.
Let $\mathcal{G}_0 = \{1, \alpha\}$ and $\mathcal{G}_1 = \{\beta\}$.
Since $d_\beta = d_{\overline{\beta}}$, we see that $\beta = \overline{\beta}$, which implies that

$$L_\beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & k \end{bmatrix}, \quad 0 \leq k \in Z,$$

so that $d_\beta = \frac{k+\sqrt{k^2+8}}{2}$. However, $N \leq 6$ implies that $d_\beta \leq 2$ and so $k \leq 1$. Hence either $k = 0$ when $d_\beta = \sqrt{2}$ and $N = 4$, or $k = 1$ when $d_\beta = 2$ and $N = 6$ (the case $k = 1$ corresponding to $\mathcal{G} = S_3^\wedge$).

(iii) Let $n_1 = 2$:
Then $\mathcal{G}_0$ is the trivial group.
Let $\mathcal{G}_1 = \{\alpha, \beta\}$. There are two cases depending on whether the elements of $\mathcal{G}_1$ are self-contragredient or not.

(a) $\overline{\alpha} = \alpha$ (then $\overline{\beta} = \beta$)

In this case, the hypergroup $\mathcal{G}$ is Hermitian, hence abelian, and we find that

$$L_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 1 & x & y \\ 0 & y & z \end{bmatrix} \text{ and } L_\beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & y & z \\ 1 & z & w \end{bmatrix},$$

for some non-negative integers x, y, z and w. For the above two matrices to represent two elements of a hypergroup, we should have, $L_\alpha L_\beta = yL_\alpha + zL_\beta$; a look at the $(2,3)$ entry of this matrix equation shows that

$$1 + xz + yw = y^2 + z^2 \tag{4.3.1}$$

From the dimension function for this hypergroup,

$$d_\alpha^2 = 1 + xd_\alpha + yd_\beta \text{ and } d_\beta^2 = 1 + zd_\alpha + wd_\beta.$$

Since $N \leq 6$, $1 + d_\alpha^2 + d_\beta^2 \leq 6$.

Thus from the above two equations,

$(x + z)d_\alpha + (y + w)d_\beta \leq 3$. Further, the fact $d_\alpha, d_\beta \geq \sqrt{2}$ together with the above inequality implies that,

$$x + y + z + w \leq 2. \tag{4.3.2}$$

We have from 4.3.1

$$y(y - w) + z(z - x) = 1. \tag{4.3.3}$$

which together with the inequality 4.3.2 implies that $max\{x,\ y,\ z,\ w\} = 1$. The equation 4.3.3 implies that either $y(y - w) = 1$ and $z(z - x) = 0$ or $y(y - w) = 0$ and $z(z - x) = 1$.

Consider the case $y(y - w) = 1$ and $z(z - x) = 0$. Here we have $y = 1$, $w = 0$ and either $z = 0$ or $z = x = 1$. The case $y = 1$, $w = 0$ and $z = 0$ is not possible as this will imply that $d_\beta = 1$ which contradicts the fact that $d_\beta \geq \sqrt{2}$. The possibility $y = 1$, $w = 0$ and $z = x = 1$ will imply that $x + y + z + w = 3$ which contradicts the inequality 4.3.2. The case $y(y - w) = 0$ and $z(z - x) = 1$ will also lead to similar contradictions.

(b) Let $\overline{\alpha} = \beta$.

Then,

$$L_\alpha = L_\beta' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & x & y \\ 0 & z & w \end{bmatrix}, \quad 0 \leq x, y, z, w \in Z.$$

Then,

$$x = < \alpha \otimes \alpha, \alpha > = < \beta \otimes \beta, \beta > = w \text{ and}$$
$$w = < \alpha \otimes \beta, \beta > = < \alpha \otimes \beta, \alpha > = y.$$

Thus $x = y = w$.

Further, for the above two matrices to form the elements of a hypergroup we should have $L_\alpha L_\beta = 1 + yL_\alpha + wL_\beta$; a look at the $(3,3)$ entry of this matrix equation and the fact that $x = y = w$ show that

$$z^2 = 1 + x^2,$$

Thus $z = 1$ and $x = 0$.

Therefore $\mathcal{G} \simeq Z_3$. Thus when $n = 3$ the only possible hypergroups are Z_3 and $Z_2 \sqcup \{\beta\}$ where,

$$L_\beta \in \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\}.$$

(4) $n = 4$:

Since $N \leq 6$, it follows that $n_1 \leq 2$.

(i) $n_1 = 0$:

Then $\mathcal{G} = \mathcal{G}_0 \simeq Z_4$ or $Z_2 \oplus Z_2$ and here $N = 4$.

(ii) $n_1 = 1$:

Then $\mathcal{G}_0 = Z_3$. Let $\mathcal{G}_1 = \{\gamma\}$. Again we see that $\gamma = \overline{\gamma}$, so L_γ is symmetric and

$$L_\gamma = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & k \end{bmatrix}, \quad 0 \leq k \in Z.$$

so that $d_\gamma = \frac{k-\sqrt{k^2+12}}{2} \geq \sqrt{3}$. Since $N \leq 6$, it follows that $d_\gamma = \sqrt{3}$. Therefore $k = 0$ and $N = 6$.

(iii) Let $n_1 = 2$:
Then $\mathcal{G}_0 = Z_2$. Let $\mathcal{G}_0 = \{1, \alpha\}$ and $\mathcal{G}_1 = \{\beta, \gamma\}$.
There are two possibilities depending on whether the elements of $\mathcal{G}_1$ are self-contragredient or not. In both cases $N \leq 6$ and $d_\beta, d_\gamma \geq \sqrt{2}$ imply that, $d_\beta = d_\gamma = \sqrt{2}$.

(a) $\overline{\beta} = \beta$ (then $\overline{\gamma} = \gamma$). As before, this implies the hypergroup is abelian, and we have,

$$\text{either } L_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ or } L_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let

$$L_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

. Then, since $\mathcal{G}$ is abelian and $d_\beta^2 = 2$, we see that

$$L_\beta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & k \end{bmatrix} \quad 0 \leq k \in Z.$$

We have $k = \langle \beta \otimes \gamma, \gamma \rangle$.
Also from the dimension function equation we have $2 = d_\beta d_\gamma = k\sqrt{2}$ which implies that $k = \sqrt{2}$ which is a contradiction.
Now let

$$L_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\text{Then, } L_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & x & y \\ 0 & 1 & y & z \end{bmatrix} \quad 0 \leq x, \quad y, \quad z \in Z.$$

We have $x = < \beta \otimes \beta, \beta >$ and from the dimension function equation, $2 = d_\beta^2 = 1 + x\sqrt{2} + y\sqrt{2}$ which implies that $(x + y) = \frac{1}{\sqrt{2}}$ which is again a contradiction.

(b) $\bar{\beta} = \gamma$. Here also the matrix L_α is either

$$L_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ or } L_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let

$$L_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Then, since $d_\beta^2 = d_\gamma^2 = 2$, it follows that

$$L_\beta = L_\gamma = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & x & 0 \\ 0 & 0 & y & 0 \end{bmatrix}, \quad 0 \leq x, \quad y \in Z.$$

Then by the dimension function equation, $2 = d_\beta^2 = (x + y)\sqrt{2}$ which implies that $x + y = \sqrt{2}$ which is a contradiction.

$$\text{Now let } L_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \text{ Then,}$$

$$L_3 = L_\gamma' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & x & z \\ 0 & 1 & y & w \end{bmatrix} \quad 0 \leq x, \quad y, \quad z \;, w \in Z.$$

Here again $2 = d_\beta^2 = 1 + (x + y)\sqrt{2}$ which implies that $x + y = \frac{1}{\sqrt{2}}$ which is a contradiction.

Thus the only hypergroups with $n = 4$ are $Z_2 \oplus Z_2$, Z_4 and $Z_3 \sqcup \{\gamma\}$ where

$$L_\gamma = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

(5) $n = 5$:

In this case $N \leq 6$ implies that $n_1 \leq 1$.

(i) $n_1 = 0$:

Then $\mathcal{G}$ is just a group of order 5 . i.e., $\mathcal{G} \simeq Z_5$.

(ii) $n_1 = 1$:

Then $\mathcal{G}_0$ is a group of order 4 , i.e., $\mathcal{G} \simeq Z_2 \oplus Z_2$ or Z_4.

Let $\mathcal{G}_0 = \{1, \alpha, \beta, \gamma\}$; and $\mathcal{G}_1 = \{\delta\}$.

Since $N \leq 6$, it follows that $d_\delta \leq \sqrt{2}$.

In either case

$$L_\delta = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & k \end{bmatrix}, \quad 0 \leq k \in Z.$$

so that $d_\delta = \frac{k+\sqrt{k^2+16}}{2} \geq 2$ which is a contradiction to the fact that $d_\alpha \leq \sqrt{2}$.

Hence the only hypergroup with $N \leq 6$ and $n = 5$ is the group Z_5.

Here $N = 5$.

(6) $n = 6$:

Since $N \leq 6$, $d_\alpha \leq 1$ for all α.

Thus $d_\alpha = 1$ for all α, $n_0 = 6$, $n_1 = 0$ and $N = 6$.

Hence the hypergroup $\mathcal{G}$ is actually a group of order 6.

Thus $\mathcal{G} \simeq Z_6$ or $Z_2 \times Z_3$ or S_3 (the permutation group of order 6). Q.E.D.

From the above discussion we have the following conclusion :

Corollary 49 *If $\mathcal{G}$ is a finite hypergroup of order less than 6, then $\mathcal{G}$ is abelian; more precisely, the only non-abelian hypergroup of order ≤ 6 is S_3.*

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Errata to the Ph.D. thesis titled *Hypergroups, Graphs and Subfactors* by A.K. Vijayarajan

Page No.	Line No.	Is	Should be
ii	-11	*possesseed*	*possessed*
5	-14	inner-product	inner product space
6	-1	upto	up to
7	-10	$\oplus_{j=1}^{\nu}$	$\oplus_{i=1}^{\nu}$
7	-5	upto	up to
8	2	nodes	edges
8	6	dimensinal	dimensional
9	13	upto	up to
9	15	Braetteli	Bratteli
16	-3	eqivalence	equivalence
17	-3	construction.	construction
18	4	non-occurence	non-occurrence
22	8	associative axiom	the associative axiom
26	15	non-occurence	non-occurrence
29	-3	till	until
29	-3	distiguish	distinguish
29	-1	*graph*	*graph are*
31	13	occuring	occurring
33	-8	occuring	occurring
33	-7	*hand only*	*hand*
37	-2	non-occurence	non-occurrence
40	4	$V$by	V by
43	3	innerproduct	inner product
43	-10	upto	up to
50	13	corrollory	corollary
50	14	occurs	occur
51	1	corollory	corollary
51	3	[P, theorem 3.8]	[Po, theorem 3.8]
51	-4	*to to*	*to*
51	-1	equation	equations
54	9	A_2 is given by,	A_2, is given by
57	4	*is*	*are*
57	5	occuring	occurring
58	-9	folowing	following
59	-5	*preeceeding*	*preceding*
63	4	finte	finite
63	-6	all positive integers	each positive integer
64	-9	*admit block*	*admit a block*
66	2	of the	of
73	-19	$1 + xz + yw = y^2 + z^2$	$1 + xz + yw = y^2 + z^2.$
76	1	,*it follows that*	, *it follows that*
78	5	3-dimenssional	3-dimensional

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