TIME-SPACE HARMONIC POLYNOMIALS FOR STOCHASTIC PROCESSES

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Contents

Chapter 1

Introduction

1.1 Statement of the problem

The sequence of polynomials of a single variable known as the Hermite polynomials

$$h_k(x) = \frac{(-1)^k}{k!} e^{x^2/2} \frac{\partial^k}{\partial x^k}(e^{-x^2/2}), \quad k \geq 1,$$

has many close links with the Normal distribution. Their association goes very deep, and extends to several connections between the two-variable Hermite polynomials

$$H_k(t, x) = t^k h_k(x/t^2), \; . \; k \geq 1,$$

and the prime example of Gaussian processes, that is Brownian motion, as well. Much of this connection stems from what we term the time-space harmonic property of these polynomials for the Brownian motion process. An exact definition of this property follows later. A natural question that arises is, for stochastic processes in general, when do there exist two-variable polynomials that are time-space harmonic for the process in question, and in case such polynomials exist, whether they carry algebraic and analytical properties analogous to those of the Hermite polynomials.

We have been able to arrive at reasonably satisfactory answers which will be described in detail. Interestingly enough, the investigation has raised several new questions some of which have been answered here while some others may well be the subject of further research.

Throughout, we shall assume as given, on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a process $M = \{M_t\}$ indexed by a partially ordered 'time-set' $(T, \leq)$ and taking values in some metric space E. We also fix notation for its natural filtration $\mathcal{F}_t = \sigma< M_t : s \leq t>$. A real-valued function f on $T \times E$ will be called *time-space harmonic* for M if $\{f(t, M_t) : t \in T\}$ is an

$\{\mathcal{F}_t\}$-martingale. Such functions clearly form a vector space $\mathcal{V}$. For functions in $\mathcal{V}$, their two arguments are referred to as the 'time' and 'space' variables, in that order. The object we shall be interested in studying in this work is the vector subspace of $\mathcal{V}$ consisting of only those functions in $\mathcal{V}$ which are *polynomials* in the two variables. We denote this subspace by $\mathcal{P}$ and call its elements *time-space harmonic polynomials* for M. It bears mention that in a majority of situations, T will be either $\{0, 1, \cdots\}$ or $[0, \infty)$ and E, the real line $\mathbb{R}$. In such cases, $\mathcal{P}$ consists merely of those functions in $\mathcal{V}$ which are polynomials in both arguments in the usual sense. Only a few situations not conforming to this setup will occur in our investigation. Foremost among these is the multivariate situation, when the time-set remains either $\{0, 1, \cdots\}$ or $[0, \infty)$ and E represents some rth Euclidean space. One can then extend the results in a natural way to the vector-valued case, that is, as for example when E is a Hilbert space.

Let us now state the main problem more explicitly in the setting to occur most frequently, that is, when T is either the discrete or the continuous time-set and $E = \mathbb{R}$. As the previous discussion suggests, the subject of this work is the study of conditions for a certain degree of richness of the class $\mathcal{P}$. The following definition makes this clearer. We shall call M *polynomially harmonisable*, or *p-harmonisable* in short, if for every positive integer k, $\mathcal{P}$ contains a polynomial of degree k in the 'space' variable. We wish to make it clear right away that this definition has no connection with the existing notions of harmonisability of a process, which refer to the representability of its covariance function, or equivalently, sample functions, as Fourier transforms in a certain sense (see [13], p. 474-476). The central question we investigate is, when is a stochastic process M p-harmonisable, and if it is, how are the various properties of M reflected in a sequence of time-space harmonic polynomials?

1.2 Chief examples known

There are many known examples of processes possessing this property, the most familiar perhaps being Brownian motion, where, as indicated earlier, the two-variable Hermite polynomials (H_k) form such a sequence. The standard Poisson process N_t is also p-harmonisable, with a sequence of time-space harmonic polynomials being given by the two-variable Charlier polynomials

$$C_k(t, x) = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \sum_{i=0}^{j} \left\{ {j \atop i} \right\} t^i,$$

where $\left\{ {j \atop i} \right\}$'s (see [11]) are the Stirling numbers of the second kind. The Gamma process turns out to be yet another example of a p-harmonisable process.

It has recently been established [14] that alongwith an additional hypothesis, the time-space harmonicicity property of any Hermite polynomial of order $k \geq 2$ (in fact, any parabolic function) for a continuous martingale actually characterizes Brownian motion. The special case $k = 2$ of this has long been well-known as Lévy's characterisation. It is also known that any simple point process M for which $M_t - t$ is a martingale, is necessarily a standard Poisson process.

The two results last mentioned can be viewed as charaterising the law of a certain process by a *finite sequence* of its time-space harmonic polynomials. However, both also depend crucially on 'path properties' of the underlying process such as continuity or pure jump nature of paths, in the sense that the first two (in fact the first three) time-space harmonic polynomials we have exhibited respectively for Brownian motion and the Poisson process agree with one another. One naturally wonders how essential these path properties are, or in other words, whether one can impose milder conditions, as for example just r.c.l.l. paths, to obtain a necessary and sufficient condition for a process to be determined upto law, by a finite sequence of time-space harmonic polynomials. We show subsequently that the above characterisation problem admits a solution in the affirmative at least in a certain class of Lévy processes. This necessary and sufficient condition is satisfied, for instance, by Brownian motion and Poisson process, but not by the Gamma process.

Among the various other connections of Hermite polynomials with Brownian motion, one of the most important is sometimes referred to by saying that Hermite polynomials play the same role in stochastic calculus involving Brownian motion to that played by the ordinary powers in usual calculus. It represents the iterated relation

$$H_k(t, B_t) = \int_0^t H_{k-1}(s, B_s) dB_s \tag{1.1}$$

with $\{B_t : t \geq 0\}$ a standard Brownian motion. This relation, incidentally, serves also as a first-glance proof of their time-space harmonic property with reference to Brownian motion.

For the standard Poisson process $\{N_t : t \geq 0\}$, a similar property is enjoyed by the Charlier polynomials described earlier. In analogy with the relation (1.1), we have here

$$C_k(t, M_t) = \int_0^t C_{k-1}(s, M_s) dM_s \tag{1.2}$$

where M is the compensated Poisson process defined as $M_t = N_t - t$.

At this point it may not be inappropriate to point out a significant difference between the two families of polynomials vis-a-vis the processes they respectively arise from, as detailed in [18]. The former has the property that if one takes a deterministic function f on [0,1] with $\int_0^1 f^2(s)ds = 1$, (1.1) holds, at time $t = 1$, with B_s replaced by $Z_s = \int_0^s f(u)dB_u$ for $0 \leq s \leq 1$. However, such a recurrence relation does *not* hold for similar constructs from the Poisson martingale with the Charlier polynomials. A heuristic justification furnished by [18] is that in general, linear combinations of Poisson random variables need not have a Poisson distribution, unlike Gaussian variables.

Equations (1.1) and (1.2) suggest a construction of time-space harmonic polynomials for general homogeneous Lévy processes along similar lines. Denoting by M any such process, if one defines recursively the functionals as in (1.1) starting with the basic 0-mean (locally) square-integrable martingale $(M_t - \mathbf{E}M_t)$, one gets the appropriate analogues of 'powers' in the sense that their images in the associated reproducing kernel Hilbert space do turn out to be usual powers, modulo constant multiples, and the expansion of the "exponential martingale" involves precisely these functionals. However, these are in general not space-time functions, let alone polynomials, of the corresponding martingales, and usually depend on the full history of the paths. For this reason, they do not enter into our consideration any further.

We now revert back to similarities between the two families of polynomials (H_k) and (C_k). An important feature common to them is that they both appear in the expansion of a certain function arising out of the exponentials for the respective processes :

$$\exp(\alpha x - \alpha^2 t) = \sum_{k=0}^{\infty} H_k(t,x)\frac{\alpha^k}{k!} \tag{1.3}$$

$$\exp\{\alpha x - t(1-e^{\alpha})\} = \sum_{k=0}^{\infty} C_k(t,x)\frac{\alpha^k}{k!} \tag{1.4}$$

In fact, this phenomenon is actually shared by all homogeneous Lévy processes and leads to a way of constructing such polynomials for them, as well as for their discrete-time counterparts, the partial sums of iid random variables. For the latter, this method is indicated by Neveu ([15], p. 80), and will be described here in brief. One starts with a sequence $\{X_n : n \geq 1\}$ of iid random variables such that the cumulant generating function (c.g.f.)

$$\varphi(\alpha) = \log \mathbf{E}\{\exp(\alpha X_1)\}$$

is defined in a neighbourhood of 0; one then expands the function $\eta(\alpha, t, x) = \exp\{\alpha x - t\varphi(\alpha)\}$ as a power series in α :

$$\eta(\alpha, t, x) = \sum_{k=0}^{\infty} P_k(t,x)\frac{\alpha^k}{k!}. \tag{1.5}$$

Now, $P_0(\cdot,\cdot) \equiv 1$, and for each $k \geq 1$, $P_k(\cdot,\cdot)$ turns out to be a time-space harmonic polynomial for $M = \{M_n = \sum_{i=1}^{n} X_j,\ n \geq 1\}$ (with $M_0 \equiv 0$), and it is of degree k in both its arguments. The harmonicity of P_k follows from successively interchanging the order of two operations - namely, that of k-th order partial differentiation at $\alpha = 0$ and of taking conditional expectation, on the *exponential martingale* $\{\eta(\alpha, n, M_n)\}$. The complete proof is provided in Chapter 2.

It is more or less obvious that the equalities (1.3) and (1.4) are due really to the expansion (1.5). Actually, this idea of expanding the exponential martingale applies also to the most immediate generalisation of the above kind of processes, that is, to processes with differences (or increments) independent but not identically distributed (or stationary), and yields a necessary and sufficient condition for p-harmonisability of these processes too. However, this approach suffers from two major lacunae. Firstly, one has to assume the existence of moment generating functions (m.g.f.), and secondly, if the differences (or increments) of M are no longer independent, it is not at all clear how, or if at all, one can modify the function φ to obtain exponential martingales. Therefore, we adopt an alternative approach. For discrete M, this approach involves writing out the martingale equality for each of the polynomials and to equate coefficients of various powers of M_n. This is where the support condition (**S**), defined in the next section, is required. We then use induction to obtain some repercussion on certain moments, or conditional moments, of M_n.

In continuous-time cases, the corresponding discrete-time results alongwith suitable approximation arguments do the needful.

1.3 Main definitions

At this point let us introduce the principal definitions and notations used in the sequel. For a real-valued process $\{M_t : t \in T\}$ where $T \subset \mathbb{R}$, we define $\mathcal{P} = \mathcal{P}(M)$ as the class of all time-space harmonic polynomials P with $P(0,0) = 0$, and for $k \geq 1$,

$$\begin{aligned} \mathcal{P}_k = \mathcal{P}_k(M) &= \{P \in \mathcal{P} : P(t,x) \text{ is of degree } k \text{ in the variable } x\}, \\ \bar{\mathcal{P}}_k = \bar{\mathcal{P}}_k(M) &= \{P \in \mathcal{P}_k : \textit{coeff}(x^k) \text{ in } P(t,x) \text{ is a constant free of } t\}, \\ \text{and} \qquad & \overline{\mathcal{P}} = \overline{\mathcal{P}}(M) = \cup_{k \geq 1} \bar{\mathcal{P}}_k. \end{aligned}$$

Thus the definition of p-harmonisability, restated in terms of these objects, merely requires each $\mathcal{P}_k$ to be nonempty. Analogously, we call a process M *restricted* p-harmonisable if $\bar{\mathcal{P}}_k(M) \neq \emptyset$ for each $k \geq 1$. Notice that there is no loss of generality in assuming that the "leading coefficient" of each element of $\bar{\mathcal{P}}_k$, as referred to in its definition, is 1. Henceforth we do assume this.

We often require M to satisfy two conditions, referred to as the *moment* and *support* conditions respectively. The first is of obvious importance in our context :

$$E|M_t^k| < \infty \; \forall \; k \geq 1, \; t \in T \qquad \textbf{(M)}.$$

Evidently, this condition is essential for any process M to have any chance of being p-harmonisable; but even when this fails, we shall investigate some instances of what we call *partial* p-harmonisability (defined in page 11). The second condition is stated in terms of sequences $\{M_n\}$. It demands that for every $k \geq 1$, there be an infinite number of n's with

$$|\text{supp}(M_n)| > k \qquad \textbf{(S)}.$$

The role of **(S)** is less apparent, being mostly technical. It will be clarified, however, when we make use of it. In brief, we make use of it through the fact that if the support of a random variable X has more points than the degree of a polynomial p, then $p(X)$ cannot be identically zero unless $p \equiv 0$. Although we state it in terms of sequences, its necessity is not limited to discrete-time situations alone. We shall usually understand by condition **(S)** for a continuous-time process (M_t) the fulfillment of **(S)** by the sequence $\{M_n\}$ of random variables. More precise conditions will be laid down in specific contexts.

1.4 Properties of the polynomials

We now list some properties of the sequence $\{P_k : k \geq 1\}$ of time-space harmonic polynomials obtained in (1.5) above for processes with iid summands (or independent stationary increments). Let us first note that by definition, in our classes $\mathcal{P}$ and $\overline{\mathcal{P}}$, we only allow polynomials P with $P(0,0) = 0$. Therefore, non-zero constant polynomials are excluded from both these classes. In other words, every non-zero element of $\mathcal{P}$ or $\overline{\mathcal{P}}$ has to be of degree at least 1 in the space variable. However, in order to state the properties of the sequence $\{P_k : k \geq 1\}$ as obtained in (1.5), we also need to have a time-space harmonic polynomial P_0 of degree 0 in the space variable, or in other words, a constant polynomial. A natural candidate for P_0 suggested by the equation (1.5) is the constant polynomial 1. Throughout, we shall follow this convention and take $P_0 \equiv 1$.

For $k \geq 0$, writing $P_k(t,x) = \sum_{j=0}^{k} p_j^{(k)}(t)x^j$, where $p_j^{(k)}$, $0 \leq j \leq k$ are polynomials of a single variable, we have,

(i) for each $k \geq 1$, $P_k \in \bar{\mathcal{P}}_k$, that is, $p_k^{(k)}(\cdot) \equiv 1$;

(ii) for every $k \geq 1$, $\frac{\partial P_k}{\partial x}(t,x) = kP_{k-1}(t,x)$, that is, $j\, p_j^{(k)}(t) = k\, p_{j-1}^{(k-1)}(t)$, $1 \leq j \leq k$;

(iii) for each $k \geq 1$, $\deg P_k(\cdot, x) = k$ and there exists a sequence $\{h_k : k \geq 1\}$ of real numbers such that for each $k \geq 1$, $\frac{\partial}{\partial t} P_k(t,x) = \sum_{i=1}^{k} \binom{k}{i} h_i P_{k-i}(t,x)$, that is, for $1 \leq j \leq k$, $\frac{d}{dt}\, p_j^k(t) = \sum_{i=1}^{k-j} \binom{k}{i} h_i p_j^{k-i}(t)$;

(iv) for each $k \geq 1$, $P_k(0,x) = x^k$, i.e. $p_j^{(k)}(0) = 0, \quad 0 \leq j < k$; and

(v) $\mathcal{P} = sp\{P_k : k \geq 1\}$, that is, the linear span of $\{P_k : k \geq 1\}$ is all of $\mathcal{P}$.

It will be convenient to attach suitable names to these properties for future reference. Accordingly, we designate property *(i)* the *restriction* property, *(ii)* the *Appell* property, *(iii)* the *pseudo-type-zero* property, *(iv)* the *uniqueness* property and *(v)*, the *spanning* property. Properties *(i)*, *(ii)*, *(iii)* and *(iv)* can be seen to follow from the representation (1.5) without much difficulty. Our alternative approach, however, will help to bring out the specific significance of each of these properties. In particular, they will be shown to be influential in determining various distributional properties of the underlying process M. These distributional properties will often be stated, in discrete time, in terms of the difference sequence $\{X_n = M_n - M_{n-1} : n \geq 1\}$ associated to M, and in continuous time, through its increments $M_{t+s} - M_t,\ t, s > 0$.

It is not difficult to see that with the convention $P_0 \equiv 1$, property *(ii)* implies property *(i)*. If, moreover, M happens to be itself a martingale, one can choose $P_1(t, x) = x$, and in this case *(ii)* actually implies a slightly stronger property than *(i)*, namely,

(i′) for each $k \geq 1$, $P_k(\cdot, x) - x^k$ has degree at most $k - 2$ (in x), that is, $p_k^{(k)}(\cdot) \equiv 1$ and $p_{k-1}^{(k)}(\cdot) \equiv 0$.

A word of explanation is in order for the nomenclature of the properties. *(i)* and *(v)* are more or less self-explanatory. To explain *(ii)* and *(iii)*, we reproduce some work of Sheffer from [20]. A sequence of (one-variable) polynomials $(p_k)_{k\geq 0}$, where p_k is of degree k, is called an *Appell set* if they satisfy $p_k' = k\,p_{k-1}$, (or sometimes $p_k' = p_{k-1}$), $k \geq 1$. In that sense, if *(ii)* is satisfied, we see that for each t our polynomials $(P_k(t, \cdot))_{k\geq 0}$ forms an Appell set. It is well-known that a sequence $(p_k)_{k\geq 0}$ is an Appell set if, and only if, there exists a formal power series $A(\alpha) = \sum_{n=0}^{\infty} a_k \alpha^k / k!$ such that

$$A(\alpha) \exp(\alpha x) = \sum_{k=0}^{\infty} p_k(x) \frac{\alpha^k}{k!}\,, \tag{1.6}$$

again as a formal power series. A is called the determining series of the sequence $(p_k)_{k\geq 0}$, or the determining function, if it actually defines one in some domain. In the case of the Appell set of polynomials $(P_k(t, \cdot))_{k\geq 0}$ obtained from (1.5), the interpretation of this function A_t, say, which now depends on t, becomes quite apparent. In this case A_t clearly represents the reciprocal of the m.g.f. of M_t.

To understand property *(iii)*, we continue to follow Sheffer [20] and recall that more generally, a sequence $(p_k)_{k\geq 0}$ of one-variable polynomials with $\deg(p_k) = k$ is said to be of *type zero* if there exists a sequence of numbers c_i, $i \geq 1$, with $c_1 \neq 0$, such that for each k, $k p_{k-1} = \sum_i c_i p_k^{(i)}$ (or sometimes $p_{k-1} = \sum_i c_i p_k^{(i)}$), where for any function f, $f^{(i)}$ denotes its i-th derivative. In analogy with the representation (1.6) for Appell polynomials, a type zero sequence $(p_k)_{k\geq 0}$ satisfies an expansion

$$\sum_{k=0}^{\infty} p_k(t)\,\frac{\alpha^k}{k!} = B(\alpha)e^{xh(\alpha)} \tag{1.7}$$

where B and h are formal power series, and $h(\alpha) = \sum_{k=1}^{\infty} h_k\alpha^k/k!$ is the (formal) inverse of the (again formal) power series $\sum_k c_k\alpha^k/k!$. In this case, the polynomials $(p_k)_{k\geq 0}$ also satisfy the relation $p_k'(t) = \sum_{i=1}^{k-j}\binom{k}{i}h_i p_{k-i}(t)$ for every $k \geq 1$. We shall designate this latter property for the sequence $(p_k)_{k\geq 0}$ as the pseudo-type-zero property. While the type zero property implies the pseudo-type-zero property, it turns out that the latter is really only slightly weaker than the former. In fact, if the pseudo-type-zero property holds with $h_1 \neq 0$, then it actually implies the type zero property, and hence these two are equivalent in this case. Actually, this is the only situation considered in [20].

Returning to our polynomials P_k now, what property *(iii)* really says is that for each fixed x, the sequence $\{P_k(\cdot, x) : k \geq 0\}$ is pseudo-type-zero, which explains the name for property *(iii)*. Thus, while the Appell property, as we call it, refers to the space-derivatives of our polynomials, the pseudo-type-zero property concerns derivatives with respect to the time variable. We shall see later that in the presence of property *(ii)*, *(iii)* is equivalent to the pseudo-type-zero property for the one-variable polynomials $p_0^{(k)}$, $k \geq 0$.

A question that deserves some attention at this point is, whether one cannot actually get the time-space harmonic polynomials $\{P_k(\cdot, x) : k \geq 0\}$ to satisfy the classical type zero property, rather than the pseudo-type-zero property as stated. The answer to this is 'no' in general. To see why, let us observe that if the sequence $\{P_k : k \geq 0\}$ arises from the expansion (1.5), the formal power series $h(\alpha) = \sum_{k=1}^{\infty} h_k\alpha^k/k!$ indeed defines a function, which is just the negative of φ, the c.g.f. of M_1. Clearly in this situation $h_1 = 0$ if, and only if, the first cumulant, or equivalently, the mean of M_1 is zero. This happens to be the case with Brownian motion for instance, so that while the two-variable Hermite polynomials defined before satisfy the pseudo-type-zero property, they are not of type zero in the classical, or Sheffer's, sense.

As for *(iv)*, we shall see that in the presence of *(i)*, it is instrumental in rendering the choice of the sequence unique. Property *(ii)* is actually stronger than both *(i)* and *(iv)*, so that when *(ii)* holds, it guarantees uniqueness by itself. However, when we do not assume *(ii)*, *(iv)* is required to achieve uniqueness. To maintain a semblance of uniformity with these cases, we actually make use of *(iv)* to obtain uniqueness in other situations also, even when *(ii)* obtains.

Another important distinction needs to be emphasised between properties *(i)* – *(iv)* and property *(v)*. While the former hold for *specific* sequences, the latter, on the contrary, is true irrespective of the choice of the sequence $P_k \in \mathcal{P}_k$. That is because the existence of any such sequence implies that for every $j \geq 1$ the dimension of $\cup_{k=1}^{j} \mathcal{P}_k$ is exactly j, and $P_1, P_2, \cdots, P_j$ are clearly linearly independent elements in it. Actually, we shall see later in Theorems 2.6 and 3.5 that property *(i)* is more or less of the same character, in that for a process satisfying the support condition **(S)**, if $\bar{\mathcal{P}}_k$ is nonempty $\forall\, k \geq 1$, then every $P \in \mathcal{P}_k$ must actually belong to $\bar{\mathcal{P}}_k$, i.e. $\mathcal{P}_k = \bar{\mathcal{P}}_k$ for all k.

1.5 Summary of later chapters

The following gives a brief overview of the material of the subsequent chapters.

In discrete time, we first sketch a formal proof of Neveu's result giving the existence of time-space harmonic polynomials of partial sums of iid random variables, and that they indeed satisfy the stated properties. Next, we generalise Neveu's work to derive a necessary and sufficient condition for p-harmonisability when the summands are only independent but not necessarily identically distributed any more. In this case, we actually exhibit a unique sequence of polynomials, which satisfies the properties *(i)*, *(ii)*, *(iv)* and *(v)* and determines the distribution of the underlying process M. This is followed by a necessary condition for general M. However, this necessary condition is for M to exhibit not simply p-harmonisability, but that property alongwith an additional hypothesis. This necessary condition is then shown to entail Markov property for M under some mild additional restrictions. Next, in analogy with p-harmonisability for processes with independent differences, a necessary and sufficient condition is furnished for general processes to be restricted p-harmonisable. As a byproduct we obtain, in this case, the equality of the restricted classes of polynomials and the corresponding unrestricted classes. We also show that in all the cases where this sequence can be chosen satisfying *(i)*, it has the spanning property *(v)*, and also determines the distribution of M upto joint moments. These results, which appeared in [7], form the crux of Chapter 2.

In Chapter 3, parallel results for continuous-time counterparts to the various discrete-time cases are first sought and obtained. The first case is that of homogeneous Lévy processes, which are continuous-time counterparts to discrete-time processes with iid differences. These are shown to always exhibit p-harmonisability, and the polynomials that arise can be chosen to satisfy the listed properties. For general Lévy processes, which we take as the analogues of partial sums of independent random variables, a condition similar to that proved for the latter is shown to be necessary and sufficient for the property under study. This condition, translated in terms of the Lévy measure of the process, gives rise to an interesting analytical problem. For general processes, an analogous necessary condition for p-harmonisability holds, and again yields Markov property as a corollary with a similar additional restriction imposed. As in Chapter 2, here too, for general processes, a necessary and sufficient condition is seen to be afforded only by restricted p-harmonisability, and also the equality between the restricted and unrestricted classes of polynomials in this case is obtained. We also study certain semi-stable Markov processes [12] in this context. The technique of intertwined semigroups as set forth in [3], e.g., allows us to obtain some not so well-known examples of p-harmonisable processes such as the Azéma martingale [1] etc.. We also briefly discuss some extensions to multivariate or more general cases.

Next, starting with a p-harmonisable process M, we study in Chapter 4 various algebraic properties of a sequence of time-space harmonic polynomials and their ramifications on the distribution of the process. Beginning with property *(ii)*, we show that this property forces the conditional moments of the differences (or increments) of the process, given the past, to be degenerate. A mild additional hypothesis is then seen to yield independence of these differences (or increments). Next, the property *(iii)*, in conjunction with *(ii)* and the same additional condition, is shown to yield furthermore their being identically distributed (or stationary). To prove this theorem, however, we adopt an alternative approach to that employed so far, for reasons explained there. Certain other properties of the polynomials such as homogeneity in the two arguments are shown to have some consequences for the process.

Chapter 5 is reserved for investigating how far the work of the earlier chapters can be carried through when the moment condition **(M)** is relaxed. Instead, we assume the existence of $\mathbf{E}|M_n|^i$ for some $i \geq 1$ and investigate conditions on M such that $\mathcal{P}_k(M) \neq \emptyset$ only for $1 \leq k \leq i$. This situation we refer to as *partial* p-harmonisability. Partial restricted p-harmonisability is analogously defined. We also try to see how far the other results obtained previously for a sequence of time-space harmonic polynomials can be transplanted to this case, where we have only a finite sequence $\{P_k : 1 \leq k \leq i\}$. We close this chapter with a partial answer a question previously raised : namely, to exhibit a necessary and sufficient condition for a Lévy process to be determined in distribution by finitely many time-space harmonic polynomials. This material in adapted from [16].

The last chapter contains miscellaneous examples and some counterexamples. The latter illustrate the efficiency of our results, in that they establish that the hypotheses imposed to prove many of these are indeed essential. The examples are of known, and less well-known, processes with the p-harmonisability property, and of processes that are finitely polynomially determined, in the sense described in Chapter 5. A somewhat exceptional case arises from the 'measure' on path space developed by Hochberg [8]. We show that this 'measure' admits time-space polynomials that are 'harmonic' in a certain sense. Finally, a few special cases are dealt with in more detail, and a few of the time-space harmonic polynomials listed in various examples.

Chapter 2

Results in Discrete Time

2.1 Brief exposition of Neveu's method

In this chapter we observe conditions for p-harmonisability of discrete-time processes. As mentioned in the previous chapter, the simplest examples of p-harmonisable processes in discrete time are processes arising out of iid summands. Our first theorem pertains to this case.

Practically all of this theorem follows from the next two as a special case. Nonetheless we give a proof chiefly because this result was already known. The method, due to Neveu, is different from our arguments in the subsequent results, and it uses an added assumption, namely, that of the existence of a certain m.g.f.. However, this assumption is not necessary for the main assertion of the theorem to hold good, as will be shown by our later work.

In the statement, we tacitly assume that the distribution of X_1 is not degenerate. Notice that this guarantees the fulfilment of the condition **(S)** by the process M.

Theorem 2.1 *Suppose $\{X_n : n \geq 1\}$ is a sequence of independent and identically distributed random variables with X_1 having finite moment generating function in a nondegenerate open neighbourhood of Γ of 0. Set $M_0 \equiv 0$ and $M_n = \sum_{l=1}^{n} X_l$, $n \geq 1$. Then the process M is p-harmonisable, and there exists a unique sequence $\{P_k \in \mathcal{P}_k(M) : k \geq 1\}$ satisfying the properties (i)– (v). Further, if N is any discrete-time p-harmonisable process for which $\mathcal{P}_k(N) = \mathcal{P}_k(M)\ \forall\ k \geq 1$, then N has same distribution as M.*

Proof : Let us denote the c.g.f. of X_1, or in other words the logarithm of its m.g.f., by φ, defined on Γ. Then we have $\varphi(\alpha) = \sum_{k=1}^{\infty} \gamma_k \alpha^k / k!$, where for $k \geq 1$, the number γ_k is the k-th cumulant of X_1.

Define, for $t \geq 0$, $x \in \mathbb{R}$ and $\alpha \in \Gamma$, the function

$$\eta(\alpha, t, x) = \exp\{\alpha x - t\varphi(\alpha)\}.$$

It is easy to see that $\forall\ \alpha \in \Gamma$, the function $\eta(\alpha, \cdot, \cdot)$ is time-space harmonic for M. In fact, $\eta(\alpha, n, M_n)$ is called the exponential martingale.

Now, expand η as power series in α :

$$\begin{aligned}
\eta(\alpha, t, x) &= \left[1+\sum_{k=1}^{\infty} \frac{x^k}{k!}\alpha^k\right] \cdot \left[1+\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} \frac{\gamma_i}{i!}\alpha^i\right)^j \frac{(-t)^j}{j!}\right] \\
&= \left[1+\sum_{k=1}^{\infty} \frac{x^k}{k!}\alpha^k\right]\left[1+\sum_{j=1}^{\infty}\left(\sum_{i_1,\dots,i_j\geq 1} \frac{\gamma_{i_1}\cdots\gamma_{i_j}}{i_1!\cdots i_j!}\alpha^{i_1+\cdots+i_j}\right)\frac{(-t)^j}{j!}\right] \\
&= \left[1+\sum_{k=1}^{\infty} \frac{x^k}{k!}\alpha^k\right]\left[1+\sum_{j=1}^{\infty}\left\{\sum_{m=j}^{\infty}\left(\sum_{\substack{i_1,\dots,i_j\geq 1\\ i_1+\cdots+i_j=m}} \frac{\gamma_{i_1}\cdots\gamma_{i_j}}{i_1!\cdots i_j!}\right)\alpha^m\right\}\frac{(-t)^j}{j!}\right] \\
&= 1+\sum_{k=1}^{\infty}\left[x^k + k!\sum_{m=1}^{k}\frac{x^{k-m}}{(k-m)!}\sum_{j=1}^{m}\left(\sum_{\substack{i_1,\dots,i_j\geq 1\\ i_1+\cdots+i_j=m}} \frac{\gamma_{i_1}\cdots\gamma_{i_j}}{i_1!\cdots i_j!}\right)\frac{(-t)^j}{j!}\right]\frac{\alpha^k}{k!} \\
&= \sum_{k=0}^{\infty} P_k(t,x)\frac{\alpha^k}{k!}.
\end{aligned}$$

For every $t \geq 0$ and $x \in \mathbb{R}$, this expansion is valid for $\alpha \in \Gamma$. One easily reads off from the above calculation that for all $k \geq 1$, the coefficient $P_k(t, x)$ of $\alpha^k/k!$ in the above expansion is a polynomial in t and x, and furthermore, that it is of degree k in both t and x.

To see that the sequence $\{P_k : k \geq 1\}$ satisfies the properties *(i)*– *(iv)*, we go back to the discussion on page 9 in section 1.4, where the expansion (1.5) led to these properties. Specifically, *(i)* and *(iv)* follow just by inspection. For *(ii)*, one differentiates the function η with respect to the variable x and collects the coefficients of α^k on both sides. In the same way, differentiation with respect to t leads to the property *(iii)*.

As far as the spanning property *(v)* and the property of determination of the distribution of M are concerned, the proofs are no different from the same properties when M has only independent but not necessarily identically distributed differences. These cases are treated subsequently in Theorem 2.3, so we skip the details for these two properties. The only difference here is that owing to the existence of the m.g.f. of M_1, the distribution of M is determined by $\{\mathcal{P}_k(M)\}$ entirely and not just upto moments, unlike in Theorem 2.3. For if N is any process as in the hypothesis, then it will follow by the arguments of Theorem 2.3

that $\forall\ n \geq 0$, $\mathbf{E}N_n^k = \mathbf{E}M_n^k$ for every $k \geq 1$, and as a result, N_n admits a m.g.f. which is indeed the same as that of M_n. It follows that N also has iid differences with the same common distribution as those of M.

Let us just show now that each P_k, $k \geq 1$, is time-space harmonic for M. First of all, by the Mean Value Theorem, one has, for every $k \geq 1$, and for any $t > 0$, $x \in \mathbb{R}$, $\alpha \in \Gamma$,

$$\eta(\alpha, t, x) - \sum_{j=0}^{k-1} P_j(t, x)\, \frac{\alpha^j}{j!} = \frac{\alpha^k}{k!}\, \eta^{(k)}\left(\alpha_k(t, x), t, x\right) \tag{2.1}$$

where $\eta^{(k)}$ denotes the k-th partial derivative (with respect to α) of η and $\alpha_k(t, x)$ lies between 0 and α. Now, it is not difficult to see that $\eta^{(k)}(\alpha, t, x) = Q_k(\alpha, t, x)\eta(\alpha, t, x)$ where Q_k is a polynomial of degree at most k in both t and x, say $Q_k(\alpha, t, x) = \sum_{i=0}^{k}\sum_{j=0}^{k} q_{i,j}(\alpha)t^i x^j$. Further, the coefficients $q_{i,j}(\alpha)$ are just some constant multiples of products of various powers of derivatives of φ upto order k.

Let us choose $\delta > 0$ such that $[-\delta, \delta] \subseteq \Gamma$. Then the function φ, along with all its derivatives is bounded on $(-\delta, \delta)$. This implies the existence of a constant $C_{t,k}$ for each $t > 0$ and $k \geq 1$, such that $t^i|q_{i,j}(\alpha)| < C_{t,k}$ and $e^{-t\varphi(\alpha)} < C_{t,k}$ for all $\alpha \in (-\delta, \delta)$ and $0 \leq i, j \leq k$.

Observe also that for every $j \geq 1$, there exists a constant D_j such that for all $\alpha \in (-\delta, \delta)$ and all $x \in \mathbb{R}$, $|x^j e^{\alpha x}| \leq D_j \max\{e^{-\delta x}, e^{\delta x}\}$. Combining all these facts, it follows that for each $n \geq 1$ and $k \geq 1$, there exists a constant $A_{n,k}$ such that for all $\alpha \in (-\delta, \delta)$, $|\eta^{(k)}(\alpha, n, M_n)| \leq A_{n,k} \cdot \max\{e^{-\delta M_n}, e^{\delta M_n}\}$, and the latter is an integrable random variable by our choice of δ.

Going back to the equation (2.1) with $k = 1$ now, we see that for $|\alpha| < \delta$,

$$\left|\frac{1}{\alpha}(\eta(\alpha, n, M_n) - 1)\right| = \left|\eta^{(1)}(\alpha_1(n, M_n), n, M_n)\right| \leq A_{n,1} \cdot \max\{e^{-\delta M_1}, e^{\delta M_1}\}.$$

Applying DCT for conditional expectations (given $\mathcal{F}_{n-1}$) and making use of the fact that $\eta(\alpha, n, M_n)$ is an $(\mathcal{F}_n)$-martingale, we get

$$\begin{aligned}\mathbf{E}\left[P_1(n, M_n)|\mathcal{F}_{n-1}\right] &= \mathbf{E}\left[\lim_{\alpha\to 0} \frac{1}{\alpha}(\eta(\alpha, n, M_n) - 1)|\mathcal{F}_{n-1}\right] \\ &= \lim_{\alpha\to 0} \frac{1}{\alpha}\left[\mathbf{E}(\eta(\alpha, n, M_n)|\mathcal{F}_{n-1}) - 1\right] \\ &= \lim_{\alpha\to 0} \frac{1}{\alpha}(\eta(\alpha, n-1, M_{n-1}) - 1) = P_1(n-1, M_{n-1}),\end{aligned}$$

that is, P_1 is time-space harmonic for M. The proof for harmonicity of each P_k, $k > 1$, can now be completed by induction using the fact that for each $k \geq 1$,

$$P_k(n, M_n) = \lim_{\alpha\to 0} \frac{k!}{\alpha^k}\left[\eta(\alpha, n, M_n) - \sum_{j=0}^{k-1} P_j(n, M_n)\, \frac{\alpha^j}{j!}\right] \qquad \blacksquare$$

2.2 Independent summands

The most immediate generalisation of processes of the above kind are processes with differences that are independent but not necessarily identically distributed. In this case, it may be noted that the *support* condition (S) means simply that infinitely many of the differences are nondegenerate. This is the case we next address. Our result describes a necessary and sufficient condition for p-harmonisability of such processes. But let us first make a remark that will be useful in the proof of the theorem and also in the sequel.

Remark 2.1 If q is any polynomial of a single variable then there exists a unique one-variable polynomial p such that for each $n \geq 1$,

$$p(n) = \sum_{l=1}^{n} q(l). \tag{2.2}$$

Moreover, this polynomial p satisfies $p(0) = 0$. Throughout, starting from a one-variable polynomial q, when we talk of defining a polynomial p by the formula (2.2), we actually refer to this unique polynomial.

Theorem 2.2 *Suppose* $M = \{M_n : n \geq 0\}$ *has independent difference sequence* $\{X_n : n \geq 1\}$ *and satisfies conditions* (M) *and* (S). *Then* $\mathcal{P}_k$ *is non-empty for all* $k \geq 1$, *if and only if for each* $k \geq 1$, $\mathbf{E}(X_n^k)$ *is a polynomial in* n.

Proof : Denote $b_k(n) = \mathbf{E}(X_n^k)\ \forall\ k,\ n$. Note first that $\forall\ j \geq 0,\ \forall\ n \geq 1$,

$$\mathbf{E}(M_n^j|\mathcal{F}_{n-1}) = \mathbf{E}[\sum_{i=0}^{j}\binom{j}{i}M_{n-1}^i X_n^{j-i}|\mathcal{F}_{n-1}] = \sum_{i=0}^{j}\binom{j}{i}M_{n-1}^i\, b_{j-i}(n).$$

Now any two-variable polynomial $P(\cdot, x)$ of degree k in x with $P(0,0) = 0$ can be written as $P(t,x) = \sum_{j=0}^{k} p_j(t)x^j$ where $p_j(\cdot)$, $0 \leq j \leq k$, are polynomials of a single variable, and $p_0(0) = 0$. Given such a polynomial P, it belongs to class $\mathcal{P}_k$ if and only if $\forall\ n \geq 1$,

$$P(n-1, M_{n-1}) = \mathbf{E}[P(n, M_n)|\mathcal{F}_{n-1}] = \sum_{j=0}^{k} p_j(n) \sum_{i=0}^{j}\binom{j}{i}M_{n-1}^i\, b_{j-i}(n) \quad a.s.,$$

that is,

$$\sum_{i=0}^{k} M_{n-1}^i p_i(n-1) = \sum_{i=0}^{k} M_{n-1}^i \sum_{j=i}^{k}\binom{j}{i} p_j(n)\, b_{j-i}(n) \quad a.s. \tag{2.3}$$

We now proceed to prove the "only if" part of the theorem; we use induction argument. Clearly $b_0(n) \equiv 1$. To show that $b_k(\cdot)$ is a polynomial for a general $k \geq 1$, we assume, as induction hypothesis, that for all $j \le k-1$, the $b_j(\cdot)$'s are polynomials. Since $\mathcal{P}_k$ is non-empty, there is a polynomial $P(t,x) = \sum_{j=0}^{k} p_j(t) x^j$ as above, so that the equation (2.3) is satisfied. But this is a polynomial equation in M_{n-1} of degree k, so in view of the "support" condition **(S)**, we must have, for infinitely many n's, the equality

$$p_i(n-1) = \sum_{j=i}^{k} \binom{j}{i} p_j(n)\, b_{j-i}(n) \tag{2.4}$$

for each i, $0 \le i \le k$. However, for each i with $1 \le i \le k$, the R.H.S. of (2.4) is a polynomial in n by induction hypothesis while the L.H.S. is so anyway, and therefore, equality (2.4) holds actually for all $n \ge 1$. We next show that for $i = 0$ also, equality (2.4) is true for all $n \ge 1$. For that consider the process $\{Q_n : n \ge 0\}$ defined as $Q_0 = 0$ and for $n \ge 1$,

$$Q_n = \left[-\sum_{l=1}^{n} \sum_{j=1}^{k} p_j(l)\, b_j(l) \right] + \sum_{j=1}^{k} p_j(n) M_n^j .$$

Using the formula for $\mathbf{E}(M_n^j | \mathcal{F}_{n-1})$ as obtained above, it is quite easy to see that $\{Q_n : n \ge 0\}$ is an $(\mathcal{F}_n)$-martingale, and hence so also is $\{P(n, M_n) - Q_n : n \ge 0\}$. But for each $n \ge 1$, $P(n, M_n) - Q_n = p_0(n) + \sum_{l=1}^{n} \sum_{j=1}^{k} p_j(l)\, b_j(l)$, which is non-random. Therefore, $P(n, M_n) - Q_n \equiv P(0, M_0) - Q_0 = 0$ for all $n \ge 1$, which means that

$$p_0(n) \quad = \quad -\sum_{l=1}^{n} \sum_{j=1}^{k} p_j(l)\, b_j(l)$$

for all n, and consequently,

$$p_0(n-1) - \sum_{j=1}^{k} p_j(n)\, b_j(n) \quad = \quad p_0(n) \quad \text{for all } n.$$

This last equality is nothing but (2.4) for $i = 0$. Thus we have proved that for every $n \ge 1$,

$$p_k(n)\, b_k(n) \quad = \quad p_0(n-1) - \sum_{j=0}^{k-1} b_j(n)\, p_j(n). \tag{2.5}$$

But appealing to equation (2.4) for $i = k$, one easily obtains that $p_k(n)$ must be a constant over n. This proves that $b_k(n)$ is a polynomial in n and completes the induction.

As for the "if" part, we show non-emptiness of the $\mathcal{P}_k$'s by constructing explicitly, for each $k \ge 1$, a polynomial $P_k \in \mathcal{P}_k$. For $k = 1$, we take $P_1(t,x) = x + p_0^{(1)}(t)$, where $p_0^{(1)}$ is defined through the relation (2.2) in Remark 2.1 by $p_0^{(1)}(n) = \sum_{l=1}^{n} [-b_1(l)]$. The two-variable function P_1 is evidently a polynomial in t and x, with degree 1 in x. Time-space harmonicity of P_1 for M is a consequence of the independence of the differences $(X_n)_{n \ge 1}$.

For $k \geq 2$, we take $P_k(t,x) = \sum_{i=0}^{k} p_i^{(k)}(t)x^i$, where the $p_i^{(k)}$, $0 \leq i \leq k$, are defined recursively as follows: $p_k^{(k)}(t) \equiv 1$, and having obtained $p_j^{(k)}$ for $i+1 \leq j \leq k$, where $0 \leq i \leq k-2$, $p_i^{(k)}$ is defined, as in Remark 2.1 by the formula

$$p_i^{(k)}(n) = \sum_{l=1}^{n}\left[-\sum_{j=i+1}^{k}\binom{j}{i} b_{j-i}(l)\, p_j^{(k)}(l)\right], \quad n \geq 1,$$

or, equivalently,

$$p_i^{(k)}(n) = -\sum_{j=i+1}^{k}\binom{j}{i}\sum_{l=1}^{n} b_{j-i}(l)\, p_j^{(k)}(l), \quad n \geq 1, \tag{2.6}$$

$P_k(t,x)$ thus defined is therefore a polynomial jointly in the variables t and x. Also $P_k(t,x)$ is of degree k in x. To see therefore that $P_k \in \mathcal{P}_k$, that is, $\{P_k(n, M_n) : n \geq 0\}$ is an $(\mathcal{F}_n)$-martingale, we merely observe that the $p_i^{(k)}$'s satisfy, by the manner in which they are defined, the relations

$$\begin{aligned} p_k^{(k)}(n-1) &= p_k^{(k)}(n), \quad \text{and for } 0 \leq i \leq k-1, \\ p_i^{(k)}(n-1) &= p_i^{(k)}(n) + \sum_{j=i+1}^{k}\binom{j}{i} b_{j-i}(n) p_j^{(k)}(n); \end{aligned}$$

that is, for all i, $0 \leq i \leq k$,

$$p_i^{(k)}(n-1) = \sum_{j=i}^{k}\binom{j}{i} b_{j-i}(n) p_j^{(k)}(n) \tag{2.7}$$

so that equation (2.3) holds as an identity for all $n \geq 1$. ∎

We now make a couple of remarks. The last two of these will facilitate the comparison of these results with their respective continuous-time counterparts. A little algebra will convince the reader of their truth.

Remark 2.2 The necessary and sufficient condition in the foregoing theorem can be seen to be trivially satisfied by partial sums of iid random variables. Here, each of the moments $\mathbf{E}X_n^k$ are simply constant polynomials.

Remark 2.3 Although the conclusion of Theorem 2.2 was stated in terms of the difference sequence X, it easily translates into the following equivalent condition on the moments of M : namely, $\mathbf{E}(M_n^k)$ be a polynomial in n for each $k \geq 1$. Putting down in this form allows us to readily compare it with the continuous-time Theorem 3.2.

Remark 2.4 Another useful point to note is that the fallout of assuming p-harmonisability can be put in terms of the moments of the *multi-step* differences $M_{m+n} - M_n$ also. In fact, an equivalent form of the conclusion of the above theorem is that for each $k \geq 1$, there is a two-variable polynomial $\bar{b}_k$ such that $\forall\, m,\, n \geq 1$, $\mathbf{E}(M_{m+n} - M_n)^k = \bar{b}_k(m, n)$. This is easy to check by the relation $b_k = \bar{b}_k(1, n-1)$ on one hand and by induction on m on the other.

At this point, the actual existence of a unique sequence of time-space harmonic polynomials with the properties listed earlier is called for.

Theorem 2.3 *If a process M with independent difference sequence and satisfying condition* (**S**) *is p-harmonisable, then there exists a unique sequence $\{P_k \in \mathcal{P}_k : k \geq 1\}$ such that the properties (i), (ii) and (iv) are satisfied. Also, this sequence $\{P_k : k \geq 1\}$ satisfies the property (v) and determines the law of M upto all joint moments of its finite-dimensional distributions.*

Proof : Let us continue to denote $b_k(n) = \mathbf{E}(X_n^k)$, $k,\, n \geq 0$. We prove that the polynomials $\{P_k : k \geq 1\}$ chosen in the proof of the direct part in Theorem 2.2 above, themselves satisfy the stated properties. We shall repeatedly make use of the relations (2.4), (2.5) etc. in the proof of that theorem.

Property *(i)* is immediate from the definition. For property *(ii)*, since $\partial P_1(t, x)/\partial x = 1 = P_0(t, x)$, we have simply to check that $\forall\, k \geq 2$, $\forall\, 1 \leq i \leq k$,

$$i p_i^{(k)}(n) = k\, p_{i-1}^{(k-1)}(n). \tag{2.8}$$

We use downward induction on i : equation (2.8) is clearly satisfied for $i = k$, since $p_k^{(k)} \equiv 1 \equiv p_{k-1}^{(k-1)}$. Now fix any i with $1 \leq i \leq k-1$ and assume, by way of induction hypothesis, that for each j with $i+1 \leq j \leq k$, (2.8) holds (with j replacing i). Then for all $n \geq 1$,

$$\begin{aligned} i p_i^{(k)}(n) &= -\sum_{j=i+1}^{k} i \binom{j}{i} \sum_{l=1}^{n} b_{j-i}(l)\, p_j^{(k)}(l), && \text{by (2.6)},\\ &= -\sum_{j=i+1}^{k} i \binom{j}{i} \sum_{l=1}^{n} b_{j-i}(l)\, \frac{k}{j}\, p_{j-1}^{(k-1)}(l), && \text{by induction hypothesis},\\ &= -k \sum_{j=i+1}^{k} \binom{j-1}{i-1} \sum_{l=1}^{n} b_{j-i}(l)\, p_{j-1}^{(k-1)}(l), &&\\ &= k\, p_{i-1}^{(k-1)}(n), && \text{again by (2.6)}. \end{aligned}$$

To show that the property *(iv)* holds, we refer to the Remark 2.1 which implies that by definition, the polynomials $p_i^{(k)}$, $0 \leq i \leq k-1$, $k \geq 1$, satisfy $p_i^{(k)}(0) = 0$. This, alongwith property *(i)*, gives property *(iv)*.

Next, we prove the spanning property *(v)* : suppose $P \in \mathcal{P}$ and write $P(t,x) = \sum_{i=0}^{k} p_i(t)x^i$. Denoting $a_i = p_i(0)$ for $1 \le i \le k$, we claim that $P = \sum_{i=1}^{k} a_i P_i$, or equivalently, that for all j with $0 \le j \le k$,

$$p_j = \sum_{i=j}^{k} a_i p_j^{(i)} \tag{2.9}$$

First of all, as argued before in the proof of Theorem 2.2, the fact that $P \in \mathcal{P}$ itself ensures that, for infinitely many n's, equation (2.4) holds for all i, $0 \le i \le k$. In particular, with $i = k$ in (2.4), we obtain $p_k(n-1) = p_k(n)$ for infinitely many n. But p_k is a polynomial, so we must actually have $p_k(n) \equiv$ constant $\equiv p_k(0)$ for all $n \ge 1$. Further, since the b_j's are known to be polynomials, therefore, for $0 \le i \le k-1$ also, both sides of (2.4) are polynomials in n, hence (2.4) holds actually for all n. This means that, for $0 \le i \le k-1$,

$$p_i(n) = p_i(0) - \sum_{j=i+1}^{k} \binom{j}{i} \sum_{l=1}^{n} b_{j-i}(l) p_j(l) \tag{2.10}$$

We now prove the claim (2.9) by downward induction on j : since $p_k(n) = p_k(0)$ for all $n \ge 1$ and $p_k^{(k)} \equiv 1$, (2.9) clearly holds for $j = k$. For $0 \le j \le k-1$, we have, by equation (2.10) and by induction hypothesis,

$$\begin{aligned}
p_j(n) &= a_j - \sum_{l=1}^{n} \sum_{i=j+1}^{k} \binom{i}{j} b_{i-j}(l) \sum_{m=i}^{k} a_m p_i^{(m)}(l) \\
&= a_j p_j^{(j)}(n) - \sum_{m=j+1}^{k} a_m \sum_{i=j+1}^{m} \binom{i}{j} \sum_{l=1}^{n} b_{i-j}(l) p_i^{(m)}(l) \\
&= a_j p_j^{(j)}(n) + \sum_{m=j+1}^{k} a_m p_j^{(m)}(n) \qquad \text{by (2.6)} \\
&= \sum_{m=j}^{k} a_m p_j^{(m)}(n),
\end{aligned}$$

completing the proof of claim (2.9) and property *(v)*.

We now show that our chosen sequence $\{P_k\}$ is indeed the only sequence satisfying the property *(iv)*. In fact, we prove that if for $k \ge 1$, $P \in \mathcal{P}_k(M)$ is such that $P(0,x) = x^k$, then $P = P_k$. By the spanning property of our sequence, we can write $P = \sum_{i=1}^{k} a_i P_i$, but then, we would have $x^k = P(0,x) = \sum_{i=1}^{k} a_i P_i(0,x) = \sum_{i=1}^{k} a_i x^i \ \forall\, x$. This means $a_k = 1$ and $a_i = 0$ for $1 \le i < k$, which implies $P = P_k$.

Finally, $\{P_k : k \geq 1\}$, determines the b_k's (which are necessarily polynomials by the "only if" part of Theorem 2.2) recursively from equation (2.5) by

$$b_k(n) = p_0^{(k)}(n-1) - \sum_{j=0}^{k-1} b_j(n) p_j^{(k)}(n) \quad \forall\, k \geq 1,\ \forall\, n \geq 1, \tag{2.11}$$

and finite-dimensional joint moments of M are but linear combinations of products of these numbers. ■

Remark 2.5 It turns out that under the hypothesis of the previous theorem, any sequence $\{P_k \in \mathcal{P}_k : k \geq 1\}$, and not just the one chosen in the proof, possesses the spanning property *(v)* and also determines the moments of the underlying process. For the former, the observation that for every $j \geq 1$, $\cup_{k=1}^{j} \mathcal{P}_k$ has dimension j suffices, as we argued in page 11 in Chapter 1. As for the latter property, owing to the support condition **(S)**, the harmonicity of $\{P_k\}$ would force equation (2.5), and consequently, equation (2.11) to be satisfied. In contrast, It is only property *(iv)* that results in the uniqueness of our chosen sequence.

2.3 The general case : Markov property

While for sums of independent variables, the previous theorem gives a necessary and sufficient condition for p-harmonisability, such a nice characterisation becomes unavailable for general discrete-time processes. Instead, we get only a necessary condition. Besides, slightly more than just p-harmonisability needs to be assumed. In the next section, however, we shall rectify the situation somewhat by considering what we call restricted p-harmonisability in the next section. First, let us describe the necessary condition for p-harmonisability of general processes.

Theorem 2.4 *Suppose for a process $M = \{M_n : n \geq 0\}$ the classes $\mathcal{P}_k(M)$, $k \geq 1$, are each nonempty. Suppose further that for all $n \geq 1$ and for each $k \geq 1$, there is a member $P \in \mathcal{P}_k$ such that $P(n, x)$, as a polynomial in x alone, is of degree exactly k.*

Then, for every $k \geq 1$ and $n \geq 1$, there exists a one-variable polynomial c_n^k of degree at most k such that $\mathbf{E}(M_n^k | \mathcal{F}_{n-1}) = c_n^k(M_{n-1})$ almost surely.

Proof : We prove the result by induction on k. We have to first prove that the conclusion is valid for k=1. Choose and fix a $n \geq 1$. By the hypothesis of the theorem, $\exists$ polynomials p_1 and p_0 with $p_1(n) \neq 0$ such that $P_1(t, x) = p_1(t)x + p_0(t)$ is in $\mathcal{P}_1(M)$. Then,

$$p_1(n-1)M_{n-1} + p_0(n-1) = \mathbf{E}\left(p_1(n)M_n + p_0(n)\,|\mathcal{F}_{n-1}\right) = p_1(n)\,\mathbf{E}(M_n|\mathcal{F}_{n-1}) + p_0(n),$$

so that

$$\mathbf{E}(M_n|\mathcal{F}_{n-1}) = \frac{1}{p_1(n)}\left[p_1(n-1)M_{n-1} - \{p_0(n) - p_0(n-1)\}\right],$$

a polynomial of degree at most 1 in M_{n-1}.

Now for $k \geq 2$ and any $n \geq 1$, choose a $P \in \mathcal{P}_k$ satisfying the hypothesis of the theorem. This means that if we write $P(t,x) = \sum_{j=0}^{k} p_j(t)x^j$, then $p_k(n) \neq 0$. By the time-space harmonicicity of P, we get,

$$\begin{aligned}
\sum_{j=0}^{k} p_j(n-1)M_{n-1}^j &= P(n-1, M_{n-1}) \\
= \mathbf{E}[P(n, M_n)|\mathcal{F}_{n-1}] &= \sum_{j=0}^{k} p_j(n)\,\mathbf{E}(M_n^j|\mathcal{F}_{n-1}) \\
= \sum_{j=0}^{k-1} p_j(n)\,c_n^j(M_{n-1}) &+ p_k(n)\,\mathbf{E}(M_n^k|\mathcal{F}_{n-1}) \quad \text{by induction hypothesis,}
\end{aligned}$$

so that $\mathbf{E}(M_n^k|\mathcal{F}_{n-1}) = c_n^k(M_{n-1})$ where $c_n^k(x)$ is the polynomial of degree at most k defined by

$$c_n^k(x) = \frac{1}{p_k(n)}\left[\sum_{j=0}^{k} p_j(n-1)x^j - \sum_{j=0}^{k-1} p_j(n)\,c_n^j(x)\right]. \tag{2.12}$$

This completes the proof. ∎

Remark 2.6 The necessary condition in the last theorem could also be put in the form : $\forall\, n \geq 1,\ k \geq 1,\ \exists$ a polynomial d_n^k of degree at most k such that

$$\mathbf{E}(X_n^k|\mathcal{F}_{n-1}) = d_n^k(M_{n-1})\ a.\ s.$$

Remark 2.7 Under the hypothesis of the previous theorem, we have, as a more general consequence, for all k, m and n the existence of a polynomial $c_{m,n}^k$ such that $\mathbf{E}(M_{m+n}^k|\mathcal{F}_n) = c_{m,n}^k(M_n)$ almost surely.

Remark 2.8 As a special case, when the same polynomial can be used for all $t > 0$, that is, when, $\forall\ k \geq 1,\ \exists\, P_k \in \mathcal{P}_k(M)$ such that $\forall\ t > 0$, degree of $P_k(t,\cdot)$ is k, we can actually claim that the sequence $\{P_k : k \geq 1\}$ also determines the law of M upto moments of f.d. distributions. That is because, from the fact $\mathbf{E}P_k(t, M_t) = 0$ for all $k \geq 1$, we have $\mathbf{E}M_n = -p_0^{(1)}(n)/p_1^{(1)}(n)$, and more generally,

$$\mathbf{E}M_n^k = -\frac{1}{p_k^{(k)}(n)}\sum_{j=1}^{k-1} p_j^{(k)}(n)\,\mathbf{E}M_n^j.$$

Thus the one-dimensional moments $\{\mathbf{E}M_n^k, k \geq 1\}$ are all determined by the $\{p_j^{(k)} : 1 \leq j \leq k, k \geq 1\}$, or equivalently, by the sequence $\{P_k : k \geq 1\}$.

To get all moments of f.d. laws of higher dimensions, one proceeds as follows. Firstly, for $1 \leq n_1 < n_2$ and any $k_1 \geq 1$, $k_2 \geq 1$, one has, using the notation of Remark 2.7,

$$\mathbf{E}\left[M_{n_1}^{k_1} M_{n_2}^{k_2}\right] = \mathbf{E}\left[M_{n_1}^{k_1}\mathbf{E}(M_{n_2}^{k_2}|\mathcal{F}_{n_1})\right] = \mathbf{E}\left[M_{n_1}^{k_1} c_{n_2-n_1,n_1}^{k_2}(M_{n_1})\right]$$

which is again determined by the sequence $\{P_k\}$. The extension to $\mathbf{E}\left[M_{n_1}^{k_1} M_{n_2}^{k_2} \cdots M_{n_r}^{k_r}\right]$ can be completed using induction on r.

As a Corollary to Theorem 2.4, we derive Markov property for M albeit under an extra condition. This condition is needed merely to enforce that for all $n \geq 1$, the conditional distribution of X_n, given $\mathcal{F}_{n-1}$, be uniquely determined by its moments, that is, by the conditional moments $\mathbf{E}(X_n^k|\mathcal{F}_{n-1})$.

Corollary 2.4.1 *Let M be a process satisfying the hypotheses of Theorem 2.4. If, moreover, for all $n \geq 1$, M_n has finite m.g.f. in the interval $(-\delta_n, \delta_n)$ for some $\delta_n > 0$, then M is Markov.*

Proof : For each $n \geq 1$, let us denote by $\mathbf{Q}_n(\cdot,\cdot)$ a version of the regular conditional distribution of M_n given $\mathcal{F}_{n-1}$. Then, we have that $\int e^{tx}\mathbf{Q}_n(\cdot, dx) = \mathbf{E}(e^{tM_n}|\mathcal{F}_{n-1}) < \infty$ a.s. for all $t \in (-\delta_n, \delta_n)$. This, by a standard result in probability theory (see e.g. [2], p. 408), guarantees that, almost surely, $\mathbf{Q}_n(\cdot, dx)$ is determined by its moments $\{\int x^k \mathbf{Q}_n(\cdot, dx) = c_n^k(M_{n-1}) : k \geq 2\}$, which are functions of M_{n-1} alone, by Theorem 2.4. This means that $\mathbf{Q}_n$ is almost surely equal to (a version of) the r.c.d. of M_n given M_{n-1} alone. Using repeatedly this fact and the smoothing property of conditional expectations one obtains the Markov property for M (not necessarily time homogeneous)! ∎

2.4 The restriction condition

Evidently, general discrete-time processes present a somewhat unhappy picture with regard to p-harmonisability, as opposed to those with independent differences. This distinction is pinpointed by the contrast between Theorems 2.2 and 2.4. While the former yields a necessary and sufficient condition for p-harmonisability at one go, in the latter, only a necessary condition is available, and that too only at the cost of an extra hypothesis. This extra hypothesis, one may note, is a condition imposed on the "leading coefficient" (in the space variable) of elements of $\mathcal{P}$. This leads us to suspect that a condition on these leading coefficients of time-space harmonic polynomials is what is required to arrive at a more satisfactory answer. Indeed, our next Theorem 2.5 bears testimony to this. Here, we consider restricted p-harmonisability, instead of p-harmonisability, for a general discrete-time process and obtain a necessary and sufficient condition.

Theorem 2.5 *For a process $M = \{M_n, n \geq 0\}$ with difference sequence $\{X_n\}$, the classes $\bar{\mathcal{P}}_k(M)$ are non-empty for all $k \geq 1$, if and only if for each $k \geq 1$, there is a two-variable polynomial $B_k(t, x)$ of degree at most $k-1$ in x, such that $\mathbf{E}(X_n^k | \mathcal{F}_{n-1}) = B_k(n, M_{n-1})$ almost surely for all n.*

In this case, if M moreover satisfies condition **(S)**, *then there exists a unique sequence $\{P_k \in \bar{\mathcal{P}}_k : k \geq 1\}$ so that the properties (iv) and (v) are satisfied. Further, this sequence $\{P_k : k \geq 1\}$ determines the law of M upto all moments of its finite dimensional distributions.*

Proof : As in Theorem 2.2, we prove the "only if" part first and do so by means of induction on k. Since $\bar{\mathcal{P}}_k$ is non-empty, there exists a $P \in \bar{\mathcal{P}}_k$ of the form $P(t, x) = x^k + \sum_{j=0}^{k-1} p_j(t) x^j$. For k=1, the fact that $\bar{\mathcal{P}}_1 \neq \emptyset$ itself implies that $(M_n - \mathbf{E}M_n)$ is a martingale, and that $\mathbf{E}M_n$ is a polynomial in n, so that we can clearly take $B_1(n, \cdot) \equiv \mathbf{E}M_n - \mathbf{E}M_{n-1} \equiv \mathbf{E}X_n$. Now for $k \geq 2$, by employing calculations similar to those in the proof of Theorem 2.2, we have, for all $n \geq 1$,

$$\begin{aligned}\sum_{j=0}^{k-1} p_j(n-1) M_{n-1}^j &= \mathbf{E}(X_n^k | \mathcal{F}_{n-1}) + \sum_{i=1}^{k-1} \binom{k}{i} M_{n-1}^i \, \mathbf{E}(X_n^{k-i} | \mathcal{F}_{n-1}) \\ &\quad + \sum_{i=0}^{k-1} p_i(n) \sum_{j=0}^{i} \binom{i}{j} M_{n-1}^j \, \mathbf{E}(X_n^{i-j} | \mathcal{F}_{n-1}),\end{aligned}$$

which by induction hypothesis yields,

$$\begin{aligned}\sum_{j=0}^{k-1} p_j(n-1) M_{n-1}^j &= \mathbf{E}(X_n^k | \mathcal{F}_{n-1}) + \sum_{i=1}^{k-1} \binom{k}{i} M_{n-1}^i \, B_{k-i}(n, M_{n-1}) \\ &\quad + \sum_{i=0}^{k-1} p_i(n) \sum_{j=0}^{i} \binom{i}{j} M_{n-1}^j \, B_{i-j}(n, M_{n-1}) \qquad a.\ s.\end{aligned}$$

that is, $\mathbf{E}(X_n^k | \mathcal{F}_{n-1}) = B_k(n, M_{n-1})$ a.s. $\forall\, n \geq 1$, where $B_k(t, x)$ is the two-variable polynomial given by

$$B_k(t, x) = \sum_{j=0}^{k-1} \left\{ p_j(t-1) - \sum_{i=j}^{k-1} \binom{i}{j} p_i(t) \, B_{i-j}(t, x) \right\} x^j - \sum_{i=1}^{k-1} \binom{k}{i} B_{k-i}(t, x) \, x^i$$

For the "if" part too, we follow the same route as in the corresponding part in Theorem 2.2. First of all, note that since for each $k \geq 1$, $B_k(t, x)$ is of degree at most $k-1$ in x, $B_1(t, x)$ does not involve x at all and thus is just a one variable polynomial in t. In other words, $B_1(t, x) \equiv B_1(t, 0)$, so that $\mathbf{E}(X_n|\mathcal{F}_{n-1}) = B_1(n, M_{n-1}) = B_1(n, 0)$ which is non-random, which means $\mathbf{E}(X_n) = \mathbf{E}(X_n|\mathcal{F}_{n-1}) = B_1(n, 0)$. If now the one-variable polynomial $p_0^{(1)}$ is defined, as in Remark 2.1, by the formula $p_0^{(1)}(n) = \sum_{l=1}^{n} [-B_1(l, 0)]$, $n \geq 1$, then clearly $p_0^{(1)}(n) = -\mathbf{E}M_n$ for all $n \geq 1$. It follows that $\left(M_n + p_0^{(1)}(n)\right)$ is an $(\mathcal{F}_n)$-martingale. So, $P_1(t, x) = x + p_0^{(1)}(t)$ is in $\bar{\mathcal{P}}_1$, and the latter is therefore non-empty.

For $k \geq 2$, we show that $\bar{\mathcal{P}}_k$ is non-empty by again explicitly constructing one-variable polynomials $p_i^{(k)}$, $0 \leq i \leq k$, such that $P_k(t, x) = \sum_{j=0}^{k} p_j^{(k)}(t)x^j \in \bar{\mathcal{P}}_k$. The definition is again recursive: $p_k^{(k)} \equiv 1$, and for $0 \leq j \leq k-1$, having obtained $p_i^{(k)}$ for $j+1 \leq i \leq k$, we define $p_j^{(k)}$, as in Remark 2.1, by the formula

$$p_j^{(k)}(n) = \sum_{l=1}^{n} \left[-\sum_{i=0}^{j} \sum_{m=j+1}^{k} \binom{m}{i} p_m^{(k)}(l)\, b_{j-i}^{m-i}(l) \right] \quad \text{for } n \geq 1,$$

where the b_j^i are the one-variable polynomials obtained from B_i, $1 \leq i \leq k$, by writing $B_i(t, x) = \sum_{j=0}^{i-1} b_j^i(t)x^j$. That the polynomial P_k thus defined does indeed belong to $\bar{\mathcal{P}}_k$ can be checked easily using the following recurrence relation :

$$p_j^{(k)}(n-1) = p_j^{(k)}(n) + \sum_{i=0}^{j} \sum_{m=j+1}^{k} \binom{m}{i} p_m^{(k)}(n)\, b_{j-i}^{m-i}(n)$$

As for the next part, let us first observe by definition, our chosen polynomials $\{P_k\}$ satisfy *(iv)*. Now, if $P(t, x) = \sum_{j=0}^{k} p_j(t)x^j$ and $P \in \mathcal{P}_k$, then the condition **(S)**, the martingale property of $\{P(n, M_n)\}$ and the condition

$$\mathbf{E}(X_n^i|\mathcal{F}_{n-1}) = \sum_{j=0}^{i-1} b_j^i(n)\, M_{n-1}^j \quad a.\ s. \quad \forall\, n \geq 1 \text{ and } i \geq 1,$$

together imply that for infinitely many n's, $p_k(n-1) = p_k(n)$; and for $0 \leq j \leq k-1$,

$$p_j(n-1) = p_j(n) + \sum_{i=0}^{j} \sum_{m=j+1}^{k} \binom{m}{i} p_m(n)\, b_{j-i}^{m-i}(n).$$

But since in each of the above equations, both sides are polynomials in n, equality must hold actually for all n. Now choosing $a_i = p_i(0)$, $i = 0, 1, \ldots, k$, one can show exactly in the same way as in the proof of *(v)* in Theorem 2.2, that $P = \sum_{i=0}^{k} a_i P_i$, where the $\{P_i : i \geq 1\}$ are as defined earlier. This proves that the sequence $\{P_k : k \geq 1\}$ chosen above spans $\mathcal{P}$.

This same argument also serves to ensure uniqueness in view of *(iv)*. For if $P = \sum_{i=1}^{k} a_i P_i$ has $P(0, x) = x^k$, then $a_k = 1$ and $a_i = 0$ for $1 \leq i < k$, so that $P = P_k$.

The proof of the characterization of finite-dimensional moments follows by the smoothing property of conditional expectations, and our previous calculations, since any such moment $\mathbf{E}[M_{n_1}^{k_1} M_{n_2}^{k_2} \ldots M_{n_m}^{k_m}]$ reduces to the expectation of a polynomial in M_{n_1} with coefficients determined by the B_k's and consequently in turn by the P_k's, and one-dimensional moments can easily be derived from the P_k's by an induction argument. ∎

Remark 2.9 Just as in Theorem 2.3, the properties *(v)* and determination of the finite-dimensional moments are shared by any sequence $\{P_k \in \bar{\mathcal{P}}_k : k \geq 1\}$ whatsoever, but it is property *(iv)* that renders the choice unique.

Remark 2.10 The conclusion of Theorem 2.5 was cast in terms of the conditional moments of the differences (X_n). Again, for convenience in comparison with the continuous-time case, we recast it in terms of the conditional moments of (M_n) itself and claim that the following is equivalent to it : for every $k \geq 1$, there exists a two-variable polynomial $C_k(\cdot, \cdot)$ with $C_k(\cdot, x)$ having x^k as leading term in x, such that $\mathbf{E}(M_n^k | \mathcal{F}_{n-1}) = C_k(n, M_{n-1}) \ \forall\, k \geq 1$.

One side of the implication is easy to see : given B_k's, we have

$$C_k(t, x) = x^k + \sum_{j=1}^{k} \binom{k}{j} B_j(t, x) x^{k-j}.$$

The other side follows by induction from the above one :

$$B_k(t, x) = C_k(t, x) - x^k - \sum_{j=1}^{k-1} \binom{k}{j} B_j(t, x) x^{k-j}.$$

Remark 2.11 Alternatively, we can state the result either in terms of the conditional moments of the multi-step differences : $\forall\, k$, $\exists\, \bar{B}_k$, a three-variable polynomial of degree at most $k-1$ in the third variable, such that $\forall\, m,\, n,\, k$,

$$\mathbf{E}((M_{m+n} - M_n)^k | \mathcal{F}_n) = \bar{B}_k(m, n, M_n) \quad a.s.;$$

or in terms of those of M as : $\forall\, k$, $\exists\, \bar{C}_k$, a three-variable polynomial, $\bar{C}_k(\cdot, \cdot, x)$ having leading term x^k, such that $\forall\, m,\, n,\, k$,

$$\mathbf{E}(M_{m+n}^k | \mathcal{F}_n) = \bar{C}_k(m, n, M_n) \quad a.s..$$

Remark 2.12 The connection between Theorem 2.4 and the converse part of Theorem 2.5 is worth comment. Clearly, the P_k's described in the latter are themselves candidates for the choice required in the former, for each n. In fact, in equation (2.12) on page 23 one has $p_k(n) \equiv 1$, and the conclusion of Theorem 2.5 can be seen to follow from this by induction on k. However, we prefer the direct calculations in spite of the partial repetition, since these, first of all, are tractable, and secondly, seem to clarify the proof better.

One fallout of the theorem just proved seems important enough to deserve a separate statement. Its proof follows trivially from the spanning property.

Theorem 2.6 *If, for a process M satisfying* **(S)**, *the classes $\bar{\mathcal{P}}_k(M)$ are non-empty for all $k \geq 1$, then for every $k \geq 1$, $\mathcal{P}_k(M) = \bar{\mathcal{P}}_k(M)$.*

Remark 2.13 The assumption that $\bar{\mathcal{P}}_k(M) \neq \emptyset$ is critical for the conclusion of the theorem; in general, one may have $\bar{\mathcal{P}}_k(M) = \emptyset$ but $\mathcal{P}_k(M) \neq \emptyset$. An example in section 6.2 in the last chapter will bear this out.

Chapter 3

Counterparts in Continuous Time

3.1 A preliminary lemma

In the previous chapter, we investigated conditions for p-harmonisability of discrete-time processes. In the present one, counterparts in similar situations in continuous time are developed. As mentioned before, our general plan is to use the discrete-time results obtained before and appropriate approximation arguments. To make these approximations work, we naturally need to impose a regularity condition of some kind on the paths of the process M under consideration. We assume throughout our process M to have r.c.l.l. (right continuous with left limits) paths. This path property, although not essential as we shall see later, will facilitate simpler proofs. As before, our process M always starts at 0, that is, $M_0 \equiv 0$.

Starting with such a process M, we define, for every $N \geq 1$, the discrete-time process

$$M^{(N)} = \{M_{n/N} : n \geq 0\},$$

and denote its natural filtration by $(\mathcal{G}^{(N)})$, that is,

$$\mathcal{G}_n^{(N)} = \sigma < M_i^{(N)} : 0 \leq i \leq n >, n \geq 0.$$

With this groundwork, we present a lemma which will be needed an essential ingredient in many of our arguments that follow.

Lemma 3.1 *Suppose $t = j/m$ is a positive rational. Then, for any integrable random variable X,*

$$\mathbf{E}(X|\mathcal{F}_t) = \lim_{l\to\infty} \mathbf{E}(X|\mathcal{G}_{tN_l}^{(N_l)})$$

where, for each $l \geq 1$, $N_l = 2^l \cdot m$.

Proof : The assertion follows as a direct consequence of Lévy's Upward Theorem ([2], p. 492), once we show that as $l \to \infty$, the σ-fields $\mathcal{G}_{tN_l}^{(N_l)}$ increase to $\mathcal{F}_t$. That is, we mean to show that $\mathcal{G}_{tN_l}^{(N_l)} \subseteq \mathcal{G}_{tN_{l+1}}^{(N_{l+1})}$ for all $l \geq 1$, and that $\mathcal{F}_t = \vee_{l\geq 1}\mathcal{G}_{tN_l}^{(N_l)}$. Since for any $j \geq 0$ we have $j/N_l = 2j/N_{l+1}$, the first part is immediate. For the second part, we observe that for any s with $0 \leq s < t$, there is a sequence of positive integers j_l with $j_l \leq tN_l$, $l \geq 1$, such that j_l/N_l decreases to s as $l \uparrow \infty$. As a consequence of the right-continuity of the paths, we have $M_s = \lim_{l\to\infty} M_{j_l/N_l}$. M_s, therefore, is measurable with respect to $\vee_{l\geq 1}\mathcal{G}_{tN_l}^{(N_l)}$, inasmuch as each $M_{\frac{j_l}{N_l}}$ is so. Of course, M_t is measurable with respect to $\mathcal{G}_{tN_l}^{(N_l)}$ for every l, and therefore also with respect to $\vee_{l\geq 1}\mathcal{G}_{tN_l}^{(N_l)}$. This shows $\mathcal{F}_t = \sigma< M_t : s \leq t>$ is contained in $\vee_{l\geq 1}\mathcal{G}_{tN_l}^{(N_l)}$. Trivially, $\vee_{l\geq 1}\mathcal{G}_{tN_l}^{(N_l)} \subseteq \mathcal{F}_t$, hence these two are equal. ∎

3.2 Lévy processes

As explained in the introductory chapter, Theorem 3.1, our first result in continuous time, is the analogue to Theorem 2.1. It applies to homogeneous Lévy processes, which are the continuous-time counterparts of partial sums of iid random variables. For us, a Lévy process is a process with independent increments with no fixed discontinuities and having r.c.l.l. (right continuous with left limits) paths. The last assumption, although not always included in the definition of Lévy processes, is not overly restrictive since such a modification always exists for separable processes with independent increments, and taking modifications leaves the property of p-harmonisability (or the lack of it) undisturbed. A homogeneous Lévy process is one which is homogeneous as a Markov process, that is, whose increments are stationary apart from being independent.

Given a continuous-time process M, we shall sometimes require property **(S)** to be satisfied by the discrete-time process (M_{t_n}) for certain sequences (t_n) of times, but unlike in discrete time, an exact equivalent condition to cover precisely for these situations is rather difficult to find. Consequently we impose a condition that suffices for our arguments to go through, namely, **(S)** be satisfied by the discrete-time process $M^{(1)} = \{M_n : n \geq 0\}$. This is what we call **(S)** in the case of continuous time processes.

Clearly, a homogeneous Lévy process, unless it is deterministic, always satisfies **(S)** defined not only in this sense, but in fact through *any* arbitrary increasing sequence of time points (t_n) whatsoever. The moment condition **(M)** remains unchanged from the discrete time situation.

Theorem 3.1 *Let* $M = \{M_t : t \geq 0\}$ *be a homogeneous Lévy process with* $M_0 \equiv 0$ *satisfying the condition* **(S)**, *such that* $\mathbf{E}\exp(\alpha M_1) < \infty$ *for all* $\alpha \in \Gamma$ *where* Γ *is* *some nonempty open neighbouhood of 0. Then there exists a unique sequence $\{P_k \in \mathcal{P}_k(M) : k \geq 1\}$ satisfying the properties (i)– (v). Moreover, any process N for which $\mathcal{P}_k(N) = \mathcal{P}_k(M)$ for all $k \geq 1$ must have the same distribution as that of M.*

Proof : Consider the discrete-time processes $M^{(N)} = \{M_{\frac{n}{N}} : n \geq 0\}$ defined earlier, with their natural filtrations $\mathcal{G}_n^{(N)}$. For every $N \geq 1$, $M^{(N)}$ satisfies, by hypothesis, the conditions in the statement of Theorem 3.1. Thus, for each $N \geq 1$, there exists a unique sequence $\{P_k^{(N)} \in \mathcal{P}_k(M^{(N)}) : k \geq 1\}$ satisfying *(i)- (v)*. This of course means that for every $N \geq 1$, $\{P_k^{(N)}(n, M_{\frac{n}{N}}) : n \geq 0\}$ is a $(\mathcal{G}_n^{(N)})$-martingale for each $k \geq 1$. We prove that the polynomials $P_k = P_k^{(1)}$ are time-space harmonic for the continuous-time M, that is, $\{(P_k(t, M_t) : t \geq 0\}$ is an $(\mathcal{F}_t)$-martingale for every $k \geq 1$.

Fix a $k \geq 1$. We first show the martingale equality $\mathbf{E}(P_k(s, M_s)|\mathcal{F}_t) = P_k(t, M_t)$, for $0 \leq t < s$, when t and s are rationals. We can always write $s = i/m$ and $t = j/m$ where i, j and m are positive integers with $i > j$. For every $l \geq 1$, let N_l be as in Lemma 3.1. Consider the polynomials $\tilde{P}_k(u, x) = P_k^{(N_l)}(uN_l, x)$, $k \geq 1$. By construction, $\tilde{P}_k \in \mathcal{P}_k(M^{(1)})$ and they also satisfy the conditions *(i)- (v)*. In particular, since *(iv)* is satisfied, uniqueness in Theorem 3.1 tells us that for all $k \geq 1$, $\tilde{P}_k = P_k$. Thus, $P_k(\frac{n}{N_l}, M_{n/N_l}) = P_k^{(N_l)}(n, M_n^{(N_l)})$ is a $(\mathcal{G}_n^{(N_l)})$-martingale; consequently, $P_k(t, M_t) = \mathbf{E}(P_k(s, M_s)|\mathcal{G}_{tN_l}^{(N_l)})$. The martingale property follows now on taking limit as $l \uparrow \infty$ and applying Lemma 3.1.

For general $0 \leq t < s$, take sequences $t_n \downarrow t$, $s_n \downarrow s$, $t_n < s_n$, with each t_n and s_n being rational. By our argument for rational times, $P_k(s_n, M_{s_n})$ is a reversed martingale. But by the hypotheses of the theorem, $\mathbf{E}P_k(s_n, M_{s_n})^2$, being some polynomial in s_n, is bounded. Hence $P_k(s_n, M_{s_n})$ is an L^2-bounded reversed martingale. Therefore it converges in L^2, so also in L^1, as $n \to \infty$. Now, by the right-continuity of paths, we have

$$P_k(s, M_s) \stackrel{a.s.}{=} \lim_{n\to\infty} P_k(s_n, M_{s_n}) \stackrel{L^1}{=} \lim_{n\to\infty} P_k(s_n, M_{s_n}).$$

Similarly, $P_k(t, M_t) \stackrel{L^1}{=} \lim_{n\to\infty} P_k(t_n, M_{t_n})$. Therefore, using the smoothing property of conditional expectations,

$$\begin{aligned}
\mathbf{E}(P_k(s, M_s)|\mathcal{F}_t) &\stackrel{a.s.}{=} \lim_{n\to\infty} \mathbf{E}(P_k(s_n, M_{s_n})|\mathcal{F}_t) \\
&= \lim_{n\to\infty} \mathbf{E}(\mathbf{E}(P_k(s_n, M_{s_n})|\mathcal{F}_{t_n})|\mathcal{F}_t) \\
&= \lim_{n\to\infty} \mathbf{E}(P_k(t_n, M_{t_n})|\mathcal{F}_t) = P_k(t, M_t).
\end{aligned}$$

By choice, the sequence $\{P_k\}$ satisfies *(i) - (iv)*. That it is the only sequence to satisfy *(iv)* is a consequence of the support condition **(S)** as we have defined it. The property *(v)* also follows since the sequence spans the obviously larger space $\mathcal{P}(M^{(1)})$.

Finally, the determination of the distribution of M by the sequence of time-space harmonic polynomials is due simply to the fact that they determine the moments of M_1. For these moments themselves determine its m.g.f. $\phi(\alpha) = \mathbf{E}\exp(\alpha M_1)$, $\alpha \in \Gamma$, and the joint m.g.f. of a typical finite-dimensional distribution is

$$\begin{aligned}
&\mathbf{E}\exp\left(\alpha_1 M_{t_1} + \alpha_2 M_{t_2} + \cdots + \alpha_r M_{t_r}\right) \\
&= \mathbf{E}\exp\left(\sum_{i=1}^{r} \alpha_i M_{t_1} + \sum_{i=2}^{r} \alpha_i (M_{t_2} - M_{t_1}) + \cdots + \alpha_r (M_{t_r} - M_{t_{r-1}})\right) \\
&= \phi(\sum_{i=1}^{r} \alpha_i t_1)\, \phi\left(\sum_{i=2}^{r} \alpha_i (t_2 - t_1)\right) + \cdots + \phi\left(\alpha_r (t_r - t_{r-1})\right).
\end{aligned}$$

where $r \geq 1$, $0 \leq t_1 < t_2 < \ldots < t_r$, and $\alpha_1, \alpha_2, \ldots, \alpha_r \in \Gamma$. This determines the joint distribution of $M_{t_1}, M_{t_2}, \ldots, M_{t_r}$, and thereby, since r, t_1, t_2, $\ldots$, t_r are arbitrary, the law of M. ∎

Remark 3.1 One could also have proved the theorem by expanding the exponential martingale like Theorem 2.1 in discrete time. We prefer, however, to give the method of approximation in full since this is what will be required in most of our subsequent results in continuous time.

Remark 3.2 Barring the determination of the distribution of M in its entirety, the condition on the existence of m.g.f. becomes redundant, since Theorem 3.1 becomes wholly a mere special case of Theorem 3.2. The finite-dimensional joint moments can still be deduced from the polynomials, in the following way. The moments of M_1 being determined by $\{P_k\}$, it follows that so are those of each M_t. This leads to the moments $\mathbf{E}((M_t - M_s)^k) = \mathbf{E}(M_{t-s}^k)$ being determined, and these in turn, to the joint moments

$$\mathbf{E}(M_{t_1}^{k_1} M_{t_2}^{k_2}) = \mathbf{E}\left[M_{t_1}^{k_1}(M_{t_1} + (M_{t_2} - M_{t_1}))^{k_2}\right] = \sum_{j=0}^{k_2} \binom{k_2}{j} \mathbf{E}(M_{t_1}^{k_1+j})\, \mathbf{E}(M_{t_2-t_1}^{j}).$$

Higher dimensional moments follow analogously. The condition on existence of the m.g.f. was used only to allow for the application of the discrete-time Theorem 2.1 directly.

The next theorem applies to general Lévy processes. This is our continuous-time analogue to Theorem 2.2. Here, let us describe one condition that suffices to ensure (S) as defined earlier, as well as by any sequence $\{M_{t_n}\}$ whenever $0 \leq t_1 < t_2 < \ldots$. This condition is that $\forall\, 0 \leq s < t$, $M_t - M_s$ is a nonconstant random variable. In other words, we demand that no increment of the process have a degenerate distribution.

Theorem 3.2 *Let $M = \{M_t : t \geq 0\}$ be a Lévy process with $M_0 \equiv 0$ satisfying conditions the conditions* **(M)** *and* **(S)**. *Then $\mathcal{P}_k(M) \neq \emptyset \ \forall\ k \geq 1$ if, and only if, $\mathbf{E}M_t{}^k$ is a polynomial in t for every $k \geq 1$. In such a case, there exists a unique sequence $\{P_k \in \mathcal{P}_k : k \geq 1\}$, satisfying the properties (i), (ii) and (iv). This sequence also satisfies the property (v) and determines the law of M upto moments of finite-dimensional distributions.*

Proof : For the "if" part, consider as before, for $N \geq 1$, the discrete-time processes $M^{(N)}$, which are all processes with independent differences. By hypothesis, each of them satisfies the hypotheses of Theorem 2.2. Hence for each $N \geq 1$, there exists a unique sequence $\{P_k^{(N)} \in \mathcal{P}_k(M^{(N)}) : k \geq 1\}$ satisfying *(i)*, *(ii)* and *(iv)*. Then, following exactly the same argument as in the proof of Theorem 3.1, we can conclude that $P_k = P_k^{(1)}$ belongs to $\mathcal{P}_k(M)$ for every $k \geq 1$.

Clearly the sequence $\{P_k\}$ satisfies *(i)*, *(ii)* and *(iv)*. *(v)* also follows since the sequence spans the obviously larger space $\mathcal{P}(M^{(1)})$. Property *(iv)* guarantees uniqueness. Also, by virtue of *(i)*, the moments of each M_t can be determined inductively from this sequence using simply the fact that $\mathbf{E}P_k(t, M_t) = 0$. This is enough to determine all the moments of finite-dimensional joint distributions, using independence of the increments.

For the converse part, one first needs to note that any sequence $\{P_k \in \mathcal{P}_k(M) : k \geq 1\}$ has to satisfy the property *(i)*. That can be seen by writing, as in Theorem 2.2, $P(t, x) = \sum_{j=0}^k p_j(t)x^j$ where $p_j(\cdot)$, $0 \leq j \leq k$, are polynomials of a single variable. Then just like the relation (2.3) from the martingale condition, one gets here for $0 \leq t < s$,

$$\sum_{i=0}^{k} M_t^i p_i(t) \;=\; \sum_{i=0}^{k} M_t^i \sum_{j=i}^{k} \binom{j}{i} p_j(s)\, \mathbf{E}(M_s - M_t)^{j-i} \quad a.s.$$

and applying the support condition **(S)**, for $1 \leq i \leq k$,

$$p_i(t) \;=\; \sum_{j=i}^{k} \binom{j}{i} p_j(s)\, \mathbf{E}(M_s - M_t)^{j-i}$$

in analogy to equation (2.4). Taking $i = k$, it follows that p_k is a constant, or that $P_k \in \bar{\mathcal{P}}_k$.

The conclusion now follows on using induction on k and using the fact that $\mathbf{E}P_k(t, M_t) = 0$ for all $t \geq 0$ and $k \geq 1$. ∎

Remark 3.3 Let us note that the point of Remark 2.3 following Theorem 2.2 now becomes clear as it renders the analogy between the discrete and continuous-time cases evident. Actually, here too, the conclusion of the previous theorem can be stated in terms of the moments of the increments of M. A little algebra allows us to put it in the equivalent form: for each $k \geq 1$, there exists a polynomial b_k in two variables such that $\mathbf{E}\left(M_{t+s} - M_t\right)^k = b_k(s,t)$.

3.3 A question in terms of the Lévy measure and its resolution

It is well-known that associated to a Lévy process is a σ-finite measure on $[0,\infty) \times \{\mathbb{R}\backslash\{0\}\}$, called its Lévy measure, which governs the distribution of the sizes of its jumps. We now put the conclusion of the previous theorem for a p-harmonisable Lévy process M in terms of this object. Denoting its Lévy measure by m, it is well-known that the characteristic function ϕ_t of M_t can be represented as follows: for $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\phi_t(\alpha) &= \mathbf{E}\exp(i\alpha M_t) \\
&= \exp\left[i\alpha\mu(t) - \frac{\alpha^2\sigma^2(t)}{2} + \int (e^{i\alpha u} - 1 - \frac{i\alpha u}{1+u^2})m([0,t]\otimes du)\right] \\
&= \exp\psi_t(\alpha), \quad \text{say},
\end{aligned}$$

for continuous functions μ and σ^2, the latter being nomenclative and increasing in addition. They represent, respectively, the mean and variance functions of the 'Gaussian part' of M. The functions μ, σ^2 and the measure m together determine the distribution M uniquely.

Now, the moments of M_t are nothing but various derivatives of the above function at 0, multiplied by powers of i. Suppose now that for every t, the m.g.f. of M_t exists in a neighbourhood of 0. One can then write the m.g.f. as $\exp\psi_t(-i\alpha)$. Denote by $\varphi_t(\alpha) = \psi_t(-i\alpha)$ the c.g.f. of M_t in this neighbourhood. One can now easily show that p-harmonisability of M, which is equivalent to each moment of M_t being a polynomial in t, translates actually to each derivative of φ_t at 0 being a polynomial in t. To prove this involves an induction on the order of the moment and making use of the fact $\phi_t(0) = 1$ appropriately. This gives us another equivalent formulation of p-harmonisability of M, namely that each of the cumulants of M_t be polynomials in t.

Now, these derivatives are given by

$$\varphi_t'(0) = \mu(t) + \int \frac{u^3}{1+u^2} m([0,t]\otimes du), \quad \varphi_t''(0) = \sigma^2(t) + \int u^2 m([0,t]\otimes du),$$

and for $k > 2$,

$$\varphi_t^{(k)}(0) = \int u^k m([0,t]\otimes du).$$

Two questions arise at this juncture. First, the above equations tell us that for a p-harmonisable Lévy process, each of the 'moments' from the third onwards of the projection of its Lévy measure on $[0, t]$ must be a polynomial in t. Since this projection is in general only a finite measure on $\mathbb{R} \setminus \{0\}$ and not necessarily a probability, 'moments' of course have to be interpreted as just the integrals of the powers $u \mapsto u^k$, $k \geq 1$. If we impose sufficient conditions to ensure that the tail of the sequence of moments determines the measure (as for example the existence of its 'moment generating function' in a neighbourhood of 0), we can ask what we can infer about the projection, and hence the Lévy measure itself, from this phenomenon? Secondly, can we, in particular, conclude in this case that the first two moments of this projection as also the integral of the function $u \mapsto u/(1+u^2)$, are also polynomials in t?

This latter question is tied up with another one in the sense that an affirmative answer to it would lead to the following result. It would mean that if a Lévy process M is p-harmonisable, the functions $\mu(t)$ and $\sigma^2(t)$ are also polynomials, and that the Gaussian and pure jump parts in the canonical decomposition of M are both individually p-harmonisable. Notice that the sole fact of their being mutually independent is not enough to assert this – indeed, although by virtue of Theorem 3.2 the sum of two independent Lévy processes is p-harmonisable if each of them is so, the converse need not hold. A simple counterexample can be obtained just by adding a non-polynomial deterministic function from one of the component processes and subtracting the same function from the other, rendering them still mutually independent Lévy processes, but no longer p-harmonisable, with the sum obviously remaining the same. Going back now to the question as to whether the Gaussian and pure jump parts of a p-harmonisable Lévy process are separately p-harmonisable, the answer turns out to be in the negative in general. We prove this using a transformation which converts the Lévy measure to what is known as 'Kolmogorov measure'. It is the Borel measure L on $[0, \infty) \times \mathbb{R}$ defined as

$$L(A) = \int_{A_0} d\sigma^2(t) + \int_A u^2 m(dt \otimes du),$$

where $A \subseteq [0, \infty) \times \mathbb{R}$ is Borel, and $A_0 = \{t \in [0, \infty) : (t, 0) \in A\}$ is the 0-section of A.

The representation of the characteristic function in terms of the Kolmogorov measure L takes the form (see [9], page 152)

$$\log \mathbf{E}(e^{i\alpha M_t}) = i\alpha\nu(t) + \int \left(\frac{e^{i\alpha u} - 1 - i\alpha u}{u^2} \right) L([0, t] \otimes du) \tag{3.1}$$

The integrand in the second line is defined at $u = 0$ by the limiting value as $u \to 0$. Here, $\nu(t) = \mathbf{E}M_t$ is the mean function. The distribution of the process is determined alternatively by the function ν and the measure L. An extremely important property of L is that $\forall\ k > 1$, the k-th cumulant of M_t equals $\int u^{k-2}L([0,t] \otimes du)$. In particular, $\mathbf{E}|M_t|^k < \infty$ if and only if this integral is finite. Applying this fact to $k = 2$, we get that the total mass $L([0,t] \otimes \mathbb{R})$ is precisely the variance of M_t. This decomposes nicely into the variances of the components of M_t as follows:

$$\sigma^2(t) = L([0,t] \otimes \{0\})$$

is the variance of the Gaussian part, while the residual $L\left([0,t] \otimes (\mathbb{R} \setminus \{0\})\right)$ accounts for the variance of the pure jump part.

We now define a Lévy process M by setting $\nu(\cdot) \equiv 0$ and

$$L(A) = \int_A \left[\frac{1}{g(t)}\delta_{g(t)}(du) + \left(1 - \frac{1}{g(t)}\right)\delta_0(du)\right] dt$$

for Borel $A \subseteq [0,\infty) \times \mathbb{R}$, where $g(t) = 1 + t^2$, $t \geq 0$ and for any $x \in \mathbb{R}$, δ_x denotes the degenerate probability at x. Essentially, this means that L is supported on the graphs of the two functions $g(t)$ and the constant function 0, and the mass on each of these has the time-derivative, or density, $(g(t))^{-1}$ and $(1 - (g(t))^{-1})$ respectively. We return in Chapter 5 to Lévy processes whose Kolmogorov measures admit densities with respect to time.

The process M defined as above is p-harmonisable, since for every $k \geq 1$, the k-th cumulant $c_k(t)$ of M_t is a polynomial in t. We have $c_1(t) \equiv 0$, $c_2(t)$ equals

$$L([0,t] \otimes \mathbb{R}) = \int_0^t \left[\frac{1}{g(s)} + \left(1 - \frac{1}{g(s)}\right)\right] ds = t,$$

and for $k > 2$,

$$\begin{aligned} c_k(t) &= \int u^{k-2}L([0,t] \otimes du) \\ &= \int_0^t (g(s))^{k-2}(g(s))^{-1}ds = \int_0^t (1+s^2)^{k-1}ds, \end{aligned}$$

a polynomial in t.

However, the Gaussian part, while having mean 0, has the non-polynomial variance function $\int_0^t dt/\left\{1 - (1/g(s))\right\}$, which means that it cannot be p-harmonisable. Naturally, the same is true automatically for can the pure jump part.

We resume the programme of finding analogues of discrete-time results in continuous time. For general processes, a parallel to Theorem 2.4 is presented in Theorem 3.3. Again, Markov property follows as a corollary, with a similar additional condition imposed. It turns out that for both these results, it is easier to give for a direct proof than the one using approximation from discrete-time.

Theorem 3.3 *Suppose for a continuous-time process M, $\mathcal{P}_k \neq \phi \ \forall\ k \geq 1$, and for all $k \geq 1$ and $t > 0$, there is a $P_{(k,t)} \in \mathcal{P}_k$ with $P_{(k,t)}(t, \cdot)$ being of degree exactly k. Then for every $t, s > 0$ and $k \geq 1$, there exists a polynomial $d^k_{t,s}$ of degree at most k such that $\mathbf{E}(M^k_{t+s}|\mathcal{F}_t) = d^k_{t,s}(M_t)$ a.s..*

Proof : Fix $t, s > 0$. Let us temporarily write as P the polynomial $P_{(k,t+s)}$ as in the statement of the theorem. Thus, if $P(u,x) = \sum_{j=0}^k p_j(u)x^j$, we have $p_k(t+s) \neq 0$. We now use induction to prove the result :

$$\begin{aligned}\sum_{j=0}^{k} p_j(t) M_t^j &= P(t, M_t) = \mathbf{E}(P(t+s, M_{t+s})|\mathcal{F}_t) = \sum_{j=0}^{k} p_j(t+s)\,\mathbf{E}(M^j_{t+s}|\mathcal{F}_t) \\ &= \sum_{j=0}^{k-1} p_j(t+s)\, d^j_{t,s}(M_t) + p_k(t+s)\,\mathbf{E}(M^k_{t+s}|\mathcal{F}_t)\end{aligned}$$

so that $\mathbf{E}(M^k_{t+s}|\mathcal{F}_t) = d^k_{t,s}(M_t)$ where

$$d^k_{t,s}(x) = \frac{1}{p_k(t+s)}\left[\sum_{j=0}^{k} p_j(t)\, x^j - \sum_{j=0}^{k-1} p_j(t+s)\, d^j_{t,s}(x)\right]$$

is a polynomial of degree at most k. ∎

Corollary 3.3.1 *Suppose M satisfies the conditions of the above theorem and for all $t > 0$, there exists $\delta_t > 0$ such that M_t has finite m.g.f. in $(-\delta_t, \delta_t)$. Then M is Markov.*

Proof : The proof is similar to the discrete-time case. The one point to note here is that the conditional moments $\mathbf{E}(M^k_{t+s}|\mathcal{F}_t)$ for each $t, s > 0$ and $k \geq 1$ depend only on M_t and hence so does the conditional distribution of M_{t+s} given $\mathcal{F}_t$, this distribution being determined by its moments; that is, the conditional moments just referred to. This is nothing but Markov property. ∎

3.4 The restriction condition in continuous time

Just as in the discrete-time case, for general continuous time processes also, we have a necessary and sufficient condition for restricted p-harmonisability, in analogy to Theorem 2.5. Here, $M = \{M_t : t \geq 0\}$ is any continuous-time process with r.c.l.l. paths.

Theorem 3.4 *If M is a process satisfying* **(M)** *and* **(S)**, *then $\bar{\mathcal{P}}_k(M) \neq \emptyset \ \forall\ k \geq 1$ if and only if for each $k \geq 1$, there exists a three-variable polynomial C_k with $C_k(\cdot, \cdot, x)$ having x^k as its leading term in x, such that for all $t,\ s > 0$, $\mathbf{E}(M^k_{t+s}|\mathcal{F}_t) = C_k(t, s, M_t)$.*

In this case, any sequence $\{P_k \in \mathcal{P}_k : k \geq 1\}$ satisfies (v), determines the distribution of M upto moments of finite-dimensional joint distributions, and imposing (iv) in addition renders it unique.

Proof : The arguments used in the proof are quite similar to those used to prove Theorem 3.2. First we define, just as in that theorem, the discrete-time processes $M^{(N)}$ and their natural filtrations $\mathcal{G}^{(N)}$ for every $N \geq 1$.

Now, for the "if" part, we take recourse to Remark 2.10 following Theorem 2.5 in the previous chapter where an equivalent condition to the necessary and sufficient condition in the statement of Theorem 2.5 was given. We claim that the hypothesis of the "if" part ensures that for each N, the process $\{M_n^{(N)} : n \geq 1\}$ satisfies this condition. Indeed, using the smoothing property of conditional expectations, we have

$$\begin{aligned} \mathbf{E}[\{M_n^{(N)}\}^k | \mathcal{G}_{n-1}^{(N)}] &= \mathbf{E}[\mathbf{E}(M^k_{\frac{n}{N}} | \mathcal{F}_{\frac{n-1}{N}}) | \mathcal{G}_{n-1}^{(N)}] \\ &= C_k\left(\frac{n-1}{N}, \frac{1}{N}, M_{\frac{n-1}{N}}\right) = D_k(n, M_{n-1}^{(N)}) \end{aligned}$$

if one defines, for N fixed, $D_k(t,x) = C_k(\frac{t-1}{N}, \frac{1}{N}, x)$, a polynomial of degree exactly k in x and the leading term in x having coefficient free of t. This establishes our claim. Thus, for each N and k, we have $\bar{\mathcal{P}}_k(M^{(N)}) \neq \emptyset$. Now, the choice $P_k^{(N)} \in \bar{\mathcal{P}}_k(M^{(N)})$, the identification of $P_k^{(N)}(t,\cdot)$ with $P_k^{(1)}(\frac{t}{N},\cdot)$ using uniqueness (note that $M^{(1)}$ satisfies **(S)**), and the time-space harmonic property of $P_k = P_k^{(1)}$ for M, first over the rational time-set and finally over all $t \in [0,\infty)$, follow exactly in the same way as in the proof of the direct part of Theorem 3.2.

For the converse part, given $\{P_k \in \bar{\mathcal{P}}_k(M)\}$, exactly as in Theorem 2.5, first observe that if $P_1(t,x) = x + p(t)$, then $\mathbf{E}P_1(t, M_t) = 0$ implies that $\mathbf{E}M_t = -p(t)$, a polynomial in t. Therefore, $\forall\, s, t > 0$,

$$\mathbf{E}\,(M_{t+s}|\mathcal{F}_t) = M_t - \mathbf{E}M_t + \mathbf{E}M_{t+s} = C_1(t,s,M_t),$$

where $C_1(t,s,x) = x + p(t) - p(t+s)$. Thus the conclusion is valid for $k = 1$. Now, for $k > 1$, writing $P_k(t,x) = x^k + \sum_{j=0}^{k-1} p_j(t) x^j$, we get from the martingale condition

$$\mathbf{E}\left(M_{t+s}^k|\mathcal{F}_t\right) = M_t^k + \sum_{j=0}^{k-1} p_j(t) M_t^j, - \sum_{j=0}^{k-1} p_j(t+s)\mathbf{E}\left(M_{t+s}^j|\mathcal{F}_t\right) \tag{3.2}$$

and our claim follows by induction on k.

It remains to show that $\{P_k : k \geq 1\}$ satisfies the required properties. Properties *(iv)* and *(v)* as also the claim on uniqueness are immediate as in Theorem 3.2.

We have left to show now just that $\{P_k\}$ determines $\mathbf{E}M_{t_1}^{k_1}M_{t_2}^{k_2}\cdots M_{t_m}^{k_m}$ whenever $m \geq 1$, $0 \leq t_1 < t_2 \ldots < t_m$ and $k_i \geq 1$. We only prove it for $m = 1$ and $m = 2$ since these two cases seem sufficiently illustrative. The first case is easily disposed of upon observing $\mathbf{E}P_j(t, M_t) = 0 \,\forall\, j$ and using induction on k_1. Now, we have

$$\begin{aligned}\mathbf{E}M_{t_1}^{k_1}M_{t_2}^{k_2} &= \mathbf{E}[M_{t_1}^{k_1}\mathbf{E}(M_{t_2}^{k_2}|\mathcal{F}_{t_1})] \\ &= \mathbf{E}[M_{t_1}^{k_1}C_{k_2}(t_2 - t_1, t_1, M_{t_1})] \text{ using Theorem 3.4,}\end{aligned}$$

and the sought equality now follows from the case $m = 1$ since the C_k's are determined completely by the polynomials P_k. This last claim can be proved by induction on k as follows. Writing $P_k(t, x) = \sum_{j=0}^{k} p_j^{(k)}(t)x^j$, we have $C_1(t, s, x) = x - (\mathbf{E}M_{t+s} - \mathbf{E}M_t)$, and for $k \geq 2$, we have

$$\begin{aligned}\sum_{j=0}^{k} p_j^{(k)}(t)M_t^j &= P_k(t, M_t) = \mathbf{E}(P_k(t+s, M_{t+s})|\mathcal{F}_t) = \sum_{j=0}^{k} p_j^{(k)}(t+s)\,\mathbf{E}(M_{t+s}^j|\mathcal{F}_t) \\ &= \sum_{j=0}^{k} p_j^{(k)}(t+s)\,C_j(t, s, M_t) \\ &= C_k(t, s, M_t) + \sum_{j=0}^{k-1} p_j^{(k)}(t+s)\,C_j(t, s, M_t), \text{ whence}\end{aligned}$$

$$C_k(t, s, x) = x^k + \sum_{j=0}^{k-1}(p_j^{(k)}x^j - p_j^{(k)}(t+s)C_j(t, s, x))$$

which can be obtained from $\{P_j : 1 \leq j \leq k\}$. ∎

We observe next the analogue of Theorem 2.6 as an immediate consequence of the last theorem.

Theorem 3.5 *If M satisfies* **(S)** *and $\bar{\mathcal{P}}_k(M) \neq \emptyset$ for all $k \geq 1$, then $\bar{\mathcal{P}}_k(M) = \mathcal{P}_k(M)$ for each $k \geq 1$.*

Proof : This is again an easy consequence of the spanning property: any $P \in \mathcal{P}(M)$, being a linear combination of the polynomials from the canonical sequence $\{P_k \in \mathcal{P}_k(M) : k \geq 1\}$, must belong to $\overline{\mathcal{P}}(M)$, that is, $\mathcal{P}(M) \subseteq \overline{\mathcal{P}}(M)$. But $\overline{\mathcal{P}}(M) \subseteq \mathcal{P}(M)$ anyway, therefore $\mathcal{P}(M) = \overline{\mathcal{P}}(M)$. It follows that for each $k \geq 1$, $\bar{\mathcal{P}}_k(M) = \mathcal{P}_k(M)$. ∎

3.5 Semi-stable Markov processes

We now observe some connections between p-harmonisability of Markov processes and what is known as the *semi-stability* property as developed in [12], and also see some examples.

Let us first recall the definition of the latter property. A process $\{M_t : t \geq 0\}$ with $M_0 \equiv 0$ is called *semi-stable of index* $\beta > 0$ if for every $c > 0$, the processes $\{M_{ct} : t \geq 0\}$ and $\{c^\beta M_t : t \geq 0\}$ have the same distribution. For homogeneous Markov processes $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (\mathbf{P}^x)_{x\geq 0})$ with the half-line $[0, \infty)$ as state space, the above definition of semi-stability implies (and in fact is equivalent to) the equality of the laws of (M_{ct}) under $\mathbf{P}^x$ and of $(c^\beta M_t)$ under $\mathbf{P}^{x/c^\beta}$ for all $c > 0$ and all x. This means, for the transition function $p_t(\cdot, \cdot)$ of M, that $p_{ct}(x, A) = p_t(x/c^\beta, A/c^\beta)$.

A fact worth recording here is that if (P_k) is a sequence of time-space harmonic polynomials for a semi-stable Markov process M, of any order β, then so is the sequence of polynomials $(\tilde{P}_k)$ defined as $\tilde{P}_k(t, x) = P_k(ct, c^\beta x)$ for any $c > 0$. Let us now assumethat conditions for uniqueness for this sequence are met (see Theorem 4.6 in the next chapter for the details). This means, comparing the coefficients of x^k on both sides using property (i), we have $P_k(t, x) = \frac{1}{c^{\beta k}} \tilde{P}_k(t, x) = \frac{1}{c^{\beta k}} P_k(ct, c^\beta x)$ for all $c > 0$. In particular, for all $t > 0$,

$$P_k(t, x) = P_k(t \cdot 1, t^\beta \frac{x}{t^\beta}) = t^{\beta k} P_k(1, \frac{x}{t^\beta}) = t^{\beta k} p_k(\frac{x}{t^\beta}),$$

where p_k is the one-variable polynomial $P_k(1, \cdot)$. A property of this type for a two-variable polynomial is usually referred to as its *homogeneity* in the two arguments, or sometimes in one argument and a power of the other, the index in the latter being called the order of homogeneity of the polynomial. It can be easily seen that a polynomial P with leading term x^k in x is homogeneous of order β, that is, homogeneous in t^β and x, if and only if $P(ct, c^\beta x) = c^{\beta k} P(t, x)$ for all c, t and x. Hence, if for a process M one has for every k, a polynomial of the above kind which is time-space harmonic for M, then by virtue of the characterization of its distribution by these polynomials, one should expect the processes $(M_{ct})_{t\geq 0}$ and $(c^\beta M_t)_{t\geq 0}$ to be identically distributed, or in other words, M to be semi-stable! This, indeed, turns out to be true under certain conditions, and we give a rigorous proof of this in the next chapter, stating the needed conditions.

Let us proceed now to describe in detail how to obtain time-space harmonic polynomials for certain semi-stable Markov processes. For this purpose, we essentially look for a function ϕ_λ, for every $\lambda > 0$, such that $\phi_\lambda(M_t)\, e^{\lambda t}$ is a martingale. Towards that end, we first seek to solve the equation

$$\mathcal{A}\phi_\lambda + \lambda \phi_\lambda = 0, \tag{3.3}$$

where $\mathcal{A}$ is the infinitesimal generator of M.

The example we treat here is that of the square (M_t) of the Bessel process with dimension $2a$, say, which forms, as a varies, the only family of semi-stable Markov processes with continuous paths as remarked in [3] (page 77). Its generator, at least on the domain of twice continuously differentiable functions f, is given by $\mathcal{A}f = 2x\, f'' + 2a\, f'$. Actually, for our purpose, solving the equation (3.3) for any one particular value of a is good enough because time-space harmonic polynomials obtained for one value of a give rise to those for the other values as well, as we shall see later. We consider the value $a = \frac{1}{2}$, in which case the function $\phi_\lambda(x) = \cos(\sqrt{2\lambda x})$ can be easily seen to be a solution to (3.3). Semi-stability of the process manifests in the fact that we have a choice satisfying $\phi_\lambda(x) = \phi_1(\lambda x)$. Now, the standard power series expansion $\cos\theta = \sum_{k=0}^{\infty}(-\theta^2)^k/k!$gives $\phi_\lambda(x) = \sum_{k=0}^{\infty} \frac{(-2\lambda x)^k}{(2k)!}$. This allows us to expand the function $\phi_\lambda(x)\, e^{\lambda t}$ as a power series in λ:

$$\phi_\lambda(x)\, e^{\lambda t} = \sum_{k=0}^{\infty} \lambda^k \sum_{j=0}^{k} \frac{(-2x)^j t^{k-j}}{(2j)!(k-j)!}$$

Now, exchanging, as before, the orders of the operations of conditional expectation and k-fold partial differentiation, we obtain that $\{P_k(t,x) = k! \sum_{j=0}^{k} \frac{(-2)^j t^{k-j}}{(2j)!(k-j)!} x^j : k \geq 1\}$ is a sequence of time-space harmonic polynomials for M.

The above construction of time-space harmonic polynomials is merely an illustration of the method outlined, since our choice of a makes M to be $\mathrm{BES}^2(1)$, or simply the square of one-dimensional Brownian motion. Thus one could have taken just the sequence $P_k(t,x) = H_{2k}(t, \sqrt{x})$, $k \geq 1$, where $\{H_k\}$ are the Hermite polynomials. However, this method is applicable for general homogeneous Markov processes too, if the solution to (3.3) is 'well-behaved' with respect to λ, in the sense of admitting a power series expansion.

Now, p-harmonisability of some other semi-stable Markov processes can be obtained from this process, using what is known as the intertwining relation between them [3]. By an intertwining relation between two Markov semigroups $(\mathbf{P}_t)$ and $(\mathbf{Q}_t)$, or the corresponding processes, we mean the existence of an operator Λ such that $\Lambda \mathbf{P}_t = \mathbf{Q}_t \Lambda\ \forall\ t$. In some cases, this operator Λ is given by the "multiplicative kernel" for a random variable Z, that is, $\Lambda f(x) = \mathbf{E} f(xZ)$. In such a case, if $P(t,x) = \sum_{j=0}^{k} p_j(t) x^j$ is a time-space harmonic polynomial for the process corresponding to the semigroup $(\mathbf{P}_t)$, then $Q(t,x) = \Lambda P(t,x) = \sum_{j=0}^{k} p_j(t) \mathbf{E} Z^j\, x^j$ is one for the process with semigroup $(\mathbf{Q}_t)$. This fact is easy to verify from the definition.

We present in the final chapter some examples of processes whose semigroups are intertwined with that of the square of the Bessel process. We also present some explicit calculations in this context.

3.6 Multivariate and vector-valued processes

The p-harmonisability question has been dealt more or less exhaustively for real-valued processes $\{M_t\}$. Now, as was indicated in the introduction, we raise the same question for processes M taking values in some $\mathbb{R}^m$ where $m > 1$, or in a more general Hilbert space

$\mathcal{H}$. We first take up the former, to be referred to in the sequel as the multivariate, or multidimensional, case.

Recall our definition of the space $\mathcal{V}$ of all time-space harmonic functions. A natural extension from the earlier set-up to this case suggests $\mathcal{P}$ to stand for the set of all polynomials P of $r+1$ variables for which clubbing together the last r coordinates into one, we have $P(\cdot,\cdot) \in \mathcal{V}$. We now introduce the analogues in this case to the objects $\mathcal{P}_k$ and $\bar{\mathcal{P}}_k$ for univariate processes, and rephrase the definitions for p-harmonisability and restricted p-harmonisability, respectively, in terms of these before proceeding further.

By the degree of an r-variable polynomial p, we mean an r-index $\mathbf{k} = (k_1, k_2, \cdots, k_r)$ such that $\deg p(x_1, \cdots, x_{i-1}, \cdot, x_{i+1}, \cdots, x_r) = k_i$ for $1 \le i \le r$ and $\text{coeff}(x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}) \neq 0$. This differs from the usual definition of degree as $\sum k_i$ for such polynomials, but allows us to present our result more succinctly. Now for an r-dimensional process M and for a multi-index $\mathbf{k} = (k_1, k_2, \cdots, k_r)$, we define

$$\mathcal{P}_{\mathbf{k}} = \{P(t, \mathbf{x}) \in \mathcal{P}(M) : \deg.P(t, \cdot) = \mathbf{k}\}.$$

Since there are polynomials to which we can not assign degrees in the above sense, it is clear that we may not have $\mathcal{P} = \cup_{\mathbf{k}} \mathcal{P}_{\mathbf{k}}$ in this case. However, all polynomials in r variables are linear combinations of polynomials to which we do assign a degree, and we deem this fact enough justification to call M p-harmonisable if $\mathcal{P}_{\mathbf{k}}$ happens to be nonvoid for each r-index $\mathbf{k}$.

A trivial, but perhaps interesting, observation is that if M is a discrete-time r-dimensional p-harmonisable process, then each of its components $M^{(i)} = \{M_{i,n} : n \ge 0\}$, $1 \le i \le r$, is so in one dimension. This is because

$$\mathcal{P}_k(M^{(i)}) = \mathcal{P}_{\underbrace{0,0,\cdots,0}_{i-1},k,0,0,\cdots,0}(M).$$

Is the converse true ? In the special case that the component processes $M^{(i)}$, $1 \le i \le r$, are mutually independent, the converse is easily seen to be true, that is, p-harmonisability for each $M^{(i)}$ implies that for M too. Indeed, if for $1 \le i \le r$, $P_{i,k_i} \in \mathcal{P}_{k_i}(M^{(i)})$, then the polynomial P defined as $P(t, x_1, x_2, \ldots, x_r) = \prod_{i=1}^{r} P_{i,k_i}(t, x_i)$ is in $\mathcal{P}_{\mathbf{k}}(M)$.

Thus, if one assumes the independence of the components $M^{(i)}$, one gets counterparts of the various results we have for one-dimensional processes, simply by rewriting the same results for the component processes, in terms of M. Processes with iid differences (or stationary independent increments) satisfying the moment condition, for instance, can be easily shown to be p-harmonisable. Similarly, for processes with simply independent differences (or increments), one has a necessary and sufficient condition for p-harmonisability. In discrete time, this condition, put in terms of the moments of the process M itself, is just that for all $k_1, k_2, \ldots, k_r \geq 0$, $\prod_{i=1}^{r} \mathbf{E}(M_{r,n}^{k_r})$ be a polynomial in n. This is because

$$\mathbf{E}(M_{1,n}^{k_1} M_{2,n}^{k_2} \cdots, M_{r,n}^{k_r}) = \mathbf{E}(M_{1,n}^{k_1})\, \mathbf{E}(M_{2,n}^{k_2}) \ldots \mathbf{E}(M_{r,n}^{k_r}),$$

a polynomial in n if each of the factors is. We have thus in analogy to Theorem 2.2 the following

Theorem 3.6 *For a process $M = \{(M_{1,n}, M_{2,n}, \cdots, M_{r,n}) : n \geq 0\}$ with independent components $M^{(i)} = (M_{i,n})_{n \geq 1}$, $1 \leq i \leq r$, and with independent differences, p-harmonisability of M is equivalent to the condition that for all $k_1, k_2, \cdots, k_r \geq 0$, $\mathbf{E}(M_{1,n}^{k_1}, M_{2,n}^{k_2}, \cdots, M_{r,n}^{k_r})$ be a polynomial in n.*

One can also develop parallels of the various algebraic conditions considered for univariate processes. One would first have an obvious analogue to the restriction condition, assuming which for a sequence, of more appropriately, an array of time-space harmonic polynomials $\{P_{\mathbf{k}} \in \mathcal{P}_{\mathbf{k}} : \mathbf{k} \text{ an } r\text{-index}\}$ leads to a counterpart of Theorem 2.5. Analogues to the other properties, whose fallouts in the one-dimensional case are studied in the next chapter, are also possible to obtain, alongwith counterparts of the corresponding results. Naturally, one also has the counterparts of all these in continuous time too.

The case that remains to be considered is when the state space E is a separable Hilbert space $\mathcal{H}$, and we briefly describe here the required modifications in the aforementioned definitions in this case. By a *pure* polynomial of degree k on $\mathcal{H}$ we mean a function of the type $h_{\mathbf{v}}^{\mathbf{k}}(x) = h_{v_1, v_2, \cdots, v_r}^{k_1, k_2, \cdots, k_r}(x) = < v_1, x >^{k_1} < v_2, x >^{k_2} \ldots < v_r, x >^{k_r}$ where $r \geq 1$, $\sum_{i=1}^{r} k_i = k$ and $v_1\ v_2, \ldots v_r \in \mathcal{H}$. Two-variable polynomials on $T \times \mathcal{H}$ now stand simply for (finite) linear combinations of products of powers of t and pure polynomials on $\mathcal{H}$. $\mathcal{P}$ is now defined in the self-evident way.

We first provide a more detailed explanation for the definition of bivariate polynomials we have adopted, in the light of [6] (page 534). A *pure* polynomial on $\mathcal{H}$ in our sense would be an example of what are termed there as *homogeneous* polynomials on $\mathcal{H}$, namely, functions of the type $H(x, x, \ldots, x)$ where H stands for a multilinear function. A general polynomial was defined there as a linear combination of these homogeneous polynomials. However, a certain fact was also implicitly mentioned, and motivated us to consider, instead of general polynomials, the functions we have denoted $h_{\mathbf{v}}^{\mathbf{k}}$. First, identify the space of the functions determined by k-linear forms with the k-fold tensor product-power of $\mathcal{H}$. It now becomes clear that the $h_{\mathbf{v}}^{\mathbf{k}}$'s form an orthonormal basis for this space if $k = \sum_{i=1}^{r} k_i$ and the v_i's are chosen from a fixed orthonormal basis of $\mathcal{H}$. Fixing such a basis Λ of $\mathcal{H}$, we now define

$$\mathcal{P}^{\mathbf{k}}_{\mathbf{v}} = \mathcal{P}^{k_1,k_2,\ldots,k_r}_{v_1,v_2,\ldots,v_r} = \{P(\cdot,\cdot) \in \mathcal{V} : P(t,x) = \sum_{\mathbf{j}\leq\mathbf{k}} p^{\mathbf{k}}_{\mathbf{j}}(t) h^{\mathbf{j}}_{\mathbf{v}}(x), p^{\mathbf{k}}_{\mathbf{j}}(\cdot) \text{ polynomial}\}$$

and call M p-harmonisable if this object $\mathcal{P}^{\mathbf{k}}_{\mathbf{v}}$ is non-empty for all $r \geq 1$, $v_i \in \Lambda$ and $k_i \geq 0$.

However, we require one more property of the orthonormal basis Λ in order to be able to extend to this case our previous results. This property, quite similar to the criterion of independence of the co-ordinates $M^{(i)}$ in the multidimensional case, is that the processes $< v, M_t >$, $v \in \Lambda$, be mutually independent. With this condition, one easily derives analogues to our results in the one-dimensional case. We mention the simplest one, when the discrete-time process M has independent differences.

Theorem 3.7 *Suppose M is an $\mathcal{H}$-valued process with independent differences and Λ is a basis such that $< v, M_n >$, $v \in \Lambda$, are independent processes. Then, for any fixed $r \geq 1$, $\mathcal{P}^{k_1,k_2,\ldots,k_r}_{v_1,v_2,\ldots,v_r} \neq \emptyset$ for all $v_i \in \Lambda$, $k_i > 0$, if and only if $\mathbf{E}h^{k_1,k_2,\cdots,k_r}_{v_1,v_2,\cdots,v_r}(M_n)$ is a polynomial in n for all $v_i \in \Lambda$, $k_i > 0$.*

The analogues of the algebraic properties *(i)* – *(v)* and their ramifications are again routine but cumbersome. All that is involved is that the corresponding properties be satisfied by the individual processes $< v, M_t >$ for all $v \in \Lambda$. One can, of course, also obtain continuous-time versions.

Chapter 4

Algebraic and Analytical Properties

4.1 The Appell condition

This chapter is devoted to bringing out the interconnections between various algebraic and analytical properties possessed by sequences of time-space harmonic polynomials and distributional properties of the process they arise from. One side of this connection we have already come across, in the earlier chapters, where specific distributional properties for the process, such as iid difference sequence, or even just independent differences, or increments, were shown to result in properties like *(i)*, *(ii)* and *(iii)* for the time-space harmonic polynomials. We also observed the ramification of the restriction property *(i)* on the process, in that a necessary and sufficient condition was derived for restricted p-harmonisable of a given process. Now we concentrate on the reverse side, that is, to determine what the listed algebraic conditions on the polynomials, other than *(i)*, mean for the process.

The first theorem in this context demonstrates the fallout of assuming the Appell condition *(ii)*. This culminates nicely in a corollary which shows that *(ii)* forces the summands to be independent in the presence of a mild additional hypothesis. Recall our adopted convention that $P_0 \equiv 1$.

Theorem 4.1 *Suppose M is a process satisfying condition* (**S**) *with difference sequence X. If $\mathcal{P}_k(M) \neq \emptyset$ for all $k \geq 1$ and a sequence $\{P_k \in \mathcal{P}_k : k \geq 1\}$ can be chosen so as to satisfy the Appell property (ii), then for each $k \geq 1$ and all $n \geq 1$, $\mathbf{E}(X_n^k|\mathcal{F}_{n-1})$ is non-random. Moreover, such a sequence $\{P_k : k \geq 1\}$ satisfies the properties (i) and (v), and, is the unique sequence to also satisfy (iv). Further, any* *sequence $\{P_k \in \mathcal{P}_k(M) : k \geq 1\}$ determines the moments of the finite-dimensional distributions of M.*

Proof : First, remember our convention that $P_0 = 1$. Now, let us write, for each $k \geq 0$, $P_k(t,x) = \sum_{j=0}^{k} p_j^{(k)}(t)x^j$ as before. Now, we recall that property *(ii)* for the sequence $\{P_k : k \geq 1\}$ is equivalent to : $\forall\, k \geq 1,\ 1 \leq j \leq k$,

$$jp_j^{(k)} = k\,p_{j-1}^{(k-1)} \Rightarrow p_j^{(k)} = \frac{k}{j}\,p_{j-1}^{(k-1)}, \text{ and therefore, by repetition, } p_j^{(k)} \equiv \binom{k}{j} p_0^{(k-j)}. \quad (4.1)$$

This means, in particular, that $p_k^{(k)} \equiv p_0^{(0)} \equiv 1$ for all $k \geq 1$. Thus the sequence $\{P_k : k \geq 1\}$ satisfies property *(i)*. Clearly, for each $k \geq 1$, $P_k \in \bar{\mathcal{P}}_k$, so that the hypothesis of Theorem 2.5 is valid. Therefore, for each $k \geq 1$, $\exists$ a polynomial $B_k(t,x) = \sum_{j=0}^{k-1} b_j^k(t)x^j$, so that $\mathbf{E}(X_n^k|\mathcal{F}_{n-1}) = B_k(n, M_{n-1})$ *a. s.* for all $n \geq 1$.

In particular, $\mathbf{E}(X_n|\mathcal{F}_{n-1}) = B_1(n, M_{n-1}) = b_0^1(n)$ a.s., that is, $\mathbf{E}(X_n|\mathcal{F}_{n-1})$ is almost surely non-random. We also note here that from the harmonicity of $P_1(t,x) = x + p_0^{(1)}(t)$, that is to say, from the martingale property of $\{P_1(n, M_n)\}$, we actually get $b_0^1(t) = p_0^{(1)}(t-1) - p_0^{(1)}(t)$. To prove now that for all $k \geq 2$, $\mathbf{E}(X_n^k|\mathcal{F}_{n-1})$ is non-random, we merely have to show that $\forall\ k \geq 2$, the polynomial B_k is free from its second argument, that is, $\forall\ i$, $1 \leq i \leq k-1$, $b_i^k \equiv 0$.

We use induction on k. For $k = 2$, this is easy to verify directly. Indeed, using the martingale property of $\{P_2(n, M_n)\}$, and the fact that $p_2^2 \equiv 1$, one gets

$$\begin{aligned}
&M_{n-1}^2 + p_1^{(2)}(n-1)M_{n-1} + p_0^{(2)}(n-1)\\
&= \mathbf{E}\left[M_n^2 + p_1^{(2)}(n)M_n + p_0^{(2)}(n)|\mathcal{F}_{n-1}\right]\\
&= M_{n-1}^2 + 2M_{n-1}\mathbf{E}(X_n|\mathcal{F}_{n-1}) + \mathbf{E}\left(X_n^2|\mathcal{F}_{n-1}\right) + p_1^{(2)}(n)\left[M_{n-1} + \mathbf{E}(X_n|\mathcal{F}_{n-1})\right] + p_0^{(2)}(n)\\
&= M_{n-1}^2 + 2M_{n-1}b_0^1(n) + b_1^2(n)M_{n-1} + b_0^2(n) + p_1^{(2)}(n)M_{n-1} + p_1^{(2)}(n)b_0^1(n) + p_0^{(2)}(n)\\
&= M_{n-1}^2 + \left[2b_0^1(n) + b_1^2(n) + p_1^{(2)}(n)\right]M_{n-1} + b_0^2(n) + p_1^{(2)}(n)b_0^1(n) + p_0^{(2)}(n).
\end{aligned}$$

In this, we can equate coefficients of M_{n-1} on both sides, in view of **(S)**. This yields $b_1^2(t) = p_1^{(2)}(t-1) - p_1^{(2)}(t) - 2b_0^1(t) \equiv 0$, using $p_1^{(2)} = 2p_0^{(1)}$ and $b_0^1(t) = p_0^{(1)}(t-1) - p_0^{(1)}(t)$.

Next, let $k > 2$ and assume, by way of induction hypothesis, that for all $l < k$ and $1 \leq j \leq l-1$, $b_j^l = 0$. This means precisely that for all $1 \leq l < k$, $\mathbf{E}\left(X_n^l|\mathcal{F}_{n-1}\right) = b_0^l$ almost surely. Then, from the martingale property of $\{P_k(n, M_n)\}$ and the fact that $p_k^{(k)} \equiv 1$, one easily deduces, in view of **(S)**, that

$$\begin{aligned}
b_{k-1}^k(n) &= p_{k-1}^{(k)}(n-1) - p_{k-1}^{(k)}(n) - k\,b_0^1(n), \qquad \text{and for } 1 \leq j \leq k-2,\\
b_j^k(n) &= p_j^{(k)}(n-1) - p_j^{(k)}(n) - \sum_{i=j+1}^{k} \binom{i}{j} p_i^{(k)}(n)\,b_0^{i-j}(n).
\end{aligned}$$

In the first equation, we use the facts that $p_{k-1}^{(k)} = kp_0^{(1)}$ and $b_0^1(t) = p_0^{(1)}(t-1) - p_0^{(1)}(t)$, to immediately get $b_{k-1}^k \equiv 0$.

For $2 \le j \le k-2$, we get from the second equation that

$$\begin{aligned} b_j^k(n) &= \frac{k}{j}\left[p_{j-1}^{(k-1)}(n-1) - p_{j-1}^{(k-1)}(n)\right] - \sum_{i=j+1}^{k}\binom{i}{j}\frac{k}{i}p_{i-1}^{(k-1)}(n)b_0^{i-j}(n) \\ &= \frac{k}{j}\left[p_{j-1}^{(k-1)}(n-1) - p_{j-1}^{(k-1)}(n) - \sum_{i-1=j}^{k-1}\binom{i-1}{j-1}p_{i-1}^{(k-1)}(n)b_0^{i-j}(n)\right] \\ &= \frac{k}{j}\,b_{j-1}^{k-1}(n) \equiv 0, \quad \text{by induction hypothesis.} \end{aligned}$$

Finally, again from the second equation above,

$$\begin{aligned} b_1^k(n) &= p_1^{(k)}(n-1) - p_1^{(k)}(n) - \sum_{i=2}^{k} i p_i^{(k)}(n)\, b_0^{i-1}(n) \\ &= p_1^{(k)}(n-1) - p_1^{(k)}(n) - \sum_{i=2}^{k} k\, p_{i-1}^{k-1}(n)\, b_0^{i-1}(n) \\ &= k\left\{p_0^{k-1}(n-1) - p_0^{k-1}(n) - \sum_{l=1}^{k-1} p_l^{k-1}(n) - b_0^l(n)\right\} = 0, \end{aligned}$$

since $b_0^{k-1}(n) = p_0^{(k-1)}(n-1) - p_0^{(k-1)}(n) - \sum_{i=1}^{k-2} p_i^{(k-1)}(n)\, b_0^i(n)$.

Next, we show that the polynomials we have chosen satisfy the properties as claimed in the statement. We have already observed that $\{P_k\}$ satisfies *(i)*. As for *(iv)*, we show that were $P_k(0,x)$ to contain any other term than x^k, that is, $p_j^{(k)}(0) \neq 0$ for some j with $0 \le j \le k-1$, then the polynomial P_{k-j} would fail to satisfy $P_j(0,0) = 0$, a condition that was imposed on any member of $\mathcal{P}$. This is because $P_{k-j}(0,0) = p_0^{(k-j)}(0) = p_j^{(k)}(0)/\binom{k}{j}$.

Uniqueness follows by virtue of *(iv)* just as in Theorem 2.5.

The proof of the spanning property runs exactly along the same lines as in Theorem 2.3. For the equation (2.10), which was crucial in that proof, was derived in turn from the equation (2.4), and in the latter, the independence of the differences was used only through the fact that the conditional moments $\mathbf{E}(X_n^k|\mathcal{F}_{n-1})$, $n,\ k \ge 1$, are degenerate, a property that has already been proved to hold in this theorem.

That the moments of the finite-dimensional joint distributions are determined by $\{P_k\}$ is proved by arguments similar to those in Theorem 2.3. First of all, by what has already been proved, it follows that $\forall\ n,\ k \ge 1$, $\mathbf{E}(X_n^k) = \mathbf{E}(X_n^k|\mathcal{F}_{n-1}) = b_0^k(n) = b_k(n)$, say, a polynomial in n. It now follows, exactly as in Theorem 2.3, that the sequence $\{P_k : k \ge 1\}$ determines the $\{b_k\}$ recursively by the equation (2.11). Also, our previous calculations imply that the finite-dimensional joint moments of M are but linear combinations of products of the b_k's. ∎

As remarked before, one consequence of the hypothesis in the above theorem is the independence of the differences. A small price has however to be paid for this in terms of an assumption on the existence of m.g.f.'s, as was the case with the Markov property following Theorem 2.4.

Corollary 4.1.1 *If, in addition to the hypotheses of Theorem 2.6, each M_n is known to have finite m.g.f. in $(-\delta_n, \delta_n)$ for some $\delta_n > 0$, then the difference sequence associated to M is independent.*

The proof of this corollary uses arguments used several times in the past, and only a sketch will suffice. It involves proving simply that for all $n \geq 1$, the conditional distribution of X_n given $\mathcal{F}_{n-1}$ is the same as the unconditional distribution. This, in turn, is a consequence of its being determined by its moments. To prove the latter requires just a straightforward repetition of the argument used in the Corollary 2.4.1 to Theorem 2.4. We only have to establish that X_n, for every $n \geq 1$, admits a finite m.g.f. in some neighbourhood of 0. For this, merely observe that for α such that $|\alpha| < \min(\delta_{n-1}, \delta_n)/2$, by Cauchy-Schwartz inequality we have

$$\mathbf{E}\left(e^{\alpha X_n}\right) \leq \sqrt{\mathbf{E}\left(e^{2\alpha M_n}\right) \cdot \mathbf{E}\left(e^{-2\alpha M_{n-1}}\right)} < \infty.$$

Remark 4.1 For future reference, it will be convenient to put the conclusion of this theorem as : for every $k \geq 1$, there exists a polynomial b_k such that for all $n \geq 1$, $\mathbf{E}(X_n^k|\mathcal{F}_{n-1}) = \mathbf{E}(X_n^k) = b_k(n)$ almost surely. Alternatively, in terms of the multi-step differences of M, this condition becomes: for each $k \geq 1$, there exists a polynomial c_k in two variables such that for all $m, n \geq 1$. $\mathbf{E}((M_{m+n} - M_n)^k|\mathcal{F}_n) = \mathbf{E}((M_{m+n} - M_n)^k) = c_k(m, n)$.

4.2 The pseudo-type-zero condition

Once we see that independence of the differences of a p-harmonisable process can be expressed in terms of a sequence of its time-space harmonic polynomials, the question that naturally arises is, does their further being identically distributed, also translate to some property of these polynomials? As we indicated in the introduction, the pseudo-type-zero property *(iii)*, in conjunction with property *(ii)* for the sequence of time-space harmonic polynomials, does exactly that. However, we do not prove this using the direct approach we have been following so far. This approach involves writing out the martingale equality for P_k and using induction to obtain some repercussion on certain moments, or conditional moments, at times by appealing to the support condition **(S)** to equate coefficients of various powers of M_n. At first glance, this approach would appear to give us the marginally stronger result that *(ii)* and *(iii)* force the conditional moments of X_n given $\mathcal{F}_{n-1}$ (which already equal the corresponding unconditional moments by virtue of *(ii)*), to be free of n. Assuming the existence of m.g.f. as in earlier cases, one could then obtain the desired conclusion. One might, moreover, expect the connection of the numbers h_i vis-a-vis these moments to be explicitly revealed.

In contrast, the original 'exponential martingale' approach due to Neveu that we follow here requires the existence of m.g.f.'s a priori. However, the connection between the numbers h_k and the moments just spoken of, is brought out by our approach also, since $\sum h_k\alpha^k/k!$ is established to be the negative of the cumulant generating function of M_1, and thus h_k turns out to be the negative of the k-th cumulant of M_1. However, the relation between the moments and the cumulants of a random variable is extremely complicated. An idea of the level of difficulty involved can be had from equation (5.2) on page 64 where in the absence of the moment condition **(M)**, the nonexistence of the m.g.f. forces us to actually exhibit the relationship. This serves to indicate that direct calculations would be intractable, or too cumbersome at any rate.

Instead, power series arguments from [20], suitably adapted to our context, allow us to derive our result more easily. The following lemma validates the representation (1.7) for a sequence $\{p_k : k \geq 1\}$ even when the sequence is pseudo-type-zero and not necessarily of type zero.

Lemma 4.1 *If $\{p_k : k \geq 0\}$ is a sequence of one-variable polynomials with p_k being of degree k, then p_k satisfies the pseudo-type-zero condition if and only if there exists a formal power series $B(\alpha) = \sum_{k=0}^{\infty} b_k\alpha^k/k!$ such that*

$$\sum_{k=0}^{\infty} \frac{p_k(t)}{k!}\,\alpha^k = B(\alpha)e^{th(\alpha)}$$

where $h(\alpha)$ stands for the formal power series $\sum_{k=1}^{\infty} h_k\alpha^k/k!$.

Proof : The if part is easy to show, for the formal calculation

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{dp_k(t)}{dt}\frac{\alpha^k}{k!} &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{p_k(t)}{k!}\,\alpha^k = \frac{d}{dt}\left(B(\alpha)e^{th\alpha)}\right) = h(\alpha)\left(B(\alpha)e^{th\alpha)}\right)\\ &= \left(\sum_{k=1}^{\infty}\frac{h_k}{k!}\,\alpha^k\right)\left(\sum_{k=0}^{\infty}\frac{p_k(t)}{k!}\,\alpha^k\right)\end{aligned}$$

holds as power series. The rest is just a matter of equating coefficients of α^k on both sides.

In the only if part, notice that B has to be the power series $B(\alpha) = \sum_{k=0}^{\infty} p_k(0)\alpha^k/k!$. The statement is thus equivalent to proving the relation

$$\sum_{k=0}^{\infty} \frac{p_k(t)}{k!}\,\alpha^k = \sum_{k=0}^{\infty} \frac{p_k(0)}{k!}\,\alpha^k e^{t\,h(\alpha)}. \tag{4.2}$$

Let us write $p_k(t) = \sum_{j=0}^{k} \rho_{k,j} t^j$. Then, $\rho_{k,j} = \frac{d^j p_k}{dt^j}(0)/j!$. But the pseudo-type-zero property can easily be seen to imply by induction on j that for $1 \le j \le k$,

$$\begin{aligned}
\frac{d^j}{dt^j}\,p_k(t) &= \sum_{i_1=1}^{k} \binom{k}{i_1} h_{i_1} \sum_{i_2=1}^{k-i_1} \binom{k-i_1}{i_2} h_{i_2} \quad \times \quad \cdots \\
&\quad \times \sum_{i_j=1}^{k-(i_1+\cdots+i_{j-1})} \binom{k-(i_1+\cdots+i_{j-1})}{i_j} h_{i_j}\, p_{k-(i_1+\cdots+i_j)} \\
&= \sum_{m=j}^{k} \left(\sum_{\substack{i_1,i_2,\ldots,i_j \ge 1 \\ i_1+i_2+\cdots+i_j=m}} \binom{k}{i_1, i_2, \ldots, i_j} h_{i_1} h_{i_2} \cdots h_{i_j} \right) p_{k-m}(t),
\end{aligned}$$

whence we get

$$\rho_{k,j} = \frac{1}{j!}\,\frac{d^j p_k}{dt^j}(0) = \frac{1}{j!} \sum_{m=j}^{k} \left(\sum_{\substack{i_1,\ldots,i_j \ge 1 \\ i_1+\cdots+i_j=m}} \binom{k}{i_1, \ldots, i_j} h_{i_1} \cdots h_{i_j} \right) \rho_{k-m,0}. \tag{4.3}$$

Now, establishing (4.2) amounts to showing that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{p_k(t)}{k!}\,\alpha^k &= \left(\sum_{k=0}^{\infty} \frac{\rho_{k,0}}{k!}\,\alpha^k \right) \left[1 + \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \frac{h_i}{i!}\,\alpha^i \right)^j \frac{t^j}{j!} \right] \\
&= \sum_{k=0}^{\infty} \left(\frac{\rho_{k,0}}{k!}\,\alpha^k \right) \left[1 + \sum_{j=1}^{\infty} \left(\sum_{i_1,\ldots,i_j \ge 1} \frac{h_{i_1} \cdots h_{i_j}}{i_1! \cdots i_j!}\,\alpha^{i_1+\cdots+i_j} \right) \frac{t^j}{j!} \right] \\
&= \sum_{k=0}^{\infty} \left(\frac{\rho_{k,0}}{k!}\,\alpha^k \right) \left[1 + \sum_{j=1}^{\infty} \left\{ \sum_{m=j}^{\infty} \left(\sum_{\substack{i_1,\ldots,i_j \ge 1 \\ i_1+\cdots+i_j=m}} \frac{h_{i_1} \cdots h_{i_j}}{i_1! \cdots i_j!} \right) \alpha^m \right\} \frac{t^j}{j!} \right].
\end{aligned}$$

To establish the equality of the two (formal) power series above, it of course suffices to show that both sides have identical coefficients of α^k for all $k \ge 0$. For $k = 0$, both are $\rho_{0,0}$. Therefore, we only have to show that $\forall\, k \ge 1$,

$$p_k(t) = \rho_{k,0} + k! \sum_{m=1}^{k} \frac{\rho_{k-m,0}}{(k-m)!} \sum_{j=1}^{m} \left(\sum_{\substack{i_1,\ldots,i_j \ge 1 \\ i_1+\cdots+i_j=m}} \frac{h_{i_1} \cdots h_{i_j}}{i_1! \cdots i_j!} \right) \frac{t^j}{j!}$$

$$= \rho_{k,0} + \frac{k!}{(k-m)!} \sum_{j=1}^{k} \sum_{m=j}^{k} \left(\sum_{\substack{i_1,\ldots,i_j \geq 1 \\ i_1+\cdots+i_j=m}} \frac{h_{i_1} \cdots h_{i_j}}{i_1! \cdots i_j!} \right) \rho_{k-m,0} \frac{t^j}{j!} ,$$

interchanging the order of summation. That this is true follows from the expression (4.3) for $\rho_{k,j}$ obtained earlier. ∎

Theorem 4.2 *Suppose a process M admits a sequence $\{P_k : k \geq 1\}$ of time-space harmonic polynomials satisfying both conditions (ii) and (iii), and that there exists a positive δ, such that for each $n \geq 1$, M_n has finite m.g.f. in $(-\delta, \delta)$. Then M has independent and identically distributed summands.*

Proof : In view of the representations (1.6) and (1.7) for Appell sets and pseudo-type-zero sets of polynomials respectively, *(ii)* and *(iii)* together enjoin, as formal power series, the representations

$$A(\alpha, t)e^{x\alpha} = \sum_{k=0}^{\infty} P_k(t,x) \frac{\alpha^k}{k!} = B(\alpha, x)e^{th(\alpha)},$$

where A and B are formal power series in α (with coefficients depending on t and x respectively). Here, $P_0 \equiv 1$, as per our earlier convention. Taking $x = 0$, we have

$$A(\alpha, t) = B(\alpha, 0)e^{th(\alpha)} = g(\alpha)e^{th(\alpha)}, \text{ say,}$$

that is,

$$\sum_{k=0}^{\infty} P_k(t,x) \frac{\alpha^k}{k!} = g(\alpha)e^{x\alpha}e^{th(\alpha)}. \tag{4.4}$$

Recall from Theorem 4.1 that since $\{P_k\}$ satisfies *(ii)*, it also satisfies *(iv)* and is indeed the only sequence to do so. Also, the existence of m.g.f.s allows us to apply Corollary 4.1.1, so that M has independent differences. We can therefore construct its exponential martingale

$$\eta(\alpha, n, M_n) = e^{\alpha M_n - \varphi(\alpha,n)}, \quad n \geq 0, \ |\alpha| < \delta,$$

where $\varphi(\alpha, n) = \log \mathbf{E} \exp(\alpha M_n)$ is the cumulant generating function of M_n, evaluated at α. But then expanding the function η above as power series in α,

$$e^{\alpha x - \varphi(\alpha,n)} = \sum_{k=0}^{\infty} Q_k(n,x) \frac{\alpha^k}{k!} ,$$

one can show, almost imitating the argument in the proof of Theorem 2.1, that $\{Q_k\}$ is also a sequence of time-space harmonic polynomials for M and that it also satisfies the conditions *(i)*, *(ii)* and *(iv)*. The uniqueness in Theorem 4.1 therefore yields $Q_k = P_k$ for each $k \geq 1$. Consequently, using the equation (4.4), we get, the formal power series identity

$$e^{\alpha x - \varphi(\alpha,n)} = g(\alpha)e^{x\alpha}e^{nh(\alpha)}.$$

But first putting $(n, x) = (0, 0)$ and subsequently, just $x = 0$, we get first that $g \equiv 1$ followed by $\varphi(\alpha, n) = -nh(\alpha)$. This form of φ, of course, immediately yields the identicality of the distribution of each member of the difference sequence associated to M. Moreover, it shows that h actually defines a function in $(-\delta, \delta)$ and that $-h$ is actually the common c.g.f. of the difference sequence. ■

Remark 4.2 If each M_n has finite m.g.f. in the same open neighbourhood Γ of 0, it is conceivable that the fallout of independent differences as a result of assuming the Appell property *(ii)* as set out in Corollary 4.1.1 can also be proved using the kind of power series arguments employed in Theorem 4.2. In this case, the representation (1.6) leads to the formal power series identity

$$A(\alpha, t)e^{\alpha x} = \sum_{k=0}^{\infty} P_k(t, x)\,\frac{\alpha^k}{k!} \tag{4.5}$$

for all $t \geq 0$, $x \in \mathbb{R}$, where for every t, $A(\alpha, t) = \sum_{k=0}^{\infty} a_k(t)\alpha^k/k!$ is a formal power series. The convention $P_0 \equiv 1$ leads to the condition $A(\alpha, 0) \equiv 1$. Let us denote the c.g.f. of M_n in Γ by $\varphi(\cdot, n)$. We then have, for $n \geq 1$ and $\alpha \in \Gamma$, $\exp(\varphi(\alpha, n)) = \sum_{k=0}^{\infty} \mu_k(n)\,\alpha^k/k!$, where $\mu_k(n) = \mathbf{E}(M_n^k)$, $k \geq 0$. But the identity (4.5) says just that $\forall\, t \geq 0$, $x \in \mathbb{R}$ and for every $k \geq 0$,

$$\sum_{j=0}^{k} \frac{a_{k-j}(t)}{(k-j)!}\,\frac{x^j}{j!} = P_k(t, x),$$

which means that $\forall\, n \geq 1$, $x \in \mathbb{R}$, the following formal calculation is valid:

$$\begin{aligned} A(\alpha, n)e^{\varphi(\alpha,n)} &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} \frac{a_{k-j}(n)}{(k-j)!}\,\frac{\mu_j(n)}{j!}\right]\frac{\alpha^k}{k!} = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} \frac{a_{k-j}(n)}{(k-j)!}\,\frac{\mathbf{E}(M_n^j)}{j!}\right]\frac{\alpha^k}{k!} \\ &= \sum_{k=0}^{\infty}\mathbf{E}\left[\sum_{j=0}^{k} \frac{a_{k-j}(n)}{(k-j)!}\,\frac{M_n^j}{j!}\right]\frac{\alpha^k}{k!} = \sum_{k=0}^{\infty}\mathbf{E}\left[P_k(n, M_n)\right]\frac{\alpha^k}{k!} = 1. \end{aligned}$$

For $\alpha \in \Gamma$, therefore, for all $n \geq 1$, $A(\alpha, n)$ is actually the well-defined function $A(\alpha, n) = \exp(-\varphi(\alpha, n)) = \{\mathbf{E}\exp(\alpha M_n)\}^{-1}$.

Going back to the equation (4.5) now, we can now claim that for $\alpha \in \Gamma$ and $n \geq 1$, this becomes a functional identity (with n replacing t). Therefore, when $\alpha \in \Gamma$ and $n \geq 1$,

$$e^{\alpha M_n - \varphi(\alpha,n)} = \sum_{k=0}^{\infty} P_k(n, M_n)\,\frac{\alpha^k}{k!} = \lim_{N\to\infty}\sum_{k=0}^{N} P_k(n, M_n)\,\frac{\alpha^k}{k!}\,.$$

Each element of the sequence on the R.H.S. is a martingale over n. The question is whether so is the limit. We have not been able to settle this question in complete generality. Clearly, a sufficient condition would be, for example, that the convergence above take place in L^1. This may be achievable by some DCT arguments. To attempt to prove uniform integrability seems another option.

If the answer to the question happens to be 'yes', then for $\alpha \in \Gamma$,

$$\mathbf{E}(e^{\alpha X_n}|\mathcal{F}_{n-1}) = A(\alpha, n-1)/A(\alpha, n)$$

would be non-random. This then would yield independence of the differences in the same way as in Corollary 4.1.1.

However, our proof enjoys the advantage of being applicable without requiring the assumption of the existence of the m.g.f. of M_n for all n in the same neighbourhood. In the case, for instance, when $\{X_n : n \geq 1\}$ are independent with X_n having the double exponential (or Laplace) distribution with density $(\sqrt{2}n)^{-1}\exp(-\sqrt{2}|x|/n)$, the process M with $M_0 \equiv 0$ and difference sequence X will be p-harmonisable, and a sequence of time-space harmonic polynomials satisfying *(ii)* will exist. However, X_n, and consequently, M_n, has the m.g.f. $\varphi_n(\alpha) = -\log(1 - \alpha^2 n^2/2)$, defined only for $|\alpha| < \sqrt{2}/n$. In fact our argument works even under milder conditions, as long as the moments determine the distribution. That apart, the degeneracy of the conditional moments of the differences as a consequence of the Appell condition seems interesting in itself.

4.3 Continuous time analogues

To describe the continuous-time analogues of the preceding results, we invoke the setting at the beginning of Chapter 3 : $M = \{M_t : t \geq 0\}$ denotes a continuous-time stochastic process with r.c.l.l. paths starting at $M_0 \equiv 0$ and satisfying **(M)** and **(S)**.

Theorem 4.3 *If a sequence $\{P_k \in \mathcal{P}_k(M) : k \geq 1\}$ satisfies the Appell condition (ii), then each of the conditional moments $\mathbf{E}((M_{t+s} - M_t)^k|\mathcal{F}_t)$ is degenerate.*

In this case, the above sequence also satisfies (i), (iv) and (v), is the unique sequence to do so, and determines the distribution of M upto moments of its finite-dimensional distributions.

Proof : As before, we prove the conclusion first when t, s are rational and then for general values. Recall, for every $N \geq 1$, as in Theorem 3.2 the discrete-time process $M^{(N)}$ and its natural filtration $\mathcal{G}^{(N)}$, and construct the polynomials $\{P_k^{(N)} \in \mathcal{P}_k(M^{(N)}) : k \geq 1\}$ by

$$P_k^{(N)}(t, x) = P_k(\frac{t}{N}, x),$$

where P_k, $k \geq 1$, are as in the hypothesis of the theorem. Clearly, the $P_k^{(N)}$'s also satisfy *(ii)*; therefore we can apply Theorem 4.1 on $M^{(N)}$. More precisely, by the Remark 4.1 following Theorem 4.1, we have the existence of two-variable polynomials $\{c_k^{(N)} : k \geq 1\}$ such that for every $k \geq 1$, $m \geq 1$ and $n \geq 1$,

$$\mathbf{E}((M_{m+n}^{(N)} - M_n^{(N)})^k | \mathcal{G}_n^{(N)}) = \mathbf{E}((M_{m+n}^{(N)} - M_n^{(N)})^k) = c_k^{(N)}(m,n).$$

In particular, if t and s are any two positive rationals, then, choosing integer $N \geq 1$ such that when tN and sN are integers, we get, for all $k \geq 1$,

$$\mathbf{E}((M_{t+s} - M_t)^k | \mathcal{G}_{tN}^{(N)}) = \mathbf{E}((M_{t+s} - M_t)^k) = c_k^{(N)}(sN, tN)$$

The term in the middle is free from N, and therefore, the R.H.S., being a function only of s and t, can be rewritten as a polynomial c_k in these two. Applying Lemma 3.1 with N_l, $l \geq 1$ chosen as indicated there, now allows us to conclude

$$\mathbf{E}((M_{t+s} - M_t)^k | \mathcal{F}_t) = c_k(s,t).$$

Next, observe that each (unconditional) moment of M_t is a polynomial in t. This follows by a simple induction argument from the fact that for all t, $\mathbf{E}P_k(t, M_t) = \mathbf{E}P_k(0,0) = 0$ and recalling that $P_k \in \bar{\mathcal{P}}_k \ \forall\, k$. This last fact is a consequence of the Appell property *(ii)*, as in the proof of Theorem 4.1.

Now, for general t and $s > 0$, take rational sequences $s_n \downarrow s$ and $t_n \downarrow t$. The observation we just made implies in particular that for every $k \geq 1$, $(M_{t_n+s_n} - M_{t_n})^k$ is a sequence of L^2-bounded random variables. Of course, they converge almost surely to $(M_{t+s} - M_t)^k$. But then, the convergence takes place also in L^1, whence we have

$$\begin{aligned}
\mathbf{E}((M_{t+s} - M_t)^k | \mathcal{F}_t) &= \lim_{n\to\infty} \mathbf{E}((M_{t_n+s_n} - M_{t_n})^k | \mathcal{F}_t) \\
&= \lim_{n\to\infty} \mathbf{E}(\mathbf{E}\left((M_{t_n+s_n} - M_{t_n})^k | \mathcal{F}t_n\right) | \mathcal{F}_t) \\
&= \lim_{n\to\infty} c_k(s_n, t_n) = c_k(s,t).
\end{aligned}$$

Among the stated properties claimed for $\{P_k\}$, *(i)* and *(iv)* as well as the uniqueness claim follow from merely identifying $P_k = P_k^{(1)} \ \forall\, k$, and applying Theorem 4.1 with the discrete-time process $M^{(1)}$. *(v)* follows from the fact that the sequence $\{P_k\}$ span the larger space $\mathcal{P}(M^{(1)})$. The proof that $\{P_k\}$ determines the distribution of M in the stated sense is similar to that in Theorem 3.4. ∎

The additive property for M upon further assuming the existence of m.g.f.s follows similarly to the discrete-time case as a corollary.

Corollary 4.3.1 *Under the hypotheses of Theorem 4.3 and the existence of the m.g.f. of M_t in some nontrivial interval around 0 for each $t \geq 0$, M has independent increments.*

This is just as before a consequence of the conditional distribution of the increments being determined by its moments.

To end the discussion on the listed properties leading to those for M we have finally

Theorem 4.4 *If M admits a sequence of time-space harmonic polynomials satisfying both the Appell and pseudo-type-zero conditions, and if $\mathbf{E}\exp(\alpha M_t) < \infty$ for each $t \geq 0$ in some nonempty interval $(-\delta, \delta)$ around 0, then M has stationary independent increments.*

Proof : The proof goes exactly along the same lines as that of Theorem 4.2 and hence a sketch will suffice. Denoting the sequence in the statement by $\{P_k : k \geq 1\}$, we construct the formal power series $\sum P_k(t,x)\,\alpha^k/k!$, which is then shown to be of the form $e^{\alpha x + th(\alpha)}$. This is where *(ii)* and *(iii)* come into play. Next, M already being known to have independent increments by Corollary 4.3.1, one constructs the exponential martingale $\exp(\alpha M_t - \varphi(\alpha, t)$, where $\varphi(\cdot, t)$ is now the c.g.f. of M_t. Then, expanding $\eta(\alpha, t, x) = \exp(\alpha x - \varphi(\alpha, t))$ as a power series in α and invoking uniqueness, we obtain the equality of $\varphi(\alpha, t) = -th(\alpha)$. This form of φ results in the conclusion of the theorem. ∎

4.4 Homogeneous sequences of polynomials

We now turn to the property that we call homogeneity for the polynomials whose acquaintance we first made in the context of semi-stable Markov processes (see Chapter 3, page 40). We recall the definition once again. A two-variable polynomial $P(t,x)$ is said to be homogeneous in t^β and x if it is of the form $P(t,x) = t^{\beta k} p(x/t^\beta)$ for some polynomial p of degree k in a single variable. In such a case, we shall call β the order of homogeneity of P.

From the above definition, it is immediate that if $P(\cdot, \cdot)$ is a homogeneous polynomial then its leading term in x must be free of t. However, one can work with a more general definition of homogeneity, which removes this restriction. This is discussed in Remark 4.3.

In the context of a sequence of time-space harmonic polynomials, we say that the whole sequence is homogeneous, if every member is so and the order of homogeneity is the same for all of them. We shall see that a sequence of time-space harmonic polynomials for a process can be homogeneous only of orders β for which 2β is an integer. When 2β is odd, it turns out that the distribution of the process is 'symmetric' in a certain sense. To be precise, the 'odd order moments' of every finite-dimensional marginal distribution vanish.

In general, under conditions elaborated earlier, homogeneity of time-space harmonic polynomials yields semi-stability of the underlying process. This is what one would expect from our study of semi-stable Markov processes in the preceding chapter. Of course, in what we have so far derived, time-space harmonic polynomials have, in general, been seen to determine the law of the corresponding process *only* upto moments of its finite-dimensional marginals. Naturally, therefore, in order to be able at all to conclude anything about the law of the process itself, we have to also assume that these moments, in turn, actually characterize the whole distribution. In other words, a condition like the existence of the m.g.f. of each M_t needs to be imposed. With this preamble, let us state the first of our two results. Since the question of the validity of its conclusion does not arise in discrete time, we raise the issue only for continuous-time processes.

Theorem 4.5 *Suppose a p-harmonisable continuous time process M satisfies condition* (S) *and admits a sequence $P_k \in \mathcal{P}_k(M)$ which is homogeneous of order β. Suppose also that the finite-dimensional joint moments of M determine its distribution. Then M is semi-stable with index β.*

Proof : From the definition of homogeneity, it is clear that we can write $P_k(t,x) = \sum_{j=0}^{k} c_{k,j} t^{\beta(k-j)} x^j$ for some constants $c_{k,j}$ with $c_{k,k} \neq 0$. Hence, dividing by $c_{k,k}$, if necessary, we can assume without loss of generality that $P_k \in \bar{\mathcal{P}}_k(M)$. Thus we can invoke Theorem 3.4 to claim the equivalence of the law of any process having the same time-space harmonic polynomials with that of M. Now, fix a $c > 0$. For each $k \geq 1$, let us look at the $(\mathcal{F}_{ct})$-martingale

$$\begin{aligned} P_k(ct, M_{ct}) &= \sum_{j=0}^{k} c_{k,j} c^{\beta(k-j)} t^{\beta(k-j)} M_{ct}^j \\ &= c^{\beta k} P_k(t, M_{ct}/c^{\beta}) . \end{aligned}$$

This means time-space harmonicity of each P_k for the process $\tilde{M} = \{M_{ct}/c^{\beta} : t \geq 0\}$ and hence, in view of our earlier observation, the equivalence of the latter in law to M. This is but semi-stability of index β of M rephrased. ∎

Remark 4.3 We have considered homogeneous polynomials only of a particular form, one that ensures that such a polynomial, if harmonic for a process M, always belongs to the restricted class $\bar{\mathcal{P}}(M)$. However, as mentioned earlier, a more general definition of homogeneity is possible. We define a two-variable polynomial P to be homogeneous if there exist a positive number β, a non-negative integer m and a one-variable polynomial p of degree k, say, such that $P(t,x) = t^{\beta k+m}p(x/t^{\beta})$ for every t and x. We still call β the order of homogeneity of P. It is clear from the definition that $p(x) = P(1,x)$. A sequence of two-variable polynomials is homogeneous of order β if each of its members is so, and they all have also the same m.

It can be seen easily that the proof of the above theorem goes through with little change even if this general definition of homogeneity is applied in the statement. The only difference here is that in order to claim that the moments of the underlying process M are determined by the sequence of time-space harmonic polynomials, we have to take recourse to the Remark 2.8 rather than Theorem 3.4. Of course, the process M no longer remains restricted p-harmonisable.

The next theorem is to the effect that the time-space harmonic polynomials of a restricted p-harmonisable semi-stable Markov process of index β can be chosen to be homogeneous of order β. In this sense, it is a converse to the previous Theorem 4.5.

Theorem 4.6 *Suppose a restricted p-harmonisable process M is semi-stable of index β. Then there exists a sequence $\{P_k \in \mathcal{P}_k(M) : k \geq 1\}$ which is homogeneous of order β.*

Proof : Suppose (P_k) is the unique sequence of time-space harmonic polynomials for M as in Theorem 3.4. Let us fix any $c > 0$. Consider the two processes $(\bar{M}_t = c^{\beta}M_t : t \geq 0)$, and $(\tilde{M}_t = M_{ct} : t \geq 0)$. We know that they are equivalent in law. Now we also know that for every $k \geq 1$, the two-variable polynomial $\tilde{P}_k$ defined as $\tilde{P}_k(t,x) = P_k(ct,x)$ is time-space harmonic for $\tilde{M}$, while $\bar{P}_k$ defined as $\bar{P}_k(t,x) = c^{\beta k}P_k(t,x/c^{\beta})$ is so for $\bar{M}$. However, the equivalence of the laws forces $\bar{P}_k \in \mathcal{P}_k(\tilde{M})$ also. But then, both the sequences satisfy *(i)*, as well as the uniqueness requirement of Theorem 3.4. Thus, it follows that for every $k \geq 1$, $\bar{P}_k = \tilde{P}_k$, that is, $P_k(ct,x) \equiv c^{\beta k}P_k(t,x/c^{\beta})$. Now, since c was arbitrary, this gives,

$$\begin{aligned} P_k(t,x) &= P_k(t\cdot 1, x) \\ &= t^{\beta k}P_k(1, \frac{x}{t^{\beta}}) = t^{\beta k}p_k(\frac{x}{t^{\beta}}) \end{aligned}$$

where p_k is the one-variable polynomial $P_k(1,\cdot)$. ■

Remark 4.4 At this point a very natural question is, whether it is possible to drop the restriction condition *(i)* on M and obtain a sequence of polynomials homogeneous in the more general sense of Remark 4.3. In contrast to Theorem 4.5, in general it is not possible.

The reason behind this is that when the sequence of time-space harmonic polynomials does not satisfy *(i)*, we do not have any condition that ensures uniqueness. As a result, the argument of Theorem 4.6 falls through.

In the final theorem in this context, we again go back to the original definition of homogeneity. This theorem demonstrates that a p-harmonisable process M, discrete-time or continuous-time, cannot admit a homogenous sequence of time-space harmonic polynomials with arbitrary order of of homogeneity. The possible orders of homogeneity are only those β's for which for which 2β is an integer. In particular, this implies that for a p-harmonisable continuous time semi-stable Markov process, the index of semi-stability can only be a β of the above kind.

When 2β is an odd integer, it turns out that the distribution of the process is symmetric, albeit only upto moments, that is, all its 'odd order moments' vanish; to be precise,

$$\mathbf{E}\left[M_{t_1}^{k_1} M_{t_2}^{k_2} \cdots M_{t_r}^{k_r}\right] \equiv 0$$

whenever $r \geq 1$, $0 < t_1 < t_2 < \ldots < t_r$, and k_i are non-negative integers with $k_1 + k_2 + \cdots k_r$ being odd. In the statement it has been tacitly assumed that M is not the constant ($\equiv M_0 \equiv 0$) process. Obviously, this can in particular be secured by the imposing the "support condition" **(S)** on M.

Theorem 4.7 *If M admits homogeneous time-space harmonic polynomials of index β, then 2β is an integer. If 2β is an odd integer, then every finite-dimensional joint distribution of M is symmetric about 0, in the sense of moments, as defined above.*

Proof : Let us write as $P_k \in \mathcal{P}_k$ the sequence of polynomials referred to in the statement of the theorem. We saw earlier that our definition of homogeneity entails each P_k to satisfy the property *(i)*, i.e. to belong to the restricted class $\bar{\mathcal{P}}_k$. We can thus write

$$P_1(t, x) = x + c_{1,0} t^{\beta} \tag{4.6}$$

So, if $c_{1,0} \not\equiv 0$, then from (4.6), β has to be an integer for P_1 to be at all a polynomial in the first place. If, on the other hand, $c_{1,0} \equiv 0$, then M is itself a martingale. Now, let us look at

$$P_2(t, x) = x^2 + c_{2,1} t^{\beta} x + c_{2,0} t^{2\beta}.$$

Again, if at least one of $c_{2,1}$ and $c_{2,0}$ is non-zero, we have nothing to prove. We simply note now that the contingency $c_{2,1} = c_{2,0} = 0$ is not possible, because in that case, (M_t^2) is also a martingale; but this is impossible unless $M_t \equiv M_0 \equiv 0$, a possibility we have already discounted.

Suppose now that 2β is odd and proceed to prove the symmetry, as asserted. Let us dispose of the discrete-time case first. We are to prove that for every $r \geq 1$, $\mathbf{E}[M_{n_1}^{k_1} \cdots M_{n_r}^{k_r}] = 0$ whenever and $k_1, \ldots, k_r$ are non-negative integers with $k_1 + \cdots + k_r$ odd. Clearly there is no loss of generality in assuming that $n_1 = 1$, $n_2 = 2, \ldots, n_r = r$ by interpolating, if necessary, a few zeroes among the k_i's. We use induction in two stages : first, fixing $r = 1$, on k_1, and then on r. Writing $P_k(t, x) = \sum_{j=0}^{k} c_{k,j} t^{\beta(k-j)} x^j$ as before, the hypothesis 2β is odd clearly implies that $c_{k,j} = 0$ whenever $k - j$ is odd because, in that case, $\beta(k - j)$ is a non-integer. In particular, when k is odd, $P_k(t, x)$ does not have any term with an even power of x, including any 'constant', or x-free, terms. The induction in the first stage now clearly follows from the mere fact that for all odd k, $\mathbf{E}P_k(1, M_1) = 0$, and on taking into account $c_{k,k} \neq 0$.

For the second stage, we need a preliminary observation. Recall (refer to Remark 2.10) the existence of two-variable polynomials $C_k(\cdot, \cdot)$ with $C_k(\cdot, x)$ having x^k as its leading term in x, satisfying $\mathbf{E}(M_n^k | \mathcal{F}_{n-1}) = C_k(n, M_{n-1})$ k, $n \geq 1$. Indeed, this is just a consequence of $\bar{\mathcal{P}}_k(M)$ being non-empty for all k. We now claim that $C_k(\cdot, x)$ does not contain any term with a power x^i of x where $k - i$ is odd, and prove this by induction. For $k = 1$, this can be seen to be true using the fact that $P_1(t, x)$ does not contain x-free term, implying that M itself is a martingale. If now $k > 1$ and if $C_j(\cdot, \cdot)$ satisfies our claim for $1 \leq j < k$, then from the martingale property of $P_k(n, M_n) = \sum_{k-j \text{ even}} c_{k,j} n^{\beta(k-j)} M_n^j$, we have

$$\begin{aligned}
\sum_{k-j \text{ even}} c_{k,j} (n-1)^{\beta(k-j)} M_{n-1}^j &= \sum_{k-j \text{ even}} c_{k,j}\, n^{\beta(k-j)} \mathbf{E}(M_n^j | \mathcal{F}_{n-1}) \\
&= \sum_{j<k,\, k-j \text{ even}} c_{k,j}\, n^{\beta(k-j)} \mathbf{E}(M_n^j | \mathcal{F}_{n-1}) + c_{k,k} \mathbf{E}(M_n^k | \mathcal{F}_{n-1}).
\end{aligned}$$

So assuming $c_{k,k} = 1$ now, which is no loss of generality, we have

$$\mathbf{E}(M_n^k | \mathcal{F}_{n-1}) = M_{n-1}^k + \sum_{j<k,\, k-j \text{ even}} c_{k,j} \left[(n-1)^{\beta(k-j)} M_{n-1}^j - n^{\beta(k-j)} \mathbf{E}(M_n^j | \mathcal{F}_{n-1})\right],$$

$$\text{or,} \quad C_k(n, x) = x^k + \sum_{j<k,\, k-j \text{ even}} c_{k,j} \left[(n-1)^{\beta(k-j)} x^j - n^{\beta(k-j)} C_j(n, x)\right],$$

which, by induction hypothesis, can be easily seen to be free of powers of x with index differing from k by an odd number. For the induction in the second stage, we can now write

$$\begin{aligned}
\mathbf{E}[M_1^{k_1} M_2^{k_2} \cdots M_r^{k_r}] &= \mathbf{E}[\mathbf{E}(M_1^{k_1} M_2^{k_2} \cdots M_{r-1}^{k_{r-1}} M_r^{k_r} | \mathcal{F}_{r-1})] \\
&= \mathbf{E}[M_1^{k_1} M_2^{k_2} \cdots M_{r-1}^{k_{r-1}} \mathbf{E}(M_r^{k_r} | \mathcal{F}_{r-1})] \\
&= \mathbf{E}[M_1^{k_1} M_2^{k_2} \cdots M_{r-1}^{k_{r-1}} C_{k_r}(r, M_{r-1})]
\end{aligned}$$

This, being a linear combination of expressions of the same kind as the L.H.S. with $r - 1$ replacing r, equals 0 by induction hypothesis.

The continuous-time case follows from the discrete-time one by the usual approximation procedure. First, when $t_1, t_2, \ldots, t_r$ are rational, then the discrete-time case applies with an appropriate $M^{(N)}$. Finally, approximation of $\mathbf{E}\left[M_{t_1}^{k_1} M_{t_2}^{k_2} \cdots M_{t_r}^{k_r}\right]$ in L^1 by expressions of the same type, but with $(t_1, t_2, \ldots, t_r)$ replaced by rational r-tuples $(t_{1,n}, t_{2,n}, \ldots, t_{r,n})$ with $t_{i,n} \downarrow t_i$ for $1 \leq i \leq r$ using L^2-boundedness, yields our result. ∎

Chapter 5

Partial p-harmonisability

5.1 Counterparts of previous results

In this chapter, we replace the assumption of the moment condition (**M**) by the condition of the existence of only finitely many moments and try to see to what extent our previous results remain valid. It turns out that in nearly all cases, the same proofs are applicable. The only point to remember is that non-emptiness, or otherwise, of each $\mathcal{P}_k$ affects, and is in turn affected by, moments or conditional moments of order upto k alone. In the sequel, the order of a moment $\mathbf{E}(M_{t_1}^{k_1} M_{t_2}^{k_2} \cdots M_{t_r}^{k_r})$ of a finite-dimensional joint distribution is defined as $\sum k_j$. Also, as always, we keep on using the convention $P_0 \equiv 1$. The counterparts of the discrete-time Theorems 2.1, 2.2, 2.4 and 2.5 are as follows. The proofs here being identical with those those in Chapter 2, are omitted here.

Theorem 5.1 *Suppose $\{X_n : n \geq 1\}$ is a sequence of independent and identically distributed random variables with X_1 having a nondegenerate distribution with first i moments finite. Set $M_0 \equiv 0$ and $M_n = \sum_{l=1}^{n} X_l$, $n \geq 1$. Then the process M admits a unique sequence $\{P_k \in \mathcal{P}_k(M) : 1 \leq k \leq i\}$ satisfying the properties (i) – (v), with the obvious appropriate interpretation. Further, if each P_k is time-space harmonic for a process N, then N must have the same moments as those of M, upto order i.*

Theorem 5.2 *If $\{M_n : n \geq 0\}$ is a process with independent difference sequence $\{X_n\}$ and satisfying the support condition* (**S**), *then a necessary and sufficient condition for each $\mathcal{P}_k$ with $1 \leq k \leq i$ to be non-empty is that $E(X_n^k)$ be a polynomial in n for all k, $1 \leq k \leq i$.*

In that case, there exists unique $\{P_k \in \mathcal{P}_k : 1 \leq k \leq i\}$ such that (i), (ii) and (iv) hold. Also, $\{P_k : 1 \leq k \leq i\}$ spans $\cup_{k=0}^{i} \mathcal{P}_k$ and determines $\{E(X_n^k) : n \geq 1,\ 1 \leq k \leq i\}$.

Theorem 5.3 *Suppose M is a process for which $\mathcal{P}_k(M)$ is non-empty for every $1 \leq k \leq i$. Suppose, moreover, that for each $n \geq 1$ and each k, $1 \leq k \leq i$, there exists $P_k \in \mathcal{P}_k$ such that $P(n, x)$, as a polynomial in x alone, is of degree exactly k. Then, for every $n \geq 1$ and k, $1 \leq k \leq i$, there exists a polynomial c_n^k of degree at most k such that $E(M_n^k|\mathcal{F}_{n-1}) = c_n^k(M_{n-1})$ almost surely.*

Theorem 5.4 *For any process M, $\bar{\mathcal{P}}_k(M)$ is non-empty for all $1 \leq k \leq i$ if, and only if, for each $1 \leq k \leq i$, there exists a two-variable polynomial $B_k(t, x)$, of degree at most $k-1$ in x, so that $E(X_n^k|\mathcal{F}_{n-1}) = B_k(n, M_{n-1})$ a. s., for all $n \geq 1$.*

If further M satisfies **(S)**, *then there exists unique $P_k \in \bar{\mathcal{P}}_k$ for $1 \leq k \leq i$ satisfying the property (iv) for $1 \leq k \leq i$. Any sequence $\{P_k \in \mathcal{P}_k : 1 \leq k \leq i\}$ spans $\cup_{k=1}^{i}\mathcal{P}_k$. In particular, $\mathcal{P}_k(M) = \bar{\mathcal{P}}_k(M)$ for every $1 \leq k \leq i$.*

In continuous time too, the Theorems 3.1, 3.2, 3.3 and 3.4 have their counterparts in this setup, whose proofs are again omitted.

Theorem 5.5 *Let $M = \{M_t : t \geq 0\}$ be a homogeneous Lévy process with $M_0 \equiv 0$ satisfying the condition* **(S)**. *Suppose $\mathbf{E}\exp(\alpha M_1) < \infty$ for all $\alpha \in \Gamma$ where Γ is some nonempty open neighbouhood of 0. Then there exists a unique sequence $\{P_k \in \mathcal{P}_k(M) : k \geq 1\}$ satisfying the properties (i)– (iv), and spanning $\cup_{k=1}^{i}\mathcal{P}_k$. Moreover, any process N for which $\mathcal{P}_k(N) = \mathcal{P}_k(M)$ for all $1 \leq k \leq i$ must have the same finite-dimensional joint moments as that of M, upto order i.*

Theorem 5.6 *If M is a Lévy process with $M_0 \equiv 0$ satisfying* **(S)**, *then $\mathcal{P}_k(M) \neq \emptyset$ for each $1 \leq k \leq i$ if and only if $\mathbf{E}M_t^k$ is a polynomial in t for $1 \leq k \leq i$.*

In this case, there exists a unique sequence $\{P_k \in \mathcal{P}_k : 1 \leq k \leq i\}$ satisfying (i) - (v). Any such sequence determines the first i moments of each M_t.

Theorem 5.7 *Let M be a continuous-time process for which $\mathcal{P}_k(M) \neq \emptyset$ for $1 \leq k \leq i$ and for each $1 \leq k \leq i$ and every $t > 0$, there exists $P \in \mathcal{P}_k$ with $P(t, \cdot)$ being of degree exactly k. Then for all $t, s > 0$, $1 \leq k \leq i$, there exists a polynomial $d_{t,s}^k$ of degree at most k such that $\mathbf{E}(M_{t+s}^k|\mathcal{F}_t) = d_{t,s}^k(M_t)$ a.s..*

Theorem 5.8 *For a process M satisfying* **(S)**, *$\bar{\mathcal{P}}_k(M) \neq \emptyset$ for every $1 \leq k \leq i$ if and only if for each $1 \leq k \leq i$, there exists a three-variable polynomial C_k with the property that for all t, $s > 0$, $C_k(t, s, \cdot,)$ has degree k and leading coefficient 1, such that $\mathbf{E}(M_{t+s}^k|\mathcal{F}_t) = C_k(t, s, M_t)$.*

In this case, any sequence $\{P_k \in \mathcal{P}_k : 1 \le k \le i\}$ *spans* $\cup_{k=1}^{i} \mathcal{P}_k$*, and consequently.* $\mathcal{P}_k(M) = \bar{\mathcal{P}}_k(M)$ *for each* $1 \le k \le i$*. Imposing (iv) in addition renders this sequence unique.*

Concerning the interplay of various properties of sequences of time-space harmonic polynomials and the underlying process, we have next the counterpart of the discrete-time Theorem 4.1, which deals with sequences satisfying the property *(ii)*.

Theorem 5.9 *If for a process* M *satisfying* (**S**), $\mathcal{P}_k(M) \neq \emptyset$ *for each* $1 \le k \le i$*. and* $P_k \in \mathcal{P}_k$, $1 \le k \le i$*, can be chosen such that property (ii) holds, then for each* $1 \le k \le i$ *and all* n, $\mathbf{E}(X_n^k|\mathcal{F}_{n-1})$ *is non-random.*

In this case, this sequence satisfies (i), spans $\cup_{k=1}^{i} \mathcal{P}_k$*, and is the unique sequence to also satisfy (iv). Any sequence* $\{P_k \in \mathcal{P}_k : 1 \le k \le i\}$ *determines the finite-dimensional joint moments of* M*, upto order* i*.*

Its continuous-time version is an analogue to Theorem 4.3.

Theorem 5.10 *If* M *satisfies* (**S**) *and* $\{P_k \in \mathcal{P}_k(M) : 1 \le k \le i\}$ *satisfies the Appell condition (ii), then each of the conditional moments* $\mathbf{E}((M_{t+s} - M_t)^k|\mathcal{F}_t)$, $1 \le k \le i$, $s, t \ge 0$*, is degenerate.*

In this case, the above sequence also satisfies (i) and (iv) for $1 \le k \le i$*, is the unique sequence to do so, and spans* $\cup_{k=1}^{i} \mathcal{P}_k$*. Any such sequence also determines the finite-dimensional joint moments of* M*, upto order* i*.*

The final discrete-time theorem pertains to the pseudo-type-zero condition *(iii)* and is thus the counterpart to Theorem 4.2. For this, however, we need to develop a little preliminary background.

Given a set $\{p_k : 1 \le k \le i\}$ of polynomials of a single variable, with p_k being of degree k, and with $p_0 \equiv 1$, satisfying the pseudo-type-zero property with the finite sequence $h_1, h_2, \ldots, h_i$, we first prove that it is possible to extend it to a full infinite sequence $\{p_k : k \ge 1\}$ still satisfying the same property. This can be done with the original finite sequence $\{h_k, 1 \le k \le i\}$ being augmented to an infinite sequence in any way whatsoever.

Suppose $p_k(t) = \sum_{j=0}^{k} \rho_{k,j} t^j$, $k \ge 1$, where the $\rho_{k,j}$, $0 \le j \le k$, are given for $k \le i$ and to be worked out for $k > i$. Recall the equation (4.3) which gives, for $1 \le j \le k$, $\rho_{k,j}$ in terms of the $\rho_{i,0}$ for $0 \le i \le k - j$. That condition is clearly both necessary and sufficient for the pseudo-type-zero property of the $\{p_k\}$. Therefore, if we arbitrarily fix, for $k > i$, the numbers $\rho_{k,0}$ and recursively define $\rho_{k,j}$ for $1 \le j \le k$ using exactly the same formula (4.3), the resulting polynomials $p_k(t) = \sum_{j=0}^{k} \rho_{k,j} t^j$, $k > i$ will continue to satisfy the pseudo-type-zero condition. It is implicit that the arbitrarily augmented sequence $\{h_k\}$ is being used in the definition (4.3) when $k > i$.

Now, given a discrete time process M and a finite sequence $\{P_k \in \mathcal{P}_k(M) : 1 \leq k \leq i\}$ satisfying both the Appell and pseudo-type-zero properties *(ii)* and *(iii)* for $1 \leq k \leq i$, consider the one-variable polynomials $p_0^{(k)}$, where $P_k(t,x) = \sum_{j=0}^{k} p_j^{(k)}(t) x^j$, $1 \leq k \leq i$. Then, *(iii)* implies that the set $\{p_0^{(k)} : 1 \leq k \leq i\}$ satisfies the pseudo-type-zero property with the same sequence $(h_k, 1 \leq k \leq i)$. Extend this set to a full sequence $\{p_0^{(k)} : k \geq 1\}$ satisfying the pseudo-type-zero property, as described above, by augmenting the partial sequence (h_k) by zeroes, that is, defining $h_k = 0 \ \forall\ k \geq i+1$. We already have, by virtue of *(ii)* and in turn by *(iv)*, that $\rho_{k,0} = 0$ for $1 \leq k \leq i$, and for $k > i$ too, we choose the numbers $\rho_{k,0} = 0$. Now set, for $k > i$, $P_k(t,x) = \sum_{j=0}^{k} p_j^{(k)}(t) x^j$, where for $1 \leq j \leq k$, $p_j^{(k)}(t) = \binom{k}{j} p_0^{(k-j)}(t)$.

This extended sequence $\{P_k : k \geq 1\}$ now satisfies both the properties *(ii)* and *(iii)*, therefore, as in the proof of Theorem 4.2 we have the formal power series representation

$$\sum_{k=0}^{\infty} P_k(t,x) \frac{\alpha^k}{k!} = e^{\alpha x} e^{t h(\alpha)}, \tag{5.1}$$

where $h(\alpha) = \sum_{k=1}^{\infty} h_k \alpha^k / k! = \sum_{k=1}^{i} h_k \alpha^k / k!$.

We know that when the m.g.f. of a random variable exists in some neighbourhood of 0, the cumulants γ_k are defined through the c.g.f.. However, the cumulants of order upto k are also algebraically expressible in terms of the first k moments, and conversely. In other words, for every $k \geq 1$, there is a one-to-one function $f_k : \mathbb{R}^k \to \mathbb{R}^k$ such that if for a random variable (equivalently, any probability distribution on $\mathbb{R}$) for which the m.g.f. exists in an open neighbourhood of 0, $\mu = (\mu_1, \ldots, \mu_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_k)$ denote respectively the moments and cumulants of order upto k, then $\gamma = f_k(\mu)$. Naturally, $\mu = f_k^{-1}(\gamma)$. Explicitly,

$$\gamma_j = \sum_{m=1}^{j} \frac{(-1)^{m+1}}{m!} \sum_{\substack{j_1,\ldots,j_m \geq 1 \\ j_1+\cdots+j_m = j}} \binom{j}{j_1,\ldots,j_m} \mu_{i_1} \cdots \mu_{i_j}, \tag{5.2}$$

$$\text{and} \quad \mu_j = \sum_{m=1}^{j} \frac{1}{m!} \sum_{\substack{j_1,\ldots,j_m \geq 1 \\ j_1+\cdots+j_m = j}} \binom{j}{j_1,\ldots,j_m} \gamma_{i_1} \cdots \gamma_{i_j}. \tag{5.3}$$

The point to note here is that this relationship is 'universal', that is, the distribution of the underlying random variable does not affect the function f_k. Let us continue to call, even in the case when a random variable has only first k moments finite, the co-ordinates of the vector $\gamma = f_k(\mu)$ the first k 'cumulants' of that random variable.

Perhaps, a more natural way to define the 'cumulants' would be through the characteristic function in the following way. In a neighbourhood of 0 where the c.f. is bounded away from -1, one could define the logarithm of the c.f. by choosing any branch of the log function. It can be proved that the resulting function, just like the c.f., has just as many derivatives at 0 as the underlying random variable has moments. These derivatives, multiplied by suitable powers of i, can be defined as the cumulants. Of course, one may show by similar, though involved, calculations that this definition would agree with the one given earlier, that is, through the relationships (5.2) and (5.3).

Next, we formalise somewhat the fact that the moments of a sum of independent random variables can be derived from those of the summands. Specifically, let $Z_1, Z_2, \ldots, Z_n$ be independent random variables, and $Z = \sum_{l=1}^{n} Z_l$, where $n \geq 1$. Let $k \geq 1$, and suppose $\mathbf{E}|Z_l|^k < \infty$ for $1 \leq l \leq n$. Then, writing $\mathbf{E}Z_l^j$ as $\mu_{l,j}$, $1 \leq l \leq n$, $1 \leq j \leq k$, we claim that there exists a function $g_{n,k} : \mathbb{R}^{nk} \to \mathbb{R}^k$ such that

$$\left(\mathbf{E}Z, \mathbf{E}Z^2, \ldots, \mathbf{E}Z^k\right) = g_{n,k}\left(\mu_{1,1}, \ldots, \mu_{1,k}, \mu_{2,1}, \ldots, \mu_{2,k}, \ldots, \mu_{n,1}, \ldots, \mu_{n,k}\right)$$

This function $g_{n,k}$ is also 'universal' in the same sense as f_k.

Now, let us observe that inasmuch as the c.g.f. of Z is precisely the sum of those of X_l, $1 \leq l \leq n$, provided they exist, the functions f_k and $g_{n,k}$ must satisfy

$$f_k \circ g_{n,k}((\mu_{l,j})) = f_k\left((\mu_{1,j})_{1 \leq j \leq k}\right) + f_k\left((\mu_{2,j})_{1 \leq j \leq k}\right) + \cdots + f_k\left((\mu_{n,j})_{1 \leq j \leq k}\right)$$

Recall now from Theorem 5.9 that the Appell property *(ii)* implies that for each $1 \leq k \leq i$ and for every n, $\mathbf{E}\left(X_n^k | \mathcal{F}_{n-1}\right) = \mathbf{E}\left(X_n^k\right) = b_k(n)$, say, a polynomial in n. It follows that whenever $1 \leq k_1, k_2, \ldots, k_n \leq k$, $\mathbf{E}(X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}) = \prod_{l=1}^{n} \mathbf{E}\left(X_l^{k_l}\right)$ provided the L.H.S. is defined. This can be seen by applying the smoothing property of conditional expectations. One further concludes, writing $m_{k,n} = \mathbf{E}M_n^k$ for $1 \leq k \leq i$ and $n \geq 1$, that

$$\mathbf{m}_n = g_{n,i}(b_1(1), \ldots, b_i(1), \ldots, b_1(n), \ldots, b_i(n))$$

where $\mathbf{m}_n = (m_{k,n})_{1 \leq k \leq i}$, $n \geq 1$. In particular, therefore, the first i 'cumulants' of M_n are given by

$$f_i(\mathbf{m}_n) = f_i \circ g_{n,i}(\mathbf{b}(1), \mathbf{b}(2), \ldots, \mathbf{b}(n)) = \sum_{l=1}^{n} f_i(\mathbf{b}(l)), \tag{5.4}$$

where $\mathbf{b}_l = ((b_k(l))_{1 \leq j \leq i}$.

Now, we know that the polynomials P_k, $1 \leq k \leq i$, determine the moments $m_{k,n}$, $1 \leq k \leq i$. This allows us to express them also in terms of the coefficients of the polynomials $p_0^{(k)}$, which, as we have shown above, are determined by h_k, $1 \leq k \leq i$ because of the choice $p_0^{(k)}(0) = \rho_{k,0} = 0$. To be specific, using the fact $\mathbf{E}P_k(n, M_n) = 0$, we get $0 = \sum_{j=0}^{k} p_j^{(k)} m_{j,n} = m_{k,n} + \sum_{j=0}^{k-1} \binom{k}{j} p_0^{(k-j)} m_{j,n}$, whence we get

$$m_{k,n} = -\sum_{j=1}^{k} \binom{k}{j} p_0^{(j)} m_{k-j,n}.$$

This recursive relation establishes what we just claimed. Thus, the four sequences $f_i(\mathbf{m}_n)$, $\mathbf{m}_n$, $(p_0^{(k)}(n))_{1\le k\le i}$, and $(h_k)_{1\le k\le i}$ are all in one-to-one correspondence with one another. We are interested in that between the first and the last of these.

But then, appealing to equation (5.1), as in Theorem 4.2, we must conclude that

$$f_i(\mathbf{m}_n) = -n\,(h_1,\, h_2,\, \ldots,\, h_i)\,.$$

This yields, in conjunction with equality (5.4), that

$$\sum_{l=1}^{n} f_i\,(\mathbf{b}(l)) = -n\,(h_1,\, h_2,\, \ldots,\, h_i)\,.$$

Since this is true for all $n \ge 1$, it follows that

$$f_i\,(\mathbf{b}(n)) = -\,(h_1,\, h_2,\, \ldots,\, h_i)\,,$$

which is free of n, and hence that so is $\mathbf{b}(n)$. We are thus led to the

Theorem 5.11 *If M is a discrete-time process with difference sequence X such that there exists a sequence $\{P_k \in \mathcal{P}_k(M) : 1 \le k \le i\}$ satisfying both the Appell and pseudo-type-zero conditions (ii) and (iii), then the conditional moments $\mathbf{E}\left(X_n^k | \mathcal{F}_{n-1}\right)$ for $1 \le k \le i$ equal the corresponding unconditional moments almost surely, and they are free of n.*

We put the conclusion in a slightly modified equivalent form for ease in visualising the counterpart in continuous time. This formulation is in terms of the multi-step differences: that for all $m,\, n \ge 1$, $1 \le k \le i$, $\mathbf{E}\left((M_{n+m} - M_n)^k | \mathcal{F}_n\right) = \mathbf{E}\,(M_{n+m} - M_n)^k = \mathbf{E}\left(M_m^k\right)$. Naturally, for all $1 \le k \le i$, the latter is a polynomial in m.

Theorem 5.12 *If $M = \{M_t : t \ge 0\}$ is a continuous-time parameter process admitting a sequence of time-space harmonic polynomials $\{P_k : 1 \le k \le i\}$ for some $i \ge 1$, satisfying both conditions (ii) and (iii), then for every $1 \le k \le i$ and $t,\ s > 0$, $\mathbf{E}\left((M_{t+s} - M_t)^k | \mathcal{F}_t\right) = \mathbf{E}\left(M_s^k\right)$ almost surely.*

The proof involves the same approximation argument used before and the idea is as follows. Theorem 5.11 applied to the discrete-time $M^{(N)}$ for a suitable N, and then applying Lemma 3.1 would mean that the statement holds when s and t are rationals. Using L^2-boundedness and right continuity of paths, the result follows. We also observe in passing that the function of s in the R.H.S. is a polynomial.

5.2 A characterisation : the homogeneous case

In page 5 in the first chapter, we raised the question of obtaining a characterisation of processes whose laws are determined by finitely many time-space harmonic polynomials. Let us consider the following more general question.

Given a class $\mathcal{C}$ of processes, let

$$\tilde{\mathcal{C}} = \left\{ \begin{array}{l} M \in \mathcal{C} : \exists\, k \geq 1 \text{ so that } \mathcal{P}_j(M) \neq \emptyset \,\forall\, 1 \leq j \leq k \text{ and} \\ \text{for any } Y \in \mathcal{C}, \mathcal{P}_j(Y) = \mathcal{P}_j(M) \,\forall\, 1 \leq j \leq k \Rightarrow Y \stackrel{d}{=} M \end{array} \right\}.$$

The members of $\tilde{\mathcal{C}}$ are called the finitely polynomially determined (fpd for short) processes in $\mathcal{C}$. We also refer to them as being k-polynomially determined, or k-pd, where $k \geq 1$ is as in the definition. A natural question is for which classes $\mathcal{C}$ is a complete description of its fpd members, or equivalently, of $\tilde{\mathcal{C}}$, available? Our aim in this and the next two sections, is to present a couple of examples of $\mathcal{C}$ for which this is possible. Both these are classes of Lévy processes, the first consisting of the homogeneous ones and the second a more general class. Although the former class is already contained in the latter, it is quite instructive, and much simpler, to treat this case separately.

Thus, we first deal with the homogeneous Lévy processes, that is, those for whom the increments are stationary and independent. Let us recall from chapter 3 the version (3.1) of the Lévy-Khintchine representation that was used to prove the non-equivalence of the p-harmonisability of a Lévy process and that of its canonical components. For homogeneous Lévy processes, the representation takes the form

$$\begin{aligned} \log(\mathbf{E}(e^{i\alpha M_t})) &= i\alpha\nu t + t\int \left(\frac{e^{i\alpha u} - 1 - i\alpha u}{u^2} \right) l(du) && (5.5) \\ &= i\alpha\mu t - \frac{\alpha^2\sigma^2 t}{2} + t\int \left(e^{i\alpha u} - 1 - \frac{i\alpha u}{1+u^2} \right) \eta(du) && (5.6) \end{aligned}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$, $\nu \in \mathbb{R}$ is the mean of the random variable M_1 and and η and l are finite measureson $\mathbb{R}$, the former having no mass at 0. The product measures $dt \otimes \eta(du)$ and $dt \otimes l(du)$ are respectively the 'Lévy measure' and 'Kolmogorov measure' of M. The integrand in (5.5) is defined at $u = 0$ by the limiting value. The explicit relation between the measures l and η is given by

$$l(A) = \sigma^2 1_A(0) + \int_{A\setminus\{0\}} u^2 d\eta(u) \tag{5.7}$$

where $A \in \mathcal{B}(\mathbb{R})$. We recall the important property of l, to be used subsequently, that $\forall\, k > 1$, the k-th cumulant of M_1 equals $\int u^{k-2} l(du)$.

Before stating out characterisation, we need to make one more observation. Given a Lévy process M, adding another independent 0-mean Lévy process leaves its mean function unchanged. This means that that there are always infinitely many Lévy processes with the same $\mathcal{P}_1$. Clearly, therefore, we can not expect to have any characterisation not using at least $\mathcal{P}_2$. We may therefore assume without loss of generality that any Lévy process M under consideration satisfies $\mathbf{E}M_t^2 < \infty$.

We may now state

Theorem 5.13 *A homogeneous Lévy process is fpd if and only if the measure η, or equivalently, the measure l, is finitely supported.*

Proof : It is immediate from the relation (5.7) that whenever the measure m has finite support, so has the measure l and conversely. In our proof, we work with the measure l.

Suppose first that supp(l) is finite. Then we can write

$$l = \sum_{i=1}^{N} p_i \delta_{r_i} \tag{5.8}$$

for some $N \geq 1$, where $p_i > 0$ for $i = 1, \ldots, N$ and r_i's are distinct real numbers. For a real a, δ_a refers to the point mass supported at a. To prove the 'if' part of the theorem, we have to exhibit a k such that the distribution of M is uniquely determined, among homogeneous Lévy processes, by $\{\mathcal{P}_j(M) : 1 \leq j \leq k\}$. Our choice is $k = 2N + 2$.

Suppose Y is a homogeneous Lévy process with $\mathcal{P}_j(Y) = \mathcal{P}_j(M)$ for each $1 \leq j \leq 2N+2$. We then prove $Y \stackrel{d}{=} M$. Clearly, for this we have to show that Y has the same mean and Kolmogorov measure as M, or in other words, that $\nu_Y = \nu_M$ and $l_Y = l_M$.

We know, from Theorem 5.6, that $\mathcal{P}_j(Y) = \mathcal{P}_j(M)$, $1 \leq j \leq 2N+2$, implies the equality of the first $2N + 2$ moments of Y_1 and M_1, which, in turn implies that of their respective first $2N + 2$ cumulants. In view of the property of l we described previously, this entails

$$\nu_M = \nu_Y \quad \text{and} \quad \int_{\mathbb{R}} u^j l_M(du) = \int_{\mathbb{R}} u^j l_Y(du) \text{ for } j = 0, 1, \ldots, 2N. \tag{5.9}$$

Notice now that for any measure L on $\mathbb{R}$, and real numbers a_1, a_2, $\ldots a_N$, the integral $\int_{\mathbb{R}} \prod_{j=1}^{N}(u - a_j)^2 L(du)$ can be expressed as a function only of the constants $a_1, \ldots, a_N$ and the "moments" (including the zeroeth) $\int_{\mathbb{R}} u^j L(du)$ of L, $0 \leq j \leq 2N$, provided they exist. This implies,

$$\int_{\mathbb{R}} \prod_{j=1}^{N} (u - r_j)^2 l_Y(du) = \int_{\mathbb{R}} \prod_{j=1}^{N} (u - r_j)^2 l_M(du) = 0,$$

where we have used equation (5.9) to obtain the first equality and equation (5.8) for the second. It follows that

$$l_Y(\{r_1, \ldots, r_N\}^c) = 0.$$

In other words, l_Y can be written as

$$l_Y = \sum_{i=1}^{N} p_i' \delta_{r_i}$$

for some non-negative numbers p_i', $1 \leq i \leq N$.

We now claim that $p_i' = p_i$ for $1 \leq i \leq N$. From the form of l_M and l_Y, equation (5.9) tells us that

$$\sum_{i=1}^{N} p_i r_i^j = \sum_{i=1}^{N} p_i' r_i^j, \quad j = 0, 1, \ldots, N-1.$$

This can be expressed as $\mathbf{A}\mathbf{p} = \mathbf{A}\mathbf{p}'$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_N \\ \vdots & \vdots & & \vdots \\ r_1^{N-1} & r_2^{N-1} & \ldots & r_N^{N-1} \end{pmatrix}, \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} \text{ and } \mathbf{p}' = \begin{pmatrix} p_1' \\ p_2' \\ \vdots \\ p_N' \end{pmatrix},$$

The matrix $\mathbf{A}$, being a Vandermonde matrix, is non-singular in particular. This yields $\mathbf{p} = \mathbf{p}'$ and completes the proof of this part of the theorem.

For the converse, suppose that for a homogeneous Lévy process M, the measure l is not finitely supported. Fix any $k \geq 1$. We shall exhibit another homogeneous Lévy process Y, with a law distinct from that of M, with $\mathcal{P}_j(Y) = \mathcal{P}_j(M)$ for $1 \leq j \leq k$.

Since by assumption, $\text{supp}(l)$ is infinite, we can certainly pick k distinct elements $r_1, r_2, \ldots, r_k$ from it. By the definition of support, there exist open neighbourhoods A_i of r_i, $i = 1, 2, \ldots, k$ with $l(A_i) > 0$ and $A_i \cap A_j = \emptyset$. Now consider the following real vector space of signed measures on the Borel σ-field $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$.

$$\mathcal{V} = \left\{ \mu : \mu(A) = \sum_{i=1}^{k} c_i l(A \cap A_i), c_i \in \mathbb{R}, A \in \mathcal{B}(\mathbb{R}) \right\}.$$

This is clearly isomorphic to $\mathbb{R}^k$ and hence has dimension k. Define now the linear map $\Lambda : \mathcal{V} \to \mathbb{R}^{k-1}$ as

$$\Lambda(\mu) = \left(\int \mu(du), \int u\mu(du), \dots, \int u^{k-2}\mu(du)\right).$$

Since the range of Λ is a subspace of $\mathbb{R}^{k-1}$, its dimension is at most $(k-1)$. The nullity of Λ must therefore be at least 1. Choose a non-zero element in the null space of Λ, say

$$\mu(A) = \sum_{i=1}^{k} c_i l(A \cap A_i).$$

We can, of course, choose μ so that $|c_i| < 1$ for all $1 \le i \le k$. We now define

$$\bar{l}(A) = l(A) + \mu(A), \quad A \in \mathcal{B}(\mathbb{R}).$$

Clearly $\bar{l}$ is a positive measure, and since μ is non-zero, $l \neq \bar{l}$. Define the homogeneous Lévy process Y by setting $\nu_Y = \nu_M$ and using the Kolmogorov measure $L(dt, du) = dt \otimes \bar{l}(du)$. Thus $Y \overset{d}{\neq} M$. But on the other hand, for each $j = 0, 1, \dots, 2N$, we have

$$\int u^j \bar{l}(du) = \int u^j l(du) + \int u^j \mu(du) = \int u^j l(du).$$

This implies that $\mathbf{E}Y_1^j = \mathbf{E}M_1^j$, and in turn, that $\mathcal{P}_j(Y) = \mathcal{P}_j(M)$, for all $1 \le j \le 2N+2$.

■

It is natural to ask what our result exactly mean for the underlying process. It means that a homogeneous Lévy process is fpd if, and only if, its jumps, when they occur, can have sizes only in a fixed finite set. For the Lévy measure (see [9], page 145) is nothing but a multiple of the "jump distribution" of the process. The latter has a complicated definition in general. However, finitely supported jump distributions conform to the definition as the distribution of the "first jump".

5.3 Preliminaries for the general case

To present the counterpart of the characterisation for general Lévy processes, we require a few preliminary results. First recapitulate the 'Lévy-Khintchine' representation (3.1)

$$\begin{aligned}\log \mathbf{E}(e^{i\alpha M_t}) &= i\alpha\mu(t) - \frac{\alpha^2\sigma^2(t)}{2} - \int (e^{i\alpha u} - 1 - \frac{i\alpha u}{1+u^2})m([0,t] \otimes du) \\ &= i\alpha\nu(t) + \int \left(\frac{e^{i\alpha u} - 1 - i\alpha u}{u^2}\right) L([0,t] \otimes du)\end{aligned}$$

where m and L are respectively the Lévy measure and Kolmogorov measure of M.

The class $\mathcal{C}$ of processes that we consider here consists of those Lévy processes for which the Kolmogorov measure admits a 'derivative' l in the sense that l is a transition measure on $[0,\infty) \times \mathcal{B}$ satisfying

$$L([0,t] \times A) = \int_0^t l(s,A)ds \quad \forall\, t \geq 0,\ A \in \mathcal{B}.$$

In this case, it can be checked that moreover,

$$\int f dL = \int f(t,u)l(t,du)dt$$

whenever $f : [0,\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that either side is well-defined. We designate l as 'derivative measure' of M.

This class $\mathcal{C}$ is fairly large, containing such examples as Gaussian Lévy processes, whose Kolmogorov measures are of the type $L(dt,du) = d\sigma^2(t)\delta_{\{0\}}(du)$. Other members of the class include nonhomogeneous compound Poisson processes (see page 84). Naturally, $\mathcal{C}$ being a vector space, any Lévy process which arises as the independent sum of two processes of the above kinds, also belongs to $\mathcal{C}$.

It should be noted that $\mathcal{C}$ contains all homogeneous Lévy processes, for which $l(t,\cdot) \equiv l(\cdot)$. The necessary and sufficient condition characterising elements of $\tilde{\mathcal{C}}$ that we get would therefore naturally be expected to be in terms of $l(\cdot,\cdot)$. It appears quite difficult even to formulate this, or a similar condition, in terms of the Kolmogorov measure L (or the Lévy measure m) instead of l. In fact, this seems to be precisely the stumbling block in obtaining a characterisation, among all Lévy processes, of those which are fpd.

We first take up the following all-important Lemma 5.1. But first let us briefly touch on a heuristic justification for expecting it to be true. If for a substantial set of points $\{t\}$, $l(t,\cdot)$ were supported on more than k points, then like in the homogeneous case, it could not be determined uniquely from the first k moments of the process. This is exactly what Lemma 5.1 formalises. However, unlike in the homogeneous case, to define the law of another Lévy process in $\mathcal{C}$, one has to handle not just the distribution of one single random variable, but that of the whole process; that is, define a different Kolmogorov measure, or equivalently, derivative measure $\bar{l}$ which would keep the first k moments of the process intact. The crux of the matter lies naturally in constructing $\bar{l}$ while retaining its measurability with respect to its first argument. This requires a variant of a certain result of Descriptive Set Theory, known as Novikov's Selection Theorem. The exact form in which we use it here is stated as

Theorem 5.14 *Suppose U is a Standard Borel space and V a σ-compact* subset of a *Polish space, and $B \subseteq U \times V$ a Borel set whose projection onto U is the whole of U, that is,*

$\Pi_U(B) = U$. *Suppose further that the sections B_x of B, $x \in U$, are all closed. Then there is a Borel measurable function $g : U \to V$ whose graph is contained in B, that is, $g(x) \in B_x \; \forall x \in U$.*

This form of the theorem can easily be derived from its usual form (see [10] page 220, Theorem 28.8), in which the condition of σ-compactness on V is dropped, and instead, it is assumed that all the sections B_x of B are compact. The passage from this result to ours is achieved through patching up the selections over countably many compact subsets.

For the rest of this chapter, a Lévy process will always mean an element of class $\mathcal{C}$, that is, one whose Kolmogorov measure admits a derivative measure in the sense described earlier. Also, when we talk of such a process being determined in distribution, we only mean that no other member of $\mathcal{C}$ shares its law.

Lemma 5.1 *Suppose the Lévy process M of class $\mathcal{C}$ is determined in distribution by $\{\mathcal{P}_j(M) : 1 \le j \le k\}$ where $k > 1$. Then for any version of l, the set $T \subseteq [0, \infty)$ defined by $T = \{t \ge 0 : |supp\,(l(t, \cdot))| > k\}$ is Borel and has zero lebesgue measure.*

Proof: Start with any version of l, and construct the set T as in the statement. Towards the first statement, define, for each $n \ge 1$ and $j \in \mathbb{Z}$, the Borel sets $A_{j,n} = \{t : l(t, (\frac{j}{2^n}, \frac{j+1}{2^n}]) > 0\}$. For each $n \ge 1$, the possibly $+\infty$-valued function f_n on $[0, \infty)$ defined as

$$f_n(t) = \sum_j 1_{A_{j,n}}(t)$$

is measurable. Clearly, for each t, $f_n(t)$ is increasing in n, hence $f = \lim_n f_n(t)$ exits and is measurable also. (Observe $f(t) = \infty$ whenever $f_n(t) = \infty$ for some n). Finally, we just have to note that $T = \{t : f(t) > k\}$, which implies that T is Borel. In fact, $f(t)$ is exactly the cardinality of the support of $l(t, \cdot)$ in case the latter is finite. The first statement is thus established.

For the second statement, suppose if possible that $\ell(T) > 0$ where ℓ denotes the lebesgue measure on the line. Now note that the hypothesis of the lemma implies that M has finite moments of orders at least upto k. Therefore, for each t, $\int_0^t \int_{\mathbb{R}} |u|^j \, l(s, du) ds < \infty$, $0 \le j \le k - 2$. In particular, the set

$$A = \cup_{j=0}^{k-2} \left\{ t : \int |u|^j \, l(t, du) = +\infty \right\}$$

is a Borel set of zero lebesgue measure. Set $\tilde{T} = T \cap A^c$. Then clearly, $\ell(\tilde{T}) = \ell(T) > 0$. We now apply Theorem 5.14 to produce a contradiction. Choose for U the set $\tilde{T}$, and for V

the following subset of the separable Banach space $C_b(\mathbb{R})$ of real-valued bounded continuous functions defined on the real line, equipped with the usual supremeum norm, to be denoted $\|\cdot\|_{\sup}$. For each $n \geq 1$, let

$$V_n = \{f \in C_b(\mathbb{R}) : \|f\|_{\text{lip}} \leq n, \ \|f\|_{\sup} \leq n\}.$$

where $\|f\|_{\text{lip}} = |f(0)| + \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$. The V_n's are each relatively compact in $C_b(\mathbb{R})$ by the Arzela-Ascoli Theorem (see [4], page 267, Theorem 6.4), and we take V to be the union of the closures (in $C_b(\mathbb{R})$) of the V_n's. To define the set B, we introduce a definition and a notation : for $t \in \tilde{T}$, define the linear map $\Lambda_t : V \to \mathbb{R}^{k-1}$ as $\Lambda_t(f) = (\int_{\mathbb{R}} f(u)u^j l(t,du))_{0 \leq j \leq k-2}$. Also, notice that $V \subset L^\infty(l(t,\cdot))$ and denote the corresponding norm by $\|\cdot\|_t$. Define now $B = \{(t,f) \in \tilde{T} \times V : \Lambda_t f = 0, \ \frac{1}{2} \leq \|f\|_t \leq 1\}$. To show both that B is Borel and that its sections B_t are closed in V, we use the following

Lemma 5.2 *Both the maps $(t,f) \mapsto \Lambda_t f$ and $\|f\|_t$ as functions on $\tilde{T} \times V$ are measurable in the first argument t keeping the second f fixed, and continuous in the latter fixing the former. In particular, they are both jointly measurable on $\tilde{T} \times V$.*

The proof of Lemma 5.2 follows after that of the present one, which we continue assuming the former. Obviously, as a result, B is the intersection of two Borel sets in the product $\tilde{T} \times V$, therefore Borel itself. Next, for every fixed $t \in \tilde{T}$, $B_t = \{f : \Lambda_t f = 0\} \cap \{f : \frac{1}{2} \leq \|f\|_t \leq 1\}$, is the intersection of two closed sets (each being the inverse image of a closed set under a continuous map), hence is itself closed.

It remains to show that $\Pi_{\tilde{T}}(B) = \tilde{T}$, or equivalently, that B_t is nonempty for all $t \in \tilde{T}$. Fix a $t \in \tilde{T}$. Denote the support of $l(t,\cdot)$ by S_t. By definition of $\tilde{T}$, S_t contains at least k points. Consider now the vector space $V_t = \{f 1_{S_t} : f \in V\}$. It is not difficult to show, using $|S_t| \geq k$, that V_t has thus dimension at least k.

We now define, for $t \in \tilde{T}$, a linear map $\zeta_t : V_t \to \mathbb{R}^{k-1}$, as $\zeta_t(v) = \Lambda_t f$ where $v = f 1_{S_t}$, $f \in V$. This map is well-defined, since if for some $g \in V$, $v = g 1_{S_t}$ also, then $\Lambda_t g = \Lambda_t f$. Further, ζ_t has a nontrivial kernel, its range being of strictly smaller dimension than its domain. Choosing any nonzero element $v = f 1_{S_t}$ in the kernel and scaling it appropriately so that $\frac{1}{2} \leq \sup|v| \leq 1$, we get $f \in B_t$. Thus $B_t \neq \emptyset \ \forall \ t \in \tilde{T}$.

Applying Theorem 5.14 therefore, we get a 'measurable selection' $h : \tilde{T} \to V$ such that $h(t) \in B_t \ \forall \ t \in \tilde{T}$. Now, the Borel σ-field on V is also generated by the 'evaluation maps' $f \mapsto f(u)$, $u \in \mathbb{R}$. Thus the measurability of h means precisely that $\forall \ u \in \mathbb{R}$, the map $t \mapsto h(t,u)$ (defined as $h(t)$ evaluated at u), is measurable. But for every t, $h(t) \in V$ is known to be continuous, therefore $h : \tilde{T} \times \mathbb{R} \to \mathbb{R}$ is jointly measurable.

Define now, using this function h, the new derivative measure $\bar{l}$ and Kolmogorov measure $\bar{L}$ as

$$\begin{aligned} \bar{l}(t,du) &= l(t,du)1_{\tilde{T}^c}(t) + (1+h(t,u))l(t,du)1_{\tilde{T}}(t), \quad \text{and} \\ \bar{L}(A) &= \int_A \bar{l}(t,du)dt \qquad \text{for Borel } A \subseteq [0,\infty)\times\mathbb{R}. \end{aligned}$$

The fact that $\forall\, t \in \tilde{T}$, $|h(t,u)| \leq 1$ for $l(t,\cdot)$-a.e. u, ensures that $\bar{L}$ is a measure. This is the Kolmogorov measure of our candidate for a Lévy process Y. We retain for Y the same mean function as that of M. Then, for all $s \geq 0$ and $0 \leq j \leq k-2$,

$$\begin{aligned} \int u^j \bar{L}([0,s]\otimes du) &= \int_0^s dt \int u^j \bar{l}(t,du) \\ &= \int_{[0,s]\cap\tilde{T}^c} dt \int u^j l(t,du) + \int_{[0,s]\cap\tilde{T}} dt \int u^j (1+h(t,u)) l(t,du) \\ &= \int_{[0,s]\cap\tilde{T}^c} dt \int u^j l(t,du) + \int_{[0,s]\cap\tilde{T}} dt \int u^j l(t,du) \\ &\quad + \int_{[0,s]\cap\tilde{T}} dt \int u^j h(t,u) l(t,du) \\ &= \int_0^s dt \int u^j l(t,du) \qquad \text{(by construction of } h) \\ &= \int u^j L([0,s]\otimes du); \end{aligned}$$

that is, Y has the same cumulants and consequently, the same moments, as M upto order k. This means $\mathcal{P}_j(Y) = \mathcal{P}_j(M)$ for $1 \leq j \leq k$. However, $\bar{L} \neq L$, which can be seen as follows. First of all, since $\ell(\tilde{T}) > 0$, there is a $t_0 > 0$, such that $\ell(\tilde{T}\cap[0,t_0]) > 0$. Denoting $\tilde{T}_0 = \tilde{T}\cap[0,t_0]$, consider the Borel set $A \subseteq [0,t_0]\times\mathbb{R}$ defined as $A = \{(t,u) : t \in \tilde{T}_0, h(t,u) > 0\}$. Since $A \subseteq [0,t_0]\times\mathbb{R}$, clearly $L(A) < \infty$. On the other hand, since for each $t \in \tilde{T}_0$, $\|h(t,\cdot)\|_t \geq 1/2$ and $\int h(t,u)l(t,du) = 0$, we must have $l(t,\{u : h(t,u) > 0\}) > 0$. This, along with $\ell(\tilde{T}_0) > 0$, implies that

$$\bar{L}(A) = L(A) + \int_{\tilde{T}_0} dt \int (h(t,u)\check{0})\, l(t,du) > L(A).$$

Thus $Y \overset{d}{\neq} M$, contradicting the hypothesis. ■

Proof of Lemma 5.2 : First, fixing $t \in \tilde{T}$, take a sequence $f_n \to f$ in V. That $\Lambda_t f_n \to \Lambda_t f$ easily follows by DCT as f_n has to be uniformly bounded, and each u^j, $1 \leq j \leq k$ is integrable with respect to $l(t,du)$. Thus $f \mapsto \Lambda_t f$ is continuous on V. For measurability with respect to t, observe that $t \mapsto \int g(u)l(t,du)$ is measurable when g is an indicator, and hence when it is simple. Now approximating both the positive and negative parts of each of the functions u^j, $1 \leq j \leq k$, by simple functions and using MCT, the conclusion follows.

For the map $(t, f) \mapsto \|f\|_t$ also, similar arguments apply : if $f_n \to f$ in V, then $f_n \to f$ in $L^\infty(l(t,\cdot))$, therefore $\|f_n\|_t \to \|f\|_t$. Next, if $f \in L^\infty(l(t,\cdot))$, then $f \in L^p(l(t,\cdot))\ \forall\, p > 1$. Denoting the norm on the latter space as $\|\cdot\|_{p,t}$, it is not too difficult to show, progressively for simple g and for general g, that the map $t \mapsto \|g\|_{p,t}$ is measurable $\forall\ g \in L^p(l(t,\cdot))$. Finally, observe that $\|f\|_t = \lim_{p\to\infty} \|f\|_{p,t}$. This yields the required measurability.

Also, using the separability of V, each of the above maps can be proved to be Borel measurable on the product space $\tilde{T} \times V$. In fact, if $\{f_n\}$ is a dense set in V, for each $j \geq 1$, let $\{D_{n,j}, n \geq 1\}$ denote the partition of V formed by disjointifying the balls $B_{n,j} = B(f_n, \frac{1}{j})$ over n. Thus for any map $g : \tilde{T} \times V \to \mathbb{R}$ measurable in the first coordinate and continuous in the second, one can write

$$g(t, f) = \lim_{j\to\infty} \sum_{n\geq 1} g(t, f_n) 1_{D_{n,j}}(f),$$

showing thereby that g is product-measurable. ∎

5.4 The characterisation : the general case

Before stating the theorem for a general Lévy process M belonging to the class $\mathcal{C}$, that is, the class of Lévy processes whose Kolmogorov measures admit a derivative measure, let us see, for such M, what $\mathcal{P}_k(M)$ being non-empty means in terms of the derivative measure l. Recall that $\mathcal{P}_j(M) \neq \emptyset$, $1 \leq j \leq k$, if and only if the cumulant functions $c_1(t)$ and $c_j(t) = \int u^{j-2} L([0,t] \otimes du)$, $2 \leq j \leq k$, are polynomials in t. It can be shown that this is equivalent to $c_1(t)$ being a polynomial, and the functions $t \mapsto \int u^j l(t, du)$, $0 \leq j \leq k-2$, being polynomials in t almost everywhere. Here and in the sequel, when we say that a function is a polynomial in t almost everywhere, we mean the existence of a polynomial in t with which the stated function agrees for almost every $t \geq 0$.

Theorem 5.15 *Let M be a Lévy process of the class $\mathcal{C}$. Then*

(a) If there exist an integer $k \geq 1$ and a measurable function $(x_1, \ldots, x_k, p_1, \ldots, p_k)$: $[0,\infty) \to \mathbb{R}^k \times [0,\infty)^k$ such that

- *for each $0 \leq j \leq 2k$, $\sum_{i=1}^k p_i(t)\{x_i(t)\}^j$ is a polynomial in t almost everywhere, and*
- *$l(t, du) = \sum_{i=1}^k p_i(t)\, \delta_{x_i(t)}(du)$, $t \geq 0$, is a version of the derivative measure,*

then M is fpd in $\mathcal{C}$.

(b) Conversely, if M is fpd in $\mathcal{C}$, and is determined by $\{P_j(M), 1 \le j \le k\}$, then there exists a measurable function $(x_1, x_2, \ldots, x_k, p_1, p_2, \ldots, p_k) : [0,\infty) \to \mathbb{R}^k \times [0,\infty)^k$ such that a version of the derivative measure associated with M is given by

$$l(t, du) = \sum_{i=1}^{k} p_i(t)\, \delta_{x_i(t)}(du).$$

Proof : The proof of part (a) is quite similar to the corresponding part in the homogeneous case. If l is indeed of the given form, then $\{\mathcal{P}_j : 1 \le j \le 2k+2\}$ determines the law of M. First of all, the two conditions imply that $\forall\, 1 \le j \le 2k+2$, the j-th cumulant of M_t is a polynomial in t. This ensures that $\mathcal{P}_j(M) \ne \emptyset$ for $1 \le j \le 2k+2$. If now Y is another Lévy process of the class $\mathcal{C}$ such that $\mathcal{P}_j(Y) = \mathcal{P}_j(M)$ for $1 \le j \le 2k+2$, then for every $t \ge 0$, the moments, or equivalently, cumulants, of order upto $2k+2$ of Y_t agree with those of M_t. Denote the mean function, Kolmogorov measure and derivative measure for Y by $\bar{\nu}$, $\bar{L}$ and $\bar{l}$ respectively. Then $\bar{\nu}(t) = c_1(t) = \nu(t)$, and $\forall\ 0 \le j \le 2k$ and $t \ge 0$, $\int u^j \bar{L}([0,t] \otimes du) = \int u^j L([0,t] \otimes du)$. It follows that for almost all $t \ge 0$, $\int u^j \bar{l}(t, du) = \int u^j l(t, du)$, $0 \le j \le 2k$, and consequently that

$$\int \prod_{j=1}^{k} (u - x_j(t))^2\, \bar{l}(t, du) = \int \prod_{j=1}^{k} (u - x_j(t))^2\, l(t, du) = 0.$$

By the same argument as in the proof of the 'if' part of Theorem 5.13, this proves $\bar{l} = l$; and therefore $\bar{L} = L$, implying $Y \overset{d}{=} M$. Thus $\{\mathcal{P}_j : 1 \le j \le 2k+2\}$ characterises M.

Let us now prove part (b). The hypothesis now entails the existence of a positive integer k such that $\{\mathcal{P}_j(M), 1 \le j \le k\}$, determines the law of M. We may thus apply Lemma 5.1. Consider any version of l and the set T as defined in its proof. For $t \in T$, redefine $l(t, \cdot)$ as an arbitrary measure $\nu = \sum_{i=1}^{k} \delta_{y_i}$ supported on k points. The resulting transition function still remains a version of l. Now recall, for $t \ge 0$, the notation S_t for the support of $l(t, \cdot)$. By our construction, $|S_t| \le k\ \forall\, t \ge 0$. Let us partition $[0,\infty)$ by the cardinality of S_t, that is, let

$$T_j = \{t \ge 0 : |S_t| = j\}, \qquad 1 \le j \le k.$$

By the same argument used to prove T (of Lemma 5.1) is Borel, one can conclude that so is each T_j. Notice that $T = (\cup_{j=1}^{k} T_j)^c$.

For $t \in T_j$, order the elements of S_t as $x_1(t) < x_2(t) < \ldots < x_j(t)$, and denote the $l(t, \cdot)$-measures of these points by $p_1(t), p_2(t), \ldots, p_j(t)$ respectively. Also, for $j+1 \le i \le k$, let $x_i(t) = x_j(t) + 1$ and $p_i(t) = 0$. For $t \in T = (\cup_{j=1}^{k} T_j)^c$, set $x_i(t) \equiv y_i$ and $p_i(t) \equiv 1$, $1 \le i \le k$. With these notations, it follows that

$$l(t, \cdot) = \sum_{i=1}^{k} p_i(t) \delta_{x_i(t)}.$$

We now need to prove that the real-valued functions $x_i(t)$, and the non-negative functions $p_i(t)$, are all measurable. It suffices to show the measurability of these functions only on $\cup_{j=1}^k T_j$. First we deal with the x_i's. It is enough to show that each x_i is measurable on each T_j, and that too, only for $i \le j$.

Fix $j \ge 1$. By the definition of support of a measure, $S_t = \text{supp}\,(l(t,\cdot))$ is closed, and, $x_1(t) = \inf S_t$ for every $t \in T_j$. Therefore,

$$\{t \in T_j : x_1(t) \ge a\} = \cap_{\substack{q\in\mathbb{Q}\\ q<a}} \{t \in T_j : l(t,(-\infty, q)) = 0\}.$$

This means that the function $x_1 : T_j \to \mathbb{R}$ is measurable. Next, if $j \ge 2$, then

$$\{t \in T_j : x_2(t) \ge a\} = \{t \in T_j : x_1(t) \ge a\} \bigcup \Big\{\{t \in T_j : x_1(t) < a\} \cap \{t \in T_j : x_2(t) \ge a\}\Big\},$$

and the second set can be written as

$$\{t \in T_j : x_1(t) < a\} \cap \bigcap_{\substack{q\in\mathbb{Q}\\ q<a}} \{\{t \in T_j : x_1(t) \ge q\} \cup \{t \in T_j : l(t,(q,a)) = 0\}\}.$$

Thus x_2 is measurable on T_j. In similar fashion, it can be shown that the rest of the functions $x_i(t)$, $1 \le i \le j$, are each measurable on T_j. Now the task remains to show that the p_i's are also measurable. But observe that

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1(t) & x_2(t) & \cdots & x_k(t) \\ x_1^2(t) & x_2^2(t) & \cdots & x_k^2(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-1}(t) & x_2^{k-1}(t) & \cdots & x_k^{k-1}(t) \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ \vdots \\ p_k(t) \end{bmatrix} = \begin{bmatrix} \int l(t,du) \\ \int u\,l(t,du) \\ \int u^2 l(t,du) \\ \vdots \\ \int u^{k-1} l(t,du) \end{bmatrix}$$

The matrix M on the left is nonsingular, being a Vandermonde matrix, and since each of its elements is a measurable function of t, so is each element of its inverse. Each element of the vector v on the right is also measurable, by approximating the functions u^j by simple functions. It follows that the elements p_i of $M^{-1}v$ are also measurable.

This concludes the proof of the theorem. ∎

Remark 5.1 The form of the derivative of m we referred to earlier will also be the same as that of l, only, whereas the former puts no mass at $[0,\infty) \times \{0\}$, the latter will in general do so, unless the Gaussian part of the process is deterministic.

Given the form of the derivative measure, it is natural to speculate what can be said about the exact nature of the functions x_i and p_i, $1 \le i \le k$. Some possible forms in the case $k = 2$ are :

1. $x_1(t)$, $x_2(t)$, $p_1(t) \geq 0$ and $p_2(t) \geq 0$ are polynomials,

2. $x_1(t) = a(t) + \sqrt{b(t)}$, $x_2(t) = a(t) - \sqrt{b(t)}$, $p_1(t) = c(t) + d(t)\sqrt{b(t)}$ and $p_2(t) = c(t) - d(t)\sqrt{b(t)}$, where a, b, c and d are polynomials so chosen that $c \pm d\sqrt{b}$ are both non-negative on $[0, \infty)$.

3. $x_1(t) = a(t)b(t)$, $x_2(t) = c(t)b(t)$, $p_1(t) = d(t)/b(t)$, and $p_2(t) = e(t)/b(t)$. Here, a, $b > 0$, c, $d \geq 0$ and $e \geq 0$ are polynomials such that $b|(d+e)$.

In all these examples, the underlying process is itself p-harmonisable. That does not mean of course that any fpd process is also p-harmonisable, as the following example, also in the case $k = 2$, bears out :

> $x_1(t) = (1+t^2)^{-1}$, $x_2(t) = (1+t^4)^{-1}$, $p_1(t) = (1+t^2)^5$, and $p_2(t) = (1+t^4)^6$. Then the cumulants of the underlying process are polynomial functions only upto the order 7 and not beyond.

In the final chapter we investigate the fulfilment, and otherwise, of our necessary and sufficient condition by some standard processes.

Chapter 6

Examples and Counterexamples

6.1 Hochberg's measure

For homogeneous Lévy processes or partial sums of iid random variables, Neveu's method of obtaining time-space harmonic polynomials, as illustrated, for example in the proof of Theorem 2.1, essentially consists of expanding the exponential martingale $\eta(\alpha, t, M_t) = \exp(\alpha M_t - t\varphi(\alpha)))$ as a power series in α, where φ is the c.g.f. of M_1. The time-space harmonic polynomials then arise arise from this expansion. In this section, we ask what happens if we try to extend this construction to the case when the function φ is not quite a valid c.g.f. but possibly a more general function. We succeed in doing so for certain functions φ which we describe in the sequel. Naturally, since there is no genuine probability distribution for which φ is the c.g.f., the polynomials so obtained do not correspond to any bonafide 'stochastic process'. However, one can still interpret them as time-space harmonic polynomials if one extends the notion of processes in a certain sense, as described by Hochberg [8]. First, we reproduce here his work in part.

Since a process is in effect characterized by a probability measure on a suitable path space, we normally take the construction of such a measure as being equivalent to defining a process. In [8], Hochberg defined, on certain subsets of the path space $(\Omega, \mathcal{F}) = (\mathbb{R}^{[0,\infty)}, \mathcal{B}^{[0,\infty)})$, a class of set functions $\mathbf{P}$ with $\mathbf{P}(\Omega) = 1$, each of which, while resemblant of a measure in many ways, is not quite one. Nonetheless, we continue to call these 'measures' following his terminology.

To explain the construction, we recall some facts for a process M with stationary independent increments starting at 0. Suppose the m.g.f. of M_1 is defined in an interval around 0, say $\mathbf{E}\exp\alpha M_1 = \exp L(\alpha)$. Then the m.g.f. of M_t is $\mathbf{E}\exp\alpha M_t = \exp\{tL(\alpha)\}$. Moreover, L extends analytically in a neighbourhood of 0, continues on the imaginary axis, and admits the representation

$$L(i\alpha) = mi\alpha - \frac{v\alpha^2}{2} + \int (e^{i\alpha u} - 1 - \frac{i\alpha u}{1+u^2})d\lambda(u), \qquad \alpha \in \mathbb{R},$$

where $m \in \mathbb{R}$, $v > 0$ and λ is a finite measure. Also,

$$\mathbf{E}\exp(i\alpha M_t) = \exp[tL(i\alpha)], \qquad \alpha \in \mathbb{R},$$

is the characteristic function of M_t. Hence, perforce $\exp[tL(i\alpha)]$ is a positive definite function on $\mathbb{R}$. Again, the law of the process can be recovered from L via the *transition function* $p(\cdot,\cdot,\cdot)$ defined as

$$p(t,x,A) = p(t,0,A-x) = \mathbf{P}\{M_t \in A - x\}, \qquad t > 0,\ x \in \mathbb{R},\ A \in \mathcal{B}(\mathbb{R}).$$

Of course, $L(\alpha) = \alpha^2/2$ is a well-known special case which corresponds to Brownian Motion. However, it is not difficult to see that $L(\alpha) = -\alpha^4$, or more generally, $L(\alpha) = (-1)^{n+1}\alpha^{2n}$, $n > 1$, do not arise in the above construction, simply because the condition of positive definiteness of $\exp[tL(i\alpha)]$ fails. Starting precisely with one of these functions as L, Hochberg defined the transition densities as the inverse Fourier transforms of $\exp(tL(i\alpha))$:

$$p_t(x) = \frac{1}{2\pi}\int e^{-i\alpha x + tL(i\alpha)}d\alpha.$$

These are real-valued functions, although not non-negative, of the Schwarz class with total integral 1 and satisfying the Chapman-Kolmogorov equation. The signed measure $\mathbf{P}$ was now consistently defined on finite-dimensional cylinder sets by

$$\mathbf{P}(A) = \int_B p_{t_1}(x_1)p_{t_2-t_1}(x_2-x_1)\cdots p_{t_k-t_{k-1}}(x_k-x_{k-1})dx_1dx_2\cdots dx_k$$
$$\text{if} \quad A = \{(X_{t_1}, X_{t_2}, \cdots, X_{t_k}) \in B\}, k \geq 1, 0 \leq t_1 < t_2 < \cdots < t_k, B \in \mathcal{B}^k,$$

where X_t, $t \geq 0$, are the coordinate functions on $\mathbb{R}^{[0,\infty)}$. However, unlike the traditional set-up, here $\mathbf{P}$ is of unbounded variation on *every* nonempty finite-dimensional cylinder. $\mathbf{P}$ is clearly countably additive on every σ-field $\mathcal{G}_{(t_1,t_2,\cdots,t_k)} = \sigma < X_{t_1}, X_{t_2}, \cdots, X_{t_k} >$ for fixed $0 \leq t_1 < t_2 < \cdots < t_k$, but only finitely additive on the fields $\mathcal{G}_t = \cup_{k\geq 1}\cup_{0\leq t_1<t_2<\cdots<t_k\leq t}\mathcal{G}_{(t_1,t_2,\cdots,t_k)}$, $t > 0$, which means it does not extend to a measure on $\mathcal{B}^{[0,\infty)}$. Thus many usual notions, in particular that of "martingales", have to be redefined appropriately. This is what we proceed to do now. We emphasise that we consider only integrals, or "expectations", of functions of finitely many coordinates, and interpret these as just the integrals with respect to the restriction of $\mathbf{P}$ on the σ-field generated by those coordinates.

Thus, we call an $(\mathcal{F}_t)$-adapted process (M_t) a martingale if

a) $\forall\ t \geq 0$, M_t is $\mathcal{G}_t$-measurable and $\mathbf{P}$-integrable,

b) $\forall\ 0 \leq t \leq s$, $A \in \mathcal{G}_t$, $\int_A M_s d\mathbf{P} = \int_A M_t d\mathbf{P}$.

$\mathcal{G}_t$-measurability here is to be interpreted as measurability with respect to a constituent σ-field $\mathcal{G}_{(t_1,t_2,\cdots,t_k)}$ of $\mathcal{G}_t$. Although in case $\mathbf{P}$ is a probability, this definition restricts the class of martingales, it is easy to see that any $(\mathcal{G}_t)$-'adapted' martingale is of this type. For example, (X_t) is itself a martingale in this sense. $\mathcal{P}_k$, $\bar{\mathcal{P}}_k$ etc. are defined in the obvious way.

We now exploit Neveu's idea [15] to exhibit time-space harmonic polynomials for $\mathbf{P}$ in the above sense. Firstly, we show that the "exponential martingale" actually qualifies to be so designated in this case, i.e. whenever $0 \leq t_1 < t_2 < \cdots < t_k \leq s < t$, $B \in \mathcal{B}^k$,

$$\int_{\{(X_{t_1},X_{t_2},\cdots,X_{t_k})\in B\}} e^{\alpha X_t - tL(\alpha)}\, d\mathbf{P} = \int_{\{(X_{t_1},X_{t_2},\cdots,X_{t_k})\in B\}} e^{\alpha X_s - sL(\alpha)} d\mathbf{P},$$

or,

$$\begin{aligned} &\int_{\mathbb{R}}\int_B p_{t_1}(x_1)p_{t_2-t_1}(x_2-x_1)\cdots p_{t_k-t_{k-1}}(x_k-x_{k-1})p_{t-t_k}(y-x_k) \\ &\qquad e^{\alpha y - tL(\alpha)} dx_1 dx_2 \cdots dx_k dy \\ = &\int_{\mathbb{R}}\int_B p_{t_1}(x_1)p_{t_2-t_1}(x_2-x_1)\cdots p_{t_k-t_{k-1}}(x_k-x_{k-1})p_{s-t_k}(x-x_k) \\ &\qquad e^{\alpha x - sL(\alpha)} dx_1 dx_2 \cdots dx_k dx. \end{aligned}$$

This equality is seen as follows: L.H.S.

$$\begin{aligned} = &\int_{\mathbb{R}}\int_B p_{t_1}(x_1)p_{t_2-t_1}(x_2-x_1)\cdots p_{t_k-t_{k-1}}(x_k-x_{k-1}) \\ &\times \left(\int_{\mathbb{R}} p_{t-s}(y-x)p_{s-t_k}(x-x_k)dx\right) \cdot e^{\alpha(y-x)-(t-s)L(\alpha)} e^{\alpha x - sL(\alpha)} \\ &\quad dx_1 dx_2 \cdots dx_k dy \\ = &\int_{\mathbb{R}}\int_B p_{t_1}(x_1)p_{t_2-t_1}(x_2-x_1)\cdots p_{t_k-t_{k-1}}(x_k-x_{k-1})p_{s-t_k}(x-x_k) \\ &\qquad e^{\alpha x - sL(\alpha)} dx_1 dx_2 \cdots dx_k dx \int_{\mathbb{R}} p_{t-s}(y-x)e^{\alpha(y-x)-(t-s)L(\alpha)} dy \\ = &\ \text{R.H.S.} \qquad \text{since the last integral is 1.} \end{aligned}$$

A routine interchange of the order of integrals and k-fold partial differentiation now establishes our claim regarding the existence of harmonic polynomials for $\mathbf{P}$.

It is clear that the above construction goes through word for word if instead of one of the candidates above, we take for L a (nonnegative) linear combination of these. Further, since the polynomials arise out of an expansion like (1.5), they possess the properties *(i)* – *(iv)* and also determine $\mathbf{P}$ through L. We summarise the final outcome in the following

Theorem 6.1 *Let L be an even polynomial of the form*

$$L(\alpha) = \sum_{k=0}^{m} c_k\, \alpha^{4k+2} - \sum_{k=1}^{n} d_k\, \alpha^{4k},$$

where c_k, $d_k \geq 0$. Denote by $\mathbf{P}$ the signed measure defined as above. Then the coordinate 'process' under $\mathbf{P}$ has the p-harmonisability property. Polynomials $P_k \in \mathcal{P}_k$, $k \geq 1$, can be chosen satisfying (i) – (iv) and they determine $\mathbf{P}$.

6.2 Counterexamples

With regard to the results proved so far in this thesis, a fairly natural question that could arise is whether the various conditions imposed in these are indeed indispensable? We give in this section a few counterxamples to show that in the absence of some of these conditions, the respective results indeed fail.

The first result in this vein relates to Theorem 2.4. In it, a necessary condition was obtained in for each $\mathcal{P}_k$ to be non-empty for a general discrete-time process. It may be wondered if the same is not a sufficient condition as well. Our aim here is to establish the contrary: that is, we produce a process M such that $\forall\, k \geq 1$, $E(M_n^k|\mathcal{F}_{n-1})$ is a polynomial in M_{n-1} of degree at most k for each n, but $\mathcal{P}_k(M)$ is empty for some (actually, for every) $k \geq 2$. The definition is via Tulcea's Theorem; we only specify $M_0 \equiv 0$, and for $n \geq 1$, the conditional distribution of M_n given $\mathcal{F}_{n-1}$ to be $N(M_{n-1}, e^n(1 + M_{n-1}^2))$. Then trite calculations show that for every $k \geq 1$ and $n \geq 1$, $E(M_n^k|\mathcal{F}_{n-1}) = c_n^k(M_{n-1})$ where each c_n^k is a polynomial of degree at most k, but that $\mathcal{P}_2(M)$, for instance, is empty.

Our next example is that of a process $M = \{M_n\}$ with independent difference sequence $\{X_n\}$ for which $\overline{\mathcal{P}}_2 = \emptyset$, but $\mathcal{P}_2 \neq \emptyset$. We define M through the independent X_n's as follows:

$$X_1 = \begin{cases} 1 & \text{with probability } 2/3 \\ -2 & \text{"} \quad \text{"} \quad 1/3 \end{cases} \quad \text{and } X_n \equiv 0 \text{ for } n \geq 2.$$

Then clearly, $M_n \equiv M_1 \equiv X_1\ \forall\, n \geq 1$. It is now easy to check that $\overline{\mathcal{P}}_2(M) = \emptyset$, but whenever $p(\cdot)$ is a polynomial with $p(0) = 0$, $P(t, x) = p(t)(x^2 + x - 2) \in \mathcal{P}_k(M)$. Of course, it is the violation of the support condition **(S)** here that makes this possible, since Theorem 2.3 tells us that for a process with independent differences, **(S)** ensures $\bar{\mathcal{P}}_k = \mathcal{P}_k$ for every k.

Our next result is to the effect that even **(S)** is not sufficient to guarantee this for processes with non-independent increments. As such, it pertains to Theorem 2.6 where **(S)** and the non-emptiness of $\{\bar{\mathcal{P}}_j : 1 \leq j \leq k\}$ was shown to imply that $\bar{\mathcal{P}}_j(M) = \mathcal{P}_j(M)$

$\forall\, 1 \leq j \leq k$. We furnish a counterexample to show that even in the presence of **(S)**, it is possible to have $\bar{\mathcal{P}}_k$ empty even though $\mathcal{P}_k$ is not. We address the case $k = 3$ and prove, in fact, that whenever p_0, p_1, p_2 and p_3 are one-variable polynomials with $p_0(0) = 0$, and p_3 having no positive integer as a root, then there exists a martingale M, with $M_0 \equiv 0$, such that the two-variable polynomial $P(t,x) = \sum_{i=0}^{3} p_i(n)x^i$ is in $\mathcal{P}_3(M)$. Note merely that $\forall\, x \in \mathbb{R}$, there is a probability measure P^x on $\mathbb{R}$, supported on two points, such that $\int y^i P^x(dy) = 0, 1$ or x, according as i =1, 2 or 3. In fact P^x is the distribution putting mass $(1+a_x^2)^{-1}$ at a_x and $a_x^2(1+a_x^2)^{-1}$ at $-a_x^{-1}$, where a_x is the strictly positive real number $a_x = (x + \sqrt{x^2+4})/2$. Now, given polynomials p_i, $0 \leq i \leq 3$ as above, define, for $n \geq 1$, the function

$$\begin{aligned} f(n,x) &= \frac{1}{p_3(n)}\{(p_3(n-1) - p_3(n))x^3 + (p_2(n-1) - p_2(n))x^2 \\ &\quad +(p_1(n-1) - p_1(n) - 3p_3(n))x + p_0(n-1) - p_0(n) - p_2(n)\}. \end{aligned}$$

Consider now the process $M = \{M_n : n \geq 0\}$ defined as $M_n = \sum_{i=0}^{n} X_i$, where $X_0 \equiv 0$ and for $n \geq 1$, $P^{f(n,M_{n-1})}$ is the conditional distribution of X_n given $\mathcal{F}_{n-1}$, with $\mathcal{F}_n$ denoting as usual the σ-field $\sigma < M_i : i \leq n >$, $n \geq 0$. The existence of such a process is again guaranteed by Tulcea's Theorem. Clearly, M satisfies the support condition **(S)**. Finally, using the fact that for every $n \geq 1$, $\mathbf{E}(X_n^i|\mathcal{F}_{n-1})$ =0, 1 or $f(n, M_{n-1})$ according as i =1, 2 or 3, it is now not difficult to check that both M as well as $\{P(n, M_n) : n \geq 0\}$ are martingales, where $P(\cdot,\cdot)$ is the polynomial as defined above. It is also easy to see that $\{M_n^2 + M_n - n : n \geq 0\}$ is a martingale, showing that $\overline{\mathcal{P}}_2(M) \neq \emptyset$ and thereby that $\overline{\mathcal{P}}_2(M) = \mathcal{P}_2(M)$.

Incidentally, this method, applied to the case with p_i's constant, say $p_1 = c$, $p_2 = b$ and $p_3 = a \neq 0$, also proves the existence of a martingale $M = \{M_n\}$ such that $\{aM_n^3 + bM_n^2 + cM_n\}$ is also a martingale.

6.3 Examples of polynomially harmonisable processes

As might be expected, the most common examples of p-harmonisable processes arise from the class of homogeneous Lévy processes. We have already discussed the cases of Brownian motion and Poisson process in detail in the first chapter. In this section we take a look at some other examples.

In discrete time, for the symmetric simple random walk (M_n), the function giving the exponential martingale is

$$\begin{aligned} \eta(\alpha,n,x) &= \exp(\alpha x)(\operatorname{sech}\alpha)^n = \exp(\alpha x)\exp(\alpha n)(\exp 2\alpha + 1)^{-n} \\ &= 2^{-n}\sum_{k=0}^{\infty} E_k^{(n)}(x+n)\alpha^k/k! \end{aligned}$$

where $E_k^{(n)}$ is the k-th Euler polynomial of order n. This allows us to identify $P_k(n,x)$ as $E_k^{(n)}(x+n)/2^{-n}$.

While for standard Brownian motion, the Hermite polynomials H_k are time-space harmonic, for Brownian motion with drift μ and diffusion σ^2, we get the polynomials $H_k(\sigma t,(x-\mu t))$.

For the Poisson process with intensity λ, our polynomial sequence is $C_k(\lambda t, x)$.

If M is the non-homogeneous Compound Poisson process with intensity function $\lambda(\cdot)$ and jump-size distribution F, then simple calculations yield

$$\mathbf{E}\left(e^{i\alpha M_t}\right) = \exp\left\{i\alpha\nu(t) + \int_0^t \lambda(s)ds \int_{\mathbb{R}} (e^{i\alpha u} - 1 - i\alpha u)F(du)\right\}$$

where $\nu(t) = \int_0^t \lambda(s)ds \int_{\mathbb{R}} uF(du)$ is the mean function. This says that in this case, the Kolmogorov measure is given by $L(dt,du) = \lambda(t)u^2F(du)dt$, which, naturally, means that L admits a derivative measure given by

$$l(t,du) = \lambda(t)u^2F(du), \qquad t \geq 0. \tag{6.1}$$

It now follows that M is p-harmonisable if and only if $\lambda(\cdot)$ is a polynomial function and all moments of F are finite. It is possible, though cumbersome, to explicitly get a sequence $\{P_k \in \mathcal{P}_k : k \geq 1\}$ in this case from the above representation.

We now consider a few examples of p-harmonisable processes arising out of intertwinings of Markov semigroups and calculate the resulting polynomials. Some examples of random variables which lead to semigroups intertwined with that of the square M of the Bessel process of dimension $2a$, say, in the way that was described in page 41 are :

- $Z = Z_{a,b} \sim \beta_{a,b}$ has a beta distribution with parameters a and b,
- $Z = 2Z_{a+b}$, where $Z_c \sim \gamma_c$ has a gamma distribution with parameter c.

In the former case, we recover the semigroup of another Bessel process, this time of dimension $2b$, and in the latter, a semigroup of a certain process detailed in [21] with "increasing saw-teeth" paths.

The above procedure also allows one to obtain time-space harmonic polynomials for Azema's martingale (see e.g. [22]), $M_t = \text{sign}(B_t)\sqrt{t-g_t}$, $t \geq 0$, where B is a Brownian motion and g_t is the last hitting time of 0 by B before time t. Its semigroup is intertwined with that of Brownian motion by the multiplicative kernel for the random variable denoted m_1, arising as the terminal value of the process $\{m_u : 0 \leq u \leq 1\}$ called "Brownian meander". m_1 has what is known as a Rayleigh distribution, with density $x\exp(-x^2/2)$, $x \geq 0$.

Chapter 15 of [22], in the context of Chaotic Representation Property, also presents an alternative, or direct, proof of the p-harmonisability of the process M, as also of each member of the class of "Emery's martingales". In fact, Theorem 15.2 there makes it possible to directly apply our Theorem 3.4.

The specific polynomials that we get from the method just outlined are:

- **Azema's martingale** : The semigroup of M_t is intertwined with that of Brownian motion by the multiplicative kernel of the random variable m_1. We have,

  $$\mathbf{E}m_1^k = 2^{k/2}\Gamma\left(\frac{k}{2}+1\right),$$

  whence

  $$P_k(t,x) = \mathbf{E}H_k(t,m_1x) = \sum_{j=0}^{k} 2^{j/2}\Gamma\left(\frac{j}{2}+1\right)h_j^k(t)x^j$$

  are a sequence of time-space harmonic polynomials for M where $H_k(t,x) = \sum_{j=0}^{k} h_j^k(t)x^j$ are the Hermite polynomials.

- **$\mathrm{BES}^2(2b)$** : M_t has semigroup intertwined with that of $\mathrm{BES}^2(1)$ by the random variable $Z_{1/2,b} \sim \beta_{1/2,b}$.

  $$\mathbf{E}Z_{1/2,b}^k = \frac{(1/2)_k}{(1/2+b)_k},$$

  where for any real number y, $(y)_k$ stands for the ascending factorial $y(y+1)\cdots(y+k-1)$. So the time-space harmonic polynomials we get for Bessel process of dimension $2b$ are given by

  $$\sum_{j=0}^{k} \frac{(-2)^j(1/2)_j t^{k-j}}{(2j)!(b+1/2)_j(k-j)!}x^j$$

- The final example in this context is that of the process M whose semigroup is intertwined with that of the square of BES(1) by $Z = 2Z_{b+1/2} \sim \gamma_{b+1/2}$. Here, $\mathbf{E}Z^k = \frac{\Gamma(k+b+1/2)}{\Gamma(k)}$, so that

  $$P_k(t,x) = \sum_{j=0}^{k} \frac{(-2)^j\Gamma(k+b+1/2)t^{k-j}}{\Gamma(k)(2j)!(k-j)!}x^j$$

At the stage of writing this thesis, it has come to our notice that in [17], the time-space harmonic polynomials of certain homogeneous Lévy processes are described in greater detail. However, the emphasis there is on describing comprehensively those examples in which these polynomials are also orthogonal, or in other words, sequences $\{Q_k\}$ such that $Q_k(t,M_t)$ and $Q_j(t,M_t)$ are uncorrelated for $j \neq k$. In brief, this method consists in a change of variable in the exponential martingale. Thus one chooses a function $\alpha = g(\beta)$ and rewrites equation (1.5) as

$$\eta(\alpha, t, x) = e^{g(\beta)x - t\varphi(g\beta))} = \sum_{k=0}^{\infty} Q_k(t, x)\frac{\beta^k}{k!},$$

say. One then seeks a suitable g such that $\{Q_k\}$ are orthogonal. Note that the sequence $\{Q_k\}$ satisfies the pseudo-type-zero property in both arguments, but not necessarily the Appell property. A description of orthogonal polynomials satisfying the type zero property, due to Meixner, is now used to characterise all such examples. Here, we satisfy ourselves with the observation that these polynomials also arise as linear combinations of those that we have described.

6.4 Finitely polynomially determined Lévy processes

We now consider examples and counterexamples of finitely polynomially determined Lévy processes. We choose our examples mostly in the homogeneous setting, and a few from the class $\mathcal{C}$ of general Lévy processes admitting a derivative measure. More general examples are not too difficult to construct. First, recall the necessary and sufficient condition for a homogeneous Lévy process to be fpd; namely, the measure η, or equivalently, the measure l guiding the jump distribution of the process, be finitely supported.

The notation here is that of the equations (5.5) and (5.6) of section 5.2.

- **2-pd Processes.** The only 2-pd Lévy processes M are those which are deterministic, that is, such that M_t identically equals a polynomial $p(t)$, $t \geq 0$. Clearly, then, $\mu(t) = \nu(t) = p(t)$, $\sigma^2(t) \equiv 0$ and $m = L = 0$. The two time-space harmonic polynomials characterising such a process are $P_1(t, x) = x - p(t)$ and $P_2(t, x) = (x - p(t))^2$.

- **Brownian Motion.** For Brownian motion, it is well-known that the measure $\eta(du) \equiv 0$, and hence $l(du) = \delta_0$, supported at the single point 0. Thus the first four Hermite polynomials

$$H_1(t, x) = x, \quad H_2(t, x) = x^2 - t, \quad H_3(t, x) = x^3 - 3tx, \quad \text{and}$$
$$H_4(t, x) = x^4 - 6tx^2 + 4t^2,$$

determine it uniquely among all homogeneous Lévy processes. Actually four is the minimum number of time-space harmonic polynomials required to do so. An example of another homogeneous Lévy process for which the first three time-space harmonic polynomials agree with those of Brownian motion, is specified by the following :

$$\nu = 0, \quad \text{and} \quad l = \frac{1}{2}(\delta_{-1} + \delta_1).$$

The point is that this homogeneous Lévy process will have the same cumulants as those of Brownian motion upto order 3.

We may note here that Brownian motion with a constant drift is also four-polynomially determined. More generally, any gaussian Lévy process, with mean and variance functions being polynomials, is 4-pd.

- **Poisson Process.** In this case, $\nu = \lambda$ and $l = \lambda\delta_1$. Here too, the first four Poisson-Charlier polynomials

$$\begin{aligned}
&C_1(t,x) = x - \lambda t, \quad C_2(t,x) = (x-\lambda t)^2 - \lambda t, \\
&C_3(t,x) = (x-\lambda t)^3 - 3\lambda t(x-\lambda t) - \lambda t, \qquad \text{and} \\
&C_4(t,x) = (x-\lambda t)^4 - 6\lambda t(x-\lambda t)^2 - 4\lambda t(x-\lambda t) + 3(\lambda t)^2 - \lambda t,
\end{aligned}$$

are necessary and sufficient to characterise the Poisson process among all homogeneous Lévy processes. A different homogeneous Lévy process with the first three time-space harmonic polynomials matching those of the Poisson process is given by

$$\nu = \lambda \quad \text{and} \quad l = \frac{\lambda}{2}(\delta_0 + \delta_2).$$

For the nonhomogeneous compound Poisson process, it can easily be seen, using the representation (6.1) in page 84, that it is fpd if, and only if, the jump-size distribution F is finitely supported and the intensity function $\lambda(t)$ is a polynomial; and that in this case, it is actually $(2k+2)$-pd where $k \geq 1$ is the cardinality of the support of F.

- In all the examples at the end of Chapter 5, the respective processes are 6-pd.

- **Gamma Process.** This is a counterexample in contrast to all the previous ones. Here $\nu = \alpha/\lambda$ and

$$l(du) = \alpha u e^{-\lambda u} du, \qquad u \geq 0$$

which is clearly not finitely supported. Therefore the Gamma process is not fpd.

- Finally, we present an example of a homogeneous Lévy process which is not even *infinitely* polynomially determined, let alone being fpd. Take $\nu = 0$ and as $l(du)$, any distribution that is not determined by its moments, e.g. (see [5], page 224)

$$l(du) = \frac{1}{24} e^{-\sqrt[4]{u}}(1 - \alpha \sin \sqrt[4]{u}) du, \ u \geq 0, \quad \text{for some } 0 < \alpha < 1.$$

If $M^{(\alpha)}$ denotes the homogeneous Lévy process defined by this, then for all $k \geq 1$, $\mathcal{P}_k(M^{(\alpha)})$ would be the same for each α, $0 < \alpha < 1$. Of course, the $M^{(\alpha)}$ have different distributions for distinct values of α.

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