Essays in Dynamic Games

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Contents

Chapter 1

Introduction

This thesis contains four essays which broadly come under the area of Dynamic Games. All the essays involve developments or applications of non-cooperative equilibrium concepts to games played over infinite horizons. The two essays in Chapter 2 and Chapter 3 involve the concept of 'renegotiation proof equilibria' in repeated games. The essay in Chapter 4 discusses how a social norm of slow building of trust in bilateral relationships can be understood as a social equilibrium even in the absence of asymmetric information problems. Chapter 5, which represents joint work with Prabal Raychaudhuri, applies non-cooperative bargaining theory to a context where the management of a firm has to bargain with the union over wages and employment levels simultaneously, in the presence of minimum wage regulation.

The issues addressed in Chapter 2, titled "Existence of Internally Renegotiation Proof Sets in Infinitely Repeated Games", are as follows. The notion of Renegotiation Proof sets in infinitely repeated games involves internal as well as external consistency. Debraj Ray (1994) has recently argued that in infinitely repeated games with discounting, Weakly Renegotiation Proof sets (WRP sets) as defined by Farrell and Maskin (1989), do not satisfy a natural internal consistency property. Ray defines an Internally Renegotiation Proof (IRP) set as a WRP set which satisfies such an internal consistency property. He shows that as the common discount factor in a two player infinitely repeated game goes to 1, the limit IRP sets can only be either of two types: a singleton or a subset of the efficient frontier of the payoff set. In other words, similar to the results of Benoit and Krishna (1993) for undiscounted finitely repeated games, renegotiation proof sets are either efficient or unique.

Chapter 2 addresses certain questions which nevertheless remained unresolved. Does an IRP set always exist? Ray presented an example where an IRP set fails to exist, but this hinges on low discount factors. He conjectured that IRP sets always exist with sufficiently high discount factors.

In section 2.3 of the chapter, I show that Ray's conjecture is false by presenting an example where an IRP set fails to exist for all high enough discount factors. This holds irrespective of whether pure or mixed strategies are used by the players. The non-existence result is robust to small perturbations of the payoffs. It is interesting to contrast this with the profusion of subgame perfect equilibrium as demonstrated by the Folk Theorem in infinitely repeated games; this illustrates how demanding the requirement of internal consistency can be. It can also be contrasted with the results of Van Damme (1989) and Farrell-Maskin (1989) that limit WRP sets on the Pareto frontier of the feasible set are typically 'large'.

The concept of IRP as examined by Ray is stationary across time. To address the non-existence problem, we introduce, in section 2.4, a modification of the definition of IRP sets to include non-stationary behavior at different histories. For mixed action spaces, our example admits (Almost) Non-Stationary Internally Renegotiation Proof (ANIRP) collections for low discounting. More generally, it is possible for each limit point renegotiation proof set of long finite games to be included in an ANIRP collection, provided that the limit point sets are sufficiently large.

In Chapter 3 titled "Renegotiation Proof Sets in Long Finitely Repeated Games with Low Discounting", I explore connections between renegotiation proof sets in (long) finitely repeated games with discounting, and IRP sets of infinitely repeated games. The notion of renegotiation-proofness in finitely repeated games is relatively uncontroversial, as noted by Benoit-Krishna (1993). If some link exists between the two notions, that would provide additional basis for the claim that IRP is the natural notion of internal consistency in infinitely repeated games. Benoit-Krishna considered undiscounted finitely repeated games and found that if a limit of the renegotiation proof sets exists as the time horizon goes to infinity, then the limit set must either be a singleton or a subset of the efficient frontier of the game. They were not able to settle the existence question although they conjectured that the limits always exist in games with finitely many actions.

The main results of Chapter 3 are the following:

(a) If for high discount factors, the limit of renegotiation proof payoffs of finitely repeated games (as the number of repetitions $T \to \infty$) exists, then the limit sets can converge, as discounting goes to zero, only to sets which are either singletons or are subsets of the efficient frontier of the payoff set. This extends the main result of Benoit-Krishna for the undiscounted case to the case of low rates of discounting.

(b) If as in (a), the limit of renegotiation proof sets for sufficiently long finite repetitions of a game (with discounting) exists, then the set of limiting payoffs must be a WRP set of the infinitely repeated game. If the limit is a singleton, then it must be an IRP set. If in addition mixed strategies are allowed, and the limit set is a "sufficiently large" non-singleton, then it is an almost IRP set of the infinitely repeated game, where an almost IRP set is a slight weakening of the definition of an IRP set.

(c) In the example in Chapter 2, where IRP sets failed to exist in the infinitely repeated context, limits of renegotiation proof sets for long finite repetitions of the stage game also do not exist for low enough discounting (irrespective of whether we consider pure or mixed strategies). Combined with (b) above, this indicates a close connection between IRP sets and renegotiation proof sets for long finite games in the case of low discounting.

(d) In the same example, however, limits of renegotiation proof sets in the undiscounted case exists, when attention is restricted to pure strategies. This is in contrast to the case of discounting, where they do not exist even for discount factors arbitrarily close to 1.

(e) Even when Renegotiation Proof sets for long finite games do not have a limit, it is possible for each limit point set to be included in an ANIRP collection, provided that the limit point sets are sufficiently large. In particular, in the example of Chapter 2, every limit point set for low discounting belongs to an NIRP collection. This provides additional insight into the connection of Renegotiation Proof concept between finitely and infinitely repeated games.

Chapter 4 is titled 'Building Trust'. Trust frequently builds slowly in most bilateral relationships, such as in friendship, credit relations, or employer-employee relations. One obvious reason why trust takes time to build is based on reputation formation, resulting from incomplete information about the characteristics of one's partner. In this chapter, I provide an alternative explanation which applies in a perfect information setting. It is based on the possibility of endogenous quit decisions by partners.

To elaborate on the role of endogenous quits, consider an infinitely repeated two player Prisoners Dilemma. It is now commonly known that if players are sufficiently patient the full cooperation outcome can be supported as a subgame perfect Nash equilibrium, where deviants are punished by suitable threats. Such punishments, however, assume that the deviator is not free to avoid them by terminating the relationship and starting afresh with a new partner. Consequently, it is difficult to support cooperative outcomes in the conventional way. We examine the question of how this possibility of 'endogenous quitting and finding new partners' affects the level of cooperation that can be supported.

We consider a matching model of a credit market along the following lines. There are large numbers of lenders and borrowers. The game takes place over an infinite time horizon and all agents discount their future payoffs identically. An exogenous process matches lenders and borrowers with one another at date 1. Whenever a player of lender is paired with a borrower, they become partners in a new relationship. The stage game played between a matched pair is common knowledge and is as follows. The lender offers a loan of a certain size (a level of trust) to his client. The borrower decides whether to default on the loan or repay it. Default generates short term gains to the borrower, and short term losses to the lender. Before period 1 ends the partners simultaneously decide either to continue or terminate the relationship. Among all relationships where both partners decided to continue, nature picks a fraction of those relationships and terminates them for exogenous reasons. At the start of the next period (date 2), another random matching amongst all the players with broken partnerships takes place and then the game continues as before between the current partners.

Important features of the model are as follows: (a) a partnership can continue if it is wished to by both partners, except that (b) there is a small probability with which a partnership can be broken for exogenous reasons ; (c) when two new partners meet, they are not aware of the past histories of the other; (d) and most importantly, they ignore their own past histories in deciding how to play with the new partner; so players play with all new partners identically. In this setting, cooperative behavior can be enforced only by the threat of terminating the relationship, provided termination is costly for the deviant. In the absence of any "unemployment" a la Shapiro-Stiglitz (1984), this requires the 'next' relationship to involve a slow rather than immediate build-up of cooperation.

We restrict our attention to equilibria where trust is never dishonored along the equilibrium path. The main results are that maximal equilibria in terms of payoffs exist for the population; along the equilibrium path of a maximal equilibrium average trust must be non-decreasing and strictly increasing between some time points; as the quit rate goes to zero and the discount factor goes to one, along a maximal equilibrium path, the level of trust offered must either attain or asymptotically approach the maximal level.

Chapter 5 is titled "Management Union Bargaining under Minimum Wage Regulation" (joint work with Prabal Raychaudhuri). This chapter addresses policy questions relating to minimum wage regulations. In many countries the government enacts minimum wage laws in an effort to raise living standards of the workers. In India, for example, the Minimum Wages Act of 1948 lays down standards of minimum wage. The objective was stated to be "not merely...the bare sustenance of life but...for some measure of education, medical requirements and amenities." In other countries as well, such laws were motivated by similar concerns.

Unfortunately, however, such laws may have a detrimental effect on the level of employment, as they may induce firms to reduce the number of workers employed. The usual argument against such an objection is that, in the presence of unionized workers, such an outcome cannot result. It is contended that the unions would prevent the employment level from declining at all, or that they would minimize the extent of any such decline. The objective of this chapter is to examine, in a formal bargaining setup and in the presence of unionized workers, the impact of changes in minimum wage laws.

We model this problem as an infinite horizon alternating offers bargaining game. Since we are concerned about the levels of both employment and the wage, we allow the management and the union to bargain over both simultaneously. Therefore, in contrast to the standard bargaining models (e.g. Rubinstein (1982)), the cake size is endogenously determined. The game starts with the management making an offer and the union accepting or rejecting it immediately. If the union rejects the offer, then it can make a counter-offer after a delay of one period. The management in its turn can now either accept the offer, or reject it. The game continues in this manner until an agreement is reached, when the agreed upon offer is implemented. We assume that there exists a minimum wage level, fixed by the government, below which a worker cannot be employed by the firm. The union maximizes the wage bill of the workers who belong to the union.

We first establish that a subgame perfect Nash equilibrium exists and that it is unique. The structure of the equilibrium outcome is rather interesting. We show that the outcome always involves a wage level equal to the minimum wage set by the government. The level of employment, however, is higher compared to what it would have been in the absence of the union (i.e., the competitive level). This suggests that although bargaining takes place over both wage and the employment level, in effect the union can affect only one of these. It is in the interests of the workers not to demand an increase in wage level over and above that set by the government.

We then carry out some comparative statics exercises. Those involving the discount factors of the management and the union (δ_1 and δ_2 respectively) are what we would expect intuitively, in the sense that an increase in δ_i leads to an increase in the payoff of the concerned party and a decrease in the payoff of the other party. If, however, the management's bargaining position is strong enough, then a small increase in δ_2 may leave the payoffs unaffected.

We now look into the effects of an increase in the minimum wage level. We show that the employment level will decline. We also demonstrate that the income of the management declines as well. The surprising part of the result is that the income of the union may decline as well. We show that a sufficient condition for this to occur is that the marginal product of labour is inelastic. Thus, somewhat paradoxically, it is the very success of the union in increasing the level of employment which ensures that the workers lose out as a result of an increase in minimum wage.

Chapter 2

Existence of Internally Renegotiation Proof Sets in Infinitely Repeated Games

2.1 Introduction

The notion of Renegotiation Proof equilibrium in infinitely repeated games involves internal[1] as well as external[2] consistency. Ray (1994) has recently argued that in infinitely repeated games with discounting, Weakly Renegotiation Proof sets (WRP sets) as defined by Farrell and Maskin (1989), do not satisfy a natural internal consistency property[3]. Ray defines an Internally Renegotiation Proof (IRP) set as a WRP set which satisfies such an internal consistency property. He shows that as the common discount factor in a two player infinitely repeated game goes to 1, the limit IRP sets can only be either of two types: a singleton, or a subset of the efficient frontier of the payoff set. In other words, similar to the results of Benoit and Krishna (1993) for undiscounted finitely repeated games, renegotiation proof sets are either efficient or unique.

However, the existing literature leaves some issues unresolved, which are addressed in this chapter. Does an IRP set always exist? Ray produces an example where an IRP set fails to exist, but this hinges on low discount rates. He conjectures that IRP sets always exist with sufficiently high discount factors.

In section 2.3, I show that Ray's conjecture is false by presenting an example where an IRP set fails to exist for all sufficiently high discount factors. This holds irrespective of whether pure or mixed strategies are used by the players. The non-existence result is also robust to small perturbations of the payoffs. It is interesting to contrast this with the profusion of subgame perfect equilibrium for high discount factors as demonstrated by the Folk Theorem in infinitely repeated games; this illustrates how demanding the requirement of internal consistency can be. Moreover, it can be contrasted with the results of Van Damme (1989) and Farrell-Maskin (1989) concerning WRP sets; for certain

[1]See Bernheim and Ray (1989), Farrell and Maskin (1989) and Ray (1994) for a discussion. Note that Pearce ((1987) has a completely different way of studying renegotiation proofness. Also see Abreu, Pearce and Stacchetti (1989).

[2]See Bernheim and Ray (1989), Farrell and Maskin (1989) and Asheim (1991) for a discussion.

[3]Roughly speaking, this requires a set B of payoffs to be the Pareto Frontier of the set of payoffs supportable by continuation payoffs from the set B itself.

classes of games they show the existence of "large" limit WRP sets on the Pareto frontier of the feasible payoff set[4].

The concept of IRP set as examined by Ray embodies a form of stationary behavior. To address the existence issue, we consider in section 2.4, a modification of the definition of IRP sets to include non-stationary[5] behavior at different histories: this is referred to as Almost Non-Stationary Internally Renegotiation Proof (ANIRP) collection. For mixed action spaces, ANIRP collections exist in our example for high discount factors. More generally, it is possible for each limit point renegotiation proof set of long finite games to be included in an ANIRP collection, provided that the limit point sets are "sufficiently large". This is shown in Theorem 2.2 and Corollary 2.1.

In the next chapter of this thesis, I explore connections between renegotiation proof sets in (long) finitely repeated games with discounting, and IRP sets of infinitely repeated games.

Section 2.2 introduces the basic framework of the model. Section 2.3 presents the example where IRP sets do not exist for sufficiently high discount factors. Section 2.4 deals with the notion of ANIRP collections. Section 2.5 concludes. We shall essentially employ the notation used in Ray (1994).

2.2 Framework

Consider a one-shot game $G \equiv (A_1, A_2, g_1, g_2)$ played by two players (denoted 1 and 2 respectively). Each player is assumed to have a finite number of pure actions and A_i denotes the set of mixed actions[6] for i, while $g_i(\cdot) : A_1 \times A_2 \to I\!R$ denotes i's payoff ($i = 1, 2$). $g_i(\cdot)$ is continuous[7] for each i. Let $A = A_1 \times A_2$.

[4]Our example also has this property.

[5]Ray makes a passing mention of this non-stationary conterpart of consistency although he does not deal with it explicitly.

[6]Certain statements are also valid for pure strategy action spaces. We will mention them as we go along. However, we assume that there are a finite number of pure actions supporting the mixed strategies.

[7]When $a = (a_1, a_2) \in A$, a_i can be viewed as a probability distribution over the pure actions of player i. For $a_1, a_1' \in A_1$ and $a_2 \in A_2$, and for $\lambda \in [0, 1]$, $(1 - \lambda)a_1 + \lambda a_1'$ denotes a new

For an action vector $a = (a_1, a_2) \in A$, let $c_1(a) \equiv \text{Max}_{\{a'_1 \in A_1\}}\ g_1(a'_1, a_2)$ and $c_2(a) \equiv \text{Max}_{\{a'_2 \in A_2\}}\ g_2(a_1, a'_2)$. The functions $c_1(\cdot)$ and $c_2(\cdot)$ are continuous and represent the maximum deviation payoffs for player 1 and 2 respectively from any action vector. Let F^* be the convex hull of the payoffs of the two players in G. Let δ be the common discount factor and (G^T, δ) denote the T-fold repeated version of G. (G^∞, δ) denotes the infinitely repeated version of G with δ as the discount factor.

For a sequence $\langle a_t \rangle_{t=0}^{T} \in A^{T+1}$ and $\delta < 1$, the normalized payoff to player i in (G^{T+1}, δ) from this action sequence is $\frac{(1-\delta)}{(1-\delta^{T+1})} \sum_{t=0}^{T} \delta^t g_i(a_t)$. For a sequence $\langle a_t \rangle_{t=0}^{\infty} \in A^\infty$ in (G^∞, δ) with $\delta < 1$, the normalized payoff to player i from this action sequence is $(1-\delta) \sum_{t=0}^{\infty} \delta^t g_i(a_t)$. Note that all normalized payoffs lie in F^*.

Definition 1. Let $B(T, \delta)$ be a set of normalized payoffs in (G^T, δ) for $\delta < 1$. Clearly $B(T, \delta) \subseteq F^* \subset \mathbb{R}^2$. Then the payoff vector $p(T+1) \in \mathbb{R}^2$ is said to be **supportable** by $B(T, \delta)$ if there exists $a \in A$ and $\hat{p}(T), p^1(T), p^2(T) \in B(T, \delta)$ such that for $i = 1, 2$:

$$p_i(T+1) = \frac{(1-\delta)}{(1-\delta^{T+1})} g_i(a) + \frac{\delta(1-\delta^T)}{(1-\delta^{T+1})} \hat{p}_i(T)$$

$$\text{and} \quad [c_i(a) - g_i(a)] \leq \frac{\delta(1-\delta^T)}{(1-\delta)} (\hat{p}_i(T) - p_i^i(T)).$$

When the above conditions hold, we will say that the payoff vector $p(T+1)$ is **supported** by $B(T, \delta)$ through the action vector a and the payoff vectors $\hat{p}(T), p^1(T), p^2(T) \in B(T, \delta)$.

As in Benoit-Krishna (1993), the payoff vector $p(T+1)$ is supported in (G^{T+1}, δ) by an action vector a in period 1, and continuation payoff $\hat{p}(T)$ on the equilibrium path, with $p^1(T)$ and $p^2(T)$ denoting the punishment continuation payoffs for players 1 and 2 respectively.

Definition 2. $p \in \mathbb{R}^2$ is said to be **supportable** by a set $B \subseteq F^*$ in (G^∞, δ)

mixed strategy for player 1. If a_{1k}, a'_{1k} denote the probabilities that a_1, a'_1 respectively place on pure action k of player 1, then $(1-\lambda)a_{1k} + \lambda a'_{1k}$ is the probability that $(1-\lambda)a_1 + \lambda a'_1$ places on pure action k. Hence, $g_i((1-\lambda)a_1 + \lambda a'_1, a_2) = (1-\lambda) g_i(a_1.a_2) + \lambda g_i(a'_1, a_2)$.

with $\delta < 1$ if there exists $a \in A$ and $\hat{p}, p^1, p^2 \in B$ such that for $i = 1, 2$:

$$p_i = (1-\delta)g_i(a) + \delta\hat{p}_i$$

and

$$[c_i(a) - g_i(a)] \leq \frac{\delta}{1-\delta}(\hat{p}_i - p_i^i).$$

When the above conditions hold, we will say that the payoff vector p is **supported** by B through the action vector a and the payoff vectors $\hat{p}, p^1, p^2 \in B$.

For a nonempty set of payoffs $B(T,\delta) \subseteq F^*$, let $\theta(B(T,\delta))$ denote the set of payoff vectors supported by $B(T,\delta)$. It follows that $\theta(B(T,\delta))$ is compact if $B(T,\delta)$ is compact. $\theta(\cdot)$ can similarly be defined for the infinitely repeated game (G^∞, δ). To avoid confusion, we will henceforth write this function as $\theta^I(\cdot)$ for the infinitely repeated game.

For any non-empty set $C \subset I\!R^2$, define $F(C) = \{x \in C |$ there is no $y \in C$ such that $y >> x\}$ where $y >> x$ if $y_i > x_i$ for $i = 1, 2$. Hence, $F(\theta(B(T,\delta)))$ is the weak Pareto Frontier of payoffs supported by $B(T,\delta)$. Similarly, in (G^∞, δ), $F(\theta^I(B))$ denotes the weak Pareto frontier of payoffs supported by B. The papers of Farrell and Maskin (1989), Benoit and Krishna (1993), Ray (1994) all consider this definition of Pareto optimality. We are now in a position to define two notions of renegotiation proofness in infinitely repeated games.

Definition 3. A set of normalized payoffs $W(\delta) \subset I\!R^2$ is WRP (Weakly Renegotiation Proof) in (G^∞, δ) if and only if
(a) $W(\delta) \subset \theta(W(\delta))$
(b) There exist no two vectors $x, y \in W(\delta)$ such that $x >> y$.

This definition of WRP set is due to Farrell and Maskin (1989). In a WRP set, every payoff vector in the set is supported by the same set and no payoff vector in the set strongly dominates any other payoff vector in the set.

Definition 4. A compact[8] set $P \subseteq I\!R^2$ is Internally Renegotiation Proof (IRP) in (G^∞, δ) if $P = F(\theta(P))$.

[8]It is not true in general that if P is an IRP set, closure(P) is also an IRP set. Nevertheless,

This is Ray's definition of an IRP set. Hence, a payoff in the IRP set P must belong to the weak efficient frontier of the set of payoffs supported by P itself. Further, the set of all payoff vectors on the weak efficient frontier of the set supported by P must identically equal P.

Let $\mathcal{P}(\delta)$ denote the collection of all compact IRP sets of the game for a given δ. The space of all compact subsets of F^* endowed with the Hausdorff distance relative to the Euclidean distance, is a compact metric space (see Theorem 1, page 17 of Hildenbrand (1974)). Define $\mathcal{P}$ to be the set of all limit points[9] of sequences $\langle P(\delta_n)\rangle_{n=1}^{\infty}$, where $\delta_n \to 1$ and $P(\delta_n) \in \mathcal{P}(\delta_n)$ for each n. The convergence is with respect to the Hausdorff metric relative to the Euclidean distance. If $P \in \mathcal{P}$, then P is closed by virtue of a property of set convergence. We will refer to a set belonging to $\mathcal{P}$ as a limit IRP set.

Theorem 2.1 (Ray, 1994) *Each element of $\mathcal{P}$ is either a singleton or is a subset of $F(F^*)$, the efficient frontier of F^*.*

Although Ray considered only the case of compact IRP sets, the result extends identically to the case of IRP sets when the compactness assumption is not imposed.

Before proceeding any further we need to define the 'minimum payoff vector for a player i' in a compact set $B \subseteq F^*$. For any such set B, consider the subset of payoff vectors in B which gives player i the minimum among all payoff vectors in B. If this subset has an unique element, then that unique element is player i's 'minimum payoff vector in B'. If the subset (note that this subset is closed) is non-unique, then take that payoff vector in this subset which gives player j ($j \neq i$) the highest payoff among all vectors in this subset. This vector will be called the ***minimum payoff vector for player i in B***. Note that a minimum payoff vector for a player i in any non-empty compact set $B \subseteq F^*$ always exists

we concentrate on compact IRP sets in examining the existence question. However, a simple modification of the non-existence proof in our example goes through even when we allow for non-compact IRP sets.

[9]A set B is a limit point of a sequence of sets $\langle B_n\rangle_{n=1}^{\infty}$ if there is subsequence $\langle B_{n_k}\rangle_{k=1}^{\infty}$ which converges to B.

and is unique. Player i's minimum payoff vector in a non-empty compact set will typically be denoted as $\bar{p}^i$.

2.3 Nonexistence of IRP sets

Consider the game with payoffs as depicted in Figure 2.1. Figure 2.2 gives the structure of the convex hull of the feasible payoff set F^* for the game. There are two pure strategy Nash equilibrium payoffs : (4,1) which results from Player 1 playing M and player 2 playing L, and (1,4) resulting from player 1 playing T and player 2 playing C. We will argue that a limit IRP set cannot exist in this game. There are four major parts in the argument. The outline of these four parts are as follows:

(a) First, we will show that for any discount factor δ, the minimum payoff vector $\bar{p}^1(\delta)$ for player 1 in any IRP set cannot give player 2 a payoff less than 4. The argument runs as follows. Suppose a IRP set exists for some discount factor and this result is false. The payoff $(1-\delta)(1,4)+\delta\bar{p}^1(\delta)$ is supportable by the IRP set. This new vector (or something that Pareto dominates it) must also belong to the IRP set. Now there are two possibilities. Either this new vector Pareto dominates $\bar{p}^1(\delta)$ which contradicts the fact that $\bar{p}^1(\delta)$ belongs to the IRP set. Or this new vector gives player 1 less than his minimum payoff vector $\bar{p}^1(\delta)$ in the IRP set, which contradicts the definition of a 'minimum' vector.

(b) Second, we will use a result (Proposition 2.1) to conclude that the minimum vector for player 1 in any IRP set for any discount factor cannot give player 2 a payoff greater than 4. Proposition 2.1 really makes statements for WRP sets. It tells us when a particular vector in the convex hull of the payoff space can never belong to any WRP set. Since an IRP set is a WRP set, we can use the proposition to conclude that no payoff vector which gives player 2 a payoff more than 4 can ever belong to an IRP set.

(c) Third, using (a) and (b), we will show that any IRP set must contain the payoff vector (1,4). (a) and (b) tells us that $\bar{p}^1_2(\delta)$ must be equal to 4. It cannot be that $\bar{p}^1_1(\delta)>1$, for this will contradict that $\bar{p}^1(\delta)$ is player 1's

Player 2's pure actions

Player 1's pure actions	L	C	R
T	-36,-36	1,4	6,0
M	4,1	0,0	0,0
B	0,6	0,0	0.8,0.8

Figure 2.1

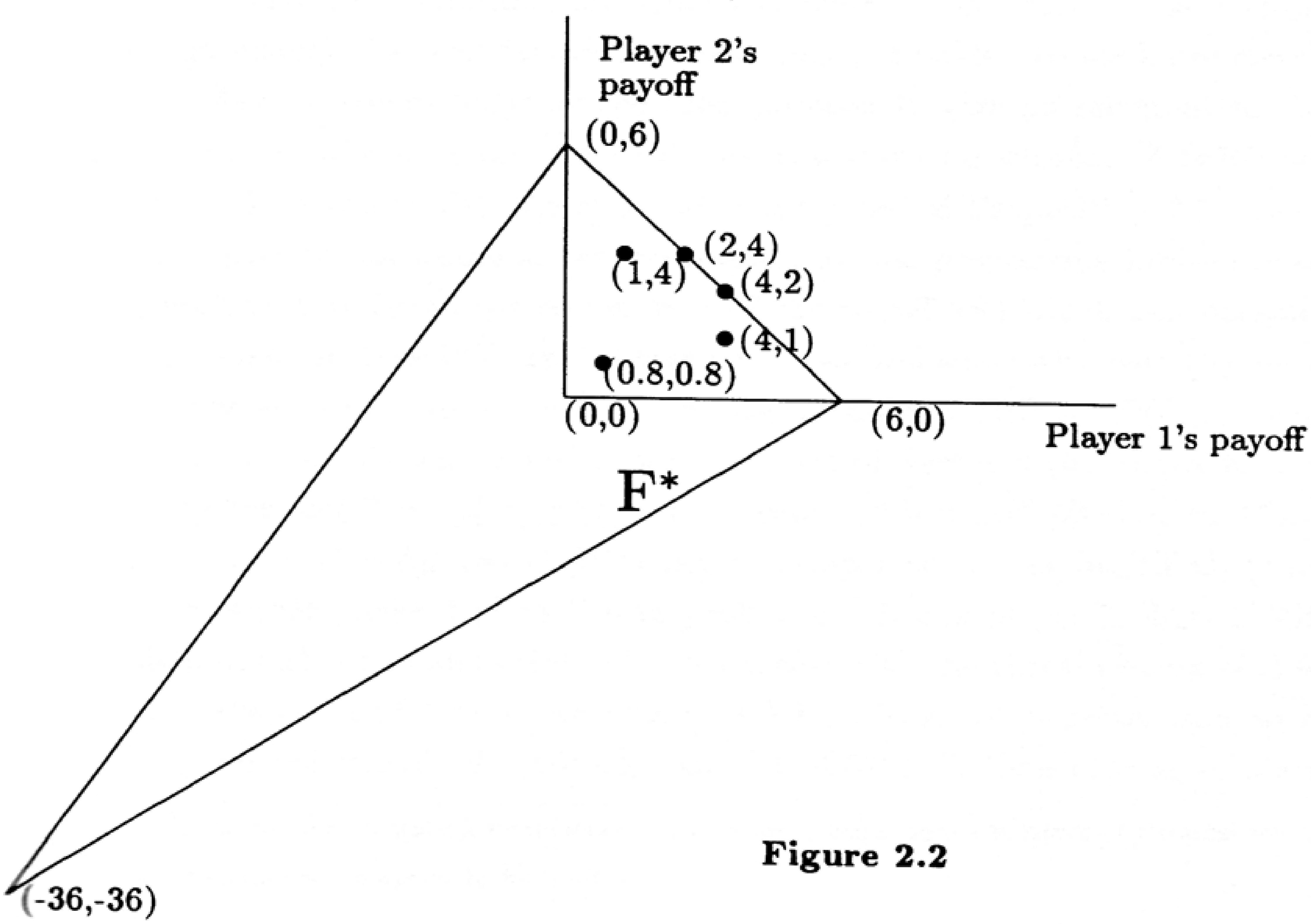

Figure 2.2

minimum payoff in the IRP set. If $\bar{p}_1^1(\delta) < 1$, then it will be argued that (1,4) is contained in the IRP set.

(d) Finally, from the symmetry of the problem (using the minimum payoff of player 2 in any IRP set) it can be argued that (4,1) must also belong to any IRP set. Since both (1,4) and (4,1) must belong to any IRP set, a limit IRP set cannot be a singleton; nor can it belong to the Pareto frontier of F^*, as required by Ray's theorem, as neither of these two vectors belong to the Pareto frontier. So, no limit IRP set can exist, which implies that for high enough discount factors IRP sets do not exist in this example.

The rest of this section is organized as follows. We will start with the formal proof of part (a). Next, for the time being, we will assume Proposition 2.1 and conclude[10] part (b). At this stage, it is enough to note that part (b) holds, since Proposition 2.1, applied to our example, tells us that no IRP set can contain a payoff vector which gives player 2 a payoff greater than 4. Following this, we will show part (c) formally. Part (d), requires no further elaboration. At this stage, we would have seen that IRP sets do not exist in the example for high enough discount factors provided part (b) holds. To see more clearly where Ray's theorem helps us, we then proceed to give an intuition to this non-existence phenomenon. After this, we will start a discussion of WRP sets which will conclude with the statement and proof of Proposition 2.1. Next, with respect to our example, we will relate how the proposition limits the set of payoff vectors (from the convex hull of the payoff set) which can possibily be contained in some IRP set; in other words we will show why part (b) holds.

Suppose part (a) does not hold; i.e., there exists an IRP set $P(\delta)$ for some discount factor δ such that the minimum payoff vector $\bar{p}^1(\delta)$ of player 1 in $P(\delta)$ (where $\bar{p}^1(\delta) = (\bar{p}_1^1(\delta), \bar{p}_2^1(\delta)))$ gives player 2 a payoff $\bar{p}_2^1(\delta) < 4$. Check that $y(1) = (1-\delta)(1,4) + \delta\bar{p}^1(\delta)$ can be supported by playing (T,C) (what I mean is that player 1 plays T with probability 1, and player 2 plays C with probability 1) today and using $\bar{p}^1(\delta)$ tomorrow with the threats being $p(\delta)$ for both players 1 and 2 (observe that (T,C) is a Nash equilibrium and so no player can unilaterally deviate and gain). If $\bar{p}_1^1(\delta) > 1$, then $y(1)$ gives player

[10] The reason for relegating Proposition 2.1 to later space is that the proof of Proposition 2.1 requires some technicalities to be built up.

1 a payoff of less than $\bar{p}_1^1(\delta)$. Since $y(1) \in F(\theta(P(\delta)))$ and $P(\delta)$ is an IRP set, either of two cases must hold: (a) $y(1) \in P(\delta)$, or (b) there exists $z \in P(\delta)$ such that $z >> y(1)$. If (a) holds, this contradicts that $\bar{p}^1(\delta)$ is the minimum payoff vector for player 1 in $P(\delta)$. If (b) holds, then note that $z_2 > \bar{p}_2^1(\delta)$. If $z_1 \leq \bar{p}_2^1(\delta)$, then $\bar{p}^1(\delta)$ cannot be the minimal vector for player 1 in $P(\delta)$. If $z_1 > \bar{p}_1^1(\delta)$, then z clearly dominates $\bar{p}^1(\delta)$ which contradicts that $\bar{p}^1(\delta)$ belongs to the IRP set $P(\delta)$. If $\bar{p}_1^1(\delta) < 1$ and $\bar{p}_2^1(\delta) < 4$, then $y(1)$ Pareto dominates $\bar{p}^1(\delta)$. This contradicts the fact that no payoff vector supported by an IRP set can Pareto dominate any other vector in the IRP set. If $\bar{p}_1^1(\delta) = 1$ and $\bar{p}_2^1(\delta) < 4$, then $y(1)$ again contradicts that $\bar{p}^1(\delta)$ (as defined) is the minimum for player 1 in the IRP set. This shows that it is not possible that $\bar{p}_2^1(\delta) < 4$.

Once we assume Proposition 2.1 and the fact that it implies in our example that player 2 cannot get more than 4 in any vector contained in the IRP set, part (b) immediately follows. It cannot be that $\bar{p}_2^1(\delta) > 4$.

We are left with the proof of part (c). If $\bar{p}_1^1(\delta) > 1$ and $\bar{p}_2^1 = 4$, then $y(1)$ is supportable by $P(\delta)$. $y(1)$ must belong to $P(\delta)$ for if any vector strongly Pareto dominates $y(1)$, then we contradict Proposition 2.1 (that no payoff vector in an IRP set can give player 2 a payoff greater than 4). If $\bar{p}_1^1(\delta) < 1$ and $\bar{p}_2^1 = 4$, then note that $y(2) = (1-\delta)(1,4)+\delta y(1)$ which results from playing the Nash equilibrium (T, C) today and $y(1)$ from tomorrow, is supportable by $P(\delta)$ and must belong to $P(\delta)$ (nothing can strongly Pareto dominate this vector when (b) holds. In this way, by repeatedly supporting payoffs with the help of $(1,4)$ and $y(k)$, we can in the limit approach $(1,4)$ which must belong to $P(\delta)$ as $P(\delta)$ is compact.

Now, applying the arguments given in the outline for part (d), we conclude that our example does not admit any IRP sets for all high enough discount factors.

Assuming the ramifications of Proposition 2.1, it is still instructive to know where Ray's theorem is helping us. The intuition behind the non-existence of a singleton limit set is simple. Such a singleton set must Pareto dominate (at least weakly) both the Nash equilibrium payoffs. Otherwise, for large δ, points close to that Nash equilibrium which is not dominated can be supported by repeatedly supporting payoffs with the help of that Nash equilibrium (note that nobody can unilaterally deviate and gain from a Nash equilibrium). However, given the structure of F^* in the example, there exists no payoff in F^* which weakly Pareto dominates both the Nash equlibria. It is in the non-existence of a non-singleton limit IRP set where we use the full power of Ray's Theorem : the argument hinges strongly on the structure of IRP set. From Proposition 2.1, no IRP set can have a payoff where player 2 gets more than 4. If IRP sets exist, (1,4) should belong to the IRP sets as shown. The proof of Ray's theorem implicitly shows that if (1,4) has to belong to a non-singleton limit IRP set, then for large discount factors, points arbitrarily close to (2,4) must also belong to each member of the sequence of IRP sets which converge to the non-singleton limit. Let $\langle \delta_n \rangle$ be a sequence of discount factors converging to 1 as $n \to \infty$. For any large discount factor δ_n, let z_n be close enough to (2,4). Let x_n be the point on the line segment joining (0,6) and z_n which gives player 2 a payoff of 4. Now x_n, for z_n sufficiently close to (2,4), is far away from (1,4). Consider the first large enough positive integer k such that $(1-\delta_n^k)(0,6)+\delta_n^k z_n$ Pareto dominates (1,4) for the first time. If K is the first such k, then note that the expression $(1-\delta_n^k)(0,6)+\delta_n^k z_n$ can be supported by the three element sets $\{z_n, (1,4), (4,1)\}$ for each $k \leq K$ with the threat for player 1 being reversion to the Nash equilibrium with payoff (1,4) if he deviates (the threat for player 2 is reversion to the other Nash equilibrium with payoff (4,1)). The future losses of player 1 for deviating is at least $\delta_n(x_n - 1)$ which for large n (since player 1's payoff from x_n is greater than and bounded away from 1) will be higher than $(1-\delta_n)4$ (the one shot deviation gains). Now, one can argue that $(1-\delta_n^K)(0,6)+\delta_n^K z_n$, or a payoff that Pareto dominates it, must belong to the IRP set. However, player 2 gets more than 4 in this payoff vector. Consequently, this violates Proposition 2.1. Hence, in this example, there can be no IRP set for high enough discount factors.

A minimal modification of this non-existence proof goes through even when we do not explicitly require IRP sets to be compact.

Now, we proceed towards the statement and proof of Proposition 2.1. To develop the idea and the notation, we will start with a discussion on WRP sets. The first part of the discussion is essentially on the intuition behind the necessary conditions of Theorem 1 of Farrell and Maskin (1989). Consider the payoff vector in a compact WRP set which gives player 1 his minimum payoff in the WRP set (once again, if this is not unique, then take that vector among player 1's minimizers which give player 2 the highest payoff). Let this vector be $\bar{p}^1(\delta)$. Let $\bar{p}^1(\delta)$ be supported by the compact WRP set with action vector b and payoff vectors p, p^1, p^2, all belonging to the WRP set such that

$$(1-\delta)(g_1(b), g_2(b)) + \delta p = \bar{p}^1(\delta)$$

$$\text{and for each } i = 1,2, (1-\delta)[c_i(b) - g_i(b)] \leq \delta[p_i - \bar{p}_i^i].$$

Note that $(g_1(b), g_2(b))$ must belong weakly to the northwest[11] of $\bar{p}^1(\delta)$ (If it belongs strictly to the northeast, then $\bar{p}^1(\delta)$ dominates the continuation vector with which it is supported and this cannot happen. If it lies strictly to the southwest of $\bar{p}^1(\delta)$, then $\bar{p}^1(\delta)$ is dominated by the vector with which it is supported and this is also not possible. If it belongs to the southeast of $\bar{p}^1(\delta)$, then the vector with which $\bar{p}^1(\delta)$ is supported contradicts the definition of $\bar{p}^1(\delta)$). Note further the action pair must satisfy $(1-\delta)c_1(b) + \delta\bar{p}_1^1 \leq \bar{p}_1^1$ if player 1 is not to deviate and gain from $\bar{p}^1(\delta)$; in other words $c_1(b) \leq \bar{p}_1^1(\delta)$. Since such an action vector b has to exist, so for any $v = (v_1, v_2)$ belonging to the WRP set, vector a belongs weakly to the northwest of v and satisfies $c_1(a) \leq v_1$ (note that v belongs weakly to the southwest of $\bar{p}^1(\delta)$).

Let A' denote the set of all action pairs (of the one shot game G) such that for each $a \in A'$ ($a = (a_1, a_2)$ where a_1 is player 1's action and a_2 is player 2's action), there exists some v_1 such that $(v_1, g_2(a)) \in F(F^*)$. In the example (see figure 2.3) A' corresponds to that section of F^* which gives player 2 at least a payoff of zero.

For each $a \in A'$, let $f(a)$ denote the maximum value of v_1 for which $(v_1, g_2(a)) \in F(F^*)$ for each $a \in A'$. As an illustration, in the example, $f(T, C) = 4$.

Consider the following problem

$$\text{Min}_{a \in A'} \ \text{Max}[c_1(a), f(a)] \tag{2.3.1}$$

[11] A vector (v_1, v_2) lies weakly to the northwest of (v_1', v_2') iff $v_1 \leq v_1'$ and $v_2 \geq v_2'$. Observe that (v_1', v_2') lies weakly to the northwest of itself. Lying 'to the northwest' indicates that at least one of the inequalities is strict. Lying 'strictly to the northwest' implies that both the inequalities are strict. Similarly, lying to the southwest, southeast, northeast can be defined.

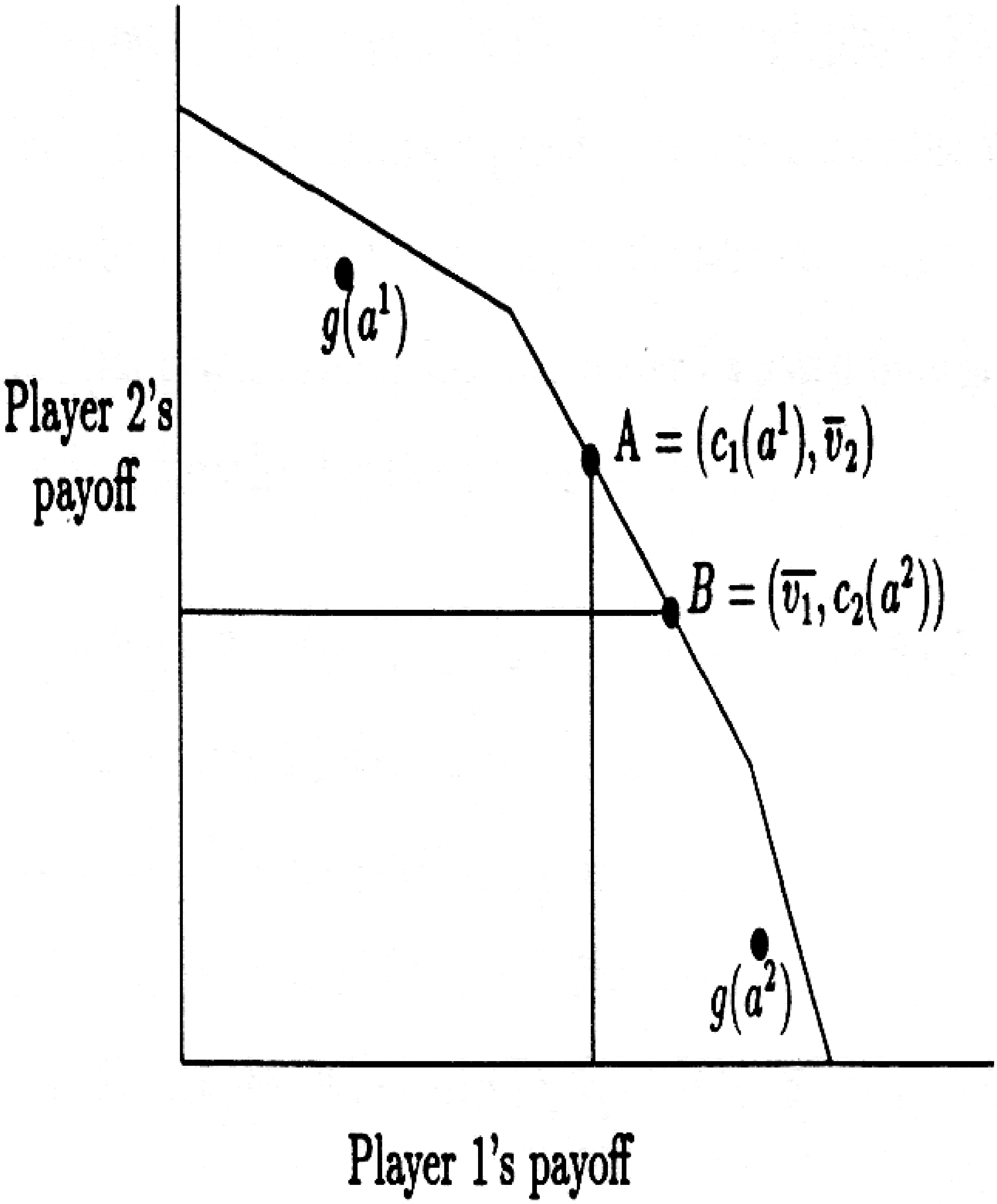
$g(a^1)$
Player 2's payoff
$A = (c_1(a^1), \overline{v}_2)$
$B = (\overline{v_1}, c_2(a^2))$
$g(a^2)$
Player 1's payoff

Figure 2.3

where it may be remembered that $c_1(a)$ denotes the maximum deviation payoff for player 1 from action vector a. Note that the solution to program (2.3.1) exists because c_1 and f are both continuous functions on a compact set A'. It can be shown (see Appendix) that irrespective of whether A_i for $i = 1, 2$, are mixed or pure action spaces, in the example, the solution to (2.3.1) is uniquely attained at $a^1 = (T, C)$ with payoffs (1,4). $c_1(T, C) = 1$ and $f(T, C) = 4$.

Program (2.3.1) helps us to identify how high player 2's best payoff vector (which is player 1's minimum payoff vector) can be in the WRP set such that we can have an action vector lying to the northwest with maximum deviation payoff (for player 1) being less than player 1's payoff from his minimum vector. All this is formalized in the next proposition.

Proposition 2.1 *Let player 1's payoff be written on the X-axis and player 2's payoff be written on the Y-axis. Suppose for a game G, $F(F^*)$ is downward sloping. Let a^1 be one solution to problem (2.3.1) and k_1 be the value of (2.3.1). Let $X = k_1$ intersect $F(F^*)$ at $\bar{v}_2$. Then for any $\delta < 1$, there cannot be a WRP set $W(\delta)$ such that there exists $(v_1, v_2) \in W(\delta)$ where $v_2 > \bar{v}_2$.*

Proof. (See Figure 2.3) If there exists a WRP set with a payoff (v_1, v_2) such that $v_2 > \bar{v}_2$, then from Farrell and Maskin (Theorem 1, 1989) $\exists\ \tilde{a}$ such that $c_1(\tilde{a}) \leq v_1 < k_1$ and $g_2(\tilde{a}) \geq v_2 > \bar{v}_2$. Now if $F(F^*)$ is downward sloping, then $f(\tilde{a}) < k_1$. So $\tilde{a}$ contradicts the fact that a^1 is a solution of (2.3.1). ■

The existence of a^2 corresponding to player 2's version of (2.3.1) is also guaranteed. In the example, $k_1 = \text{Max}\ [c_1(T, C), f(T, C)] = 2$; where $c_1(T, C) = 1$ and $f(T, C) = 2$. So, $\bar{v}_2 = 4$. To see the implication of the Proposition 2.1 on our example, note that if $P(\delta)$ is an IRP set, then it is a WRP set. So, it is not possible that $\bar{p}^1_2(\delta) > 4$. A symmetric argument rules out $\bar{p}^2_1(\delta)$ to be greater than 4.

The game in our example has no dominated strategies. A small perturbation of the payoff structure keeps the non-existence result valid. This is because the solution to program (2.3.1) still remains a Nash equilibrium for small perturbations. There will be another Nash equilibrium far apart and there will be no payoff in F^* which will dominate both the equilibria simultaneously. This is enough for non-existence of limit IRP set. Further, even for pure strategies an IRP set does not exist (the same proof). The non-existence remains even when we consider the weak Pareto dominance criterion.

2.4 Almost Non-Stationary IRP Collection

We wish to examine if existence can be restored in our example with a modification of the concept of an IRP set. Specifically, we modify the stationarity (across histories) assumption in the definition of IRP set to include non-stationary behavior at different histories [12]. If the theory of tomorrow is that any payoff from the payoff set P' can occur, then consistency should require that P, the theory of what payoffs can accrue to the players today given tomorrow's theory, should be the Pareto frontier of the set of payoffs supportable by P'. In other words, today's theory P should equal $F(\theta^I(P'))$. Further, consistency also requires that the theory P' of tomorrow should be consistent with the theory of day after tomorrow (say P''); i.e. $P' = F(\theta^I(P''))$. Similarly, day after tomorrow's theory should be consistently derived from "day after day after tomorrow's" theory and so on. Note that the concept of IRP set goes beyond this kind of requirement of consistency. The definition of IRP set requires that not only should the theories at different dates be derived consistently, but that they should all be the same; i.e., it imposes an additional stationarity property. We now dispense with this stationarity property in the following notion of a Non-Stationary Internally Renegotiation Proof collection.

Definition 5. We call a collection N of compact subsets of F^*, a Non-Stationary Internally Renegotiation Proof Collection (**NIRP** collection) if for any subset $P_t \in N$, there exists some subset $P_{t+1} \in N$ such that $P_t = F(\theta^I(P_{t+1}))$.

In other words, if P_{t+1} is the theory of tomorrow, then today's admissible payoff set P_t is the efficient frontier of payoffs supported by P_{t+1}. Note that P_{t+1} must itself satisfy $P_{t+1} = F(\theta^I(P_{t+2}))$ for some $P_{t+2} \in N$ and so on. When

[12]Ray has mentioned this non-stationary concept in his paper although he has not elaborated on its properties. Also, see Bergin and Macleod (1993) for a discussion on 'efficient arrangements'.

a NIRP collection is a singleton, then the singleton element is an IRP set.

Nonetheless, it is difficult to show the existence of NIRP collections in most classes of games (except when there is a unique Nash equilibrium). What exists more generally is a marginally weaker concept which we will call an "almost" NIRP collection. If P_{t+1} is the theory tomorrow, then the definition of an NIRP collection requires that today's theory P_t be consistent with tomorrow's theory, and therefore be exactly equal to $F(\theta^I(P_{t+1}))$. In an almost NIRP collection everything that belongs to P_t must belong to $F(\theta^I(P_{t+1}))$. However, there could be some payoff vectors in $F(\theta^I(P_{t+1}))$ which may not belong to today's theory P_t, if each such vector is weakly Pareto dominated by some other payoff vector (however, only one player should get a higher payoff in this dominating vector) from today's theory P_t.

Definition 6. We call a collection N of compact subsets of F^*, an Almost Non-Stationary Internally Renegotiation Proof Collection (**ANIRP** collection) if for any $P_t \in N$, there exists $P_{t+1} \in N$ such that $P_t \subseteq F(\theta^I(P_{t+1}))$ and for any $p \in F(\theta^I(P_{t+1}))$, there exists $p' \in P_t$ such that $p' \geq \dot{p}$ where both co-ordinates of p' are not higher than the corresponding co-ordinates of p.

In an ANIRP collection, we have weakened the requirement that $F(\theta^I(P_{t+1})) \subseteq P_t$. All that we require is that for $p \in F(\theta^I(P_{t+1}))$, if p itself is not in P_t, then some other p' must belong to P_t where $p_i = p'_i$ for some i and $p_j < p'_j$ for $j \neq i$.

In this section, we concentrate on mixed strategy action spaces. We first show that, if for some discount factor, a game has certain properties, then it must admit an ANIRP collection for that discount factor. We show, later, that for all high enough discount factors, our example satisfies the property; so, an ANIRP collection exists in our example. But before we can show this, we need to develop some additional notation. Recursively, define $R(T, \delta)$ as follows: Let $P(1, \delta)$ denote the set of Nash equilibrium payoffs[13] of G. Moreover, let $R(1, \delta)$ denote $F(P(1, \delta))$ and $R(2, \delta)$ denote $F(\theta(R(1, \delta)))$. Continuing in this

[13] Since G is played once, δ plays no part. Nevertheless to keep notation simple we put δ in the notation $P(1, \delta)$.

way, for $T > 2$, we can define[14] $R(T,\delta)$ as denoting $F(\theta(R(T-1,\delta)))$.

Since $P(1,\delta)$ is nonempty and compact and $F(\cdot)$ and $\theta(\cdot)$ always map compact sets onto compact sets, $R(T,\delta)$ is non-empty and compact for each T and δ. $R(T,\delta)$ as defined is nothing but the set of Renegotiation Proof equilibrium in (G^T,δ) as defined by Benoit-Krishna. Nevertheless, we presently ignore the significance of the above recursive expression in finitely repeated games. This will be attended to in Chapter 3 in greater detail.

Theorem 2.2 portrays a situation in which every limit point set of $\langle R(T,\delta)\rangle_{T=1}^{\infty}$ belongs to an ANIRP collection. More generally, Theorem 2.2 says that if for a discount factor, the sets $R(T,\delta)$ are large for all large enough repetitions, then any limit point set of $\langle R(T,\delta)\rangle_{T=1}^{\infty}$ must belong to an ANIRP collection.

Let $\vec{p}^i(T,\delta)$ be the minimum payoff vector of player i among all vectors in $R(T,\delta)$.

Theorem 2.2 *Let $A_i, i = 1,2$ denote mixed strategy action spaces. Let δ be such that there exists a positive integer $T'(\delta) > 1$ and a $\beta > 0$ such that whenever $T > T'(\delta)$, there exists $x(T,\delta) \in R(T,\delta)$ which satisfies, for each $i = 1,2$ and any action vector $a \in A$, the following inequality:*

$$[c_i(a) - g_i(a)] + \beta < \frac{\delta}{1-\delta}[x_i(T,\delta) - \vec{p}_i^i(T,\delta)]. \tag{2.4.2}$$

Then any limit point set $R^1(\delta)$ of $\langle R(T,\delta)\rangle_{T=1}^{\infty}$ must belong to an almost NIRP collection.

Observe that expression (2.4.2) can hold only for discount factors sufficiently higher than zero. Put another way, if expression (2.4.2) holds for all $a \in A$ and all repetitions T large enough, then, for all i, for all $a \in A$ and all large repetitions $\tilde{T}$, the following expression must also hold:

$$[c_i(a) - g_i(a)] + \beta < \frac{\delta(1-\tilde{T})}{(1-\delta)}[x_i(\tilde{T},\delta) - \vec{p}_i^i(\tilde{T},\delta)].$$

[14]This recursive definition of $R(T,\delta)$ is similar to the definition of Coalition Proof Nash equilibrium in Bernheim and Whinston (1987).

The above expression indicates how large $R(\tilde{T}, \delta)$ must be for large $\tilde{T}$. It must include a point $x(\tilde{T}, \delta)$ such that, firstly,

$$\frac{(1-\delta)}{(1-\delta^{\tilde{T}+1})} g_i(a) + \frac{\delta(1-\delta^{\tilde{T}})}{(1-\delta^{\tilde{T}+1})} x(\tilde{T}, \delta)$$

is supportable in the $\tilde{T}$-times repeated game by the three point set $\{x(\tilde{T}, \delta), \bar{p}^1(\tilde{T}, \delta), \bar{p}^2(\tilde{T}, \delta)\}$ for all $a \in A$. Secondly, while supporting the above payoff vector, the incentive constraints for both players must be satisfied with a slack higher than some positive number β.

We want to show that if $R^1(\delta)$ is a limit for some sequence $\langle R(T_k, \delta)\rangle_{k=1}^{\infty}$ and $R^2(\delta)$ is a limit point set of $\langle R(T_{k-1}, \delta)\rangle_{k-1=1}^{\infty}$, then $R^1(\delta)$ is derived consistently (in the sense of the definition of ANIRP collection) from $R^2(\delta)$. The formal proof can be outlined as a sequence of claims as follows: our first claim is that $R^1(\delta) \subseteq \theta^I(R^2(\delta))$; i.e., every vector in $R^1(\delta)$ belongs to the set supported by $R^2(\delta)$ in the infinite repeated game. Our second claim is that for any payoff vector $r(\delta)$ belonging to the set supported by $R^2(\delta)$, there is some vector in $R^1(\delta)$ which gives both player at least as high as from $r(\delta)$. Before proving the second claim, we will look at the ramifications that the second claim can have when it is true. This leads to our third claim, which says that whenever the second claim is true, any payoff vector in $R^1(\delta)$ must belong to the efficiency frontier of the set of payoffs supported by $R^2(\delta)$. Our fourth claim says that whenever our second claim is true, for any payoff vector p belonging to the efficiency frontier of the payoff set supported by $R^2(\delta)$, there exists some payoff vector in $R^1(\delta)$ which is no worse (for each player) than p but which cannot give both players strictly higher payoff than p. After proving the fourth claim, we will try to prove the second claim. In the proof of the second claim, we will need mixed strategies (to move smoothly around any action vector) and also the largeness of the sets $R(T, \delta)$, as required by the statement of the theorem.

Proof of Theorem 2.2

Let $\langle T_k\rangle_{k=1}^{\infty}$ be an increasing sequence such that $R(T_k, \delta) \to R^1(\delta)$ as $k \to \infty$. Consider the sequence $\langle T_k - 1\rangle_{k=2}^{\infty}$. We can always extract a subsequence $\langle T_{k_n} - 1\rangle_{n=1}^{\infty}$ from $\langle T_k - 1\rangle_{k=2}^{\infty}$ such that $R(T_{k_n} - 1, \delta) \to R^2(\delta)$ as $n \to \infty$. This follows from the fact that the space of all compact subsets of F^* endowed with the Hausdorff metric is a compact metric space. Note that $R^1(\delta)$ and $R^2(\delta)$ are compact from properties of set convergence. Since no payoff in $R(T,\delta)$ Pareto dominates any other payoff in $R(T,\delta)$, this property holds also for the limit point sets; i.e., $R^1(\delta) = F(R^1(\delta))$ and $R^2(\delta) = F(R^2(\delta))$. Let $\bar{p}^i(\delta)$ be the minimum payoff vector for player i in $R^2(\delta)$.

Claim 1. $R^1(\delta) \subseteq \theta^I(R^2(\delta))$.

Note that $R(T_{k_n},\delta) \subseteq \theta(R(T_{k_n}-1,\delta))$. If $R(T_{k_n},\delta) \rightarrow R^1(\delta)$, then for any $z \in R^1(\delta)$, there must be sequence $\langle z_n \rangle_{n=1}^{\infty}$ where each $z_n \in R(T_{k_n},\delta)$ such that $z_n \rightarrow z$ as $n \rightarrow \infty$. Let z_n be supported by $R(T_{k_n}-1,\delta)$ through the action vector a_n and the payoff vectors $p_n, \bar{p}^1(T_{k_n}-1,\delta), \bar{p}^2(T_{k_n}-1,\delta) \in R(T_{k_n}-1,\delta)$. As $n \rightarrow \infty$, let the action vector converge to a and the payoff vectors converge (taking converging subsequences if necessary) to $p, p^1(\delta), p^2(\delta)$. Now, the payoff vectors $p, p^1(\delta), p^2(\delta)$ must belong to $R^2(\delta)$ and from the continuity (see the incentive constraints in Definition 1 and 2) of $c_i(\cdot) - g_i(\cdot)$, z must be supported by $R^2(\delta)$ with the help of action vector a and these payoff vectors. This proves Claim 1.

Claim 2. For any vector $r(\delta) \in \theta^I(R^2(\delta))$, then there exists $q(\delta) \in R^1(\delta)$ such that $q(\delta) \geq r(\delta)$.

We will postpone the proof of Claim 2 until we have shown our other claims.

Claim 3. If claim 2 is true, then $R^1(\delta) \subseteq F(\theta^I(R^2(\delta)))$.

We know from claim 1 that if $r \in R^1(\delta)$, then $r \in \theta^I(R^2(\delta))$. So, if r does not belong to $F(\theta^I(R^2(\delta)))$, then there exists $z \in \theta^I(R^2(\delta))$ such that $z >> r$. Then, whenever claim 2 is true, there exists $q \in R(\delta)$ such that $q \geq z$. However, this implies that $q >> r$ which violates $R^1(\delta) = F(R^1(\delta))$. Hence, it must be that $r \in F(\theta^I(R^2(\delta)))$.

Claim 4. If claim 2 is true, then for any $p \in F(\theta^I(R^2(\delta)))$, there exists $p' \in R^1(\delta)$ such that $p' \geq p$ with p' not strongly Pareto dominating p.

If claim 2 is true, and since $p \in \theta^I(R^2(\delta))$, there exists $p' \in R^1(\delta)$ such that $p' \geq p$. Now, from claim 3, we know that $p' \in F(\theta^I(R(\delta)))$. p' cannot strongly dominate p, since otherwise, there are two vectors p' and p in $F(\theta^I(R^2(\delta)))$ one of which strictly dominating the other, which cannot be true.This proves claim 4.

Note that if we can show that claim 2 is true, then we can conclude with the help of claim 3 and claim 4 that $R^1(\delta)$ indeed is consistently derived (in the sense of ANIRP collection) from the next day's theory $R^2(\delta)$. We now start proving claim 2.

Following the statement of claim 2, let $r(\delta) \in \theta^I(R^2(\delta))$; then $r(\delta)$ be supported by $R^2(\delta)$ by some action vector $b \in A$ and payoff vectors $p(\delta), \bar{p}^1(\delta), \bar{p}^2(\delta) \in R^2(\delta)$. So, for $i = 1, 2$,

$$r_i(\delta) = (1 - \delta) g_i(b) + \delta p_i(\delta)$$

and

$$[c_i(b) - g_i(b)] \leq \frac{\delta}{1-\delta}[p_i(\delta) - \bar{p}_i^i(\delta)].$$

Extract a subsequence from $\langle T_{k_n} - 1 \rangle_{n=1}^{\infty}$, such that, along the subsequence, there exists payoff vectors $x(T_{k_n} - 1, \delta), p(T_{k_n} - 1, \delta)$ and $\bar{p}^i(T_{k_n} - 1, \delta)$, all belonging to $R(T_{k_n} - 1, \delta)$, such that
$x(T_{k_n} - 1, \delta), p(T_{k_n} - 1, \delta), \bar{p}^i(T_{k_n} - 1, \delta) \longrightarrow x(\delta), p(\delta), \bar{p}^i(\delta)$ as $T_{k_n} \to \infty$ for some $x(\delta), \bar{p}^i(\delta) \in R^2(\delta)$. For large $T_{k_n} - 1$, the vector $x(T_{k_n} - 1, \delta)$ is the vector from the statement of this theorem. Note that extending expression (2.4.2) to the limit, for each $a \in A$ and $i = 1, 2$, the following conditions must be satisfied:

$$[c_i(a) - g_i(a)] < \frac{\delta}{1-\delta}[x_i(\delta) - \bar{p}_i^i(\delta)].$$

Now check that $p_i(\delta) \geq x_i(\delta)$ for some i. Otherwise, $x(\delta)$ will Pareto dominate $p(\delta)$ which will contradict $R^2(\delta) = F(R^2(\delta))$. Let without loss of generality $p_2(\delta) \geq x_2(\delta)$, in which case

$$[c_2(b) - g_2(b)] < \frac{\delta}{1-\delta}[p_2(\delta) - \bar{p}_2^2(\delta)].$$

Case 1 :
Let $[c_1(b) - g_1(b)] < \frac{\delta}{(1-\delta)}[p_1(\delta) - \bar{p}_1^1(\delta)]$. Then from continuity of $[c_i(\cdot) - g_i(\cdot)]$, for large enough $T_{k_n} - 1$, for each $i = 1, 2$, $[c_i(b) - g_i(b)] < \frac{\delta(1-\delta^{T_{k_n}-1})}{(1-\delta)}[p_i(T_{k_n} - 1, \delta) - \bar{p}_i^i(T_{k_n} - 1, \delta)]$. So, $r(T_{k_n}, \delta) \in \theta(R(T_{k_n} - 1, \delta))$ where $r(T_{k_n}, \delta) = (r_1(T_{k_n}, \delta), r_2(T_{k_n}, \delta))$ is defined for $i = 1, 2$ as

$$r_i(T_{k_n}, \delta) = \frac{(1-\delta)}{(1-\delta^{T_{k_n}})} g_i(a) + \frac{\delta(1-\delta^{T_{k_n}-1})}{(1-\delta^{T_{k_n}})} p_i(T_{k_n} - 1, \delta).$$

Now, from definition of $R(T_{k_n}, \delta_n)$, there exists vector $q(T_{k_n}, \delta) = (q_1(T_{k_n}, \delta), q_2(T_{k_n}, \delta)) \in R(T_{k_n}, \delta)$ such that for $i = 1, 2$, $q_i(T_{k_n}, \delta) \geq r_i(T_{k_n}, \delta)$. Note that $r(T_{k_n}, \delta) \overset{T_{k_n}-1\to\infty}{\longrightarrow} r(\delta)$. We now extract a subsequence from $\langle q(T_{k_n}, \delta)\rangle_{n=1}^{\infty}$ which converges to some $q \in R^1(\delta)$. It must be that $q \geq r(\delta)$.

Case 2 :

Let $[c_1(b) - g_1(b)] = \frac{\delta}{(1-\delta)}[p_1(\delta) - \bar{p}_1^1(\delta)]$. Note that if $c_1(a) = g_1(a)$, the proof in Case 1 goes through since $[c_1(b) - g_1(b)] \leq \frac{\delta(1-\delta^{T_{k_n}-1})}{(1-\delta)}[p_1(T_{k_n} - 1, \delta) - \bar{p}_1^1(T_{k_n} - 1, \delta)]$ as player 1 cannot deviate and gain. That player 2 should not deviate follows for large n is easy to see.

If $c_1(a) > g_1(a)$, it may no longer be true that for large n, $r(T_{k_n}, \delta)$ (where $r_i(T_{k_n}, \delta) = \frac{(1-\delta)}{(1-\delta^{T_{k_n}})} g_i(a) + \frac{\delta(1-\delta^{T_{k_n}-1})}{(1-\delta^{T_{k_n}})} p_i(T_{k_n} - 1, \delta)$ for some $p(T_{k_n} - 1, \delta) \in R(T_{k_n} - 1, \delta)$) is supportable by $R(T_{k_n} - 1, \delta)$ because the incentive constraints may not be satisfied. Nevertheless, we take a different route. We claim the following : there exists a sequence $\langle a_S\rangle_{S=1}^{\infty}$ of mixed action vectors such that $b_S \to b$, and for each b_S and $i = 1, 2$,

$$[c_i(b_S) - g_i(b_S)] < \frac{\delta}{(1-\delta)}[p_i(\delta) - \bar{p}_i^i(\delta)]$$

To show this, let $b = (b_1, b_2)$; here b_1 and b_2 are probability distributions (mixed strategies) over the pure action spaces of player 1 and 2 respectively. Let b_1' denote the maximum deviation strategy for player 1 from action vector b. So, $c_1(b) = g_1(b_1', b_2)$ (without loss of generality b_1' is a degenerate distribution at a pure action). Let $b_1(\lambda) = (1-\lambda)b_1 + \lambda b_1'$ for $\lambda \in [0, 1]$ denote a mixed strategy action for player 1. Note that $g_1(b_1(\lambda), b_2) = (1-\lambda)g_1(b) + \lambda c_1(b)$, and $c_1(b_1(\lambda), b_2) = c_1(b)$. Further, $g_1(b_1(\lambda), b_2)$ is an increasing function of λ and correspondingly, $c_1(b_1(\lambda), b_2) - g_1(b_1(\lambda), b_2)$ is a decreasing function of λ. For λ close enough to zero, it must be the case that $c_1(b_1(\lambda), b_2) - g_1(b_1(\lambda), b_2) < \frac{\delta}{(1-\delta)}[p_1(\delta) - \bar{p}_1^1(\delta)]$. From continuity of $c_2(\cdot) - g_2(\cdot)$, it follows that $c_2(b_1(\lambda), b_2) - g_2(b_1(\lambda), b_2) < \frac{\delta}{(1-\delta)}[p_2(\delta) - \bar{p}_2^2(\delta)]$. So, it is possible to get a sequence $\langle b_S = (b_1(\lambda_S), b_2)\rangle_{S=1}^{\infty}$ where for each S, λ_S is close enough to zero, such that $(b_1(\lambda_S), b_2) \to b$ as $S \to \infty$ and for each b_S for $i = 1, 2$,

$$[c_i(b_S) - g_i(b_S)] < \frac{\delta}{(1-\delta)}[p_i(\delta) - \bar{p}_i^i(\delta)] \qquad (2.4.3)$$

For each b_S we apply the proof of Case 1. So, there exists $r(b_S, \delta) \in$

$\theta^I(R^2(\delta))$ and $q(b_S) \in R(\delta)$ such that $q(b_S) \geq r(b_S, \delta)$, where for $i = 1, 2$, $r_i(b_S, \delta) = (1-\delta)g_i(b_S) + \delta_n p_i(\delta)$. Note that $r(b_S, \delta) \to r(\delta)$ as $b_S \to b$ (as $\lambda \to 0$). Extract a subsequence $\langle b_{S_l} \rangle_{l=1}^{\infty}$ from $\langle b_S \rangle_{S=1}^{\infty}$ such that $q(b_{S_l}) \to q$ as $l \to \infty$. Note that $q \in R^1(\delta)$ and $q \geq r(\delta)$.

This completes the proof of claim 2. Now, to see how $R^1(\delta)$ belongs to an ANIRP collection, let us note that $R^2(\delta)$ must itself be almost (in the sense of the definition of ANIRP collection) the efficient frontier of payoffs supportable by some other limit point set $R^3(\delta)$ by an identical argument, and so on and so forth. The collection $\{R^n(\delta) | n = 1, 2, \cdots, \infty\}$ is an ANIRP collection. We started with an arbitrary $R^1(\delta)$. Taking different limit point sets as the initial $R^1(\delta)$ shows that any limit point set belongs to some ANIRP collection. ∎

Note that in Case 2 of Theorem 2.2, we are using properties of mixed strategy action spaces. The problem with pure strategies in this case is the following. Suppose pure action vector $a \in A$ satisfies the hypothesis of case 2. So, $[c_1(b) - g_1(b)] = \frac{\delta}{(1-\delta)}[p_1(\delta) - \bar{p}_1^1(\delta)]$ and the corresponding expression for player 2 is attained with strict inequality. We cannot directly rule out that $[c_1(b) - g_1(b)] > \frac{\delta(1-\delta^{T_{k_n}-1})}{(1-\delta)}[p_1(T_{k_n} - 1, \delta) - \bar{p}_1^1(T_{k_n} - 1, \delta)]$ for all large enough $T_{k_n} - 1$ since the magnitude of the right hand side expression is not known. We only know it's limit as n goes to infinity to be $[c_1(b) - g_1(b)]$. Moreover, if we restrict attention to pure strategy action spaces, unlike in the mixed strategy case, we may not be able to action vectors b_S close enough to b which satisfies $[c_i(b_S) - g_i(b_S)] < \frac{\delta}{(1-\delta)}[p_i(\delta) - \bar{p}_i^i(\delta)]$ for each $i = 1, 2$. So we can no longer apply the arguments of case 1 and case 2 in the proof of claim 2. This is where Theorem 2.2 may fail to hold for pure strategy action spaces.

Can the collection of all limit point sets of $\langle R(T, \delta) \rangle_{T=1}^{\infty}$ be an ANIRP collection? In a setting as in Theorem 2.2, this is true. In the proof of theorem 2.2, we started with an arbitrary limit point set $R^1(\delta)$ and showed that there must exist some other limit point set $R^2(\delta)$ such that $R^1(\delta)$ is consistently derived from $R^2(\delta)$. Since the initial $R^1(\delta)$ is arbitrary, the collection of limit point sets is an ANIRP collection. This is formally stated in Corollary 2.1. The setting in which this corollary works is enjoyed by all games which satisfy the setting of Theorem 2.2.

Corollary 2.1 *Let $A_i, i = 1,2$ denote mixed strategy action spaces. Consider the collection of all limit point sets of $\langle R(T,\delta)\rangle_{T=1}^{\infty}$. Suppose, for any set R belonging to this collection, there exists a vector $x(R) \in R$ such that for each $i = 1,2$ and any action vector $a \in A$, $x(R)$ satisfies the following inequality*

$$[c_i(a) - g_i(a)] < \frac{\delta}{1-\delta}[x_i(R) - \bar{p}_i^i(R)]. \tag{2.4.4}$$

Then this collection of all limit point sets is an almost NIRP collection. Above, $\bar{p}^i(R)$ is the minimum payoff vector for player i in R.

Note that the main virtue of condition (2.4.2) in Theorem 2.2 is to guarantee that in the limit, condition (2.4.4) holds for the limit point set R. So, the proof of Corollary 2.1 follows immediately from the proof of Theorem 2.2 (through the arguments in Case 1 and Case 2) and hence, will be omitted.

We now show that our example of the previous section satisfies the conditions of Theorem 2.2 for large enough δ.

Example

We start by establishing that in the game considered in section 2.3, (see figure 2.2) it must be the case that for all T, and $i = 1,2$, $\bar{p}^1(T,\delta)$ is weakly to the north-west of (1,4), and $\bar{p}^2(T,\delta)$ is weakly to the north-west of (4,1). This will further imply that for all T, $\bar{p}^1(T,\delta)$ will always be on or above the straight line passing through (6,0) and (1,4). Similarly, $\bar{p}^2(T,\delta)$ will always be on or above the straight line passing through (0,6) and (4,1). So, we claim that for all T and $i = 1,2$, $\bar{p}^i(T,\delta)$ must satisfy $\bar{p}_j^i(T,\delta) \geq 4$ for $j \neq i$ and $1 \leq \bar{p}_i^i(T,\delta) \leq 2$. Without loss of generality, we show that this is true for $\bar{p}^1(T,\delta)$ for all T. Otherwise, let T^* be the first T such that either of these three situations occur: (a) $\bar{p}_2^1(T,\delta) < 4$; (b) $\bar{p}_1^1(T,\delta) > 2$ or (c) $\bar{p}_1^1(T,\delta) < 2$. For all $T < T^*$, $\bar{p}_2^1(T,\delta) \geq 4$ and $1 \leq \bar{p}_1^1(T,\delta) \leq 2$. Note that $T^* > 1$, for at $T = 1$, $\bar{p}_i^i(T,\delta) = 1$ and $\bar{p}_j^i(T,\delta) = 4$. Note that $\frac{(1-\delta)}{(1-\delta^{T^*})}(1,4) + \frac{\delta(1-\delta^{T^*-1})}{(1-\delta^{T^*})}\bar{p}^1(T^*-1,\delta)$ is supportable by $R(T^*-1,\delta)$ with the help of the Nash equilibrium (1,4) and gives a payoff of less than 2 to player 1 and a payoff of at least 4 to player 2. Consequently, given the structure of the feasible set of payoffs in this game, cases (a) and (b) cannot occur. If case (c) occurs, then consider the action vector $\tilde{a}$ through which (we also require three payoff vectors to support) $\bar{p}^1(T^*, \delta)$ is supported by $R(T^* - 1, \delta)$. $(g_1(\tilde{a}), g_2(\tilde{a}))$ must occur to the north-west of (1,4). However, from our analysis in section 2.3, we know that $c_1(\tilde{a}) > 2$. So, if player 1 deviates from playing $\tilde{a}$, he gets $\frac{(1-\delta)}{(1-\delta^{T^*})} c_1(\tilde{a}) + \frac{\delta(1-\delta^{T^*-1})}{(1-\delta^{T^*})} \bar{p}^1(T^* - 1, \delta)$ which is greater than 1 (current payoff is greater than 1 while continuation payoff is at least 1); this implies that player 1 should deviate since by not deviating he will get less than 1. So, case (c) also cannot occur. This implies that $\bar{p}^1(T, \delta)$ lies weakly to the north-west of (1,4). A similar argument shows that for all T, $\bar{p}^2(T, \delta)$ must lie weakly to the north-west of (4,1).

Let $d = \text{Max}_{i=1,2} \text{Max}_{a \in A}[c_i(a) - g_i(a)]$. d denotes the maximum one shot deviation gain possible by one of the players. We will focus on the line segment L joining points (2.3,2.7) and (2.7,2.3). Note that (2.5,2.5) lies on L. Also, note that for all T, there exists $\bar{p}^1(T, \delta)$ lying strictly to the north-west of (2.3,2.7) (actually to the north-west of (2,4)) and $\bar{p}^2(T, \delta)$ lying strictly to the south-east of (2.7,2.3). Let δ be high enough such that δ satisfies **(a)** $2.5\delta > 2.3$ and **(b)** $\frac{\delta}{(1-\delta)}[2.5 - 2.3] > d + \beta$ for some $\beta > 0$. For δ in the requisite range, let $T(\delta)$ be such that **(a')** $\frac{\delta(1-\delta^{T(\delta)})}{(1-\delta^{T(\delta)+1})} 2.5 > 2.3$ and **(b')** $\frac{\delta(1-\delta^{T(\delta)})}{(1-\delta)}[2.5 - 2.3] > d + \beta$. For δ in the given range, we will show that there exists $T'(\delta) > T(\delta)$ such that for all $T > T'(\delta)$, there exist $x(T, \delta) \in R(T, \delta)$ and $y(T, \delta) \in L$ where $x(T, \delta)$ satisfy the conditions of Theorem 2.2 and $x(T, \delta) \geq y(T, \delta)$. We build an algorithm to get hold of such a $T'(\delta)$. While going through the algorithm, always remember that the algorithm can only pick such points which lie on or above the straight line passing through (6,0) and (1,4).

Step 1. For δ in the requisite range and $T(\delta)$ as defined, we know that $\bar{p}^1(T(\delta), \delta)$ lies on or above the line passing through (6,0) and (1,4) and to the north-west of (2,4). Check that for our δ and $T(\delta)$, and the fact that player 1 cannot deviate and gain,

$$r^{(1)}(\delta) = \frac{(1-\delta)}{(1-\delta^{T(\delta)+1})}(6,0) + \frac{\delta(1-\delta^{T(\delta)})}{(1-\delta^{T(\delta)+1})} \bar{p}^1(T(\delta), \delta) \in \theta(R(T(\delta), \delta)).$$

So, there exists $q^{(1)}(\delta) \in R(T(\delta)+1, \delta)$, such that $q^{(1)}(\delta) \geq r^{(1)}(\delta)$. If ${q_2}^{(1)}(\delta) > 2.5$, then go to Step 2. If ${q_2}^{(1)}(\delta) \leq 2.5$, then go to step 3. Note that $q^{(1)}(\delta)$ has to lie above the straight line passing through (6,0) and (1,4). So, if ${q_2}^{(1)}(\delta) \leq 2.5$, then ${q_1}^{(1)}(\delta) \geq 11.5/4$ where (11.5/4,2.5) is a point lying on the line segment joining (6,0) and (1,4).

Step 2. If for some $K \geq 1$, ${q_2}^{(K)}(\delta) > 2.5$, then generate $q^{(K+1)}(\delta)$ in the following manner. Check that

$$r^{(K+1)}(\delta) = \frac{(1-\delta)}{(1-\delta^{T(\delta)+K+1})}(6,0) + \frac{\delta(1-\delta^{T(\delta)+K})}{(1-\delta^{T(\delta)+K+1})}q^{(K)}(\delta) \in \theta(R(T(\delta)+K,\delta)).$$

So, there exists $q^{(K+1)}(\delta) \in R(T(\delta)+K+1,\delta)$ such that $q^{(K+1)}(\delta) \geq r^{(K+1)}(\delta)$. If ${q_2}^{(K+1)}(\delta) > 2.5$, go to Step 2. If ${q_2}^{(K+1)}(\delta) \leq 2.5$, then go to Step 3.

Step 3. If for some $K \geq 1$, ${q_2}^{(K)}(\delta) \leq 2.5$, then generate $q^{(K+1)}(\delta)$ in the following manner. Using the fact that ${q_1}^{(K)} \geq 11.5/4$, and that player 2 cannot deviate and gain, check that

$$r^{(K+1)}(\delta) = \frac{(1-\delta)}{(1-\delta^{T(\delta)+K+1})}(0,6) + \frac{\delta(1-\delta^{T(\delta)+K})}{(1-\delta^{T(\delta)+K+1})}q^{(K)}(\delta) \in \theta(R(T(\delta)+K,\delta)).$$

So, there exists $q^{(K+1)}(\delta) \in R(T(\delta)+K+1,\delta)$ such that $q^{(K+1)}(\delta) \geq r^{(K+1)}(\delta)$. If ${q_2}^{(K+1)}(\delta) > 2.5$, go to Step 2. If ${q_2}^{(K+1)}(\delta) \leq 2.5$, then go to Step 3.

From the way we have built our algorithm, it is easy to see the following facts. As K becomes large $q^{(K)}(\delta)$ approaches the line segment joining (0,6) and (6,0). Further, there exists a large K' such that for all $K > K'$, $q^{(K)}(\delta) \geq y(K)$ for some $y(K) \in L$. Let $T'(\delta) = T(\delta) + K'$, and for $T > T'(\delta)$ define $x(T,\delta) = q(T - T(\delta))$. We will show that for $T > T'(\delta)$, $x(T,\delta)$ satisfies the condition of Theorem 2.2. Given the β in the construction, we require to show that for any action vector $a \in A$, for each $i = 1,2$ we must have $[c_i(a) - g_i(a)] + \beta < \frac{\delta(1-\delta^T)}{(1-\delta)}[x_i(T,\delta) - \vec{p}_i^{\,i}(T,\delta)]$. Note, from the fact that $\vec{p}_i^{\,i}(T,\delta) \leq 2$, for $T > T'(\delta)$ we must have

$$\frac{\delta(1-\delta^T)}{(1-\delta)}[x_i(T,\delta) - \vec{p}_i^{\,i}(T,\delta)] \geq \frac{\delta(1-\delta^T)}{(1-\delta)}[2.3-2] > \frac{\delta(1-\delta^{T(\delta)})}{(1-\delta)}[2.5-2.3].$$

Observe that the expression on the extreme right is greater than $d+\beta$. So, as required in the statement of Theorem 2.2, in the limit, $[c_i(a) - g_i(a)] < \frac{\delta}{(1-\delta)}[x_i(T,\delta) - \vec{p}_i^{\,i}(T,\delta)]$ for all action vector $a \in A$. From Theorem 2.2, it now follows that an ANIRP collection exists for our example.

2.5 Conclusion

In this chapter, we have shown that IRP sets may not always exist, even with high discount factors. Nevertheless, in many occasions, an almost NIRP collection may exist. In particular, in the example where no IRP sets exist for low discounting, almost NIRP collections do exist for low discounting. One area of my future research is to try and characterize games which admit ANIRP collections. Another future research direction includes the question about the structure of limiting ANIRP (also NIRP) collections as discounting vanishes. In particular, do these limits look like the IRP set limits; i.e., are they either singletons or efficient?

In the next chapter, we explore the connections between renegotiation proof sets of long finitely repeated games for low discounting and internally renegotiation proof sets in infinitely repeated games.

2.6 Appendix

We will show that the solution to program (2.3.1) is uniquely attained at $a' =$ (T,C). We will focus on mixed strategies (the proof for pure strategies is easy to see).

Note that $c_1(a') = 1$ and $f(a') = 2$. So, $\text{Max}[c_1(a'), f(a')] = 2$. We will show that there exists no other action vector $\tilde{a}$ such that $\text{Max}[c_1(\tilde{a}), f(\tilde{a})] \leq 2$. We focus on action vectors $\tilde{a} \in A'$ such that $f(\tilde{a}) \leq 2$ (otherwise $\text{Max}[c_1(\tilde{a}), f(\tilde{a})] >$ 2). So, it must be that $g_2(\tilde{a}) \geq 4$.

The only two ways that there can be any action pair $\tilde{a} \in A$ such that $g_2(\tilde{a}) \geq 4$ are

(a) $\tilde{a}$ is (T,C).

(b) $\tilde{a}$ assigns at least $\frac{2}{3}$ probability to (B,L) being played.

We will show that if (b) holds, then $\text{Max}[c_1(\tilde{a}), f(\tilde{a})] \geq \frac{8}{3}$. Let $\tilde{a} = (\tilde{a}_1, \tilde{a}_2)$. Note that if $\tilde{a}$ has to assign at least probability $\frac{2}{3}$ to (B,L) being played, then $\tilde{a}_2$ must place a probability of at least $\frac{2}{3}$ on L. Consider now the action pair (M,$\tilde{a}_2$), where player 1 plays the pure strategy Mwith probability 1 and player

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2 plays $\tilde{a}_2$. Check that $c_1(\tilde{a}) \geq g_1(\text{M},\tilde{a}_2) = \frac{8}{3}$. So, clearly $\text{Max}[c_1(\tilde{a}), f(\tilde{a})] \geq \frac{8}{3}$. Consequently, program (2.3.1) must be attained uniquely at (T,C). ■

Chapter 3

Renegotiation Proof Sets in Long Finitely Repeated Games with Low Discounting

3.1 Introduction

In this chapter, I explore connections between renegotiation proof sets in (long) finitely repeated games with discounting, and IRP sets of infinitely repeated games. The notion of renegotiation proofness in finitely repeated games is relatively uncontroversial, as noted by Benoit-Krishna (1993). If some link exists between the two notions, that would provide additional basis for the claim that IRP is the natural notion of internal consistency in infinitely repeated games. Benoit-Krishna considered undiscounted finitely repeated games and found that if a limit of the renegotiation proof sets exists as the time horizon goes to infinity, then the limit set must either be a singleton or a subset of the efficient frontier of the game. They were not able to show if such limits always exist, although they conjectured the answer to be in the affirmative, at least in games with finite number of actions. Do their results extend to the case of low discounting? Do such limits exist for low but positive discounting? What form do these limits take; in particular, do they bear any relation to internally renegotiation proof sets (perhaps the non-stationary version presented in the previous chapter)? These issues are addressed in this chapter.

The main results are the following:

(a) If for high discount factors, the limit of renegotiation proof payoffs of finitely repeated games (as the number of repetitions $T \to \infty$) exists, then the limit set is arbitrarily close to either of two types of sets as discounting goes to zero: a singleton, or a subset of the efficient frontier of the payoff set. This extends the main result of Benoit-Krishna (1993) to the case of low rates of discounting.

(b) If as in (a), the limit of renegotiation proof sets for sufficiently long finite repetitions of a game (with discounting) exists, then the set of limiting payoffs must be a WRP set of the infinitely repeated game. If the limit is a singleton, then it must be an IRP set. If in addition mixed strategies are allowed, and the limit set is a "sufficiently large" non-singleton, then it is an almost IRP (ANIRP) set of the infinitely repeated game, where an almost IRP set (which has been considered in chapter 2) is a slight weakening of the definition of an IRP set.

(c) In the example in chapter 2, where IRP sets failed to exist in the infinitely repeated context, limits of renegotiation proof sets for long finite repetitions of the stage game also do not exist for low enough discounting (irrespective of whether we consider pure or mixed strategies). Combined with (b) above, this indicates a close connection between IRP sets and renegotiation proof sets for long finite games in the case of low discounting.

(d) In the same example, however, limits of renegotiation proof sets in the undiscounted case exist, when attention is restricted to pure strategies. This is in contrast to the case of discounting, where they do not exist even for discount factors arbitrarily close to 1.

(e) Even when Renegotiation Proof sets for long finite games do not have a limit, it is possible for each limit point set to be included in an almost NIRP collection, provided that the limit point sets are sufficiently large. In particular, in the example of chapter 2, every limit point set for low discounting belongs to an ANIRP collection. This provides additional insight into the connection of Renegotiation Proof concept between finitely and infinitely repeated games.

Section 3.1 introduces some basic definitions. The basic model is identical to the one in Chapter 2 and will not be repeated. The definitions of a WRP set, IRP set, an NIRP collection and an almost NIRP collection are as in chapter 2. Section 3.2 discusses the case of long finitely repeated discounted games. Section 3.3 discusses the non-existence (for low discounting) of limit renegotiation proof sets for long finitely repeated version of the game considered in the example of chapter 2. Section 3.4 briefly discusses the case of almost NIRP collections. Section 3.5 concludes. We will follow the notation used in chapter 2.

3.2 Framework

We use the same basic model of Chapter 2 and most of it will not be repeated. Let δ be the common discount factor and (G^T, δ) denote the T-time repeated version of G.

For a sequence $\langle a_t \rangle_{t=0}^{T} \in A^{T+1}$ and $\delta < 1$, the normalized payoff to player i in (G^{T+1}, δ) from this action sequence is $\frac{(1-\delta)}{(1-\delta^{T+1})} \sum_{t=0}^{T} \delta^t g_i(a_t)$, whereas if $\delta = 1$, the normalized payoff is $\frac{1}{T+1} \sum_{t=0}^{T} g_i(a_t)$. Note that all normalized payoffs lie in F^*.

Definition 1. Let $B(T,\delta)$ be a set of normalized payoffs in (G^T,δ). Clearly $B(T,\delta) \subseteq F^* \subset I\!R^2$. Then payoff vector $p(T+1) \in I\!R^2$ is said to be **supportable** by $B(T,\delta)$ if there exists $a \in A$ and $\hat{p}(T), p^1(T), p^2(T) \in B(T,\delta)$ such that for $i = 1,2$

$$p_i(T+1) = \frac{1}{\sum_{t=0}^{T}\delta^t} g_i(a) + \frac{\delta(\sum_{t=0}^{T-1}\delta^t)}{(\sum_{t=0}^{T}\delta^t)} \hat{p}_i(T)$$

and

$$[c_i(a) - g_i(a)] \leq \delta(\sum_{t=0}^{T-1}\delta^t)(\hat{p}_i(T) - p_i^i(T)).$$

When the above conditions hold for a payoff vector $p(T+1)$, we will say that $p(T+1)$ is supported by $B(T,\delta)$ through the action vector a and the payoff vectors $\hat{p}(T), p^1(T), p^2(T)$.

So for $\delta < 1$, the above conditions reduce to

$$p_i(T+1) = \frac{(1-\delta)}{(1-\delta^{T+1})} g_i(a) + \frac{\delta(1-\delta^T)}{(1-\delta^{T+1})} \hat{p}_i(T)$$

$$\text{and} \quad [c_i(a) - g_i(a)] \leq \frac{\delta(1-\delta^T)}{(1-\delta)}(\hat{p}_i(T) - p_i^i(T)).$$

On the other hand, for $\delta = 1$, they are equivalent to

$$p_i(T+1) = \frac{1}{T+1} g_i(a) + \frac{T}{T+1} \hat{p}_i(T)$$

$$\text{and} \quad [c_i(a) - g_i(a)] \leq T(\hat{p}_i(T) - p_i^i(T)).$$

As in Benoit-Krishna (1993), the payoff vector $p(T+1)$ is supported in (G^{T+1},δ) by an action vector a in period 1, and continuation payoff $\hat{p}(T)$ on the equilibrium path, with $p^1(T)$ and $p^2(T)$ being the punishment continuation payoffs for players 1 and 2 respectively.

For a nonempty set of payoffs $B(T,\delta) \subseteq F^*$, let $\theta(B(T,\delta))$ denote the set of payoff vectors $p(T+1)$ supported by $B(T,\delta)$. It follows that $\theta(B(T,\delta))$ is compact if $B(T,\delta)$ is compact.

For any non-empty set $C \subset I\!R^2$, define $F(C) = \{x \in C |$ there is no $y \in C$ such that $y >> x\}$ where[1] $y >> x$ if $y_i > x_i$ for $i = 1,2$. Hence, $F(\theta(B(T,\delta)))$ is the weak Pareto Frontier of payoffs supported by $B(T,\delta)$.

[1] As in Chapter 2, the notation $y \geq x$ will mean that $y_i \geq x_i$ for $i = 1,2$.

We will also require a modification of the definitition of an IRP set. We will call this modification an ***almost IRP set***.

Definition 2

A compact set $P \subseteq I\!R^2$ is an 'almost IRP set' in G^∞ if the following two conditions hold;

(a) $P \subseteq F(\theta(P))$,

(b) If $p \in F(\theta(P))$, then there exists $p' \in P$ such that $p'_i \geq p_i$ for $i = 1, 2$, but p' does not strongly dominate p.

An almost IRP set is much like an IRP set. What has been weakened is the requirement $F(\theta(P)) \subseteq P$. If p belongs to $F(\theta(P))$, then p may belong to P. If not, then there must exist some $p' = (p'_1, p'_2)$ in P such that $p'_i = p_i$ for some i and $p'_j > p_j$ for $j \neq i$. An almost IRP set is obviously a WRP set. An IRP set is also an almost IRP set.

We now consider our analysis of renegotiation proof sets for discounted finitely repeated games.

3.3 Long Finitely Repeated Games and Renegotiation Proofness

The set of Renegotiation Proof Equlibria of (G^T, δ) is defined recursively[2] as follows:

Let $P(1, \delta)$ denote the set of Nash equilibrium payoffs of G. Let $R(1, \delta)$ denote $F(P(1, \delta))$, and let $R(2, \delta))$ denote $F(\theta(R(1, \delta)))$. Continuing in this way, for any $T > 2$, $R(T, \delta)$ denotes $F(\theta(R(T-1, \delta)))$.

$R(T, \delta)$ will be defined as the set of Renegotiation Proof payoff vectors in (G^T, δ). As shown in chapter 2, $R(T, \delta)$ will be non-empty and compact[3].

Here is some notation which we will use throughout this chapter.

[2] This recursive definition was also required in section 2.4 of chapter 2.

[3] Remember that the way we have defined the map $\theta(\cdot)$ guarantees us that payoffs in $R(T, \delta)$ are normalized. $R(T, 1)$ here, is what Benoit and Krishna call $\frac{R(T)}{T}$.

Notation. $R(T,\delta)$, as defined above, will denote the renegotiation proof set in (G^T,δ). If the sequence of $R(T,\delta)$ has a limit as the number of repetitions T go to infinity, then this limit will be denoted as $R(\delta)$. $\mathcal{R}$ will denote the set of all limit points[4] of sequences $\langle R(\delta_n)\rangle_{n=1}^{\infty}$ where $\delta_n \to 1$ and where $R(\delta_n) = \lim_{T\to\infty} R(T,\delta_n)$. A typical element of $\mathcal{R}$ will be R. Similarly, $\mathcal{R}(\delta)$ will denote the set of all limit points of the sequence $\langle R(T,\delta)\rangle_{T=1}^{\infty}$ with a typical element of the set being $R^1(\delta)$. By virtue of a property of set convergence, note that each $R \in \mathcal{R}$ and each $R^1(\delta) \in \mathcal{R}(\delta)$ is a closed set. Just as in chapter 2, the ***minimum payoff vector*** of player i in a compact set $B \subseteq F^*$ is that vector which gives player i the minimum payoff in B. If the set of minimizers is non-unique, then call that vector in the set of minima as 'player i's minimum vector' which gives player j (for $j \neq i$) the highest payoff in the set. In any compact set, the minimum vector for any player i is unique. $\vec{p}^i(T,\delta)$, $\vec{p}^i(\delta)$ and $\vec{p}^i$ will denote the minimum vector of player i in $R(T,\delta)$, $R(\delta)$ and R respectively.

Our first lemma is a technical reqirement which we will keep needing throughout the chapter. This lemma states that when the limit of renegotiation proof (RP) sets of finitely repeated game (as time horizon goes to infinity) exists, the sequence of minimum payoff vectors for any player i in the RP sets of the finitely repeated games, converges to the minimum payoff vector (for the same player) of the limit set. A similar argument holds for the minimum payoff vector sequence of any player from a sequence $\langle R(\delta_n)\rangle_{n=1}^{\infty}$ converging to a set R as $\delta_n \to 1$.

Lemma 3.1 *(a) If $R(\delta)$ exists for $\delta < 1$, then for $i = 1,2$, $\vec{p}^i(T,\delta) \to \vec{p}^i(\delta)$ as $T \to \infty$.*
(b) If $\langle R(\delta_n)\rangle_{n=1}^{\infty}$ converges to a set R as $\delta_n \to 1$, then for $i = 1,2$, $\vec{p}^i(\delta_n) \to \vec{p}^i$.

[4] P' is a limit point set of a sequence of compact sets $\langle P_k\rangle_{k=1}^{\infty}$ if there exists a subsequence $\langle k_n\rangle_{n=1}^{\infty}$ such that $P_{k_n} \to P'$ as $k_n \to \infty$. Since the space of all compact subsets of F^* when endowed with the Hausdorff metric is a compact metric space, any sequence of compact sets in F^* will have a converging subsequence. Note that this implies that if $\langle P_k\rangle_{k=1}^{\infty}$ does not have a limit as $k \to \infty$, then it must have at least two limit points.

Proof.

Since the arguments for both (a) and (b) are similar, we will only show that (a) holds. Suppose $\bar{p}^i(T,\delta) \to p \neq \bar{p}^i(\delta)$. Then, by a property of set convergence, there must exist a sequence $\langle p(T,\delta)\rangle_{T=1}^{\infty}$ where $p(T,\delta) \in R(T,\delta)$, such that $p(T,\delta) \to \bar{p}^i(\delta)$ as $T \to \infty$. Now, for large T, it must be the case that either $p(T,\delta)$ gives player i less than his minimum vector (or if it gives i the same payoff, then it gives j a higher payoff) or it Pareto dominates $\bar{p}^i(\delta)$. In either case we have a contradiction. ∎

We now show, in Theorem 3.1, that if the limit of renegotiation proof sets in (G^T,δ) exists as the time horizon goes to infinity, then the limit set must be a WRP set of the infinite repeated game. Later, we will examine the connection of this limit set with the IRP notion.

To establish theorem 3.1, we need the following preliminary lemma: no payoff in the limit set (when it exists) can Pareto dominate any other payoff in the limit set.

Lemma 3.2 *(a) Let* $\lim_{T\to\infty} R(T,\delta) = R(\delta)$ *for* $\delta < 1$. *Then* $R(\delta) = F(R(\delta))$.
(b) If $\langle R(\delta_n)\rangle_{n=1}^{\infty}$ *converges to a set* R *as* $\delta_n \to 1$, *then for* $i = 1, 2$,

Proof.

We will only prove (a) since the two proofs are very similar. Suppose (a) does not hold. Then there exists $x, y \in R(\delta)$ such that $x \gg y$. So, for any $\epsilon > 0, \exists\ \bar{T}(\epsilon)$ s.t. $\forall\ T > \bar{T}(\epsilon),\ \exists\ x^T, y^T \in R(T,\delta)$ s.t. $x^T \in B_\epsilon(x)$ and $y^T \in B_\epsilon(y)$, where $B_\epsilon(x)$ and $B_\epsilon(y)$ are ϵ-neighbourhoods around x and y respectively. If $x \gg y$, and ϵ is be taken small enough, then $x^T \gg y^T$ for all $T > \bar{T}(\epsilon)$. This contradicts the fact that no vector in $R(T,\delta)$ can dominate any other vector in $R(T,\delta)$. ∎

Theorem 3.1 *Let* $\lim_{T\to\infty} R(T,\delta) = R(\delta)$ *for* $\delta < 1$. *Then* $R(\delta)$ *is a WRP set.*

Proof.

We will show that $R(\delta) \subseteq \theta^I(R(\delta))$, i.e., if $x \in R(\delta)$, then there exist $a \in A$

and $\hat{p}, p^1, p^2 \in R(\delta)$ such that for $i = 1, 2$,

$$x_i = (1-\delta)g_i(a) + \delta\hat{p}_i$$

and

$$[c_i(a) - g_i(a)] \leq \frac{\delta}{1-\delta}(\hat{p}_i - p_i^i)$$

This combined with Lemma 3.2 gives us the proof.

Consider a sequence $\langle x(T)\rangle_{T=1}^{\infty}$ such that $x(T) \in R(T,\delta)$ and $x(T) \to x$ as $T \to \infty$. Now, since $x(T) \in R(T,\delta)$, there exists $a(T) \in A$ and payoff vectors $\hat{p}(T-1), p^1(T-1), p^2(T-1) \in R(T-1)$ such that

$$x_i(T) = \frac{(1-\delta)}{(1-\delta^T)}g_i(a(T)) + \frac{\delta(1-\delta^{T-1})}{(1-\delta^T)}\hat{p}_i(T-1)$$

$$\text{and} \quad [c_i(a(T)) - g_i(a(T))] \leq \frac{\delta(1-\delta^{T-1})}{(1-\delta)}(\hat{p}_i(T-1) - p_i{}^i(T-1))$$

Since F^* and A are compact it is always possible to extract a subsequence $\{T_n\}_{n=1}^{\infty}$ such that along the subsequence, $a(T_n) \to a$; $\hat{p}_i(T_n - 1) \to \hat{p}_i$, $p_i{}^j(T_n - 1) \to p_i{}^j$ for each $i, j = 1, 2$ as $n \to \infty$. From the fact that $R(T_n - 1, \delta) \longrightarrow R(\delta)$, it follows that $\hat{p}, p^1, p^2 \in R(\delta)$.

From properties of limits and continuity of g_i it follows that

$$\frac{(1-\delta)}{(1-\delta^{T_n})}g_i(a(T_n)) + \frac{\delta(1-\delta^{(T_n-1)})}{(1-\delta^{T_n})}\hat{p}_i(T_n - 1) \overset{T_n\to\infty}{\longrightarrow} (1-\delta)g_i(a) + \delta\hat{p}_i = x_i$$

Further,

$$\frac{\delta(1-\delta^{(T_n-1)})}{(1-\delta)}[\hat{p}_i(T_n - 1) - p_i{}^i(T_n - 1)] \overset{T_n\to\infty}{\longrightarrow} \frac{\delta}{(1-\delta)}(\hat{p}_i - p_i{}^i)$$

since

$$\frac{\delta(1-\delta^{(T_n-1)})}{(1-\delta)} \overset{T_n\to\infty}{\longrightarrow} \frac{\delta}{(1-\delta)}$$

From continuity of $[c_i(\cdot) - g_i(\cdot)]$, it follows that in the limit,

$$[c_i(a) - g_i(a)] \leq \frac{\delta}{1-\delta}(\hat{p}_i - p_i^i)$$

So, x is supported by $R(\delta)$.

■

The arguments of Theorem 3.1 essentially follow from the continuity of the the function $c_i(\cdot) - g_i(\cdot)$ and the compactness of the action set and the feasible payoff set.

We now proceed to examine the connection of the limit (when such a limit exists) of renegotiation proof sets for long finitely repeated games to the IRP concept in infinitely repeated games.

In any game, let us consider the strong[5] Pareto Frontier of the set of Nash equilibrium payoffs which we will call *the set of undominated Nash equilibria* payoffs. If in some game this set is a singleton, then in any finitely repeated version of this game, the renegotiation proof set is a singleton containing just this solitary payoff. Further, if the limit set $R(\delta)$ exists and is a singleton, then the stage game must have a unique Nash equilibrium payoff which is undominated by any other Nash equilibrium payoffs (i.e., the strong Pareto Frontier of the set of Nash equilibrium payoffs is a singleton) and $R(\delta)$ must be this singleton. This is our next theorem.

Theorem 3.2 *In a game G, 'the set of undominated Nash equilibria' payoffs is a singleton if and only if $R(\delta)$ exists and is this singleton. This is true for all δ.*

Proof.
If $R(\delta)$ is a singleton, then the singleton has to be a WRP set (from theorem 3.1); so, it must be a Nash equilibrium payoff. It is easy to see that $R(\delta)$ cannot be a Nash equilibrium payoff which is dominated by some other Nash equilibrium, as then for large T, $R(T, \delta)$ will have payoff vectors close to the dominating equilibrium. Suppose, without loss of generality, the stage game has at least two distinct undominated (each not dominated by any other Nash equilibrium) Nash equilibria payoff x and y, and let $R(\delta)$ be $\{x\}$. Now $R(T, \delta) \to R(\delta)$ as $T \to \infty$. Without loss of generality, let $x_2 > y_2$ (so $x_1 < y_1$ from strong Pareto frontier property). Let $c(T) = \bar{p}^2(T, \delta)$ for some large T. Let $b(T) = (1 - \delta)y + \delta c(T)$. $b(T)$ is supportable by $R(T, \delta)$ as y is a Nash

[5] The strong Pareto frontier $\tilde{F}(B)$ of a set $B \in I\!R^2$ is the subset of all vectors in B such that there are no two vectors $x, y \in \tilde{F}(B)$ such that for $i = 1, 2$, $x_i \geq y_i$ with at least one strict inequality.

equilibrium payoff. If $b(T) \in R(T+1,\delta)$, define $c(T+1) = b(T)$. Otherwise, if there exists some $p(T+1) \in R(T+1,\delta)$ which Pareto dominates $b(T)$, then let $c(T+1) = p(T+1)$. Continuing in this way, for an arbitrary positive integer k, define $b(T+k) = (1-\delta)y + \delta c(T+k)$ where $c(T+k) = b(T+k-1)$ if $b(T+k-1) \in R(T+k,\delta)$; otherwise, if there exists some $p(T+k) \in R(T+k,\delta)$ which Pareto dominates $b(T+k-1)$, then $c(T+k) = p(T+k)$. Note that for each k, $b(T+k)$ is supportable by $R(T+k,\delta)$ and $c(T+k) \in R(T+k,\delta)$. However $c(T+k)$ moves away from x as $k \to \infty$. So, it cannot be that $R(\delta) = \{x\}$.

The 'only if' part of the proof is trivially true once we observe that $R(T,\delta)$ is the unique undominated Nash equilibrium for all T. ∎

If the stage game has a unique Nash equilibrium, then Theorem 3.2 implies that the limit set $R(\delta)$ is a singleton containing only the Nash equilibrium payoff.

We now investigate a non-singleton limit set $R(\delta)$. We find that $R(\delta)$, when it exists, can be an almost IRP set provided it is 'sufficiently' large. The requirement of the 'extent of largeness' is reminiscent of a similar requirement in Corollary 2.1 in chapter 2. Corollary 2.1 was an easy application of theorem 2.2 in chapter 2. The proof of Theorem 3.3 draws heavily on the proof of Theorem 2.2 and therefore, a lot of detail will be omitted.

Theorem 3.3 *Let A be mixed strategy action space. Let there exist some vector $x \in R(\delta)$ such that for any action vector $a \in A$ and $i = 1,2$, $[c_i(a) - g_i(a)] < \frac{\delta}{(1-\delta)}[x_i - \bar{p}_i^i(\delta)]$. Then, $R(\delta)$ is an almost IRP set.*

Proof. :
Any $p(\delta) \in R(\delta)$, must satisfy, for some $i = 1,2$, $p_i(\delta) \geq x_i$; otherwise, x will Pareto dominate $p(\delta)$, which is not possible by Lemma 3.2. So, for any such $p(\delta) \in R(\delta)$, there exists some i, such that for any action vector $a \in A$, $[c_i(a) - g_i(a)] < \frac{\delta}{(1-\delta)}[p_i(\delta) - \bar{p}_i^i(\delta)]$.

The proof follows from the following four claims.

Claim 1. $R(\delta) \subseteq \theta^I(R(\delta))$.

Claim 2. For any vector $r(\delta) \in \theta^I(R(\delta))$, there exists $q(\delta) \in R(\delta)$ such that $q(\delta) \geq r(\delta)$.

Claim 3. If claim 2 is true, then $R(\delta) \subseteq F(\theta^I(R(\delta)))$.
Claim 4. If claim 2 is true, then for any $p \in F(\theta^I(R(\delta)))$, there exists $p' \in R(\delta)$ such that $p' \geq p$ with p' not strongly Pareto dominating p.

Note that when Claim 2 is true, Claim 3 and Claim 4 together tells us that $R(\delta)$ must indeed be an almost IRP set. The proofs of the four claims are copies of the proofs of four similar claims in theorem 2.2 of chapter 2; hence, they will not be repeated (in particular, claim 2 is proved by taking an identical approach as in Case 1 and Case 2 in the proof of theorem 2.2). ■

We now turn to the result of Benoit-Krishna (1993) which establishes that when $\delta = 1$, $\lim_{T\to\infty} R(T,1)$ is either a singleton or is a subset of the efficient frontier. We show that this extends to the case of low discounting as well, using a modification of the analogous result of Ray (1994) for IRP sets (see Theorem 2.1 in Chapter 2).

Theorem 3.4 *If $R \in \mathcal{R}$, then either (a) R is a singleton or (b) R is a subset of $F(F^*)$.*

The reader can verify that Theorem 3.4 is established by an argument which mimics Ray's proof; hence, the proof will be omitted. We will give the argument very briefly. We require the following notation. For each $i = 1, 2$, define $g^i \in F^*$ by first maximizing p_i over $p \in F^*$ and then minimizing p_j over the set of maximizers. Observe that g^i is unique and is the payoff ensuing from a pure action vector in G. Let $R \in \mathcal{R}$. Suppose for some p', $p'' \in R$ and for some $i \in \{1, 2\}$, we have $p'_i < p''_i$. Define the following subset of $\mathbb{R}^2$:

$$L(i, p', p'') = \{p | \exists \lambda \in [0,1] \text{ s.t. } p = (1-\lambda)g^j + \lambda p'',\ j \neq i, \text{ and } p_i > p'_i\}$$

To prove Theorem 3.4, the following two Lemmas (3.3 and 3.4) are required. They are analogous to two lemmas in Ray's (1994) paper (the difference is that R has to be read as a limit IRP set in Ray's paper) and can be established by similar arguments; so the proofs are omitted.

Lemma 3.3 *For each $p \in L(i, p', p'')$, there exists $q \in F^*$ such that $q \geq p$ and $q \in R$.*

equilibrium payoff. If $b(T) \in R(T+1,\delta)$, define $c(T+1) = b(T)$. Otherwise, if there exists some $p(T+1) \in R(T+1,\delta)$ which Pareto dominates $b(T)$, then let $c(T+1) = p(T+1)$. Continuing in this way, for an arbitrary positive integer k, define $b(T+k) = (1-\delta)y + \delta c(T+k)$ where $c(T+k) = b(T+k-1)$ if $b(T+k-1) \in R(T+k,\delta)$; otherwise, if there exists some $p(T+k) \in R(T+k,\delta)$ which Pareto dominates $b(T+k-1)$, then $c(T+k) = p(T+k)$. Note that for each k, $b(T+k)$ is supportable by $R(T+k,\delta)$ and $c(T+k) \in R(T+k,\delta)$. However $c(T+k)$ moves away from x as $k \to \infty$. So, it cannot be that $R(\delta) = \{x\}$.

The 'only if' part of the proof is trivially true once we observe that $R(T,\delta)$ is the unique undominated Nash equilibrium for all T. ■

If the stage game has a unique Nash equilibrium, then Theorem 3.2 implies that the limit set $R(\delta)$ is a singleton containing only the Nash equilibrium payoff.

We now investigate a non-singleton limit set $R(\delta)$. We find that $R(\delta)$, when it exists, can be an almost IRP set provided it is 'sufficiently' large. The requirement of the 'extent of largeness' is reminiscent of a similar requirement in Corollary 2.1 in chapter 2. Corollary 2.1 was an easy application of theorem 2.2 in chapter 2. The proof of Theorem 3.3 draws heavily on the proof of Theorem 2.2 and therefore, a lot of detail will be omitted.

Theorem 3.3 *Let A be mixed strategy action space. Let there exist some vector $x \in R(\delta)$ such that for any action vector $a \in A$ and $i = 1,2$, $[c_i(a) - g_i(a)] < \frac{\delta}{(1-\delta)}[x_i - \bar{p}_i^i(\delta)]$. Then, $R(\delta)$ is an almost IRP set.*

Proof. :

Any $p(\delta) \in R(\delta)$, must satisfy, for some $i = 1,2$, $p_i(\delta) \geq x_i$; otherwise, x will Pareto dominate $p(\delta)$, which is not possible by Lemma 3.2. So, for any such $p(\delta) \in R(\delta)$, there exists some i, such that for any action vector $a \in A$, $[c_i(a) - g_i(a)] < \frac{\delta}{(1-\delta)}[p_i(\delta) - \bar{p}_i^i(\delta)]$.

The proof follows from the following four claims.

Claim 1. $R(\delta) \subseteq \theta^I(R(\delta))$.

Claim 2. For any vector $r(\delta) \in \theta^I(R(\delta))$, there exists $q(\delta) \in R(\delta)$ such that $q(\delta) \geq r(\delta)$.

Lemma 3.4 *Suppose that R is not a singleton and that R is not a subset of $F(F^*)$. Then there exists $p \in R$, $q \in F^*$, such that $q \gg p$ and for $i = 1, 2$, $p_i > \bar{p}_i^{\,i}$.*

The argument in the proof of Theorem 3.4 runs in the following manner. Lemma 3.3 helps to show that any $R \in \mathcal{R}$ must be a closed and connected set (see Lemma 3.5 stated and proved in the Appendix). Now, lemma 3.4 tells us that if R is a non-singleton which is not a subset of the Pareto frontier, then there will be a payoff vector $p \in R$ which is not on the Pareto frontier and which gives each player a payoff higher than from his minimum payoff vector in R. There exist sequences $p(\delta_n), p^1(\delta_n), p^2(\delta_n) \in R(\delta_n)$, converging, as $\delta_n \to 1$, to $p, \bar{p}^1$ and $\bar{p}^2$ respectively. Similarly, there exist sequences $p(T, \delta_n), p^1(T, \delta_n), p^2(T, \delta_n) \in R(T, \delta_n)$, converging, as $T \to \infty$, to $p(\delta_n), \bar{p}^1(\delta_n)$ and $\bar{p}^2(\delta_n)$ respectively. Now, note that all these payoffs are normalized; for large δ_n and all large enough (depending on how large is δ_n) T, the actual payoffs for each player i from $p(\delta_n)$ may be very far away from the payoff that the same person is getting from $\bar{p}^i(\delta_n)$. Consequently, there should be no incentive problem to support payoffs which in the limit (as $T \to \infty$) Pareto dominate $p(\delta_n)$, which is a contradiction. So, no point in the relative interior of R can be Pareto inefficient. Connectedness of R helps us to extend the same conclusion to the endpoints of R.

Our next result holds for mixed action spaces. Theorem 3.5 states that if the Pareto frontier of F^* is strictly downward sloping, and $R(\delta_n)$ converges to a nonsingleton R along a sequence $\langle \delta_n \rangle_{n=1}^{\infty}$ converging to one, then for large δ_n, $R(\delta_n)$ must be an almost IRP set. In other words, it is not possible for a sequence of $R(\delta_n)$ to hit a non-singleton (on the Pareto frontier from Theorem 3.4) limit as discounting vanishes, without sets from the sequence already becoming almost IRP sets for high discount factors.

We need some further notation before we can embark on Theorem 3.5. For any closed set $S \subset \mathbb{R}^2$, define $B_\epsilon(S) = \{y | \operatorname{dist}(y, S) < \epsilon\}$ to be the open ball of radius ϵ around S.

Theorem 3.5 *For $i = 1,2$, let A_i denote the mixed strategy action space for player i in G. Let there be a sequence $\langle\delta_n\rangle_{n=1}^{\infty}$ such that $\delta_n \to 1$, $R(\delta_n)$ exists for each n and $R(\delta_n) \to R$ as $n \to \infty$. If R is a non-singleton such that $Min\{\bar{p}_1^2 - \bar{p}_1^1, \bar{p}_2^1 - \bar{p}_2^2\} > \beta$ for some $\beta > 0$, then there exists $\bar{\delta} < 1$, such that if $\delta_n > \bar{\delta}$, $R(\delta_n)$ is an almost IRP set.*

Proof.
Let the conditions of the theorem hold. So, there exists a sequence $\langle\delta_n\rangle_{n=1}^{\infty}$ such that $\delta_n \to 1$, $R(\delta_n)$ exists for each n and $R(\delta_n) \to R$ where R is a non-singleton subset of $F(F^*)$ (from theorem 3.4) satisfying $\text{Min}\{\bar{p}_1^2 - \bar{p}_1^1, \bar{p}_2^1 - \bar{p}_2^2\} > \beta$ for some $\beta > 0$. It follows from Lemma 3.5 that R is closed and connected.

Can $\bar{p}^1 \in F(F^*)$ belong to a vertical segment of $F(F^*)$, i.e. can there be $p' \in F(F^*)$ such that $p' \neq \bar{p}^1$ and $p'_1 = \bar{p}_1^1$? Note that if such a p' exists, then the set $X = \{p \in F(F^*) | p_2 \leq \bar{p}_2^1\}$ must have for each element $x \in X$, $x_1 = \bar{p}_1^1$. Note further that $\bar{p}^2$ belongs to X. So, $\bar{p}_1^2 = \bar{p}_1^1$. This will contradict that $\text{Min}\{\bar{p}_1^2 - \bar{p}_1^1, \bar{p}_2^1 - \bar{p}_2^2\} > \beta > 0$. So, $\bar{p}^1$ cannot belong to a vertical segment of $F(F^*)$. Similarly, $\bar{p}^2$ cannot belong to a horizontal segment of $F(F^*)$.

Let $\epsilon > 0$ be such that the intersection of $B_\epsilon(\bar{p}^1)$ and $B_\epsilon(\bar{p}^2)$ is empty. Note that such an ϵ always exists (any $\epsilon < \beta/2$ will do) given that $\text{Min}\{\bar{p}_1^2 - \bar{p}_1^1, \bar{p}_2^1 - \bar{p}_2^2\} > \beta$. Consider ϵ/m for any positive integer m. Since $R(\delta_n) \to R$, it follows (from the definition of convergence in Hausdorff metric[6]) that there exists $\delta(m) < 1$ such that whenever $\delta_n > \delta(m)$, $R(\delta_n) \subset B_{\epsilon/m}(R)$. Now R is a connected set where if $p \in R$ and $p \notin B_\epsilon(\bar{p}^i)$ for some $i = 1,2$, then there exists a number $2\gamma > 0$ such that $p_i - \bar{p}_i^i > 2\gamma$ (remember from the previous paragraph that $\bar{p}^1$ cannot belong to a vertical section of $F(F^*)$, nor can $\bar{p}^2$ belong to horizontal section of $F(F^*)$). It now must be the case that there exists a m sufficiently large (say M), such that if for some $\delta_n > \delta(M)$, if $p(\delta_n) \in R(\delta_n)$ but $p(\delta_n) \notin B_\epsilon(\bar{p}^i)$ for some $i = 1,2$, then $p_i(\delta_n) - \bar{p}_i^i(\delta_n) > \gamma$. If the above does not happen, then $R(\delta_n)$ cannot converge to the set R. Moreover, since $B_\epsilon(\bar{p}^1)$ is disjoint from $B_\epsilon(\bar{p}^2)$, the above must be true for each $p(\delta_n) \in R(\delta_n)$ for large enough δ_n. Let $\bar{\delta} \in (\delta(M), 1)$ be such that for all $\delta_n > \bar{\delta}$, and for $i = 1,2$,

[6] See Hildenbrand (1974) for a discussion.

$[c_i(a) - g_i(a)] < \frac{\delta}{(1-\delta)}\gamma$ for each $a \in A_1 \times A_2$. In other words for some $i = 1, 2$,

$$[c_i(a) - g_i(a)] < \frac{\delta}{(1-\delta)}[p_i(\delta_n) - \bar{p}^i_i(\delta_n)] \tag{3.3.1}$$

We now have to show that for $\delta_n > \bar{\delta}$, $R(\delta_n)$ is an almost IRP set. One can check that the four claims of the proof of Theorem 3.3 holds; in particular the second claim holds from (3.3.1) through the arguments of Case 1 and Case 2 of theorem 2.2 in chapter 2. ■

When $\text{Min}\{\bar{p}^2_1 - \bar{p}^1_1, \bar{p}^1_2 - \bar{p}^2_2\} = 0$, the above theorem may not go through. To see why, let $\{\bar{p}^1_2 - \bar{p}^2_2\} = 0$ without loss of generality. Off-hand, we cannot rule out the case that for large δ_n, punishments for player 2 are not very severe (in other words $\frac{\delta_n}{(1-\delta_n)}[p^1{}_2(\delta_n) - \bar{p}^2_2(\delta_n)]$ is very small). Consider such a scenario for some large δ_n. Suppose $r(\delta_n)$ is a payoff vector belonging to the 'strict' Pareto frontier of the payoff set supported by $(R(\delta_n))$. Consider this extreme perverse case where the only way $r(\delta_n)$ is supported by $R(\delta_n)$ is through a 'bad' action vector a, a 'bad' continuation vector $p(\delta_n) \in R(\delta_n)$ and punishment vectors $\bar{p}^i(\delta_n)$ for $i = 1, 2$. The vectors are called 'bad' if, firstly, each player i's incentive constraint

$$[c_i(a) - g_i(a)] = \frac{\delta_n}{(1-\delta_n)}[p_i(\delta_n) - \bar{p}^i_i(\delta_n)]$$

is attained with equality; and secondly, for any three sequences $p(T, \delta_n), p^1(T, \delta_n), p^2(T, \delta_n) \in R(T, \delta_n)$, which converge, as $T \to \infty$, to $p(\delta_n), \bar{p}^1(\delta_n), \bar{p}^2(\delta_n)$ respectively, there exists a small neighbourhood around a such that for any b in that neighbourhood, and for all large T, the following is satisfied for some player i:

$$[c_i(b) - g_i(b)] > \frac{\delta_n(1-\delta_n{}^T)}{(1-\delta_n)}[p_i(T, \delta_n) - \bar{p}^i_i(T, \delta_n)].$$

(Note that such a case cannot occur when we consider mixed strategies and the condition of the theorem holds). Consequently, it may not be possible to construct a sequence $p'(T+1, \delta_n) \in R(T+1, \delta_n)$ such that $p'(T+1, \delta_n) \to r(\delta_n)$ as $T \to \infty$. Now $r(\delta_n)$, which had been supported by the bad a and the bad $p(\delta_n)$, belongs to the Pareto frontier of the payoff set supported by $R(\delta_n)$. However, it is neither in $R(\delta_n)$, nor is there any other payoff in $R(\delta_n)$ which weakly (in the sense of almost IRP) dominates $r(\delta_n)$. This is the reason why the limit set may not be an almost IRP set when the 'largeness' condition of Theorem 3.5 is violated.

3.4 Example

We consider the example in chapter 2, in which we showed that IRP sets cannot exist for high discount factors in the infinitely repeated game context. If there were a close connection between the limit of renegotiation proof sets for long finitely repeated games, we might expect a similar non-existence result pertaining to the former. This turns out to be precisely the case: we show that there exists $\bar{\delta} > 0$ such that for all $\delta > \bar{\delta}$, $R(\delta)$ does not exist.

Suppose not (see figures 2.1 and 2.2 in chapter 2). Let $\langle \delta_n \rangle_{n=1}^{\infty}$ such that $\delta_n \to 1$, $R(\delta_n)$ exists for each n and that $R(\delta_n) \to R$. By Theorem 3.4, R is either a singleton or is a subset of the Pareto efficient Frontier of F^*.

That R cannot be a singleton follows from a logic very similar to the proof of the 'if' part of Theorem 3.2. The same logic works as there are two 'undominated' (among Nash) Nash equilibrium payoffs.

If R is a non-singleton it must be a subset of $F(F^*)$. Note that $F(F^*)$ in this example does not have any horizontal or vertical sections. So, if A_i for $i = 1, 2$, are mixed action spaces of the stage game, R must satisfy the conditions of Theorem 3.5. Consequently, for all large δ_n (greater than some $\bar{\delta} > 0$), $R(\delta_n)$ is an almost IRP set. Although we have not said anything about the existence of almost IRP sets in infinitely repeated games, we will show, in our example, that an almost IRP set which is a limit of renegotiation proof sets for long finite repetitions of the game, cannot exist for large discount factors.

From Theorem 3.1, $R(\delta_n)$ is a WRP set for each n. Further, since $R(\delta_n)$ is a limit set, $R(\delta_n)$ is closed and so compact. $\bar{p}^1(\delta_n)$, $\bar{p}^1(T, \delta_n)$ and $\bar{p}^1$ as usual denote the minimum payoff vectors of player 1 in $R(\delta_n)$, $R(T, \delta_n)$ and R respctively. Can $\bar{p}^1_2(\delta_n) < 4$ for large n? For such an n, let $c(T) = \bar{p}^1(T, \delta_n)$ for some large T. Let $b(T) = (1 - \delta_n)(1, 4) + \delta_n c(T)$. $b(T)$ is supportable by $R(T, \delta_n)$ as $(1, 4)$ is a Nash equilibrium payoff. If $b(T) \in R(T + 1, \delta_n)$, define

$c(T+1) = b(T)$. Otherwise, if there exists some $p(T+1) \in R(T+1, \delta_n)$ which Pareto dominates $b(T)$, then let $c(T+1) = p(T+1)$. For an arbitrary positive integer k, define $b(T+k) = (1-\delta_n)(1,4)+\delta_n c(T+k)$ where $c(T+k) = b(T+k-1)$ if $b(T+k-1) \in R(T+k, \delta_n)$; otherwise, if there exists some $p(T+k) \in R(T+k, \delta_n)$ which Pareto dominates $b(T+k-1)$, let $c(T+k) = p(T+k)$. Note that each $b(T+k)$ is supportable by $R(T+k, \delta_n)$ and $c_2(T+k) \to 4$ as $k \to \infty$. Moreover, it must be that $\bar{p}^1_2(T+k, \delta_n) \geq c_2(T+k)$. From Lemma 3.1, $\bar{p}^1(T+k, \delta_n) \to \bar{p}^1(\delta_n)$; so, it cannot be that $\bar{p}^1_2(\delta_n) < 4$.

$R(\delta_n)$ is a compact WRP set (Theorem 3.1). so, from Proposition 2.1 of chapter 2, note that $\bar{p}^1_2(\delta_n)$ cannot be greater than 4.

Can $\bar{p}^1_2(\delta_n) = 4$ and $\bar{p}^1_1(\delta_n) > 1$? Since $R(\delta_n)$ is WRP it follows from Theorem 1 of Farrell-Maskin (also see discussion in chapter 2 before Proposition 2.1) that $\bar{p}^1(\delta_n)$ is supportable in $R(\delta_n)$ by using an action vector $a \in A$ such that $g_2(a) \geq 4$ and $c_1(a) \leq \bar{p}^1_1(\delta_n) \leq 2$. The only such action vector available is the Nash equilibrium (T, C) with payoff $(1,4)$. So, it must be that for large n, $\bar{p}^1(\delta_n)$ is supported as $(1-\delta_n)(1,4) + \delta_n(p(\delta_n))$ for some payoff vector $p(\delta_n) \in R(\delta_n)$ where $p_1(\delta_n) > 1$ and $p_2(\delta_n) = 4$. Now from Lemma 3.1, we know that $\bar{p}^1(T, \delta_n) \to \bar{p}^1(\delta_n)$ as $T \to \infty$. Let $\bar{p}^1(T, \delta_n)$ be supported in $R(T, \delta_n)$ as

$$\bar{p}^1(T, \delta_n) = \frac{(1-\delta_n)}{(1-\delta_n^T)} g(a_T) + \frac{\delta_n(1-{\delta_n}^{T-1})}{(1-{\delta_n}^T)} y_{T-1}$$

where $y_{T-1} \in R(T-1, \delta_n)$. If $\bar{p}^1(T, \delta_n)$ has to converge to $\bar{p}^1(\delta_n)$, then it has to be the case that $a_T \to (T, C)$, for otherwise there will be at least two distinct subsequential limits $a(1)$ and $a(2)$ which can support $\bar{p}^1(\delta_n)$; this contradicts the fact that the only such vector is the Nash equilibrium vector (T, C). Also, $y_{T-1} \in R(T-1, \delta_n)$, for otherwise, there will two distinct subsequential limits from $\bar{p}^1(T, \delta_n)$ which converge to $\bar{p}^1(T, \delta_n)$; thus contradicting $\bar{p}^1(T, \delta_n) \to \bar{p}^1(\delta_n)$. Can for all large T, $g_2(a_T) > 4$? If $g_2(a_T) > 4$ for all large T, then note that $c_1(a_T) - g_1(a_T)$ is bounded away from zero for large T. Extract a subsequence such that a_T converges to some a' and y_{T-1} converges to some y' along the subsequence. Since a' is a subsequential limit, note that $c_1(a') - g_1(a')$ must be bounded away from zero (from continuity of $c_i(\cdot) - g_i(\cdot)$); so a' cannot be (T, C), which is a contradiction. This implies that it must be the case that $g_2(a_T) \leq 4$ for all large T (greater than some $T' > 0$). Define

$$p'(T,\delta_n) = \frac{1-\delta_n}{(1-{\delta_n}^T)}(1,4) + \frac{\delta_n(1-{\delta_n}^{T-1})}{(1-{\delta_n}^T)}\bar{p}^1(T-1,\delta_n).$$

We claim that for all $T > T'+1$, $\bar{p}^1(T,\delta_n) = p'(T,\delta_n)$. Check that for all T, $p'(T+1,\delta_n) \in \theta(R(T,\delta_n))$. For $T > T'+1$, $p'(T,\delta_n)$ as defined must either lie weakly to the north-west of $p(T,\delta_n)$ or must Pareto dominate $\bar{p}^1(T,\delta_n)$ (as $g_2(a_T) \leq 4$ and y_{T-1} lies weakly to the south-east of $\bar{p}^1(T-1,\delta_n)$). If $\bar{p}^1(T,\delta_n) \neq p'(T,\delta_n)$, then $\bar{p}^1(T,\delta_n)$ cannot be the minimum vector of player 1 in $R(T,\delta_n)$. So, for all $T > T'+1$, $\bar{p}^1(T,\delta_n) = p'(T,\delta_n)$. So, for $T > T'+1$, the following recursive relation holds:

$$\bar{p}^1(T,\delta_n) = \frac{(1-\delta_n)}{(1-\delta_n)^T}(1,4) + \frac{\delta_n(1-{\delta_n}^{T-1})}{(1-{\delta_n}^T)}\bar{p}^1(T-1,\delta_n).$$

From the above recursive relation, it must be the case that $\bar{p}^1(T,\delta_n) \to (1,4)$. So, $\bar{p}^1(\delta_n) = (1,4)$ for all discount factors higher than $\bar{\delta}$ and cannot converge to the Pareto frontier; in other words, a non-singleton R cannot exist.

The non-existence of IRP set in the example for low discounting therefore extends in a certain sense to long finite games. Small perturbations of the payoff structure maintains the non-existence. The example also does not admit limit Renegotiation Proof sets (IRP or long finite games, as the case may be) in pure strategies (the proof of which is similar). However, the undiscounted long finite game with pure strategies, admits a limit renegotiation proof set as the horizon goes to infinity. The limit can be calculated through the techniques employed by Benoit-Krishna and the explicit calculations are being omitted. The limit works out to be the convex hull of {(1.8,4.2),(4.2,1.8)} which lies on the efficient frontier. Note that this limit set contains payoffs (e.g. (1.8,4.2)) which does not satisfy the necessary condition (see proposition 2.1 of chapter 2) of a payoff vector to belong to a limit WRP set vanishing discounting (no payoff vector giving player 2 more than 4 can belong to a WRP set). This fact tend to point out that renegotiation proof sets for long finitely repeated games with vanishing discounting may not have much in common with the case where there is no discounting.

3.5 Existence restored

We wish to examine if in the example considered in the previous section, existence can be restored with some modifications of the concept of renegotiation proof sets for long finitely repeated games. We already know that, in the example, $R(\delta)$ does not exist for large discount factors. When such limits do not exist, can we conclude that limit point sets (which will always exist) will correspond to something like a NIRP set of the infinitely repeated game? We will show that this indeed is the case. In this section, we concentrate on mixed strategy action spaces and show that if the discount factor is such that $R(T,\delta)$ is 'sufficiently large' for all large enough T then any limit point set of the sequence $\langle R(T,\delta)\rangle_{T=1}^{\infty}$ will belong to some almost NIRP collection. Theorem 3.6 portrays such a situation. More generally, if the above conditions hold, the collection $\mathcal{R}(\delta)$, of all limit point sets will be an ANIRP collection (Corollary 3.1). This suggests that under certain situations, when $R(T,\delta)$ do not converge to some $R(\delta)$, almost NIRP set may be a good concept of extending the concept of renegotiation proof sets from long finitely repeated games to infintely repeated games. Our example satisfies the conditions of Theorem 3.6 and Corollary 3.1 for large discount factors. So almost NIRP sets exist in the example.

Theorem 3.6 *Let $A_i, i = 1,2$ denote mixed strategy action spaces. Let δ be such that there exists a positive integer $T'(\delta) > 1$ and a $\beta > 0$ such that whenever $T > T'(\delta)$, there exists $x(T,\delta) \in R(T,\delta)$, which satisfies, for each $i = 1,2$ and any action vector $a \in A$, the following inequality:*

$$[c_i(a) - g_i(a)] + \beta < \frac{\delta}{1-\delta}[x_i(T,\delta) - \vec{p}_i^{\,i}(T,\delta)].$$

Then any limit point set $R^1(\delta)$ of $\langle R(T,\delta)\rangle_{T=1}^{\infty}$ must belong to an almost NIRP collection.

Theorem 3.6 is exactly the same as Theorem 2.2 in chapter 2; the proof is therfore omitted.

Corollary 3.1 *If the conditions of Theorem 3.6 hold, then the collection of all limit point sets of $\langle R(T,\delta)\rangle_{T=1}^{\infty}$ is an NIRP collection.*

Corollary 3.1 is again a restated version of Corollary 2.1 in chapter 2 and so the proof will be omitted.

That our example satisfies the conditions of theorem 3.6 has been shown following Theorem 2.2 in chapter 2.

3.6 Conclusion

In this chapter, we have focused on the behavior of Renegotiation Proof sets for discounted finitely repeated games as the time horizon goes to infinity and the discounting vanishes. We started by looking at the limiting behaviour of the renegotiation proof sets $R(T, \delta)$ for T-times repeated games, as the time horizon goes to infinity for a given factor $\delta < 1$. Such limits are always WRP sets (as shown in Theorem 3.1). Theorem 3.2 told us that this limit can be a singleton only if it has an unique undominated Nash equilibrium, in which case, it is an IRP set of the infinitely repeated version of the game. Theorem 3.3 gave some sufficient conditions under which a non-singleton limit can be almost like an IRP set of the infinitely repeated game (discount factors are also required to be high enough). As discounting vanishes, such limits can converge (may be subsequentially) to sets which are either singletons or subsets of the Pareto frontier (Theorem 3.4); moreover for most games, if the limits converge to a non-singleton set, then these limits must be almost IRP sets for higher discount factors (Theorem 3.5). However, do such limits always exist? Our example of chapter 2 does not admit such limits for all high enough discount factors. However, all games admit limit point sets even if they do not admit a limit. Do these limit points look like the non-stationary version of the IRP sets which we have discussed in chapter 2? The answer is provided in the affirmative for certain class of games for high discount factors (Theorem 3.6). In particular, for this class of of games, the collection of all limit points (for reasonably large discount factors) is an almost NIRP collection (Corollary 3.1). Our example belongs to this class of games.

This chapter has left several issues unresolved.

Will any limit point set of long finite repetitions of a game belong to some almost NIRP collection of the infinite repeated game when discounting is sufficiently low? Where do these almost NIRP collections, which are limit points for long repetitions of the stage game, converge as discounting vanishes? Do they also exhibit properties as in Ray (1994) and Benoit-Krishna (1993); namely, that they are either singletons or are subsets of the efficiency frontier? Further my results on NIRP collections depend heavily on mixed strategies. Do these results remain true if we consider pure strategies?

3.7 Appendix

Lemma 3.5 *If $R \in \mathcal{R}$, then R is a closed and connected set.*

We will require some additional notation to conclude that R is connected. For two vectors $x, y \in I\!R^2$, let $x|y$ denote $x_1 \leq y_1$ and $x_2 \geq y_2$. If $R \in \mathcal{R}$ and if $x, y \in R$, define $R(x,y) = \{z \in R \text{ such that } x|z|y\}$

Proof.

Since R is a limit of sets, R has to be closed from a property of set convergence. Trivially, R is non-empty. If R is a non-singleton which is not connected, then there exist $x, y \in R$ such that $R(x,y) = \{x,y\}$. Without loss of generality, let $x_1 < y_1$. Now, from Lemma 3.3, for small enough $\lambda > 0$, there exists $p \in L(1,x,y)$, where $p = (1-\lambda)g^2 + \lambda y$ and a $q \in F^*$ such that $q \geq p$ and $q \in R$. Further, since $R \subseteq F(F^*)$, $q \in F(F^*)$. Note that if λ is large enough, $x|q|y$, and this contradicts that $R(x,y) = \{x,y\}$. ∎

Chapter 4

Building Trust

4.1 Introduction

Friendship involves give and take, and good friendship rarely starts as soon as two persons meet. Typically, people take time to nurture their friendship till the friends assess that they can fully trust each other. Credit relations involve trust building. Rarely will a lender give a lot of credit to a borrower whom he has recently met. However, if the borrower consistently keeps a good credit record, then the lender can feel confident in course of time to increase the scale of lending. Slow trust building is also visible in most employer-employee relationships. An employer will very rarely give a very responsible job to a freshly recruited employee, or a large purchase order to a new contractor. In countries (like India) where domestic hands are often employed in houses (by housewives), a new domestic hand often faces intense monitoring by the employer. If the domestic hand is a good worker, then, only with time will the level of monitoring diminish.

One obvious reason why trust takes time to build is that partners may be incompletely informed about each other's characteristics (e.g. whether the partner is honest or dishonest; or about how long the partner intends the partnership to last) which are gradually revealed. This is the subject of the 'reputation' literature (see Fudenberg and Kreps (1987), and Fudenberg and Levine (1989)). In this chapter, we provide an alternative explanation which operates even in a perfect information scenario. Our explanation relies, instead, on the possibility of endogenous quits.

To elaborate, consider an infinitely repeated Prisoners' Dilemma played by two players. It is well known that if players are sufficiently patient, then the full cooperation outcome can be supported as a subgame perfect Nash equilibrium, where deviations are punished by suitable threats. The effectivenrss of such punishments is however, predicated on the assumption that the deviator is not free to avoid them by terminating the current relationship and seeking a new partner. If players could change partners, a switch to non-cooperation (a typical punishment strategy to support cooperation) would then fail to punish a deviator, as he could quit from the ongoing relationship and find a new partner. Indeed, in a population where all relationships involve full cooperation from the very beginning, a player could take advantage of others, by repeatedly defecting and then changing partners. Consequently, it is difficult to support cooperative outcomes in the conventional way. We examine the question of how the possibility of 'endogenous quitting and finding new partners' affects the level of cooperation that can be supported.

We develop a matching model for the credit market of the following kind. There is a large population of players (or agents) belonging to either of two types of equal size, 'Lenders' and 'Borrowers'. The game takes place over an infinite time horizon and all players discount their future identically. An exogenous process matches players of the two types with one another at date 1. Whenever a lender is paired with a borrower, they become partners in a new relationship. The stage game played between a matched pair is common knowledge and is as follows. The lender offers a level of trust (loan size) to his partner borrower. The borrower decides whether to respond honestly (repay the loan) or dishonestly (defaulting on the loan). Whenever the loan size is positive, defaulting gives short term gains to the borrower and short term losses to the lender. On the other hand, conditional on a honest response, the welfare of both players increase in the loan size. Before the period ends the partners simultaneously decide either to continue or terminate the relationship. Among all relationships where both partners decided to continue, nature picks a fraction of those relationships and terminates them for exogenous reasons. At the start of the next date, another random matching amongst all the players with broken partnerships takes place and then the game continues as before between the matched partners.

We assume that players in any relationship condition their actions 'only' on the history of the current relationship, and we focus on symmetric strategy profiles (where all members of a given type select the same strategy). Suppose we restrict attention to socially self-sustainable play where trust is never dishonored along the equilibrium path[1]. Along such paths involving "maximal" trust the following property is established in this chapter: "average" trust should build gradually along the course of any partnership. The only threat against dishonoring trust in the relationship is termination (since any other threat can be avoided by changing partners); this can be effective only if trust

[1]Note that trust can be dishonored in out of equilibrium behavior

builds slowly in a new relationship. Borrowers are deterred from dishonoring (defaulting on the loan) as any change of partner will lead to a fresh phase of slow trust building. Hence the slow build up of trust can be thought of as a social norm to induce cooperation when endogenous quits are possible.

Our model is related to that of Shapiro-Stiglitz (1984), where the fear of involuntary unemployment acts as a deterrent to deviation. Shapiro and Stiglitz argue that with imperfect monitoring and full employment, termination of the employee's contract (upon defection of shirking) does not serve to punish the worker since he can be immediately rehired in the market. They go on to show how involuntary unemployment can solve this incentive problem. In this chapter, however, there is no involuntary unemployment: defectors are re-matched in the very next date. Consequently, the only way in which a non-myopic "social norm of behavior" can be sustained is if the play invoves slow build-up of cooperation among new partners. Defection is thus discouraged by the fact that it will take time to build up a new cooperative relationship. In paticular, there is no difference between "on" and "off" equilibrium paths; the norm itself must serve as its own punishment.[2]

Okuno-Fujiwara (1990) introduced the notion of local information processing in matching games. Kandori (1992) considers a situation where agents change their partners over time (at every time point there is random matching). Amongst other things, Kandori gave an example where a community can sustain cooperation through the threat of an epidemiological process of defection, even when each agent knows nothing more than his personal experience. Our model differs from Kandori's in the following respects: (a) A partnership can continue if mutually desired, except that (b) there is a small probability with which a partnership can be broken for exogenous reasons ; (c) when two new partners meet, each does not know whether the other partner has quit an old relationship by choice or for exogenous reasons; and most im-

[2]Other related papers in the literature of contracts include the following: Stiglitz and Weiss (1983) discuss why it can be better for banks to terminate a credit relationship with a defaulter rather than charging a higher interest rate in the future; Lazear (1979) argues that it is preferable to pay workers less than their marginal value product when they are young and more when they are old. Then he argues why firms should have mandatory retirement.

portantly, (d) players treat all new partners identically; in other words no one will want 'not to cooperate' with a new partner just because of a bitter experience with someone before. If (d) did not hold and the exogenous quit rate was low enough, then Kandori's epidemiological process of defection could have ensured full cooperation right from the start. Now, consider what happens if somebody defects. Condition (b), now, would lead us to the unhappy situation where after some time there will be matched partners who have exogenously quit their initial partnership (for no fault of their own) and have to forever play the inefficient (minimal trust) course of play. With time almost all (with probability one) partnerships and levels of cooperation would break down for exogenous reasons.

One should emphasize that this chapter has nothing to do with the building of trust as is commonly known in the literature (in particular, there is no incomplete information). The point is that social norms which slowly build up cooperation (and which look like trust building) are relatively more immune to defections.

Section 4.2 simplifies the search for an equilibrium (we call 'social equilibrium') which is maximal for that type of the population which offers trust. Section 4.3 analyzes the model.

The main results are that a maximal equilibrium among all social equilibria in the honesty regime exist for all discount factors and quit rates (Theorem 4.1) ; along a maximal equilibrium path, average trust must be non-decreasing and strictly increasing between some time points (Theorem 4.2 and 4.4) ; for high quit rates and low discount factors, positive trust is never offered along a maximal equilibrium. However, for high discount factors and low quit rates, the full trust level will eventually be offered along a maximal equilibrium (Theorem 4.3).

4.2 The Model

There is a large population of players (agents) belonging to either of two types of equal size, Type 1 and Type 2, where the number of players are equal across the two types. A Type 1 player will be thought of as a lender and a Type 2

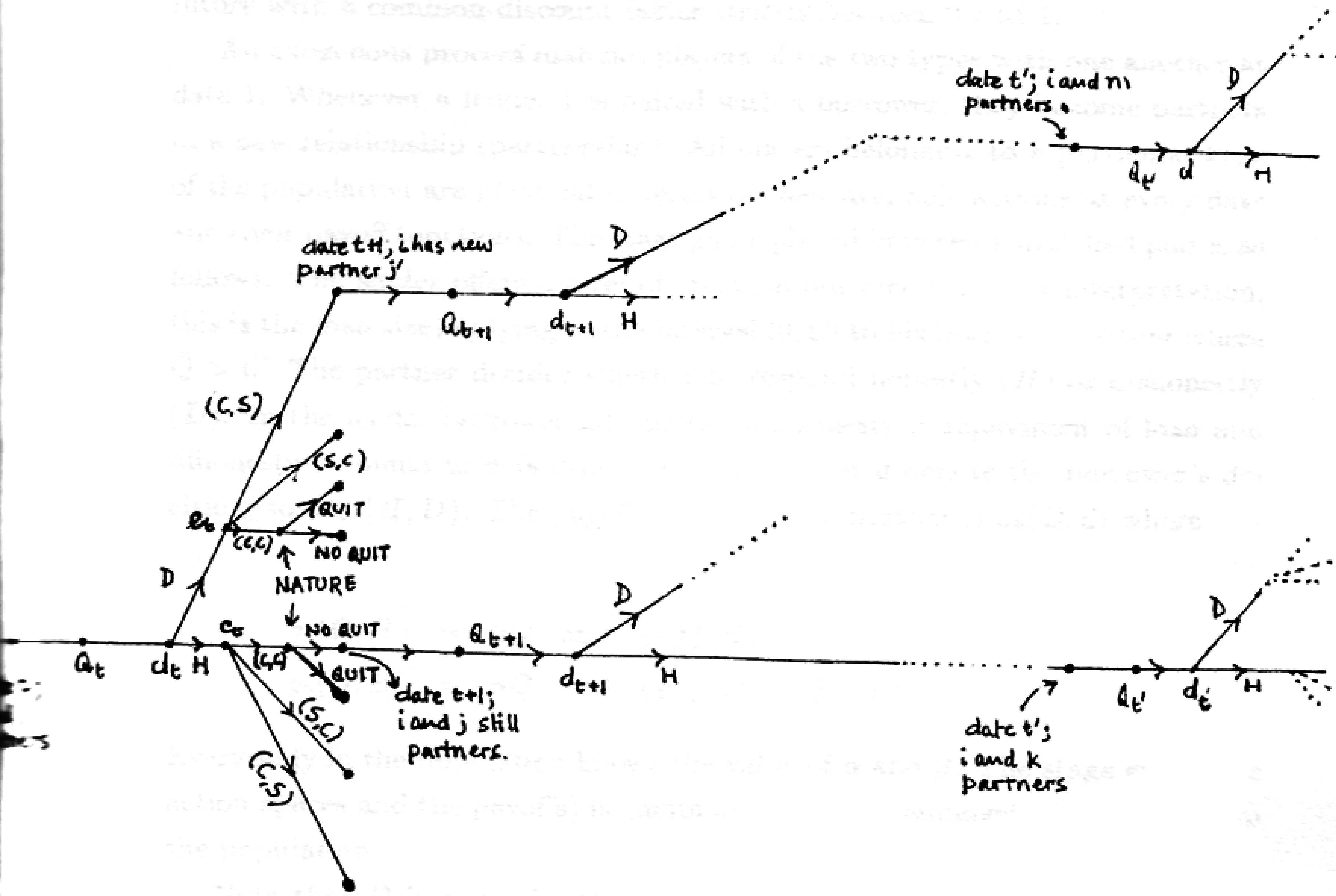

FIGURE 4.1 : TIMING SEQUENCE WITH REFERENCE TO PLAYER i FROM TYPE 1

player will be thought of as a borrower. The game takes place over a discrete infinite time horizon $t = 1, 2, \cdots \infty$. We assume that all players discount their future with a common discount factor strictly between 0 and 1.

An exogenous process matches players of the two types with one another at date 1. Whenever a lender 1 is paired with a borrower, they become partners in a new relationship (partnership). All players belonging to a particular type of the population are identical in terms of their available actions at every date and their payoff functions. The stage game played between a matched pair is as follows. The lender offers a level of trust (in our credit market interpretation, this is the loan size) Q lying in the interval $[0, \bar{Q}]$ to his borrower partner where $\bar{Q} > 0$. The partner decides whether to respond honestly (H) or dishonestly (D). In the lender-borrower interpretation, honesty is repayment of loan and dihonesty amounts to defaulting on the loan. Let d denote the borrower's decision; so $d \in \{H, D\}$. The payoff to the Type k partner is $g_k(Q, d)$ where

$$
\begin{aligned}
g_1(Q,H) &= \beta Q \quad \text{and} \quad g_1(Q,D) = -Q, \quad \text{where } \beta > 0 \text{ and} \\
g_2(Q,H) &= \alpha Q \quad \text{and} \quad g_2(Q,D) = Q, \quad 1 > \alpha > 0
\end{aligned}
$$

Everybody in the population knows the value of α and β. The stage game (the action spaces and the payoffs) is common knowledge amongst all the players in the population.

Note that D is a dominant strategy in the stage game for the borrower. The only Nash equilibrium in the stage game is where a lender offers trust $Q = 0$ (zero loan) and the borrower player plays D. The Nash equilibrium is inefficient. The only efficient action choice is where a lender offers trust $\bar{Q}$ and the borrower plays H, in which case they get $\beta\bar{Q}$ and $\alpha\bar{Q}$ respectively.

The game starts at date 1 with random pairing between lenders and borrowers (see Rosenthal (1979) for issues in modelling 'matching') such that each lender gets matched with exactly one borrower and vice versa. The partners proceed to play the stage game as described above at date 1. Once the partners have played the stage game, the extensive form (see figure 4.1) proceeds as follows. Just before date 1 ends the partners simultaneously decide either to 'continue' (C) the relationship, or 'terminate' (S for 'stop') it. We will call this decision c; so $c \in \{C, S\}$. If either of the partners play S, then the relationship is 'broken', in which case both the partners have to look for fresh partners in the next date. If they both decide to continue (both plays C) with the relationship, nature comes into play before the next date starts. Among all relationships which have decided to 'continue', nature picks a fraction q $(0 < q < 1)$ of those relationships and terminates them for exogenous reasons. At the start of the next date (date 2), another random matching amongst all the players with broken partnerships (broken by choice of either of the partners, or by nature) takes place and then the game continues as before between the current partners.

For notational convenience, let i represent a generic 'lender' and j represent a generic 'borrower' in the population. (i, j) denotes a generic partnership at time t if i and j are partners at time t. Figure 4.1 is an illustration of the timing and course of actions when lender i and borrower j are matched at time t.

The following definition tells us when a partnership will be labelled 'new' or 'old'. It also tells us what we mean by 'current player history' of a partnership at some date.

Definition 1. A partnership (i, j) is labelled 'new' at date $t > 1$ if both i and j had different partners at date $t - 1$. At date 1, every partnership is new. A partnership (i, j) is labelled 'old' at date $t > 1$ if i and j had been partners at date $t - 1$. If at date t, the partner of lender i is borrower j, then j is the 'current' partner of lender i at date t. Moreover, if j is the 'current' partner of i at date $t + k$ where $t, k \geq 1$, and the (i, j) partnership was 'new' at date t, with the course[3] of play between date t and date $t + k - 1$ being $(Q_t, d_t, Q_{t+1}, d_{t+1}, \cdots, Q_{t+k-1}, d_{t+k-1})$, then the

[3]When we say that the course of play between date t and date $t+k-1$ (of current partners (i, j) at date $t + k$ who were 'new' at date t) is $(Q_t, d_t, Q_{t+1}, d_{t+1}, \cdots, Q_{t+k-1}, d_{t+k-1})$, what we mean is that at date l, Q_l is the level of trust offered, d_l is the decision of player j from $\{H, D\}$ to honor or dishonor trust, and $c_l = C$ for both players i and j. Note that the decision c_l at date l from $\{C, S\}$ to continue or stop the relationship does not feature explicitly in the course of play as the very fact the relationship is on at date $t + k$ implies that both partners opted for C (continuation) between date t and date $t + k - 1$.

Current Player History (CPH) of the partnership (i, j) at date $t + k$ is $h_k = (Q_t, d_t, Q_{t+1}, d_{t+1}, \cdots, Q_{t+k-1}, d_{t+k-1})$. When a partnership (i, j) is new at date t, the current player history of the partnership is $h_1 = \{\phi\}$, the null set. At date 1, all partnerships have current player history h_1.

Note that the current player history of a partnership at date $t + k$ does not include play at date $t + k$. Further, note that it is possible that a lender i meets a 'new' partner j at date $t > 2$ who had also been i's 'current' partner at date $t' < t - 1$, in which case j is not really a fresh partner. Nevertheless, this likelihood can be ignored as the population of players to be matched at any date is large. Moreover, in some contexts a 'new' partner for lender i, who is not a fresh partner for lender i in the above sense, can suitably disguise himself such that lender i cannot recognise him as any different from a really new partner.

In Definition 1, we have considered 'current player history at a date $t + k$ of a partnership'. Noting that players have possibly different information when they take their respective actions, what we ought to define is the current player history of a player at a particular date. If h_t is a current player history of a partnership at some date T, then the current player history of lender i at date T when he decides on the level of trust to offer, is h_t itself. The current player history of lender i at date T when he is to make his continuation decision at date T is the history h_t followed by the trust level Q_T that he had offered at T, followed by the honesty decision d_T of his partner j at date T, i.e. (h_t, Q_T, d_T). A representative current player history for lender i at a decision node where he has to move following a partnership current player history h_t will be denoted as ${h^{1s}}_t$ where the superscript 1 is for type 1 and s is for the kind of decision node (the two kinds are Q, the 'level of trust to offer' and c,the continuation decision of lender i) that lender i has reached. Similarly, the current player history of borrower j at all nodes where he has to move can be defined with the help of partnership current player history at a particular date. A representative current player history of borrower j after partnership CPH h_t will be ${h^{2s}}_t$, where s is either a kind d node or kind c (continuation node for type 2) node. The continuation decisions of players i and j will be referred as c^1 and c^2 respectively.

We focus our attention on strategies of a player in which, actions taken by the player at any date can be contingent only on the current partner history of the player at that date. In other words, we assume that players ignore their experiences with previous partners. Let $\mathcal{H}^1$ be the set of all current player histories for lender i. The set of all current player histories for lender i where he has to take a decision on the level of trust is denoted by $\mathcal{H}^{1Q}$. The set of all current player histories for lender i where he takes his decision to continue is $\mathcal{H}^{1c}$. So, $\mathcal{H}^1 = \mathcal{H}^{1Q} \cup \mathcal{H}^{1c}$. Similarly, the set of all current player histories of borrower j is $\mathcal{H}^2$ which is the disjoint union of $\mathcal{H}^{2d}$ and $\mathcal{H}^{2c}$ where $\mathcal{H}^{2d}$ is the set of all current player histories of lender j where the lender has to make his decision d and $\mathcal{H}^{2c}$ is the set of all current player histories where lender j has to make his continuation decision c^2.

Definition 2. σ^i is a **current player history contingent** (CPHC) strategy for player i if σ^i is a function which maps $\mathcal{H}^1$ into the set $[0, \bar{Q}] \cup \{C, S\}$ such that (a) $\sigma^i({h_t}^{1Q}) \in [0, \bar{Q}]$ for ${h_t}^{1Q} \in \mathcal{H}^{1Q}$ and (b) $\sigma^i({h_t}^{1c}) \in \{C, S\}$ for ${h_t}^{1c} \in \mathcal{H}^{1c}$. A similar CPHC strategy σ^j can be defined for borrower j.

When a player plays a CPHC strategy, he takes on every 'new' partner with an identical contingency plan. In a CPHC, at all dates when the current player history of the player is the same, the actions taken are also the same.[4] Also note that a player has information only on that part of the game which he has actually played. He cannot see the course of play in partnerships in which he

[4]We elaborate on the notation used. Let h_t be a partnership current player history at date $T+t$ of the form $(Q_T, d_T, Q_{T+1}, d_{T+1}, \cdots, Q_{T+t-1}, d_{T+t-1})$. Let $Q_{T+t}, d_{T+t}, c^i_{T+t}, c^j_{T+t}$ respectively be the trust offered, the decision d, the continuation decision c^i of lender i and the continuation decision c^j of i's current partner j at date $T+t$. The current player history of lender i at date $T+t$ when lender i is to make his trust level decision is $(Q_T, d_T, Q_{T+1}, d_{T+1}, \cdots, Q_{T+t-1}, d_{T+t-1}) = (h_t) \equiv h_t$ (by convention). Then the current player history for borrower j when he is to make his choice d at date $T+t$ is $(Q_T, d_T, Q_{T+1}, d_{T+1}, \cdots, Q_{T+t-1}, d_{T+t-1}, Q_{T+t}) = (h_t, Q_{T+t})$ where Q_{T+t} is the level of trust offered at date $T+t$. Similarly, the current player history of lender i at date $T+t$ when i is to make his continuation decision is $(Q_T, d_T, Q_{T+1}, d_{T+1}, \cdots, Q_{T+t-1}, d_{T+t-1}, Q_{T+t}, d_{T+t}) = (h_t, Q_{T+t}, d_{T+t})$.

himself had no part. In particular, when lender i meets a 'new' borrower j at date $T > 1$, lender i does not know whether borrower j is only seeking a fresh partner after cheating somebody in the previous date, or whether borrower j's old relationship had been broken exogenously by nature.

We will be interested in social norms represented by 'symmetric' strategies which are self-enforcing in the sense that no individual player has an incentive to deviate in his self-interest to some other current partner history contingent strategy. In other words, we shall consider strategy profiles where all members of the same type employ the same strategy. To formalize the notion of 'self-enforcement', we need to consider the outcome of individual deviations. Accordingly, we introduce the notion of 'population strategy profile' with the help of which we define a 'social equilibrium'.

$(\sigma_l^k, \sigma^1, \sigma^2)$ for $k = 1$ or 2, will be called a **population strategy profile** where σ_l^k is the current partner history contingent (CPHC) strategy of member l from Type k of the population, when everybody else in Type k uses the CPHC strategy σ^k and everybody of the other Type m, uses CPHC strategy σ^m. A 'symmetric (population) strategy profile' will usually be written as (σ^1, σ^2) where every lender in the population uses CPHC strategy σ^1 and every borrower in the population uses CPHC strategy σ^2. σ will indicate a typical symmetric strategy profile.

If CPH h_t is the partnership history at some date T, and Q, d, c^1, c^2 are the actions taken at date T, then the partnership CPH $h(T+1, Q, d, c^1, c^2)$ that evolves at date $T+1$ according to population strategy profile $(\sigma_i^1, \sigma^1, \sigma^2)$ is as follows:

$$
\begin{aligned}
\text{when } c^k = C \text{ for } k &= i, j, \\
h(T+1, Q, d, c^1, c^2) &= (h_t, Q, d) \quad \text{with probability } (1-q) \\
&= h_1 \text{ with probability } q \qquad (4.2.1) \\
\text{when } c^k = S \text{ for } k &= i \text{ or } j, \\
h(T+1, Q, d, c^1, c^2) &= h_1 \qquad (4.2.2)
\end{aligned}
$$

Let $\sigma = (\sigma_i^1, \sigma^1, \sigma^2)$ be a population strategy profile and $h^{1s}{}_t$ be CPH for a lender i with the corresponding partnership CPH being h_t. Let $V_i^1(h^{1s}{}_t, \sigma_i^1, \sigma^1, \sigma^2)$ be the expected present value payoff to lender i after his

CPH $h^{1*}{}_t$. We use the following notation to derive an expression for the expected present value payoff of lender i. We will use the following notation throughout the rest of the chapter. Let $\tilde{Q}_t$ denote the level of trust that lender i will offer after his CPH $h^{1Q}{}_t$ if he is using the CPHC σ_i^1; so $\tilde{Q}_t = \sigma^1{}_i(h_t)$. For an arbitrary level of trust Q offered after $h^{1Q}{}_t$, let $d_t{}^*(Q) = \sigma^2(h^{1Q}{}_t, Q)$. Similarly, following $h^{1Q}{}_t$, for an arbitrary trust level Q followed by an arbitrary decision d, define the continuation decisions $c^{*i}{}_t = \sigma^1{}_i(h^{1Q}{}_t, Q, d)$ and $c^{*j}{}_t = \sigma^2(h^{1Q}{}_t, Q, d)$. Let $\tilde{d}_t = \sigma^2(h_t, \tilde{Q}_t)$, $\tilde{c}_t^i = \sigma^1{}_i(h_t, \tilde{Q}_t, \tilde{d}_t)$ and $\tilde{c}_t^j = \sigma^2(h_t, \tilde{Q}_t, \tilde{d}_t)$. In other words, $\tilde{Q}_t$, $\tilde{d}_t$, $\tilde{c}_1$, $\tilde{c}_2$ is the course of actions taken following $h^{1Q}{}_t$ according to σ. The expected present discounted payoffs for lender i at $h^{1Q}{}_t$ according to σ, i.e. $V_i^1(h^{1Q}{}_t, \sigma_i^1, \sigma^1, \sigma^2)$, is as follows:

$$\begin{aligned}
\text{when } \tilde{c}_t^k = C \text{ for } k &= i, j, \\
V_i^1(h^{1Q}{}_t, \sigma_i^1, \sigma^1, \sigma^2) &= g_1(\tilde{Q}_t, \tilde{d}_t) + \delta(1-q)V_i^1((h_t, \tilde{Q}_t, \tilde{d}_t), \sigma_i^1, \sigma^1, \sigma^2) \\
&\quad + \delta q V_i^1(h^{1Q}{}_1, \sigma_i^1, \sigma^1, \sigma^2) \qquad (4.2.3) \\
\text{when } \tilde{c}_t^k = S \text{ for } k &= i \text{ or } j, \\
V_i^1(h^{1Q}{}_t, \sigma_i^1, \sigma^1, \sigma^2) &= g_1(\tilde{Q}_t, \tilde{d}_t) + \delta V_i^1(h^{1Q}{}_1, \sigma_i^1, \sigma^1, \sigma^2) \qquad (4.2.4)
\end{aligned}$$

A similar expression $V_j^2(h^{1Q}{}_t, \sigma_j^2, \sigma^1, \sigma^2)$ for any borrower j's expected present discounted payoff can be found by replacing V_i^1 by V_j^2 and g_1 by g_2 in (4.2.3) and (4.2.4).

Similarly, after the CPH $h^{1c}{}_t = (h^{1Q}{}_t, Q, d)$ of lender i, the present discounted (expected) payoff $V_i^1(h^{1c}{}_t, \sigma_i^1, \sigma^1, \sigma^2)$ for lender i is as follows:

$$\begin{aligned}
\text{when } c^{*k}{}_t(Q, d) = C \text{ for } k &= i, j, \\
V_i^1(h^{1c}{}_t, \sigma_i^1, \sigma^1, \sigma^2) &= g_1(Q, d) + \delta(1-q)V_i^1((h_t, Q, d), \sigma_i^1, \sigma^1, \sigma^2) \\
&\quad + \delta q V_i^1(h^{1Q}{}_1, \sigma_i^1, \sigma^1, \sigma^2) \qquad (4.2.5) \\
\text{when } c^{*k}{}_t = S \text{ for } k &= i \text{ or } j, \\
V_i^1(h^{1c}{}_t, \sigma_i^1, \sigma^1, \sigma^2) &= g_1(Q, d) + \delta V_i^1(h^{1Q}{}_1, \sigma_i^1, \sigma^1, \sigma^2) \qquad (4.2.6)
\end{aligned}$$

For any borrower j, his payoff $V_j^2(h^{1c}{}_t, \sigma_i^1, \sigma^1, \sigma^2)$ after CPH $h^{1c}{}_t$ of lender i can be found by replacing V_i^1 by V_j^2 and g_1 by g_2 in (4.2.5) and (4.2.6). Payoffs at other decision nodes can also be derived as above. In general, if a player m belongs to type l of a population and $h^{l*}{}_t$ for $l = 1, 2$, is a CPH of this player at some decision node s, then his present discounted expected payoff $V_m^l(h^{ls}{}_t, \sigma_m^l, \sigma^1, \sigma^2)$ from the population strategy profile $(\sigma_m^l, \sigma^1, \sigma^2)$ following $h^{ls}{}_t$ can be derived in a manner analogous to the derivation of $V_i^1(h^{1Q}{}_t, \sigma_i^1, \sigma^1, \sigma^2)$. $V_i^1(h^{1Q}{}_1, \sigma_i^1, \sigma^1, \sigma^2)$ and $V_j^2(h^{1Q}{}_1, \sigma_i^1, \sigma^1, \sigma^2)$ will denote the payoffs to lender i and j respectively in the game from σ.

Definition 3. A **symmetric social equilibrium** in CPHC strategies is a symmetric population strategy profile (σ^1, σ^2), such that for any player m of any type k and any CPH $h^{ks}{}_t$ of that player, $V_m^k(h^{ks}{}_t, \sigma^k, \sigma^1, \sigma^2) \geq V_m^k(h^{ks}{}_t, \sigma_m^k, \sigma^1, \sigma^2)$ for all CPHC strategies σ_m^k of player m. The payoff for player m from the social equilibrium is $V_m^k(h^{1Q}{}_1, \sigma^1, \sigma^1, \sigma^2)$.

In the rest of the chapter, whenever we have a symmetric strategy profile we will drop the subscript m from V_m^k since all players of the same type will obtain the same payoffs in a symmetric social equilibrium. Also, for any symmetric strategy profile $\sigma = (\sigma_1, \sigma_2)$, if $h^{ks}{}_t$ is a CPH of player m of type k, his payoff $V^k(h^{ks}{}_t, \sigma^1, \sigma^1, \sigma^2)$ will in future be referred simply as $V^k(h^{ks}{}_t, \sigma)$.

Note that, in spirit, the definition of social equilibrium is similar to the usual definition of a subgame perfect equilibrium. The problem in using the concept of subgame perfect equilibrium here is that the only proper subgame is the whole game (since a player does not observe the course of play in other partneships).

4.3 Simplification of the search for social equilibrium

In this section, we try to develop the concept of a 'Principal outcome Path strategy profile'. A principal outcome path (POP) strategy profile is a symmetric strategy profile which is very simple. With a POP strategy profile, the continuation play after any deviation history is just a restart of the original path from date 1. Moreover, every symmetric strategy profile has a corresponding POP strategy profile which generates the same payoff for every player. Also, any social equilibrium strategy profile induces a POP strategy profile social equilibrium (see Proposition 4.1 below). We want to see if the set of social equilibrium has a maximal element in terms of players' payoffs (actually among those equilibrium in which along the equilibrium path, trust is never dishonored) and if one does exist we seek to characterize its properties. The simplicity of POP strategy profiles help us confine our analysis to the set of all Principal outcome path social equilibrium. The use of Principal Outcome Paths is analogous to the use of 'Optimal Penal Code' in Abreu (1988) where he characterized subgame perfect equilibrium paths with the help of this code.

We now proceed to define a 'principal outcome path' (POP) from a symmetric (CPHC) strategy profile $\sigma = (\sigma^1, \sigma^2)$. A POP helps us track down the outcome path from a social equilibrium $\sigma = (\sigma^1, \sigma^2)$, however complicated, by a relatively simple procedure. However, before defining a POP, we need to define an *outcome path* generated by a symmetric strategy profile. In simple language, an outcome path generated by a symmetric strategy profile is the course of play that will result from a 'newly' matched partnership in which nature never intervenes to terminate.

Definition 4. Given a symmetric strategy profile σ, consider a 'new' partnership (i, j) with CPH as h_1. Suppose nature never breaks the partnership (i, j), i and j play according to σ and the course of play in the partnership is consequently given by $\langle Q_t(\sigma), d_t(\sigma)\rangle_{t=1}^{T}$. If T is infinity, then the partnership lasts forever. Note that the continuation decisions c are not mentioned in the course of play as it is implied that the partners both choose to continue upto T. If T is finite, then the partnership chooses to continue along the course of play until T dates, when at least one of the partners decides to 'stop' the relationship. $\langle Q_t(\sigma), d_t(\sigma)\rangle_{t=1}^{T}$ will be called the **outcome path** generated by σ.

Whenever we mention an 'outcome path', it will be implied that the path has been generated by some symmetric strategy profile σ even if the strategy profile is not explicitly mentioned. Further, any arbitrary sequence (finite or infinite) is an outcome path generated by some symmetric strategy profile.

Definition 5. Let $\sigma = (\sigma^1, \sigma^2)$ be a symmetric strategy profile, and $A =$

$\langle a_t(\sigma)\rangle_{t=1}^{T} = \langle Q_t(\sigma), d_t(\sigma)\rangle_{t=1}^{T}$ be the outcome path generated by σ. If $T = \infty$, then the **Principal Outcome Path** (POP) generated by σ is the 'outcome path' A itself. If $T < \infty$, then the **Principal Outcome Path** (POP) generated by σ is $A^{\infty} = \langle A(n)\rangle_{n=1}^{\infty}$ where each $A(n)$ is a copy of A. Whenever the partnership comes to the end of A, a new relationship between the same partners commences with a fresh copy of A.

It is not difficult to see that a POP is also an infinite length outcome path generated by some symmetric strategy profile. Further any symmetric strategy profile generates a POP. For notational convenience, if σ is a symmetric strategy profile, then $P(\sigma)$ will denote the POP that σ generates.

Definition 6. Let σ be a symmetric strategy profile and $P(\sigma)$ be the POP generated by σ. Then the **principal outcome path strategy profile** (POPSP) induced by σ is as follows. Players always take the decision C at all nodes where they have to make the continuation decision c. Any new partnership starts with partners playing along $P(\sigma)$. If i deviates from $P(\sigma)$ in the level of trust at any point of date, j plays D, following which both players decide to play C and then, in the next date, if nature has not already broken their partnership, they restart $P(\sigma)$. Any deviation by partner j from his stated decision d will lead to continuation decisions C by both partners and then restarting $P(\sigma)$, if they are still matched in the next date. Any deviation from any restarted $P(\sigma)$ is dealt with exactly as above.

Note that a POPSP induced by a symmetric strategy profile σ is itself a symmetric strategy profile. In a POPSP all that any player can do by changing partners (by deviating and playing S) is to restart the POP. The way the POPSP has been defined makes the set of payoffs available at any date, following some history along the path of actual play of σ, from unilaterally deviating from σ in that date (and then changing partners) equal to the set of payoffs available by making the same deviant decision following a corresponding history in the POPSP and restarting the POP. Whenever there is no confusion, if $P(\sigma)$ is a POP, $P(\sigma)$ will also be called the POPSP. The payoffs for any player in the game along σ and $P(\sigma)$ must be same given the definition of POPSP.

This is the next remark.

Remark 4.1. If σ is a CPHC symmetric strategy profile and $P(\sigma)$ is the POPSP it generates, and $h_1(\sigma)$ and $h_1(P(\sigma))$ are partnership CPH for 'new' partners along the two strategy profiles, then the payoff for any player along the equilibrium path from σ following $h_1(\sigma)$ will be the same as the payoff for that player along the equilibrium path from $P(\sigma)$ following $h_1(P(\sigma))$. In other words, for any player m belonging to any type k, $= 1,2$, $V^k(h_1,\sigma) = V^k(h_1, P(\sigma))$.

For two symmetric strategy profiles $\gamma = (\gamma_1, \gamma_2)$ and $\sigma = (\sigma_1, \sigma_2)$, we will say that partnership CPH $h_t(\gamma)$ in γ **corresponds** to partnership CPH $h_{t'}(\sigma)$ in σ, if the following three conditions are true:

(a) For $m = 1,2$, $V^m(h_t(\gamma), \gamma^m, \gamma^1, \gamma^2) = V^m(h_{t'}(\sigma), \sigma^m, \sigma^1, \sigma^2)$.

(b) $\gamma^1(h_t(\gamma)) = \sigma^1(h_{t'}(\sigma))$ and
$\gamma^2(h_t(\gamma), Q(\gamma^1(h_t(\gamma)))) = \sigma^2(h_{t'}(\sigma), Q(\sigma^1(h_{t'}(\sigma))))$.

(c) For any sequence $\{Q, d, c^k\}$ where c^k is the continuation decision for some player of type k, there exists some continuation action $\tilde{c}^k$ which depends on $h_t(\gamma), Q, d, c^k$, such that
$V^k((h_t(\gamma), Q, d, c^k), \gamma^k, \gamma^1, \gamma^2) = V^k((h_{t'}(\sigma), Q, d, \tilde{c}^k), \sigma^k, \sigma^1, \sigma^2)$

When a partnership CPH of one strategy profile 'corresponds' to a partnership CPH of some other strategy profile, the situations, in terms of payoffs, as viewed from the stage games immediately following the two histories are very much alike. Condition (a) says that any player should get the same payoff from the games following the two histories according to the respective strategy profiles. Condition (b) says that following both the partnership histories, lender i will offer identical trust according to the respective strategy profiles. Moreover, if lender i does not deviate in his trust level, then borrower j should also take the same decision d following both the partnership CP histories according to the respective strategy profiles. Condition (c) asserts the following: whatever payoffs are possible by a player following any stage game interaction immediately after $h_t(\gamma)$, is also possible for the same player following $h_{t'}(\sigma)$ by, at most, a suitable modification of only his continuation decision in the stage game.

What we hope to gain from the above definition of correspondence of partnership histories from two different strategy profiles is roughly as follows. Firstly, we will show that any partnership CPH of the POPSP $P(\sigma)$ induced by σ corresponds to some partnership CPH of σ along the outcome path of σ (i.e., no partner has deviated along such history). This is Lemma 4.1. Next, we show in Proposition 4.1 that if σ is a social equilibrium, then $P(\sigma)$ must also be a social equilibrium. To see why, let us for the time being replace γ by $P(\sigma)$ in the definition of 'corresponds' in partnership CP histories. Roughly speaking, if any player can deviate in the stage game following any partnership CPH of $P(\sigma)$, then for the 'corresponding' partnership CPH in σ the same player can make a similar deviation by using his S option and gain. We are exploiting the possibility of endogenous termination here.

Lemma 4.1 *Let σ be a symmetric profile and $P(\sigma) = (\gamma^1, \gamma^2)$ be the symmetric strategy profile POPSP induced by σ. Then every partnership CPH of $P(\sigma)$ corresponds to a partnership CPH of σ along the actual course of play of σ (i.e, from the set of partnership CP histories of σ which can result when no player ever deviates from σ).*

Proof.

Let $A = \langle a_t \rangle_{t=1}^{T}$ (where T can be infinity) be the outcome path generated by σ. Let $h_t(P(\sigma))$ be a typical partnership CPH of $P(\sigma)$. We first show that $h_1(P(\sigma))$ corresponds to $h_1(\sigma)$. Condition (a) holds good from Remark 1. Condition (b) holds from the way the POPSP $P(\sigma)$ has been defined from σ. To check condition (c) for lender i, let $\{Q_1, d_1, c_1^1\}$ be an arbitrary sequence of actions following $h_1(P(\sigma))$. If it is the case that no player has deviated from $P(\sigma)$ while actions Q_1, d_1 and c_1^1 were being played, then, from the way the POPSP $P(\sigma)$ is defined from σ, it must be that continuation decisions following $h_1(\sigma)$ (according to σ) must be identical to that following $h_1(P(\sigma))$ (according to $P(\sigma)$). So, (c) must hold. If some partner has deviated following $h_1(P(\sigma))$ along $P(\sigma)$ while decisions Q_1, d_1 or c_1^1 were being taken, then according to $P(\sigma)$, the POP ought to get restarted from the next date. This is payoff equivalent for lender i to the situation along σ, where following his CPH $(h_1(\sigma), Q_1, d_1)$, lender i plays $\tilde{c}_1^1 = S$. The stage game payoffs are the same (they depend only on Q_1 and d_1) and the same POP is going to be restarted at the next date. So, (c) must hold true. A similar argument can be made to show that (c) holds for borrower j too. In the rest of the proof, this above argument will be considered standard and will not be repeated.

Consider $h_t(P(\sigma))$ when $t > 1$ where $h_t(P(\sigma)) = (h_{t-1}(P(\sigma)), \tilde{Q}, \tilde{d}, C, C)$ for some partnership CPH $h_{t-1}(P(\sigma))$ and actions $\tilde{Q}, \tilde{d}$. If there has been a deviation from $P(\sigma)$ by any partner while playing $\tilde{Q}$ and $\tilde{d}$, a check through the standard argument shows that $h_t(P(\sigma))$ corresponds to $h_1(\sigma)$. If there has been no such deviation, then there must be a maximum positive integer T', where $1 \le T' \le T$ (if T is infinity, then just $1 \le T'$), such that the last T' dates play in $h_t(P(\sigma))$ is identical to $\langle a_t \rangle_{t=1}^{T'}$. If $T' = T$, then standard checking shows that $h_t(P(\sigma))$ corresponds to $h_1(\sigma)$. If $T' < T$, then $h_t(P(\sigma))$ corresponds to the partnership CPH $h_{T'}(\sigma) = \langle a_t \rangle_{t=1}^{T'}$ of σ. ∎

Proposition 4.1 *Consider a social equilibrium $\sigma = (\sigma^1, \sigma^2)$. Then $P(\sigma) = (\gamma^1, \gamma^2)$, the symmetric strategy profile POPSP generated by σ is also a social equilibrium which gives each player the same payoff as σ.*

Proof.

That σ and $P(\sigma)$ generate the same payoff as σ follows from Remark 1. To check that $P(\sigma)$ is a social equilibrium, we use the Principal of Unimprovability in Dynamic Programming. All that we have to show is that no player can gain by a 'single stage' deviation[5] from his prescribed strategy.

We will show first that in $P(\sigma)$, if borrower j can deviate (stage game) after some CPH $h_t^{2s}(P(\sigma))$, and then follow the prescribed $P(\sigma)$ and gain, then borrower j can also gain by stage game deviating even in the symmetric strategy

[5] A single stage deviation by player k (in any symmetric strategy profile) at date T following a CPH history of player k is a one stage deviation by player k, following which all players return to play according to the original strategy profile. In this one stage deviation, player k can deviate in any of his decision nodes at date T which follows the CPH. If he has two decision nodes left at date T, he can deviate, if he chooses to, at both his nodes from his prescribed actions.

profile σ after some CP history. Let $h_t^{2s}(P(\sigma))$ be of the form $(h_t(P(\sigma)), Q_t)$ for some partnership CPH $h_t(P(\sigma))$ following which lender i played Q_t. Let $h_{t'}(\sigma)$ be a partnership CPH in σ along the play of σ, as demonstrated in the proof of Lemma 4.1, such that $h_t(P(\sigma))$ corresponds to $h_{t'}(\sigma)$. Now, borrower j has to take two successive decisions d and c^2. If Q_t itself is a deviation by lender i from $P(\sigma)$, then borrower j cannot stage game deviate and gain, for, (a) he cannot get in this date anything more than what he gets from playing D; and (b) irrespective of his continuation action, the POP is going to get restarted at the next date, because lender i deviated in his trust level. So, let Q_t be not a deviation for i. Now, if borrower j plays d_t and c_t^2, at least one of which is a deviation, he can mimic the payoff at CPH $(h_{t'}(\sigma), Q_t)$ by playing the same d_t followed by S and get more than he would have got by not deviating, which is $V^2((h_{t'}(\sigma), Q_t), \sigma^2, \sigma^1, \sigma^2)$. This contradicts that σ is a social equilibrium.

Now, we check that lender i cannot make a stage game deviation and gain. Suppose, in $P(\sigma)$, lender i unilaterally deviates in the stage game following some CPH $h^{1Q}{}_t(P(\sigma))$ (with partnership CPH as $h_t(P(\sigma)) = h^{1Q}{}_t(P(\sigma))$) and gains. So, the left hand side of (c) is greater than the left hand side of (a) for some Q, d, c^1 where d is played by borrower j according to $P(\sigma)$. Let $h_{t'}(\sigma)$ be the partnership CPH in σ along the course of actual play of σ such that h_t corresponds to $h_{t'}(\sigma)$. If there is no deviation by i in Q, then his deviation must be in playing $c^1 = S$ instead of the required C. Check that at the CPH $(h_{t'}(\sigma), Q)$ in σ, j will still offer d (according to condition (b)). So, if i plays S at $(h_{t'}(\sigma), Q, d)$, there will be a change of partner at the next date which is like restarting the POP. Correspondingly, since right hand side of (c) must be greater than right hand side of (a), i should gain in σ which contradicts that σ is a social equilibrium. On the other hand, if there is a deviation in i's trust level when he plays Q at $h^{1Q}{}_t(P(\sigma))$ in $P(\sigma)$, then according to $P(\sigma)$ d must be D. Without loss of generality, c^1 can be taken as S because the POP is anyway getting restarted (as there already has been a deviation in Q). Let $\tilde{d}$ be the honesty decision of j in σ at the CPH $(h_{t'}(\sigma), Q)$. Now, if S is the continuation decision of i at CPH $(h_{t'}(\sigma), Q, \tilde{d})$, then the stage payoff cannot be less than the stage game payoff from Q, D and the POP is going to be restarted from the next date. So lender i gets more in $V^1((h_{t'}(\sigma), Q, \tilde{d}, S), \sigma^1, \sigma^1, \sigma^2)$ than following σ at $h_{t'}(\sigma)$. This again contradicts that σ is a social equilibrium. ■

The above proposition helps us to restrict our attentions to Principal Outcome Path strategy profiles while we search for a 'payoff maximal' social equilibria. In a POPSP equilibrium, there is no difference between "on" and "off" equilibrium paths. The norm itself must act as its own punishment. We study the existence issue of maximal paths and their time behavior in the next section.

4.4 Analysis of the Model

In the rest of the chapter, by a social equilibrium with some trust, what we mean is an equilibrium, along the POP of which, there is some date t where lender j plays H. Moreover, if there is a t such that $Q_t > 0$ and $d_t = H$ along the POP generated by the equilibrium, then we will say that the equilibrium is characterized by positive trust. In other words, for us to say that the equilibrium has positive trust, borrower j must honor a positive level of trust offered at some date t.

We will focus attention on 'honesty regime equilibria' where the honesty decision will always be H along the equilibrium outcome path. Note that this does not mean that dishonesty cannot occur 'off' the equilibrium path. From now on $\Omega = \langle Q_t \rangle_{t=1}^{\infty}$ will denote a typical POP along which the honesty decision is always H (in the notation we will not mention the continuation decisions which are always C). $V^k(\Omega, t, q, \delta)$ will denote the payoff for a player from type k after partnership CPH h_t arising along actual play of the corresponding POPSP, where the quit rate is q and the discount factor is δ. We will drop q, δ in the notation of payoffs when it creates no confusion. Similarly, Ω will also be dropped in the notation of payoffs when the POP is understood.

The next Proposition states that a social equilibrium always exists. Further, an equilibrium with positive trust (all positive loans that have been offered have been repayed along the equilibrium path) can also exist when players are patient enough and the exogenous quit rate is low. More particularly, we show that the POP with $(Q = 0, H)$ for a fixed number of dates t followed by $(\bar{Q}, H)$ forever, constitutes a POPSP equilibrium for large enough t when players are very patient and the exogenous quit rate is very low.

Proposition 4.2 *An honesty regime social equilibrium always exist. There exists δ^* and q^* such that for all $\delta > \delta^*$ and $p < p^*$, there is a social equilibrium with positive trust.*

Proof.
The POP where trust level 0 is always offered by a lender and borrower always play H induces a POPSP which is an equilibrium for all q and δ. This is easy to see.
We will prove the second part of the proposition.

Let us construct the principal outcome path strategy profile where the principal outcome path is to play $(Q = 0, H)$ for a fixed number of dates $t > 2$ and thereafter play $(\bar{Q}, H)$ forever. Let $V^i(k, q, \delta)$be the payoff of the player from Type i along the principal outcome path strategy profile after partnership CPH h_k. Check that $V^i(t+n, q, \delta) = V^i(t+1, q, \delta) \quad \forall n \geq 1$. This implies $V^1(t+1, q, \delta) = \beta\bar{Q} + \delta q V^1(1, q, \delta) + \delta(1-q)V^1(t+1, q, \delta)$ So,

$$V^1(t+1, q, \delta) = \frac{\beta\bar{Q} + \delta q V^1(1, q, \delta)}{1 - \delta(1-q)} \tag{4.4.7}$$

Again,

$$\begin{aligned} V^1(1, q, \delta) &= 0 + \delta q V^1(1, q, \delta) + \delta(1-q)V^1(2, q, \delta) \\ &= \delta q V^1(1, q, \delta)\{0 + \delta(1-q)V^1(1, q, \delta)\} + \delta^2(1-q)^2 V^1(3, q, \delta) \\ &= \delta q \frac{1 - \delta^t(1-q)^t}{1 - \delta(1-q)} V^1(1, q, \delta) + \delta^t(1-q)^t V^1(t+1, q, \delta) \quad (4.4.8) \end{aligned}$$

A similar expression can be found for $V^2(1, q, \delta)$. Replacing $V^1(t+1, q, \delta)$ in (4.4.8) with the help of (4.4.7),

$$\begin{aligned} V^1(1, q, \delta) &= \delta q \frac{1 - \delta^t(1-q)^t}{1 - \delta(1-q)} V^1(1, q, \delta) + \frac{\delta^t(1-q)^t}{1 - \delta(1-q)}\{\beta\bar{Q} + \delta q V^1(1, q, \delta)\} \\ &= \frac{\delta q}{1 - \delta(1-q)} V^1(1, q, \delta) + \frac{\delta^t(1-q)^t}{1 - \delta(1-q)}\beta\bar{Q} \end{aligned}$$

So,

$$V^1(1, q, \delta) = \frac{\delta^t(1-q)^t}{(1 - \delta q)}\beta\bar{Q} \tag{4.4.9}$$

$$\text{and} \quad V^1(t+1,q,\delta) = \frac{\beta\bar{Q}[(1-\delta q)+\delta q\delta^t(1-q)^t]}{\{1-\delta(1-q)\}\{1-\delta q\}} \tag{4.4.10}$$

Similarly,

$$V^2(1,q,\delta) = \frac{\delta^t(1-q)^t}{(1-\delta q)}\alpha\bar{Q} \tag{4.4.11}$$

$$V^2(t+1,q,\delta) = \frac{\alpha\bar{Q}[(1-\delta q)+\delta q\delta^t(1-q)^t]}{\{1-\delta(1-q)\}\{1-\delta q\}} \tag{4.4.12}$$

We will show that after any partnership CPH, borrower j cannot deviate in a stage game once and gain (a similar argument applies to i). Without loss of generality, we will focus on deviation oppotunities of j when $\bar{Q}$ has been offered after partnership CPH h_t. His deviation payoff is $\bar{Q} + \delta\frac{\delta^t(1-q)^t}{(1-\delta q)}\alpha\bar{Q}$. If j has to gain, then j has to receive higher than (4.4.10). This, reduces to the requirement that $1 > \frac{\alpha[1-\delta^t(1-q)^t(1-\delta)\frac{\delta}{(1-\delta q)}]}{\{1-\delta(1-q)\}}$. Note that the numerator of the right hand type is positive and bounded away from zero as $q \longrightarrow 0$ and $\delta \longrightarrow 1$, while the denominator goes to zero. So, the right hand type must go to infinity. The requirement thus cannot be satisfied for low enough q and high enough δ.

■

The next lemma says that in any social equilibrium, there cannot be a last date in the induced POP in which positive trust is offered and honored.

Lemma 4.2 *Consider any POPSP social equilibrium with principal outcome path Ω along which there is positive trust. Then positive trust must be offered and honored infinitely often along the path.*

Proof.

Let T be the last date with positive trust, $Q_T > 0$ denoting the trust level which was H (honored). Now, it cannot be that after date T, positive trust is ever offered by i along the equilibrium path because borrower j must respond by playing D (otherwise T would not be the last date positive trust is honored); so, lender i will do well to offer zero trust after date T. Then, at date T, borrower j would be better off by dishonoring trust when Q_T has been offered and restarting the POP, since following the original POP from date $T+1$ offers him no positive payoff. ■

We proceed to investigate if there is a best social equilibrium in the honesty regime for any type of the population. Note that as long as we restrict our attention to honesty regime equilibria, the best equilibrium for both types coincide, by virtue of the structure of the payoff functions. We shall be interested in the properties of this maximal (best) equilibrium. It would have been more natural to look at the the best equilibrium for the 'lenders' in the population where we do not restrict our attention to honesty regime equilibria. However, a general analysis of such an equilibrium turns out to be rather complicated and will be addressed in future research.[6]

We need to define a 'maximal outcome path' in an honesty regime. By a *maximal outcome path*, we refer to a POP induced by that honesty regime POPSP social equilibrium which gives the highest payoff (expected) to a player of Type k amongst all honesty regime social equilibrium POPSP. Any such equilibrium which maximizes payoff will be referred to as a maximal POPSP equilibrium. The next theorem tells us that a maximal POPSP social equilibrium exists in the honesty regime. The main idea of the proof is that the set of honesty regime equilibrium POPSP is compact and the payoff functions are continuous.

Theorem 4.1 *A maximal POPSP social equilibrium exists (given any q and δ).*

Proof

Let Ω be a typical honesty regime POPSP social equilibrium for some q and δ such that $\Omega = \langle Q_t \rangle_{t=1}^{\infty}$. Let $V^l(\Omega, t)$ be the expected payoff for player of Type l at date t along the path while following the principal outcome path social equilibrium strategy. Let $\bar{V}^l = \sup_{\Omega \in \mathcal{P}}\{V^l(\Omega, 1)\}$ where $\mathcal{P}$ is the set of all honesty regime POPSP social equilibria. This is the maximum payoff that a type l player can get from a honesty regime POPSP social equilibrium if a maximum exists. From the discussion just before the statement of this theorem,

[6]In particular, it remains as yet, an unresolved question if lenders will strictly do better in the best 'lender' equilibrium in the unrestricted regime as compared to the best 'lender' equilibrium in the honesty regime.

we know that the best equilibrium for lenders is also the best equilibrium for borrowers as long as we restrict our attention to the honesty regime.

Note that the space $B = [0, \bar{Q}] \times \{H, D\}$ is a compact subspace of $\mathbb{R}^2$ (interpret $\{H, D\}$ as $\{0, 1\}$ where 0 stands for H and 1 stands for D). B^∞, the space of all POPSP, is compact in the product topology by Tychonoff's Theorem. Moreover, as the stage game payoffs $g_1(.)$ and $g_2(.)$ are continuous, $V^k(h^{l_*}, .)$ is continuous in B^∞ for any $k, l = 1, 2$ and any player history h^{l_*}. All that is required to show that maximal POPSP exists is that $\mathcal{P}$ is compact. Given that B^∞ is compact, it is enough to show that $\mathcal{P}$ is closed.

Let $\langle \Omega^n \rangle_{n=1}^\infty$ be a sequence of honesty regime POPSP social equilibrium such that $\Omega^n \longrightarrow \hat{\Omega}$ as $n \to \infty$. Let, $\Omega^n = \langle Q_t^n \rangle_{t=1}^\infty$ and $\hat{\Omega} = \langle \hat{Q}_i \rangle_{i=1}^\infty$. It is enough to show that $\hat{\Omega}$ is a POPSP social equilibrium. Let us see that the borrower should not be able to deviate and gain from this POPSP. Let, on the contrary there be a partnership CPH h_t along the actual path of play of $\hat{\Omega}$ (without loss of generality), such that after lender i offered some trust Q^*_t following h_t, borrower j stage game deviated and gained. Without loss of generality, let

$$g_2(Q^*_t, D) + \delta V^2(\hat{\Omega}, 1) > V^2((h_t, Q^*_t), \hat{\Omega}, t)$$

where D denotes the deviation (dishonesty) of player 2 at date t. From the fact that $\Omega^n \longrightarrow \hat{\Omega}$ as $n \to \infty$, and the fact that V^2 is continuous, there will exist, for sufficiently large n, a history $h'_t(\Omega_n)$ such that if j plays D at his decision node $h'_t(\Omega_n, Q^*_t)$, then

$$g_2(Q^*_t, D) + \delta V^2(\Omega^n, 1) > V^2((h'_t, Q^*_t), \Omega^n, t).$$

So, Ω^n cannot be a social equilibrium POPSP for large enough n. This is a contradiction. A similar argument can be given to show that lender i cannot deviate after any of his CPH and gain. This proves that $\hat{\Omega}$ is a social equilibrium. That $\hat{\Omega}$ is of the honesty regime is easy to see. So, $\mathcal{P}$ is a closed set.

A continuous function on a compact set attains it's maximum. So, a maximal POPSP social equilibrium exists. ■

For the rest of the section, we will use the following notation. $\Omega = \langle Q_t \rangle_{t=1}^\infty$ will denote a typical honesty regime POPSP where all levels of trust have been honored along the the principal outcome path. Along Ω,

$$\begin{aligned}
V^2(\Omega,t) &= \alpha Q_t + \delta q V^2(\Omega,1) + \delta(1-q)V^2(\Omega,t+1) \\
&= \alpha Q_t + \delta q V^2(\Omega,1) + \delta(1-q)\alpha Q_{t+1} \\
&\quad + \delta^2 q(1-q)^2 V^2(\Omega,1) + \delta^2(1-q)^2 V^2(\Omega,t+2) \\
&= \sum_{\tau=t}^{\infty} \delta^{\tau-t}(1-q)^{\tau-t}\alpha Q_\tau \\
&\quad + \frac{\delta q}{\{1-\delta(1-q)\}} V^2(\Omega,1)
\end{aligned}$$

Replacing $V^2(\Omega,t)$ in the left hand side of above by $V^2(\Omega,1)$, we have,

$$V^2(\Omega,1) = \frac{\{1-\delta(1-q)\}}{(1-\delta)} \sum_{\tau=1}^{\infty} \delta^{\tau-1}(1-q)^{\tau-1}\alpha Q_\tau. \tag{4.4.13}$$

So,

$$\begin{aligned}
V^2(\Omega,t) &= \sum_{\tau=t}^{\infty} \delta^{\tau-t}(1-q)^{\tau-t}\alpha Q_\tau \\
&+ \frac{\delta q}{(1-\delta)} \sum_{\tau=1}^{\infty} \delta^{\tau-1}(1-q)^{\tau-1}\alpha Q_\tau \\
&= A(\Omega,t) + \frac{\delta q}{(1-\delta)} A(\Omega,1)
\end{aligned} \tag{4.4.14}$$

where $A(\Omega,t) = \sum_{\tau=t}^{\infty} \delta^{\tau-t}(1-q)^{\tau-t}\alpha Q_\tau$.

Let us normalize the payoffs by multiplying both sides of (4.4.14) by $\{1-\delta(1-q)\}$. Let, $V^{*2}(\Omega,t)$ be the normalized right hand side of (4.4.14). So,

$$\begin{aligned}
V^{*2}(\Omega,t) &= \{1-\delta(1-q)\}A(\Omega,t) \\
&+ \frac{(1-\delta(1-q))\delta q}{(1-\delta)} A(\Omega,1)
\end{aligned} \tag{4.4.15}$$

We now turn to an important lemma. In any maximal honesty regime POPSP social equilibrium, the amount of trust offered at any date along the POP cannot be higher than the average level of trust offered from that date. Otherwise, we can replace the social equilibrium with some other POPSP social equilibrium and increase payoffs.

Lemma 4.3 *Let $\Omega = \langle Q_t \rangle_{t=1}^{\infty}$ be the sequence of trust levels offered in a maximal social equilibrium POPSP. Then, $\forall t$,*

$$Q_t \leq \{1 - \delta(1-q)\} \sum_{\tau=t}^{\infty} \delta^{\tau-t}(1-q)^{(\tau-t)} Q_\tau \tag{4.4.16}$$

Proof. Let the right hand side of (4.4.16) be denoted by Q_0. Suppose, (4.4.16) does not hold, i.e., $Q_0 < Q_t$ for some $t \geq 1$.

Consider the POPSP $\hat{\Omega} = \langle \hat{Q}_t \rangle_{t=1}^{\infty}$ where

$$\begin{aligned} \hat{Q}_\tau &= Q_\tau \ \forall\ \tau < t, \\ &= Q_0 \ \forall \tau \geq t \end{aligned}$$

i clearly cannot deviate from $\hat{\Omega}$ and gain. Can j deviate and gain? Clearly for all $\tau < t$, j's deviation opportunities remain identical as in Ω. So, there is no incentive to deviate for $\tau < t$.

At date $\tau = t$, since Ω is a social equilibrium, we have $\{1 - \delta(1 - q)\}Q_t + \delta V^{*2}(\Omega, 1) \leq V^{*2}(\Omega, t)$. Since $\hat{Q}_t = Q_0 < Q_t, V^{*2}(\hat{\Omega}, 1) = V^{*2}(\Omega, 1)$ and $V^{*2}(\hat{\Omega}, t) = V^{*2}(\Omega, t)$, we must have

$$\{1 - \delta(1-q)\}\hat{Q}_t + \delta V^{*2}(\hat{\Omega}, 1) < V^{*2}(\hat{\Omega}, t) \tag{4.4.17}$$

Further, for all $\tau > t$, as $\hat{Q}_\tau = \hat{Q}_t$, $V^{*2}(\hat{\Omega}, \tau) = V^{*2}(\hat{\Omega}, t)$, it must be true that

$$\{1 - \delta(1-q)\}\hat{Q}_\tau + \delta V^{*2}(\hat{\Omega}, 1) < V^{*2}(\hat{\Omega}, \tau) \tag{4.4.18}$$

So $\hat{Q}$ is a social equilibrium POPSP.

Consider a new POPSP $\Omega^* = \langle Q_t^* \rangle_{t=1}^{\infty}$ where

$$\begin{aligned} Q_\tau^* &= \hat{Q}_\tau \ \forall \tau < t, \\ &= \hat{Q}_t + \epsilon \ \forall \tau \geq t. \end{aligned}$$

Now from the strict inequalities in (4.4.17) and (4.4.18), it is easy to see that for low enough $\epsilon > 0$, Ω^* is a social equilibrium POPSP with higher payoff for any player than Ω. Consequently, Ω^* contradicts that Ω is maximal. So it must be that (4.4.16) holds. ■

Note that the right hand side of expression (4.4.16) can be interpreted as the average trust honored (and kept) from date t. The following theorem extends Lemma 4.3 in an obvious way to conclude that average trust must be non-decreasing in the duration of a relationship, in a maximal social equilibrium.

Theorem 4.2 *Along a maximal social equilibrium POPSP, average trust is non-decreasing in the duration of a relationship; i.e. for all t,* $\{1-\delta(1-q)\}\sum_{\tau=t}^{\infty}\delta^{\tau-t}(1-q)^{(\tau-t)}Q_\tau \leq \{1-\delta(1-q)\}\sum_{\tau=t+1}^{\infty}\delta^{\tau-(t+1)}(1-q)^{\tau-(t+1)}Q_\tau$.

Proof.
If the above does not hold for some t, then re-arranging terms it must be the case that for that t,

$$\{1-\delta(1-q)\}Q_t > \{1-\delta(1-q)\}^2\sum_{\tau=t+1}^{\infty}\delta^{\tau-(t+1)}(1-q)^{\tau-(t+1)}Q_\tau.$$

Note that whenever the above is true, $Q_t > 0$ and

$$Q_t > \{1-\delta(1-q)\}\sum_{\tau=t+1}^{\infty}\delta^{\tau-(t+1)}(1-q)^{\tau-(t+1)}Q_\tau$$

However, this implies that $Q_t > \{1-\delta(1-q)\}\sum_{\tau=t}^{\infty}\delta^{\tau-t}(1-q)^{\tau-t}Q_\tau$. This contradicts Lemma 4.3.

■

Let $\Omega = \langle Q_t\rangle_{t=1}^{\infty}$ be a maximal honesty regime social equilibrium POPSP. Denote Q^s to the supremum of trust offered along time along Ω, i.e., $Q^s = \sup_t Q_t$. Note that Lemma 4.3 implies that if Q^s is attained at some finite date T, then $Q_\tau = Q_T$ for all $\tau \geq T$. The next theorem states how high the supremum trust level can be in a maximal honesty regime social equilibrium. If the discount factor is low and the quit rate is high such that $\{1-\delta(1-q)\} \geq \alpha$, then $Q^s = 0$. When discount factor is high and quit rate is low enough such that $\{1-\delta(1-q)\} < \alpha$, then the supremum of offered trust is $\bar{Q}$. So, α, the rate of returns from co-operation acts as the benchmark.

Theorem 4.3 *Let* $\Omega = \langle Q_t\rangle_{t=1}^{\infty}$ *be a maximal honesty regime social equilibrium POPSP.*
(a) If $\{1-\delta(1-q)\} \geq \alpha$*, then* $Q^s = 0$.
(b) If $\{1-\delta(1-q)\} < \alpha$*, then* $Q^s = \bar{Q}$.

Proof
Since Ω is a social equilibrium, a borrower should not be able to deviate and

gain; in other words for all t, we have $\{1-\delta(1-q)\}Q_t+\delta V^{*2}(\Omega,1) \leq V^{*2}(\Omega,t)$. Rearranging terms, this boils down to the requirement that

$$\{1-\delta(1-q)\}[A(\Omega,t)-Q_t] \geq \delta(1-q)\{1-\delta(1-q)\}A(\Omega,1) \qquad (4.4.19)$$

As $Q_t \to Q^s$, note from Lemma 4.3 that the left hand side of expression (4.4.19) goes to $[\alpha - \{1-\delta(1-q)\}]Q^s$ and this should be greater than equal to the right hand side of (4.4.19). Also, note that the right hand side of (4.4.19) is non-negative. If $[\alpha - \{1-\delta(1-q)\}] < 0$, then Q^s must be zero as otherwise the left hand side will be negative. If $[\alpha - \{1-\delta(1-q)\}] = 0$, then Q^s must be zero as otherwise the left hand side of (4.4.19) will be zero whereas the right hand side will be positive. This proves part (a) of the theorem. Now, we prove part (b) of the theorem. Suppose, $Q^s < \bar{Q}$ when $\{1-\delta(1-q)\} < \alpha$. Construct an alternate honesty regime POPSP $\hat{\Omega} = \langle \hat{Q}_t \rangle_{t=1}^{\infty}$ such that $\hat{Q}_t = (1+\epsilon)Q_t$ where $\epsilon > 0$ is chosen small enough such that $\hat{Q}^s < \bar{Q}$. Note that $\{1-\delta(1-q)\}[A(\hat{\Omega},t)-\hat{Q}_t] = (1+\epsilon)\{1-\delta(1-q)\}[A(\Omega,t)-Q_t]$. Further $\delta(1-q)\{1-\delta(1-q)\}A(\hat{\Omega},1) = (1+\epsilon)\delta(1-q)\{1-\delta(1-q)\}A(\Omega,1)$. So, if (4.4.19) holds, then $\{1-\delta(1-q)\}[A(\hat{\Omega},t)-Q_t] \geq \delta(1-q)\{1-\delta(1-q)\}A(\hat{\Omega},1)$ So, borrower j cannot deviate and gain from $\hat{\Omega}$. A similar check for lender i shows that lender i cannot deviate and gain from $\hat{\Omega}$. So, $\hat{\Omega}$ is a social equilibrium. Note that $\hat{\Omega}$ gives a higher payoff to each player. This contradicts that Ω is maximal. ∎

The next theorem tells us that the average trust must be strictly increasing between certain dates when the maximal social equilibrium gives a positive payoff to either of the types. The intuition was provided in the introduction with the Prisoners' Dilemma. If there is no initial trust building, then borrower j can gainfully cheat lender i, change partners thereafter and gain. The proof of Theorem 4.4 formalizes this.

Theorem 4.4 *If $\{1-\delta(1-q)\} < \alpha$, then $Q^s = \bar{Q}$, then average trust must be strictly increasing between some dates.*

Proof.
From Theorem 4.2, we know that average trust must be non-decreasing along time along a maximal POP. From Theorem 4.3, we know that when $\{1-\delta(1-$

$q)\} < \alpha$, we have $Q^* = \bar{Q}$. If the average trust were constant along time, then using Lemma 4.3 it must be the case that $Q_t = \bar{Q}$ for all t. So, for all t, $\{1 - \delta(1 - q)\}A(\Omega, t) = \{1 - \delta(1 - q)\}A(\Omega, 1) = \alpha\bar{Q}$. We know that (4.4.19) has to be satisfied for Ω to be a social equilibrium. However, using the fact that average trust is constant at $\alpha\bar{Q}$, this would imply that $\alpha\bar{Q} < \bar{Q}$ which is a contradiction since $\alpha < 1$ and $\bar{Q} > 0$. So, average trust must be strictly increasing between some dates.

■

4.5 Conclusion

Although our model is asymmetric in the sense that the actions available to the two types are different, the same qualitative results hold for symmetric version, e.g. when the two types play a symmetric Prisoners' Dilemma.

We conjecture that a maximal POPSP social equilibrium with positive payoff is evolutionarily stable in the following sense: if the population is invaded by small fractions of mutants with different strategies of both types, then these mutants will be driven out of the population. The reason is as follows: the mutants, being small in number, have a high probability, of meeting a partner from the original population. Since the mutants' strategy is different from the population's, there will be problem of co-ordinating their strategies. Then, either this partnership will not reach any reasonable level of cooperation, or, there will be a certain date at which the partnership with the current partner will break down; in either case the mutant will have a smaller expected payoff, and so will become extinct. This is another reason why the maximal POPSP equilibrium might be an appropriate social norm.

Another question which might arise is that, given the slow trust building in a maximal POPSP equilibrium, whether partners can profitably renegotiate bilaterally, and cooperate from the very beginning. However, if every member of the population thinks the same way, then such renegotiation may not be credible. A borrowers can argue in the following way: if his current partner is willing to renegotiate and immediately cooperate, his future partners should also be willing to renegotiate similarly. Then he can cheat his current partner after renegotiating, change partners, again renegotiate and cheat the new partner and so on. However, all lenders should be able to foresee this possibility and refrain from renegotiating.

Chapter 5

Management Union Bargaining under Minimum Wage Regulation

5.1 Introduction

This chapter represents joint work with Prabal Raychaudhuri. In this chapter we address some policy questions relating to minimum wage regulations. In many countries the government resorts to minimum wage laws in a bid to raise the living standards of the workers. In India, for example, the Minimum Wages Act of 1948 lays down standards of minimum wage. The objective was "not merely...the bare sustenance of life but...for some measure of education, medical requirements and amenities"[1]. In other countries also, such laws and regulations were motivated by similar concerns.

Unfortunately, however, there is little concern about the possible detrimental effects of such laws on the level of employment, as these laws may induce the firms to cut down on the number of workers employed. The usual argument against such an objection is that, in the presence of unionized workers, such an eventuality cannot come about. It is contended that the unions would prevent the employment level from declining at all, or they would minimize the extent of any such decline.

The objective of this chapter is to examine, in a formal bargaining setup and in the presence of unionized workers, the impact of changes in minimum wage laws.

We model the bargaining problem as an infinite horizon alternating offers game. Since we are concerned about the levels of both employment and the wage, we allow the management and the union to bargain over both simultaneously. Therefore, *in contrast to the standard* bargaining models (e.g. Rubinstein (1982)), the cake size in this model is endogenously determined. The game starts with the management making an offer and the union accepting or rejecting it immediately. If the union rejects the offer, then it can make a counter-offer after a delay of one period. The management in its turn can now either accept the offer, or reject it. The game continues in this manner until an agreement is reached, when the game ends and the agreed upon offer is implemented. We assume that there exists a minimum wage level, fixed by the government, below which a worker cannot be employed by the firm. Further,

[1] Committee on Fair Wages appointed by the Government of India, 1948.

we focus on the case where there is widespread unemployment in the economy. This would imply that if a worker does not find employment in a firm, then he could be unemployed. In other words, the actual alternative wage of a worker could be much lower than the minimum wage[2]. In particular, we would concentrate on the case where the alternative wage of a worker, whenever he does not find employment in a firm with minimum wage regulation, is zero. The conclusions we draw from zero alternative wage remain valid for low, positive alternative wages.

We first establish that a subgame perfect Nash equilibrium exists and that it is unique. The structure of the equilibrium outcome is rather interesting. We show that the outcome always involves a wage level equal to the minimum wage set by the government. The level of employment, however, is higher compared to what it would have been in the absence of the union (i.e., the competitive level). This suggests that though bargaining takes place over both the wage and the employment level, in effect the union can affect only one of these. It is in the interests of the workers not to demand an increase in wage level over and above that set by the government.

We then look into some comparative statics results of our model. Those involving the discount factors of the management and the union (δ_1 and δ_2 respectively) are what we would expect intuitively, in the sense that an increase in δ_i leads to an increase in the payoff of the concerned party and a decrease in the payoff of the other party. If, however, the management's bargaining position is strong enough, then a small increase in δ_2 may leave the payoffs unaffected.

We now look into the effects of an increase in the minimum wage level. We show that the employment level is going to decline. We also demonstrate that the income of the management declines. The surprising part of the result is that the income of the union may decline as well. We show that a sufficient

[2]Another reason why the alternative wage of a worker could be lower than the minimum wage applicable is because it could be difficult for a government to monitor minimum wage regulation in all the sectors of the economy. In countries like India, there is a huge unorganized sector where minimum wage regulation, by and large, do not prevail despite the country's regulation.

condition for this to occur is that the marginal product of labour be sufficiently inelastic. The elasticity condition is identical to that for a profit maximizing competitive firm in the absence of any bargaining. Under both competitive and bargaining outcome, an increase in minimum wage leads to a decline in the level of employment. However, since the level of employment is higher under the bargaining outcome, the decline in employment would be greater in this case. This follows since the concavity of the production function implies that under a bargaining setup, a decline in employment is less costly in terms of output. Therefore the sufficient condition under the competitive outcome is sufficient under bargaining as well. Thus, somewhat paradoxically, it is the very success of the union in increasing the level of employment which ensures that the workers lose out as a result of an increase in minimum wage.

This demonstrates that our apprehension regarding the possible harmful effects of minimum wage regulations are well founded. Not only does the management lose out, so may the workers. Furthermore, since the level of output declines, this implies that the consumers are adversely affected as well.

There has been quite a few theoretical papers trying to deal with problems of management-union interaction e.g. strikes[3], unemployment[4], formation of unions[5] etc. However, most of these deal with bargaining over wages alone, so that the cake size is exogenous. Macdonald and Solow (1981) does deal with the case where bargaining takes place over both wages and employment, though in a cooperative framework. Moreover, the exisiting theoretical literature does not appear to deal with the problem of minimum wage legislations. To the best of our knowledge, ours is the only work that deals with the problem in a non-cooperative framework, where bargaining takes place over both employment and wages.

The rest of the chapter is organized as follows. In section 5.2, we set up the model. We then go on to discuss existence and uniqueness of equilibrium, as well as the various comparative statics results. Section 5.3 concludes.

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[3]See Admati and Perry (1987), Fernandez and Glazer (1991), Haller (1988), Holden(1988) and Rubinstein (1985).

[4]See Shaked and Sutton (1984).

[5]See Horn and Wolinsky (1988), Jun (1989) and Ben-Porath (1989).

5.2 The Model

We begin by introducing some notations and definitions. Let $X(N)$ be the production function of the employer, where N denotes the level of employment and X the level of output. We normalize the price of output to one.

The minimum wage level fixed by the government is denoted by $\underline{w}$, where $\underline{w}$ is strictly positive. It is not an alternative wage that the workers can obtain elsewhere. In fact we assume that the alternative wage available to the workers is zero.[6] In the context of developing countries, the presence of large scale unemployment implies that such an assumption is quite realistic, and is more than just a simplifying device. The discount factors of the management and the union are denoted by δ_1 and δ_2 respectively, where $0 < \delta_1, \delta_2 < 1$.

We make the following assumptions on $X(N)$, the production function.

Assumption 1. The production function $X(N)$ is twice differentiable, strictly increasing and strictly concave in the level of employment N.

Assumption 2. $X(0) = 0$, $X'(0) = \infty$, $X'(\infty) = 0$.

It is easy to check that assumption 2 implies that the production function $X(N)$ intersects $\underline{w}N$ at some positive employment level, $\underline{N}$. The assumption that $X'(0) = \infty$ is, however, not crucial for our analysis. It merely ensures that $X(N)$ and wN intersect for all possible positive wage levels, however high.

We assume that both the players are risk neutral and that the workers have zero disutility of effort. This allows us to represent the utility functions of the two parties by the present discounted value of the payoffs accruing to them.

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[6]Towards the end of this section we briefly discuss the case where the alternative wage can be positive.

The utility function[7] of the union is assumed to be utilitarian, i.e. a sum of the payoffs of the employed workers. Since the alternative wage is zero, we need concern ourselves only with the wage bill of the employed workers. The management is a residual claimant.

The structure of the bargaining game is as follows. The management and the union bargain over the wage and employment level at which production will take place. At time $t = 0$, the management offers a wage-employment vector (w, N) to the union. Immediately, the union decides whether to accept the offer, or reject it. If accepted, the offer is implemented and production takes place at the offered wage level w and employment level N. So the union obtains wN and the management obtains $X(N) - wN$. The game ends here and we assume that production takes place only once at the agrred vector.[8] If the union rejects the offer, then the game moves to the next time period. At time $t = 1$, the union offers a wage-employment vector (w, N) to the management, which again can be either accepted or rejected. If accepted, the offer is implemented and the present discounted value of the management's payoff is $\delta_1(X(N) - wN)$ and that of the union is $\delta_2 wN$. If rejected, the game moves to time $t = 2$, where it is again the management's turn to make an offer.

Offers continue to alternate in the above manner so that at every even time-period it is the management's turn to make an offer and at every odd time-period it is the union's turn to make an offer. If, in the course of the game, neither of the parties ever accepts an offer, both the parties obtain a payoff of zero.

---

[7]Let L be the number of labourers who form the union. We assume that $L > \underline{N}$,so that some labourers will never get employed by the firm. If a labourer is not employed by the firm, then the outside (alternative) wage rate he can get is w_0, where $w_0 < \underline{w}$. The objective of the union is to maximize $wN + (L - N)w_0$ where at a wage $w \geq \underline{w}$, N people are employed by the firm, and the remaining labour force are employed outside the firm at the wage rate w_0. Since we are in the case where $w_0 = 0$,. the objective of the union is to maximize wN.

[8]An alternative formulation could be to assume that once an offer is agreed upon, production takes place at every period, though only at the agreed upon levels of employment and wage. However, the two formulations are essentially similar, and our formulation possesses the virtue of simplicity.

Clearly, the payoff of the employer can be written as follows:

$$u_1 = \begin{cases} \delta_1^{T-1}[X(N) - wN] & \text{if bargaining ends in finite time T,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.1)$$

And the union's payoff can be expressed as:

$$u_2 = \begin{cases} \delta_2^{T-1} wN & \text{if bargaining ends in finite time T,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.2)$$

The equilibrium concept used is that of the subgame perfect Nash equilibrium. Since the formal definition of subgame perfect Nash equilibrium in a bargaining context is quite standard (see e.g. Rubinstein (1982)); we omit it. In what follows in the chapter, instead of writing subgame perfect equilibrium all the time, sometimes we will simply write 'equilibrium' instead.

Before we begin our formal analysis let us introduce one more piece of notation. Define $N(\underline{w})$ as follows:

$$\underline{w} = X'(N(\underline{w})). \quad (5.2.3)$$

Clearly, $N(\underline{w})$ is the profit maximizing level of employment for the firm in the competitive case i.e. in the absence of any bargaining.

We begin by introducing two dynamic reaction[9] functions, $d_1(w_1, N_1)$ and $d_2(w_2, N_2)$. These define the optimum offers of the union and the employer respectively, when they look only one period ahead. The interesting fact is that the (unique) intersection of these two functions defines the (unique) subgame perfect equilibrium of the infinite horizon game.

The function $d_1(w_1, N_1)$ can be motivated as follows. Suppose it is the union's turn to make an offer, and it is known that in case of disagreement, the employer and the union would agree to implement the wage-employment vector (w_1, N_1) in the next period. Then $d_1(w_1, N_1)$ defines the wage-employment bundle which maximizes the union's payoff subject to the constraint that the employer obtains at least $X(N_1) - w_1N_1$ discounted to the present period.

[9]This is exactly as in Rubinstein (1982).

Thus $d_1(w_1, N_1) = (w_2, N_2)$ where (w_2, N_2)

$$\begin{aligned} \text{Maximizes} \quad & wN \\ \text{such that } w \ & \geq \ \underline{w}, \\ \text{and } X(N) - wN \ & \geq \ \delta_1[X(N_1) - w_1 N_1] \geq 0, \end{aligned} \tag{5.2.4}$$

provided the feasible set is non-empty. Otherwise, $(w_2, N_2) = (\underline{w}, 0)$.

The maximum occurs at (w^*, N^*) where $w^* = \underline{w}$ and N^* is the maximum N such that $X(N) - wN = \delta_1[X(N_1) - w_1 N_1]$. (See Figure 5.1)

The formal proof follows from Lemma 5.1, the statement and proof of which has been relegated to the Appendix. The argument is as follows: let $\tilde{N}(w)$ be the maximum N such that $X(N) - \underline{w}N = \delta_1[X(N_1) - w_1 N_1]$. If the union were constrained to offer the wage level w^*, the union will offer the minimum feasible payoff $\delta_1[X(N_1) - w_1 N_1]$ to the management. Since the union is, in effect, the residual claimant to the surplus after paying of the management, it moves to the highest possible level of employment (and hence output) which by definition is $\tilde{N}(w^*)$. Moreover, if $w^* > \underline{w}$, then the union can reduce the wage level, w, and increase the level of employment, N, thereby increasing his payoff, as the amount to be paid out to the management remains the same.

The function $d_2(w_2, N_2)$ defines the optimum response of the employer to the symmetric problem.

Therefore, we can similarly define $d_2(w_2, N_2) = (w_1, N_1)$ where (w_1, N_1)

$$\begin{aligned} \text{Maximizes} \quad & X(N) - wN \\ \text{such that } w \ & \geq \ \underline{w}, \\ \text{and } wN \ & \geq \ \delta_2 w_2 N_2, \end{aligned} \tag{5.2.5}$$

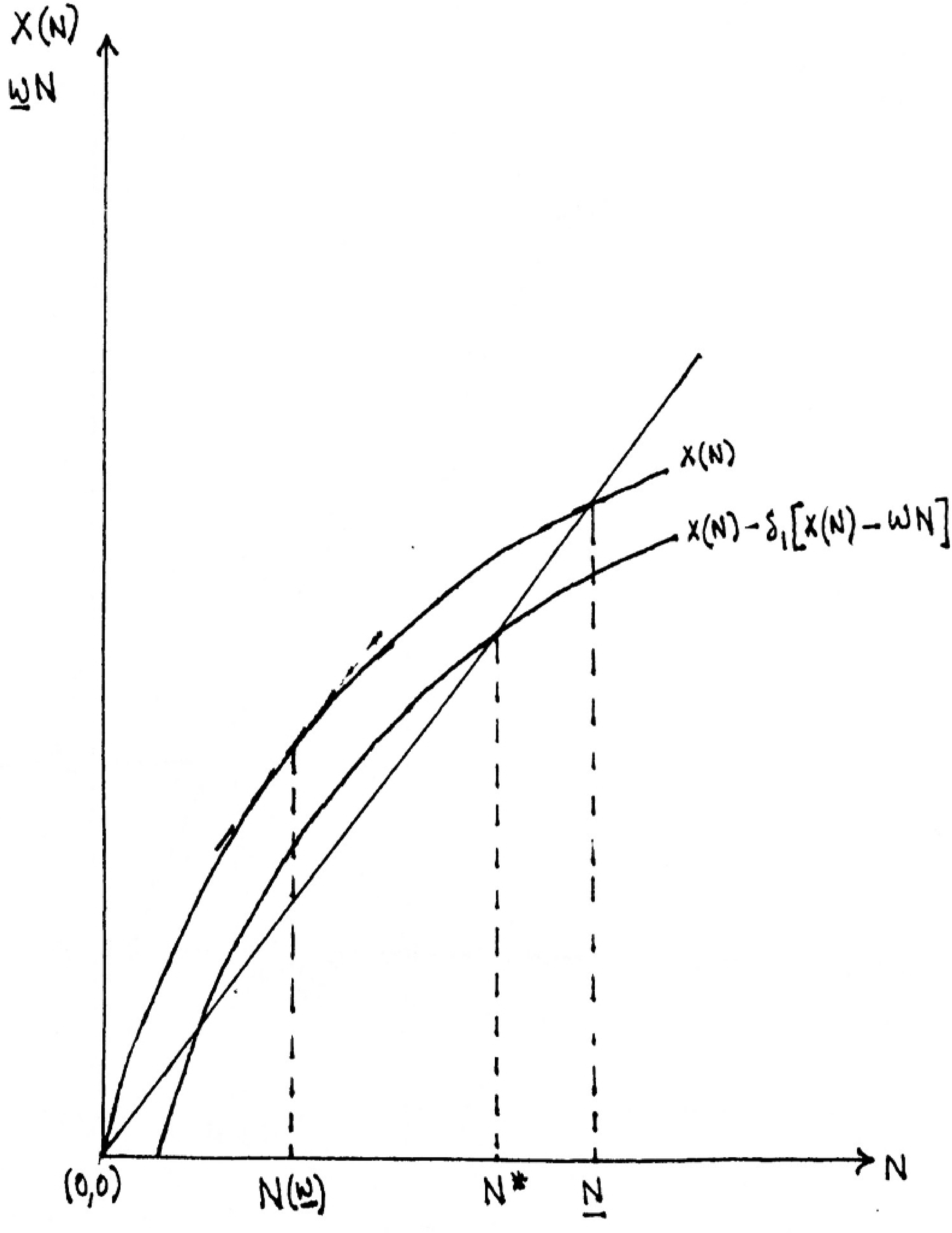
provided the feasible set is non-empty. Otherwise, $(w_1, N_1) = (\underline{w}, 0)$.

The solution in this case looks as follows (see figure 5.2):

Case 1. If $\delta_2 w_2 N_2 \leq \underline{w}N(\underline{w})$, then $w^* = \underline{w}$ and $N^* = N(\underline{w})$.

Case 2. If $\delta_2 w_2 N_2 > \underline{w}N(\underline{w})$, then $w^* = \underline{w}$ and N^* satisfies $\underline{w}N^* = \delta_2 w_2 N_2$. (See Figure 5.2)

The formal proof follows from Lemma 5.2 which has been stated and proved in the Appendix.



X(N)
$\underline{w}N$
X(N)
$X(N) - \delta_1[X(N) - wN]$
(0,0)
$N(\underline{w})$
N*
$\underline{N}$
N

FIGURE 5.1

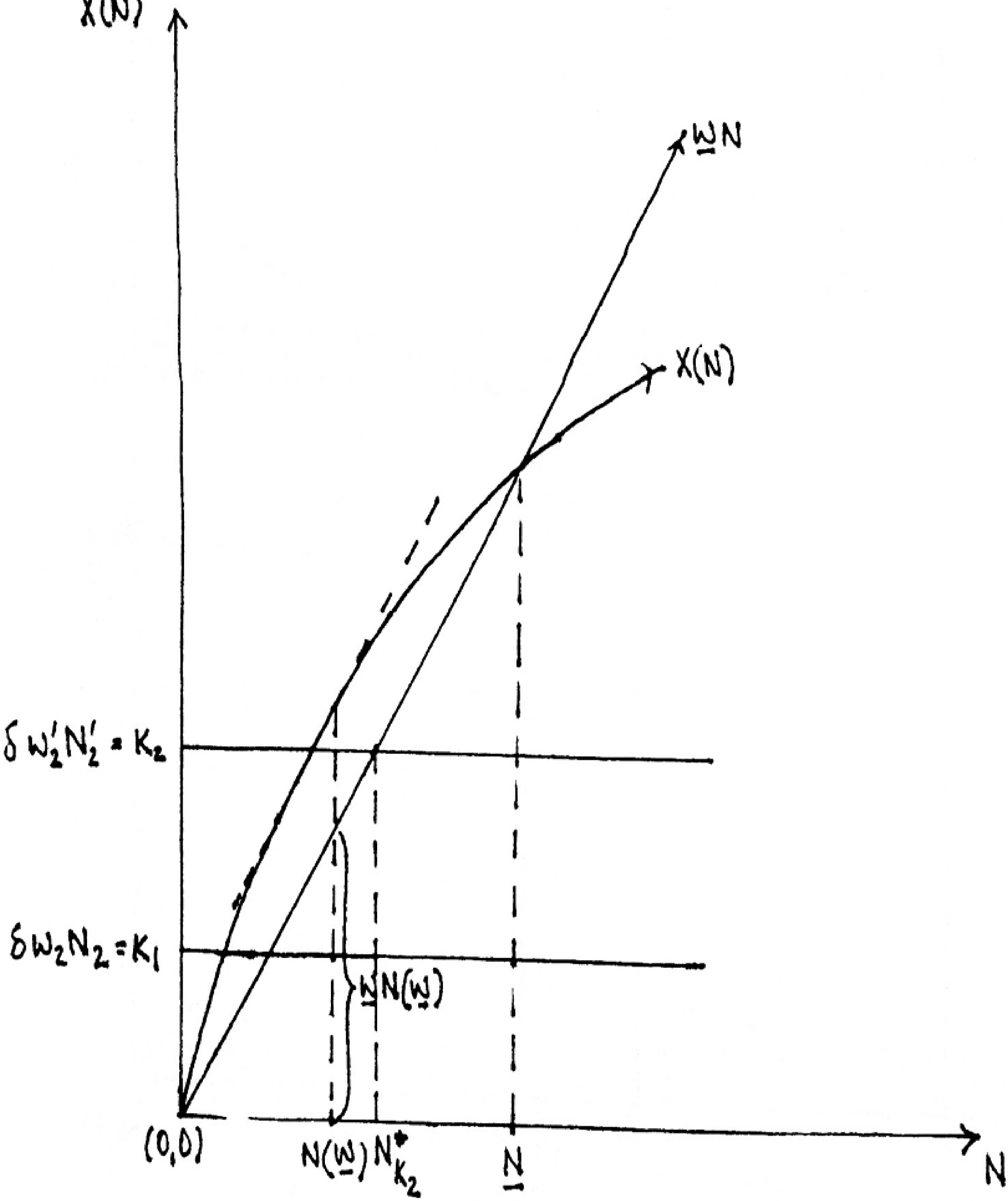


FIGURE 5.2

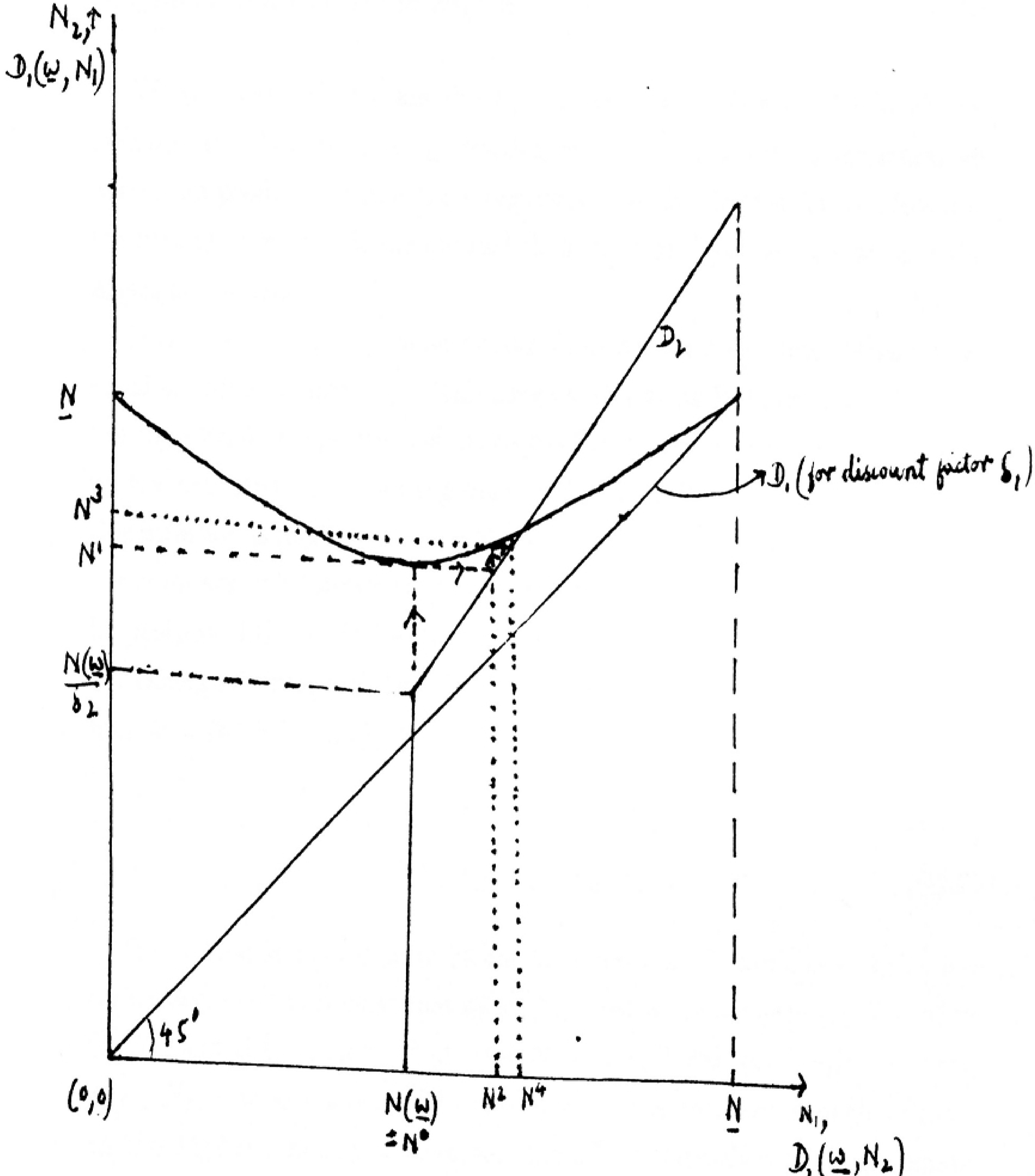


FIGURE 5·3: INTERSECTION OF D_1 & D_2

We consider the two cases briefly. In case 1, since $\delta_2 w_2 N_2 \leq \underline{w}N(\underline{w})$, the management chooses $(\underline{w}, N(\underline{w}))$ which gives the management it's unconstrained maximum profits. For case 2, the argument is similar to that for d_1. Here it is the management who is the residual claimant, and thus tries to maximize the aggregate surplus.

Note that both the dynamic reaction functions involve setting the wage level equal to the minimum wage. This suggests (following Rubinstein (1982)) that the equilibrium wage level will also equal the minimum wage level as both the parties are interested in setting wage at that level. Proposition 5.1 shows that this intuition is indeed correct. This fact will have important consequences for our comparative statics results as well. The discussion of that, however, must be postponed till we derive Proposition 5.2.

Taking advantage of the above mentioned property we now define two new functions $D_1(N_1)$ and $D_2(N_2)$ as follows:

$$D_1(N_1) = d_1(\underline{w}, N_1), \tag{5.2.6}$$

$$D_2(N_2) = d_2(\underline{w}, N_2). \tag{5.2.7}$$

The rationale behind introducing these functions is as follows. From our earlier argument it is clear that $d_i(w_i, N_i)$ must be of the form $(\underline{w}, N_j)$. Thus $D_1(N_1)$ and $D_2(N_2)$ intersect at some N_1 and N_2 if and only if $d_1(w_1, N_1)$ and $d_2(w_2, N_2)$ intersect at $(\underline{w}, N_1)$ and $(\underline{w}, N_2)$. It is therefore enough to focus on the $D_i(N_i)$ functions. Moreover, these lend themselves to diagrammatic representations that are useful for our analysis.

The shape of these two functions are demonstrated in figure 5.3. (The shapes follow from observations 5.1 to 5.6 given in the Appendix.) From the diagram it is easy to see that D_1 and D_2 has a unique intersection, which is defined by the following two equations, $D_2D_1(N_1) = N_1$ and $D_1(N_1) = N_2$. From our earlier argument this implies that $d_2(d_1(\underline{w}, N_1)) = N_1$ and $d_1(\underline{w}, N_1) = N_2$. Consider the solutions to the following functional equations:

$$X(N_2) - \underline{w}N_2 = \delta_1[X(N_1) - \underline{w}N_1], \tag{5.2.8}$$

$$N_1 = \delta_2 N_2, \tag{5.2.9}$$

$$\text{s.t. } N_2 \geq N(\underline{w}). \tag{5.2.10}$$

Let $\tilde{N}_1(\underline{w})$ and $\tilde{N}_2(\underline{w})$ solve the two functional equations. It is clear that the intersection of D_1 and D_2 coincides with $\tilde{N}_1(\underline{w})$ and $\tilde{N}_2(\underline{w})$ whenever $\tilde{N}_1(\underline{w}) \geq N(\underline{w})$. Otherwise, the intersection of D_1 and D_2 occurs at $N_1 = N(\underline{w})$ and $N_2 = D_2(N(\underline{w}))$.

We now prove that this intersection of D_1 and D_2 in fact defines the unique subgame perfect equilibrium of this game. Our proof is on the lines of the proof in Binmore (1986). This proves that any equilibrium outcome involves setting the wage level equal to the minimum wage level. This is somewhat surprising as in case of departures from competitive conditions, we usually find that none of the first best conditions hold. Here the result follows as both the parties have an interest in setting the wage level equal to $\underline{w}$.

Proposition 5.1 *If N_1 and N_2 denotes the intersection of D_1 and D_2, then the unique subgame perfect outcome involves the management making an offer of $(\underline{w}, N_1)$ at time 0 and the union accepting the offer.*

The formal proof of this proposition is in the Appendix. Here, we provide an intuitive sketch of the argument. To begin with note[10] that $(\underline{w}, N_1)$ can indeed be supported as a subgame perfect equilibrium with the help of the following strategies for the two parties.

The employer's strategy is to offer $(\underline{w}, N_1)$ whenever it is his turn to make an offer and to accept an offer if and only if the offered wage-employment vector yields him at least as much as $\delta_1[X(N_1) - \underline{w}N_1]$.

The union's strategy is to offer $(\underline{w}, N_2)$ whenever it is his turn to make an offer and to accept an offer if and only if the payoff accruing to him is greater than or equal to $\delta_2\underline{w}N_2$.

The argument of the proof of uniqueness (formal proof in Appendix) is as follows. First we show that at any subgame perfect equilibrium the management receives a payoff equal to $X(N_1) - \underline{w}N_1$ and the union receives a payoff equal to $\underline{w}N_1$. Consider the maximum payoff that the management can obtain in any subgame perfect equilibrium. This is nothing but the competitive level of profits. Denote this by A. Clearly at any subgame where the union is the

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[10] There are certain technicalities which will be resolved in the formal proof in the Appendix.

proposer, the union will offer at most $\delta_1 A$ to the management. The union in the process obtains at least B (say). Next consider any subgame where the management is the proposer. Obviously, the management must offer at least $\delta_2 B$ (if it has any reasonable hope of acceptance) to the union and itself obtain at most say A_1 in the process. Arguing in this manner we can show that in the limit the management receives a payoff of at most $X(N_1) - \underline{w}N_1$ and the union receives at least $\underline{w}N_1$. It is easy to visualize what is going on in terms of figure 5.3. Note that from our earlier discussion the payoffs $A, B, \cdots$ correspond to offers of the form $(\underline{w}, N)$. Thus corresponding to $A, B, A_1, \cdots$ we obtain a sequence of the form $N^0, N^1, N^2, \cdots$. From figure 5.3 it is easy to see that this converges to the intersection of D_1 and D_2. Note that at each step we use the fact that the subgame perfect strategies of a game remain subgame perfect for any subgame. We can employ a symmetric argument to show that the management's payoff is at least $X(N_1) - \underline{w}N_1$ and the union's payoff is at most $\underline{w}N_1$. This proves our contention that the management receives a payoff equal to $X(N_1) - \underline{w}N_1$ and the union receives a payoff equal to $\underline{w}N_1$. We then use this to show that there can be no delay in equilibrium and that the vector at which equilibrium occurs is precisely $(\underline{w}, N_1)$.

We next look at some comparative statics results of our model. The comparative statics results relating to the discount factors can be derived from figure 5.3. As the discount factor of the union increases, the kink in D_2 shifts downwards and the slope of the positively sloped section also decreases. Both these effects causes the equilibrium level of employment to increase (except when the intersection occurs in the vertical part of D_2 and continue to do so even afterwards).

Whereas, for an increase in the discount factor of the employer, D_1 shifts downwards, though always remaining above the 45^0 line. This leads to a decrease in the equilibrium level of employment when the union is the first mover. If the employer is the first mover, then the equilibrium level of employment declines only if the intersection of D_1 and D_2 is in the positively sloped section of D_2. Since all equilibria involve $w = \underline{w}$, an increase in equilibrium employment implies a lower payoff for the management and a higher payoff for the union, though output itself is going to increase.

Next, we consider the effect of an increase in the minimum stipulated wage. Clearly, as the minimum wage level increases we can restrict attention to one of the three situations:

(a) The equilibrium is originally on the positively sloped section of D_2 and remains so as $\underline{w}$ increases.

(b) The equilibrium is originally on the vertical section of D_2, however as $\underline{w}$ increases we move to the positively sloped section of D_2.

(c) The equilibrium is originally on the vertically sloped section of the D_2 and remains so as $\underline{w}$ increases.

It is easy to check that the equilibrium employment and the employers payoff decreases as the minimum stipulated wage increases. The proof involves straightforward differentiation of the solution to the two functional equations (5.8) and (5.9) subject to (5.10), whenever $\tilde{N}_1(\underline{w}) \geq N(\underline{w})$ (otherwise use $N_1 = N(\underline{w})$ and differentiate) and has been omitted. We then come to the all important question of the impact of a change in the minimum wage level on the income of the workers. Proposition 5.2 is the heart of this chapter and provides a sufficient condition for the income of the union to be decreasing in the minimum wage $\underline{w}$.

Proposition 5.2 *If the marginal product of labour is inelastic, then the equilibrium payoff of the union is non-increasing in* $\underline{w}$.

Proof.
Clearly, the marginal product of labour is $X'(N)$. Let e be the elasticity of marginal product of labour. So $e = -\frac{N}{X'(N)}\frac{dX'(N)}{dN} = -\frac{NX''(N)}{X'(N)}$. Clearly, $e \leq 1$ if and only if $NX'(N)$ is non-decreasing in N, i.e. $NX''(N) + X'(N) \geq 0$.

Case 1. Let $\underline{w}$ be such that $\tilde{N}_1(\underline{w}) \geq N(\underline{w})$. So the union's payoff is $\underline{w}\tilde{N}_1(\underline{w})$ and

$$\begin{aligned}\frac{d}{d\underline{w}}\underline{w}\tilde{N}_1(\underline{w}) &= \underline{w}\frac{d\tilde{N}_1(\underline{w})}{d\underline{w}} + \tilde{N}_1(\underline{w}) \\ &= \tilde{N}_1(\underline{w})[\frac{\underline{w}(1-\delta_1\delta_2)}{X'(\frac{\tilde{N}_1(\underline{w})}{\delta_2}) - \underline{w} - \delta_1\delta_2(X'(\tilde{N}_1(\underline{w})) - \underline{w})} + 1].\end{aligned}$$

So $\frac{d}{d\underline{w}}\underline{w}\tilde{N}_1(\underline{w}) \leq 0$ if

$$\frac{\underline{w}(1-\delta_1\delta_2)}{X'(\frac{\tilde{N}_1(\underline{w})}{\delta_2}) - \underline{w} - \delta_1\delta_2(X'(\tilde{N}_1(\underline{w})) - \underline{w})} \leq -1,$$

i.e. if $X'(\frac{\tilde{N}_1(\underline{w})}{\delta_2}) \geq \delta_1\delta_2 X'(\tilde{N}_1(\underline{w}))$.

It is easy to check that the above condition is satisfied whenever $NX'(N)$ is non-decreasing.

Case 2. Let $\underline{w}$ be such that $\tilde{N}_1(\underline{w}) < N(\underline{w})$. So equilibrium is attained at employment level $N(\underline{w})$. In this region the union's equilibrium payoff is $\underline{w}N(\underline{w})$. Now $\frac{d}{d\underline{w}}\underline{w}N(\underline{w}) = \underline{w}\frac{d}{d\underline{w}}N(\underline{w}) + N(\underline{w}) = \frac{\underline{w}}{X''(N(\underline{w}))} + N(\underline{w})$.

If $NX'(N)$ is non-decreasing in N, then $NX''(N) + X'(N) \geq 0$ $\forall N$ i.e. $\frac{X'(N)}{X''(N)} + N \leq 0$ $\forall N$. In particular, $\frac{\underline{w}}{X''(N(\underline{w}))} + N(\underline{w}) \leq 0$. ■

To see that the sufficient condition is not vacuous, it is enough to note that this condition is satisfied for the production function $X(N) = N^{\frac{1}{2}}$, whenever $N \geq \frac{1}{4}$.

Note that this is the same condition as that for a competitive firm when there is no bargaining. The level of employment under the bargaining setup is, however, different from that under the competitive outcome. Even so we find that the same sufficient condition prevails. The intuition is as follows. Note that despite bargaining over both employment and wages, it is in the interest of both parties to set the wage level equal to $\underline{w}$. This, however, implies that following an increase in the level of the minimum wage, a reduction in the level of employment is called for. Given the concavity of the production function and the fact that the level of employment is higher under bargaining (compared to the competitive outcome), this implies that reducing the level of employment is less costly (in terms of output) under the bargaining outcome. Thus any increase in the minimum wage will tend to have a more severe effect in terms of employment under the bargaining outcome. Thus the sufficient condition for the competitive case will be sufficient in this case as well. Hence the result.

Finally, notice that in this chapter we restrict attention to the case where the alternative wage (i.e. the outside option) of the workers is zero. Thus it is natural to ask whether our results depend on this assumption. Our investigations (not reported in this chapter) suggest that as long as the level of the available alternative wage is not too high (compared to the minimum wage), our results are not affected. We still obtain uniqueness of equilibrium. More importantly, a version of Proposition 2.2 still goes through. The sufficient condition, though, is more complicated and involves the alternative wage as well. Another interesting feature is that even in this case, the union can only affect either employment or wage levels, but not both. Depending on the parameter values, the outcome involves either the competitive wage or the competitive employment levels.

5.3 Conclusion

The argument in this chapter demonstrates the pitfalls involved in an indiscriminate increase of the level of minimum wages. We show that even in the presence of unionized workers it is possible that an increase in minimum wages leads to a decline in the income of the workers. Such an increase also reduces the income of the management besides causing a reduction in the level of employment and hence output.

Thus from a policy point of view an increase in the minimum wage is not always in the best interest of the workers, especially if the marginal product curve of labour is inelastic. In that case the government may have to resort to other measures, namely, increasing the bargaining power of the workers. However, even in this case the effort may prove futile, if the increase is small enough and the management is strong enough.

Finally, notice that though our chapter is restricted to management-union interactions, the basic framework of variable surplus bargaining is likely to have a wider application. For instance, consider the problem where there are two agents who possess two different factors of production needed to produce some economic good. They want to form a firm for the production of the good. Clearly, we can employ a similar framework to analyse the division of surplus in this case.

5.4 Appendix

Lemma 5.1 *Consider the problem :*

$$\begin{aligned} \textit{Maximize} \quad & wN \\ \textit{such that } w \ \geq \ & \underline{w} \textit{ and } X(N) - wN \geq K \\ \textit{where } 0 \ \leq \ & K \leq X(N(\underline{w})) - \underline{w}N(\underline{w}) \ . \end{aligned} \tag{5.4.11}$$

The maximum occurs at (w^*, N^*) *where* $w^* = \underline{w}$ *and* N^* *is maximum* N *s.t.* $X(N) - wN = K$.

Proof.

(See figure 5.1) Since wN is a continuous function over a compact feasible set (easy to check), the maximum exists. Let the maximum be attained at (w^*, N^*). Then $N^* = \tilde{N}(w^*)$ where $\int N(w) =$ Max N s.t. $X(N) - wN = K$.

If $N^* > \tilde{N}(w^*)$, then we contradict the second feasibility constraint.

If $N^* < \tilde{N}(w^*)$, then $w^*N^* < w^*\tilde{N}(w^*)$. So wN can be increased through an increase in N^*.

So $N^* = \tilde{N}(w^*)$.

Next we show that $w = \underline{w}$.

Suppose $w^* > \underline{w}$. Observe that $\tilde{N}(\underline{w}) > \tilde{N}(w^*)$. [11]

$$\begin{aligned} \text{So } \underline{w}\tilde{N}(\underline{w}) \ &= \ X(\tilde{N}(\underline{w})) - K, \\ &> \ X(\tilde{N}(w^*)) - K \\ &= \ w^*\tilde{N}(w^*). \end{aligned} \tag{5.4.12}$$

So the union's payoff can be increased. ■

In figure 5.2, $\delta_1[X(N_1) - w_1N_1]$ represents a value of K.

[11] As $X(\tilde{N}(\underline{w})) - \underline{w}\tilde{N}(\underline{w}) = K$, so $X(\tilde{N}(\underline{w})) - w^*\tilde{N}(\underline{w}) < K$ i.e $K + w^*\tilde{N}(\underline{w}) > X(\tilde{N}(\underline{w}))$. If $\tilde{N}(\underline{w}) \leq \tilde{N}(w^*)$ then from the concavity of $X(N)$ and from the definition of $\tilde{N}$ it follows that L.H.S. increases more than R.H.S. when $\tilde{N}(\underline{w})$ is replaced by $\tilde{N}(w^*)$. So we can't have equality.

Lemma 5.2 *Let the management solve the problem:*

$$\begin{aligned} \textit{Maximize} \quad & X(N) - wN \\ s.t.\ w \ & \geq \ \underline{w} \\ \textit{and}\ wN \ & \geq \ K \ \textit{where}\ 0 \leq K \leq \underline{wN} \end{aligned} \tag{5.4.13}$$

Let (w^*, N^*) *be the vector which solves this maximization problem. Then the solution is as follows.*

Case 1. *If* $K \leq \underline{w}N(\underline{w})$, *then* $w^* = \underline{w}$ *and* $N^* = N(\underline{w})$.

Case 2. *If* $K > \underline{w}N(\underline{w})$ *then* $w^* = \underline{w}$ *and* N^* *is such that* $\underline{w}N^* = K$.

Proof. (See figure 5.2)

Case 1. Since $K \leq \underline{w}N(\underline{w})$ the management chooses $(\underline{w}, N(\underline{w}))$ which gives the management it's unconstrained maximum profits.

Case 2. Here $(\underline{w}, N(\underline{w}))$ is not feasible. Let N^* be such that $K = \underline{w}N^*$. We will show that $\underline{w}$ and this N^* solves the maximization problem. Clearly $N^* > N(\underline{w})$.

Now $w^* = \underline{w}$ for, if $w^* > \underline{w}$ then w^* can be decreased and employment chosen at N^* such that the union still receives K. The management will never give more than K. So, if $w^* > \underline{w}$, then w^* can be decreased to $\underline{w}$ with the union still getting K. Now the management's payoff will be higher as $X(N)$ has increased.

Next we show that $N = N^*$.

For any $N < N^*$, $X(N)$ is less compared to $X(N^*)$ while the payoff of the worker remains the same, so the payoff of the management is reduced.

Again if $N > N^*$, then from the concavity of $X(N)$ the closer we move to $N(\underline{w})$ the higher is $X(N) - \underline{w}N$ (which denotes the management's payoff for $N \geq N^*$). So payoff can be increased.

So the solution is $w^* = \underline{w}$ and employment at N^*. ■

In figure 5.2, K_1 and K_2 correspond to different levels of $\delta_2 w_2 N_2$ for Case 1 and Case 2 respectively.

Now we establish the shape of the two functions, $D_1(N_1)$ and $D_2(N_2)$, through a series of observations. (See figure 5.3)

Observation 5.1. $D_1(N_1)$ is strictly decreasing in the level of employment N_1, for $N_1 \in [0, N(\underline{w})]$.

As N_1 increases, we find that $\delta_1[X(N_1) - \underline{w}N_1]$, the payoff to be given to the management, increases. Thus at the previous level of employment it is not possible to satisfy the management's demands, and N_1 must be reduced. Observation 5.2 follows from a similar argument.

Observation 5.2. $D_1(N_1)$ is strictly increasing in N_1 for $N_1 \in [N(\underline{w}), \underline{N}]$.

Observation 5.3. $D_1(N) > N$, $\forall N \in [N(\underline{w}), \underline{N})$

It is easy to see why. For $N \in [N(\underline{w}), \underline{N})$, the management's current period payoff is more than the discounted value of his next period payoff. Hence N can be increased by the union.

Observation 5.4. For a fixed $N_1 \in (0, \underline{N})$, $D_1(N_1)$ is decreasing in δ_1.

As δ_1 increases, for the earlier value of N_2 the management obtains less than the discounted value of his next period payoff, so N_2 must be decreased.

Observation 5.5. For

$$N_2 \leq \frac{N(\underline{w})}{\delta_2}, D_2(N_2) = N(\underline{w})$$

and for

$$N_2 > \frac{N(\underline{w})}{\delta_2}, \ D_2(N_2) = \delta_2 N_2.$$

This follows straightaway from the definition of d_2, as does observation 5.6.

Observation 5.6. Slope of D_2 is δ_2. As δ_2 increases, the slope increases.

Existence of equilibrium.

We will show that the strategies mentioned in the text constitutes an equilibrium. Note that if the intersection of the curves occurs at their positively sloped sections, then $\delta_1[X(N_1) - \underline{w}N_1] = [X(N_2) - \underline{w}N_2]$ and $\delta_2\underline{w}N_2 = \underline{w}N_1$. Here the proof is easy. So we focus on the situation where the intersection occurs on the vertical part of d_2, i.e. where

$$N_1 = N(\underline{w}) \text{ and } \delta_2\underline{w}N_2 < \underline{w}N_1 \tag{5.4.14}$$

First we check player 1's strategy. Consider his offer. Since player 1's payoff is maximized at $N(\underline{w})$ and it is being accepted he cannot gain by offering anything else. Now we look at his acceptance decision. Suppose he rejects some offer where he gets more than $\delta_1[X(N_1) - \underline{w}N_1]$. Then he gets at most $\delta_1[X(N_1) - \underline{w}N_1]$ if he makes an acceptable offer next period. Otherwise he gets at most $\delta_1^2[X(N_2) - \underline{w}N_2]$ if he makes an unacceptable offer next period. Player 1 can deviate only if $\delta_1[X(N_2) - \underline{w}N_2] > [X(N_1) - \underline{w}N_1]$.

However we know that $[X(N_2) - \underline{w}N_2] = \delta_1[X(N_1) - \underline{w}N_1] \Rightarrow$ $\delta_1[X(N_2) - \underline{w}N_2] < [X(N_1) - \underline{w}N_1]$ for $\delta_1 \leq 1$. So the management should accept $\delta_1[X(N_1) - \underline{w}N_1]$.

Consider player 2's offer. If he offers anything that gives player 1 more than what player 1 is willing to accept now, then player 2 loses by definition of d_1. If he offers anything that gives player 1 less than what player 1 is willing to accept, then player 1 rejects and player 2 gets atmost $\delta_2\underline{w}N(\underline{w})$ if player 2 makes an acceptable offer next period. So player 2 offers less only if $\delta_2\underline{w}N(\underline{w}) > \underline{w}N_2 \Rightarrow \delta_2\underline{w}N_2 > \underline{w}N(\underline{w})$ as $N_2 > N(\underline{w})$. This contradicts equation (5.4.14).

Consider player 2's acceptance-rejection decision. He accepts any offer which yields him a payoff less than or equal to what he gets from $\delta_2\underline{w}N_2$,if he rejects he gets at most $\delta_2\underline{w}N_2$,or by making an unacceptable offer he gets $\delta_2^2\underline{w}N(\underline{w})$. Player 2 can gain only if $\delta_2^2\underline{w}N(\underline{w}) > \delta_2\underline{w}N_2, \Rightarrow \delta_2\underline{w}N_2 > \underline{w}N(\underline{w})$ as $N_2 > N(\underline{w})$. This contradicts equation (5.4.14).

Similarly accepting an offer which gives him strictly less than $\delta_2\underline{w}N_2$ cannot be best because by rejecting he can do better. ∎

We proceed to the proof of uniqueness of equilibrium; the line of proof follows Binmore (1986).

Uniqueness of equilibrium.

We start with a series of claims.

Claim 1. In any equilibrium of a game where the union is the first mover, the union cannot get anything less than his payoff from the vector $d_1(\underline{w}, N(\underline{w}))$.

Proof. If the union offers $(\underline{w}, D_1(\underline{w}, N(\underline{w})))$, the management should accept as he cannot get any higher payoff than that from $(\underline{w}, N(\underline{w}))$ tomorrow. (Note that assumption 4 is needed here).

Claim 2. In any equilibrium of a game where the management is first mover, the management cannot get anything with a higher payoff than that from

$d_2 d_1(\underline{w}, N(\underline{w}))$.

Proof. We use the property that equilibrium strategies for a game remain equilibrium strategies for any subgame of the game. If in an equilibrium (at time 0) the management (player 1) gets more than what he gets from $(\underline{w}, D_2 d_1(\underline{w}, N(\underline{w})))$, then the union must be getting a payoff less than $\delta_2[\underline{w} D_1(\underline{w}, N(\underline{w}))]$ from definition of D_2. However the union knows (from claim 1) that in any subgame starting at time 1, he can get at least as much payoff as that from $d_1(\underline{w}, N(\underline{w}))$. So he will reject the offer.

Claim 3. In any equilibrium of a game where union is the first mover, the union gets at least as much as that of the bundle $d_1 d_2 d_1(\underline{w}, N(\underline{w}))$.

Proof. Follows from claim 2 as before.

Continuing, we can get sequences $\{x_i\}_{n=0}^{\infty}$ and $\{y_i\}_{n=0}^{\infty}$ such that $x_0 = N(\underline{w}), y_0 = D_1(\underline{w}, N(\underline{w}))$, and $x_i = D_2(\underline{w}, y_{i-1}), y_i = D_1(\underline{w}, x_i)$ for $i \geq 1$.

Also the union, if he is the first mover gets at least as much as that of $(\underline{w}, y_n)$ for each $n \geq 0$, and the management, if he is the first mover gets a payoff less than equal to that from $(\underline{w}, x_n)$ for each $n \geq 0$.

Now we prove the rest of the uniqueness argument. We start with the following two cases.

Case 1. Let $\frac{N(\underline{w})}{\delta_2} \geq D_1(\underline{w}, N(\underline{w}))$. Clearly $x_i = N(\underline{w})$ and $y_i = D_1(\underline{w}, N(\underline{w}))$, $\forall i \geq 0$. Since $N_1 = N(\underline{w}), N_2 = D_1(\underline{w}, N(\underline{w}))$ our claim goes through.

Case 2. Let $\frac{N(\underline{w})}{\delta_2} < D_1(\underline{w}, N(\underline{w}))$. Now $N_1 > N(\underline{w}) = x_0$ and $N_2 = D_1(\underline{w}, N_1) > D_1(\underline{w}, N(\underline{w})) = D_1(\underline{w}, x_0) = y_0$ as D_1 is strictly increasing for $N \in [N(\underline{w}), \underline{N}]$. Again $x_1 = D_2(\underline{w}, y_0) > N(\underline{w}) = x_0$ since $D_2(N)$ is strictly increasing for $N \in [N(\underline{w}), \underline{N}]$. Similarly $x_1 = D_2(\underline{w}, y_0) < D_2(\underline{w}, N_2) = N_1$. Also $y_1 = D_1(\underline{w}, x_1) > D_1(\underline{w}, x_0)$ as D_1 is strictly increasing on $[N(\underline{w}), \underline{N}]$. Further $y_1 = D_1(\underline{w}, x_1) < D_1(\underline{w}, N_1) = N_2$ as D_1 is increasing to the right of $N(\underline{w})$. So

$$x_0 < x_1 < N_1 \tag{5.4.15}$$

and

$$y_0 < y_1 < N_2. \tag{5.4.16}$$

We now prove that if for some n,

$$x_0 < x_1 \cdots < x_n < N_1 \tag{5.4.17}$$

and

$$y_0 < y_1 < \cdots < y_n < N_2 \tag{5.4.18}$$

where $x_i = D_2(\underline{w}, y_{i-1})$ and $y_i = D_1(\underline{w}, x_i)$, then

$$x_0 < x_1 \cdots < x_n < x_{n+1} < N_1 \tag{5.4.19}$$

and

$$y_0 < y_1 < \cdots < y_n < y_{n+1} < N_2. \tag{5.4.20}$$

Now $x_{n+1} = D_2(\underline{w}, y_n) > D_2(\underline{w}, y_{n-1}) = x_n$ as D_2 is increasing. Also $y_{n+1} = D_1(\underline{w}, x_{n+1}) > D_1(\underline{w}, x_n) = y_n$ as $D_1(N)$ is increasing for $N \in [N(\underline{w}), \underline{N}]$. Also $x_{n+1} = D_2(\underline{w}, y_n) < N_1$ and $y_{n+1} = D_1(\underline{w}, x_{n+1}) < N_2$. So $\{x_i\}$ and $\{y_i\}$ are increasing sequences bounded above by N_1 and N_2 respectively. So $\{x_i\} \to \overline{x} \leq N_1$ and $\{y_i\} \to \overline{y} \leq N_2$. Also $\{D_1(\underline{w}, x_i)\} \to D_1(\underline{w}, \overline{x})$ from continuity of D_1, i.e. $\{y_i\} \to D_1(\underline{w}, \overline{x})$ i.e. $\overline{y} = D_1(\underline{w}, \overline{x})$. Similarly $\overline{x} = D_2(\underline{w}, \overline{y})$. However this is only true if $\overline{x} = N_1, \overline{y} = N_2$.

Now, very similar to the above derivation, we build sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ such that $a_0 = \underline{N}$, $b_0 = D_2(\underline{w}, \underline{N})$, $a_n = D_1(\underline{w}, b_{n-1})$ and $b_n = D_2(\underline{w}, a_n)$. The union if he is the first mover, always gets a payoff less than or equal to what he gets from $(\underline{w}, a_n)$, for each $n \geq 0$, and the management if he is the first mover always gets a payoff which is greater than than equal to what he gets from $(\underline{w}, b_n)$, for each $n \geq 0$. Now, $\{a_n\} \to N_2$ and $\{b_n\} \to N_1$.

Combining the two derivations with the two different pairs of sequences, it is the case that in any equilibrium where the management is the first mover, the management must get $X(N_1) - \underline{w}N_1$ and the union must get $\underline{w}N_1$. Now, it follows from standard Rubinstein (1982) techniques that there cannot be any delay in bargaining; this implies that the equilibrium is the unique vector $(\underline{w}, N_1)$. ■

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