A FURTHER NOTE ON NON-ISOMORPHIC SOLUTIONS OF INCOMPLETE BLOCK DESIGNS

By HARIKINKAR NANDI
Statistical Laboratory, Calcutta

INTRODUCTION

In a previous paper (Nandi 1944), a method has been given for enumerating all possible non-isomorphic solutions (and, of course, for the construction) of a 'balanced incomplete block' design. The method which may be termed the 'method of structural analysis' consists in filling up the blocks of a design with the desired number of treatments, step by step, keeping in mind the structure of the design or configuration. It has also been seen that, given two unsymmetrical configurations with parameters

(i) $v' = v - k,\ b' = b - 1,\ r' = r,\ k' = k - \lambda,\ \lambda' = \lambda$

and (ii) $v'' = k,\ b'' = b - 1,\ r'' = r - 1,\ k'' = \lambda,\ \lambda'' = \lambda - 1,$

when and how we can construct a symmetrical configuration with parameters $v = b,\ r = k,\ \lambda$. (The two unsymmetrical configurations are really the designs obtained by block-section and block-intersection (Bose 1939) respectively from the symmetrical design. The first one will be called 'residual configuration' and the second, the 'derived configuration' (Ayyanger 1943). This procedure of block-adjunction to enumerate the symmetrical configuration is a very convenient one inasmuchas the unsymmetrical configurations containing lesser number of treatments admit of easier enumeration.

Let us consider the symmetrical configuration : $v = b = 15,\ r = k = 7,\ \lambda = 3$. The residual configuration has the parameters ; $v = 8,\ b = 14,\ r = 7,\ k = 4,\ \lambda = 3$ whereas for the derived confinuation : $r = 7,\ b = 14,\ r = 6,\ k = 3,\ \lambda = 2$ both of which are enumerated with less labour.

2. ENUMERATION OF THE RESIDUAL AND DERIVED CONFIGURATIONS

Consider first the configuration : $v = 8,\ b = 14,\ r = 7,\ k = 4,\ \lambda = 3$. Let us call any block of this as the initial block and let 'x' be a variable giving the number of treatments common to the initial block and any other block. Then there are thirteen values of 'x' and denoting by f_i the frequency of $x = i$, it is easily seen (Nandi 1944) that the following two sets of values of f_i's only are admissible for the case under consideration, viz.,

(a) $f_0 = 0,\ f_1 = 3,\ f_2 = 9,\ f_3 = 1$

(b) $f_0 = 1,\ f_1 = 0,\ f_2 = 12,\ f_3 = 0$

Suppose we start with an initial block which follows the law (a). Then filling up the first block with treatments 1, 2, 3, 4 we get the unique pattern :

(1, 2, 3, 4) ; (1, 2, 3) ; (1, 2) ; (1, 3) ; (1, 4) ; (1, 4) ; (1) ; (2, 3) ; (2, 4) ; (2, 4) ; (2) ; (3, 4) ; (3, 4) ; (3).

Placing of 5 in the above pattern is also unique and yields

(1, 2, 3, 4) ; (1, 2, 3, 5) ; (1, 2) ; (1, 3) ; (1, 4, 5) ; (1, 4) ; (1, 5) ; (2, 3) ; (2, 4, 5) ; (2, 4) ; (2, 5) ; (3, 4, 5) ; (3, 4) ; (3, 5).

Complete the fifth block with 6, and then 6 is to be placed in six two-treatment blocks (i.e. six blocks containing two treatments each). There are only two non-isomorphic ways of selecting these, viz.,

[α] (3, 5) ; (3, 4) ; (2, 5) ; (2, 3) ; (1, 4) ; (1, 2).

[β] (3, 5) ; (3, 4) ; (2, 5) ; (2, 4) ; (1, 3) ; (1, 2).

Thus we get two types of patterns :

[α] (1, 4, 5, 6) ; (1, 2, 3, 5) ; (1, 2, 3, 4) ; (3, 5, 6) ; (3, 4, 6) ; (3, 4, 5) ; (2, 5, 6) ; (2, 4, 5) ; (2, 3, 6) ; (1, 4, 6) ; (1, 2, 6) ; (2, 4) ; (1, 5) ; (1, 3).

[β] (1, 4, 5, 6) ; (1, 2, 3, 5) ; (1, 2, 3, 4) ; (3, 5, 6) ; (3, 4, 6) ; (3, 4, 5) ; (2, 5, 6) ; (2, 4, 6) ; (2, 4, 5) ; (1, 3, 6) ; (1, 2, 6) ; (2, 3) ; (1, 5) ; (1, 4).

Completing the third block with 7, we find that 7, which is to be placed in three three-treatment blocks and three two-treatment blocks, can be placed in the following blocks in [α] and [β] types :

Case of [α]

[α_1] (3, 5, 6) ; (3, 4, 6) ; (2, 4, 6) ; (1, 2, 6) ; (2, 4) ; (1, 5) ; (1, 3).

[α_2] (3, 4, 6) ; (3, 4, 5) ; (2, 5, 6) ; (1, 2, 6) ; (2, 4) ; (1, 5) ; (1, 3).

Case of [β]

[β_1] (3, 5, 6) ; (3, 4, 6) ; (2, 4, 5) ; (1, 2, 6) ; (2, 3) ; (1, 5) ; (1, 4).

[β_2] (3, 5, 6) ; (3, 4, 5) ; (2, 4, 6) ; (1, 2, 6) ; (2, 3) ; (1, 5) ; (1, 4).

But out of these, placing of 7 in $[\beta_1]$ is isomorphic with that of 6 in $[\alpha]$. Hence we get three solutions, starting from an initial block which follows law (a) :

$[\alpha_1]$ (3, 5, 6, 7) ; (3, 4, 6, 7) ; (3, 4, 5, 8) ; (2, 5, 6, 8) ; (2, 4, 7, 8) ; (2, 4, 5, 7) ;
(2, 3, 6, 8) ; (1, 5, 7, 8) ; (1, 4, 6, 8) ; (1, 4, 5, 6) ; (1, 3, 7, 8) ; (1, 2, 6, 7) ;
(1, 2, 3, 5) ; (1, 2, 3, 4).

$[\alpha_2]$ (3, 5, 6, 8) ; (3, 4, 6, 7) ; (3, 4, 5, 7) ; (2, 5, 6, 7) ; (2, 4, 7, 8) ; (2, 4, 5, 8) ;
(2, 3, 6, 8) ; (1, 5, 7, 8) ; (1, 4, 6, 8) ; (1, 4, 5, 6) ; (1, 3, 7, 8) ; (1, 2, 6, 7) ;
(1, 2, 3, 5) ; (1, 2, 3, 4).

$[\beta_1]$ (3, 5, 6, 7) ; (3, 4, 6, 7) ; (3, 4, 5, 8) ; (2, 5, 6, 8) ; (2, 4, 6, 8) ; (2, 4, 5, 7) ;
(2, 3, 7, 8) ; (1, 5, 7, 8) ; (1, 4, 7, 8) ; (1, 4, 5, 6) ; (1, 3, 6, 8) ; (1, 2, 6, 7) ;
(1, 2, 3, 5) ; (1, 2, 3, 4).

If we look into the solutions minutely, the following characteristics of the above three designs will be revealed :

$[\alpha_1]$ Twelve blocks follow law (a) and two follow law (b)

$[\alpha_2]$ All the fourteen blocks follow law (a)

$[\beta_1]$ Eight blocks follow law (a) and six follow law (b).

This proves the non-isomorphism of the three solutions obtained and leads us to seek a solution where all the blocks follow law (b). There is really a single solution like this :

$[\gamma]$ (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 2, 7, 8) ; (1, 3, 5, 7) ; (1, 3, 6, 8) ; (1, 4, 5, 8) ;
(1, 4, 6, 7) ; (2, 3, 5, 8) ; (2, 3, 6, 7) ; (2, 4, 5, 7) ; (2, 4, 6, 8) ; (3, 4, 5, 6) ;
(3, 4, 7, 8) ; (5, 6, 7, 8).

$[\gamma]$ is obviously non-isomorphic with the other three solutions and hence in all we get four solutions $[\alpha_1]$, $[\alpha_2]$, $[\beta_1]$, $[\gamma]$ for the configuration : $v=8$, $b=14$, $r=7$, $k=4$, $\lambda=3$.

The blocks of the derived configuration : $v=7$, $b=14$, $r=6$, $k=3$, $\lambda=2$, obey the following laws :

$$\text{(a}') \quad f_0=1, \quad f_1=0, \quad f_2=3, \quad f_3=0$$
$$\text{(b}') \quad f_0=0, \quad f_1=12, \quad f_2=0, \quad f_3=1.$$

Proceeding in the above lines, we get the four solutions :

$[\alpha'_1]$ (9, 10, 11) ; (9, 10, 12) ; (9, 11, 13) ; (10, 11, 14) ; (9, 12, 15) ; (9, 13, 14) ;
(9, 14, 15) ; (10, 12, 14) ; (10, 13, 15) ; (10, 13, 15) ; (11, 12, 13) ; (11, 12, 15) ;
(11, 14, 15) ; (12, 13, 14).

$[\alpha'_2]$ (9, 10, 11) ; (9, 10, 12) ; (9, 11, 13) ; (10, 11, 14) ; (9, 12, 14) ; (9, 13, 15) ;
(9, 14, 15) ; (10, 12, 15) ; (10, 13, 14) ; (10, 13, 15) ; (11, 12, 13) ; (11, 12, 15) ;
(11, 14, 15) ; (12, 13, 14).

$[\beta'_1]$ (9, 10, 11) ; (9, 10, 12) ; (9, 11, 13) ; (10, 11, 14) ; (9, 12, 13) ; (9, 14, 15) ;
(9, 14, 15) ; (10, 12, 14) ; (10, 13, 15) ; (10, 13, 14) ; (11, 12, 15) ; (11, 12, 15) ;
(11, 13, 14) ; (12, 13, 14).

$[\gamma']$ (9, 10, 11) ; (9, 10, 11) ; (9, 12, 13) ; (9, 12, 13) ; (9, 14, 15) ; (9, 14, 15) ;
(10, 12, 14) ; (10, 12, 14) ; (10, 13, 15) ; (10, 13, 15) ; (11, 12, 15) ; (11, 12, 15) ;
(11, 13, 14) ; (11, 13, 14) ;

These four solutions are characterised by the following properties :

$[\alpha_1']$ Twelve blocks obey law (a') and two (b')

$[\alpha'_2]$ All blocks obey law (a')

$[\beta'_1]$ Eight blocks obey law (a') and six law (b')

$[\gamma']$ All blocks obey law (b').

3. Enumeration of the symmetrical configuration

It is obvious that all symmetrical configurations can be obtained from the suitable adjunction of blocks of derived configuration to those of the residual configuration with the addition of a new block containing the treatments involved in the derived configuration. But for adjunction, the following necessary condition must be satisfied ;—if 't' blocks of the residual configuration follow law (a) and the rest obey law (b), then in the derived configuration to be adjoined to it, 't' blocks must obey the complementary law (a') and the rest, the complementary law (b'). Hence we must try the adjunction of (i) $[\alpha_1]$ and $[\alpha'_1]$, (ii) $[\alpha_2]$ and $[\alpha'_2]$, (iii) $[\beta_1]$ and $[\beta'_1]$ and (iv) $[\gamma]$ and $[\gamma']$.

Let us consider the first case. There are two different ways of adjunction as shown below, all other ways of adjunction being obtained from these two by simply renaming the blocks and treatments of the adjoined derived configuration. The method followed is like this :

$[\alpha_1]$	$[\alpha'_1]_1$	$[\alpha'_1]_2$
(1, 2, 3, 4)	(9, 10, 11)	(9, 10, 11)
(1, 2, 3, 5)	(12, 13, 14)	(12, 13, 14)
(1, 2, 6, 7)	(10, 13, 15)	(10, 13, 15)
(1, 3, 7, 8)	(9, 12, 15)	(11, 14, 15)
(1, 4, 5, 6)	(11, 12, 15)	(11, 12, 15)
(1, 4, 6, 8)	(9, 13, 14)	(9, 13, 14)
(1, 5, 7, 8)	(10, 11, 14)	(9, 10, 12)
(2, 3, 6, 8)	(11, 14, 15)	(9, 12, 15)
(2, 4, 5, 7)	(9, 14, 15)	(9, 14, 15)
(2, 4, 7, 8)	(11, 12, 13)	(11, 12, 13)
(2, 5, 6, 8)	(9, 10, 12)	(10, 11, 14)
(3, 4, 5, 8)	(10, 13, 15)	(10, 13, 15)
(3, 4, 6, 7)	(10, 12, 14)	(10, 12, 14)
(3, 5, 6, 7)	(9, 11, 13)	(9, 11, 13)
	{9, 10, 11, 12, 13, 14, 15}	

To the first block of $[\alpha_1]$ which follows law (a) any block of $[\alpha_1']$ which follows law (a) is associated. For the second block of $[\alpha_1]$ we must choose a block of $[\alpha'_1]$ such that the two blocks, after adjunction, have just three treatments in common. This procedure is repeated; if at any stage, more than one blocks of $[\alpha'_1]$ can be chosen, it is seen whether a renaming of the treatments of $[\alpha'_1]$ makes the differently chosen blocks identical, keeping the already adjoined blocks the same and the whole $[\alpha'_1]$ the same.

In the same way it has been found that $[\alpha'_2]$ can be adjoined to $[\alpha_2]$ in one manner only:

$[\alpha_2]$	$[\alpha'_2]$	$[\alpha_2]$	$[\alpha'_2]$
(1, 2, 3, 4)	(9, 10, 11)	(2, 3, 6, 8)	(10, 12, 15)
(1, 2, 3, 5)	(12, 13, 14)	(2, 4, 5, 8)	(9, 14, 15)
(1, 2, 6, 7)	(9, 13, 15)	(2, 4, 7, 8)	(11, 12, 13)
(1, 3, 7, 8)	(11, 14, 15)	(2, 5, 6, 7)	(10, 11, 14)
(1, 4, 5, 6)	(11, 12, 15)	(3, 4, 5, 7)	(10, 13, 15)
(1, 4, 6, 8)	(10, 13, 14)	(3, 4, 6, 7)	(9, 12, 14)
(1, 5, 7, 8)	(9, 10, 12)	(3, 5, 6, 8)	(9, 11, 13)
	{9, 10, 11, 12, 13, 14, 15}		

$[\beta_1]$ and $[\beta'_1]$ can be adjoined in six different ways but except the first which is given below, the other five adjunctions lead to configurations isomorphic with $[\alpha_2\alpha'_2]$ and $[\alpha_1\alpha'_1]_2$ already obtained, as is easily seen that if the first completed block is taken as the initial configuration, the derived configuration is $[\alpha'_1]$ or $[\alpha'_2]$.

$[\beta_1]$	$[\beta'_1]$	$[\beta_1]$	$[\beta'_1]$
(1, 2, 3, 4)	(9, 10, 11)	(2, 3, 7, 8)	(9, 14, 15)
(1, 2, 3, 5)	(12, 13, 14)	(2, 4, 5, 7)	(11, 12, 15)
(1, 2, 6, 7)	(10, 13, 15)	(2, 4, 6, 8)	(11, 13, 14)
(1, 3, 6, 8)	(11, 12, 15)	(2, 5, 6, 8)	(9, 10, 12)
(1, 4, 5, 6)	(9, 14, 15)	(3, 4, 5, 8)	(10, 13, 15)
(1, 4, 7, 8)	(9, 12, 13)	(3, 4, 6, 7)	(10, 12, 14)
(1, 5, 7, 8)	(10, 11, 14)	(3, 5, 6, 7)	(9, 11, 13)
	{9, 10, 11, 12, 13, 14, 15}		

$[\gamma]$ and $[\gamma']$ admit of twelve different adjunctions but except the one given below, all leads to configurations isomorphic with $[\alpha_1\alpha'_1]_1$, $[\alpha_1\alpha'_1]_2$, $[\beta_1\beta'_1]$.

$[\gamma]$	$[\gamma']$	$[\gamma]$	$[\gamma']$
(1, 2, 3, 4)	(9, 10, 11)	(2, 3, 5, 8)	(11, 13, 14)
(1, 2, 5, 6)	(9, 12, 13)	(2, 3, 6, 7)	(11, 12, 15)
(1, 2, 7, 8)	(9, 14, 15)	(2, 4, 5, 7)	(10, 13, 15)
(1, 3, 5, 7)	(10, 12, 14)	(2, 4, 6, 8)	(10, 12, 14)
(1, 3, 6, 8)	(10, 13, 15)	(3, 4, 5, 6)	(9, 14, 15)
(1, 4, 5, 8)	(11, 12, 15)	(3, 4, 7, 8)	(9, 12, 13)
(1, 4, 6, 7)	(11, 13, 14)	(5, 6, 7, 8)	(9, 10, 11)
	{9, 10, 11, 12, 13, 14, 15}		

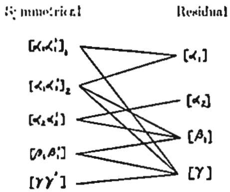
All the five symmetrical configurations, viz., $[\alpha_1\alpha'_1]_1$, $[\alpha_1\alpha'_1]_2$, $[\alpha_2\alpha'_2]$, $[\beta_1\beta'_1]$, $[\gamma\gamma']$ are non-isomorphic as can be seen by considering all possible derived configurations of each.

$(\alpha_1\alpha'_1)_1$ gives fourteen derived configurations $[\alpha'_1]$ and one $[\gamma']$.
$(\alpha_1\alpha'_1)_2$ gives eight $[\alpha_1']$, six $[\beta'_1]$, and one $[\gamma']$
$(\alpha_2\alpha'_2)$ gives eight $[\alpha'_2]$ and seven $[\beta'_1]$
$(\beta_1\beta'_1)$ gives twelve $[\beta'_1]$ and three $[\gamma']$
$(\gamma\gamma')$ gives fifteen $[\gamma']$

4. CONCLUSIONS

The symmetrical configuration : $v=b=15$, $r=k=7$, $\lambda=3$ has five non-isomorphic solutions given in the paper as $[\alpha_1\alpha'_1]_1$, $[\alpha_1\alpha'_1]_2$, $[\alpha_2\alpha'_2]$, $[\beta_1\beta'_1]$ and $[\gamma\gamma']$. The solutions for the same given in Fisher and Yates' table (1938) and Bose' paper (1939) are isomorphic with one another and with $[\gamma\gamma']$ of this paper. The residual configuration as well as the derived configuration has each four non-isomorphic solutions given in the paper respectively as $[\alpha_1]$, $[\alpha_2]$, $[\beta_1]$, $[\gamma]$ and $[\alpha'_1]$, $[\alpha'_2]$, $[\beta'_1]$, $[\gamma']$. The solutions given for the residual configuration by Fisher and Yates (1938) and Bose (1939) are isomorphic with the solution $[\gamma]$ of this paper.

The most interesting point is the relation between the symmetrical configuration studied and its residuals or derivatives. The correspondence can be shown diagrammatically like this :



The bond in the above diagram indicates that we can pass from one to another. It reveals that from a single symmetrical design, we can obtain more than one non-isomorphic residual or derived configurations, and also that from two non-isomorphic symmetrical configurations we can obtain the identical residual or derivative if the initial blocks are chosen properly i.e. there is no one-to-one correspondence, in general, between the symmetrical configurations and their residuals or denrivatives.

REFERENCES

1. AYYANGER, A. A. KRISHNASWAMY (1943) A general theory of tactical configurations. *Jr. Mysore University*, 3, 103-113.

2. BOSE, R. C. (1939) On the construction of balanced incomplete block, design. *Ann. Eug.*, 9, 358-399.

3. COX, GERTRUDE, M. (1940) Enumeration and construction of balanced incomplete configurations. *Ann. Math. Stat.*, 9, 72-85.

4. FISHER, R. A. (1940) An Examination of the different possible solution of a problem in incomplete block. *Ann. Eug.*, 10, 52 75.

5. FISHER & YATES (1938) Stat. Tables for Biological, Agricultural, and Medical Research.

6. HUSAIN, Q. M. (1944) On the Totality of solutions of symmetrical incomplete block designs. *Proc. Ind. Sci. Congress.*

7. NANDI, H. K. (1944) Note on the enumeration of all possible non-isomorphic solutions of balanced incomplete block designs. *Proc. Ind. Sci. Congress.*

Paper received 20 August, 1945