Component Importance In A Consecutive-k-out-of-n:F System

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Thesis submitted to the Indian Statistical Institute
for the degree of Doctor of Philosophy

ACKNOWLEDGEMENTS

I am thankful to Dr. K. G. Ramamurthy, Indian Statistical Institute, for introducing me to this subject and the interdisciplinary approach to problem solving. He has always been a source of motivation and encouragement in my endeavours. I take this opportunity to express my special debt of gratitude for his excellent guidance and inspiring supervision of my thesis.

I am grateful to Dr. S. R. Mohan, Indian Statistical Institute, for going through my thesis and offering helpful suggestions and comments.

I would like to thank all my colleagues in the SQC & OR unit and other friends for creating a stimulating environment and extending a fine spirit of cooperation.

Finally, I thank my parents and family for their continued support and encouragement.

Contents

A Synopsis of the Dissertation on

Component Importance In a Consecutive-k-out-of-n:F System

Reliability theory has acquired a special significance during the last few decades. It is mainly because of rapid advancement in technology which has given rise to complex and sophisticated systems. These systems suffer from design flaws and weaknesses and their failures not only result in monetary loss but also pose a serious threat to human life and national security. Hence product reliability and safety is of paramount importance to us. Since system effectiveness can be optimised during design and development phases, it therefore calls for a detailed reliability engineering program during initial stages. Often we find that due to lack of non-availabilty of reliability data, the design and reliability engineers are handicapped and no quantitative assessment is possible during system development phases. In such situations, it is of a great practical significance to know the relative importance of components of the system so that proper allocation of the resources can be made with a view to optimise system effectiveness.

Different measures have been proposed in the reliability theory to quantify the relative importance of components in the system. These measures can be classified as structural importance measures or reliability importance measures. Structural importance measures require only the knowledge of the structure function of a system whereas reliability importance measures require additional information about component reliabilities.

A similar problem is encountered in other fields like game theory. In game theory, simple games are often used for modelling voting situations. The problem of quantifying the power of a player in simple games was first considered by Shapley and Shubik [47] in 1954 and they defined the Shapley-Shubik power index which was rediscovered in reliability theory by Barlow and Proschan [4] in 1975 as a measure of structural importance of components. Banzhaf [2] while analysing legal and constitutional problems defined another power index of a player in 1965 which was rediscovered by Birnbaum [8] in 1969 as a measure of structural importance of components.

In reliability theory, a consecutive-k-out-of-n:F system has been studied since 1980. It consists of n linearly ordered and interconnected components. The system fails if and only if it has at least k consecutive failed components. This system finds applications in telecommunication, pipeline network, design of integrated circuits etc. Griffith and Govindarajulu [26] first considered the problem of calculating Birnbaum's measure of reliability importance of components in a consecutive-k-out-of-n:F system. Papastavridis [39] also studied this problem and incorrectly asserted that for the i.i.d. case, the most important components are located in the middle of the sequence of components. We give a counterexample to show that his result is not correct. It can be shown that in a consecutive-2-out-of-6:F system component 2 has more Birnbaum reliability importance than component 3. This provided the main background and motivation for studying the component importance in a consecutive-k-out-of-n:F system and consequently resulted in this dissertation. It is divided into five chapters.

Chapter 1 covers the preliminaries needed for understanding the work done. Section 2 includes coherent systems, dual structures and related results. Section 3 provides a brief introduction to the simple games and highlights its similarities with semi-coherent structures. Section 4 defines reliability function and presents related results. Shapley-Shubik power index (or Barlow-Proschan structural measure of component importance) and Banzhaf power index (or Birnbaum structural measure of component importance) of a player in case of simple games are defined in Section 5. In addition, we also describe Birnbaum reliability and Barlow-Proschan reliability importance measures and Butler 's cut importance ranking [15].

Chapters 2, 3, 4 and 5 mainly present the work of the author.

In Chapter 2 we examine the general problem of structural importance of components in a coherent system and present an unified and a new approach for calculating different measures. This is done by defining a structural matrix using the simple form of a coherent structure in Section 2. We also show how the structural matrix of a dual structure can be obtained from a given structural matrix. The two matrices are connected by a transformation matrix which is also defined here. Section 3 studies some properties of the transformation matrix including its eigenvalues and eigenvectors. Section 4 shows how the different structural measures of importance can be obtained using the structural matrix.

Chapter 3 starts with a review of a consecutive-k-out-of-n:F system and Section 2 provides a necessary and sufficient condition for a path set to be a minimal path set. The problem of obtaining recurrence relationship for the structure and reliability functions is considered in the next Section. We provide here a simple approach for determining them. In Section 4 we examine the incorrect result of Papastavridis [39], provide a counterexample and point out the mistakes in his proof. Section 5 presents a recursive procedure for obtaining the structural matrix of the dual structure of a consecutive-k-out-of-n:F system. Section 6 considers the computational efforts and space requirements needed for calculating the structural matrix of the dual of a consecutive-k-out-of-n:F system.

Chapter 4 deals with the study of component importance in a consecutive-2-out-of-n:F system. Section 2 examines the path sets and swings and it is shown that they follow Fibonacci sequences with different initial conditions. We prove a number of results on path sets and swings of components. In Section 3 we study Birnbaum reliability and Birnbaum structural importance and also give a general formula for calculating system reliability. The main result that follows is: Let $\beta_2(i,n)$ represent the Birnbaum structural importance of component i in a consecutive-2-out-of-n:F system and further assume that n_2 and n_1 are the largested even and odd integers smaller than or equal to $(n+1)/2$, respectively. Then

(i) $\beta_2(i,n) = \beta_2(n+1-i,n)$ for $i = 1, 2, \ldots, n$ and $n \geq 2$

(ii) $\beta_2(2,n) > \beta_2(4,n) > \cdots > \beta_2(n_2,n)$ for $n \geq 7$

(iii) $\beta_2(1,n) < \beta_2(3,n) < \cdots < \beta_2(n_1,n)$ for $n \geq 5$

(iv) $\beta_2(2t,n) > \beta_2(2t-1,n)$ for $t \geq 1, 2t \leq n_2$ and $n \geq 3$

(v) $\beta_2(2t,n) > \beta_2(2t+1,n)$ for $t \geq 1, 2t+1 \leq n_1$ and $n \geq 5$.

This implies that component 2 has the maximum Birnbaum structural importance and component 1 the minimum Birnbaum structural importance. It also provides a complete ranking of the components. Section 4 deals with the study of Barlow-Proschan structural importance measure and provides a procedure for calculating it. It is shown that Barlow-Proschan structural importance ranking is identical to the Birnbaum structural importance ranking. In Section 5 it is further shown that Butler cut importance ranking and Birnbaum reliability importance ranking (for i.i.d. components) are also same as Birnbaum structural importance ranking. Hence it follows that in a consecutive-2-out-of-n:F system component rankings provided by the Birnbaum reliability importance measure, Birnbaum structural importance measure, Barlow-Proschan structural importance measure and Butler cut importance ranking are all identical.

In Chapter 5 we study the importance of components in a consecutive-3-out-of-n:F system. Section 2 highlights the properties of path sets and their relationship with Tribonacci sequences (or Fibonacci sequences of order 3). Section 3 gives the properties of swings and considers the difference in swings between two consecutive components and provides conditions under which component i will have more swings than component $i+1$. Section 4 deals with Birnbaum structural importance and gives a general formula for computing the path sets and using this we develop a necessary and sufficient condition for a component to have more Birnbaum structural importance than other components. It is also shown that component 3 and component 1 have the maximum and the minimum Birnbaum structural importance, respectively. We also provide a heuristic procedure for ranking of components which matches with the Birnbaum structural ranking under certain assumptions. Section 5 is devoted to the study of Barlow-Proschan structural importance and gives a procedure for calculating it. It is shown that in this case also component 3 has the maximum and component 1 the minimum Barlow-Proschan structural importance. In Section 6 it is further proved that component 3 and component 1 have the maximum and the mimimum cut importance ranking, respectively and that the same result also holds for Birnbaum reliability importance measure in case of i.i.d. components. In the last section we mention some results of this chapter which can be easily extended for $k \geq 4$. In particular one can show that component k has the maximum and component 1 the minimum importance as per Birnbaum structural, Barlow-Proschan structural and Birnbaum reliability importance measures (for i.i.d. components) and cut importance ranking in a consecutive-k-out-of-n:F system.

Chapter 1

Preliminaries

1.1 Introduction

Because of rapid advancement in technology more and more complex and sophisticated systems or equipments are being researched, designed, manufactured and put to use in different fields like defence, space exploration and commercial and domestic applications. The pace is fast growing due to stiff competition, consumer awareness and stringent legal requirements. These man-made systems suffer from design flaws, mistakes made during manufacture and lack of perfection and often result in failures. Most failures have economic consequences in the form of downtime cost, repair or replacement costs, penalties and delays apart from other intangible losses. But in case of nuclear power plants, aircraft systems, medical equipments and safety devices the loss is not only monetary but also poses hazardous and dangerous conditions to human life. In view of this, product safety has become a major topic of public debate and issue. It is therefore absolutety essential and necessary to carry out detailed reliability analysis of the product in order to identify causes behind the failures and to eliminate or control their effect [6].

In fact the need for reliable system was first recognised during World War II because of problems arising out of unreliable equipments supplied to defence forces . In the late 1940s and early 1950s, reliability engineering appeared on the scene and came to focus when reliability was precisely defined and it became possible to measure and evaluate system reliability. Theory of statistics and probability provided the foundation for its development.

Reliability is essentially a design parameter and can be optimised during product development phases. At the beginning of system development, reliability and design engineers are required to translate overall system performance targets into individual component reliability requirements, make allocation of resources and to evaluate system reliability. Initially, reliability achieved is much lower than what is needed. Different approaches and techniques are used to identify design weaknesses and component failures that could have critical effect on system functioning. However, these investigations for system improvement heavily depend on the past experience of the personnel designing the system because generally no reliability data are available for quantitative assessment. In a nutshell, we observe that in the initial stages of system development a reliability engineer, is faced with the up hill task of allocating available resources to optimise measures of system effectiveness and design parameters in the absence of reliability data. Measures of system effectiveness may be related to its reliability, maintenance cost and safety of operations etc. For a highly complex system, formal optimisation may not be possible. In such cases, it would to be appropriate to concentrate the limited resources on a small subset of components that are considered most critical to system performance. It is based on the intuitive reasoning that not all components are equally important to system performance and some components play a bigger role than others.

Different measures have been proposed to quantify the relative importance of components with respect to the given system and to provide component ranking in the order of importance. These measures can be classified as structural importance measures and reliability importance measures. Structural measures require only the knowledge about the way different component states affect the state of the system, that is, we need to know only the structure function of the system. On the other hand, the reliability importance measures require the additional information about component reliabilities. Structural measures of importance are more suitable during system design and development phases. Birnbaum [8] in 1969 first suggested two measures of importance. One is purely structural and the other reliability based. Barlow and Proschan [4] in 1975 also proposed structural and reliability measures. Butler [14] and [15] has recommended component ranking procedure using cut sets when component reliabilities are high. Several other authors have suggested different measures [35], [36] and [4].

Although system reliability is primarily concerned with physical systems to improve the performance of mechnical, electrical and electronic systems, it is now serving as an input to the development of systems like, agricultural, biological, economic, social, political and computer software etc. In game theory, simple games are used for modelling voting situations. It is well known that conceptually semi-coherent structures and simple games are the same [16]. In fact, Shapley and Shubik [47] in 1954 first considered the problem of quantifying the power of a player and developed the Shapley-Shubik power index which was rediscovered by Barlow and Proschan in 1975 in reliability theory as a measure of structural importance. Similarily, Banzhaf [2] in 1965 defined another power index for simple games which was rediscovered by Birnbaum in 1969 as a measure of structural importance.

1.2 System Representation

A system is assumed to be composed of a number of components (or subsystems) that work together to achieve a specific objective or perform a specific function. For making reliability analysis of a system, we need to know how the performance of the components affects the performance of the system. We shall assume that a system can either perform or fail to accomplish a given task. Similarly, components also can either perform or fail to perform an assigned task. Thus we are concerned with a situation where the system as well as its components can be only in either of the two possible states–functioning or failed. We assume that the state of the system deterministically depends on the states of its components. To model this, we shall use Boolean functions which are referred to as structure functions in reliability theory.

Structure Function : Consider a system or structure of n interconnected components with serial numbers assigned to them from the set,

$\mathcal{N} = \{1, 2, \ldots, n\}$. To represent the state of components, we assign a binary indicator variable x_i to component i :

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is failed} \end{cases}$$

for $i = 1, 2, \ldots, n$. Similarly, we define the binary variable y to indicate the state of the system

$$y = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed.} \end{cases}$$

The assumption that the state of the system is completely determined by the state of the components implies the existence of a Boolean function $f : \mathcal{B}^n \rightarrow \mathcal{B}$ such that $y = f(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the vector of component states and $\mathcal{B} = \{0, 1\}$. The function f is called a structure function.

Each n-tuple, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with $x_i = 0$ or 1 for $i = 1, 2, \ldots, n$ corresponds to a vector of component states or a state vector and can assume any one of the 2^n values represented by the vertices of the unit n-cube, that is, $\mathbf{x} \in \mathcal{B}^n$. We shall use the following notations:

$$(1_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

$$(0_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

$$(\cdot_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$$

$$\mathbf{0} = (0, 0, \ldots, 0) \text{ and } \mathbf{1} = (1, 1, \ldots, 1).$$

Definition 1: Component $i \in N$ is called irrelevant to the system if and only if $f(1_i, \mathbf{x}) = f(0_i, \mathbf{x})$ for all $(\cdot_i, \mathbf{x})$ otherwise, it is called relevant. We note that an irrelevant component can never directly cause the system failure.

Since the knowledge of the structure function is equivalent to the knowledge of the structure of the system, we shall be using the term 'structure f on N' to mean the structure function f of the system consisting of n components from the set $N = \{1, 2, \ldots, n\}$.

1.2.1 Coherent System

We expect a system to function when all its components function and fail when all its components fail, that is, $f(\mathbf{1}) = 1$ and $f(\mathbf{0}) = 0$. Also it is expected that if the performance of a component is improved the system performance does not deteriorate. This leads us to the concept of monotone structure.

Monotone structure : Let f be a structure on N. We say f is monotone if $\mathbf{x}, \mathbf{y} \in B^n$ and $\mathbf{x} \geq \mathbf{y} \Longrightarrow f(\mathbf{x}) \geq f(\mathbf{y})$ where $\mathbf{x} \geq \mathbf{y}$ means co-ordinate-wise inequality with at least one strict inequality. That is, f is non-decreasing in each co-ordinate.

Semi-coherent structure : A structure f on N is called semi-coherent if

(i) f is monotone on B^n

(ii) $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$.

Coherent structure : A semi-coherent structure f on N is called a coherent structure if all its components are relevant. The number of components in the coherent structure is called the order of the system. Coherent structures were first introduced by Birnbaum, Esary and Saunders [9]. For detailed exposition see Barlow and Proschan[3], Kaufmann [32] and Ramamurthy [43].

Minimal Path Sets and Cut Sets

Corresponding to any state vector $\mathbf{x} \in B^n$, we define the sets

$$C_1(\mathbf{x}) = \{i : i \in N \text{ and } x_i = 1\} \text{ and}$$

$$C_0(\mathbf{x}) = \{i : i \in N \text{ and } x_i = 0\}$$

Definition 2: A vector $\mathbf{x} \in B^n$ such that $f(\mathbf{x}) = 1(0)$ is called a path vector (cut vector) and $C_1(\mathbf{x})$ ($C_0(\mathbf{x})$) is called the corresponding path set (cut set).

A minimal path vector is a path vector $\mathbf{x}$ such that $\mathbf{y} < \mathbf{x} \Rightarrow f(\mathbf{y}) = 0$. The corresponding minimal path set is $C_1(\mathbf{x})$. Physically, a minimal path set is a minimal set of components whose functioning is sufficient to ensure the functioning of the system.

A minimal cut vector is a cut vector $\mathbf{x}$ such that $\mathbf{y} > \mathbf{x} \Rightarrow f(\mathbf{y}) = 1$. The corresponding minimal cut set is $C_0(\mathbf{x})$. Physically, a minimal cut set is the minimal set of components whose failure causes the system failure. We denote by $\alpha(f)$ and $\gamma(f)$ the collections of minimal path and cut sets of f, respectively.

Dual Structure :

Definition 3: Let f be a structure on N. Its dual f^D is another structure on N defined by

$$f^D(\mathbf{x}) = 1 - f(\mathbf{1} - \mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{B}^n.$$

Example 1: A series structure functions if and only if each component functions. For this system the structure function is given by

$$f(\mathbf{x}) = \prod_{i=1}^{n} x_i \quad \text{for all } \mathbf{x} \in \mathcal{B}^n.$$

Example 2: A parallel structure functions if any only if at least one of its component functions. Its structure function is given by

$$f(\mathbf{x}) = 1 - \prod_{i=1}^{n}(1 - x_i) \quad \text{for all } \mathbf{x} \in \mathcal{B}^n.$$

Example 3: A k-out-of-n structure functions if and only if at least k of its components function. The structure function is given by

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{if } \sum_{i=1}^{n} x_i < k. \end{cases}$$

Example 4: The dual of a series (parallel) structure is a parallel (series) structure. The dual of a k-out-of-n structure is a (n-k+1)-out-of-n structure.

We now state, without proofs, some relevant results of coherent and semi-coherent structures which we shall be using. Proofs are given in [3], [32] and [43].

Theorem 1 *Let f be a semi-coherent structure on N. A subset S of N is a path (cut) set of f if and only if $S \cap T \neq \phi$ for every $T \in \gamma(f)$ $(T \in \alpha(f))$.*

Theorem 2 *If f is a coherent structure on N then*

$$N = \bigcup_{P \in \alpha(f)} P = \bigcup_{Q \in \gamma(f)} Q.$$

Theorem 3 *Let f be a semi-coherent structure on N. Minimal path (cut) sets of a coherent system completely determine its structure function f and vice versa.*

Theorem 4 *For every semi-coherent structure f on N we have*

$$f(\mathbf{x}) = 1 - \prod_{S \in \alpha(f)} (1 - \prod_{i \in S} x_i) = \prod_{S \in \gamma(f)} (1 - \prod_{i \in S} (1 - x_i))$$

for all $\mathbf{x} \in B^n$.

Simple Form of a Structure

Definition 4: A function $g : R^n \rightarrow R$ is called multilinear (linear in each variable) if it can be expressed in the form

$$g(y_1, y_2, \ldots, y_n) = \sum_{S \subseteq N} C_S \prod_{j \in S} y_j$$

where $N = \{1, 2, \ldots, n\}$ and C_S's are constants. When $S = \emptyset$ we take $\prod_{j \in S} y_j = 1$.

Theorem 5 *A structure f on N can be expressed as*

$$f(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j$$

where a_S's are some integer constants.

The expression $\sum_{S \subseteq N} a_S \prod_{j \in S} x_j$ is known as the simple form of the structure function on N.

Theorem 6 *The simple form of a structure f on N is unique.*

Theorem 7 *Any structure f of order n is a linear composition of two structures of order at most equal to $(n-1)$, that is,*

$$f(\mathbf{x}) = x_i f(1_i, \mathbf{x}) + (1 - x_i) f(0_i, \mathbf{x})$$

for all $\mathbf{x} \in B^n$ and $i = 1, 2, \ldots, n$.

This representation is called the pivotal decomposition of the structure function. By repeated application of this theorem, we obtain

$$f(\mathbf{x}) = \sum_{\mathbf{y} \in B^n} \prod_{j=1}^{n} x_j^{y_j} (1 - x_j)^{1 - y_j} f(\mathbf{y}).$$

Theorem 8 *If f is any structure function on N and f^D is its dual structure, then*

(i) f is monotone $\Longleftrightarrow$ f^D is monotone

(ii) f is semi-coherent $\Longleftrightarrow$ f^D is semi-coherent

(iii) f is coherent $\Longleftrightarrow$ f^D *is coherent.*

Theorem 9 *Let f be a structure function on N. If* $A \subseteq N$, *then A is a path (cut) set of f* $\Longleftrightarrow$ *A is cut (path) set of* f^D.

Theorem 10 *For any structure f on N we have*

$$\alpha(f^D) = \gamma(f) \text{ and } \gamma(f^D) = \alpha(f).$$

1.3 Simple Games

Game theory is concerned with analysis of conflicting situations of interest to various parties and designing optimum strategies. It is of a great interest to Political and Social scientists. In game theory, the concept of simple games was first introduced by John Von Neumann and Oskar Morgensten [50] in 1944. Basically, a simple game is a competitive situation in which two or more participants called players are required to make a decision in a conflicting situation and co-operation between the players is permitted. Any group of players called coalition can freely enter into a binding agreement. It is known which coalitions are capable of winning. For example, a group of individuals must collectively decide whether to accept or reject a bill or proposal. Each individual either votes in favour (yes) or against the proposal (no). It is known [16] that there exists a close analogy between reliability system and simple games. The functioning or failure of a system is equivalent to accepting or rejecting the bill. Similarly functioning or failure of a component is equivalent to voting yes or no by a player. Let $N = \{1, 2, \ldots, n\}$ be the set of players and let 2^N denote the power set of N. We use a characteristic function to represent a simple game.

Definition 5: A simple game on N is represented by a characteristic function $\Theta : 2^N \rightarrow \{0, 1\}$ such that

(i) $\Theta(\emptyset) = 0$

(ii) $\Theta(N) = 1$

(iii) $\Theta(T) \geq \Theta(S)$ whenever $N \supseteq T \supseteq S$.

A subset $S \subseteq N$ is called a coalition. A coalition S is said to be a winning (losing) if $\Theta(S) = 1(0)$. A coalition S is called blocking if $\Theta(N - S) = 0$. A winning (blocking) coalition S is called a minimal winning (blocking) coalition if $T \subset S$ implies $\Theta(T) = 0$ ($\Theta(N - T) = 1$).

Dual Simple Game

Given a simple game on N, we associate with it another simple game as in the case of reliability theory.

Definition 6: Let Θ be a simple game on N. The dual Θ^D of Θ is another simple game on N defined by

$$\Theta^D(S) = 1 - \Theta(N - S) \text{ for all coalitions S.}$$

A player i in a simple game Θ on N is called a dummy if $\Theta(S \cup \{i\}) = \Theta(S - \{i\})$ for all coalitions S.

Simple games are essentially semi-coherent structures stated in a different format. The correspondence between the terminology of reliability and game theory [44] is as follows:

	Reliability Theory	Game Theory
(i)	Component	Player or Voter
(ii)	Structure function	Characteristic function
(iii)	Path set	Winning coalition
(iv)	Cut set	Blocking coalition
(v)	Irrelevant component	Dummy player

1.4 Reliability Function

We consider the stochastic nature of the components of the system and describe how system reliability can be evaluated using component reliabilities. Reliability of a system is defined as the probability that the system will perform its intended function for at least a specified time period under specified environmental conditions. we assume that the system as well as its components cannot be repaired.

At any given instant of time t, we consider the state of the component $i \in N$ and define the binary random variable $X_i(t)$ for $i = 1, 2, \ldots, n$ by

$$X_i(t) = \begin{cases} 1 & \text{if component } i \text{ functions at time t} \\ 0 & \text{if component } i \text{ is failed at time t.} \end{cases}$$

Let $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))$, be the vector of component states at time t. If we assume that the component i has the life distribution $G_i(t)$, then the reliability of this component at time t is the probability that the component is functioning at time t and is given by

$$P[X_i(t) = 1] = E[X_i(t)] = 1 - G_i(t) = \overline{G}_i(t) = p_i$$

where $E[X_i(t)]$ represents the expected value of the random variable $X_i(t)$, $P[A]$ denotes the probability of an event A and p_i's are called the component reliabilities. When the design of a system is known, then the state vector $\mathbf{X}(t)$ determines the state of the system at time t. Let $f(\mathbf{X}(t))$ be the random variable representing the state of the system and defined as

$$f(\mathbf{X}(t)) = \begin{cases} 1 & \text{if the system functions at time } t \\ 0 & \text{if the system is failed at time } t. \end{cases}$$

We shall assume that the components are independent, that is, $X_i(t)$, for $i = 1, 2, \ldots, n$ are independent binary random variables, then the system reliability, $h(\mathbf{p})$ is a function of component reliabilities, and it is given at time t by

$$P[f(\mathbf{X}(t)) = 1] = E[f(\mathbf{X}(t))] = h(\overline{\mathbf{G}}(t)) = h(\mathbf{p})$$

where $\overline{\mathbf{G}}(t) = (\overline{G}_1(t), \overline{G}_2(t), \ldots, \overline{G}_n(t))$, $p_i = \overline{G}_i(t)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is the vector of component reliabilities. To simplify the notation, hereafter, we shall avoid the use of direct reference to time point t and hence suppress t and write only X_i to mean $X_i(t)$ and so on. We shall throughout assume that $X_1, X_2, \ldots, X_n$ are independent binary random variables unless stated otherwise. If $p_1 = p_2 = \cdots = p_n = p$ we shall denote the system reliability function by $h(p)$.

Definition 7: The reliability function of a structure f on N is the function $h : [0,1]^n \rightarrow [0,1]$ defined as

$$h(\mathbf{p}) = P[f(\mathbf{X}) = 1] = E[f(\mathbf{X})].$$

Example 5: For a series structure

$$h(\mathbf{p}) = E(X_1 \cdot X_2 \cdots X_n) = p_1 \cdot p_2 \cdots p_n.$$

Example 6: For a parallel structure

$$h(\mathbf{p}) = E\left[1 - \prod_{i=1}^{n}(1 - X_i)\right] = 1 - (1-p_1)(1-p_2)\cdots(1-p_n).$$

Theorem 11 *The reliability function, $h(\mathbf{p})$ of a structure f on N can also be defined as the multilinear extension of f over the entire unit n-cube [37].*

In game theory, the reliability function is referred as Owen's multilinear extension [37] of the characteristic function of a game.

Theorem 12 *The following identity holds for the reliability function, $h(\mathbf{p})$ of any structure f on N for $i = 1, 2, \ldots, n$ and $\mathbf{p} \in [0,1]^n$*

$$h(\mathbf{p}) = p_i h(1_i, \mathbf{p}) + (1 - p_i) h(0_i, \mathbf{p})$$

where $h(1_i, \mathbf{p}) = E[f(1_i, \mathbf{X})] = E[f(X_1, X_2, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)]$ and $h(0_i, \mathbf{p}) = E[f(0_i, \mathbf{X})] = E[f(X_1, X_2, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n)]$. This representation is called the pivotal decomposition of the reliability function.

Theorem 13 *Let $h(\mathbf{p})$ be the reliability function of a structure f on N. We then have for $i \in N$ and $\mathbf{p} \in (0,1)^n$.*

$$\begin{aligned} \frac{\partial h(\mathbf{p})}{\partial p_i} &= E\left[f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})\right] \\ &= h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}). \end{aligned}$$

Theorem 14 *Reliability function $h(\mathbf{p})$ of a monotone structure is monotone.*

1.5 Power and Importance

In simple games a problem of considerable interest is that of obtaining numerical indices to represent the voting power of an individual player or the amount of influence a player has on the outcome of the game. Intuitively, the power of a player is his ability to change the outcome of the game by changing his voting pattern. In reliability theory also a similar problem is encountered while identifying components that are considered important from the viewpoint of system performance. Quantification of relative importance of components on the basis of their contribution to system performance is extremely useful for design and reliability engineers. Measures which are based only on the structural form of the system are called structural importance measures or time independent measures and if additional information on the component reliabilities is available we can compute what are known as reliability importance measures. We now describe some of the frequently used measures in simple games and reliability theory and point out the similarities between them.

1.5.1 Power of a Player

The two most traditional and widely used quantitative measures of power in game theory and political sciences are Shapley-Shubik power index and Banzhaf power index.

Shapley-Shubik Power Index

In the context of simple games, Shapley and Shubik [47] for the first time in 1954 developed a numerical index which can be interpreted directly in terms of the a priori ability of the player to affect the outcome of a game.

Definition 8: Let Θ be a simple game on N and consider the ordering of N as representing the order in which players of N will join a coalition in support of some issue. The player whose joining turns the developing coalition from a losing coalition into a winning coalition is called pivotal for that ordering. Shapley-Shubik power index $\Psi(i)$, for player i, is the probability that player i is pivotal under the assumption that all the possible $n!$ ordering are equiprobable.

Absolute Banzhaf Power Index

John F.Banzhaf [2] suggested a different power index in 1965 in connection with studies of legal and constitutional issues arising out of U.S. Supreme Court verdict of *One Person One Vote* in 1960s. His index gained wide acceptance and helped in settling many legal problems.

Definition 9: Let Θ be a simple game on N. A pair of sets of the form $(T, T-\{i\})$ such that $\Theta(T)=1$ and $\Theta(T-\{i\})=0$ is called a swing for player i. Let $S(i)$ denote the number of swings for player $i \in N$, then absolute Banzhaf power index $\beta(i)$, for player $i \in N$ is defined by [45] and [38]

$$\beta(i) = \frac{S(i)}{2^{n-1}}.$$

It can be interpreted as the probability that player i will be a swinger, that is, his vote will make a difference between winning and losing under the assumption that all coalitions are equally likely.

Remarks :

(i) Obviously for a dummy player $\beta(i) = \Psi(i) = 0$

(ii) $\sum_{i=1}^{n} \Psi(i) = 1$

Shapley-Shubik power index, $\Psi(i)$ can also be expressed in terms number of swings for player i, if the size of the swing is taken into account as can be seen from the next theorem and for its proof see [43] and [4].

Theorem 15 *Let Θ be a simple game on N. Then for $i \in N$ the Shapley-Shubik power index $\Psi(i)$ is given by*

$$\begin{aligned}\Psi(i) &= \sum_{t=1}^{n} \frac{(t-1)!(n-t)!}{n!} S(i,t) \\ &= \frac{1}{n}\sum_{t=1}^{n} \frac{S(i,t)}{\binom{n-1}{t-1}}\end{aligned}$$

where $S(i,t)$ denotes the number of swings of size t for player i.

The above theorem shows that Shapley-Shubik power index. $\Psi(i)$ is a weighted linear function of the number of swings of different sizes for player i and the weights depend on the size of the swing whereas absolute Banzhaf power index attaches a common weight of $\frac{1}{2^{n-1}}$ to all swings of different sizes for player i.

1.5.2 Relative Importance of Components

Different measures have been proposed for quantification of component importance. We describe here some relevant measures. Theses measures can be classified as structural importance and reliability importance measures.

Structural Importance Measures

Structural measures do not require any information about component reliabilities and can be calculated using only the structure function of the system. Shapley-Shubik and absolute Banzhaf power indices were rediscovered in reliability as structural measures of component importance.

Let f be a semi-coherent structure on N. We say $(1_i, \mathbf{x})$ is a critical path vector for component i if $f(1_i, \mathbf{x}) = 1$ and $f(0_i, \mathbf{x}) = 0$ or equivalently $f(1_i, \mathbf{x}) - f(0_i, \mathbf{x}) = 1$.

Birnbaum Structural Measure

Birnbaum [8] suggested the following structural importance measure for component $i \in N$

$$\beta(i) = \sum [f(1_i, \mathbf{x}) - f(0_i, \mathbf{x})] \frac{1}{2^{n-1}}$$

where summation is over all $(x_1, x_2 \ldots, x_{i-1}, x_{i+1} \ldots, x_n)$ belonging to B^{n-1}.

It can be easily seen that it is the same as the absolute Banzhaf power index for player i in a simple game.

Barlow-Proschan structural Measure

Assume that initially all components are functioning. Consider an ordering (i.e., a permutation) of the elements of N as representing the order in which components fail. The component whose failure causes the failure of the system (that is, a transition from the functioning state to the failed state) is called the pivotal component of that ordering. The Barlow-Proschan measure of structural importance is the probability that component i is pivotal under the assumption that all $n!$ orderings are equiprobable. We note that Barlow-Proschan structural measure is the same as the Shapley-Shubik power index.

Remark : In fact Barlow and Proschan [4] have defined structural importance of component i, as the probability that it causes system failure under the assumption that component life distributions are i.i.d. random variables.

Reliability Importance Measures

Relative importance of components can also be determined by using the structure function and the component reliabilities. Birnbaum [8] and Barlow and Proschan [4] have each proposed reliability importance measures. These measures make use of probabilistic information about the components of the system.

Let f be a semi-coherent structure on N and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be the vector of component reliabilities. Suppose $h(\mathbf{p})$ represents the reliability function of a structure f on N.

Birnbaum Reliability Importance Measure

Let $\beta(i,h)$ denote the Birnbaum reliability importance of component i. It is defined by

$$\begin{aligned}\beta(i,h) &= E\left[f(1_i,\mathbf{X}) - f(0_i,\mathbf{X})\right] \\ &= h(1_i,\mathbf{p}) - h(0_i,\mathbf{p}).\end{aligned}$$

Because of Theorem 13, we note that

$$\beta(i,h) = \frac{\partial h(\mathbf{p})}{\partial p_i}$$

Obviously, for a coherent system $0 < \beta(i,h) < 1$ for $i = 1,2,\ldots,n$ and $n > 1$.

Example 7: Consider the series system. We then have

$$h(\mathbf{p}) = p_1 \cdot p_2 \cdots p_n \text{ and } \beta(i,h) = \frac{\partial h(\mathbf{p})}{\partial p_i} = \frac{h(\mathbf{p})}{p_i}.$$

Hence the component having the minimum reliability is the most important to the system and the most reliable component has the smallest reliability importance.

Example 8: For the parallel system, we have

$$\begin{aligned}h(\mathbf{p}) &= 1 - \prod_{i=1}^{n}(1-p_i) \\ \text{and } \beta(i,h) &= \prod_{j\neq i}^{n}(1-p_j)\end{aligned}$$

$$= \frac{\prod_{j=1}^{n}(1-p_j)}{(1-p_i)} .$$

Hence the component having the maximum reliability is the most important to system functioning.

Both the structural measures described earlier are related to Birnbaum reliability measures and can be obtained as follows [4] and [38]:

(i) $\Psi(i) = \int_0^1 [h(1_i,p) - h(0_i,p)]\, dp$

(ii) $\beta(i) = \frac{\partial h(\mathbf{p})}{\partial p_i}\Big|_{p_1=\cdots=p_n=\frac{1}{2}}$.

Barlow-Proschan Reliability Importance Measure

This requires the life time distribution of each component. Let G_i be the life time distribution of component i for $i = 1, 2 \ldots, n$, then Barlow-Proschan reliability measure $\Psi(i,h)$ of component i is

$$\begin{aligned} \Psi(i,h) &= P[\text{ Component } i \text{ causes system failure }] \\ &= \int_0^\infty \left[h(1_i, \overline{\mathbf{G}}(t)) - h(0_i, \overline{\mathbf{G}}(t))\right] dG_i(t) \\ \text{where } \overline{\mathbf{G}}(t) &= (\overline{G}_1(t), \overline{G}_2(t), \ldots, \overline{G}_n(t)). \end{aligned}$$

Butler Cut Importance Ranking

Both the structural measures described so far are in some sense unbiased and can be derived from Birnbaum reliability importance measure if we assume that all components have the same reliability p and in one case we take $p = 0.5$ and the other average over [0,1]. In practice, component reliability may not be close to 0.5 or even 0.5 on an average. Thus, these two measures may present a misleading picture about relative importance of components if the component reliability is high or low. Keeping this in view, Butler [14] & [15] developed a ranking that is biased in favour of high reliability. It is not a measure, but provides a complete ranking of components. It is based on cut sets of the system. It is also related to Birnbaum reliability ranking for high values of p.

Definition 10: Let f be a coherent structure on N and $\gamma(f)$ be the collection of all its minimal cut sets. Suppose the cardinality of the set $\gamma(f)$ is m. For each $u \in N$, let $t_{ij}^{(u)}$ denote the number of collections of i distinct minimal cuts such that the union of each collection contains exactly j components and includes component u for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We define the vector $\mathbf{b}^{(u)} = (b_1^{(u)}, b_2^{(u)}, \ldots, b_n^{(u)})$ where

$$b_j^{(u)} = \sum_{i=1}^{m} (-1)^{i-1} t_{ij}^{(u)}.$$

We say component u is more cut-important than component k denoted by $u >_c k$ if and only if $\mathbf{b}^{(u)} \succ \mathbf{b}^{(k)}$ where $\succ$ is the lexicographically greater than sign and components u and k are said to be equally cut-important, denoted by $u =_c k$ if and only if $\mathbf{b}^{(u)} = \mathbf{b}^{(k)}$. Thus relative cut importance ranking of components in the order of importance is given by the lexicographic ordering of $\mathbf{b}^{(\cdot)}$.

Example 9: Consider a coherent structure f on $N = \{1, 2, 3, 4, 5\}$ with

$$\gamma(f) = \{\{3\}, \{1, 2\}, \{4, 5\}\}.$$

(i) For $u = 1$ or 2

$$t^{(u)}_{12} = 1,\ t^{(u)}_{23} = 1,\ t^{(u)}_{24} = 1,\ t^{(u)}_{35} = 1 \text{ and } \mathbf{b}^{(u)} = (0,1,-1,-1,1).$$

(ii) For $u = 3$

$$t^{(u)}_{11} = 1,\ t^{(u)}_{23} = 2,\ t^{(u)}_{35} = 1 \text{ and } \mathbf{b}^{(u)} = (1,0,-2,0,1).$$

(iii) For $u = 4$ or 5

$$t^{(u)}_{12} = 1,\ t^{(u)}_{23} = 1,\ t^{(u)}_{24} = 1,\ t^{(u)}_{35} = 1 \text{ and } \mathbf{b}^{(u)} = (0,1,-1,-1,1).$$

This implies that $3 >_c 1 =_c 2 =_c 4 =_c 5$.

We give the relationship between cut-importance ranking and Birnbaum reliability importance ranking [15].

Theorem 16 *Let f be a coherent structure on N. Assuming that all components have the same reliability p, then for each $i \in N$, we have*

$$\beta(i, h(p)) = \sum_{j=1}^{n} b_j^{(i)}(1-p)^{j-1}$$

where

$$\beta(i, h(p)) = h(1_i, p) - h(0_i, p) = E[f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})].$$

The above theorem implies that Birnbaum reliability importance can be written as a polynomial in $(1-p)$ and for high values of p the lowest order terms in the polynomial dominate the rest. Since $i >_c k \Longleftrightarrow b^{(i)} - b^{(k)} \succ 0 \Longleftrightarrow \beta(i, h(p)) > \beta(k, h(p))$ for all p sufficient close to one, it follows that

Birnbaum reliability importance ranking is identical to Butler 's cut importance ranking when p is sufficient close to one. In most cases, components ranking can be known by determining the first non-zero element in $\mathbf{b}^{(\cdot)}$ and other elements in $\mathbf{b}^{(\cdot)}$ are computed only if necessary.

1.5.3 Other Measures of Importance

In reliabiliy theory some more measures of component importance are available, for details see [4], [35], [36] and [1].

In this dissertation, we shall be mainly concerned with Birnbaum structural importance measure, Barlow-Proshan measure of structural importance, Butler's cut importance ranking and Birnbum reliability importance measure for i.i.d. components.

Chapter 2

Structural Importance

2.1 Introduction

When the component reliabilities are not known structural measures play a crucial role in identification of weak points in the system design. These measures are useful during system development stages where lack of adequate reliability data is a common feature. In such situations, component ranking is done using the structure function only and the corresponding measures are called measures of structural importance. Different structural measures of importance have been suggested in the literature using pivotal components, critical path vectors and swings etc. We present an unified and a new approach for calculating the different structural measures using the structural matrix. In Section 2 we introduce the concept of a structural matrix of a structure function and establish the relationship between the structural matrix of a structure and its dual using a transformation matrix. Section 3 gives the properties of the transformation matrix. Section 4 explains how the structural importance of components can be obtained from the structural matrix.

2.2 Structural Matrix

Recall that a measure of structural importance of a component in a system is the extent to which the functioning or non-functioning of the component affects the functioning or non-functioning of the system. Various structural measures have been proposed in the literature. To present an unified treatment of theses measures, we consider the probabilistic approach and define the relative importance of component i in a semi-coherent structure f on N as

$$\Pi_i(f) = E[f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})].$$

Though different structural measures can be directly studied by considering $f(1_i, \mathbf{X})$, $f(0_i, \mathbf{X})$ and the probabilistic framework assumed, for elegant presentation and easy analysis, we introduce the concept of a structural matrix.

In this section, we first define the structural matrix of a structure using its simple form and show how the structural matrix of the dual structure can be obtained by making a linear transformation on the structural matrix of a structure.

Let f be a structure on $N = \{1, 2, \ldots, n\}$ and suppose its simple form

$$f(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j \quad \text{for all } \mathbf{x} \in \{0,1\}^n.$$

We define the collections

$$A_{ij} = \{S | S \subseteq N, i \in S \text{ and } |S| = j\}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Obviously, A_{ij} denotes the collection of subsets of N which contain the component i and are of cardinality j.

Definition 1:. We denote the structural matrix of f by $M(f)$ and define it as a square matrix of order n with elements given by

$$M(f) = ((m(f)_{ij})) = \sum_{S \in A_{ij}} a_S$$

Example 1 : Consider the system of components described in Figure 2.1.

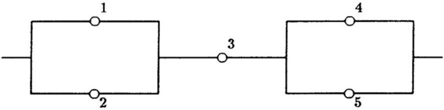

Figure 2.1:

Its simple form is

$$f(x) = x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_4 + x_2x_3x_5 - x_1x_2x_3x_5$$

$$-x_1x_2x_3x_4 - x_1x_3x_4x_5 - x_2x_3x_4x_5 + x_1x_2x_3x_4x_5.$$

Consider the case when $i = 1$ and $j = 3$. It follows from the simple form that $a_S = 1$ for $S = \{1, 3, 4\}$ or $\{1, 3, 5\}$ and $a_S = 0$ for other $S \subseteq N$. Hence we have $m(f)_{13} = 2$. Similarly for $i = 2$ and $j = 4$, we have $a_S = -1$ for $\{1, 2, 3, 5\}$ or $\{1, 2, 3, 4\}$ or $\{2, 3, 4, 5\}$ and $a_S = 0$ for remaining $S \subseteq N$. This gives us $m(f)_{24} = -3$. In fact the complete structural matrix $M(f)$ is given by

$$M(f) = \begin{bmatrix} 0 & 0 & 2 & -3 & 1 \\ 0 & 0 & 2 & -3 & 1 \\ 0 & 0 & 4 & -4 & 1 \\ 0 & 0 & 2 & -3 & 1 \\ 0 & 0 & 2 & -3 & 1 \end{bmatrix}$$

Simple form of the dual structure

To get the structural matrix of the dual of a structure, we first show how to obtain its simple form from the simple form of a structure f on N.

Theorem 1 *Let f be a semi-coherent structure on $N = \{1, 2, \ldots, n\}$ with its simple form as*

$$f(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j$$

and let the simple form of the dual of structure f be

$$f^D(\mathbf{x}) = \sum_{S \subseteq N} b_S \prod_{j \in S} x_j.$$

We then have

$$b_S = \begin{cases} 0 & \text{if} \quad S = \emptyset \\ (-1)^{|S|+1} \sum_{T \supseteq S} a_T & \text{if} \quad S \neq \emptyset \end{cases}$$

Proof : We know that

$$f^D(\mathbf{x}) = 1 - f(\mathbf{1} - \mathbf{x}) \text{ for each } \mathbf{x} \in \{0,1\}^n$$

where $\mathbf{1} = (1, 1, \ldots, 1)$. It follows that

$$\sum_{S \subseteq N} b_S \prod_{j \in S} x_j = 1 - \sum_{T \subseteq N} a_T \prod_{j \in T} (1 - x_j)$$

$$\begin{aligned} f^D(\mathbf{x}) &= 1 - \sum_{T \subseteq N} a_T \sum_{S \subseteq T} (-1)^{|S|} \prod_{j \in S} x_j \\ &= 1 - \sum_{S \subseteq N} \left[(-1)^{|S|} \prod_{j \in S} x_j \right] \sum_{T \supseteq S} a_T \\ &= 1 + \sum_{S \subseteq N} (-1)^{|S|+1} \sum_{T \supseteq S} a_T \prod_{j \in S} x_j. \end{aligned}$$

Hence the required result holds.

Example 2 :

We can easily verify that the simple form of the dual of the structure considered in Example 1 is

$$\begin{aligned} f^D(\mathbf{x}) = \; & x_3 + x_1 x_2 + x_4 x_5 - x_1 x_2 x_3 \\ & - x_3 x_4 x_5 - x_1 x_2 x_4 x_5 + x_1 x_2 x_3 x_4 x_5. \end{aligned}$$

2.2.1 Structural Matrix of the dual Structure

We can obtain the structural matrix of the dual structure f^D from its simple form. We now show how it can be obtained from $M(f)$. Before doing this, we first define a transformation matrix T.

Transformation Matrix

It is a square matrix, T_n of order n such that $T_n = ((t_{ij}))$ where

$$t_{ij} = \begin{cases} (-1)^{j-1}\binom{i-1}{j-1} & \text{for } i \geq j \\ 0 & \text{for } i < j \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Example 3 :

We give here transformation matrices for n=2 to 5

$$T_2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & -1 \end{bmatrix} \qquad T_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

Theorem 2 *If $M(f)$ and $M(f^D)$ are the structural matrices of the structure functions f and f^D respectively, then*

$$M(f^D) = M(f)T\ .$$

Proof : Let

$$f(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j \quad \text{and} \quad f^D(\mathbf{x}) = \sum_{S \subseteq N} b_S \prod_{j \in S} x_j .$$

we then have from Theorem 1

$$m(f^D)_{ij} = \sum_{S \in A_{ij}} b_S = \sum_{S \in A_{ij}} (-1)^{j-1} \sum_{L \supseteq S} a_L =$$

$$\sum_{S \in A_{ij}} (-1)^{j-1} \left[\sum_{L \in A_{S:j}} a_L + \sum_{L \in A_{S:j+1}} a_L + \cdots + \sum_{L \in A_{S:j+(n-j)}} a_L \right]$$

where $A_{S:r} = \{L : L \supseteq S \text{ and } |L| = r\}$ and $|A_{S:r}| = \binom{n-j}{r-j}$, and total number of elements being summed is $2^{n-j}\binom{n-1}{j-1}$. We can write

$$m(f^D)_{ij} = \sum_{r=0}^{n-j} \sum_{S \in A_{i,j+r}} \frac{(-1)^{j-1}\binom{n-1}{j-1}\binom{n-j}{r}}{\binom{n-1}{j+r-1}} a_S$$

$$= \sum_{r=0}^{n-j} (-1)^{j-1} \binom{r+j-1}{j-1} \sum_{S \in A_{i,j+r}} a_S .$$

Here $\binom{n-1}{j-1}\binom{n-j}{r}$ gives the total number of super sets of S containing i and of cardinality $(r+j)$ that occur in $\sum_{S \in A_{ij}} \sum_{L \supseteq S} a_L$ taking repetition into account and $\binom{n-1}{j+r-1}$ gives the number of distinct super sets of S with $i \in S$ and of cardinality $(j+r)$. It follows that

$$m(f^D)_{ij} = \sum_{r=0}^{n-j} \binom{r+j-1}{j-1} (-1)^{j-1} \, m(f)_{i,\,r+j}$$

$$= \sum_{k=j}^{n} m(f)_{ik} \binom{k-1}{j-1} (-1)^{j-1}$$

and since $\binom{k-1}{j-1} = 0$ *for* $k < j$, hence

$$m(f^D)_{ij} = \sum_{k=1}^{n} m(f)_{ik} (-1)^{j-1} \binom{k-1}{j-1}$$

$$= \sum_{k=1}^{n} m(f)_{ik} t_{kj}.$$

This implies that $M(f^D) = M(f)T$.

Example 4 : We can easily check that for Example 2

$$M(f^D) = \begin{bmatrix} 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

and also that $M(f^D) = M(f)T$.

2.3 Properties of T_n

In this section we give some properties of the transformation matrix.

Theorem 3 *$T_n^2 = I_n$ for $n > 0$ where I_n is the identity matrix of order n.*

Proof: Let $T_n^2 = ((\bar{t}_{ij}))$, we then have

$$\bar{t}_{ij} = \sum_{k=1}^{n} (-1)^{k-1} \binom{i-1}{k-1} \binom{k-1}{j-1} (-1)^{j-1}.$$

Case 1: When $i = j$, we have

$$\bar{t}_{ii} = \sum_{k=1}^{n} (-1)^{k-1} \binom{i-1}{k-1} \binom{k-1}{i-1} (-1)^{i-1}$$

and since for $k > i$, $\binom{i-1}{k-1} = 0$ and for $k < i$, $\binom{k-1}{i-1} = 0$.
Hence $\bar{t}_{ii} = (-1)^{2i-2} = 1$.

Case 2: When $j > i$, $k \le i \Longrightarrow k < j$ and $\binom{k-1}{j-1} = 0$ and for $k > i$, we have $\binom{i-1}{k-1} = 0$, hence $\bar{t}_{ij} = 0$.

Case 3: when $i > j$, we have

$$\bar{t}_{ij} = \sum_{k=1}^{n} (-1)^{k-1} (-1)^{j-1} \binom{i-1}{k-1} \binom{k-1}{j-1}$$

(a) $j > k \Longrightarrow \binom{k-1}{j-1} = 0$ and hence $\bar{t}_{ij} = 0$.

(b) $j \le k \Longrightarrow$

$$\begin{aligned} \bar{t}_{ij} &= \sum_{k=j}^{i} (-1)^{k-1} (-1)^{j-1} \binom{i-1}{k-1} \binom{k-1}{j-1} \\ &= \binom{i-1}{j-1} \sum_{k=j}^{i} \binom{i-j}{k-j} (-1)^{k-j} \\ &= \binom{i-1}{j-1} \sum_{r=0}^{i-j} \binom{i-j}{r} (-1)^{r} = 0. \end{aligned}$$

Hence the theorem is established.

Remark : We have from the above theorem

1. $T_n^{-1} = T_n$
2. $M(f) = M(f^D)\,T_n$.

Theorem 4 *Let $[x]$ denote the largest integer less than or equal to x. We then have*

1. $\det T_n = (-1)^{[n/2]}$
2. *The characteristic polynomial $P_{T_n}(x)$ of T_n is given by*

$$(x+1)^{[n/2]}\,(x-1)^{n-[n/2]}$$

3. $\sigma(T_n) = \{1, -1\}$ *where $\sigma(T_n)$ is the spectrum of T_n.*
4. *The minimal polynomial $Q_{T_n}(x)$ of T_n is given by*

$$Q_{T_n}(x) = (x-1)(x+1)$$

5. *The algebraic and the geometric multiplicity of $+1$ and -1 are $n - [n/2]$ and $[n/2]$, respectively.*

Proof :

1. Since T_n is a lower triangular matrix, we have $\det(T_n) = \prod_{i=1}^{n} t_{ii}$ and we know that $t_{ii} = (-1)^{i-1}$, hence $\det(T_n) = (-1)^{[n/2]}$.

2. Since the characteristic polynomial, P_{T_n} of T_n, is given by

$$P_{T_n}(x) = \det(xI_n - T_n) .$$

$$\text{Hence} \quad P_{T_n}(x) = (x-1)^{n-[n/2]}\,(x+1)^{[n/2]}.$$

3. Trivally $\sigma(T_n) = \{1, -1\}$.

4. Consider the polynomial $(x-1)(x+1) = x^2 - 1$ and since $T_n^2 - I_n = 0$, it follows that the polynomial $x^2 - 1$ annihilates T_n, and the required result follows from Horn & Johnson [30].

5. Since every root of $Q_{T_n}(x) = x^2 - 1 = 0$ has multiplicity 1, it follows from Horn & Johnson [30] that T_n is diagonalizable. This implies that T_n is non-defective or equivalently the geometric multiplicity is the same as the algebraic multiplicity for each eigenvalue [30].

Remark: The eigenspace of $T_n{}^t$ corresponding to the eigenvalue $+1(-1)$ is orthogonal to the eigenspace of T_n corresponding to the eigenvalue $(-1)(+1)$.

Theorem 5 *Let $R(A)$ denote the row space of a matrix A. We then have*

(i) $R(T_n + I_n) =$ *left eigenspace of T_n corresponding to the eigenvalue* 1.

(ii) $R(T_n - I_n) =$ *left eigenspace of T_n corresponding to the eigenvalue* -1.

(iii) $R(T_n + I)^t =$ *right eigenspace of T_n^t corresponding to the eigenvalue* 1.

(iv) $R(T_n - I)^t =$ *right eigenspace of T_n^t corresponding to the eigenvalue* -1.

Proof :

(i) Let $\mathbf{x} \in R(T_n+I_n) \Longleftrightarrow \exists$ a row vector $\mathbf{y} \in R^n$ such that $\mathbf{y}(T_n+I_n) = \mathbf{x}$. Since

$$\mathbf{x}(T_n - I_n) = \mathbf{y}(T_n + I_n)(T_n - I_n) = \mathbf{0}$$

as $T_n^2 = I_n$ by Theorem 3, hence we have $\mathbf{x}(T_n - I_n) = \mathbf{0}$ or $\mathbf{x}T_n = \mathbf{x}$ which implies that $\mathbf{x}$ belongs to left eigenspace of T_n corresponding to the eigenvalue 1. Converse part follows from the fact that

$$n - \text{rank}(T_n - I_n) = \text{dimension of } R(T_n + I_n)$$

(ii) The remaining proofs follow on the same lines.

2.4 Importance and Structural Matrix

We have defined the relative importance of component i in a structure f on N to be

$$\Pi_i(f) = E[f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})].$$

We now study the relationship between the relative importance of a component in a structure f and its dual structure f^D.

Theorem 6 *Let f be a structure on N. Then the relative importance of a component is the same in f and f^D.*

Proof : We have by definition

$$\begin{aligned}\Pi_i(f) &= E[f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})] \\ \Pi_i(f^D) &= E[f^D(1_i, \mathbf{X}) - f^D(0_i, \mathbf{X})].\end{aligned}$$

Since for the dual structure we have

$$
\begin{aligned}
f^D(1_i, \mathbf{X}) &= 1 - f(0_i, \mathbf{1} - \mathbf{X}) \text{ and} \\
f^D(0_i, \mathbf{X}) &= 1 - f(1_i, \mathbf{1} - \mathbf{X})
\end{aligned}
$$

hence it follows that $\Pi_i(f) = \Pi_i(f^D)$.

Structural measures of component importance have been studied using different approaches based on pivotal components, critical path vectors and swings etc. We now show how different measures can be studied and calculated using structural matrix.

Birnbaum Structural Measure

Let f be a structure on N and $\beta(i)$ be the Birnbaum measure of structural importance of component i in f.

Theorem 7 *$M(f)\mu$ represents the vector of Birnbaum structural importance of components where $\mu \in R^n$ is a column vector whose j^{th} co-ordinate is given by $\mu_j = (1/2)^{j-1}$ for $j = 1, 2, \ldots, n$.*

Proof : Let $h(\mathbf{p})$ be the reliability function of structure f on N where $\mathbf{p}=(p_1, p_2, \ldots, p_n)$ is the vector of component reliabilities, p_i's. We know that the Birnbaum structural importance of component i, $\beta(i)$, is given by

$$
\beta(i) = \left. \frac{\partial h(\mathbf{p})}{\partial p_i} \right|_{\mathbf{p}=(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})}
$$

$$
\text{Let } f(x) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j \text{ be the simple form of } f.
$$

We then have

$$
h(\mathbf{p}) = \sum_{S \subseteq N} a_S \prod_{j \in S} p_j \text{ and}
$$

$$\frac{\partial h(\mathbf{p})}{\partial p_i} = \sum_{\substack{S \subseteq N \\ i \in S}} a_S \prod_{\substack{j \in S \\ j \neq i}} p_j$$

$$\left.\frac{\partial h(\mathbf{p})}{\partial p_i}\right|_{\mathbf{p}=(\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2})} = \sum_{\substack{i \in S \\ S \subseteq N}} a_S \, (1/2)^{|s|-1}$$

$$= \sum_{j=1}^{n} (1/2)^{j-1} \sum_{\substack{i \in S \subseteq N \\ |S|=j}} a_S$$

$$= \sum_{j=1}^{n} m(f)_{ij} \, (1/2)^{j-1}.$$

Hence we have $\beta = M(f)\mu$ where β is the vector of Birnbaum structural importance of components in f. Note that μ is an eigenvector of T_n corresponding to eigenvalue 1.

Remark : If the binary random variables $X_1, X_2, \ldots, X_n$ are exchangeable [27] & [31], then $M(f)\mu$ gives the vector of Birnbaum measure of reliability importance where $\mu = (\mu_0 = 1, \mu_1, \ldots, \mu_{n-1})^t$ is a column vector and $\mu_i = P[X_1 = X_2 = \cdots = X_i = 1]$. Note that μ is an eigenvector of T_n corresponding to eigenvalue $+1$.

Barlow-Proschan Structural Measure

Let f be a semi-coherent structrure on N. Barlow-Proschan measure of structural importance $\Psi(i)$ of component i is the probability that component i is pivotal under the assumption that all orderings are equally likely to occur. It can be calculated as follows (see Chapter 1 or [4] and [37])

$$\Psi(i) = \int_0^1 [h(1_i, p) - h(0_i, p)] \; dp$$

where $h(1_i, p) = E[f(1_i, \mathbf{X})]$ and $h(0_i, p) = E[f(0_i, \mathbf{X})]$ under the assumption that $X_1, X_2, \ldots X_n$ are i.i.d. binary random variables with $P[X_i = 1] = p$ for all i $\in N$.

Theorem 8 *Let the column vector $\mu \in R^n$, be defined as $\mu_j = 1/j$ for $j = 1, 2, \ldots, n$. Then $M(f)\mu$ gives the vector of Barlow-Proschan measure of structural importance of components in f.*

Proof : Let the simple form of f be

$$f(x) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j$$

we then have

$$\begin{aligned}
\Psi(i) &= \int_0^1 [h(1_i, p) - h(0_i, p)]\, dp \\
&= \int_0^1 \Big(\sum_{\substack{i \in S \\ S \subseteq N}} a_S\, p^{|S|-1} \Big)\, dp \\
&= \sum_{\substack{i \in S \\ S \subseteq N}} a_S \frac{1}{|S|} = \sum_{j=1}^{n} \frac{1}{j} \sum_{\substack{i \in S \subseteq N \\ |S|=j}} a_S \\
&= \sum_{j=1}^{n} m(f)_{ij}\, (1/j).
\end{aligned}$$

Hence the result follows.

Remark : In this case also the column vector μ is an eigenvector of T_n corresponding to the eigenvalue 1.

Cut Importance Ranking

Butler [15] introduced a structural ranking of the components in terms of the minimal cut sets of the system. It is applicable when component reliabilities are high. Let f be a coherent structure on N and $\gamma(f)$ be the collection of all minimal cut sets of f with cardinality m. Component i is said to be more cut important than component j if $\mathbf{b}^{(i)} = (b_1^{(i)}, b_2^{(i)}, \ldots, b_n^{(i)})$ is lexicographically greater than $\mathbf{b}^{(j)} = (b_1^{(j)}, b_2^{(j)}, \ldots, b_n^{(j)})$ where $b_l^{(k)} = \sum_{u=1}^{m}(-1)^{u-1}t_{ul}^{(k)}$ and $t_{ul}^{(k)}$ denotes the number of collections of u distinct minimal cuts such that the union of each collection contains exactly l components including the component k. It can be calculated using the relationship (see Theorem 16 of Chapter 1)

$$E\left[f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})\right] = \sum_{j=1}^{n} b_j^{(i)}(1-p)^{j-1}.$$

Theorem 9 *Butler's cut importance ranking of components in a structure f on N is equivalent to the lexicographic ordering of the rows of the structural matrix of the dual of a structure f on N.*

Proof: Let $M(f^D)$ denote the structural matrix of f^D and let the simple form of f^D be

$$f^D(x) = \sum_{S \subseteq N} b_S \prod_{j \in S} x_j.$$

and hence we have

$$f(\mathbf{x}) = 1 - \sum_{S \subseteq N} b_S \prod_{j \in S}(1 - x_j) = 1 - \sum_{S \subseteq N} b_S \prod_{j \in S} \overline{x}_j$$

where $\bar{x}_j = 1 - x_j$, for $j = 1, 2, \ldots, n$. This implies that

$$\begin{aligned} f(1_i, \mathbf{x}) - f(0_i, \mathbf{x}) &= 1 - \left[\sum_{S \subseteq N - \{i\}} b_S \prod_{j \in S} \bar{x}_j \right] - \left[1 - \sum_{S \subseteq N} b_S \prod_{\substack{j \in S \\ j \neq i}} \bar{x}_j \right] \\ &= \sum_{S \subseteq N} b_S \prod_{\substack{j \in S \\ j \neq i}} \bar{x}_j - \sum_{S \subseteq N - \{i\}} b_S \prod_{j \in S} \bar{x}_j \\ &= \sum_{\substack{S \subseteq N \\ i \in S}} b_S \prod_{\substack{j \in S \\ i \notin S}} \bar{x}_j . \end{aligned}$$

This gives us

$$\begin{aligned} E\left[f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})\right] &= \sum_{\substack{S \subseteq N \\ i \in S}} b_S \, q^{|S|-1} \\ &= \sum_{j=1}^{n} q^{j-1} \sum_{\substack{S \subseteq N \\ i \in S \\ |S| = j}} b_S \\ &= \sum_{j=1} q^{j-1} m(f^D)_{ij} . \end{aligned}$$

Hence the result follows.

Remark : Let $X_1, X_2, \ldots, X_n$ be independent binary random variables with $P[X_i = 1] = p$, all $i \in N$. Then $M(f)\mu$ gives the vector of Birnbaum measure of reliability importance of components in a structure f on N if we take $\mu_i = p^{i-1}$ for $i = 1, 2, \ldots, n$.

Chapter 3

A Consecutive-k-out-of-n:F System

3.1 Introduction

A consecutive-k-out-of-n:F system consists of n linearly ordered and interconnected components such that the system fails if and only if at least k consecutive components fail. This system finds applications in telecommunication and pipeline network [19], design of integrated circuits [12], and process control techniques etc. For example, consider a telecommunication system made-up of a sequence of n relay stations and suppose that a signal transmitted by station i can be received by the next k stations. If there are less than k failures at different points in the sequence the signal can still be transmitted from station 1 to station n and no signal can be transmitted if there are k or more consecutive failures in the sequence. Similarly, in an oil pipeline system with n pumping stations, if a pump fails oil flow will not be discontinued because the neighbouring pumping stations can take the load. A conceptual model of a consecutive-k-out-of-n:F system has a number of applications in the field of Statistical Quality Control. Consider a control chart application. If a control chart shows a run of eight points out of nine points on one side of the central line, then the process is declared as out of statistical control. Similarly, other applications are available in the field of acceptance sampling. Bollinger and Salvia [12] have given an application of this model in the design of integrated circuits.

This system is more general than a conventional r-out-of-n:F system in which the system fails if and only if at least r components fail. In a consecutive-k-out-of-n:F system, if we take $k = 1$, this system reduces to the usual series system and for $k = n$ it becomes the parallel system.

A consecutive-k-out-of-n:F system was first introduced by Kontoleon [33]. He gave a computer algorithm for calculating system reliability. Chiang and Nu [19] gave a recursive formula for computing system reliability in $O(n^2k)$ time and also developed upper and lower bounds for system reliability. Bollinger and Salvia [12] and Bollinger [10] have given a direct combinatorial method for determining probability of system failure. Derman, Lieberman and Ross [21] gave a new set of recursive equations for system reliability and their method requires $O(n^2)$ computing time. They also considered the problem of sequencing of components in the system so as to maximise system reliability.

J.George Shantikumar [46] was the first one to give an efficient recursive algorithm for computing system reliability when component reliabilities are not the same. His algorithm requires a total of $(4n - 3k - 1)$ multiplications/divisions and $(2n - 2k + 1)$ additions/subtractions. Hwang [28] gave two different recursive equations using different but simple arguments when component reliabilities can be different. His methods need $O(nk)$ and $O(n)$ computing time. His second recursive equation is similar to that of Shantikumar [46]. Chao and Lin [18] considered the i.i.d. case and developed a Markov of chain model using the concept of taboo probability and gave a general closed formula for system reliability. They also showed that sytem reliability tends to exp $(-n(1-p)^k)$ for $1 \leq k \leq 4$ as $n \to \infty$ and used this limit formula for designing a large optimum system. They conjectured that this limiting result also holds for $k > 4$. Fu [24] gave a simple proof to show that their conjecture actually holds for all $k \geq 1$.

Bollinger and Salvia [13] studied the consecutive-k-out-of-n:F system when component life has exponential distribution and taking into account the actual order of individual failures leading to system failure and developed an interesting approach for computing moments of life time.

Chen and Hwang [20] gave the following direct formula for computing system failure probability

$$\sum_{j \geq k} \sum_{A(k,j-k)} \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k} p^{\sum n_i} q^{j - \sum_i^k n_i}$$

where $A(k, j-k) = \{(n_1, n_2, \ldots n_k) : n_i'{}s$ are non-negative integers such that $\sum_{i=1}^{k} i\, n_i = j - k\}$.

Papastavridis and Lambiris [41] assumed that the probability that a component is working or not working depends on the state of the previous component and considering the Markov chain model gave a recurrence relationship for computing system reliability. Fu and Hu [25] studied the system when the component failure has $(k-1)$ step Markov dependence and also deduced the result of chao and Lin [18]. Papastavridis and Hadjichristos [40] assumed the i.i.d. model and gave a general formula for mean time to system failure and applied this to the case when components have Weibull life distributions.

Section 2 provides a necessary and sufficient condition for a path set to be a minimal path set. In Section 3, we provide a simple approach for determining the reliability function. In Section 4, we examine the incorrect result of Papastavridis [39]. We give a counterexample and also the mistake in his proof. Section 5 presents a recursive procedure for obtaining the structural matrix of the dual structure of a consecutive-k-out-of-n:F system. Section 6 considers the computational efforts and space requirements needed for calculating the structural matrix of the dual system.

3.2 Minimal Path and Cut Sets

Suppose $\Phi_{k,n}$ denotes the structure function of a consecutive-k-out-of-n:F system and let $\alpha(k;n)$ and $\gamma(k;n)$ represent the collection of minimal path sets and cut sets of a consecutive-k-out-of-n:F system, respectively. Since a consecutive-k-out-of-n:F system fails if and only if at least k consecutive components fails its minimal cut sets are :

$$\gamma(k;n) = \{\{i, i+1, \ldots, i+k-1\} : i = 1, 2, \ldots, n-k+1\}$$

obviously, we have $|\gamma(k;n)| = n-k+1$. We give the necessary and sufficient conditions for a minimal path set.

Theorem 1 *Let R be a subset of $N = \{1, 2, \ldots, n\}$, of cardinality r and let $a_1, a_2, \ldots, a_r$ be its ordered elements, that is, $a_1 < a_2 < \cdots < a_r$ then $R \in \alpha(k;n)$ if and only if we have*

(i) $a_i - a_{i-1} \leq k$ *for* $i = 1, 2, \ldots, r$

(ii) $a_{i+1} - a_{i-1} \geq k+1$ *for* $i = 1, 2, \ldots, r$

where $a_0 = 0$ and $a_{r+1} = n+1$.

Proof: We know from Theorem 1 of Chapter 1 that for a semi-coherent structure on N, a subset $R \subseteq N$ is a path set if and only if it has non-empty intersection with every cut set, that is ,

$$\alpha(k;n) = \{R : R \cap Q \neq \emptyset \text{ for each } Q \in \gamma(k;n) \text{ and } R \text{ is minimal with this property } \}$$

Let $R \subseteq N$ be such that condition (i) holds $\Longleftrightarrow R \cap Q \neq \emptyset$ for all $Q \in \gamma(k;n) \Longleftrightarrow R$ is a path set of $\Phi_{k,n}$.

Suppose that condition (ii) also holds that is, $a_{i+1} - a_{i-1} \geq k+1$ for $i = 1, 2, \ldots, r \Longleftrightarrow$ if component i is deleted from the path set then $R - \{i\}$ is not a path set.

In principle, Theorem 1 gives us a simple algorithm to determine the minimal path set from a given path set if we can check that (ii) holds.

Corollary 1:

$$\alpha(k;k) = \{\{1\},\{2\},\ldots,\{k\}\}$$

$$\alpha(k;2k) = \{\{1,k+1\},\{2,k+2\},\ldots,\{k,2k\}\}$$

Corollary 2: If $R \in \alpha(k;n)$ then

1. $a_r \geq n-k+1$ and $a_{r-1} < n-k+1$.

2. $n-k \notin R \Longrightarrow a_{r-1} \leq (n-k-1) \Longrightarrow n \notin R$.

Corollary 3:

If $n-k \notin R$, then $R \in \alpha(k;n) \iff R \in \alpha(k;n-1)$.

Corollary 4:

If $n-k \in R$, then $R \in \alpha(k;n) \iff R - \{n-k+1,\ldots,n\} \in \alpha(k;n-1)$.

Chan, Chan and Lin [17] have given the following necessary and sufficient condition for a state vector to be a minimal path vector

1. $x_i + x_{i+1} + \cdots + x_{i+k-1} \geq 1, \quad i = 1,2,\ldots,n-k+1$

2. $x_{i-1}(\sum_{i \leq j_1 < j_2 \leq i+k-1} x_{j_1} x_{j_2}) = 0 \quad$ for $i = 1,2,\ldots,n$

where $x_0 = 1, x_{n+1} = 1$ and $x_{n+i} = 0$, for $i = 2,\ldots,k-1$. Condition 1 implies that $\mathbf{x}$ is a path vector and condition 2 ensures that it has no redundant functioning components. These conditions are difficult to apply and less tractable.

3.3 Structure and Reliability Functions

Various approaches have been used for obtaining reliability function of a consecutive-k-out-of-n:F system when components function independently and have the same reliability. Several authors have obtained recursive formula to compute the exact system reliability. The case of unequal component reliabilities was first studied by Shanthikumar [46]. He developed a recursive algorithm to compute system reliability. Hwang [28] also obtained exact system reliability. Chan, Chan, & Lin [17] have given an algebraic approach for obtaining the reliability function.

We give here a simple approach using dual structure to obtain the system reliability function when component reliabilities are not same.

Let $\Phi_{k,n}$ and $\Phi_{k,n}^D$ be the structure functions of a consecutive-k-out-of-n:F system and that of its dual structure, respectively. Suppose that components function independently and assume that the reliability of component i is p_i, for $i = 1, 2, \ldots, n$. Let $h_{k,n}$ and $h_{k,n}^D$ be the reliability functions associated with $\Phi_{k,n}$ and $\Phi_{k,n}^D$, respectively.

Theorem 2 *We have the reliability recurrence relationship for the dual of a consecutive-k-out-of-(n+1):F system*

$$h_{k,n+1}^D(\mathbf{p}_{n+1}) = h_{k,n}^D(\mathbf{p}_n) + \prod_{j=n-k+2}^{n+1} p_j (1 - p_{n-k+1}) \left(1 - h_{k,n-k}^D(\mathbf{p}_{n-k})\right)$$

where $\mathbf{p}_i = (p_1, p_2, \ldots, p_i)$ *and* $n \geq k$.

Proof: Consider the dual structure of a consecutive-k-out-of-(n+1):F system and let A_i for $i = 1, 2, \ldots n - k + 2$ be the event that components

$i, i+1, \ldots, i+k-1$ work. Suppose that $B = A_1 \cup A_2 \cup \cdots \cup A_{n-k+1}$, we then have

$$h^D_{k,n}(\mathbf{p}_n) = P(B) \quad \text{and}$$

$$\begin{aligned} h^D_{k,n+1}(\mathbf{p}_{n+1}) &= P(B \cup A_{n-k+2}) \\ &= P(B) + P(A_{n-k+2}) - P(B \cap A_{n-k+2}) \\ &= h^D_{k,n}(\mathbf{p}_n) + \prod_{j=n-k+2}^{n+1} p_j - \prod_{j=n-k+1}^{n+1} p_j \\ &\quad -(1-p_{n-k+1}) h^D_{k,n-k}(\mathbf{p}_{n-k}) \prod_{j=n-k+2}^{n+1} p_j \end{aligned}$$

since

$$P(B \cap A_{n-k+2}) = (1-p_{n-k+1}) h^D_{k,n-k}(\mathbf{p}_{n-k}) \prod_{j=n-k+2}^{n+1} p_j + \prod_{j=n-k+1}^{n+1} p_j.$$

Hence we have

$$\begin{aligned} h^D_{k,n+1}(\mathbf{p}_{n+1}) &= h^D_{k,n}(\mathbf{p}_n) + \prod_{j=n-k+2}^{n+1} p_j [(1-p_{n-k+1}) - (1-p_{n-k-1}) h^D_{k,n-k}(\mathbf{p}_{n-k})] \\ &= h^D_{k,n}(\mathbf{p}_n) + \prod_{j=n-k+2}^{n+1} p_j (1-p_{n-k+1})(1-h^D_{k,n-k}(\mathbf{p}_{n-k})). \end{aligned}$$

Remark: It follows on the same lines as Theorem 2 that

$$\Phi^D_{k,n+1}(\mathbf{x}_{n+1}) = \Phi^D_{k,n}(\mathbf{x}_n) + (1-x_{n-k+1})(1-\Phi^D_{k,n-k}(\mathbf{x}_{n-k})) \prod_{j=n-k+2}^{n+1} x_j$$

where $\mathbf{x}_n = (x_1, x_2, \ldots x_n)$ and $n \geq k$.

Theorem 3 *The structure function of a consecutive-k-out-of-(n+1):F system is given by*

$$\Phi_{k,n+1}(\mathbf{x}_{n+1}) = \Phi_{k,n}(\mathbf{x}_n) - x_{n-k+1}\Phi_{k,n-k}(\mathbf{x}_{n-k}) \prod_{j=n-k+2}^{n+1} (1 - x_j).$$

Proof: This follows from the above remark and the fact that for any structure f on N and its dual structure f^D, we have

$$f(\mathbf{x}) = 1 - f^D(\mathbf{1} - \mathbf{x}).$$

Remark 1: Theorems 2 and 3 are also given in [46] and [17]. We have presented a different approach for proving them.

Remark 2: For the i.i.d. case with $p_i = p$ for all i and $q = 1 - p$, the reliability recurrence relationship reduces to

$$h_{k,n}(p) = \begin{cases} h_{k,n-1}(p) - q^k p h_{k,n-k-1}(p) & \text{for} \quad n > k \\ 1 - q^k & \text{for} \quad n = k \\ 1 & \text{for} \quad k > n \geq 0. \end{cases}$$

3.4 Component Importance

Griffith and Govindarajulu [26] for the first time examined the problem of reliability importance of components in a consecutive-k-out-of-n:F system. They considered the i.i.d. case and derived an exact expression for system reliability using a Markov chain model. They then used this expression for obtaining Birnbaum reliability importance. As an example, they computed the Birnbaum reliability importance of components in a consecutive-3-out-of-8:F system. Their procedure is lengthy and a direct comparison of component importance is not possible. This problem was next consideded by Parastavridis [39]. For the special case of i.i.d. components, he incorrectly asserted that the most important components are located in the middle of the sequence of the components. He gave the intuitive reasoning that there are more number of k-tuples of consecutive components containing a specific component located nearer to the centre and it is more likely that failure of this particular component will contribute to the failure of a consecutive k-tuples of the components. We now give the results mentioned in [39].

Let $\beta_{k,p}(i,n)$ denote the Birnbaum reliability importance measure of component i in a consecutive-k-out-of-n:F system with i.i.d. components ($p_1 = p_2 = \cdots = p_n = p$) and $q = 1 - p$. Parastavridis [39] first showed that

(i) $\beta_{k,p}(i,n) > \beta_{k,p}(j,n)$ if and only if $h_{k,i-1}(p)h_{k,n-i}(p) > h_{k,j-1}(p)h_{k,n-j}(p)$.

He then goes to prove the following assertion :

(ii) $\beta_{k,p}(i,n) > \beta_{k,p}(j,n)$ if and only if $|n-2i| < |n-2j|$.

We claim that the assertion (ii) is incorrect. We give the following counterexample in support of our claim:

Example : Consider a consecutive-k-out-of-n: F system with $k = 2$

and $n = 6$. Reliability function, $h_{2,n}(p)$ for $n \leq 4$ is given by:

$$h_{2,n}(p) = \begin{cases} 1 & \text{if} \quad 0 \leq n < 2 \\ 1 - q^2 & \text{if} \quad n = 2 \\ 1 - 2q^2 + q^3 & \text{if} \quad n = 3 \\ 1 - 3q^2 + 2q^2 & \text{if} \quad n = 4. \end{cases}$$

For $i = 3$ and $j = 2$, we have

$$|n - 2i| = |6 - 6| < |n - 2j| = |6 - 4| \text{ whereas}$$

$$h_{2,2}(p) \times h_{2,3}(p) - h_{2,1}(p) \times h_{2,4}(p) =$$

$$(1 - q^2)(1 - 2q^2 + q^3) - (1 - 3q + 2q^3) = -q^3p^2 < 0.$$

Hence (ii) is not correct. In fact it can be verified that component 2 has the maximum Birnbaum reliability importance.

Parastavridis [39] has used induction on n and the following reliability recurrence relationship to prove his assertion

$$h_{k,n}(p) = h_{k,n-1}(p) - q^k p h_{k,n-k-1}(p) \text{ for } n \geq 0.$$

To prove the induction hypothesis, he assumes that his assertion is true for the consecutive-k-out-of-m:F system for $m < n$. We claim that this initial assumption is not correct. Since for $m = k < n$, the consecutive-k-out-of-k:F system reduces to a parallel system with k i.i.d. components and we know that in a parallel system with i.i.d. components all components have the same Birnbaum reliability importance. Hence the induction is not valid in this case. Moreover the reliability recurrence relationship used is not correct. The correct reliability recurrence relationship is given in Remark 2 of Theorem 3.

In the remaining chapters, we study the component importance and ranking in a consecutive-k-out-of-n:F system in details.

3.5 Structural Matrix

For studying the structural importance of components, we require the structural matrix of the consecutive-k-out-of-n:F system. We present a recursive procedure for obtaining the structural matrix of the dual of a consecutive-k-out-of-n:F system.

Let $\Phi^D_{k,n}$ denote the structure function of the dual of a consecutive-k-out-of-n:F system and suppose that its simple form is given by

$$\Phi^D_{k,n}(\mathbf{x}_n) = \sum_{S \subseteq N} b_S^{(n)} \prod_{j \in S} x_j$$

where $\mathbf{x}_n \in \{0,1\}^n$. Its reliability function when $p_1 = p_2 = \cdots = p_n = p$ can written as

$$h^D_{k,n}(p) = \sum_{r=1}^{n} a_r^{(n)} p^r \quad \text{where } a_r^{(n)} = \sum_{\substack{S \subseteq N \\ |S|=r}} b_S^{(n)}.$$

The structural matrix, $M(\Phi^D_{k,n}) = ((\mathrm{d}_{ij}^{(n)}))_{n \times n}$ of $\Phi^D_{k,n}$ is by definition

$$\mathrm{d}_{ij}^{(n)} = \sum_{\substack{i \in S \subseteq N \\ |S|=j}} b_S^{(n)}.$$

We first obtain a recursive formula to compute $a_r^{(n+1)}$ for $r = 1, 2, \ldots, n+1$, using the recurrence relationship for $h^D_{k,n}$. We have $n \geq k$

$$
\begin{aligned}
h^D_{k,n+1}(p) &= \sum_{r=1}^{n+1} a_r^{(n+1)} \, p^r \\
&= h^D_{k,n}(\mathbf{p}_n) + \prod_{j=n-k+2}^{n+1} p_j - \prod_{j=n-k+1}^{n+1} p_j \\
&\quad -(1 - p_{n-k+1}) h^D_{k,n-k}(\mathbf{p}_{n-k}) \prod_{j=n-k+2}^{n+1} p_j \\
&= \sum_{r=1}^{n} a_r^{(n)} \, p^r + p^k - p^{k+1} - \sum_{r=1}^{n-k} a_r^{(n-k)} \, (p^{r+k} - p^{r+k+1}) \\
&= \sum_{r=1}^{k-1} a_r^{(n)} \, p^r + (a_k^{(n)} + 1) \, p^k + (a_{k+1}^{(n)} - a_1^{(n-k)} - 1) \, p^{k+1} \\
&\quad + \sum_{r=k+2}^{n} \left[a_r^{(n)} - a_{r-k}^{(n-k)} + a_{r-k-1}^{(n-k)} \right] p^r + \, a_{n-k}^{(n-k)} \, p^{n+1}.
\end{aligned}
$$

This gives us

$$
a_r^{(n+1)} = \begin{cases}
a_r^{(n)} & \text{if } \; 1 \leq r < k \\
a_r^{(n)} + 1 & \text{if } \; r = k \\
a_r^{(n)} - a_1^{(n-k)} - 1 & \text{if } \; r = k+1 \\
a_r^{(n)} - a_{r-k}^{(n-k)} + a_{r-k-1}^{(n-k)} & \text{if } \; k+2 \leq r \leq n \\
a_{n-k}^{(n-k)} & \text{if } \; r = n+1
\end{cases}
$$

We shall now obtain a recursive relationship for the elements $((d_{ij}^{(n+1)}))$ of the structural matrix of $\Phi^D_{k,n+1}$, using the recurrence equation for $\Phi^D_{k,n+1}$

and $a_r^{(n+1)}$. Consider the recursive relationship for $\Phi^D_{k,n+1}$

$$\begin{aligned}\Phi^D_{k,n+1}(\mathbf{x}_{n+1}) &= \Phi^D_{k,n}(\mathbf{x}_n) + \prod_{j=n-k+2}^{n+1} x_j - \prod_{j=n-k+1}^{n+1} x_j \\ &\quad -\Phi^D_{k,n-k}(\mathbf{x}_{n-k}) \prod_{j=n-k+2}^{n+1} x_j + \Phi^D_{k,n-k}(\mathbf{x}_{n-k}) \prod_{j=n-k+1}^{n+1} x_j.\end{aligned}$$

Case 1: For $j < k$

$\mathrm{d}_{ij}^{(n+1)} = 0$ for all i and n

For $n \geq k$ and $k \geq 2$, we have

Case 2: $j = k$

$$\mathrm{d}_{ij}^{(n+1)} = \begin{cases} \mathrm{d}_{ij}^{(n)} & \text{if } 1 \leq i \leq n-k+1 \\ \mathrm{d}_{ij}^{(n)} + 1 & \text{if } n-k+2 \leq i \leq n \\ 1 & \text{if } i = n+1. \end{cases}$$

Case 3: $j = k+1 < n$

$$\mathrm{d}_{ij}^{(n+1)} = \begin{cases} \mathrm{d}_{ij}^{(n)} - \mathrm{d}_{i1}^{(n-k)} & \text{if } 1 \leq i \leq n-k \\ \mathrm{d}_{ij}^{(n)} - 1 & \text{if } i = n-k+1 \\ \mathrm{d}_{ij}^{(n)} - 1 - a_1^{(n-k)} & \text{if } n-k+2 \leq i \leq n \\ -1 - a_1^{(n-k)} & \text{if } i = n+1. \end{cases}$$

Case 4: $j = k + 2 < n$

$$d_{ij}^{(n+1)} = \begin{cases} d_{ij}^{(n)} - d_{i2}^{(n-k)} + d_{i1}^{(n-k)} & \text{if } 1 \le i \le n-k \\ d_{ij}^{(n)} + a_1^{(n-k)} & \text{if } i = n-k+1 \\ d_{ij}^{(n)} - a_2^{(n-k)} + a_1^{(n-k)} & \text{if } n-k+2 \le i \le n \\ -a_2^{(n-k)} + a_1^{(n-k)} & \text{if } i = n+1 \end{cases}$$

Case 5: $j = k + r \le n$ and $r \ge 3$

$$d_{ij}^{(n+1)} = \begin{cases} d_{ij}^{(n)} - d_{ir}^{(n-k)} + d_{i,r-1}^{(n-k)} & \text{if } 1 \le i \le n-k \\ d_{ij}^{(n)} + a_{r-1}^{(n-k)} & \text{if } i = n-k+1 \\ d_{ij}^{(n)} - a_r^{(n-k)} + a_{r-1}^{(n-k)} & \text{if } n-k+2 \le i \le n \\ -a_r^{(n-k)} + a_{r-1}^{(n-k)} & \text{if } i = n+1 \end{cases}$$

Case 6: j=n+1

$$d_{ij}^{(n+1)} = \begin{cases} d_{i,n-k}^{(n-k)} & \text{if } 1 \le i \le n-k \\ a_{n-k}^{(n-k)} & \text{if } n-k+1 \le i \le n+1 \end{cases}$$

Remark : We can obtain $M(\Phi_{k,n})$, the structural matrix of a consecutive-k-out-of-n:F system from $M(\Phi_{k,n}^D)$ by the using the relationship $M(\Phi_{k,n}) = M(\Phi_{k,n}^D)T_n$ as proved in Chapter 2.

Theorem 4 *Let $M(\Phi_{k,n}) = ((f_{ij}))$ be the structural matrix of a consecutive-k-out-of-n:F system, then $f_{ij} = f_{n+1-i,j}$ for any $i = 1, 2, \ldots, n$ and all $j = 1, 2, \ldots, n$.*

Proof : Suppose $\beta_{k,p}(i, n)$ represents the Birnbaum reliability importance of component i in a consecutive-k-out-of-n:F system with i.i.d. components $(p_1 = p_2 = \cdots = p_n = p)$. It can be shown that $\beta_{k,p}(i, n) = \beta_{k,p}(n+1-i, n)$. We know from Remark of Theorem 9 of Chapter 2 that

$$\beta_{k,p}(i, n) = \sum_{j=1}^{n} f_{ij}\mu_j \text{ where } \mu_j = p^{j-1} \text{ for } j = 1, 2, \ldots, n.$$

It follows that

$$\sum_{j=1}^{n} f_{ij}p^{j-1} = \sum_{j=1}^{n} f_{n+1-i,j}p^{j-1}. \text{ This implies that}$$

$$\sum_{j=1}^{n}(f_{ij} - f_{n+1-i,j})p^{j-1} = 0 \text{ for all } p \in [0, 1].$$

Hence we have $f_{ij} = f_{n+1-i,j}$ for all $j = 1, 2, \ldots, n$.

Remark : Using the structural matrix of a consecutive-k-out-of-n:F system, we can compute the Birnbaun, Barlow-proschan structural importances and others measures considered as proved in Chapter 2.

Example : We give here the structural matrices of the dual structure of a consecutive-k-out-of-n:F system for (i) $k = 2, n = 2$ to 8 and (ii) $k = 3, n = 3$ to 9.

Structural Matrices of
the Dual Structure of a Consecutive-2-out-of-n:F system

Case:1 k=2

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}_{n=2} \quad \begin{bmatrix} 0 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}_{n=3} \quad \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}_{n=4}$$

$$\begin{bmatrix} 0 & 1 & -1 & -1 & 1 \\ 0 & 2 & -2 & -1 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & 2 & -2 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 \end{bmatrix}_{n=5} \quad \begin{bmatrix} 0 & 1 & -1 & -2 & 3 & -1 \\ 0 & 2 & -2 & -3 & 4 & -1 \\ 0 & 2 & -3 & -1 & 3 & -1 \\ 0 & 2 & -3 & -1 & 3 & -1 \\ 0 & 2 & -2 & -3 & 4 & -1 \\ 0 & 1 & -1 & -2 & 3 & -1 \end{bmatrix}_{n=6}$$

$$\begin{bmatrix} 0 & 1 & -1 & -3 & 5 & -2 & 0 \\ 0 & 2 & -2 & -5 & 8 & -3 & 0 \\ 0 & 2 & -3 & -3 & 7 & -3 & 0 \\ 0 & 2 & -3 & -2 & 5 & -2 & 0 \\ 0 & 2 & -3 & -3 & 7 & -3 & 0 \\ 0 & 2 & -2 & -5 & 8 & -3 & 0 \\ 0 & 1 & -1 & -3 & 5 & -2 & 0 \end{bmatrix}_{n=7} \quad \begin{bmatrix} 0 & 1 & -1 & -4 & 7 & -2 & -2 & 1 \\ 0 & 2 & -2 & -7 & 12 & -4 & -2 & 1 \\ 0 & 2 & -3 & -5 & 12 & -6 & -1 & 1 \\ 0 & 2 & -3 & -4 & 9 & -3 & -2 & 1 \\ 0 & 2 & -3 & -4 & 9 & -3 & -2 & 1 \\ 0 & 2 & -3 & -5 & 12 & -6 & -1 & 1 \\ 0 & 2 & -2 & -7 & 12 & -4 & -2 & 1 \\ 0 & 1 & -1 & -4 & 7 & -2 & -2 & 1 \end{bmatrix}_{n=8}$$

Case 2: K=3 (Structural Matrices of the Dual Structure of a Consecutive-3-out-of-n:F System)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}_{n=3} \quad \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{bmatrix}_{n=4} \quad \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}_{n=5}$$

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}_{n=6} \quad \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 & 0 & -1 & 1 \\ 0 & 0 & 3 & -3 & 0 & -1 & 1 \\ 0 & 0 & 3 & -4 & 0 & 0 & 1 \\ 0 & 0 & 3 & -3 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}_{n=7}$$

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 2 & -2 & 0 & -3 & 4 & -1 \\ 0 & 0 & 3 & -3 & 0 & -3 & 4 & -1 \\ 0 & 0 & 3 & -4 & 0 & -1 & 3 & -1 \\ 0 & 0 & 3 & -4 & 0 & -1 & 3 & -1 \\ 0 & 0 & 3 & -3 & 0 & -3 & 4 & -1 \\ 0 & 0 & 2 & -2 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & -1 & 0 & -2 & 3 & -1 \end{bmatrix}_{n=8} \quad \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & -3 & 5 & -2 & 0 \\ 0 & 0 & 2 & -2 & 0 & -5 & 8 & -3 & 0 \\ 0 & 0 & 3 & -3 & 0 & -6 & 9 & -3 & 0 \\ 0 & 0 & 3 & -4 & 0 & -3 & 7 & -3 & 0 \\ 0 & 0 & 3 & -4 & 0 & -2 & 5 & -2 & 0 \\ 0 & 0 & 3 & -4 & 0 & -3 & 7 & -3 & 0 \\ 0 & 0 & 3 & -3 & 0 & -6 & 9 & -3 & 0 \\ 0 & 0 & 2 & -2 & 0 & -5 & 8 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 & -3 & 5 & -2 & 0 \end{bmatrix}_{n=9}$$

3.6 Complexity

We examine the space requirements and computational efforts needed for calculating the structural matrix, $M(\Phi^D_{k,n})$. In view of Theorem 4, it is enough to consider the first $\lfloor n/2 \rfloor$ rows of the structural matrix where $\lfloor x \rfloor$ denotes the smallest integer greater or equal to x.

Space Requirements : To calculate $((d_{ij}^{(n+1)}))$, we require the following information

(*i*) The matrices $((d_{ij}^{(n)}))$ and $((d_{ij}^{(n-k)}))$

(*ii*) The vectors $(a_1^{(n)}, \ldots, a_n^{(n)})$ and $(a_1^{(n-k)}, \ldots, a_{n-k}^{(n-k)})$

Hence the space requirements are $\lfloor n/2 \rfloor n + \lfloor (n-k)/2 \rfloor (n-k) + n + (n-k)$, and the maximum storage space required is $O(n^3)$ since at most $n+1$ bits are required to represent an element of the structural matrix.

Computational Complexity : To calculate $\lfloor (n+1)/2 \rfloor (n+1)$ elements of the matrix $((d_{ij}^{n+1}))$, we require a maximum of two additions/substractions for each element. Hence the computational steps needed is equal to $2\lfloor (n+1)/2 \rfloor (n+1)$, counting each addition/substraction as a step. To calculate the $(n+1)$ elements of the vector $(a_1^{(n+1)}, \ldots, a_{n+1}^{(n+1)})$ computational efforts needed is $2(n+1)$. Hence if we consider a structure of size n, the total complexity efforts needed is bounded by

$$(1^2 + \ldots + n^2) + 2(1 + \ldots n) = \frac{(2n+1)(n+1)n}{6} + \frac{2(n+1)n}{2}$$

that is, the complexity is $O(n^3)$.

Chapter 4

Component Importance in a Consecutive-2-out-of-n:F System

4.1 Introduction

We consider a consecutive-2-out-of-n:F system made-up of n linearly ordered components with serial numbers assigned from the set $N = \{1, 2, \ldots, n\}$ and this system fails if and only if there are at least two consecutive failed components. We are mainly interested in ranking the components using different measures of importance when component reliabilities are not known. Measures of importance considered are Birnbaum structural, Barlow-Proschan structural, Birnbaum reliability and Butler's cut importance ranking. For calculating Birnbaum structural importance, we need to study the properties of path sets and swings, which is done in Section 2. In Section 3, we study Birnbaum reliability and Birnbaum structural importance measures and provide a general formula for calculating system reliability. We also obtain Birnbaum structural ranking of components. Section 4 is devoted to the study of Barlow-Proschan structural importance and provides a procedure for calculating it. Section 5 is concerned with the study of Butler's cut importance ranking. An interesting result that follows from this Chapter is that component rankings provided by Birnbaum measure of structural importance, Barlow-Proschan structural measure of importance, Butler's cut importance ranking and Birnbaum reliability importance measure (for i.i.d. components) are all identical in a consecutive-2-out-of-n:F system.

4.2 Path Sets and Swings

Let $\Phi_{2,n}$ denote the structure function of a consecutive-2-out-of-n:F system . Suppose $\mathbf{x}=(x_1, x_2, \ldots, x_n) \in \mathcal{B}^n$ where $\mathcal{B} = \{0, 1\}$, denotes the state vector of components. A path set is a subset of components whose functioning ensures that the system functions. Let $P_2(n)$ represent the number of path sets of a consecutive-2-out-of-n:F system . Furthermore, let $P_2(1_i, n)$ and $P_2(0_i, n)$ denote the number of path sets of a consecutive-2-out-of-n:F system with i^{th} component working and not working, respectively. We then have the result

Theorem 1

$$P_2(n) = P_2(1_i, n) + P_2(0_i, n).$$

Proof : It follows from the fact that a path set of the system either contains component $i \in N$ or it does not contain component i.

Fibonacci Sequence:

A sequence of integers $\{f_n\}$ which is determined by the difference equation

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 3$$

with initial conditions $f_1 = a$ and $f_2 = b$ is called a Fibonacci sequence [48]. The numbers a and b are called a starting pair . We derive a formula for obtaining the n^{th} term of the Fibonacci sequence for the sake of completeness though it is given in [7].

Theorem 2 *Let f_n be a Fibonacci sequence with $f_1 = a$ and $f_2 = b$. We then have*

$$f_n = \frac{c_1^{n-1}}{\sqrt{5}}\left[a(c_1 - 1) + b\right] + \frac{c_2^{n-1}}{\sqrt{5}}\left[a(1 - c_2) - b\right]$$

where

$$c_1 = \frac{1+\sqrt{5}}{2} \quad \textit{and} \quad c_2 = \frac{1-\sqrt{5}}{2}.$$

Proof : Let $g(x)$ denote the generating function of f_n, we then have

$$\begin{array}{rcl} g(x) &=& f_1 + xf_2 + x^2f_3 + \cdots + x^{n-1}f_n + \cdots \\ xg(x) &=& \quad + xf_1 + x^2f_2 + \cdots + x^{n-1}f_{n-1} + \cdots \\ x^2g(x) &=& \quad\quad + x^2f_1 + \cdots + x^{n-1}f_{n-2} + \cdots \end{array}$$

Or $\quad (1 - x - x^2)g(x) = f_1 + x(f_2 - f_1)$ since $f_{n+1} = f_n + f_{n-1}$ for $n > 3$.

This gives us

$$g(x) = \frac{a + x(b-a)}{1 - x - x^2}.$$

Let

$$c_1 = \frac{1+\sqrt{5}}{2} \text{ and } c_2 = \frac{1-\sqrt{5}}{2}$$

we can then write

$$\begin{aligned}
\frac{1}{1-x-x^2} &= \frac{1}{(1-c_1x)(1-c_2x)} \\
&= \frac{A}{(1-c_1x)} + \frac{B}{(1-c_2x)} \\
&= \frac{(A+B) - x(Ac_2 + Bc_1)}{(1-c_1x)(1-c_2x)}
\end{aligned}$$

this gives $A + B = 1$ and $Ac_2 + Bc_1 = 0$ or

$$A = \frac{c_1}{c_1 - c_2} \quad \text{and} \quad B = \frac{-c_2}{c_1 - c_2}.$$

Hence we have

$$\begin{aligned}
g(x) &= A\,[a + x(b-a)][1 - c_1x]^{-1} + B\,[a + x(b-a)][1 - c_2x]^{-1} \\
&\quad \text{for sufficiently small } x \\
&= A[a + x(b-a)]\,[\sum_{n=0}^{\infty} c_1^n x^n] + B\,[a + x(b-a)]\,[\sum_{n=0}^{\infty} (c_2x)^n].
\end{aligned}$$

Coefficient of x^{n-1} in $g(x)$ =

$$\begin{aligned}
&A\,[\,ac_1^{n-1} + (b-a)c_1^{n-2}\,] + B\,[\,ac_2^{n-1} + (b-a)c_2^{n-2}\,] \\
&= \frac{c_1^{n-1}}{\sqrt{5}}\,[a(c_1 - 1) + b] + \frac{c_2^{n-1}}{\sqrt{5}}[a(1 - c_2) - b\,].
\end{aligned}$$

Remark :

When $a = 2$ and $b = 3$, we denote the Fibonacci sequence$\{f_n\}$ by $\{F_n\}$ and is given by $F_1 = 2,\ F_2 = 3,\ F_3 = 5,\ F_4 = 8, \cdots$

Theorem 3 *The number of path sets, $P_2(n)$ of a consecutive-2-out-of-n:F system follows a Fibonacci sequence with initial conditions $a = 2$, and $b = 3$, that is, $P_2(n) = F_n$.*

Proof: Every path set of a consecutive-2-out-of-n:F system must have either component 1 working or not working. The number of path sets with component 1 working is $P_2(1_1, n)$ which equals $P_2(n-1)$. If component 1 is not contained in a path set then component 2 must be present in the path set. Hence the number of path sets with component 1 not working is $P_2(0_1, n)$ and since $P_2(0_1, n) = P_2(n-2)$, for $n \geq 3$ hence we have

$$\begin{aligned} P_2(n) &= P_2(1_1, n) + P_2(0_1, n) \\ &= P_2(n-1) + P_2(n-2) \text{ for } n \geq 3. \end{aligned}$$

Obviously, we have $P_2(1) = 2$, $P_2(2) = 3$. This implies that $P_2(n) = F_n$.

Remark : It is convenient to extend the Fibonacci sequence $\{F_n\}$ to include the terms for $n = 0, -1, -2$ where $F_{-2} = 0, F_{-1} = 1, F_0 = 1$ without loss of generality and we take $P_2(n) = F_n$ for $n \geq -2$. We now show how the number of path sets, when component i is working or not working can be calculated using the Fibonacci sequence $\{F_n\}$.

Theorem 4 *The number of path sets, $P_2(1_i, n)$ of a consecutive-2-out-of-n:F system with component i functioning is given by $F_{i-1}F_{n-i}$.*

Proof : Consider the sets

$$S_1 = \{\mathbf{y} : \mathbf{y} \in \mathcal{B}^{i-1} \text{ and } \Phi_{2,i-1}(\mathbf{y}) = 1\}$$

$$S_2 = \{\mathbf{z} : \mathbf{z} \in \mathcal{B}^{n-i} \text{ and } \Phi_{2,n-i}(\mathbf{z}) = 1\}$$

where $\Phi_{2,i-1}$ and $\Phi_{2,n-i}$ represent the structure functions of a consecutive-2-out-of-(i-1):F subsystem with components $\{1, 2, \ldots, i-1\}$ and a consecutive-2-out-of-(n-i):F subsystem with components $\{i+1, \ldots, n\}$ of the original system, respectively. Now if $\mathbf{y} \in S_1$ and $\mathbf{z} \in S_2$ this implies that $(\mathbf{y},1,\mathbf{z})$ can be written as $(1_i, \mathbf{x})$ and we have $\Phi_{2,n}(1_i, \mathbf{x}) = 1$. Hence $(1_i, \mathbf{x})$ is a path vector of a consecutive-2-out-of-n:F system with i^{th} component working. And if $\Phi_{2,n}(1_i, \mathbf{x}) = 1$, we then define $\mathbf{y}=(x_1, \ldots, x_{i-1})$ and $\mathbf{z} = (x_{i+1}, \ldots, x_n)$ and we have $\Phi_{2,i-1}(\mathbf{y}) = 1$, and $\mathbf{y} \in \mathcal{B}^{i-1}$ and also $\Phi_{2,n-i}(\mathbf{z}) = 1$ and $\mathbf{z} \in \mathcal{B}^{n-i}$. Hence $\mathbf{y} \in S_1$ and $\mathbf{z} \in S_2$ and it implies that $P_2(1_i, n) = P_2(i-1)P_2(n-i) = F_{i-1}F_{n-i}$ using Theorem 3.

Remark : This result can also be obtained from [26].

Theorem 5 *The number of path sets, $P_2(0_i, n)$ of a consecutive-2-out-of-n:F system with i^{th} component not working equals $F_{i-2}F_{n-i-1}$.*

Proof : Let $\mathbf{x}$ be a path vector of a consecutive-2-out-of-n:F system with i^{th} component not working. Hence $\mathbf{x} = (x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ and since $\mathbf{x}$ is a path vector it follows that $x_{i-1} = 1$, $x_{i+1} = 1$, $x_j + x_{j+1} \neq 0$ for $j = 1, 2, \ldots, i-2$ and $x_j + x_{j+1} \neq 0$ for $j = i+1, 2, \ldots, n-1$. This shows that $(x_1, x_2, \ldots, x_{i-1})$ is a path vector for a consecutive-2-out-of-(i-1):F subsystem with $(i-1)^{th}$ component working and the number of such path vectors equals $P_2(1_i, i-1)$. Similarly, $\mathbf{z} = (x_{i+1}, \ldots, x_n)$ is a path vector for a consecutive-2-out-of-(n-i):F subsystem with first component working and the number of such path vectors is given by F_{n-i-1}. Conversally, if $\mathbf{y} = (x_1, x_2, \ldots, x_{i-1})$ with $x_{i-1} = 1$ is a path vector for a consective-2-out-of-(i-1):F system with the last component working and $\mathbf{z} = (x_{i+1}, \ldots, x_n)$

with $x_{i+1} = 1$ is a path vector for a consecutive-2-out-of-(n-i):F subsystem with the first component working, then $\mathbf{x} = (\mathbf{y}, 0, \mathbf{z})$ is a path vector of a consecutive-2-out-of-n:F system with i^{th} component not working. Hence the result.

We now examine the number of swings a component has in a consecutive-2-out-of-n:F system and show how it can be calculated using the extended Fibonacci sequence $\{F_n\}$.

4.2.1 Swings

A subset $D \subseteq N - \{i\}$ is called a swing for component i if $N - D$ is a cut set but $D \cup \{i\}$ is a path set of the system. The term *Swing* is taken from the terminology of game theory. Let $S_2(i; n)$ denote the number of swings for component $i \in N$ in a cosecutive-2-out-of-n:F system. We then have the following results:

Theorem 6 *We have for* $1 \leq i \leq n$

$$\begin{aligned} (i) \quad S_2(i;n) &= F_{i-1}F_{n-i} - F_{i-2}F_{n-i-1} \\ (ii) \quad S_2(i;n) &= F_n - 2F_{i-2}F_{n-i-1} \\ (iii) \quad S_2(i,n) &= 2F_{i-1}F_{n-i} - F_n. \end{aligned}$$

Proof: We recall that $(1_i, \mathbf{x})$ is a critical path vector or a swing vector for component i if $\Phi_{2,n}(1_i, \mathbf{x}) = 1$ and $\Phi_{2,n}(0_i, \mathbf{x}) = 0$. Hence the total number of swings for component i equals

Theorem 9 *For a given $i \geq 1$, $S_2(i;n)$ is a Fibonacci sequence for $n \geq i$ with initial conditions $S_2(i;i) = S_2(1;i)$ and $S_2(i;i+1) = S_2(2;i+1)$.*

Proof: We have from Theorem 6 part (iii), $S_2(i,n) = 2F_{i-1}F_{n-i} - F_n$ and since F_{i-1} is a constant for a given i it follows from Theorem 8, that $S_2(i,n)$ is a Fibonacci sequence with the initial conditions given by when $n = i$ and $n = i+1$ and we have $S_2(i;i) = S_2(1;i)$ and $S_2(i;i+1) = S_2(2;i+1)$ because of mirror image property.

Remark : For $i = 1$, it follows that $S_2(1;n)$=F_{n-3}, $S_2(1;1) = 0$ and $S_2(1;2) = S_2(2;2) = 1$.

Theorem 10 *For a given $i \geq 0$, $S_2(n;i+n)$ is a Fibonacci sequence with initial conditions $S_2(1;i+1)$=F_{i-2} and $S_2(2;i+2)$=$F_i + F_{i-2}$.*

Proof: We have from Theorem 6, part(iii) $S_2(n;i+n)$ =$2F_{n-1}F_i - F_{i+n}$ and it follows from Theorem 8 that $S_2(n;i+n)$ is a Fibonacci sequence as F_i is a constant for a given i and the initial conditions are $S_2(1;i+1)$=F_{i-2} and $S_2(2;2+i)$=$F_i + F_{i-2}$.

Theorem 11 *If $\{F_n\}$ is the extended Fibonacci sequence with $F_1 = 2$ and $F_2 = 3$, then it satisfies the relationship*

$$F_{i-1}F_{i+j} - F_iF_{i+j-1} = (-1)^i F_{j-2} \quad \textit{for } j \geq 0.$$

Proof: We use induction on i and j to prove this. We first prove that it is true for $j = 0, 1$ and all i. For $j = 0$, it is trivally true for all i as $F_{-2} = 0$. Now consider the case when $j = 1$, we need to show that

$$F_{i-1}F_{i+1} - F_iF_i = (-1)^i F_{-1} \quad \text{for all } i$$

$$(i) \qquad S_2(i;n) = \sum_{\mathbf{J}}[\Phi_{2,n}(1_i,\mathbf{x}) - \Phi_{2,n}(0_i,\mathbf{x})]$$
$$= P_2(1_i,n) - P_2(0_i,n)$$
$$= F_{i-1}F_{n-i} - F_{i-2}F_{n-i-1}.$$

where $\mathbf{J} = \{(x_1,\ldots x_{i-1},x_{i+1},\ldots,x_n) : x_p \in \mathcal{B}\}$.

$$(ii) \qquad S_2(i;n) = \sum_{\mathbf{J}}[\Phi_{2,n}(1_i,\mathbf{x}) - \Phi_{2,n}(0_i,\mathbf{x})]$$
$$= P_2(1_i,n) - P_2(0_i,n)$$
$$= P_2(1_i,n) - P_2(0_i,n) + P_2(0_i,n) - P_2(0_i,n)$$
$$= P_2(n) - 2P(0_i,n) = F_n - 2F_{i-2}F_{n-i-1}.$$

$$(iii) \qquad S_2(i;n) = P_2(1_i,n) - P_2(0_i,n)$$
$$= 2P_2(1_i,n) - P_2(n) = 2F_{i-1}F_{n-i} - F_n.$$

Remark : Number of swings of components can also be obtained from the structural matrix. Let $M(\Phi_{2,n})$ denote the structural matrix of a consecutive-2-out-of-n:F system, then $M(\Phi_{2,n})\mu$ gives the vector of component swings where μ is a column vector with $\mu_i = 2^{n-i}$ for $i = 1,2,\ldots,n$.

Mirror Image Component

For each component $i \in N$, we define its mirror image component to be the component $(n-i+1)$. Note that mirror image of a component $(n+1)/2$ is identical to itself when n is odd. We now show the relationship between a component and its mirror image.

Theorem 7 *In a consecutive-2-out-of-n:F system component i and its mirror image component $(n+1-i)$, have the same number of swings.*

Proof : It follows from Theorem 6 (iii) since for any i ($1 \le i \le n$)

$$\begin{aligned} S_2(i;n) &= 2F_{i-1}F_{n-i} - F_n \qquad \text{and} \\ S_2(n-i+1;n) &= 2F_{n-i}F_{i-1} - F_n. \end{aligned}$$

It is interesting to note that number of swings for component i is again a Fibonacci sequence but with different initial conditions. Before showing this we first prove a simple result regarding a linear integer combination of two Fibonacci sequences . In fact, it holds for more than two Fibonacci sequences as well.

Theorem 8 *If $\{g_n\}$ and $\{h_n\}$ are two Fibonacci sequences, then $t_n(m) = \alpha g_n + \beta h_{m+n}$, where α and β are integers, and m is a non-negative integer, is a Fibonacci sequence with initial conditions $t_1(m) = \alpha g_1 + \beta h_{m+1}$ and $t_2(m) = \alpha g_2 + \beta h_{m+2}$.*

Proof : We have by definition for $n \ge 3$

$$\begin{aligned} g_n &= g_{n-1} + g_{n-2}\ , \ \text{and} \\ h_{m+n} &= h_{m+n-1} + h_{m+n-2}, \ \text{for } m \ge 0 \text{ and } n \ge 3 \\ t_{n-1}(m) &= \alpha g_{n-1} + \beta h_{m+n-1} \\ t_{n-2}(m) &= \alpha g_{n-2} + \beta h_{m+n-2}, \ \text{which gives us} \\ t_{n-1}(m) + t_{n-2}(m) &= \alpha g_n + \beta h_{m+n} = t_n(m). \end{aligned}$$

Hence $\{t_n(m)\}$ is a Fibonacci sequence.

or,

$$\det M_i = [\det A]^{i+2} \qquad \text{for all i}$$

where

$$M_i = \begin{bmatrix} F_{i+1} & F_i \\ F_i & F_{i-1} \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

For i=1,

$$M_1 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \Longrightarrow \det M_1 = -1 \text{ and also } \det A = -1.$$

Hence the result is true for $i = 1$. Now assume that it is true for $i = p$, that is

$$\det M_p = [\det A]^{p+2}.$$

$$\text{Since} \quad M_pA = \begin{bmatrix} F_{p+1} & F_p \\ F_p & F_{p-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{p+2} & F_{p+1} \\ F_{p+1} & F_p \end{bmatrix} = M_{p+1}$$

this gives us

$$\det M_{p+1} = \det[M_pA] = \det M_p \det A = [\det A]^{p+2} \det A = [\det A]^{p+3}.$$

Hence it is true for all i and $j = 1$.

Let us now assume that theorem is true up to $j = r$ and for all i, that is

$$F_{i-1}F_{i+r} - F_iF_{i+r-1} = (-1)^i F_{r-2}.$$

Take the case for $j = r + 1$ and we have

$$F_{i-1}F_{i+r+1} - F_iF_{i+r} \quad = \quad F_{i-1}F_{i+r} + F_{i-1}F_{i+r-1} - F_iF_{i+r-1} - F_iF_{i+r-2}$$

$$
\begin{aligned}
&= F_{i-1}F_{i+r} - F_iF_{i+r-1} + F_{i-1}F_{i+r-1} - F_iF_{i+r-2} \\
&\qquad \text{since the result is true for } j = r-1 \text{ and } j = r \\
&= (-1)^r F_{r-2} + (-1)^r F_{r-3} \\
&\qquad \text{and as } \{F_n\} \text{ is the extended Fibonacci sequence} \\
&= (-1)^r F_{r-1}.
\end{aligned}
$$

Hence the claim is true for $j = r + 1$ and the theorem follows.

Remark : The particular case of the above theorem for $j = 1$ is well known in the literature [34].

Theorem 12 *For $n \geq 1$, we have $S_2(n; 2n) = F_{n-1}^2$.*

Proof : Using Theorem 6 parts (ii) and (iii), we have

$$
\begin{aligned}
S_2(n; 2n) &= 2F_{n-1}F_n - F_{2n} \\
&= F_{2n} - 2F_{n-2}F_{n-1}.
\end{aligned}
$$

Hence we can write

$$
\begin{aligned}
S_2(n; 2n) &= F_{n-1}F_n - F_{n-2}F_{n-1} \\
&= F_{n-1}[F_n - F_{n-2}] \\
&= F_{n-1}^2.
\end{aligned}
$$

Remark : Using the above result, we have $2F_{n-1}F_n - F_{2n} = F_{n-1}^2$. This result indicates that for an even number $(2n)$ of components in the system, the number of swings for component n or $n + 1$ equals the square of the number of the path sets for a system with $(n-1)$ components. Furthermore, we have for the system having an odd number of components the absolute difference in the number of swings between the middle most components is a constant quantity. This result is proved in the next theorem.

Theorem 13 *We have for $n \geq 2$*

$$|S_2(n+1; 2n+1) - S_2(n; 2n+1)| = 2.$$

Proof : From Theorem 6 it follows that

$$\begin{aligned} S_2(n+1; 2n+1) - S_2(n; 2n+1) &= 2F_nF_n - F_{2n+1} - 2F_{n-1}F_{n+1} + F_{2n+1} \\ &= 2[F_n^2 - F_{n-1}F_{n+1}] = -2(-1)^n. \end{aligned}$$

The last step follows from Theorem 11 for $j = 1$ and $i = n$, hence the result holds.

For obtaining the relative ranking of the components, we need to compare the number of swings of different components .

Theorem 14 *We have for $n \geq 2$*

$$S_2(1; n) - S_2(2; n) = -2S_2(1; n-1).$$

Proof : An application of Theorem 6 gives us

$$\begin{aligned} S_2(1; n) - S_2(2; n) &= 2F_0F_{n-1} - F_n - 2F_1F_{n-2} + F_n \\ &= 2[F_{n-1} - 2F_{n-2}] \\ &= \quad - 2F_{n-4} = -2S_2(1, n-1). \end{aligned}$$

The last step follows from Remark of Theorem 9.

Theorem 15 *For a fixed i in a consecutive-2-out-of-n:F system, we have*

(i) *$S_2(i,n) - S_2(i+1;n)$ is a Fibonacci sequence for $1 \le i \le n-1$ and $n \ge 2$*

(ii) *$S_2(i,n) - S_2(i+1;n) = -[S_2(i-1;n-2) - S_2(i;n-2)]$ for $2 \le i \le n-2$ and $n \ge 4$.*

Proof :

(i) Since for a fixed i, $S_2(i,n)$ is a Fibonacci sequence and $S_2(i+1;n)$ is also a Fibonacci sequence, it follows from Theorem 8 that the result is true.

(ii) We have

$$\begin{aligned} S_2(i;n) - S_2(i+1;n) &= 2[F_{i-1}F_{n-i} - F_iF_{n-i-1}] \\ &= 2[F_{i-1}(F_{n-i-1} + F_{n-i-2}) - (F_{i-1} + F_{i-2})F_{n-i-1}] \\ &= 2[F_{i-1}F_{n-i-2} - F_{i-2}F_{n-i-1}] \\ &= -[S_2(i-1;n-2) - S_2(i;n-2)]. \end{aligned}$$

Theorem 16 *Let m be the smallest integer greater than or equal to $n/2$. We then have for $2 \le r \le m$ and $n \ge 3$*

$$S_2(r;n) - S_2(1;n) = 2S_2(1;r)\, S_2(1;n-r+1).$$

Proof : For $r = 2$, using Theorem 14, we have

$$S_2(2;n) - S_2(1;n) = 2S_2(1;n-1)$$

and since $S_2(1,2) = 1$, it follows that the result is true for $r = 2$ and assume that it is true for $r \le p < m$. Let $r = p+1 \le m$, and we have

$$S_2(p+1;n) = S_2(p;n-1) + S_2(p-1;n-2) \text{ from Theorem 10 or 6.}$$

Since the hypothesis is true for p and $p-1$

$$= S_2(1;n-1) + 2S_2(1;p)S_2(1;n-1-p+1)$$

$$+ S_2(1;n-2) + 2S_2(1;p-1)S_2(1;n-2-(p-1)+1)$$

$$= S_2(1;n) + 2[S_2(1;p) + S_2(1;p-1)]S_2(1;n-(p+1)+1)$$

$$= S_2(1;n) + 2S_2(1;p+1)S_2(1;n-(p+1)+1).$$

This implies that result is true for $p+1$ and hence our claim holds.

Remark : Because of mirror image property, it is enough to consider the first m components of the system for purpose of comparison and ranking.

Theorem 17 *Let m be the smallest integer greater than or equal to $n/2$. Then for $1 \leq r \leq m-1$ and $n \geq 4$, we have*

$$S_2(r;n) - S_2(r+1;n) = 2(-1)^r S_2(1;n-2r+1) = 2(-1)^r F_{n-2r-2}.$$

Proof : The claim is true for $r = 1$ according to Theorem 14 and by using Theorem 15, we have for $r \geq 2$ and $n \geq 4$

$$S_2(r;n) - S_2(r+1;n) = -[S_2(r-1;n-2) - S_2(r;n-2)].$$

By repeated application of Theorem 15 we have

$$\begin{aligned} S_2(r;n) - S_2(r+1;n) &= (-1)^{r-1}[S_2(r-(r-1);n-2(r-1)) \\ &\quad - S_2(r-(r-1)+1;n-2(r-1))] \\ &= (-1)^{r-1}[S_2(1;n-2r+2) - S_2(2;n-2r+2)] \\ &= (-1)^r 2\, S_2(1;n-2r+1). \end{aligned}$$

The last step follows from Theorem 14.

Theorem 18 *Let m be the smallest integer greater than or equal to $n/2$.*

$$S_2(r;n) = S_2(1;n+1) + S_2(r-2;n-4) \text{ for } 3 \le r \le m \text{ and } n \ge 5.$$

Proof : Using Theorem 16 and Theorem 6, we get

$$\begin{aligned} S_2(r;n) &= S_2(1;n) + 2S_2(1;r)\, S_2(1;n-r+1) \\ &= F_{n-3} + 2F_{r-3}\, F_{n-r-2} \text{ as } (S_2(1,n) = F_{n-3}) \\ &= F_{n-2} - F_{n-4} + 2F_{(r-2)-1}\, F_{n-4-(r-2)} \\ &= S_2(1;n+1) + S_2(r-2;n-4). \end{aligned}$$

Theorem 19 *Let m be the smallest integer greater than or equal to $n/2$. Then*

(i) *For odd integer r such that $3 \le r \le m$ and $n \ge 5$*

$$S_2(1;n) - S_2(r;n) = -2[S_2(1;n-2) + S_2(1;n-6) + \cdots + S_2(1;n-2(r-2))].$$

(ii) *For even integer r such that $4 \le r \le m$ and $n \ge 7$.*

$$S_2(1;n) - S_2(r;n) =$$
$$-2[S_2(1;n-2) + S_2(1;n-6) + \cdots + S_2(1;n-2(r-2))] + S_2(1,n-2r+3)$$

Proof : Since by Theorem 17, we have for $1 \le j < m$

$$S_2(j;n) - S_2(j+1;n) = 2(-1)^j S_2(1;n-2j+1)$$

adding them for $j = 1, 2, \ldots, (r-1)$ such that $r \le m$, we have

$$\sum_{j=1}^{r-1} [S_2(j;n) - S_2(j+1;n)] = -2\sum_{j=1}^{r-1} (-1)^j S_2(1;n-2j+1).$$

This gives us the required results after simplification.

Theorem 20 *Let m be the smallest integer greater than or equal to $n/2$, then the following results hold:*

(i) $S_2(r+1;n) - S_2(r;n) > 0$ *for odd* $r < m$ *and* $n \geq 3$.

(ii) $S_2(r+1;n) - S_2(r;n) < 0$ *for even* $r < m$ *and* $n \geq 5$.

(iii) $S_2(1;n) < S_2(3;n) < \cdots < S_2(m_1;n)$ *where* m_1 *is the largest odd number less than or equal to* m *and* $n \geq 5$.

(iv) $S_2(2;n) > S_2(4;n) > \cdots > S_2(m_2;n)$ *where* m_2 *is the largest even number less than or equal to* m *and* $n \geq 7$.

Proof :

(i) and (ii) trivially follow from Theorem 17.

(iii) It is enough to show that

$$S_2(2d+3;n) - S_2(2d+1;n) > 0 \text{ where } 2d+3 \leq m_1$$

using Theorem 17 for $r = 2d+1 < m_1$ and $r = 2d+2 < m_2$, we have

$$\begin{aligned} S_2(2d+1;n) - S_2(2d+2;n) &= (-1)^{2d+1} 2S_2(1;n-2(2d+1)+1) \\ S_2(2d+2;n) - S_2(2d+3;n) &= (-1)^{2d+2} 2S_2(1;n-2(2d+2)+1). \end{aligned}$$

Adding them we get

$$\begin{aligned} S_2(2d+1;n) - S_2(2d+3;n) &= 2[S_2(1;n-4d-3) - S_2(1;n-4d-1)] \\ &= 2[F_{n-4d-6} - F_{n-4d-4}] < 0. \end{aligned}$$

(iv) Using Theorem 17 for $r = 2d < m_2$ and $r = 2d + 1 < m_1$, we have

$$S_2(2d;n) - S_2(2d+1;n) = (-1)^{2d}\, 2S_2(1:n-4d+1)$$

$$S_2(2d+1;n) - S_2(2d+2;n) = (-1)^{2d+1}\, 2S_2(1;n-4d-1)$$

Adding them we get

$$S_2(2d;n) - S_2(2d+2;n) = 2[S_2(1;n-4d+1) - S_2(1;n-4d-1)] > 0.$$

An Alternative Approach

We now consider the Fibonacci sequence $\{t_n\}$ given by the starting pair $a = 1$ and $b = 1$, the sequence generated is $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ and is connected to the $\{F_n\}$ by the relation $\{F_n\} = \{t_{n+2}\}, n \geq 1$. We first study some related properties of $\{t_n\}$ and show its relevance to the number of swings and provide an alternative proof for Theorem 20

Let $g_n = \frac{t_{n+1}}{t_n}$ and $\lim_{n\to\infty} g_n = G$ where $G = \frac{1+\sqrt{5}}{2}$ and is called the Golden Ratio [48]. We prove the following results.

Theorem 21 *We have*

(i) $g_n < G$ *for odd* n *and* $g_n > G$ *for even* n

(ii) $|g_n - G|$ *is a strictly decreasing function of* n

(iii) $g_{2d} > g_n$ *for all* $n > 2d$ *and*

(iv) $g_{2d-1} < g_n$ *for all* $n > 2d$.

Proof :

(i) From Theorem 2, we know that explicit formula for t_n is given by

$$t_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$\text{let } \alpha = \frac{1-\sqrt{5}}{1+\sqrt{5}}, \text{ we can then write}$$

$$t_n = \frac{1}{\sqrt{5}}[G^n(1-\alpha^n)].$$

It follows that

$$g_n = \frac{t_{n+1}}{t_n} = G\left[\frac{1-\alpha^{n+1}}{1-\alpha^n}\right]$$

$$\text{or } g_n - G = G(1-\alpha)\frac{\alpha^n}{1-\alpha^n}.$$

Now since $-1 < \alpha < 0$, it implies that $g_n < G$ for all odd n and $g_n > G$ for all even n. Hence the result holds.

(ii)

$$\text{Since } \left|\frac{\alpha^n}{1-\alpha^n}\right| = \frac{|\alpha|^n}{1-\alpha^n} \text{ and}$$

$$\frac{|\alpha|^n}{1-\alpha^n} - \frac{|\alpha|^{n+1}}{1-\alpha^{n+1}} = \frac{|\alpha|^n(1-\alpha^{n+1}-|\alpha|+\alpha^n|\alpha|)}{(1-\alpha^n)(1-\alpha^{n+1})}$$

$$= \frac{|\alpha|^n(1-2\alpha^{n+1}-|\alpha|)}{(1-\alpha^n)(1-\alpha^{n+1})}$$

Obviously, R.H.S. is greater than zero, when n is even. Consider the case when n is odd, i.e., $n = 2d+1$, for $d \geq 0$

$$\frac{|\alpha|^n}{1-\alpha^n} - \frac{|\alpha|^{n+1}}{1-\alpha^{n+1}} = \frac{|\alpha|^n(1-2|\alpha|^{2d+2}-|\alpha|)}{(1-\alpha^n)(1-\alpha^{n+1})}$$

$$> \frac{|\alpha|^n(1-2|\alpha|^2-|\alpha|)}{(1-\alpha^n)(1-\alpha^{n+1})}$$

$$> 0.$$

The above inequality follows because

$$1-2|\alpha|^2-|\alpha| > 1-2\times(0.4)^2-0.4 = 1-0.72 > 0$$

as $|\alpha| < 0.4$. Hence it follows that $|g_n - G|$ is a strictly decreasing function of n.

(iii) & (iv)

$$\text{Since } g_n - G = G(1-\alpha)\frac{\alpha^n}{1-\alpha^n} \text{ and } -1 < \alpha < 0$$

it follows that for any positive integer i, we have

$$g_{2i} - g_{2(i+1)} = \frac{G(1-\alpha)\alpha^{2i}(1-\alpha^2)}{(1-\alpha^{2i})(1-\alpha^{2i+2})} > 0 \text{ and}$$

$$g_{2i+1} - g_{2i-1} = \frac{G(1-\alpha)\alpha^{2i-1}(\alpha^2-1)}{(1-\alpha^{2i+1})(1-\alpha^{2i-1})} > 0$$

Hence it implies that

$$(i) \quad g_{2i} > g_{2(i+r)} > G \quad \text{for } r \geq 1 \text{ and } i \geq 1$$

$$(ii) \quad G > g_{2(i+r)-1} > g_{2i-1} \text{ for } r \geq 1 \text{ and } i \geq 1.$$

It follows that

1. $g_{2i} > g_n$ for $n > 2i$ and $i \geq 1$
2. $g_{2i-1} < g_n$ for $n > 2i$ and $i \geq 1$.

Remark : Since $g_n = \frac{t_{n+1}}{t_n} = \frac{F_{n-1}}{F_{n-2}}$ and $g_n \to G$, it implies that the ratio of number of path sets of a consecutive-2-out-of-n+1:F system to that of a consecutive-2-out-of-n:F system tends to the Golden ratio.

We now show that the problem of ranking of components using swings is the same as ranking of terms of the sequence $\{g_n\}$.

Theorem 22 *Let m be the smallest integer greater than or equal to $n/2$. Then ranking of the first m components in a consecutive-2-out-of-n:F system based on number of swings of a component, is the same as ranking of first m terms of the sequence $\{g_n\}$.*

Proof : We have from Theorem 6 (ii)

$$\begin{aligned} S_2(i,n) - S_2(i+1,n) &= 2(F_{i-1}F_{n-i-2} - F_{i-2}F_{n-i-1}) \\ &= 2t_{i+1}t_{n-i} - t_i t_{n-i+1} \\ &= 2t_i t_{n-i}\left(\frac{t_{i+1}}{t_i} - \frac{t_{n-i+1}}{t_{n-i}}\right) \\ &= 2t_i t_{n-i}(g_i - g_{n-i}). \end{aligned}$$

Thus $S_2(i,n)$ is greater than or equal to or less than $S_2(i+1,n)$ accordingly, as g_i is greater than or equal to or less than g_{n-i}. For $1 \le i < m \Rightarrow n-i > i$ hence it follows from Theorem 21 that $g_i > g_{n-i}$ for even $i < m$ and $g_i < g_{n-i}$ for odd $i < m$. Also note that

$$S_2(2i,n) - S_2(2i+2,n) = 2t_{2i}t_{n-2i-1}(g_{2i} - g_{n-2i-1}) \text{ for } 2i+2 \le m$$

$$S_2(2i+1,n) - S_2(2i-1,n) = 2t_{2i-1}t_{n-2i}(g_{n-2i} - g_{2i-1}) \text{ for } 2i+1 \le m.$$

and $2i+2 \le m \Rightarrow n-2i-1 > 2i$. Hence the required result follows.

(ii) This follows on the same lines as part(i) above.

We now show how to calculate the Birnbaum reliability importance measure using the reliability function, $h_{2,n}(\mathbf{p})$.

Remark : When $p_i = p$ for $i = 1, 2, \ldots, n$, we shall denote $\beta_{2,\mathbf{p}}(i,n)$ by $\beta_{2,p}(i,n)$. We take $h_{2,n}(p) = 1$ for $-2 \leq n \leq 0$.

Theorem 24 *We have for a consecutive-2-out-of-n:F system*

(i) $h_{2,n}(1_i, \mathbf{p}) = h_{2,i-1}(p_1, p_2, \ldots, p_{i-1})\, h_{2,n-i}(p_{i+1}, p_{i+2}, \ldots, p_n)$

(ii) $h_{2,n}(0_i, \mathbf{p}) = h_{2,i-2}(p_1, p_2, \ldots, p_{i-2})\, h_{2,n-i-1}(p_{i+2}, p_{i+3}, \ldots, p_n)$

where $h_{2,i-1}(p_1, p_2, \ldots, p_{i-1})$ denote the reliability function of a consecutive-2-out-of-(i-1):F subsystem consisting of first $(i-1)$ components of the original system and $h_{2,n-i}(p_{i+1}, p_{i+2}, \ldots, p_n)$ represents the reliability function of a consecutive-2-out-of-(n-i):F subsystem composed of last $(n-i)$ components of the original system. $h_{2,i-2}(p_1, p_2, \ldots, p_{i-2})$ and $h_{2,n-i-1}(p_{i+2}, p_{i+3}, \ldots, p_n)$ are defined similarly.

Proof :

(i) Since a consecutive-2-out-of-n:F system with i^{th} component functioning, will work when and only when the consecutive-2-out-of-(i-1):F subsystem consisting of first $(i-1)$ components of the original system, functions as well as the consecutive-2-out-of-(n-i):F subsystem consisting of last $(n-i)$ components of the original system, functions.

4.3 Birnbaum Importance Measure

In this section we examine the Birnbaum reliability importance and the Birnbaum structural importance measures.

Birnbaum Reliability Importance :

Let $h_{2,n}(\mathbf{p})$ denote the reliability function of a consecutive-2-out-of-n:F system where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is the vector of component reliabilities. Obviously, $E[\Phi_{2,n}(\mathbf{x})] = h_{2,n}(\mathbf{p})$, $E[\Phi_{2,n}(1_i, \mathbf{x})] = h_{2,n}(1_i, \mathbf{p})$, and $E[\Phi_{2,n}(0_i, \mathbf{x})] = h_{2,n}(0_i, \mathbf{p})$

Theorem 23 *Let $\beta_{2,\mathbf{p}}(i, n)$ denote the Birnbaum measure of reliability importance of component i in a consecutive-2-out-of-n:F system, we then have for $p_i \in (0,1)$*

$$(i) \qquad \beta_{2,\mathbf{p}}(i,n) = \frac{1}{(1-p_i)}\left[h_{2,n}(1_i,\mathbf{p}) - h_{2,n}(\mathbf{p})\right]$$

$$(ii) \qquad \beta_{2,\mathbf{p}}(i,n) = \frac{1}{p_i}\left[h_{2,n}(\mathbf{p}) - h_{2,n}(0_i,\mathbf{p})\right].$$

Proof :

(i) We have from Theorem 7 of Chapter 1 (the pivotal decomposition Theorem of a structure function),

$$\begin{aligned}
\Phi_{2,n}(\mathbf{x}) &= x_i\Phi_{2,n}(1_i,\mathbf{x}) + (1-x_i)\Phi_{2,n}(0_i,\mathbf{x}). \text{ Hence} \\
h_{2,n}(\mathbf{p}) &= p_i h_{2,n}(1_i,\mathbf{p}) + (1-p_i)h_{2,n}(0_i,\mathbf{p}) \text{ and} \\
h_{2,n}(\mathbf{p}) - h_{2,n}(1_i,\mathbf{p}) &= (1-p_i)[h_{2,n}(0_i,\mathbf{p}) - h_{2,n}(1_i,\mathbf{p})]
\end{aligned}$$

and the result follows since $\beta_{2,\mathbf{p}}(i,n) = h_{2,n}(1_i,\mathbf{p}) - h_{2,n}(0_i,\mathbf{p})$.

Hence we have under the assumption of independent functioning of the components .

$$h_{2,n}(1_i, \mathbf{p}) = h_{2,i-1}(p_1, p_2, \ldots, p_{i-1})\, h_{2,n-i}(p_{i+1}, p_{i+2}, \ldots, p_n)$$

Remark : It also follows from [26] and [39].

(ii) A consecutive-2-out-of-n:F system with i^{th} component not working will function if and only if the following two conditions hold:

1. The consecutive-2-out-of-(i-1):F subsystem works, and component $(i-1)$ works . This reliability is given by $h_{2,i-1}(1_{i-1}, p_1, p_2, \ldots, p_{i-2})$.
2. The consecutive-2-out-of-(n-i):F subsystem consisting of components $\{i+1, \ldots, n\}$ works and component $(i+1)$ works. The reliability of such a subsystem equals $h_{2,n-i}(1_{i+1}, p_{i+2}, \ldots, p_n)$. Since we have

$h_{2,i-1}(1_{i-1}, p_1, p_2, \ldots, p_{i-2}) = h_{2,i-2}(p_1, p_2, \ldots, p_{i-2})$ and

$h_{2,n-i}(1_{i+1}, p_{i+2}, \ldots, p_n) = h_{2,n-i-1}(p_{i+2}, \ldots, p_n)$, the result follows.

Reliability Function :

The reliability function of a consecutive-2-out-of-n:F system with i.i.d. components can be calculated using the theorem stated next. Let p be the common reliability of components and $q = 1 - p$.

Theorem 25 *we have for a consecutive-2-out-of-n:F system*

$$h_{2,n}(p) = \frac{1}{p\sqrt{p^2+pq}}\left[\left\{\frac{2pq}{\sqrt{p^2+4pq}-p}\right\}^{n+2} - \left\{\frac{-2pq}{\sqrt{p^2+4pq}+p}\right\}^{n+2}\right].$$

Proof : Let $H(x,p)$ be the generating function of $h_{2,n}(p)$, we then have

$$\begin{array}{rlllll} H(x,p) = & h_{2,0}(p) + & h_{2,1}(p)x & + h_{2,2}(p)x^2 & + h_{2,3}(p)x^3 & + \cdots \\ pxH(x,p) = & & h_{2,0}(p)xp & + h_{2,1}(p)x^2p & + h_{2,2}(p)x^3p & + \cdots \\ pqx^2H(x,p) = & & & h_{2,0}(p)x^2pq & + h_{2,1}(p)x^3pq & + \cdots \\ \hline \end{array}$$

$$\text{Or} \quad H(x,p)[1 - px - pqx^2] = 1 + x(1-p) = 1 + xq$$

since $h_{2,n+2}(p) = ph_{2,n+1}(p) + pqh_{2,n}(p)$ and $h_{2,i}(p) = 1$ for $0 \le i < 2$.

$$\text{Hence } H(x,p) = \frac{1 + xq}{1 - px - pqx^2}.$$

Roots of the equation $pqx^2 + px - 1 = 0$ are:

$$\lambda_1 = \frac{-p + \sqrt{p^2 + 4pq}}{2pq}, \ \lambda_2 = \frac{-p - \sqrt{p^2 + 4pq}}{2pq} \text{ and}$$

$$\lambda_1 + \lambda_2 = \frac{-1}{q}, \quad \lambda_1\lambda_2 = \frac{-1}{pq},$$

$$\lambda_1 - \lambda_2 = \frac{\sqrt{p^2 + 4pq}}{pq} \text{ and as in Feller [22]}$$

$$h_{2,n}(p) = \frac{a}{\lambda_1^{n+1}} + \frac{b}{\lambda_2^{n+1}}$$

where

$$a = \frac{1 + q\lambda_1}{2pq\lambda_1 + p} = \frac{1/p(1/q + \lambda_1)}{\lambda_1 + (\lambda_1 + 1/q)} = \frac{\lambda_2}{(\lambda_2 - \lambda_1)p}$$

and

$$b = \frac{1 + q\lambda_2}{2pq\lambda_2 + p} = \frac{-\lambda_1}{p(\lambda_2 - \lambda_1)}. \text{ Hence}$$

$$\begin{aligned} h_{2,n}(p) &= \frac{pq}{p\sqrt{p^2 + 4pq}} \left[\frac{-\lambda_2}{\lambda_1^{n+1}} + \frac{\lambda_1}{\lambda_2^{n+1}} \right] \\ &= \frac{1}{p\sqrt{p^2 + 4pq}} \left[\left\{ \frac{2pq}{\sqrt{p^2 + 4pq} - p} \right\}^{n+2} - \left\{ \frac{-2pq}{\sqrt{p^2 + 4pq} + p} \right\}^{n+2} \right]. \end{aligned}$$

Remark : It follows from the above theorem that

$$h_{2,n}(1/2) = \frac{4}{\sqrt{5}}\left[\left\{\frac{1}{\sqrt{5}-1}\right\}^{n+2} - \left\{\frac{-1}{\sqrt{5}+1}\right\}^{n+2}\right].$$

Birnbaum Structural Importance

We can use Birnbaum measure of structural importance when component reliabilities are not known to determine importance of components. This measure can be expressed in terms of the number of swings. Let $\beta_2(i,n)$ represent the Birnbaum measure of structural importance of component i in a consecutive-2-out-of-n:F system.

Theorem 26 *We have for a consecutive-2-out-of-n:F system*

$$(i) \qquad \beta_2(i,n) = \frac{S_2(i,n)}{2^{n-1}}$$

$$(ii) \qquad \beta_2(i,n) = \frac{[2F_{i-1}F_{n-i} - F_n]}{2^{n-1}} = \frac{[F_n - 2F_{i-2}F_{n-i-1}]}{2^{n-1}}$$

$$(iii) \qquad \beta_2(i,n) = \frac{\beta_2(i,n-1)}{2} + \frac{\beta_2(i,n-2)}{2^2} \quad \textit{for } n > 2.$$

Proof :

(i) By definition, we have

$$\beta_2(i,n) = \sum_{\mathbf{J}}[\Phi_{2,n}(1_i,\mathbf{x}) - \Phi_{2,n}(0_i,\mathbf{x})]\frac{1}{2^{n-1}}$$

using Theorem 6 we have

$$= \frac{S_2(i,n)}{2^{n-1}}$$

where $\mathbf{J} = \{(x_1,\ldots.x_{i-1},x_{i+1},\ldots,x_n) : x_i = 0 \text{ or } 1\}$.

(ii) This follows from part (i) and Theorem 6 since we have

$$\begin{aligned} S_2(i,n) &= 2F_{i-1}F_{n-i} - F_n \\ &= F_n - 2F_{i-2}F_{n-i-1}. \end{aligned}$$

(iii) Since we know that $S_2(i,n)$ for a fixed i, $1 \le i \le n$ and $n \ge 3$ is a Fibonacci sequence, it follows that

$$S_2(i,n) = S_2(i,n-1) + S_2(i,n-2), \text{ hence}$$
$$\beta_2(i,n) = \frac{1}{2}\beta_2(i,n-1) + \frac{1}{2^2}\beta_2(i,n-2).$$

Theorem 27 *For components $i, j \in N$ $(j \neq i)$ of a consecutive-2-out-of-n:F system, we have*

$$\begin{aligned} &(i) & \tfrac{1}{2}[\beta_2(i,n) - \beta_2(j,n)] &= h_{2,n}(1_i, 1/2) - h_{2,n}(1_j, 1/2) \\ &(ii) & &= h_{2,n}(0_j, 1/2) - h_{2,n}(0_i, 1/2) \\ &(iii) & &= h^D_{2,n}(1_i, 1/2) - h^D_{2,n}(1_j, 1/2) \\ &(iv) & &= h^D_{2,n}(0_j, 1/2) - h^D_{2,n}(0_i, 1/2). \end{aligned}$$

where $h^D_{2,n}(p)$ denotes the reliability function of the dual of a consecutive -2-out-of-n:F system.

Proof: Since we know that $\beta_2(i,n) = \beta_{2,1/2}(i,n)$ (from Chapter 1) and from Theorem 23, we have

$$\begin{aligned} \beta_2(i,n) &= \frac{1}{1/2}[h_{2,n}(1_i, 1/2) - h_{2,n}(1/2)] \\ &= \frac{1}{1/2}[h_{2,n}(1/2) - h_{2,n}(0_i, 1/2)] \end{aligned}$$

(i) and (ii) follow immediately for this.

(iii) and (iv) follow from the fact that a component i has the same relative importance in a structure and in its dual structure or directly from the definition of the dual structure.

Theorem 28 *For $n \geq 3$ we have*

$$\beta_2(i,n) = \frac{\beta_2(i-1,n-1)}{2} + \frac{\beta_2(i-2,n-2)}{2^2} \quad \textit{for } 3 \leq i \leq n.$$

Proof : It follows from Theorems 10 and 26 for $i > 2$.

Theorem 29 *Let m be the smallest integer greater than or equal to $n/2$, we then have*

$$\beta_2(i,n) - \beta_2(i+1,n) = \frac{(-1)^i}{2^{2i}}\left[h_{2,n-2i-2}(1/2)\right] \textit{ for } i < m.$$

Proof : From Theorems 17 and 26, we have for $i < m$

$$\begin{aligned}
\beta_2(i,n) - \beta_2(i+1,n) &= \frac{(-1)^i\, 2S_2(1,n-2i+1)}{2^{n-1}} \\
&= (-1)^i \frac{\beta_2(1,n-2i+1)}{2^{2(i-1)}} \\
&\quad \text{using Theorems 23 and 24} \\
&= \frac{(-1)^i}{2^{2i-3}}\left[h_{2,n-2i}(1/2) - h_{2,n-2i+1}(1/2)\right] \\
&= \frac{(-1)^i}{2^{2i}} h_{2,n-2i-2}(1/2).
\end{aligned}$$

Remark : Feller [22] has given a very good approximation for the reliability function, $h_{k,n}(p)$ which is quite appropriate even for small n. Using this, we can write

$$\beta_2(i,n) - \beta_2(i+1,n) \approx \frac{(-1)^i}{2^{2i} x^{n-2i-1}} \left[\frac{1-0.5x}{1.5-x}\right]$$

where $x = 1.236067$(a root of the equation $x^3 - 8x + 8 = 0$)

Theorem 30 *Let n_2 and n_1 be the largest even and odd integers smaller than or equal to $(n+1)/2$, respectively. We then have*

$$\begin{array}{rrcl} (i) & \beta_2(2,n) & > & \beta_2(4,n) > \cdots > \beta_2(n_2,n) \text{ for } n \geq 7 \\ (ii) & \beta_2(1,n) & < & \beta_2(3,n) < \cdots < \beta_2(n_1,n) \text{ for } n \geq 5 \\ (iii) & \beta_2(2t,n) & - & \beta_2(2t-1,n) > 0 \text{ for } t \geq 1, 2t \leq n_2 \text{ and } n \geq 3 \\ (iv) & \beta_2(2t,n) & - & \beta_2(2t+1,n) > 0 \text{ for } 2t+1 \leq n_1 \text{ and } n \geq 5 \\ (v) & \beta_2(i,n) & = & \beta_2(n+1-i,n). \end{array}$$

Proof: These results follow from Theorems 20 or 22 and 23 and the mirror image property.

This theorem shows that component 2 (or its mirror image component) has the maximum Birnbaum structural importance and component 1 (or its mirror image component) the minimum Birnbaum structural importance in a consecutive-2-out-of-n:F system whereas Papastavridis [39] has incorrectly asserted that in a consecutive-k-out-of-n:F system the component nearer to the centre has more Birnbaum importance. It also provides a complete Birnbaum structural ranking of components.

Table 4.1 gives the Birnbaum measure of structural importance for components in a consecutive-2-out-n:F system for n=2 to 20.

4.4 Barlow-Proschan Structural Measure

So far we have studied the path sets and swings without any reference to the number of components in the path sets that is, ignoring the size of path sets and here we take this aspect into account and study the Barlow-Proschan structural importance measure.

Theorem 31 *Let $g_2(n, r)$ denote the number of path sets of a consecutive-2-out-of-n:F system with r failed components or equivalently $(n - r)$ components functioning, that is, path sets of size $(n - r)$. We then have*

$$g_2(n, r) = \binom{n - r + 1}{r}.$$

Proof: It is well known in combinatorics theory [6] and [7] that the number of ways in which s plus signs and t minus signs can be placed in a row such that no two minus signs are adjacent is, $\binom{s + 1}{t}$. Hence, it follows that $g_2(n, r) = \binom{n - r + 1}{r}$.

Remark : Since the number of path sets of a consecutive-2-out-of-n: F system follows the Fibonacci sequence $\{F_n\}$ with $F_0 = 1, F_1 = 2, F_2 = 3$ hence trivially, we have

$$F_n = \sum_{r=0}^{[(n+1)/2]} \binom{n - r + 1}{r}$$

where [x] denotes the largest integer not greater than x.

Theorem 32 *Let $L_n(x) = \sum_{r=0}^{\infty} x^r \binom{n-r+1}{r}$ be the generating function of $\binom{n-r+1}{r}$ we then have the recurrence relationship*

$$L_{n+2}(x) = L_{n+1}(x) + xL_n(x), \text{ for } n \geq 0.$$

Proof : Let $g_2(n,r)_1$= the number of path sets with r failed components and the n^{th} component working of a consecutive-2-out-of-n:F system and $g_2(n,r)_0$ = the number of path sets with r failed components and n^{th} component not working of a consecutive-2-out-of-n:F system. Obviously,

$$\begin{aligned} g_2(n+2,r) &= g_2(n+2,r)_1 + g_2(n+2,r)_0 \\ &= g_2(n+1,r) + g_2(n+1,r-1)_1 \\ &= g_2(n+1,r) + g_2(n,r-1). \end{aligned}$$

we have

$$\begin{aligned} L_{n+2}(x) &= \sum_{i=0}^{n+2} g_2(n+2,i)\, x^i, \text{ since } g_2(n+2,i)=0, \text{ for } i>n+2 \\ &= \sum_{i=0}^{n+2} [g_2(n+1,i)x^i + g_2(n,i-1)x^i] \\ &= \sum_{i=0}^{n+1} g_2(n+1,i)x^i + x\sum_{i=1}^{n+1} g_2(n,i-1)x^{i-1} \\ &= L_{n+1}(x) + xL_n(x) \text{ hence the result holds.} \end{aligned}$$

Remark : Obviously $L_0(x)=1$, $L_1(x)=1+x$. We define $L_{-1}(x)=1$ and $L_i(x)=0$ for $i\le -2$. We then have $L_n(x)=L_{n-1}(x)+xL_{n-2}(x)$ for $n\ge 0$.

Theorem 33 *We have for a consecutive-2-out-of-n:F system*

$$h_{2,n}(p) = p^n L_n(q/p)$$

where $h_{2,n}(p)$ denotes the reliability function of a consecutive-2-out-of-n:F system and p is the component reliability and it is assumed that all components have the same reliability and function independently and $q=1-p$.

Proof: Since $\binom{n-r+1}{r}$ gives the number of path sets with r failed components, we have

$$h_{2,n}(p) = \sum_{r=0}^{n} \binom{n-r+1}{r} p^{n-r} q^r$$

$$= p^n \sum_{r=0}^{n} \binom{n-r+1}{r} (q/p)^r = p^n L_n(q/p).$$

Theorem 34

$$L_n(x) = \frac{1}{\sqrt{1+4x}} \left[\left(\frac{2x}{\sqrt{1+4x}-1} \right)^{n+2} - \left(\frac{-2x}{\sqrt{1+4x}+1} \right)^{n+2} \right].$$

Proof: Let $H(y,x) = \sum_{n=0}^{\infty} L_n(x) y^n$ be the generating function of $L_n(x)$. Hence we have

$$\begin{array}{llllll} H(y,x) & = L_0(x) & + L_1(x)y & + L_2(x)y^2 & + L_3(x)y^3 & + \cdots \\ H(y,x)y & = & L_0(x)y & + L_1(x)y^2 & + L_2(x)y^3 & + \cdots \\ H(y,x)xy^2 & = & & L_0(x)xy^2 & + L_1(x)xy^3 & + \cdots \\ \hline \end{array}$$

Using Theorem 32, we have $H(y,x)[1-y-xy^2] = 1 + xy$

$$H(y,x) = \frac{1+xy}{1-y-xy^2}.$$

The two roots of the quadratic equation $xy^2 + y - 1 = 0$ are:

$$\lambda_1 = \frac{-1+\sqrt{1+4x}}{2x}, \quad \lambda_2 = \frac{-1-\sqrt{1+4x}}{2x}$$

and we have

$$\lambda_1 + \lambda_2 = -\frac{1}{x}, \quad \lambda_1 - \lambda_2 = \frac{\sqrt{1+4x}}{x},$$

and $\lambda_1 \lambda_2 = -\frac{1}{x}$.

Let $H(y,x) = \dfrac{1+xy}{1-y-xy^2} = \dfrac{-(1/x+y)}{y^2+y/x-1/x} = \dfrac{A}{\lambda_1 - y} + \dfrac{B}{\lambda_2 - y}$

this gives us

$$A(\lambda_2 - y) + B(\lambda_1 - y) = -(\frac{1}{x} + y)$$

$$A(\lambda_2 - \lambda_1) = -(\frac{1}{x} + \lambda_1) = \lambda_2 \text{ or } A = \lambda_2/(\lambda_2 - \lambda_1)$$

$$B(\lambda_1 - \lambda_2) = -(\frac{1}{x} + \lambda_2) = \lambda_1 \text{ or } B = \lambda_1/(\lambda_1 - \lambda_2)$$

$$H(y, x) = \frac{1}{\lambda_1 - \lambda_2}\left[\frac{-\lambda_2}{\lambda_1 - y} + \frac{\lambda_1}{\lambda_2 - y}\right]. \text{ Hence}$$

It follows [48]

$$L_n(x) = \frac{x}{\sqrt{1+4x}}\left[\frac{-\lambda_2}{\lambda_1^{n+1}} + \frac{\lambda_1}{\lambda_2^{n+1}}\right] = \frac{1}{\sqrt{1+4x}}\left[\frac{1}{\lambda_1^{n+2}} - \frac{1}{\lambda_2^{n+2}}\right] \text{ or}$$

$$L_n(x) = \frac{1}{\sqrt{1+4x}}\left[\left\{\frac{2x}{\sqrt{1+4x}-1}\right\}^{n+2} - \left\{\frac{-2x}{\sqrt{1+4x}+1}\right\}^{n+2}\right]$$

Remark: It follows from Theorems 33 and 34 that

$$h_{2,n}(1/2) = \frac{1}{\sqrt{5}}\left[\left\{\frac{2}{\sqrt{5}-1}\right\}^{n+2} - \left\{\frac{-2}{\sqrt{5}+1}\right\}^{n+2}\right](1/2)^n$$

Theorem 35 *Let $\Psi_2(i, n)$ be the Barlow-Proschan measure of structural importance of component i ($i = 1, 2 \ldots, n$) in a consecutive-2-out-of-n:F system. We then have*

$$\Psi_2(i, n) = a + \int_0^1 \frac{1}{q}\left[p^{n-1} L_{i-1}(q/p) L_{n-i}(q/p)\right] dp$$

where the constant a is chosen such that $\sum_{i=1}^n \Psi_2(i, n) = 1$.

Proof: It is well known (refer Chapter 1) or see Owen [38] and Barlow-Proschan [4] that

$$\Psi_2(i,n) = \int_0^1 [h_{2,n}(1_i,p) - h_{2,n}(0_i,p)]dp$$

and from Theorem 23 we know that

$$\beta_{2,p}(i,n) = h_{2,n}(1_i,p) - h_{2,n}(0_i,p) = [h_{2,n}(1_i,p) - h_{2,n}(p)]\frac{1}{1-p}.$$

By Theorem 24, we have $h_{2,n}(1_i,p) = h_{2,i-1}(p)h_{2,n-i}(p)$. Now using Theorem 33, we obtain

$$\begin{aligned}\Psi_2(i,n) &= \int_0^1 \frac{1}{q}\left[p^{i-1}L_{i-1}(q/p)L_{n-i}(q/p)p^{n-i} - p^n L_n(q/p)\right] dp \\ &= a + \int_0^1 \frac{1}{q}\left[p^{n-1}L_{i-1}(q/p)L_{n-i}(q/p)\right] dp\end{aligned}$$

Since $p^n/qL_n(q/p)$ does not depend on i and using the fact that $\sum_{i=1}^n \Psi_2(i,n) = 1$, we can determine the constant a.

Remark : Obviously $\Psi_2(i,n) = \Psi_2(n+1-i,n)$.

Example : Calculation of Barlow-Proschan structural importance.

We have

$L_0(x) = 1,$ $\quad L_1(x) = 1 + x,$ $\quad L_2(x) = 1 + 2x$

$L_3(x) = 1 + 3x + x^2$ $\quad L_4(x) = 1 + 4x + 3x^2$

$L_5(x) = 1 + 5x + 6x^2 + x^3,$ $\quad L_6(x) = 1 + 6x + 10x^2 + 4x^3$

Let the number of components in the system be five that is, $n = 5$. We then have

$$L_0(x)L_4(x) = 1+4x+3x^2$$

$$L_1(x)L_3(x) = 1+4x+4x^2+x^3$$

$$L_2(x)L_2(x) = 1+4x+4x^2$$

$$\text{Since } \Psi_2(i,n) = a + \int_0^1 1/q\left[p^{n-1}L_{i-1}(q/p)L_{n-i}(q/p)\right]dp$$

we obtain

$$\begin{aligned}
\Psi_2(1,5) &= a+\textstyle\int_0^1(4p^3+3p^2q)dp &&= a+\tfrac{5}{4}\\
\Psi_2(2,5) &= a+\textstyle\int_0^1(4p^3+4p^2q+pq^2)dp &&= a+1+\tfrac{1}{3}+\tfrac{1}{12}\\
\Psi_2(3,5) &= a+\textstyle\int_0^1(4p^3+4p^2q)dp &&= a+1+\tfrac{1}{3}.
\end{aligned}$$

As $\Psi_2(i,n) = \Psi_2(n+1-i,n)$, we can calculate a from $\sum_1^5 \Psi_2(i,n) = 1$. This gives us $a = -17/15$

$$\Psi_2(1,5) = \Psi_2(5,5) = \frac{7}{60}, \Psi_2(2,5) = \Psi_2(4,5) = \frac{17}{60} \text{ and } \Psi_2(3,5) = \frac{12}{60}.$$

Table 4.2 presents the Barlow-Proschan structural importance of components for $n = 2$ to 20.

In order to compare Barlow-Proschan structural importance of different components, we first study the nature and properties of the polynomial $L_{i-1}(x)L_{n-i}(x)$.

Theorem 36 *Let*

$$P_{i,n}(x) = L_{i-1}(x)L_{n-i}(x) = \sum_{j=1}^{d} a_{ij}x^j \text{ for } i = 1,2,\ldots,n$$

where d denotes the degree of the polynomial $P_{i,n}(x)$. We then have

(i) $d =$ $n/2$ *when n is even*

(ii) $d =$ $[n/2]$ *when i is odd and n is also odd*

(iii) $d =$ $[(n+1)/2]$ *when i is even and n is odd*

(iv) $a_{ij} =$ 1 *for j=0 and all i*

(v) $a_{ij} =$ $n-1$ *for j=1 and all i*

(vi) $a_{ij} =$ $a_{n+1-i,j}$ *all j and i*

where [x] denotes the largest integer less than or equal to x.

Proof : Since $L_n(x) = \sum_{r=0}^{m} \binom{n-r+1}{r} x^r$ where $m = [\frac{n+1}{2}]$, hence the degree of the polynomial $P_{i,n}(x) = L_{i-1}(x)L_{n-i}(x)$ is equal to

$$[\frac{i}{2}] + [\frac{n-i+1}{2}].$$

(i) When $n = 2t$ and $i = 2p$, it implies that

$$d = [\frac{i}{2}] + [\frac{2t-i+1}{2}] = p + t - p = t = n/2$$

When $n = 2t$ and $i = 2p - 1$, we then have

$$d = [\frac{2p-1}{2}] + [\frac{2t-(2p-1)+1}{2}] = p - 1 + t - p + 1 = t = n/2.$$

(ii) When $n = (2t-1)$ and $i = (2p-1)$, we have

$$d = [\frac{2p-1}{2}] + [\frac{2t-1-2p+1+1}{2}] = p - 1 + t - p = t - 1 = [n/2]$$

(iii) When $n = 2t - 1$ and $i = 2p$, we have

$$d = [\frac{2p}{2}] + [\frac{2t-1-2p+1}{2}] = p + t - p = [\frac{n+1}{2}].$$

(iv) Since $P_{i,n}(x) = L_{i-1}(x)L_{n-i}(x)$ we have from Theorem 31

$$P_{i,n}(x) = \sum_{j=0}^{d} a_{ij}x^j = \left\{\sum_{r=0}^{[i/2]} \binom{i-r}{r} x^r\right\} \left\{\sum_{r=0}^{[(n-i+1)/2]} \binom{n-i+1-r}{r} x^r\right\}.$$

It follows that

$$a_{ij} = \sum_{r=0}^{j} \binom{i-r}{r} \binom{n-i+1-(j-r)}{j-r}$$

and for j=0 , we have for $i = 1, 2, \ldots, n$

$$a_{i0} = \binom{i}{0}\binom{n-i+1}{0} = 1.$$

(v) For j=1, we have from part (iv) for $i = 1, 2, \ldots, n$

$$\begin{aligned} a_{i1} &= \sum_{r=0}^{1} \binom{i-r}{r}\binom{n-i+r}{1-r} = n-1 \text{ for a all i} \\ &= \binom{i}{0}\binom{n-i}{1} + \binom{i-1}{1}\binom{n-i+1}{0} \\ &= n-1 \text{ for all } i. \end{aligned}$$

(vi) Since $P_{i,n}(x) = P_{n-i+1,n}(x)$ hence the required result follows.

Theorem 37 *For integer* $i \geq 2$ *and* $i \leq \frac{n+1}{2}$ *we have*

$$L_{i-2}(x)L_{n-i-1}(x) - L_{i-1}(x)L_{n-i-2}(x) = (-1)^{i+1}x^i L_{n-2i-2}(x).$$

Proof: We have from Theorem 32 for integer $i \geq 2$ and $i \leq \frac{n+1}{2}$

$$L_{i-2}(x)L_{n-i-1}(x) - L_{i-1}(x)L_{n-i-2}(x) =$$
$$L_{i-2}(x)\left[L_{n-i-2}(x) + xL_{n-i-3}(x)\right] - \left[L_{i-2}(x) + xL_{i-3}(x)\right]L_{n-i-2}(x)$$
$$= x\left[L_{i-2}(x)L_{n-i-3}(x) - L_{i-3}(x)L_{n-i-2}(x)\right]$$
$$= x\left[\{L_{i-3}(x) + xL_{i-4}(x)\}L_{n-i-3}(x) - L_{i-3}(x)\{L_{n-i-3}(x) + xL_{n-i-4}(x)\}\right]$$
$$= x^2\left[L_{i-4}(x)L_{n-i-3}(x) - L_{i-3}(x)L_{n-i-4}(x)\right]$$
$$= x^3\left[L_{i-4}(x)L_{n-i-5}(x) - L_{i-5}(x)L_{n-i-4}(x)\right].$$ Repeated use of this gives

Case 1: When $i = 2t$ (even)$\leq (n+1)/2$, we have

$$L_{i-2}(x)L_{n-i-1}(x) - L_{i-1}(x)L_{n-i-2}(x)$$
$$= x^{2(t-1)}\left[L_{i-2-2(t-1)}(x)L_{n-i-1-2(t-1)}(x) - L_{i-1-2(t-1)}(x)L_{n-i-2-2(t-1)}(x)\right]$$
$$= x^{2(t-1)}\left[L_0(x)L_{n-2i+1}(x) - L_1(x)L_{n-2i}(x)\right]$$
$$= x^{2(t-1)}\left[L_{n-2i+1}(x) - L_{n-2i}(x) - xL_{n-2i}(x)\right]$$
$$= x^{2(t-1)}\left[xL_{n-2i-1}(x) - xL_{n-2i}(x)\right]$$
$$= x^{2t-1}\left[L_{n-2i-1}(x) - L_{n-2i}(x)\right] = x^{i-1}\left[-xL_{n-2i-2}(x)\right]$$
$$= -x^{i}L_{n-2i-2}(x).$$

Case 2 : When $i = 2t+1$ (odd) $\leq (n+1)/2$, we have

$$L_{i-2}(x)L_{n-i-1}(x) - L_{i-1}(x)L_{n-i-2}(x)$$
$$= x^3\left[L_{i-4}(x)L_{n-i-5}(x) - L_{i-5}(x)L_{n-i-4}(x)\right]$$
$$= x^{1+2(t-1)}\left[L_{i-2-2(t-1)}(x)L_{n-i-3-2(t-1)}(x) - L_{i-3-2(t-1)}(x)L_{n-i-2-2(t-1)}(x)\right]$$
$$= x^{2t-1}\left[(1+x)L_{n-2i}(x) - L_0(x)L_{n-2i+1}(x)\right]$$

$$= x^{2t-1}\left[L_{n-2i}(x) - L_{n-2i+1}(x) + xL_{n-2i}(x)\right]$$

$$= x^{2t-1}\left[-xL_{n-2i-1}(x) + xL_{n-2i}(x)\right] = x^{2t}\left[L_{n-2i}(x) - L_{n-2i-1}(x)\right]$$

$$= x^{2t+1}L_{n-2i-2}(x) = x^{i}L_{n-2i-2}(x).$$

Theorem 38 *We have*

$$a_{ij} = a_{jj} \text{ for } i \geq j \geq 2 \text{ and integer } i \leq (n+1)/2.$$

Proof: Since we have

$$\begin{aligned} L_i(x)L_{n-i-1}(x) &= \left[L_{i-1}(x) + xL_{i-2}(x)\right]\left[L_{n-i}(x) - xL_{n-1-2}(x)\right] \\ &= L_{i-1}(x)L_{n-i}(x) + \\ &\quad x\left[L_{i-2}(x)L_{n-i-1}(x) - L_{i-1}(x)L_{n-i-2}(x)\right] \end{aligned}$$

and using the Theorem 37, we have

$$L_i(x)L_{n-i-1}(x) = L_{i-1}(x)L_{n-i}(x) + (-1)^{i+1}x^{i+1}L_{n-2i-2}(x).$$

This implies that

$$P_{i+1,n}(x) = P_{i,n}(x) + (-x)^{i+1}L_{n-2i-2}(x).$$

Hence it follows that $a_{i+1,j} = a_{i,j}$ for $j \leq i$ and $i + 1 \leq (n+1)/2$.

Remark: It follows from the above theorem that for $(n+1)/2 \geq i \geq j \geq 2$ and i and j integers

$$\sum_{r=0}^{j} \binom{i-r}{r}\binom{n-i+1-(j-r)}{j-r} = \sum_{r=0}^{j} \binom{j-r}{r}\binom{n-j+1-(j-r)}{j-r}.$$

Theorem 39 *Let $\Psi_2(i,n)$ be the Barlow-Proschan measure of structural importance for component i ($i = 1, 2, \ldots, n$) in a consecutive-2-out-of-n:F system. We then have*

(i) $\Psi_2(i,n) = \Psi_2(n+1-i,n)$ *for all* $i = 1, 2, \ldots, n$

(ii) $\Psi_2(2t,n) > \Psi_2(2t-1,n)$ *for* $t \geq 1, 2t \leq n_2$ *and* $n \geq 3$

(iii) $\Psi_2(2t,n) > \Psi_2(2t+1,n)$ *for* $2t+1 \leq n_1$ *and* $n \geq 5$

(iv) $\Psi_2(2,n) > \Psi_2(4,n) > \cdots > \Psi_2(n_2,n)$ *for* $n \geq 7$

(v) $\Psi_2(1,n) < \Psi_2(3,n) < \cdots < \Psi_2(n_1,n) < \Psi_2(n_2,n)$ *for* $n \geq 5$

where n_2 (n_1) is the largest even (odd) number less than or equal to $(n+1)/2$.

Proof:

(i) We know from Theorem 35 that

$$\Psi_2(i,n) = a + \int_0^1 \frac{1}{q}\left[p^{n-1} L_{i-1}(q/p) L_{n-i}(q/p)\right] dp$$

and since $P_{i,n}(x) = P_{n+1-i,n}(x)$, the required result follows.

(ii)& (iii) Using Theorem 35, we get

$$\Psi_2(i+1,n) - \Psi_2(i,n) = \int_0^1 \frac{p^{n-1}}{q}\left[L_i(q/p)L_{n-i-1}(q/p) - L_{i-1}(q/p)L_{n-i}(q/p)\right] dp$$

and according to Theorem 38, we have for integer i such that $i+1 \leq (n+1)/2$

$$L_i(q/p)L_{n-i-1}(q/p) - L_{i-1}(q/p)L_{n-i}(q/p) = (-1)^{i+1}(q/p)^{i+1}L_{n-2i-2}(q/p).$$

Hence it follows that for $i+1 \leq (n+1)/2$, we have

$$\Psi_2(i+1,n) - \Psi_2(i,n) = \begin{cases} > 0 & \text{for odd } i \\ < 0 & \text{for even } i. \end{cases}$$

(iv) & **(v)** We have from Theorem 38 for $i+1 \leq (n+1)/2$

$$\begin{aligned} L_i(x)L_{n-i-1}(x) - L_{i-1}(x)L_{n-i}(x) &= (-x)^{i+1}L_{n-2i-2}(x) \\ L_{i-1}(x)L_{n-i}(x) - L_{i-2}(x)L_{n-i+1}(x) &= (-x)^i L_{n-2i}(x). \end{aligned}$$

And adding these two equations, we get

$$L_i(x)L_{n-i-1}(x) - L_{i-2}(x)L_{n-i+1}(x) = (-x)^i L_{n-2i-1}(x).$$

Hence for $2 \leq$ and $i+1 \leq (n+1)/2$

$$\Psi_2(i+1,n) - \Psi_2(i-1,n) = \begin{cases} > 0 & \text{for even } i \\ < 0 & \text{for odd } i. \end{cases}$$

since $x^i L_{n-2i-1}(x) > 0$ if $x = q/p$ and $p \in (0,1)$

It follows that in a consecutive-2-out-of-n:F system, Barlow-Proschan structural ranking is identical to Birnbaum structural ranking.

4.5 Cut Importance Ranking

We consider here, Butler's structural ranking of component in a consecutive-2-out-of-n:F system. This ranking is based on minimal cut sets and provides a complete ordering of all components relative to their importance. It is applicable when the component reliability is high. This ranking is the same as lexicographic ordering of the rows of the structural matrix of the dual of a given structure (see Theorem 9 of Chapter 1)

When component i is more (less) cut important than component j in a consecutive-2-out-of-n:F system, we represent this fact by $i >_c j$ ($i <_c j$) and if both are equally cut important by $i =_c j$. The next theorem provides a complete structural cut importance ranking of all components.

Theorem 40 *In a consecutive-2-out-of-n:F system, we have*

$$\begin{array}{ll} (i) & i =_c (n-i+1) \text{ for } n \geq 2 \\ (ii) & 2 >_c 4 >_c 6 >_c \cdots >_c n_2 \text{ for } n \geq 7 \\ (iii) & 1 <_c 3 <_c 5 <_c \cdots <_c n_1 \text{ for } n \geq 5 \\ (iv) & r >_c r-1 \text{ for even } r \leq n_2 \text{ and } n \geq 3 \\ (v) & r >_c r+1 \text{ for even } r \text{ and } r+1 \leq n_1 \end{array}$$

where n_2 (n_1) denotes the largest even (odd) number less than or equal to $(n+1)/2$.

Proof : Since the cut-importance ranking is same as Birnbaum reliability importance ranking for a sufficiently high value of component reliability (see Theorem 16 of Chapter 1) and we know that Birnbaum reliability importance of component i in a consecutive-2-out-of-n:F system with i.i.d. components is given by Theorem 23

$$\begin{aligned} \beta_{2,p}(i,n) &= \frac{1}{(1-p)}\left[h_{2,n}(1_i,p) - h_{2,n}(p)\right] \\ &= \frac{1}{(1-p)}\left[p^{n-1}L_{i-1}(q/p)L_{n-i}(q/p) - p^n L_n(q/p)\right] \end{aligned}$$

from Theorems 24 and 33. Hence the required results follow on the same lines as Theorem 39.

We now summarise the main results of this Chapter.

Theorem 41 *For a consecutive-2-out-of-n:F system the rankings of components provided by the following are identical*

(1) *Birnbaum structural importance measure*

(2) *Barlow-Proschan structural importance measure*

(3) *Butler's cut importance ranking*

(4) *Birnbaum reliability importance measure in case of i.i.d. components.*

Proof: It follows from Theorems 23, 30, 33, 39 and 40.

Chapter 5

Component Importance in a Consecutive-3-out-of-n:F System

5.1 Introduction

This chapter is devoted to the study of structural importance of components in a consecutive-3-out-of-n:F system. Section 2 highlights the properties of path sets and establishes their relationships with Tribonacci sequences. Section 3 studies the properties of swings and the difference in swings between two consecutive components in the system and develops conditions under which component i will have more swings than component $i+1$. Section 4 is concerned with Birnbaum structural importance and gives a general formula for calculating number of path sets. It also gives necessary and sufficient conditions for a component i to have more power than component j. It is proved that component 3 has the maximum and component 1 the minimum Birnbaum structural importance. We also give a heuristic procedure for ranking of components based upon proportionate error which matches with the Birnbaum structural ranking under certain assumptions. In Section 5, we analyse Barlow-Proshan structural importance measure and provide a procedure for its calculation. It shown that in this case also, component 3 has the maximum and component 1 the minimum Barlow-Proschan structural importance. Butler's cut importance ranking is covered in Section 6. Using the structural matrix, we provide the complete cut importance ranking of components for n up to 20. We also cover Birnbaum reliability importance measure in this Section. In the last Section we mention results of this chapter which can be easily generalised to any $k \geq 4$.

5.2 Path Sets

Let $\Phi_{3,n}(x)$ represent the structure function of a consecutive-3-out-of-n:F system. Suppose that $P_3(n)$ denotes the number of path sets of the system and let $P_3(1_i, n)$ and $P_3(0_i, n)$ represent the number of path sets with the i^{th} component working and not working of a consecutive-3-of-out-n:F system, respectively.

Theorem 1 *We have*

$$P_3(n) = P_3(1_i, n) + P_3(0_i, n).$$

Proof: Same as that of Theorem 1 of Chapter 4.

Tribonacci Sequence

We now define a Tribonacci sequence [23] and show that $P_3(n)$ is a Tribonacci sequence.

Definition : A sequence of integers $\{f_n\}$ determined by the difference equation

$$f_n = f_{n-1} + f_{n-2} + f_{n-3}, \text{ for } n > 3$$

with the initial conditions $f_1 = \alpha, f_2 = \beta$ and $f_3 = \gamma$ is called a Tribonacci sequence or a Fibonacci sequence of order 3 [11] and [49]. We consider the Tribonacci sequence $\{T_n\}$ with $T_1 = 2, T_2 = 4$ and $T_3 = 7$.

Theorem 2 *The number of path sets, $P_3(n)$ of a consecutive-3-out-of-n: F system follow the Tribonacci sequence $\{T_n\}$.*

Proof : Let the number of path vectors with the first component working be $P_3(1_1, n)$, the number of path vectors with the first component not working and the second component working be $P_3(0_1, 1_2, n)$ and the number of path vectors with the first and the second component not working but the third component working be $P_3(0_1, 0_2, 1_3, n)$. Then obviously, we have for $n > 3$

$$\begin{aligned} P_3(n) &= P_3(1_1, n) + P_3(0_1, 1_2, n) + P_3(0_1, 0_2, 1_3, n) \\ &= P_3(n-1) + P_3(n-2) + P_3(n-3). \end{aligned}$$

Trivially, we have $P_3(1) = 2, P_3(2) = 4, P_3(3) = 7$. Hence $P_3(n) = T_n$

Without loss of generality, we extend the Tribonacci sequence $\{T_n\}$ by adding the terms $T_{-2} = 0$, $T_{-1} = 1$, and $T_0 = 1$. We now use the Tribonacci sequence $\{T_n\}$ to determine $P_3(1_i, n), P_3(0_i, n)$ and the number of swings.

Theorem 3 *For* $i \geq 1$, *we have*

(i) $P_3(1_i, n) = T_{i-1}T_{n-i}$.

(ii) $P_3(0_i, n) = T_{i-2}T_{n-i-1} + T_{i-3}T_{n-i-1} + T_{i-2}T_{n-i-2}$.

Proof:

(i) The proof follows on the same lines as Theorem 4 of Chapter 4.

(ii) Consider a vector of component states $(x_1, x_2, \ldots, x_n)$ with $x_i = 0$. This vector can be a path vector of the system if and only if

(a) $(x_1, \ldots, x_{i-1})$ is a path vector with $x_{i-1} = 1$ of a consecutive-3-out-of-(i-1):F subsystem made-up of components $\{1, \ldots, i-1\}$ and $(x_{i+1}, \ldots, x_n)$ is a path vector, with $x_{i+1} = 1$, of a consecutive-3-out-of-(n-i):F subsystem made-up of components $\{i+1, \ldots, n\}$ of the original system. Hence the number of path vectors of a consecutive-3-out-of-n:F system of the type $x_{i+1} = 1, x_i = 0$ and $x_{i-1} = 1$ equals $T_{i-2}T_{n-i-1}$.

Or

(b) $(x_1, \ldots, x_{i-1})$ with $x_{i-1} = 0$ and $x_{i-2} = 1$ is a path vector of a consecutive-3-out-of-(i-1):F subsystem and $(x_{i+1}, \ldots, x_n)$ with $x_{i+1} = 1$ is a path vector of a consecutive-3-out-of-(n-i):F subsystem, respectively. Hence there are $T_{i-3}T_{n-i-1}$ path vectors of a consecutive-3-out-of-n:F system with $x_{i-2} = 1, x_{i-1} = 0, x_1 = 0$ and $x_{i+1} = 1$.

Or

(c) $(x_1, \ldots, x_{i-1})$ with $x_{i-1} = 1$ is a path vector of a consecutive-3-out-of-(i-1):F subsystem, and $(x_{i+1}, \ldots x_n)$ with $x_{i+1} = 0$ and $x_{i+2} = 1$ is a path vector of a consecutive-3-out-of- (n-i):F subsystem. Hence a consecutive 3-out-of-n:F system has $T_{i-2}T_{n-i-2}$ path vectors with $x_{i-1} = 1, x_i = 0, x_{i+1} = 0, x_{i+2} = 1$.

It follows from the above that $P_3(0_i, n) = T_{i-2}T_{n-i-1} + T_{i-3}T_{n-i-1} + T_{i-2}T_{n-i-2}$.

Remark : Obviously, we have for $1 \le i \le n$

$$T_n = T_{i-1}\,T_{n-i} + T_{i-2}\,T_{n-i-1} + T_{i-3}\,T_{n-i-1} + T_{i-2}\,T_{n-i-2}.$$

5.3 Swings

Let $S_3(i, n)$ denote the number of swings of component $i \in N$ in a consecutive-3-out-of-n:F system. As in the case of a consecutive-2-out-of-n:F system, we have

Theorem 4 $S_3(i, n) = 2T_{i-1}T_{n-i} - T_n$.

Proof : We know that

$$S_3(i, n) = \sum_{\mathbf{x} \in \mathcal{B}^n} [(1 - x_i)][\Phi_{3,n}(1_i, \mathbf{x}) - \Phi_{3,n}(0_i, \mathbf{x})]$$

(by pivotal decomposition theorem)

$$
\begin{aligned}
&= \sum_{\mathbf{x}\in\mathcal{B}^n} [\Phi_{3,n}(1_i,\mathbf{x}) - \Phi_{3,n}(\mathbf{x})] \\
&= 2P_3(1_i,n) - P_3(n) \\
&= 2T_{i-1}T_{n-i} - T_n.
\end{aligned}
$$

It trivially, follows from the above theorem that component i and $n-i+1$ have the same number of swings. Hence the mirror image property holds in this case also.

Theorem 5 *For $n \geq i$ and $i \geq 1$, $S_3(i,n)$ is a Tribonacci sequence with the initial conditions $S_3(1,i), S_3(2,i+1)$ and $S_3(3,i+2)$.*

Theorem 6 *For any $t \geq 0$, $S_3(n,n+t)$ follows a Tribonacci sequence with the initial conditions $S_3(1,1+t), S_3(2,2+t)$ and $S_3(3,3+t)$.*

Theorem 7 *For $i > j \geq 1$, and $n \geq i$, $S_3(i,n) - S_3(j,n)$ is a Tribonacci sequence.*

Proof : Theorems 5, 6 and 7 follow from the fact that a linear integer combination of two or more Tribonacci sequences is also a Tribonacci sequence.

Theorem 8 *For $n \geq i$ and $i > 3$, we have*

$$S_3(i,n) = S_3(i-1,n-1) + S_3(i-2,n-2) + S_3(i-3,n-3).$$

Proof : Since $S_3(i,n) = 2T_{i-1}T_{n-i} - T_n$, $i \in N$ and using the fact that $\{T_n\}$ is the Tribonacci sequence, the required result implies immediately.

Remark : Theorem 8 also follows from Theorem 6.

We now consider the swings for component i and $(i+1)$ and show how this difference in the swing can be calculated using Tribonacci sequence. We also give the conditions under which component i has more swings than component $(i+1)$. To do this, we first define the difference matrix D.

Definition : We define the difference matrix D as

$$D = ((\mathrm{d}_{ij})) \text{ where } \mathrm{d}_{ij} = S_3(i, i+j) - S_3(i+1, i+j)$$

for $j \geq 1$ and $i \geq 1$.
The elements of the matrix can be obtained using the theorem stated next.

Theorem 9 *The elements of the difference matrix have the following properties :*

(i)	$d_{i,i+1}$	$= 2[T_{i-1}T_{i+1} - T_iT_i]$
(ii)	$d_{i-1,i}$	$= 2[T_{i-2}T_i - T_{i-1}T_{i-1}]$
(iii)	d_{ji}	$= -d_{ij}$
(iv)	d_{ii}	$= 0$
(v)	$d_{i+3,j}$	$= d_{i+2,j} + d_{i+1,j} + d_{ij}$
(vi)	$d_{i,j+3}$	$= d_{i,j+2} + d_{i,j+1} + d_{ij}$
(vii)	d_{1j}	$= -2T_{j-4}$
$(viii)$	d_{2j}	$= -4T_{j-4}$
(ix)	d_{3j}	$= -2[-4T_{j-4} + T_{j-1}]$.

Proof:

(i) By definition, we have

$$d_{i,i+1} = S_3(i, 2i+1) - S_3(i+1, 2i+1) = 2[T_{i-1}T_{i+1} - T_iT_i].$$

(ii) It easily follows from **part** (i).

(iii) We have by definition

$$\begin{aligned} d_{ji} &= S_3(j, i+j) - S_3(j+1, i+j) \\ &= S_3(i+1, i+j) - S_3(i, i+j) \text{ (due to mirror-image property)} \\ &= -d_{ij}. \end{aligned}$$

(iv) It follows from **part** (iii) that $d_{ii} = 0$.

(v) By definition, we have

$$\begin{aligned} d_{i+2,j} + d_{i+1,j} + d_{ij} &= S_3(i+2, i+j+2) - S_3(i+3, i+j+2) \\ &\quad + S_3(i+1, i+j+1) - S_3(i+2, i+j+1) \\ &\quad + S_3(i, i+j) - S_3(i+1, i+j) \end{aligned}$$

Using Theorem 8 we get

$$\begin{aligned} d_{i+2,j} + d_{i+1,j} + d &= S_3(i+3, i+j+3) - S_3(i+4, i+j+3) \\ &= d_{i+3,j}. \end{aligned}$$

(vi) We have

$$\begin{aligned} d_{i,j+2} + d_{i,j+1} + d_{ij} &= S_3(i, i+j+2) - S_3(i+1, i+j+2) \\ &\quad + S_3(i, i+j+1) - S_3(i+1, i+j+1) \\ &\quad + S_3(i, i+j) - S_3(i+1, i+j) \\ &= S_3(i, i+j+3) - S_3(i+1, i+j+3) \\ &= d_{i,j+3}. \end{aligned}$$

(vii) Since

$$\begin{aligned} d_{1j} &= S_3(1, j+1) - S_3(2, j+1) \text{ and using Theorem 4} \\ &= 2[T_0 T_j - T_1 T_{j-1}] = -2T_{j-4}. \end{aligned}$$

(viii) We have by definition

$$\begin{aligned} d_{2j} &= S_3(2, j+2) - S_3(3, j+2) \\ &= 2[T_1 T_j - T_2 T_{j-1}] \\ &= 2[2T_j - 4T_{j-1}] \\ &= -4T_{j-4}. \end{aligned}$$

(ix) We have by definition

$$\begin{aligned} d_{3j} &= S_3(3, j+3) - S_3(4, j+3) \\ &= 2[T_2 T_j - T_3 T_{j-1}] = 2[4T_j - 7T_{j-1}] \\ &\qquad \text{since } \{T_n\} \text{ is a Tribonacci sequence} \\ &= 2[-4T_{j-4} + T_{j-1}]. \end{aligned}$$

Remark : The above theorem implies that diagonal elements of the matrix, D are zero and its row and column elements form a Tribonacci sequence with different initial conditions. The complete matrix can be obtained from the following generator matrix

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -4 \\ 2 & 4 & 0 \end{bmatrix}$$

From this, we can easily write the $D_{7\times 8}$ and it is given by:

$$D = \begin{bmatrix} 0 & 0 & -2 & -2 & -4 & -8 & -14 & -26 \\ 0 & 0 & -4 & -4 & -8 & -16 & -28 & -52 \\ 2 & 4 & 0 & 6 & 10 & 16 & 32 & 58 \\ 2 & 4 & -6 & 0 & -2 & -8 & -10 & -20 \\ 4 & 8 & -10 & 2 & 0 & -8 & 6 & -14 \\ 8 & 16 & -16 & 8 & 8 & 0 & 16 & 24 \\ 14 & 28 & -32 & 10 & 6 & -16 & 0 & -10 \end{bmatrix}$$

Here, each row and column represents a Tribonacci sequence with initial conditions given by the first three elements of a row or a column.

Theorem 10 *We have*

(*i*) $d_{i+2,j+1} = -(d_{ij} + d_{i+1,j} + d_{i+1,j-1})$ *for* $j \geq 1$

(*ii*) $\sum_{r=0}^{i-1} d_{i-r,1+r} = 0$ *for* $i \geq 1$

Proof:

(i) It follows from the definition of d_{ij} that

$$d_{i+2,j+1} = -[S_3(i+3, i+j+3) - S_3(i+2, i+j+3)]$$

and using Theorems 5 and 8, that is

$$\begin{aligned}
d_{i+2,j+1} &= -[S_3(i+2,i+j+2)+S_3(i+1,i+j+1)+S_3(i,i+j) \\
&\quad -S_3(i+2,i+j+2)+S_3(i+2,i+j+1)+S_3(i+2,i+j)] \\
&= -[S_3(i,i+j)+S_3(i+1,i+j+1)-S_3(i+2,i+j+1) \\
&\quad -S_3(i+2,i+j)] \\
&= -[S_3(i,i+j)-S_3(i+1,i+j)+S_3(i+1,i+j+1) \\
&\quad -S_3(i+2,i+j+1)+S_3(i+1,i+j)-S_3(i+2,i+j)\,] \\
&= -[d_{ij}+d_{i+1,j}+d_{i+1,j-1}].
\end{aligned}$$

(ii) $\sum_{r=0}^{i-1} d_{i-r,1+r} = 0$ since $d_{ij} = -d_{ji}$ according to Theorem 9.

Theorem 11 *For $t \geq 0$, we have the result*

$$d_{i,i+t} = d_{i+1,i+t+1} + d_{i+2,i+t+2} + d_{i+3,i+t+3}.$$

Proof: It follows from the definition of d_{ij} and Theorem 8 that

$$\begin{aligned}
d_{i+3,i+t+3} &= S_3(i+3,2i+t+6)-S_3(i+4,2i+t+6) \\
&= S_3(i+3,2i+t+5)+S_3(i+3,2i+t+4)+S_3(i+3,2i+t+3) \\
&\quad -[S_3(i+3,2i+t+5)+S_3(i+2,2i+t+4)+S_3(i+1,2i+t+3)]
\end{aligned}$$

and

$$d_{i+2,i+t+2} = S_3(i+2,2i+t+4)-S_3(i+3,2t+t+4). \text{ Hence}$$

$$d_{i+2,i+t+2}+d_{i+3,i+t+3} = S_3(i+3,2i+t+3)-S_3(i+1,2i+t+3)$$

using Theorem 8, we get

$$\begin{aligned} &= S_3(i+2,2i+t+2)+S_3(i+1,2i+t+1) \\ &\quad +S_3(i,2i+t)-S_3(i+1,2i+t+2) \\ &\quad -S_3(i+1,2i+t+1)-S_3(i+1,2i+t) \end{aligned}$$

Since $d_{i+1,i+t+1} = S_3(i+1,2i+t+2) - S_3(i+2,2i+t+2)$, this implies that

$$\begin{aligned} d_{i+1,i+t+1} + d_{i+2,i+t+2} + d_{i+3,i+t+3} &= S_3(i,2i+t) - S_3(i+1,2i+t) \\ &= d_{i,i+t}. \end{aligned}$$

Hence the theorem follows.

Remark : The above theorem implies that each upper diagonal of the difference matrix represents a Tribonacci sequence in the backward direction, that is a sequence of the type $\{t_n\}$ where $t_n = t_{n+1} + t_{n+2} + t_{n+3}$.

Theorem 12 *Let* $G_i = d_{i,i+1}$, *we then have for* $j \geq 0$

$$d_{i,i+j} = T_{j-2}G_i - T_{j-3}G_{i-1}$$

where $\{T_n\}$ *is the extended Tribonacci sequence.*

Proof : Since we have

$$G_i = d_{i,i+1} = S_3(i,2i+1) - S_3(i+1,2i+1) = 2\left[T_{i-1}T_{i+1} - T_i^2\right]$$

and as per Theorem 9 $d_{ii} = 0$, and $d_{i,i-1} = -d_{i-1,i} = -G_{i-1}$. It follows from Theorem 9 (iv) and (vi) that

$$d_{i,i+2} = d_{i,i+1} + d_{ii} + d_{i,i-1} = -G_{i-1} + G_i.$$

Hence the result is true for $j = 0, 1$ and 2 as $T_{-3} = 0, T_{-2} = 0, T_{-1} = 1$ and $T_0 = 1$. Assume that result is true for $j \leq t$ and consider the case when $j = t+1$. We have from Theorem 9 (vi)

$$\begin{aligned} d_{i,i+t+1} &= d_{i,i+t} + d_{i,i+t-1} + d_{i,i+t-2} \\ &\quad \text{since the hypothesis is true for } j \leq t \\ &= T_{t-2}G_i - T_{t-3}G_{i-1} + T_{t-3}G_i - T_{t-4}G_{i-1} \\ &\quad + T_{t-4}G_i - T_{t-5}G_{i-1} \\ &= G_i(T_{t-2} + T_{t-3} + T_{t-4}) - G_{i-1}(T_{t-3} + T_{t-4} + T_{t-5}) \\ &= G_iT_{t-1} - G_{i-1}T_{t-2}. \end{aligned}$$

Since $\{T_n\}$ is the Tribonacci sequence, the result is true for $j = t+1$. Hence the theorem follows.

Theorem 13 *We have*

(i) $d_{ij} = 0$, *for* $j \geq 1$ *if* $G_i = G_{i-1} = 0$

(ii) $dij > 0$ *for* $j \geq i+2$, *if* $G_{i-1} \leq 0$ *and* $G_i \geq 0$ *but not both equal to 0*

(iii) $d_{ij} > 0$ *for* $j \geq i+2$, *if* $G_i \geq G_{i-1} \geq 0$ *but both not zero.*

Proof : This follows trivially from Theorem 12

We now prove that component 3 or its mirror image component has the maximum number of swings and component 1 or n has the minimum of swings.

Theorem 14 *We have for a consecutive-3-out-of-n:F system*

(i) $S_3(1,n) < S_3(i,n)$ *for* $2 \le i \le n-1$, *and* $n > 3$.

(ii) $S_3(3,n) > S_3(i,n)$ *for* $i \neq 3,\ n-2,\ 1 \le i \le n$ *and* $n > 3$.

Proof :

(i) Since for a fixed i ($\neq 1, n$) $S_3(i,n) - S_3(1,n)$ is a Tribonacci sequence with the following initial conditions:

$$
\begin{aligned}
(a) \qquad S_3(i,i) - S_3(1,i) &= 0 \\
(b) \quad S_3(i,i+1) - S_3(1,i+1) &= S_3(2,i+1) - S_3(1,i+1) \\
&= -d_{1i} > 0 \\
(c) \quad S_3(i,i+2) - S_3(1,i+2) &= S_3(3,i+2) - S_3(1,i+2) \\
&= S_3(3,i+2) - S_3(2,i+2) \\
&\quad + S_3(2,i+2) - S_3(1,i+2) \\
&= -d_{2i} - d_{1,i+1} > 0
\end{aligned}
$$

since d_{2i} and $d_{1,i+1}$ are negative (Theorem 9), the result follows.

(ii) Since $S_3(3,n) - S_3(i,n)$ is a Tribonacci sequence for a given i ($\neq 3, n-2$) and its initial conditions are:

$$
\begin{aligned}
(a) \qquad S_3(3,i) - S_3(i,i) &= S_3(3,i) - S_3(1,i) > 0 \text{ (from part (i))} \\
(b) \quad S_3(3,i+1) - S_3(i,i+1) &= S_3(3,i+1) - S_3(2,i+1) \\
&= -d_{2,j-1} > 0 \\
(c) \quad S_3(3,i+2) - S_3(i,i+2) &= S_3(3,i+2) - S_3(3,i+2) = 0
\end{aligned}
$$

Since $S_3(i,n) = S_3(n+1-i,n)$.

As all the initial conditions are non-negative and at least one condition is positive the required result is immediate.

5.4 Birnbaum Structural Importance

Here we study the Birnbaum structural importance of components in a consecutive-3-out-of-n:F system and develop a necessary and sufficient condition for component i to have more importance than component j. We also provide an efficient heuristic procedure for ranking components based on proportionate error. We first give a general formula for calculating the number of path sets

Theorem 15 *We have for a consecutive-3-out-of-n:F system*

$$T_n =]t_n[\quad \textit{where } t_n = \frac{\rho}{\theta}(1/\theta)^n$$

where $]x[$ *denotes the closest integer to* x, θ *is the real root of the equation* $1 - x - x^2 - x^3 = 0$ *and* $\rho = \dfrac{\theta - 1}{6\theta - 4}$.

Proof : Consider the Tribonacci sequence $\{T_n\}$ with the initial conditions, $T_0 = 1, T_1 = 2$ and $T_2 = 4$. Let its generating function be $G(y)$. We then have

$$\begin{array}{lclclclclclc}
G(y) & = & T_0 & + & T_1 y & + & T_2 y^2 & + & T_3 y^3 & + \cdots + & T_n y^n & + \cdots \\
G(y)y & = & & & T_0 y & + & T_1 y^2 & + & T_2 y^3 & + \cdots + & T_{n-1} y^n & + \cdots \\
G(y)y^2 & = & & & & & T_0 y^2 & + & T_1 y^3 & + \cdots + & T_{n-2} y^n & + \cdots \\
G(y)y^3 & = & & & & & & & T_0 y^3 & + \cdots + & T_{n-3} y^n & + \cdots \\
\hline
\end{array}$$

$$\text{Or } (1 - y - y^2 - y^3)G(y) = T_0 + (T_1 - T_0)y + (T_2 - T_1 - T_0)y^2$$

since $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n > 0$. This gives us

$$G(y) = \frac{1 + y + y^2}{1 - y - y^2 - y^3}.$$

Let y_1, y_2 and y_3 represent the roots of the equation $(1-y-y^2-y^3) = 0$. It then follows from Feller [22] that

$$G(y) = \frac{\rho_1}{(y_1 - y)} + \frac{\rho_2}{(y_2 - y)} + \frac{\rho_3}{(y_3 - y)}$$

$$\text{where } \rho_i = \frac{-(1 + y_i + y_i^2)}{-1 - 2y_i - 3y_i^2}$$

and since $y_i \neq 1$ for any i we can write

$$\begin{aligned} \rho_i &= \frac{-(1 + y_i + y_i^2)(1 - y_i)}{(-1 - 2y_i - 3y_i^2)(1 - y_i)} \\ &= \frac{1 - y_i^3}{1 + y_i + y_i^2 - 3y_i^3} \quad \text{and} \\ & \quad (\text{ as } 1 - y_i - y_i^2 - y_i^3 = 0) \\ &= \frac{1 - y_i^3}{2 - 4y_i^3}. \end{aligned}$$

Since $(1 - y_i)(1 - y_i - y_i^2 - y_i^3) = 0$, this gives us that $1 - 2y_i + y_i^4 = 0$ or $y_i^3 = \frac{2y_i - 1}{y_i}$. Hence we can write

$$\rho_i = \left[\frac{y_i - 1}{6y_i - 4}\right] \qquad \text{for } i = 1, 2, 3.$$

We have for sufficiently small y

$$\frac{1}{y_i - y} = \frac{1}{y_i}\left[1 + \frac{y}{y_i} + (\frac{y}{y_i})^2 + \cdots\right]$$

it follows that we can express

$$T_n = \frac{\rho_1}{y_1^{n+1}} + \frac{\rho_2}{y_2^{n+1}} + \frac{\rho_3}{y_3^{n+1}}.$$

Let $f(y) = 1 - y - y^2 - y^3$. It can be easily seen that the equation $f(y) = 0$ has one positive root and the remaining roots are not real. Suppose $y_1 = \theta$ is the positive root. Then $.5436 < \theta < .5437$ since $f(.5436) > 0$ and $f(.5437) < 0$. Let the remaining two roots be $y_2 = a + ib$ and $y_3 = a - ib$. It follows that

$$y_1 y_2 y_3 = \theta(a^2 + b^2) = 1 \text{ or } a^2 + b^2 = 1/\theta$$

$$y_1 y_2 + y_2 y_3 + y_1 y_3 = 2a\theta + a^2 + b^2 = 1$$

$$\text{and} \quad y_1 + y_2 + y_3 = \theta + 2a = -1.$$

This gives us

$$a = -\frac{1+\theta}{2} \quad \text{and} \quad b^2 = \frac{3-\theta^2}{4\theta}.$$

It implies that we can now write

$$T_n = \rho_1 \left(\frac{1}{\theta}\right)^{n+1} + \frac{\rho_2}{(a+ib)^{n+1}} + \frac{\rho_3}{(a-ib)^{n+1}}$$

where

$$a = -\frac{1+\theta}{2}, \quad b = \sqrt{\frac{3-\theta^2}{4\theta}}.$$

Since $\rho_2 = \dfrac{y_2 - 1}{6y_2 - 4}$ and $y_2 = a + ib$, we have

$$\begin{aligned} 6\rho_2 - 1 &= \frac{-2}{6(a+ib) - 4} = \frac{-2}{(6a-4) + i6b} \\ &= \frac{-2\left[(6a-4) - i6b\right]}{(6a-4)^2 + 36b^2} \end{aligned}$$

Similarly, we have $6\rho_3 - 1 = \dfrac{-2[(6a-4)+i6b]}{(6a-4)^2+36b^2}$.

Let

$$\begin{aligned}
e_n &= \frac{\rho_2}{y_2^{n+1}} + \frac{\rho_3}{y_3^{n+1}} \\
&= \frac{1}{6}\left[\frac{-2(6a-4)+i12b}{(6a-4)^2+36b^2}+1\right](a+ib)^{-(n+1)} \\
&\quad +\frac{1}{6}\left[\frac{-2(6a-4)-i12b}{6a-4)^2+36b^2}+1\right](a-ib)^{-(n+1)} \\
&= \frac{1}{6}\left[\frac{-2(6a-4)+i12b}{(6a-4)^2+36b^2}+1\right]\left[\frac{\cos(n+1)\phi - i\sin(n+1)\phi}{r^{n+1}}\right] \\
&\quad +\frac{1}{6}\left[\frac{-2(6a-4)-i12b}{(6a-4)^2+36b^2}+1\right]\left[\frac{\cos(n+1)\phi + i\sin(n+1)\phi}{r^{n+1}}\right].
\end{aligned}$$

This implies that we have

$$e_n = \frac{\rho_2}{y_2^{n+1}} + \frac{\rho_3}{y_3^{n+1}} = [C_2\cos(n+1)\phi + C_3\sin(n+1)\phi]\frac{1}{r^{n+1}} = h_n\frac{1}{r^{n+1}}$$

where $h_n = C_2\cos(n+1)\phi + C_3\sin(n+1)\phi$,

$$C_2 = \frac{1}{6}\left[\frac{-4(6a-4)}{(6a-4)^2+36b^2}+2\right],\ C_3 = \frac{4b}{(6a-4)^2+36b^2}$$

and $\phi = \tanh(b/a)$, $a = -(1+\theta)/2$, $b^2 = \dfrac{3-\theta^2}{4\theta}$ and $r^2 = a^2+b^2$.

$$\begin{aligned}
\text{We have } (6a-4)^2+36b^2 &= \left[\theta(3\theta+7)^2+9(3-\theta^2)\right]\frac{1}{\theta} \\
&= \left[9\theta^3+33\theta^2+49\theta+27\right]\frac{1}{\theta} \\
&= \left[24\theta^2+40\theta+36\right]\frac{1}{\theta}
\end{aligned}$$

since $\theta^3 = 1 - \theta - \theta^2$ we can write

$$C_2 = \frac{1}{3}\left[\frac{(6\theta + 14)\theta}{24\theta^2 + 40\theta + 36} + 1\right]$$

$$= \frac{5\theta^2 + 9\theta + 6}{12\theta^2 + 20\theta + 18} > 0$$

$$\text{and } C_3 = \frac{4b\theta}{24\theta^2 + 40\theta + 36} > 0$$

We now obtain a bound on h_n. we have

$$|C_2 \cos(n+1)\phi + C_3 \sin(n+1)\phi| \le C_2 + C_3 < C_2 + bC_3 \text{ as } b > 1$$

and consider

$$C_2 + bC_3 = \left[\frac{10\theta^2 + 18\theta + 12}{24\theta^2 + 40\theta + 36} + \frac{3 - \theta^2}{24\theta^2 + 40\theta + 36}\right]$$

$$= \frac{9\theta^2 + 18\theta + 15}{24\theta^2 + 40\theta + 36}.$$

Let $g(x) = \dfrac{9x^2 + 18x + 15}{24x^2 + 40x + 36}$ with $.5436 < x < .5437$. We have

$$\frac{dg(x)}{dx} = \frac{24(-3x^2 - 3x + 2)}{(24x^2 + 40x + 36)^2}$$

and $\dfrac{dg(x)}{dx}|_{x=.5436} < 0$ and also $\dfrac{dg(x)}{dx}|_{x=.5437} < 0$.

It implies that $g(x)$ is a decreasing function on $(.5436, .5437)$ and $g(x) < g(.5436) = .4233$ for $x \in (.5436, .5437)$.

Since, we can write

$$T_n = t_n + e_n = t_n + \sqrt{\theta}\,(\sqrt{\theta})^n\, h_n \quad \text{as } r^2 = \frac{1}{\theta}$$

where $t_n = \frac{\rho_1}{\theta}(1/\theta)^n$ and $h_n = C_2 \cos(n+1)\phi + C_3 \sin(n+1)\phi$, it follows that

$$|T_n - t_n| = \sqrt{\theta}\,(\sqrt{\theta})^n\,|h_n| \le C_2 + bC_3 < 0.4233 \text{ as } \theta < 1.$$

Hence we have $T_n =]t_n[$ and the theorem is established.

Let $\beta_3(i,n)$ denote the Birnbaum structural importance of component i in a consecutive-3-out-of-n:F system.

Theorem 16 *We have for a consecutive-3-out-of-n:F system*

$$\begin{aligned}\beta_3(i,n) &= \frac{S_3(i,n)}{2^{n-1}} \\ &= \frac{2T_{i-1}T_{n-i} - T_n}{2^{n-1}}.\end{aligned}$$

Proof : It easily follows from the definition of Birnbaum structural importance and Theorem 4.

Theorem 17 *For $i, j(\neq i) \le m$, $\beta_3(i,n) > \beta_3(j,n)$ if and only if for $n \ge 5$*

$$h_{i-1}\delta^{i-1} + h_{n-i}\delta^{n-i} + \frac{h_{i-1}h_{n-i}}{\lambda}\delta^{n-1} > h_{j-1}\delta^{j-1} + h_{n-j}\delta^{n-j} + \frac{h_{j-1}h_{n-j}}{\lambda}\delta^{n-1}$$

where m denotes the smallest integer greater than or equal to $n/2$.

Proof: We have from Theorem 15

$$T_n = t_n + \sqrt{\theta}\,(\sqrt{\theta})^n\,h_n$$

$$= t_n \left\{ 1 + \frac{\sqrt{\theta}(\sqrt{\theta})^n h_n}{\frac{\rho_1}{\theta}(\frac{1}{\theta})^n} \right\}$$

$$= t_n \left\{ 1 + \frac{h_n}{\lambda} \delta^n \right\}$$

where $\lambda = \frac{\rho_1}{\delta}$ and $\delta = \sqrt{\theta}\,\theta$.

Let $h(x) = \frac{x-1}{6x-4}$ and we get

$$\frac{dh(x)}{dx} = \frac{2}{(6x-4)^2} > 0$$

it follows that $h(\theta) = \frac{\theta - 1}{6\theta - 4} = \rho_1$ is an increasing function on $(.5436, .5437)$ and $0.61809 < \rho_1 < 0.6185$. Since $.5436 < \theta < .5437$, it implies that $0.4007 < \delta < 0.40091$ and $0.61809 < \rho_1 < 0.6185$ and hence

$$\lambda > \frac{0.61809}{0.40091} = 1.5417$$

We know from Theorem 16 that

$$\beta_3(i,n) > \beta_3(j,n) \iff T_{i-1}T_{n-i} > T_{j-1}T_{n-j}.$$

That is

$$t_{i-1}\, t_{n-i} \left\{ 1 + \frac{h_{i-1}\delta^{i-1}}{\lambda} \right\} \left\{ 1 + \frac{h_{n-i}\delta^{n-i}}{\lambda} \right\} >$$

$$t_{j-1} t_{n-j} \left\{ 1 + \frac{h_{j-1}\delta^{j-1}}{\lambda} \right\} \left\{ 1 + \frac{h_{n-j}\delta^{n-j}}{\lambda} \right\} \iff$$

$$\left\{ 1 + \frac{h_{i-1}\delta^{i-1}}{\lambda} \right\} \left\{ 1 + \frac{h_{n-i}\delta^{n-i}}{\lambda} \right\} > \left\{ 1 + \frac{h_{j-1}\delta^{j-1}}{\lambda} \right\} \left\{ 1 + \frac{h_{n-j}\delta^{n-j}}{\lambda} \right\}$$

$$\iff h_{i-1}\delta^{i-1} + h_{n-i}\delta^{n-i} + \frac{h_{i-1}h_{n-i}}{\lambda}\delta^{n-1} > h_{j-1}\delta^{j-1} + h_{n-j}\delta^{n-j} + \frac{h_{j-1}h_{n-j}}{\lambda}\delta^{n-1}.$$

Theorem 18 *In a consecutive-3-out-of-n:F system*

(i) $\beta_3(3,n) > \beta_3(i,n)$ *for* $i \neq 3, i \leq m$ *and* $n > 4$

(ii) $\beta_3(1,n) < \beta_3(i,n)$ *for* $2 \leq i \leq m$ *and* $n > 4$

where m *denotes the smallest integer greater than or equal to* $n/2$.

Proof : It follows from Theorem 14.

Theorem 18 implies that component 3 or its mirror image component has the maximum Birnbaum structural importance and component 1 the minimum Birnbaum structural importance. For $n = 3$ all components have the same Birnbaum structural importance since for $n = k = 3$ the system reduces to a parallel structure. For $n = 4$ component 2 and component 3 have the same Birnbaum structural importance. Table 5.1 gives the Birnbaum structural importance of components for $1 \leq i \leq m$ and $n = 3$ to 20 where m denotes the smallest integer greater than or equal to $n/2$. Table 5.3 presents the Birnbaum structural ranking of components. It can be seen that for $n = 21$ and 22 components 8 and 11 have the same rank. Similarly components 11 and 12 have the same rank for n=25 and 26 and in case of n=29 and 30, components 14 and 15 have the same rank. This shows that for $k = 3$ ties occur in the ranks of first m components of the system whereas such a thing was not found for the case of $k = 2$.

Heuristic Ranking

Exact Birnbaum structural ranking of components can be obtained by using Theorem 17 or 16. However it would involve the number of components in the system n. When $n-i$ and $n-j$ are large and $i, j \leq m$, the dominating terms in the necessary and sufficient condition of Theorem 17 are $h_{i-1}\delta^{i-1}$ and $h_{j-1}\delta^{j-1}$, as $0 < \delta < 1$ and $|h_n| < 0.43$ where m denotes the smallest integer greater than or equal to $n/2$. Hence $\beta_3(i, n)$ is expected to be more than $\beta_3(j, n)$ if $h_{i-1}\delta^{i-1} > h_{j-1}\delta^{j-1}$. It follows that ranking of the first m components based on $h_{i-1}\delta^{i-1}$ (or proportionate error $=h_{i-1}\delta^{i-1}\lambda^{-1}$) for lagre $n-i$ should match with the exact Birnbaum structural ranking.

Comparison with exact ranking

Table 5.4 presents the heuristic ranking of components for n up to 30. It can be seen that the component ranking based on proportionate error is in agreement with the exact ranking in almost all cases and in case of extreme ranks the agreement between the heuristic and exact ranking is complete. The difference in ranks is of at most one step.

5.5 Barlow-Proschan Structural Importance

Having studied the Birbaum structural importance, we now wish to study the Barlow-Proschan structural importance of components. Let

$g_3(n,i)$ denote the number of path sets with i failed components (or $n-i$ components working) in a consecutive-3-out-of-n:F system. We obtain a recurrence relationship for $g_3(n,i)$ and use its generating function to calculate system reliability and Barlow-Proschan structural importance measure. It is also proved that component 3 has the maximum and component 1 the minimum Barlow-Proschan structural importance.

Theorem 19 *Let $g_3(n,i)$ denote the number of path sets with i failed components (or path sets of size $n-i$) of a consecutive-3-out-of-n:F system, we then have for $n \geq 4$ and $i \geq 3$*

$$g_3(n,i) = g_3(n-1,i) + g_3(n-2,i-1) + g_3(n-3,i-2)$$

with the initial conditions $g_3(n,0) = 1$, $g_3(n,1) = n$ and $g_3(n,2) = \binom{n}{2}$.

Proof: For $n \geq 4$, let $g_3(n,i)_u$ denote the number of path sets of a consecutive-3-out-of-n:F system with i failed components, u being the largest index of the component that is working in all these path sets. Obviously, we have for $i \geq 3$

$$\begin{aligned} g_3(n,i) &= g_3(n,i)_n + g_3(n,i)_{n-1} + g_3(n,i)_{n-2} \\ &= g_3(n-1,i) + g_3(n-2,i-1) + g_3(n-3,i-2). \end{aligned}$$

Trivially, we have $g_3(n,i) = \binom{n}{i}$ for $0 \leq i \leq 2$ and $g_3(n,i) = 0$ for $i > n$.

Remark: Without loss of generality we take $g_3(0,0) = 1.$ and $g_3(n,i) = 0$ for $i < 0$. We have

$$\begin{aligned}
(i) \quad g_3(n,i) &= g_3(n-1,i) + g_3(n-2,i-1) + g_3(n-3,i-2), n \geq 3 \\
(ii) \quad T_n &= \sum_{i=0}^{n} g_3(n,i) \text{ for } n \geq 0
\end{aligned}$$

Theorem 20 *Let $L_n(x) = \sum_{i=0}^{\infty} g_3(n,i)x^i$, we then have*

$$L_{n+3}(x) = L_{n+2}(x) + xL_{n+1}(x) + x^2 L_n(x) \quad n \geq 0$$

with the initial conditions $L_i(x) = (1+x)^i$ for $0 \leq i \leq 2$.

Proof: We have for $n \geq 0$

$$\begin{aligned}
L_{n+3}(x) &= \sum_{i=0}^{n+3} g_3(n+3,i)x^i \text{ as } g_3(n+3,i) = 0 \text{ for } i > n+3 \\
&= \sum_{i=0}^{n+3} [g_3(n+2,i) + g_3(n+1,i-1) + g_3(n,i-2)]\ x^i \\
&= \sum_{i=0}^{n+2} g_3(n+2,i)x^i + x\sum_{i=1}^{n+2} g_3(n+1,i-1)x^{i-1} \\
&\quad + x^2 \sum_{i=0}^{n+2} g_3(n,i-2)x^{i-2} \\
&= \sum_{i=0}^{n+2} g_3(n+2,i)x^i + x\sum_{i=0}^{n+1} g_3(n+1,i)x^i \\
&\quad + x^2 \sum_{i=0}^{n} g_3(n,i)x^i \\
&= L_{n+2}(x) + xL_{n+1}(x) + x^2 L_n(x)
\end{aligned}$$

as $g_3(n,i) = 0$ for $i > n$ or $i < 0$. Obviously, we have

$$L_n(x) = \sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n \quad \text{for } 0 \le n \le 2.$$

Without loss of generality, we take $L_{-1}(x) = 1$ and $L_i(x) = 0$ for $i \le -2$. This implies that $L_n(x) = L_{n-1}(x) + xL_{n-2}(x) + x^2 L_{n-2}(x)$ for $n \ge 0$.

Example : For $0 \le n \le 8$, $g_3(n,i)$ is given by the $(n,i)^{th}$ element of the matrix

$$\begin{bmatrix} 1 & & & & & & & & \\ 1 & 1 & & & & & & & \\ 1 & 2 & 1 & & & & & & \\ 1 & 3 & 3 & 0 & & & & & \\ 1 & 4 & 6 & 2 & 0 & & & & \\ 1 & 5 & 10 & 7 & 1 & 0 & & & \\ 1 & 6 & 15 & 16 & 6 & 0 & 0 & & \\ 1 & 7 & 21 & 30 & 19 & 3 & 0 & 0 & \\ 1 & 8 & 28 & 50 & 45 & 16 & 1 & 0 & 0 \end{bmatrix}$$

Obviously upper diagonal elements are zero.

Theorem 21

$$\begin{aligned} \textit{The degree of } L_n(x) &= 2r && \textit{if} \quad n = 3r \\ &= 2r+1 && \textit{if} \quad n = 3r+1 \\ &= 2r+2 && \textit{if} \quad n = 3r+2 \end{aligned}$$

Proof: Since we have $L_0(x) = 0,\ L_1(x) = 1+x,\ L_2(x) = 1+2x+x^2$, and $L_3(x) = 1+3x+3x^2$, trivially, the result is true for $n = 0, 1, 2, 3$ and suppose that it is true for $n+2$ and consider the case for $n+3$.

(i) Let $n = 3r$. By the induction hypothesis we have

$$\text{degree of } L_{n+2}(x) = 2r+2$$

$$\text{degree of } xL_{n+1}(x) = 2r+2$$

$$\text{degree of } x^2L_{n+2}(x) = 2r+2$$

hence degree of $L_{n+3}(x) = 2r+2 = 2(r+1)$.

(ii) Let $n = 3r+1$. Becuase of the induction hypothesis

$$\begin{aligned} \text{degree of } L_{n+3}(x) &= Max\{2(r+1), 2r+3, 2r+1+2\} \\ &= 2r+3 = 2(r+1)+1 \end{aligned}$$

(iii) Let $n = 3r+2$. It follows from the induction hypothesis that

$$\begin{aligned} \text{degree of } L_{n+3}(x) &= \text{Max } \{2(r+1)+1, 2(r+1)+1, 2(r+1)+2\} \\ &= 2(r+1)+2 \end{aligned}$$

Theorem 22 *Let $h_{3,n}(\mathbf{p})$ represent the reliability function of a consecutive-3-out-of-n:F system where $\mathbf{p}$ is the vector of component reliabilities. We then have*

(i) $h_{3,n}(1_i, \mathbf{p}) = h_{3,i-1}(p_1, p_2, \ldots, p_{i-1}) \times h_{3,n-i}(p_{i+1}, \ldots, p_n)$.

(ii) $\beta_{3,\mathbf{p}}(i,n) = 1/(1-p_i)\,[h_{3,n}(1_i,\mathbf{p}) - h_{3,n}(\mathbf{p})]$ *for* $p_i \in (0,1)$ *where* $\beta_{3,\mathbf{p}}(i,n)$ *denotes the Birnbaum reliability importance of component* i *in a consecutive-3-out-of-n:F system.*

(iii) $h_{3,n}(p) = p^n L_n(q/p)$ *for the i.i.d. case* ($p_1 = p_2 = \cdots = p_n = p$).

Theorem 23 *Let* $\Psi_3(i,n)$ *denote the Barlow-Proschan structural importance of component* i *in a consecutive-3-out-of-n:F system. Then*

$$\Psi_3(i,n) = \int_0^1 (1/q)\left[p^{n-1}L_{i-1}(q/p)L_{n-i}(q/p) - p^n L_n(q/p)\right] dp.$$

Proof of Theorem 22 and 23 follows on the same lines as that of the corresponding Theorems for the consecutive-2-out-of-n:F system.

5.5.1 Calculation of Barlow-Proschan Structural Measure

We can calculate Barlow-Proschan measure of structural importance for different components using Theorem 23. We have

$$\begin{aligned}
\Psi_3(i,n) &= \int_0^1 (1/q)\left[p^{n-1}L_{i-1}(q/p)L_{n-i}(q/p) - p^n L_n(q/p)\right] dp \\
&= a + \int_0^1 (1/q)p^{n-1}L_{i-1}(q/p)L_{n-i}(q/p)dp \\
&= a + \int_0^1 [\sum_{j=1}^{d(i,n)} a_{ij}p^{n-(j+1)}q^{j-1}\,]dp
\end{aligned}$$

where a is a constant to be determined such that $\sum_{i=1}^{n} \Psi_3(i,n) = 1$ and $d(i,n)$ is the degree of the polynomial

$$L_{i-1}(x)L_{n-i}(x) = \sum_{j=0}^{d(i,n)} a_{ij}x^j$$

Note that $a_{i0} = 1$ for all i and n. Also we have from beta function

$$\int_0^1 p^{n-(j+1)}q^{j-1}\,dp = \frac{(n-j-1)!\,(j-1)!}{(n-1)!}.$$

Example : Let $n = 7$. We have

$$\begin{aligned}
L_0(x) &= 1,\ L_1(x) = 1 + x,\ \text{and } L_2(x) = 1 + 2x + x^2 \\
L_3(x) &= 1 + 3x + 3x^2, \quad L_4(x) = 1 + 4x + 6x^2 + 2x^3 \\
L_5(x) &= 1 + 5x + 10x^2 + 7x^3 + x^4 \\
L_6(x) &= 1 + 6x + 15x^2 + 16x^3 + 6x^4 \\
L_7(x) &= 1 + 7x + 21x^2 + 30x^3 + 19x^4 + 3x^5.
\end{aligned}$$

For $i = 1$

$$L_0(x)L_6(x) = 1 + 6x + 15x^2 + 16x^3 + 6x^4$$

and this gives us

$$\begin{aligned}
\Psi_3(1,7) &= a + \int_0^1 \left[6p^5 + 15p^4q + 16p^3q^2 + 6p^2q^3\right] dp \\
&= a + \frac{1}{6!}[6 \times 5! + 15 \times 4! \times 1! + 16 \times 3! \times 2! \\
&\quad +6 \times 2! \times 3!] \\
&= a + \frac{1}{6!}[720 + 360 + 192 + 72] \\
&= a + \frac{1344}{720}.
\end{aligned}$$

For $i = 2$

$$\begin{aligned}
L_1(x)L_5(x) &= 1 + 6x + 15x^2 + 17x^3 + 8x^4 + x^5 \text{ and} \\
\Psi_3(2,7) &= a + \frac{1}{6!}\int_0^1 \left[6p^5 + 15p^4q + 17p^3q^2 + 8p^2q^3 + pq^4\right] dp \\
&= a + \frac{1}{720}[6 \times 5! + 15 \times 4! + 17 \times 3! \times 2! \\
&\quad + 8 \times 2!3! + 4!] \\
&= a + \frac{1404}{720}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\Psi_3(3,7) &= a + \frac{1}{6!}\int_0^1 \left[6p^5 + 15p^4q + 18p^3q^2 + 10p^2q^3 + 2pq^4\right] dp \\
&= a + \frac{1}{6!}[6 \times 6! + 15 \times 4! + 18 \times 3!\ 2! \\
&\quad +10 \times 2!3! + 2 \times 4!] \\
&= a + \frac{1}{720}(1464)
\end{aligned}$$

and

$$\begin{aligned}
\Psi_3(4,7) &= a + \int_0^1 \left[6p^5 + 15p^4q + 18p^3q^2 + 9p^2q^3\right] dp \\
&= a + \frac{1404}{720}
\end{aligned}$$

since $\sum_{i=1}^{7} \Psi_3(i,7) = 1$ and $\Psi_3(i,n) = \Psi_3(n-i+1,n)$ hence, we can calculate the constat a from

$$7a + \frac{2688 + 2808 + 2928 + 1404}{720} = 1$$

This gives us

$$a = -\frac{9108}{7 \times 720} = -1.80714 \text{ and hence we have}$$

$$\begin{aligned}
\Psi_3(1,7) &= \Psi_3(7,7) &= 300/5040 \\
\Psi_3(2,7) &= \Psi_3(6,7) &= 720/5040 \\
\Psi_3(3,7) &= \Psi_3(5,7) &= 1140/5040 \\
\Psi_3(4,7) &= 720/5040. &
\end{aligned}$$

Table 5.2 gives the Barlow-Proschan structural importance of components for n=3 to 20.

Theorem 24 *Consider the polynomial*

$$P_{i,n}(x) = L_{i-1}(x)L_{n-i}(x) = \sum_{j=0}^{d(i,n)} a_{ij}x^j.$$

We then have

(i) $a_{ij} = \binom{n-1}{j}$ *for* $j = 0, 1, 2$ *and for all* $i = 1, 2, \ldots, n$.

(ii) $a_{ij} = a_{jj}$ *for* $i \geq j \geq 3$, $i \leq m$ *and* $n > 5$

wehere m is smallest integer greater than or equal to $n/2$.

Proof :

(i) Let $L_{i-1}(x) = \sum_{j=0}^{d(i)} a_j x^j$ and $L_{n-i}(x) = \sum_{j=0}^{d(n-i)} b_j x^j$.

we have $P_{i,n}(x) = L_{i-1}(x)L_{n-i}(x) = \sum_{j=0}^{d(i,n)} a_{ij}x^j$. This implies

$$a_{ij} = \sum_{r=0}^{j} a_r b_{j-r} \text{ for } i = 1, 2, \ldots, m.$$

Hence

$$a_{i0} = a_0 b_0 = 1$$

$$a_{i1} = a_0 b_1 + a_1 b_0 = n - i + i - 1 = n - 1$$

$$\begin{aligned} a_{i2} &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\ &= \binom{i-1}{0}\binom{n-1}{2} + \binom{i-1}{1}\binom{n-i}{1} + \binom{i-1}{2}\binom{n-i}{0} \\ &= \binom{n-1}{2} \end{aligned}$$

Since we have $a_0 = b_0 = 1$,

$$a_1 = \binom{i-1}{1}, \quad b_1 = \binom{n-i}{1} \text{ and}$$

$$a_2 = \binom{i-1}{2}, \quad b_2 = \binom{n-i}{2}$$

(ii) Since the coefficients of the polynomial $P_{i,n}(x)$ depend on the values of i and n, we therefore prefer to use the notation

$$P_{i,n}(x) = L_{i-1}(x)L_{n-i}(x) = \sum_{j=0}^{d(i,n)} a_{ij}^{(n)} x^j$$

For $n = 5$, we have

$$P_{i,5}(x) = 1 + 4x + 6x^2 + 2x^3, \text{ if } i = 3$$

For $n = 6$

$$P_{i,6}(x) = 1 + 5x + 10x^2 + 9x^3 + 3x^4, \text{ if } i = 3$$

For $n = 7$

$$P_{i,n}(x) = 1 + 6x + 15x^2 + 18x^3 + 10x^4 + 2x^5, \text{ if } i = 3$$

$$P_{i,n}(x) = 1 + 6x + 15x^2 + 18x^3 + 9x^4, \text{ if } i = 4$$

It follows from the above that the result is true for $n = 5, 6$ (vacuously) and 7. Assume that it is true for all values of n upto $r-1$ ($r \geq 8$). Consider

$$\begin{aligned} P_{i,r}(x) &= L_{i-1}(x)L_{r-i}(x) \text{ for integer } i \leq (r+1)/2 \\ &= L_{i-1}(x)[L_{r-i-1}(x) + xL_{r-i-2}(x) + x^2 L_{r-i-3}(x)] \\ &= P_{i,r-1}(x) + xP_{i,r-2}(x) + x^2 P_{i,r-3}(x). \end{aligned}$$

Hence it follows that

$$a_{ij}^{(r)} = a_{ij}^{(r-1)} + a_{i,j-1}^{(r-2)} + a_{i,j-2}^{(r-3)}.$$

As the induction hypothesis is true up to $(r-1)$, we get for $j \geq 5$

$$a_{ij}^{(r)} = a_{jj}^{(r-1)} + a_{j-1,j-1}^{(r-2)} + a_{j-2,j-2}^{(r-3)}$$

and also for $i = j$

$$\begin{aligned} a_{jj}^{(r)} &= a_{jj}^{(r-1)} + a_{j,j-1}^{(r-2)} + a_{j,j-2}^{(r-3)} \\ &= a_{jj}^{(r-1)} + a_{j-1,j-1}^{(r-2)} + a_{j-2,j-2}^{(r-3)} \end{aligned}$$

For $j = 3, 4$ the same argument works as $a_{i1}^{(r)} = r - 1$ and $a_{i2}^{(r)} = \binom{r-2}{2}$. Hence the result is true for all $n \geq 5$.

We now show that component 3 has the maximum and component 1 the minimum Barlow-Proschan structural importance in a consecutive-3-out-of-n:F system.

Theorem 25 *Let* $W(i,n,x) = L_2(x)L_{n-3}(x) - L_{i-1}(x)L_{n-i}(x)$. *We then have for a positive integer,* $n \geq 4$ *and* $x > 0$

$$W(i,n,x) \geq 0 \text{ for any } i \neq 0 \text{ or } n+1$$

and the strict inequality holds for $i \neq 3,\ n-2$.

Proof : We have from Theorem 20 for $n \geq 0$

$$L_n(x) = L_{n-1}(x) + xL_{n-2}(x) + x^2 L_{n-3}(x)$$

where $L_i(x) = (1+x)^i$ for $0 \leq n \leq 2$, $L_{-1}(x) = 1$ and $L_i(x) = 0$ for $i \leq -2$. We use induction on n to prove the result.

For $n = 4$, $W(i,n,x) = L_2(x)L_1(x) - L_{i-1}(x)L_{4-i}(x)$. This gives us

$$W(i,4,x) = \begin{cases} (1+x)^3 & \text{if } \ i \leq -1 \text{ and } i \geq 6 \\ -x - 3x^2 - x^3 & \text{if } \ i = 0, 5 \\ x^3 & \text{if } \ i = 1, 4 \\ 0 & \text{if } \ i = 2, 3 \end{cases}$$

For $n = 5$, $W(i,n,x) = L_2(x)L_2(x) - L_{i-1}(x)L_{5-i}(x)$. It follows that

$$W(i,5,x) = \begin{cases} (1+x)^4 & \text{if } \ i \leq -1 \text{ and } i \geq 7 \\ -x - 4x^2 - 3x^3 & \text{if } \ i = 0, 6 \\ 2x^3 + x^4 & \text{if } \ i = 1, 5 \\ x^3 + x^4 & \text{if } \ i = 2, 4 \\ 0 & \text{if } \ i = 3 \end{cases}$$

For $n = 6$, $W(i,n,x) = L_2(x)L_3(x) - L_{i-1}(x)L_{6-i}(x)$ and we have

$$W(i,6,x) = \begin{cases} L_2(x)L_3(x) & \text{if} \quad i \leq -1 \text{ and } i \geq 8 \\ -x - 5x^2 - 7x^3 - 3x^4 & \text{if} \quad i = 0,7 \\ 2(x^3 + x^4) & \text{if} \quad i = 1,6 \\ x^3 + x^4 & \text{if} \quad i = 2,5 \\ 0 & \text{if} \quad i = 3,4 \end{cases}$$

For $n = 7$, $W(i,n,x) = L_2(x)L_4(x) - L_{i-1}(x)L_{7-i}(x)$ and we have

$$W(i,7,x) = \begin{cases} L_2(x)L_4(x) & \text{if} \quad i \leq -1 \text{ and } i \geq 9 \\ -x - 6x^2 - 12x^3 - 9x^4 - x^5 & \text{if} \quad i = 0,8 \\ 2x^3 + 4x^4 + 2x^5 & \text{if} \quad i = 1,7 \\ x^3 + 2x^4 + x^5 & \text{if} \quad i = 2,6 \\ 0 & \text{if} \quad i = 3,5 \\ x^4 + 2x^5 & \text{if} \quad i = 4 \end{cases}$$

It follows from the above that the stated result is true for $n = 4,5,6,7$ and suppose that it is true all values of n upto $r-1$ ($r \geq 8$) and consider $W(i,r,x)$ for integer $i \geq 1$ and $i \neq r+1$

$$\begin{aligned} W(i,r,x) &= L_2(x)L_{r-3}(x) - L_{i-1}(x)L_{r-i}(x) \\ &= L_2(x)[L_{r-4}(x) + xL_{r-5}(x) + x^2L_{r-6}(x)] \\ &\quad - [L_{i-2}(x) + xL_{i-3}(x) + x^2L_{i-4}(x)]L_{r-i}(x) \end{aligned}$$

$$
\begin{aligned}
&= [L_2(x)L_{r-4}(x) - L_{i-2}(x)L_{r-i}(x)] + x[L_2(x)L_{r-5}(x) \\
&\quad -L_{i-3}(x)L_{r-i}(x)] + x^2[L_2(x)L_{r-6}(x) - L_{i-4}(x)L_{r-i}(x)] \\
&= W(i-1, r-1, x) + xW(i-2, r-2, x) \\
&\quad +x^2W(i-3, r-3, x)
\end{aligned}
$$

since the induction hypothesis is true for upto $r-1$ it follows that $W(i,r,x) \geq 0$ for $i \neq 1, 2,$ and 3. For $i = 1, 2$ and 3 we prove this separately . For $i = 1$ we have

$$
\begin{aligned}
W(i,r,x) &= L_2(x)L_{r-3}(x) - L_0(x)L_{r-1}(x) \\
&= L_2(x)L_{r-3}(x) - L_{r-1}(x) \\
&= (1+2x+x^2)L_{r-3}(x) - L_{r-2}(x) - xL_{r-3}(x) - x^2L_{r-4}(x) \\
&= (1+x+x^2)L_{r-3}(x) - L_{r-2}(x) - x^2L_{r-4}(x) \\
&= (1+x+x^2)L_{r-3}(x) - L_{r-3}(x) - xL_{r-4}(x) \\
&\quad -x^2L_{r-5}(x) - x^2L_{r-4}(x) \\
&= x(1+x)L_{r-3}(x) - x(1+x)L_{r-4}(x) - x^2L_{r-5}(x) \\
&= x(1+x)[L_{r-3}(x) - L_{r-4}(x)] - x^2L_{r-5}(x) \\
&= x(1+x)[xL_{r-5}(x) + x^2L_{r-6}(x)] - x^2L_{r-5}(x) \\
&= x(1+x)x^2L_{r-6}(x) + x^3L_{r-5}(x) > 0 (x > 0).
\end{aligned}
$$

For $i = 2$

$$
W(i,r,x) = L_2(x)L_{r-3}(x) - L_{i-1}(x)L_{r-i}(x)
$$

$$= L_2(x)L_{r-3}(x) - L_1(x)L_{r-2}(x)$$
$$= (1+x)[(1+x)L_{r-3}(x) - L_{r-2}(x)]$$
$$= (1+x)[x^3 L_{r-6}(x)] > 0 \ (r > 8 \text{ and } x > 0)$$

For $i = 3$ trivially, we have $W(i,r,x) = 0$. And for $i \leq -1$

$$W(i,r,x) = L_2(x)L_{r-3}(x) > 0 \ (\text{as } x > 0)$$

and $L_j(x) = 0$ for $j \leq -2$.

Since by hypothesis for $t \leq r-1$, $W(i,t,x) > 0$ for $i \neq 3, t-2$ it follows that

$$W(i,r,x) = W(i-1,r-1,x) + xW(i-2,r-2,x) + x^2 W(i-3,r-3,x) > 0$$

for $i \neq 3, r+1$. This completes the proof.

Theorem 26 *Let* $U(i,n,x) = L_{i-1}(x)L_{n-i}(x) - L_{n-1}(x)$. *For any* $n \geq 4$, *we then have*

$$U(i,n,x) > 0 \quad \textit{for } 2 \leq i \leq n-1.$$

Proof : For $n = 4$

$$U(i,n,x) = L_{i-1}(x)L_{4-i}(x) - L_3(x)$$
$$= L_1(x)L_2(x) - L_3(x) = x^3 \quad \text{if } i = 2,3$$

For n=5

$$U(i,n,x) = L_{i-1}(x)L_{5-i}(x) - L_4(x)$$
$$U(i,n,x) = L_1(x)L_3(x) - L_4(x) = x^3 \quad \text{if } i = 2,4$$
$$U(i,n,x) = L_2(x)L_2(x) - L_4(x) = 2x^3 + x^4 \quad \text{if } i = 3$$

For $n = 6$

$$U(i,n,x) = L_{i-1}(x)L_{6-i}(x) - L_5(x)$$
$$U(i,n,x) = L_1(x)L_4(x) - L_5(x) = x^3 + x^4 \quad \text{if } i = 2,5$$
$$U(i,n,x) = L_2(x)L_3(x) - L_5(x) = 2x^3 + 2x^4 \quad \text{if } i = 3,4$$

For $n = 7$

$$U(i,n,x) = L_{i-1}(x)L_{7-i}(x) - L_6(x)$$
$$U(i,n,x) = L_1(x)L_5(x) - L_6(x) = x^3(1+x)^2 \quad \text{if } i = 2,6$$
$$U(i,n,x) = L_2(x)L_4(x) - L_6(x) = 2x^3 + 4x^4 + 2x^5 \quad \text{if } i = 3,5$$
$$U(i,n,x) = L_3(x)L_3(x) - L_6(x) = 2x^3 + 3x^4 \quad \text{if } i = 4$$

Hence the theorem is true for $n = 4, 5, 6$ and 7. Assume that it is true for $n = r - 1$ and consider the case for $n = r > 8$

$$\begin{aligned} U(i,r,x) &= L_{i-1}(x)L_{r-i}(x) - L_{r-1}(x) \\ &= L_{i-1}(x)[L_{r-i-1}(x) + xL_{r-i-2}(x) + x^2L_{r-i-3}(x)] \\ &\quad -[L_{r-2}(x) + xL_{r-3}(x) + x^2L_{r-4}(x)] \\ &= [L_{i-1}(x)L_{r-1-i}(x) - L_{r-2}(x)] + x[L_{i-1}(x)L_{r-2-i}(x) \\ &\quad -L_{r-3}(x)] + x^2[L_{i-1}(x)L_{r-3-i}(x) - L_{r-4}(x)] \\ &= U(i,r-1,x) + xU(i,r-2,x) + x^2U(i,r-3,x). \end{aligned}$$

Since by the induction hypothesis the result is true upto $r - 1$, this implies

$$U(i,r,x) > 0 \quad \text{for } 2 \geq i \geq r-2$$

For $i = r-1$ we have $U(r-1, r, x) = L_{r-2}(x)L_1(x) - L_{r-1}(x) = x^3 L_{r-5}(x) = U(2, r, x)$. Hence $U(i, r, x) > 0$ for $1 \le i \le r-1$ and the theorem follows.

Theorem 27 *In a consecutive-3-out-of-n:F system for* $n > 4$

(i) component 3 or its mirror image component has the maximum Barlow-Proschan structural importance

(ii) component 1 or its mirror image component has the minimum Barlow-Proschan structural importance.

Proof: In view of Theorem 23, we have

$$\begin{aligned}\Psi_3(i,n) &= \int_0^1 (1/q)[p^{n-1}L_{i-1}(q/p)L_{n-i}(q/p) - p^n L_n(q/p)]dp \\ &= a + \int_0^1 (1/q)p^{n-1}[L_{i-1}(q/p)L_{n-i}(q/p)]dp\end{aligned}$$

where a is a constant such that $\sum_{i=1}^{n} \Psi_3(i,n) = 1$. The required results follow from Theorems 25 and 26.

5.6 Cut Importance Ranking

Cut importance ranking of components is based on the minimal cut sets and provides a complete ranking of components in the system. It is applicable for when component reliability is high.

Theorem 28 *For a consecutive-3-out-of-n:F system, we have*

(i) $i =_c (n+1-i)$

(ii) $i >_c 1$ *for* $2 \le i \le m$ *and* $n > 4$

(iii $3 >_c i$ for $i \neq n-2$ and $n > 4$

where $i >_c j$ ($i =_c j$) denotes that component i is more (equal) cut important than component j and m denotes the smallest integer greater than or equal to $n/2$.

Proof : It follows from Theorems 23,25 and 26 and the result the cut importance ranking is the same as Birnbaum reliability ranking for high value of components reliability (see Theorem 16 of Chapter 1).

Thus component 3 has the maximum cut importance and component 1 the minimum cut importance ranking for $n \geq 5$. The complete cut importance ranking for all components can be obtained by lexicographically ordering the rows of the structural matrix of the dual of a consecutive-3-out-of-n:F system as proved in Chapter 2. We obtained the recurrence relationship for calculating the structural matrix of the dual of a consecutive-k-out-of-n:F system in Chapter 3. Using this we can find the cut importance ranking of all components. The ranking so obtained for n upto 20 is given in Table 5.6. This ranking does not completely match with the ranking given by the other two structural importance measures considered.

Theorem 29 *Let a consecutive-3-out-of-n:F system have i.i.d. components ($p_1 = p_2 = \cdots = p_n = p$). Then component 3 has the maximum and component 1 the minimum Birnbaum reliability importance for $n > 4$ and all $p \in (0,1)$.*

Proof : It follows from Theorems 22, 25 and 26.

For $k = 3$ Tables 5.3, 5.5 and 5.6 provide the Birnbaum structural importance ranking (n=3 to 30), Barlow-Proschan structural ranking (n=3 to 30) and cut importance ranking (n=3 to 20), respectively. Comparing them we find that the three ranking are not the same for all n. For example, in case of $n = 13$ Birnbaum and Barlow-Proschan structural importance rankings are identical whereas cut importance ranking for component 5 and 7 does not match other two rankings. Similarly for $n = 18$ Birnbaum structural and cut importance rankings are the same for all components but components 8 and 9 have different Barlow-Proschan structural ranking. However all the three structural importance rankings have the following pattern for $5 < n \leq 30$

$$i_1 = 1, i_2 = 2 \text{ and } i_{n-1} = 6, i_n = 3$$

where i_1 denotes the minimum component importance rank. We haved proved that $i_1 = 1$ and $i_n = 3$ as per all measures considered and it can be easily checked that $\beta_{3,p}(2, n) > \beta_{3,p}(1, n)$. Hence we make the conjecture that for $k = 3$ the second most important component is 6 or its mirror image component for $n \geq 12$.

5.7 General Results

In this section, we mention some results of a consecutive-3-out-of-n:F system which can be easily extended to a consecutive-k-out-of-n:F system for $k \geq 4$. By defining a Fibonacci sequence of order k [11] and [49], we can generalize all results of section 5.2 and 5.3 but for Theorem 3 part (ii). Similarly Theorems 16, 18, 22, 24, 27, 28 and 29 can be generalised for $k \geq 4$.

This would imply that component k has the maximum and component 1 the minimum, Birnbaum structural importance, Barlow-Proschan structural importance, Birnbaum reliability importance in case of i.i.d. components and Butler cut importance ranking.

[redacted]
IN A CONSECUTIVE - 2 -OUT -OF- n:F SYSTEM

	COMPONENT									
n	1	2	3	4	5	6	7	8	9	10
2	.500000									
3	.250000	.750000								
4	.250000	.500000								
5	.187499	.437499	.312499							
6	.156249	.343749	.281249							
7	.125001	.281251	.218751	.250001						
8	.101562	.226562	.179687	.195312						
9	.082031	.183593	.144531	.160156	.152343					
10	.066406	.148438	.117188	.128906	.125000					
11	.053711	.120118	.094727	.104493	.100586	.102539				
12	.043457	.097168	.076660	.084472	.081543	.082519				
13	.035156	.078613	.062012	.068359	.065918	.066895	.066406			
14	.028442	.063599	.050171	.055298	.053345	.054077	.053833			
15	.023010	.051452	.040588	.044739	.043152	.043762	.043518	.043640		
16	.018616	.041626	.032837	.036194	.034912	.035400	.035217	.035278		
17	.015061	.033676	.026566	.029282	.028244	.028641	.028488	.028549	.028519	
18	.012184	.027244	.021492	.023689	.022850	.023170	.023048	.023094	.023079	
19	.009857	.022041	.017387	.019165	.018486	.018745	.018646	.018684	.018669	.018677
20	.007975	.017832	.014067	.015505	.014956	.015165	.015085	.015116	.015104	.015108

TABLE 4.2: BARLOW - PROSCHAN STRUCTURAL IMPORTANCE OF COMPONENTS IN A CONSECUTIVE - 2 -OUT -OF- n:F SYSTEM

	COMPONENT									
n	1	2	3	4	5	6	7	8	9	10
2	.500000									
3	.166667	.666667								
4	.166667	.333333								
5	.116667	.283333	.200000							
6	.100000	.216667	.183333							
7	.083333	.183333	.150000	.166667						
8	.072619	.155952	.132143	.139286						
9	.063889	.136508	.116270	.123413	.119841					
10	.057143	.121032	.104365	.109524	.107936					
11	.051623	.108766	.094480	.098845	.097258	.098052				
12	.047078	.098701	.086364	.089935	.088781	.089141				
13	.043254	.090332	.079510	.082540	.081566	.081927	.081746			
14	.039999	.083253	.073668	.076254	.075461	.075724	.075641			
15	.037193	.077192	.068623	.070862	.070194	.070416	.070332	.070374		
16	.034751	.071945	.064225	.066181	.065615	.065795	.065734	.065754		
17	.032607	.067358	.060355	.062080	.061594	.061746	.061694	.061714	.061704	
18	.030709	.063315	.056925	.058457	.058037	.058164	.058123	.058137	.058133	
19	.029017	.059726	.053864	.055234	.054868	.054976	.054941	.054953	.054949	.054951
20	.027500	.056517	.051114	.052347	.052025	.052118	.052089	.052098	.052095	.052096

TABLE 5.1: BIRNBAUM STRUCTURAL IMPORTANCE OF COMPONENTS
IN A CONSECUTIVE-3-OUT- OF- n:F SYSTEM

	COMPONENT									
n	1	2	3	4	5	6	7	8	9	10
3	.2500000	.2500000								
4	.1250000	.3750000								
5	.1250000	.2500000	.5000000							
6	.1250000	.2500000	.3750000							
7	.1093750	.2343750	.3593750	.2656250						
8	.1015625	.2109375	.3359375	.2578125						
9	.0937500	.1953125	.3046875	.2421875	.2500000					
10	.0859375	.1796875	.2812500	.2187500	.2343750					
11	.0791016	.1650391	.2587891	.2021484	.2119141	.2197266				
12	.0727539	.1518555	.2377930	.1860352	.1958008	.1987305				
13	.0668945	.1396484	.2187500	.1708984	.1801758	.1835938	.1796875			
14	.0615234	.1284180	.2011719	.1572266	.1655273	.1689453	.1660156			
15	.0565796	.1181030	.1849976	.1445923	.1522827	.1552124	.1527710	.1533813		
16	.0520325	.1086121	.1701355	.1329651	.1400452	.1427917	.1403503	.1411438		
17	.0478516	.0998840	.1564636	.1222839	.1287842	.1313171	.1291199	.1296692	.1298828	
18	.0440063	.0918579	.1438904	.1124573	.1184387	.1207581	.1187439	.1192932	.1193237	
19	.0404701	.0844765	.1323280	.1034203	.1089211	.1110573	.1091957	.1097069	.1097755	.1096230
20	.0372181	.0776882	.1216946	.0951099	.1001682	.1021328	.1004238	.1008854	.1009541	.1008511

TABLE [illegible] BARLOW PROSCHAN STRUCTURAL IMPORTANCE
IN A CONSECUTIVE- 3-OUT-OF- n:F SYSTEM

	COMPONENT									
n	1	2	3	4	5	6	7	8	9	10
3	.33333333	.33333333								
4	.08333333	.41666667								
5	.08333333	.16666667	.50000000							
6	.08333333	.16666667	.25000000							
7	.05952381	.14285714	.22619048	.14285714						
8	.05357143	.11309524	.19642857	.13690476						
9	.04761905	.10119048	.16071429	.12500000	.13095238					
10	.04166667	.08928571	.14285714	.10714286	.11904762					
11	.03773449	.07940115	.12702020	.09805195	.10321068	.10916306				
12	.03430736	.07204185	.11370851	.08946609	.09462482	.09585137				
13	.03138528	.06569264	.10342713	.08189033	.08654401	.08827561	.08556999			
14	.02896825	.06035354	.09466089	.07582973	.07947330	.08120491	.07950938			
15	.02687313	.05584138	.08722666	.07050172	.07373460	.07496115	.07377067	.07418137		
16	.02505411	.05192724	.08089549	.06586053	.06868271	.06982046	.06862998	.06912948		
17	.02346542	.04851954	.07539266	.06181457	.06427184	.06527500	.06431763	.06458403	.06471861	
18	.02206127	.04552670	.07058081	.05822650	.06041042	.06127900	.06046315	.06072955	.06072261	
19	.02081307	.04287434	.06633977	.05503179	.05697829	.05775807	.05703796	.05726845	.05729742	.05720168
20	.01969653	.04050960	.06257087	.05217027	.05391525	.05461355	.05398918	.05417643	.05420541	.05415290

TABLE – 5.3

Birnbaum Structural Importance Ranking of Components in a Consecutive-3-out-of-n:F System

n	Increasing order of importance														
5	1	2	3												
6	1	2	3												
7	1	2	4	3											
8	1	2	4	3											
9	1	2	4	5	3										
10	1	2	4	5	3										
11	1	2	4	5	6	3									
12	1	2	4	5	6	3									
13	1	2	4	7	5	6	3								
14	1	2	4	5	7	6	3								
15	1	2	4	5	7	8	6	3							
16	1	2	4	5	7	8	6	3							
17	1	2	4	5	7	8	9	6	3						
18	1	2	4	5	7	8	9	6	3						
19	1	2	4	5	7	10	8	9	6	3					
20	1	2	4	5	7	10	8	9	6	3					
21	1	2	4	5	7	10	8	=11	9	6	3				
22	1	2	4	5	7	10	8	=11	9	6	3				
23	1	2	4	5	7	10	11	8	=12	9	6	3			
24	1	2	4	5	7	10	12	11	8	9	6	3			
25	1	2	4	5	7	10	13	11	=12	8	9	6	3		
26	1	2	4	5	7	10	13	11	=12	8	9	6	3		
27	1	2	4	5	7	10	13	14	12	11	8	9	6	3	
28	1	2	4	5	7	10	13	14	12	11	8	9	6	3	
29	1	2	4	5	7	10	13	14	=15	12	11	8	9	6	3
30	1	2	4	5	7	10	13	14	=15	12	11	8	9	6	3

TABLE – 5.4

Heuristic Ranking of Components in a Consecutive-3-out-of-n:F System

n		Increasing order of importance														
13	B	1	2	4	7	5	6	3								
	H	1	2	4	5	7	6	3								
21	B	1	2	4	5	7	10	8	=11	9	6	3				
	H	1	2	4	5	7	10	11	8	9	6	3				
22	B	1	2	4	5	7	10	8	=11	9	6	3				
	H	1	2	4	5	7	10	11	8	9	6	3				
23	B	1	2	4	5	7	10	11	8	=12	9	6	3			
	H	1	2	4	5	7	10	12	11	8	9	6	3			
25	B	1	2	4	5	7	10	13	11	=12	8	9	6	3		
	H	1	2	4	5	7	10	13	12	11	8	9	6	3		
26	B	1	2	4	5	7	10	13	11	=12	8	9	6	3		
	H	1	2	4	5	7	10	13	12	11	8	9	6	3		
29	B	1	2	4	5	7	10	13	14	=15	12	11	8	9	6	3
	H	1	2	4	5	7	10	13	15	14	12	11	8	9	6	3

For remaining values of $n \leq 30$ the heuristic ranking is identical to Birnbaum structural importance ranking.

Where

B: denotes the Birnbaum structural importance ranking

H: denotes the Heuristic ranking.

TABLE – 5.5

Barlow-Proschan Structural Importance Ranking of Components in a Consecutive-3-out-of-n:F System

n	Increasing order of importance														
5	1	2	3												
6	1	2	3												
7	1	2	4	3											
8	1	2	4	3											
9	1	2	4	5	3										
10	1	2	4	5	3										
11	1	2	4	5	6	3									
12	1	2	4	5	6	3									
13	1	2	4	7	5	6	3								
14	1	2	4	5	7	6	3								
15	1	2	4	5	7	8	6	3							
16	1	2	4	7	5	8	6	3							
17	1	2	4	5	7	8	9	6	3						
18	1	2	4	5	7	9	8	6	3						
19	1	2	4	5	7	10	8	9	6	3					
20	1	2	4	5	7	10	8	9	6	3					
21	1	2	4	5	7	10	11	8	9	6	3				
22	1	2	4	5	7	10	11	8	9	6	3				
23	1	2	4	5	7	10	11	12	8	9	6	3			
24	1	2	4	5	7	10	12	11	8	9	6	3			
25	1	2	4	5	7	10	13	12	11	8	9	6	3		
26	1	2	4	5	7	10	13	12	11	8	9	6	3		
27	1	2	4	5	7	10	13	14	12	11	8	9	6	3	
28	1	2	4	5	7	10	13	14	12	11	8	9	6	3	
29	1	2	4	5	7	10	13	14	=15	12	11	8	9	6	3
30	1	2	4	5	7	10	13	15	14	12	11	8	9	6	3

TABLE – 5.6

Butler's Cut Importance Ranking of Components in a Consecutive-3-out-of-n:F System

n	Increasing order of Importance									
5	1	2	3							
6	1	2	3							
7	1	2	4	3						
8	1	2	4	3						
9	1	2	4	5	3					
10	1	2	4	5	3					
11	1	2	4	5	6	3				
12	1	2	4	5	6	3				
13	1	2	4	5	7	6	3			
14	1	2	4	5	7	6	3			
15	1	2	4	5	7	8	6	3		
16	1	2	4	5	7	8	6	3		
17	1	2	4	5	7	8	9	6	3	
18	1	2	4	5	7	8	9	6	3	
19	1	2	4	5	7	8	10	9	6	3
20	1	2	4	5	7	8	10	9	6	3

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