Essays on First Best Implementable Incentive Problems

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Contents

Chapter 1

Introduction

The theory of mechanism design originated in the mid 1930s with the work of Lange[27], Lerner [28] and Hayek [20] on market socialism[1]. Further rigour was added to their ideas by Arrow and Hurwicz [1]. Hurwicz extended them to the general problem of mechanism design. An important aspect of mechanism design is asymmetric information. Information asymmetry typically imposes constraints on the goals which can be attained. For example, in the classic pure public goods problem, mechanisms that achieve truthful revelation of private information are Pareto sub-optimal i.e. these mechanisms lead to a welfare loss (see Hurwicz [24]). When can mechanism design lead to no welfare loss is addressed here. The question that we address in the three essays in this thesis is the following: *Can we identify decision problems for which mechanisms can be designed where information shortages and asymmetries do not impose any welfare loss.* In other words, do there exists "interesting" incentive problems where the "first best" is attainable? We provide a broadly affirmative answer to this question. We begin by discussing some theoretical results on implementation relevant for this work. We also give a brief sketch of the specific problems to be addressed in the next few

[1]The theory of market socialism deals with replicating the workings of a perfectly competitive market through a decentralized planning process in which incomes can be controlled.

chapters.

1.1 Games and equilibrium concepts

To describe the mechanism design problem we first define a game in normal form and the relevant equilibrium concepts. Formally, a non-cooperative game in normal form is written as $\Gamma \equiv \langle \mathbf{N}, \{S_j\}_{j\in\mathbf{N}}, \{u_j\}_{j\in\mathbf{N}}\rangle$, where $\mathbf{N} = \{1,\ldots,n\}$ is the (finite) set of players, $\{S_j\}$ is the strategy set of player j and $u_j : \prod_{j=1}^n S_j \longrightarrow \mathbf{R}$ is the payoff of player j[2]. Let $\mathbf{S} \equiv \prod_{j=1}^n S_j$ and $\mathbf{S}_{-j} \equiv \prod_{l\neq j} S_l$. A generic element of $\mathbf{S}_{-j}$ is denoted by s_{-j}. An **equilibrium concept** E for the game Γ is a selection from $\mathbf{S}$ and is denoted by $E(\Gamma)$.

DEFINITION 1.1.1 The strategy profile $s^* \in \mathbf{S}$ is a **Nash equilibrium** of the game Γ if $u_j(s_j^*, s_{-j}^*) \geq u_j(s_j, s_{-j}^*)$ for all $s_j \in S_j$ and all $j \in \mathbf{N}$. The set of Nash equilibria of Γ is denoted by $NE(\Gamma)$.

A Nash equilibrium has the property that it is a best response to itself. In other words, if a player believes that other players will play Nash equilibrium strategies (corresponding to a particular Nash equilibrium), then he is happy to play his Nash equilibrium strategy. This equilibrium notion has some significant weaknesses from a decision-theoretic point of view (see van Damme [39] and Mailath [29]). A more robust equilibrium notion is a dominant strategy equilibrium which we describe below.

DEFINITION 1.1.2 The strategy $s_j^* \in S_j$ is said to be a dominant strategy for player $j \in \mathbf{N}$ if $u_j(s_j^*, s_{-j}) \geq u_j(s_j, s_{-j})$ for all $s_j \in S_j$ and all $s_{-j} \in \mathbf{S}_{-j}$.

If a player has a dominant strategy then no matter what others play, he has an incentive to play his dominant strategy. His beliefs about the strategic choices of other players becomes irrelevant.

[2] $\mathbf{R}$ represents the real line.

DEFINITION 1.1.3 The strategy profile $s^* \in \mathbf{S}$ is a **dominant strategy equilibrium** of the game Γ if for each $j \in \mathbf{N}$, s_j^* is a dominant strategy for j. The set of dominant strategy equilibria of Γ is denoted by $DS(\Gamma)$.

Thus, the notion of dominant strategy equilibrium is very robust in the sense that it makes very weak assumption about players' behaviour. It is worth mentioning that a game in strategic form may not have a dominant strategy equilibrium. Further, a dominant strategy, if it exists, belongs to the best response set of a player irrespective of what others play. Therefore, a dominant strategy equilibrium is a Nash equilibrium i.e. $DS(\Gamma) \subset NE(\Gamma)$.

1.2 Dominant Strategy Implementation

A typical implementation problem involves a group of agents and a planner. Each individual has a preference which is known **only** to him[3]. The planner has to make a certain decision. However, his decision choice depends on the private information of all agents. For example, his decision of whether or not to build a bridge could depend on the value that each member of the society places on the bridge. Therefore, the planner has to find a way to elicit this privately held information. One way of solving this problem is to design a game that gives individuals the incentive to reveal their information truthfully. This forms the core of implementation theory.

We now try to develop a formal theory of implementation following Moore [31] and Green, Mas-Colell and Whinston [16]. Consider a situation where there are $\mathbf{N} = \{1, \ldots, n\}$ individuals or agents and a set of feasible outcomes or decisions denoted by A. Let $\Theta \equiv \Theta_1 \times \cdots \times \Theta_n$ be the set of possible states of nature. In a given state, the profile of individuals' preferences is given by $\theta = (\theta_1, \ldots, \theta_n) \in \Theta$ where $\theta_j \in \Theta_j$ for all $j \in \mathbf{N}$. A **social choice function** associates a decision $f(\theta)$ to each $\theta \in \Theta$ i.e. $f : \Theta \longrightarrow A$. A **mechanism** $M = \langle \mathbf{S}, \Gamma \rangle$ is an $n+1$ tuple $(\mathbf{S} \equiv S_1 \times \ldots \times S_n; \Gamma)$ where S_j is the message set (or strategy set) of player $j \in \mathbf{N}$ and $\Gamma : \mathbf{S} \rightarrow A$.

[3] This is the so called incomplete information framework.

DEFINITION 1.2.4 The mechanism $M = \langle \mathbf{S}, \Gamma \rangle$ **implements** f in $E(\Gamma)$ if there exists an equilibrium strategy vector $s^*(\theta) = (s_1^*(\theta), \ldots s_n^*(\theta)) \in \mathbf{S}$, such that $\Gamma(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.

The natural question to ask in this context is the following: given a social choice function f, does there exist a mechanism $M = \langle \mathbf{S}, \Gamma \rangle$ such that when agents with preference profile θ play the corresponding game, the unique equilibrium outcome is $f(\theta)$? The answer to this question depends on the extent of informational asymmetry across agents and the equilibrium concept $E(\Gamma)$ used in solving the problem. If each agent knows the preferences of all other agents then we have a mechanism design problem under **complete information** and the equilibrium concept that can be used is Nash equilibrium. However, if each agent knows only his preference and has some or no information about the preferences of the other agents then we have a mechanism design problem under **incomplete information**. Under incomplete information it is not very meaningful to talk about Nash equilibrium since the validity of the notion depends critically on the assumption that payoffs of all players are common knowledge to the players. The dominant strategy equilibrium is a more appropriate equilibrium concept under incomplete information since truth-telling leads to a maximal payoff irrespective of others' strategies. The three essays in this thesis deal with mechanism design problems under incomplete information with dominant strategy equilibrium as the equilibrium concept.

DEFINITION 1.2.5 The strategy profile $s^*(.) = (s_1^*(.), \ldots s_n^*(.)) \in \mathbf{S}$ is a **dominant strategy equilibrium of a mechanism** $M = \langle \mathbf{S}, \Gamma \rangle$ if, for all $j \in \mathbf{N}$ and for all $\theta_j \in \Theta_j$,

$$U_j(\Gamma(s_j^*(\theta_j), s_{-j}), \theta_j) \geq U_j(\Gamma(s_j', s_{-j}), \theta_j)$$

for all $s_j' \in S_j$ and for all $s_{-j} \in \mathbf{S}_{-j}$.

Given the notion of dominant strategy equilibrium of a mechanism, one can now define dominant strategy implementability.

DEFINITION 1.2.6 The mechanism $M = \langle \mathbf{S}, \Gamma \rangle$ **implements** f **in dominant strategies** if there exists a dominant strategy equilibrium of Γ, $s^*(.) = (s_1^*(.), \dots s_n^*(.)) \in \mathbf{S}$, such that $\Gamma(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.

What gives the implementation problem its bite is that the same M has to cope with all profiles in Θ. To identify whether a social choice function f is implementable in $E(\Gamma)$, we need, in principle to consider all possible mechanisms. Much of the complexity of mechanism design problem under incomplete information is reduced by the "revelation principle". The revelation principle states that a planner can concentrate only on "direct revelation mechanisms" where the game form is such that the strategy set of each individual is his type space. The revelation principle has been used among others by Gibbard [13], Green and Laffont [15], Dasgupta, Hammond and Maskin [7] and Myerson [33][4].

DEFINITION 1.2.7 A **direct revelation mechanism** is a mechanism in which $S_j = \Theta_j$ for all $j \in \mathbf{N}$ and $\Gamma(\theta) = f(\theta)$ for all $\theta \in \Theta$.

To state the "revelation principle for dominant strategy implementation" we need the following definition.

DEFINITION 1.2.8 The social choice function f is **truthfully implementable in dominant strategies** or **dominant strategy incentive compatible** or **strategyproof** if for all $j \in \mathbf{N}$ and for all $\theta_j \in \Theta_j$,

$$U_j(f(\theta), \theta_j) \geq U_j(f(\theta'_j, \theta_{-j}), \theta_j)$$

for all $\theta'_j \in \Theta_j$ and for all $\theta_{-j} \in \prod_{l \neq j} \Theta_l$.

The revelation principle for dominant strategy implementation states the following: *If a social choice function f is implementable in dominant strategies then f is strategyproof.* Due to the revelation principle, one can restrict

[4] Also see Fudenberg and Tirole [11].

attention to direct mechanisms which are strategyproof. All the essays in this thesis use the notion of dominant strategy implementability. Therefore, the mechanisms considered are all direct revelation mechanisms. We now provide a brief survey of the main issues addressed in the thesis.

1.3 Main Issues

The notion of dominant strategy implementability, though very robust in terms decision theoretic foundations, is impossible to achieve if domain of preferences is unrestricted. With unrestricted domain of preferences we have an impossibility result, better known in the literature as the Gibbard-Satterthwaite Theorem. The theorem states the following: Suppose Θ includes all possible strict preference orderings over the finite set of alternatives A. Then for $|A| \geq 3$, there is no social choice function f that can be truthfully implemented in dominant strategy equilibrium unless it is dictatorial[5]. For a formal proof of this theorem see Gibbard [13] or Satterthwaite [35].

The Gibbard-Satterthwaite theorem holds for unrestricted domain of preferences. There are however, environments where domain restrictions are natural. One such natural restriction is quasi-linear preferences. In a quasi-linear framework, the utility of an agent $j \in \mathrm{N}$ is of the form $v_j(d_i, \theta_j) + t_j$ where d_i is an outcome or a decision and t_j is his transfer. This is an important domain restriction where money can be thought of as the private good and there exists a possibility of agents compensating each other via monetary transfers. The literature on dominant strategy implementability with quasi-linear preferences is vast. We mention some of the results on this topic that are relevant for this thesis.

In a quasi-linear framework, a powerful result on dominant strategy implementability with a finite set of decisions over which preferences are unrestricted, was established by Roberts [34]. However, in general we may have other objectives over and above dominant strategy implementability like "ef-

[5] $|A|$ represents the cardinality of A.

ficiency of decisions". The following discussion is in this direction.

Groves [17] and Clarke [3] have discovered that for one important class of social decision problems for which there is a class of mechanisms, popularly known as "Groves" mechanism, that are dominant strategy incentive compatible. The most appealing problem of implementation in this context is the non-excludable pure public goods problem. In this framework the set of agents have preferences over two decisions- public goods and no public goods. The preferences of each individual for the public good is private information. The planner has to decide on whether or not to produce the public good based on the preferences of the agents. The planners' goal is to achieve efficiency of decision i.e. the planner wants to maximise the total surplus of all agents from the public decision in all states of the world. This is the classic "free-rider" problem where Groves mechanisms ensure dominant strategy implementation and efficiency of decisions.

In a Groves mechanism, the transfer of a particular agent is selected in such a way that his payoff is the total surplus from the public decision in all states plus a constant that depends on the announcements of all other individuals. The transfer is strategyproof and achieves efficiency of decisions because this transfer puts the planner's optimization problem into the utility function of each individual. Green and Laffont [14] has proved the uniqueness of Groves mechanism in the public goods problems. Holmström [21] has proved that if the domain of preferences in a quasi-linear framework is smoothly connected, then Groves mechanisms are the only mechanisms that are dominant strategy incentive compatible and efficient in terms of the decision. More recently Suijs [36] has proved that for finite decision problems Groves mechanism are unique mechanisms that achieve efficiency of decisions and strategyproofness provided the quasi-linear domain of preferences is graph connected. These results suggest that for a very broad class of quasilinear preferences, Groves mechanisms are the only mechanisms that achieve efficiency of decision and strategyproofness.

The main drawback of Groves mechanism is that it is, in general, not balanced. This means that there are preference realizations where the sum of transfers differ from zero. In the pure public goods problem Hurwicz [23], Green and Laffont [15] and Walker [41] proved the budget imbalance of a Groves scheme. The implication of budget imbalance is that Groves mechanisms are not Pareto-optimal. The damaging nature of budget imbalance of Groves mechanism, in the public goods context, was analysed by Groves and Ledyard [18]. They proposed, using a very simple model that an alternative procedure based on majority rule voting may lead to an allocation of resources which is Pareto-superior to the one produced by Groves mechanism.

What can be done in the light of this difficulty? Different remedies for dealing with this problem exists in the literature. Some of these are specified below.

- We can search for Groves mechanism that generates a budget surplus for the planner in every state. Mechanisms that ensure a budget surplus for the planner are called feasible mechanisms. It is desirable that the sum of Groves transfers is non-positive so that the planner can achieve efficiency of decision and truth-telling in dominant strategies without incurring a budget-deficit. This has been the most standard way out of this problem in this literature.

- We can look for a "second-best" Groves mechanisms. For example one can try and identify the Groves transfer among the class of feasible Groves transfers that minimizes the surplus for the planner in every state. Moulin argued that the "pivotal" mechanism, belonging to the Groves family of mechanisms, achieves this goal. In a pivotal mechanism, each agent is taxed to the extent of the exact cost. that his presence imposes upon the rest of the agents. One obvious feature of this mechanism is feasibility since the agents are always taxed and never subsidized. Moulin [32] proved that in the non-excludable pure public goods problem with finite set of decisions, for all profiles, no other feasible strategyproof mechanism has a smaller budget surplus than the pivotal mechanism. More recently, Deb and Seo [9] derived a closed form solution for the maximal surplus generated by pivotal mechanism in the case of non-excludable binary public good. Their formula can be used to compare pivotal mechanism with other mechanisms.

- We can weaken the dominant strategy to Bayesian incentive compatibility and look for mechanisms that are Bayesian incentive compatible, efficient in terms of decision and budget-balanced. A classic paper in this direction is by d'Aspremont and Gerard-Varet [8]. This paper proposes a mechanism which is now known as AGV mechanism. An AGV mechanism is a Groves mechanism in expectations. In this mechanism each agent is paid the expected value of the other agents' surplus conditional upon his own report. They prove that there exists AGV transfer schemes that are balanced.

- We can relax the efficiency assumption and look for mechanisms that are dominant strategy incentive compatible and budget balancing (see e.g. Gary-Bobo and Jaaidane [12]). One possible approach is the so called "sampling" mechanism. The idea of a "sampling" mechanism is to select a sample from the set of agents and give them the Groves transfer. The surplus or deficit is distributed among the agents outside the sample. Clearly, this mechanism is dominant strategy incentive compatible and budget balancing. However, it is not efficient in terms of the decision due to the sampling error. It is clear that under appropriate assumptions, as the number of agents become large, the sampling error becomes arbitrarily small.

- We can add domain restrictions and look for first best implementable decision problems. In other words, we can try to find decision problems for which it is possible to find Pareto optimal or "fully efficient" mechanisms. In the non-excludable public goods problem this has led to partial success. Groves and Loeb [19] show that, with quadratic preferences it is possible to find balanced Groves transfers. Laffont and Maskin [25], by assuming continuity and differentiability of preferences, derive a necessary and sufficient condition for the existence of balanced Groves mechanism. Tian [37] obtained a sufficient class of preferences for which balanced Groves mechanism exists. This class includes, as a special case, the quadratic preferences derived by Groves and Loeb. In all these results, on first best implementability for the public goods problem, the decision set is not finite. For finite decision problems, Suijs [36] shows that a "sequencing" problem is first best implementable.

In this thesis, we adopt an approach similar to Suijs. We explore the possibility of identifying decision problems where the first best can be implemented. Clearly, the non-excludable public good problem does not fall within this class but we attempt to demonstrate nonetheless, that there are other interesting incentive problems which have this property.

The mechanism design approach is meaningful as long as the agents agree to voluntarily participate in the mechanism. An agent will want to participate in the mechanism if under all profiles, the payoff he receives is not less than what he would have received if he did not participate. This condition of acceptance or rejection by agents is captured by the individual rationality condition. For example, if an agent has an outside option, independent of his preferences within the mechanism such that it gives him a utility of zero, then individual rationality condition means that the utility of the agent, by participating in the mechanism, should be no less than zero. There can be other ways of looking at the individual rationality condition. For example, the outside option can be state dependent or it may not be zero for all individuals. We do not go into such details and concentrate only on the simplest of individual rationality condition-that with zero reservation utility for all agents. With this definition of individual rationality in the pure public goods problem, Green and Laffont [15] proved that Groves mechanisms are individually rational under a sufficient condition on the transfer provided there is no feasibility restriction. However, this condition results in huge budget deficit for the planner. If there are n agents in the economy, then the sufficient condition on individual rationality requires that the budget deficit be at least as large as $(n-1)$ times the total surplus in each state. Laffont and Maskin [26] has proved that for pure public goods problem one cannot find feasible Groves mechanism that satisfy individual rationality.

The broad conclusion that follows from this survey is that it is in general not possible to find decision problems that are first best i.e. decision problems that satisfy dominant strategy incentive compatibility, efficiency in terms of decision and budget balancedness. Moreover, it is even more difficult to find first best decision problems that satisfy individual rationality as well. In the three essays we identify decision problems for which first best can be achieved and we analyse the distinct nature of such decision problems. We also address the individual rationality issue in this context and show that in some special cases first best decision problems are also individually rational. We now proceed to a more detailed discussion of the three essays.

1.4 Essay 1: Public Decisions

The first essay considers an economy with a public good and a private good where agents have quasi-linear preferences. A public decision d_i is a $n \times 1$ vector and the utility of the jth agent from this decision is his private valuation multiplied by the jth component of the vector d_i, that is $\theta_j d_{ji}$. Here, θ_j is the valuation for the public good to individual $j \in \mathbf{N}$ and d_{ji} is the jth component of the ith public decision d_i. In this framework, we first derive the necessary and sufficient condition for first best implementability. An important observation is made in this context. If the decision set satisfies the dummy property i.e. there exists an individual for whom all decision lead to the same utility, then a decision problem is always first best implementable. The idea is quite simple-the dummy player absorbs the entire surplus or deficit and hence balances the budget. This result is not quite interesting and we rule out this possibility by assuming "symmetry". Symmetry means that if d_i is a particular public decision then all possible permutations of the vector d_i belongs to the decision set.

The first model of a public decision is a queueing model. The queueing model is derived from a situation where there are n agents who have jobs to be processed through a single machine. We can think of a computer server who has to serve a set of individuals or agents. Each agent has a cost function that depends on his completion time, i.e. the time he has to wait in the queue and the time it takes to serve him. It is assumed that there is only one server who needs one unit of time to serve any agent (homogeneous length of time). The per period waiting cost of each individual is private information. The decision problem for the server, who is the planner here, is to elicit this private information for determining the efficient queue for serving the set of individuals. The queueing model with at least three individuals is shown to be first best implementable.

We illustrate the main idea behind this result in the case where there are three individuals. Observe that efficiency implies that in every state the individual who announces the highest cost is served first followed by the individual who announces the second and lowest cost respectively. We claim that if the individual who is served first pay the amount of the second highest cost to the individual who goes last, then all individuals have the incentive to reveal their information truthfully. There is a striking resemblance that this scheme bears to the second price auction (see Vickrey [40]). In the auction, the highest bidder pays an amount equal to the second highest bid to an outsider (the seller). In the queueing model this amount is transferred to the individual who gets the last position. It not only balances transfers in the aggregate but also provides incentives for the individual who is served last.

The next situation is a particular class of decision problems called allotment problems. In an allotment problem the sum of the elements of a particular decision vector is unity. For example, if there are n individuals and if the decision vector is $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $\sum_{i=1}^{n} \alpha_i = 1$. A special case of this problem is the decision vector of an indivisible good auction without the possibility of no sale. For $n = 3$, the decision of an auction is a public decision problem consisting of all permutations of the vector $\alpha^a = (1,0,0)$. In the $(1,0,0)$ case, it is possible to ensure truth-telling in dominant strategies by giving the object to the highest bidder at a price which is equal to the valuation of the second highest bidder. However, this transfer scheme is not balanced because the payment of the highest bidder "leaks" out of the system. In contrast consider an auction with the decision vector $\hat{\alpha} = (\frac{2}{3}, \frac{1}{3}, 0)$ for $n = 3$. This can be thought of a special type of auction where efficiency means that the highest bidder gets $\frac{2}{3}$ of the good, the second highest bidder gets $\frac{1}{3}$ and the lowest bidder does not get anything. Consider a transfer scheme where the highest bidder pays $\frac{1}{3}$ of the second highest bid to the lowest bidder. This transfer scheme is balanced. The first part of this transfer scheme is that the highest bidder pays the marginal loss incurred by the second highest bidder. The margin being the difference between best decision $\frac{2}{3}$ and the second best decision $\frac{1}{3}$. This part of the transfer is like the second price auction α^a, where the highest bidder pays the second highest bid which is the marginal loss, i.e. $1 - 0 = 1$, the second highest bidder incurs. However, the payment of $\frac{1}{3}$ of the second highest bid by the highest bidder in $\hat{\alpha}$ does not go to an outsider because unlike the auction α^a, in $\hat{\alpha}$, there is an incentive problem for the second and third position also. The amount of money the lowest bidder receives is the marginal gain of the second bidder, the margin being the difference between the second and third decision i.e. $\frac{1}{3} - 0 = \frac{1}{3}$. By considering all possible deviations it is easy to check that this transfer scheme for $\hat{\alpha}$ is dominant strategy incentive compatible. The important thing to observe is the appropriate "spread of incentives" that this allotment problem gives rise to. The incentives are spread in such a way that the individuals getting the better bundles can compensate the individuals not getting the good bundles, such that the transfer is both incentive compatible as well as budget balanced. The appropriate "spread of incentives" is captured by a combinatorial property which we call "Property T". This sort of spread is absent in the single good auction models. Consider a particular decision vector and arrange them in a decreasing order. "Property T" is a combinatorial restriction on this set of ordered decisions. The example of the decision $\hat{\alpha}$ for three agents, suggested above, satisfies "Property T" while α^a does not.

In this framework, we assume one property of the set of decisions that gives a complete characterisation result. This property is called "non-trivial range or NTR" property. An allotment problem satisfies the "NTR" property if for all decisions there exists an open neighbourhood of states for which the decision is efficient and strictly dominates all other decisions in the decision set. For example, a symmetric allotment problem with decisions α^a and $\hat{\alpha}$ violates the non-trivial range property because there is no open neighbourhood where $\hat{\alpha}$ is strictly preferred to α^a. The main result is that the class of allotment problem satisfying "symmetry" and "NTR" property are first best implementable if and only if they are "simple" and satisfy "Property T". An allotment problem is "simple" if it includes one and only one decision vector and all its permutations in the set of decisions.

The final class of decisions generalizes the non-excludable pure public goods problem by including the possibility of exclusion. In the classic public goods problem, the public good, if produced, is enjoyed by all the agents. We allow for the possibility of exclusion. Like the non-excludable public goods case, each decision vector now has only two distinct elements, 0 and 1 where 0 is the case where the individual cannot enjoy the public goods and 1 the case where the individual can enjoy the public good. We show that in this model a public goods problem satisfying the "symmetry" property is first best implementable if and only if we allow for full exclusion. In other words, the public good is available to the set of individuals having non-negative valuation. Individuals having negative valuation do not get the public good. This implies that a public good problem is FB implementable if and only if the possibility of exclusion transforms it into a private good. The if part of this result is quite easy to establish. The converse is non-trivial because we allow for all possible intermediate levels of exclusions. In this framework, the reason for first best implementability of fully excludable public goods problem is different from that of allotment and queueing models. Here, it is the full excludability aspect that drives the result, because the incentive problem ceases to exist. In fact, it is easy to argue, in this context, that the transfers are zero for all individuals and for all states.

Finally, the issue of individual rationality is addressed. It is proved that a queueing model is not individually rational. The class of first best allotment problem is individually rational under a sufficient restriction on the decision. The fully excludable first best implementable public goods problem is also individually rational.

1.5 Essay 2: Queueing Models

The second essay considers a generalization of the queueing model of the previous essay. It focuses on the conditions that lead to first best implementability in the queueing framework. To do so, queueing models with different cost structures are explored. Think of a computer server who has to serve a set of individuals in a queue. The servers role is like the planner who wants to minimise aggregate cost or find an efficient queue given the cost parameters. As before, in all the queueing models considered in this essay, it is assumed that the server needs one unit of time to serve an agent. The individual's waiting cost for all periods is private information. These costs need not be linear. A special case of queueing model is the linear cost model that was dealt with in the previous essay. We generalise the queueing model by taking different cost structures. In this framework, individuals, in order to reduce their own queueing cost, can announce their cost strategically. This gives rise to a mechanism design problem for the server.

The unique feature of the queueing models we consider is that all agents are served by the server. Therefore, once an individual is in the queue, he cannot leave the queue and the server cannot throw him out of the queue. This is necessary for first best implementability. There are a number of papers that look at queueing models, where agents have the option of not joining the queue. Papers by Crés and Moulin [4], [5] explore the queueing model in this framework. It is quite apparent from the structures of these models that "full efficiency" or first best cannot be achieved. Therefore, they compare the relative inefficiency of different mechanisms. They also focus on other nice properties of these mechanisms like group strategyproofness i.e. whether these mechanisms are immune to group deviation or not.

The first model in this essay is a general queueing model. In this model a complete characterization result is obtained by making minimal assumptions about the cost vector of the agents. A cost vector of an agent is a row vector with the kth element measuring the cost at the kth queue position. The cost is assumed to be non-decreasing in queue positions. A combinatorial property CP on the cost vector of each agent and the independence property IP on group preferences are both necessary and sufficient for first best implementability. Independence property is a restriction on the externality that an agent can impose on the remaining set of agents. In other words, IP means that if in some state agent j is served before agent l, then for all queueing situations where agent $i \in \mathrm{N}/\{j,l\}$ is eliminated, agent j continues to be served before agent l. The cost vector must be such that, if agent j is a predecessor of agent l, in some state, then j must continue to be a predecessor of agent l, independent of the elimination of any $i \in \mathrm{N}/\{j,l\}$ from the queue. The elimination of an agent $i \in \mathrm{N}$, from the queue is a way of measuring the externality agent i imposes on the remaining set of agents in the queue with all individuals.

The second type of queueing models looks at agents with identical cost functions. The form of the cost function is common knowledge. The cost function is of the form $g(\theta_j)f(k)$. The first function depends on the cost parameter θ_j of an individual $j \in \mathrm{N}$. The second function $f(k)$ depends on the queueing position $k \in \{1,\ldots,n\}$. The parameter θ_j which is private information. In this framework, the independence property or IP is satisfied for all states. This class of separable cost queueing models is proved to be first best implementable if and only if the queue dependent cost function

$f(k)$ satisfies CP. The class of separable cost queueing models satisfying CP includes the polynomials of degree $(n-2)$ that are non-decreasing in k. The queueing model, shown to be first best implementable in the first essay, is a special case of this class of first best implementable queueing models. Moreover, the class of queueing models with cost function of the form $\sum_{p=1}^{M} g^p(\theta_j)f(k) + \beta_j(\theta_j)$ with $f^p(.)$ satisfying CP and some restrictions on $g^p(.)$ for all $p = 1, 2, \ldots, M$, is shown to be first best implementable. We call this type of queueing models the general first best implementable queueing models. These results confirms the fact that there exists a fairly large class of first best implementable queueing models. Finally, another type of queueing model is introduced where, the cost is measured by the discounted value of the benefit from the service. This model is called the discounted cost queueing model and it violates CP at all but boundary preferences. Therefore, this model is not first best implementable.

The results obtained from different queueing models highlight the importance of CP and IP for first best implementability of any queueing model. The critical element for possibility results is the externality structure of the first best implementable queueing models captured by IP. The IP condition limits the externality that an individual imposes on the remaining set of individuals. To get a better understanding of the externality structure one can compare the first best implementable queueing model with that of the classic pure public goods model. In the classic pure public goods problem with non-excludability, an agent can, by changing his announcement, affect the decisions of all other individuals. For example, in a state where the efficient public decision is to produce the public good, an individual, by changing his announcement strategically, can change the efficient decision to one where no public good is produced. This will affect all individuals. It is this severe externality that causes budget imbalance in the pure public goods model[6]. In the queueing models it is necessary that the externality that can be imposed by an individual on the remaining set of individuals is less severe. In

[6]See Green and Laffont [15].

particular, if an individual has position k in the queue, then, by changing his announcement he can affect individuals either in his predecessor set or his successor set. He cannot simultaneously affect both the predecessor as well as the successor set of individuals. This type of externality, that results due to IP, is less severe than the one in pure public goods problem.

Finally, the question of individual rationality is explored. In this context, the gross benefit that an individual receives from the service plays an important role. The queueing model in its most general form is not individually rational. The first best implementable separable cost queueing models and the general first best implementable class of queueing models are proved to be individually rational under sufficient restrictions on the benefit. The general conclusion that we can arrive at is that with a sufficiently high gross benefit from the service for all agents, a first best implementable queueing model is also individual rational.

1.6 Essay 3: Sequencing Models

Sequencing models are a special type of queueing models which differ from queueing models of the first two essays in two important ways:the model is a continuous time model and different agents may differ in the amount of time they require to be serviced. The framework followed in this essay is the same as that in Suijs [36]. An example of a sequencing model is a large multi-unit firm with each unit in need of the facility provided by a particular repair and maintenance unit. On the occasions when a number of units of the firm ask for this facility, each unit incurs a cost for the time it is down. In this scenario, the firm's role is that of a planner who has to serve the units by forming a queue that minimises the total cost of waiting. Each unit has a waiting cost which is private information and a servicing cost which is common knowledge.

The sequencing model was studied as an incentive problem by Dolan [10]. Dolan provided an incentive compatible but not budget balanced mechanism. Suijs proved that if costs are linear, the sequencing model satisfies first best implementability. Suijs conjectured that it is the linearity of the cost function that is the driving force behind first best implementability. In this essay, we generalize, the sequencing model to show that this is indeed the case. The servicing cost function is restricted to be a sub-class of weakly convex functions. Suijs' sequencing model which looks at linear time cost function is a special case of this class of sequencing models. For this class of sequencing models we prove that if there are more than three units, first best implementability is achieved if only if the time cost function is linear. Linearity of the time cost function implies that the relative queue position of any two units in a state is independent of the preferences of all other units. It is this type of independence that is both necessary and sufficient for first best implementability of sequencing models. The independence of relative queue positions for linear cost queueing model is identical to the independence property or IP defined for queueing models. Unlike discrete time queueing models, where we have a fairly large class of first best implementable queueing models, in the continuous time queueing models or sequencing models, first best implementability can be achieved only for one model-that with linear cost.

Chapter 2

Public Decisions

2.1 Introduction

A classical problem in the theory of incentives is the design of a scheme of taxes and transfers which can solve the "free rider" problem in the provision of public goods. In the standard formulation , agents have quasi-linear utility and their valuation for the public good is private information. The planners objective is to construct a mechanism which will achieve efficiency. The pioneering work of Groves [17] and Clarke [3] has established that there exists a class of mechanisms, now called Groves mechanisms where all individuals have a dominant strategy to reveal their valuations; moreover, the truth-telling outcome leads to an efficient amount of the public good being produced[1]. Under relatively weak assumptions on the domain of preferences, Groves mechanisms have been shown (Green and Laffont [14], Holmström [21] and more recently by Suijs [36]) to be the only ones which satisfy these properties.

A difficulty with the Groves mechanisms is that they are not **balanced**. There are preference realizations where aggregate transfers are non-zero. This is an irksome problem not just because it entails a loss of overall ef-

[1]See Green and Laffont [14] for a comprehensive account of these mechanisms and their properties.

ficiency but also because it is no longer clear whether in these circumstances, efficiency with regard to the public decision remains desirable. Groves and Loeb [19] show that in the special case where the utility function is quadratic, aggregate transfers are always zero. Tian [37] generalizes the class of preferences where this property holds. However, Walker [41] shows that these domains must be non-generic. There are other approaches to this balancedness issue. Green and Laffont [15] (Chapter 9) demonstrates that the magnitude of the budget imbalance vanishes asymptotically as the number of agents become large, under appropriate "sampling" hypothesis. Gary-Bobo and Jaaidane [12] constructs a random mechanism which is balanced and induces truth-telling in dominant strategies but which is approximately efficient. In the classic paper of d'Aspremont and Gerard-Varet [8], it is shown that efficiency and budget balance can be reconciled provided the incentive compatibility requirement is weakened from dominant strategy to Bayesian incentive compatibility. The inescapable conclusion from this body of work is that the requirements of dominant strategies, generic domains, efficiency and budget balance (and indeed, individual rationality) are mutually incompatible; however, it is possible to make trade-offs between one requirement and the others. Makowski and Mezzeti [30], provides a unified approach to some of these tradeoffs.

In this essay, we reconsider this issue in a different perspective. We define a more general class of problems which we call **public decision problems** and show that there are members of this class for which all the objectives discussed previously, can be reconciled. We believe that the possibility results are of some interest in themselves. In addition, the approach yields some insights into the question of why the public goods and other problems are "insoluble".

We illustrate the notion of a public decision problem by means of an example in queueing. Suppose there are n individuals each of whom wishes to use a computer terminal for a unit period of time. Each individual has a constant per unit time waiting cost which is private information. A public decision in this context is the order in which the individuals will be allowed use. There is a significant sense in which decisions of this type differ from public good decisions. In the latter case, all individuals are "affected" in the same way; if the decision is whether or not to built a bridge, all individuals are forced to consume its services if it is built but not if it isn't. In contrast, changes in decisions regarding queues could affect individuals in a variety of ways. A change which affects an individual need not affect all individuals. For example, if the positions of two individuals in the queue were interchanged, then they would be the only individuals affected. Thus the nature of the "externality" imposed by a player on other players is not as severe as that in a public good model. One of our main results is that efficient queueing decisions can be implemented by means of balanced transfers. We illustrate this possibility result by means of a simple example. Consider the case where there are three individuals. Efficiency requires the individual announcing the highest cost to be served first followed by the individual who announces the second highest cost with the individual who announces the lowest cost being served last. Consider the transfer scheme where the individual who is served first compensates the individual who goes last by the amount which is the cost announcement of the "middle" individual (who pays and receives nothing). The mechanism is obviously balanced. It also bears resemblance two the well-known second price auction (see Vickrey [40]). It is fairly easy to verify that truth-telling is a dominant strategy for all individuals. This example is presented in greater detail as Example 2.4.5.

Our basic objective is to characterize the class of public decision problems where efficient public decisions can be attained by means of balanced transfers (which we call first-best or FB implementable) and to show that there are "interesting" problems which fall within this class. We have deliberately chosen a model of linear utility functions to demonstrate our possibility results. We are thereby able to attribute these results to the **structure** of the public decision problems rather than to domain restrictions such as quadratic preferences. Our main results are as follows. We identify a condition which is necessary and sufficient for FB implementability. We then apply this condition to show that the queueing problem is FB implementable; we also use it to characterize the class of public goods problems with excludability and the class of allotment problems which are FB implementable. For the latter results we also need to assume a symmetry condition in order to rule out trivial possibilities and in the case of allotment problems, an assumption on the range is also required. The results for the public goods problem are a trifle disappointing- FB implementability is possible only when "full excludability" is permitted. On the other hand, the results from the allotment problem are far more positive. We show that an allotment problem is FB implementable if and only if its associated matrix consists of all permutations of a single vector which satisfies a certain combinatorial property. Finally, we consider the question of public decision problems which are FB implementable and also satisfy ex-post individual rationality. Here we show that although the queueing problem cannot be FB implemented without violating individual rationality, there is a sub-class of allotment problems which can. We provide a sufficient condition for this property to hold.

A paper that is related to ours is Suijs [36]. That paper is primarily concerned in extending the result in Holmström [21] on the characterization of domains where Groves transfers are unique in the class of mechanisms which are efficient and satisfy dominant strategy incentive compatibility. The author goes on to show that a problem which he calls the scheduling problem which is similar to the queueing problem can be implemented by balanced transfers. Our work formulates the issue he raises in a general framework: and our results are therefore an extension and a refinement of his.

This essay is organised as follows. In section 2, we describe the model and in section 3, we present general results relating to implementability and first-best implementability. Section 4 is the heart of the essay and consists of applying the general results of the previous section to the public goods, queueing and allotment problems. Section 5 discusses the individual rationality condition while section 6 is the conclusion.

2.2 The Model

A public decision problem is a triple $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ where

- $\mathbf{N}$ is the initial segment of the n integers denoting the set of players.
- $D = (d_{ji})$ is an $n \times M$ matrix whose entries are all non-negative. With slight abuse of notation its ith column d_i will be referred to as the ith public decision.
- $\Theta \subset \mathbf{R}$. For any player $j \in \mathbf{N}$, $\theta_j \in \Theta$ denotes the type of j. In addition we shall assume that Θ is an interval.

The utility derived by individual j in state θ with decision d_i is given by

$$U_j(d_i, t_j; \theta_j) = d_{ji}\theta_j + t_j$$

where t_j is the transfer payment to individual j.

A vector $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$ will be referred to as a profile or a state of the world.

We illustrate our notion of public decision problem by means of three examples.

EXAMPLE 2.2.1 The **queueing problem** is a triple $\Omega^Q = \langle \mathbf{N}, \mathbf{D}^Q, \mathbf{R}_- \rangle$ where $\mathbf{D}^Q$ is the $n \times n!$ matrix whose columns are distinct permutations of the vector $\{0, 1, \ldots, n-1\}$. One interpretation of this formulation is as follows. There are n users each of whom want access to a single computer terminal for a unit time period. Users can only be served one at a time. A public decision is the order in which users are served. If $d_{ji} = k$, then the jth user has to wait k periods in the ith public decision. Here, $\theta_j \leq 0$ denotes j's waiting cost per time period. Thus, j's utility from the ith public decision is $d_{ji}\theta_j$.

EXAMPLE 2.2.2 An **allotment problem** is a triple $\Omega^A = \langle \mathbf{N}, \mathbf{D}^A, \mathbf{R}_+ \rangle$ where $\mathbf{D}^A$ is a matrix whose elements lie between 0 and 1 and whose column sums are all 1. The n individuals wish to share a fixed amount of resource and $\theta_j \geq 0$ denotes j's unit valuation. A special case of this model is the case where $\mathbf{D}^A$ is the $n \times n$ identity matrix. This is the case where an indivisible commodity has to be allocated to one of the individuals.

EXAMPLE 2.2.3 A **public good problem** is a triple $\Omega^G = \langle \mathbf{N}, \mathbf{D}^G, \mathbf{R} \rangle$ where $\mathbf{D}^G$ is a matrix whose elements are either 0 or 1. In the standard pure public good case, the public good is either provided or not provided so that $\mathbf{D}^G$ in this case is an $n \times 2$ matrix with one column consisting entirely of 0's and the other one of 1's. We wish to include within our analysis situations where individuals may be **excluded** from the consumption of the public good. Thus, if the (j, i)th element of $\mathbf{D}^G$ is 0, then the jth individual is excluded in the ith public decision, if it is 1, then she is included. In this class of problems, individual j's type is her valuation of the public good which can either be positive or negative.

We now introduce some basic definitions.

Let Ω be a public decision problem. The **efficiency correspodence** Σ_Ω associates a non-empty subset of the set of column vectors of $\mathbf{D}$ with every profile $\theta \in \Theta^n$ as follows:

$$\Sigma_\Omega(\theta) = \{d_i \in \mathbf{D} \mid d_i \in argmax \sum_{j \in \mathbf{N}} d_{ji}\theta_j\}$$

An **efficient rule** d^*_Ω is a single valued selection from Σ_Ω that is for all $\theta \in \Theta^n$, $d^*_\Omega(\theta)$ is a singleton and belongs to $\Sigma_\Omega(\theta)$.

We assume that for all j, θ_j is private information for player j. The planner's problem is to design a **mechanism** that will elicit this information truthfully. Formally a mechanism $\mathbf{M}$ is a pair $\langle d, \mathbf{t} \rangle$ where $d : \Theta^n \rightarrow \mathbf{D}$ and $\mathbf{t} = (t_1, \ldots, t_n) : \theta^n \rightarrow \mathbf{R}^n$.

Under $\mathbf{M} = \langle d, \mathbf{t} \rangle$ the utility of player j of type θ_j who announces $\theta_j{}'$ is given by

$$U_j(d(\theta_j{}', \theta_{-j}), t_j(\theta_j{}', \theta_{-j}), \theta_j) = d_j(\theta_j{}', \theta_{-j})\theta_j + t_j(\theta_j{}', \theta_{-j})$$

DEFINITION 2.2.9 A public decision problem $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ is **implementable** if there exists an efficient rule $d^*_\Omega : \Theta^N \to \mathbf{D}$ and a mechanism $\mathbf{M} = \langle d^*_\Omega, \mathbf{t} \rangle$ such that for all $j \in \mathbf{N}$, for all $(\theta_j, \theta_j') \in \Theta^2$, and for all $\theta_{-j} \in \Theta^{n-1}$,

$$U_j(d^*_\Omega(\theta), t_j(\theta); \theta_j) \geq U_j(d^*_\Omega(\theta_j', \theta_{-j}), t_j(\theta_j', \theta_{-j}); \theta_j)$$

This definition says that for any given θ_{-j} individual j cannot benefit by reporting anything other than his true type. In other words, truth-telling is a dominant strategy for all players.

DEFINITION 2.2.10 A public decision problem $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ is **first best implementable** or **FB implementable** if there exists a mechanism $\mathbf{M} = \langle d^*_\Omega, \mathbf{t} \rangle$ which implements it and such that, for all $\theta \in \Theta^n$, $\sum_{j \in \mathbf{N}} t_j(\theta) = 0$.

Thus, a public decision problem is first-best implementable if, it can be implemented in a manner such that aggregate transfers are zero in every state of the world. In such problems, incomplete information does not impose any welfare cost.

2.3 Preliminary Results

Our primary objective in this section is to characterize the class of first-best implementable public decision problems. We also explore some implications of these results. Our first result states that all public decision problems are implementable. Moreover, the associated transfers must be of the so called 'Groves' type.

DEFINITION 2.3.11 A mechanism $\mathbf{M} = \langle d, \mathbf{t} \rangle$ is a Groves mechanism, if the transfer t_j is of the form

$$t_j(\theta) = \sum_{l \neq j} d_l(\theta)\theta_l + \gamma_j(\theta_{-j}) \tag{2.3.1}$$

PROPOSITION 2.3.1 Let $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ be a public decision problem. Then Ω is implementable uniquely by the class of Groves mechanisms.

REMARK 2.3.1 We only state this result as it follows directly from Holmström (1979) where it is proved that for the domain that are "convex" the only transfer schemes that ensures implementability are the Groves transfers (see Theorem 2 in Holmström [21]). It is quite easy to check that the domain of the public decision problems specified in this essay satisfy Holmström's definition of "convex" domains. Thus Groves mechanisms are versatile enough to implement all public decision problems.

We now turn to the question of first-best implementability. It is clear that not all public decision problems are FB implementable. The two properties described below characterize the class of such public decision problems.

DEFINITION 2.3.12 The public decision problem $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ satisfies Property **S** if there exists an efficient rule d^* and functions $f_{jl} : \Theta^{n-1} \to \mathbf{R}$, $j, l \in \mathbf{N}, j \neq l$ such that for all $\theta \in \Theta^n$,

$$d_j(\theta) = \sum_{l \neq j} f_{jl}(\theta_{-l})$$

Property **S** requires the existence of an efficient decision rule which satisfies a separability property.

To define the next property we introduce some more notation. Let $S \subseteq \mathbf{N}/\{j\}$ and let $\theta_{-j}, \theta'_{-j} \in \Theta^{n-1}$. Define $\theta_l(S) = \theta_l$ if $l \notin S$ and $\theta_l(S) = \theta_l{}'$ if $l \in S$ and $\theta_{-j}(S) = (\theta_1(S), \ldots, \theta_{j-1}(S), \theta_{j+1}(S), \ldots, \theta_n(S)) \in \Theta^{n-1}$.

DEFINITION 2.3.13 The public decision problem $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ satisfies Property CA if $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} d_j^*(\theta_j, \theta_{-j}(S)) = 0$ for all $j \in \mathbf{N}$, for all $\theta_j \in \Theta$ and for all $\theta_{-j}, \theta'_{-j} \in \Theta^{n-1}$.

Property CA is similar to the Cubical Array Lemma in Walker [41]. It is a restriction on the change in the decision of an individual when type of all other individual changes in a particular way.

THEOREM 2.3.1 The following statements are equivalent

1. Ω is FB implementable.

2. Ω satisfies Property **S**.

3. Ω satisfies Property CA.

PROOF: We first show that (1) $\Rightarrow$ (2). Since Ω is FB implementable, there must exist a mechanism $\mathbf{M} = \langle d^*, \mathbf{t} \rangle$ which is balanced and induces truth-telling (where d^* is an efficient rule). Applying Proposition 2.3.1 it follows that there exists functions $\gamma_j : \Theta^{n-1} \to \mathbf{R}, j \in \mathbf{N}$ such that $t_j(\theta) = \sum_{l \neq j} d_l^*(\theta)\theta_l + \gamma_j(\theta_{-j})$, for all θ. Since aggregate transfers are zero, we obtain by summing both sides of the equation over the index j, that

$$\sum_{j \in \mathbf{N}} d_j^*(\theta)\theta_j = \sum_{j \in \mathbf{N}} F_j(\theta_{-j}) \tag{2.3.2}$$

where $F_j(\theta_{-j}) = -\frac{\gamma_j(\theta_{-j})}{n-1}$. Now consider the generic term $d_j^*(\theta)\theta_j$ on the left hand side of (2.3.2). It must be the case that this term does not depend on some θ_l (If it depends on all the θ's then (2.3.2) will be violated). There are only three possibilities-(i)$l \neq j$, (ii) $l = j$ and (iii) $d_j^*(\theta) = z_j(\theta)$ and $\sum_{j \in \mathbf{N}} z_j(\theta)\theta_j = 0$. Therefore, it must be the case that there exist functions $\bar{f}_{jl} : \Theta^{n-1} \to \mathbf{R}$ for all $j, l \in \mathbf{N}, j \neq l$, $h_j : \Theta^{n-1} \to \mathbf{R}$ for all $j \in \mathbf{N}$ and $z_j : \Theta^n \to \mathbf{R}$ with $\sum_{j \in \mathbf{N}} z_j(\theta)\theta_j = 0$ for all $j \in \mathbf{N}$ such that

$$d_j^*(\theta) = \sum_{l \neq j} \bar{f}_{jl}(\theta_{-l}) + h_j(\theta_{-j})/\theta_j + z_j(\theta) \tag{2.3.3}$$

Since the possible values of $d_j^*(\theta)$ is finite (the matrix $\mathbf{D}$ has a finite number of columns), it must be true that there exist functions $f_{jl} : \Theta^{n-1} \to \mathbf{R}$. $j, l \in \mathbf{N}, l \neq j$ which take only a finite number of values and

$$\sum_{l \neq j} \bar{f}_{jl}(\theta_{-l}) = \sum_{l \neq j} f_{jl}(\theta_{-l}) - h_j(\theta_{-j})/\theta_j$$

Therefore,

$$d_j^*(\theta) = \sum_{l \neq j} f_{jl}(\theta_{-l}) + z_j(\theta) \tag{2.3.4}$$

where $\sum_{j \in \mathbf{N}} z_j(\theta)\theta_j = 0$. Define $d_j^{**}(\theta) = \sum_{l \neq j} f_{jl}(\theta_{-l})$. Observe using the definition and $\sum_{j \in \mathbf{N}} z_j(\theta)\theta_j = 0$ that $\sum_{j \in \mathbf{N}} d_j^{**}(\theta)\theta_j = \sum_{j \in \mathbf{N}} d_j^*(\theta)\theta_j$. Since d^* is an efficient rule, so is d^{**}. Thus Ω satisfies Property **S** (with respect to the efficient decision rule d^{**}).

We now show that (2) $\Rightarrow$ (1). Let d^* and f_{jl} be the efficient rule and functions specified in the definition of Property **S**. Let $\mathbf{M} = \langle d^*, \mathbf{t} \rangle$ be a mechanism where

$$t_j(\theta) = \sum_{l \neq j} d_l^*(\theta)\theta_l - (n-1)\sum_{l \neq j} f_{lj}(\theta_{-j})\theta_l \tag{2.3.5}$$

It follows immediately from Proposition 2.3.1 that **M** induces truth-telling. We only need to check that the transfers are balanced. From (2.3.5) we get

$$\begin{aligned}
&\sum_{j \in \mathbf{N}} t_j(\theta) \\
&= \sum_j \sum_{l \neq j} d_l^*(\theta)\theta_l - (n-1)\sum_j \sum_{l \neq j} f_{lj}(\theta_{-j})\theta_l \\
&= (n-1)\sum_l d_l^*(\theta)\theta_l - (n-1)\sum_l \sum_{j \neq l} f_{lj}(\theta_{-j})\theta_l \\
&= (n-1)\sum_l d_l^*(\theta)\theta_l - (n-1)\sum_l d_l^*(\theta)\theta_l \\
&= 0.
\end{aligned}$$

The equivalence of (2) and (3) is straightforward and its proof is omitted. It is clear that if the function d_j has separable form for all $j = 1, \ldots, n$, then it must satisfy an appropriate restriction on the $(n-1)$th order cross-partial derivative. The condition in Property CA is analogous of this derivative for finite changes. ∎

REMARK 2.3.2 A subtle feature of Property S is that it only requires the existence of **an** efficient rule which satisfies Property **S**. It does not require all efficient rules to satisfy the separability result. It is easy to construct examples where some efficient rules satisfy the property but others do not. The critical issue is the selection of an efficient rule when more than one gives rise to the same level of surplus at a given profile. It is possible to design a tie-breaking rule in a manner which destroy the separability property of the efficient rule.

Are there public decision problems which satisfy Property **S**? We examine this question in the case of Examples 2.2.1, 2.2.2 and 2.2.3 in the subsequent section. In the remainder of this section, we make two points. The first is that there is a relatively uninteresting class of public decisions which satisfy Property **S**. The second, is that it is easy to construct non-trivial public decision problems which satisfy Property **S** and which are not covered by Examples 2.2.1, 2.2.2 and 2.2.3 nor by the class described in the previous point.

DEFINITION 2.3.14 The public decision problem $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ satisfies the **dummy property** if the matrix $\mathbf{D}$ has a constant row, that is there exists $j \in \mathbf{N}$ and $a \in \mathbf{R}_+$ such that $d_{ji} = a$ for all $i = 1, 2, \ldots, M$.

PROPOSITION 2.3.2 If a public decision problem satisfies the dummy property, then it satisfies Property **S**.

PROOF: Let Ω be a public decision problem that satisfies the dummy property. Let $j \in \mathbf{N}$ be such that $d_{ji} = a$ for all i. It follows from the definition of an efficient rule that

$$\sum_{l \in \mathbf{N}} d_l^*(\theta)\theta_l \geq \sum_{l \in \mathbf{N}} d_l(\theta)\theta_l$$

for all decision rules d. That is

$$\sum_{l \neq j} d_l^*(\theta)\theta_l + a\theta_j \geq \sum_{l \neq j} d_l(\theta)\theta_l + a\theta_j$$

$$\Rightarrow \sum_{l \neq j} d_l^*(\theta)\theta_l \geq \sum_{l \neq j} d_l(\theta)\theta_l.$$

This implies that the efficient decision rule does not depend on θ_j. That is for all $l \neq j$, there exists functions $f_l : \Theta^{n-1} \to \mathbf{R}$ such that $d_l^*(\theta) = f_l(\theta_{-j})$. This immediately implies that Ω satisfies Property S. ■

The intuition behind the FB implementability of decision problems which satisfy the dummy property is clear. Since the dummy player j gets the same utility from all public decisions, his announcement has no bearing on the choice of an efficient decision. The remaining players can design a mechanism for themselves which induces them to reveal their types truthfully (this is possible because of Proposition 2.3.1) and the aggregate transfer can be paid to the dummy. Since the dummies role is entirely passive, he has no incentive problems.

The dummy property assumption is obviously unsatisfactory. In order to eliminate decision problems which satisfy this assumption from consideration in the rest of the essay, we shall typically require decision problems to satisfy a symmetry property.

DEFINITION 2.3.15 The public decision problem $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ satisfies **symmetry** if the matrix $\mathbf{D}$ has the following property: Let $\delta = (\delta_1, \ldots, \delta_n)$ be a column vector of $\mathbf{N}$ and let $\sigma : \mathbf{N} \to \mathbf{N}$ be a $1-1$ function. Then $\delta_\sigma = (\delta_{\sigma(1)}, \ldots, \delta_{\sigma(n)})$ is also a column vector in $\mathbf{D}$.

The symmetry requires that all permutations of a column vector of the matrix $\mathbf{D}$ are also column vectors in $\mathbf{D}$. Under this assumption, all individuals are treated symmetrically. Note that unlike the usual anonymity assumption, symmetry is a condition on the structure of the problem and not based on any profile considerations. Clearly, symmetry rules out the existence of a dummy player except, of course in the completely trivial case where all players are dummies.

We now provide an example of an FB implementable decision problem. It is not covered by the examples in the next section; nor it satisfies the dummy property.

EXAMPLE 2.3.4 $\Omega = \langle \mathbf{N}, \mathbf{D}, \Theta \rangle$ where $|\mathbf{N}| = 3$ and

$$\mathbf{D} = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Observe that the problem is symmetric. We claim that Ω satisfies Property **S**. In particular, $d_j^*(\theta) = f_{jl}(\theta_j, \theta_m) + f_{jm}(\theta_j, \theta_l)$ for all θ, for all $j \neq l \neq m \neq j$ where

$$f_{jm}(\theta_j, \theta_l) = \begin{cases} 1 & \text{if } \theta_j + \theta_l \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(Here $\mathbf{N} = \{j, l, m\}$). In view of the symmetry of the problem, it suffices to verify by direct computation that the following relationships holds:

(i) $d^*(\theta) = (2,2,2) \Leftrightarrow \theta_1 + \theta_2 \geq 0,\ \theta_1 + \theta_3 \geq 0,\ \theta_2 + \theta_3 \geq 0$

(ii) $d^*(\theta) = (2,1,1) \Leftrightarrow \theta_1 + \theta_2 \geq 0,\ \theta_1 + \theta_3 \geq 0,\ \theta_2 + \theta_3 < 0$

(iii) $d^*(\theta) = (1,1,0) \Leftrightarrow \theta_1 + \theta_2 \geq 0,\ \theta_1 + \theta_3 < 0,\ \theta_2 + \theta_3 < 0$

(iv) $d^*(\theta) = (0,0,0) \Leftrightarrow \theta_1 + \theta_2 < 0,\ \theta_1 + \theta_3 < 0,\ \theta_2 + \theta_3 < 0$.

We omit the details of this calculations which are routine.

Other examples of a similar nature can be constructed. An interesting question is the characterization of all symmetric decision problems which are FB implementable. We do not pursue this matter but examine instead, the less abstract question of whether there are instances of queueing problem, allotment problems and public good problems which are FB implementable.

2.4 Applications

In this section, we consider the issue of FB implementability in the context of the "economic" examples described in section 2.

2.4.1 Queueing

In this subsection, we demonstrate a possibility result. Consider a queueing problem $\Omega^Q = \langle \mathbf{N}, \mathbf{D}^Q, \mathbf{R}_- \rangle$.

THEOREM 2.4.2 Ω^Q is FB implementable provided $|\mathbf{N}| \geq 3$.

PROOF: We shall show that Ω^Q satisfies Property **S**. For all $S \subseteq \mathbf{N}$, $j \in S$ and $\theta \in \Theta$, let

$\sigma(j, S, \theta) = |\{l \in S/\{j\} \mid \theta_l > \theta_j\}| + |\{l \in S/\{j\} \mid \theta_l = \theta_j \text{ and } l < j\}|$

In other words, $\sigma(j, S, \theta)$ is the number of individuals who have valuations greater than θ_j in the profile θ. It also includes those individuals with the same valuation as j but with a lower index. For all $\theta \in \Theta^n$, $j \in \mathbf{N}$, let $d_j^*(\theta) = n - \sigma(j, \mathbf{N}, \theta) - 1$. It is clear that d^* is an efficient decision rule. We now construct the f functions specified in Property **S**. For all $j, l \in \mathbf{N}$ and $\theta \in \Theta^n$, let

$$f_{jl}(\theta_{-l}) = \frac{n - 2 - \sigma(j, \mathbf{N}/\{l\}, \theta_{-l})}{n-2}$$

Observe that $\forall j \in \mathbf{N}$,

$$\sigma(j, \mathbf{N}/\{l\}, \theta_{-l}) = \begin{cases} \sigma(j, \mathbf{N}, \theta) - 1 & \text{if } \theta_l > \theta_j \text{ or } \theta_l = \theta_j \text{ and } l < j \\ \sigma(j, \mathbf{N}, \theta) & \text{otherwise} \end{cases}$$

Therefore,

$$\sum_{l \neq j} f_{jl}(\theta_{-l})$$

$$= (n - \sigma(j, \mathbf{N}, \theta) - 1)\tfrac{(n - \sigma(j, \mathbf{N}, \theta) - 2)}{n-2} + \sigma(j, \mathbf{N}, \theta)\tfrac{(n - \sigma(j, \mathbf{N}, \theta) - 1)}{n-2}$$

$$= \tfrac{(n - \sigma(j, \mathbf{N}, \theta) - 1)}{n-2}(n - \sigma(j, \mathbf{N}, \theta) - 2 + \sigma(j, \mathbf{N}, \theta))$$

$$= (n - \sigma(j, \mathbf{N}, \theta) - 1) = d_j^*(\theta).$$

Therefore, Property **S** is satisfied. ■

It is easy to write an explicit formula for the transfers. For this purpose, it will be convenient to consider the "inverse" of the order σ, that is given

$\theta \in \Theta^n$ suppose μ is a permutation such that

$$\theta_{\mu(1)} \geq \theta_{\mu(2)} \geq \ldots \geq \theta_{\mu(n)}$$

Furthermore, if $\theta_j = \theta_l$ and if $j < l$ then $\sigma(j, \mathbf{N}, \theta) < \sigma(l, \mathbf{N}, \theta)$. Elementary computations yield,

$$t_{\mu(r)}(\theta) = \sum_{q=1}^{r-1} (\frac{q-1}{n-2})\theta_{\mu(q)} - \sum_{q=r+1}^{n} (\frac{n-q}{n-2})\theta_{\mu(q)}$$

$j \in \mathbf{N}^2$. Since the θ_j's are all negative, the first term in the expression on the RHS is a tax while the second term is a receipt. Thus, the $\mu(r)$th person in the queue pays the amount represented by the first expression, to all individuals behind him in the queue and receives the amount represented by the second expression from those who will be served before him.

EXAMPLE 2.4.5 Consider, the queueing model for three individuals i.e. $\Omega^Q = \langle \mathbf{N} = \{1, 2, 3\}, \mathbf{D}^Q, \mathbf{R}_{-} \rangle$. Observe, from the above specification of the transfer that $t_{\mu(1)}(\theta) = -\theta_{\mu(2)}$, $t_{\mu(2)}(\theta) = 0$ and $t_{\mu(3)}(\theta) = \theta_{\mu(2)}$. We start by showing that this transfer is dominant strategy incentive compatible. Consider a state $\theta = (\theta_1, \theta_2, \theta_3)$ where $0 > \theta_1 > \theta_2 > \theta_3$. Here $d^*(\theta) = (d_1^*(\theta) = 2, d_2^*(\theta) = 1, d_3^*(\theta) = 0)$ and $(t_1(\theta) = -\theta_2, t_2(\theta) = 0, t_3(\theta) = \theta_2)$. Individual 3 is of the lowest type. Hence he has the highest cost and consequently receives the service immediately (i.e. $d_3^*(\theta) = 0$) and pays θ_2. If individual 3 announces $\hat{\theta}_3 \in (\theta_2, \theta_1)$, then $d_3^*(\theta_1, \theta_2, \hat{\theta}_3) = 1$ and his transfer is $t_3(\theta_1, \theta_2, \hat{\theta}_3) = 0$. His benefit from this deviation is $\theta_3 - \theta_2 < 0$. Alternatively, if individual 3 announces $\tilde{\theta}_3 \in (\theta_1, 0)$, then $d_3^*(\theta_1, \theta_2, \tilde{\theta}_3) = 2$ and his transfer is $t_3(\theta_1, \theta_2, \tilde{\theta}_3) = -\theta_1$. The benefit from this deviation is $2\theta_3 - \theta_2 - \theta_1 < 0$. Thus, individual 3 has no incentive to deviate. We can apply similar arguments to verify that neither of the individuals 2 nor 1 has any incentive to deviate. Therefore, this transfer scheme is strategyproof, satisfies efficiency and budget balancedness.

[2]Transfer are taken to be of the form given by equation 2.3.5. There are other transfers which will also work.

REMARK 2.4.3 For $|\mathbf{N}| = 4$, $t_{\mu(1)}(\theta) = -\theta_{\mu(2)} - \frac{1}{2}\theta_{\mu(3)}$, $t_{\mu(2)}(\theta) = -\frac{1}{2}\theta_{\mu(3)}$, $t_{\mu(3)}(\theta) = \frac{1}{2}\theta_{\mu(2)}$ and $t_{\mu(4)}(\theta) = \frac{1}{2}\theta_{\mu(2)} + \theta_{\mu(3)}$. In this case as well the individuals with the first two positions compensate the individuals with the last two positions. We can easily verify that this transfer scheme is first best.

REMARK 2.4.4 In the case where $|\mathbf{N}| = 2$, Ω^Q is not FB implementable. Suppose Property **S** were satisfied. It would follow immediately that $d_j^*(\theta) = f_j(\theta_j)$ for some function f_j. On the other hand we know that efficiency implies that $d_j^*(\theta) = 0$ whenever $\theta_j < \theta_l$ and 1 whenever $\theta_j > \theta_l$. We therefore have a contradiction.

2.4.2 Allotment Problems

Our objective in this section is to characterize completely, the class of allotment problems which are FB implementable. Recall that an allotment problem is a triple $\Omega^A = \langle \mathbf{N}, \mathbf{D}^A, \mathbf{R}_+ \rangle$ where $\mathbf{D}^A$ is a matrix whose elements lie between 0 and 1 and whose column sums are all 1. We shall restrict attention to symmetric allotment problems only. In this case the matrix $\mathbf{D}^A$ can be written as $\mathbf{D}^A = ([\alpha^1], [\alpha^2], \ldots, [\alpha^M])$ where α^k, $k = 1, \ldots, M$ is an $n \times 1$ vector whose ith element, denoted by α_i^k, lies between 0 and 1 with $\sum_i \alpha_i^k = 1$. Thus $\mathbf{D}^A$ is the collection of all permutations of the vectors $\alpha^1, \alpha^2, \ldots, \alpha^M$. We will assume, without loss of generality that $\alpha_1^k \geq \alpha_2^k \geq \ldots \geq \alpha_n^k$ for all $k = 1, \ldots, M$. We shall make a further assumption on the class of admissible allotment problems.

DEFINITION 2.4.16 The allotment problem $\Omega^A = \langle \mathbf{N}, \mathbf{D}^A, \mathbf{R}_+ \rangle$ satisfies the **non-trivial range** (NTR) property, if for all $\alpha^k, k = 1, \ldots, M$, there exists an open neighbourhood $N_k \subset \mathbf{R}_+^n$ such that, $\alpha^k \in \Sigma_{\Omega^A}(\theta)$ for all $\theta \in N_k$.

All column vectors in $\mathbf{D}^A$ must be in the efficient correspondence associated with $\mathbf{D}^A$ for values of θ in some open neighbourhood of $\mathbf{R}_+^n$, in order for an allotment problem to satisfy the NTR property. We will illustrate this idea by the following example.

EXAMPLE 2.4.6 Let $\Omega^A = \langle \mathbf{N} = \{1,2,3\}, \mathbf{D}^A = ([\alpha^1],[\alpha^2]), \mathbf{R}_+\rangle$ where $\alpha^1 = (0.50, 0.30, 0.20)$ and $\alpha^2 = (0.45, 0.32, 0.23)$. We claim that this allotment problem does not satisfy the NTR property. To see this let $\theta \in \mathbf{R}^3$ and suppose $\theta_1 \geq \theta_2 \geq \theta_3$. Let $\epsilon_1 = \theta_1 - \theta_2$ and $\epsilon_2 = \theta_2 - \theta_3$. Observe that α^2 is efficient at θ only if $-0.05\theta_1 + 0.02\theta_2 + 0.03\theta_3 \geq 0$ which on simplification yields, $-0.02\epsilon_1 - 0.03(\epsilon_1 + \epsilon_2) \geq 0$. Since $\epsilon_1, \epsilon_2 \geq 0$, this is possible only if $\epsilon_1 = \epsilon_2 = 0$, that is $\theta_1 = \theta_2 = \theta_3$. Clearly, there does not exist an open neighbourhood of valuations where α^2 is efficient.

We introduce some further definitions

DEFINITION 2.4.17 An allotment problem $\Omega^A = \langle \mathbf{N}, \mathbf{D}^A, \mathbf{R}_+\rangle$ is **simple** if $\mathbf{D}^A = ([\alpha])$.

DEFINITION 2.4.18 A simple allotment problem $\Omega^A = \langle \mathbf{N}, \mathbf{D}^A, \mathbf{R}_+\rangle$ (where $\mathbf{D}^A = [\alpha]$) satisfies Property **T** if $\sum_{r=1}^{n}(-1)^{r-1}\binom{n-1}{r-1}\alpha_r = 0$[3].

Thus, a simple allotment problem satisfies Property **T** if the vector α whose permutations comprise the matrix $\mathbf{D}^A$, satisfies a certain combinatorial property. We shall discuss this condition at greater length after the next result.

THEOREM 2.4.3 Let Ω^A be a symmetric allotment problem satisfying the NTR property. Then Ω^A is FB implementable if and only if Ω^A is simple and satisfies Property **T**.

PROOF: We first prove the necessity, i.e. we assume that Ω^A is symmetric, satisfies NTR and is FB implementable. We will show that Ω^A is simple and satisfies Property **T**.

STEP 1: We claim that Ω^A is simple. Suppose it is not, i.e. assume $\mathbf{D}^A = ([\alpha^1], \ldots, [\alpha^M])$, $M > 1$. Let L be the minimum integer such that it is not the case that $\alpha^1_L = \ldots = \alpha^M_L$. Obviously $L < n$. For all integers

[3] $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, i.e. the coefficient of x^r in the expansion $(1+x)^n$.

r, $1 \leq r \leq n-L$, let $\mu(r) \subset \{1,\ldots,M\}$ be defined such that, for all $j \in \mu(r), \sum_{i=1}^{L+r} \alpha_i^j \geq \sum_{i=1}^{L+r} \alpha_i^k$, $k = 1\ldots, M$. Define $\mu(n-L) = \mu(1)$.

LEMMA 2.4.1 For all integers r, $1 \leq r \leq n-L$, there exists $j(r) \in \mu(r)$ such that $\sum_{r=1}^{n-1}(-1)^{r-1}\binom{n-L}{r-1}\alpha_L{}^{j(r)} = 0$.

PROOF: Pick $a, b, c, d \in \mathbf{R}_+$ such that $d > a > b > c > 0$. Define profiles $\theta, \theta' \in \mathbf{R}_+$ as follows
$\theta_L = \theta'_L = a$, $\theta_1 = \ldots \theta_{L-1} = \theta_{L+1} = \ldots = \theta_n = c$, $\theta_1{}' = \ldots, \theta_{L-1}{}' = d$ and $\theta_{L+1}{}' = \ldots = \theta_n{}' = b$. For all $S \subseteq \mathbf{N}/\{L\}$, define $\theta(S) \in \mathbf{R}^n$ in the usual way, i.e. $\theta_L(S) = a$, $\theta_j(S) = \theta_j{}'$ if $j \in S$ and θ_j if $j \notin S$.
Since Ω^A is FB implementable, we know from Theorem 3.2 that it must satisfy Property **S**. Applying Property CA, we have $\sum_{S \subseteq \mathbf{N}/\{L\}} (-1)^{|S|} d_L^*(\theta(S)) = 0$.
Our objective is to compute on the LHS for specific values of a, b, c and d.

Let $i_1, \ldots, i_k < L$ be K distinct integers with $K < L-1$. Let $\mathbf{T}^1 = \{T \mid i_1, \ldots, i_k \in T \text{ and } j \in T/\{i_1, \ldots, i_k\} \Rightarrow j > L\}$. In other words, $\mathbf{T}^1$ is the collection of sets T with the property that the only integer in T less than L is exactly $\{i_1, \ldots, i_k\}$. Observe that for all such sets T, $d_L^*(\theta(T)) = \alpha_{k+1}$ where $\alpha_{k+1}^1 = \ldots = \alpha_{k+1}^M$ by assumption. Moreover, there are 2^{n-k-1} such sets and it is easy to verify that $\sum_{T \in \mathbf{T}^1} (-1)^{|T|} = 0$ (signs "alternate" for different cardinalities). Therefore, $\sum_{T \in \mathbf{T}^1} (-1)^{|T|} d_L^*(\theta(T)) = 0$. Now let $\mathbf{T}^2$ be the collection of all subsets of $\mathbf{N}/\{L\}$ with the property that for all $T \in \mathbf{T}^2$, $1, \ldots, L-1 \in T$. An immediate implication of this argument is that $\sum_{T \notin \mathbf{T}^2} (-1)^{|T|} d_L^*(\theta(T)) = 0$. Therefore, $\sum_{T \in \mathbf{T}^2} (-1)^{|T|} d_L^*(\theta(T)) = 0$ and we now restrict attention to evaluating the expression on the LHS.

Let $T \in \mathbf{T}$ with $|T| = L - r - 1$. Observe that $\theta_L(T)$ is the Lth highest value amongst all valuations; thus $d_L^*(\theta(T)) = \alpha_L^j$ for some $j = 1, \ldots, M$. Now pick $k, p \in \{1, \ldots, M\}$ and note that

$$\sum_{i=1}^n \alpha_i^k - \sum_{i=1}^n \alpha_i^p$$
$$= a(\alpha_L^k - \alpha_L^p) + b[(\alpha_{L+1}^k + \ldots + \alpha_{L+r}^k) - (\alpha_{L+1}^p + \ldots + \alpha_{L+r}^p)]$$

$$+c[(\alpha^k_{L+r+1} + \ldots + \alpha^k_n) - (\alpha^p_{L+r+1} + \ldots + \alpha^p_n)]$$
$$= (\alpha^k_L - \alpha^p_L)(a-b) + (\sum_{i=1}^{L+r} \alpha^k_i - \sum_{i=1}^{L+r} \alpha^p_i)(b-c).$$

By fixing $(a-b)$ and increasing $(b-c)$ sufficiently it is clear that the second term in the expression above can be made to dominate the first, i.e. it is possible to ensure that $d^*(\theta(T)) = \alpha^j \Rightarrow j \in \mu(r)$ where $|T| = L+r-1$. The number of the sets $T \in \mathbf{T}$ with this cardinality is exactly $\binom{n-L+1}{r-1}$. Therefore, $\sum_{T \in \mathbf{T}} (-1)^{|T|} d^*_L(\theta(T)) = \sum_{r=1}^{n-L} (-1)^{r-1} \binom{n-L}{r-1} \alpha_L^{\mu(r)} = 0$. ■

LEMMA 2.4.2 $\alpha_L^{j(1)} = \alpha_L^{j(2)} = \ldots = \alpha_L^{j(n-1)}$.

PROOF: Assume without loss of generality that $L = 1$. In the proof of the previous Lemma, we found $\theta, \theta' \in \mathbf{R}^n$ such that $d^*_1(\theta(S)) = \alpha_1^{j(r)}$ where $|S| = r$. In view of the finiteness of the number of public decisions, we can find an η neighbourhood of θ and θ' such that $d^*_1(\bar{\theta}) = \alpha_1^{j(r)}$ for all $\bar{\theta} \in \mathbf{R}^n$ where $\bar{\theta}_j \in (\theta_j(S) - \eta, \theta_j(S) + \eta)$, $j = 1, 2, \ldots, n$ and $|S| = r$. In other words, we can "perturb" the values of θ and θ' while ensuring that the same public decisions are chosen at $\theta(S)$ for each S. Now we progressively decrease the values of θ'_2 from b to c (these were defined in the proof of the previous Lemma). Let $\theta'_2 : [0,1] \to \mathbf{R}$ be defined as follows: $\theta'_2(t) = c + (b-c)t$ where $t \in [0,1]$. Define $\theta^t(S)$ as follows $\theta^t_2(S) = \theta'_2(t)$ if $2 \in S$ and θ_2 if $2 \notin S$; for all $j \notin 2$, $\theta^t_j(S) = \theta'_j$ if $j \in S$ and θ_j if $j \notin S$. We wish to compute $d^*_1(\theta^t(S))$ for particular values of t.

Let $S = \{2\}$. Observe that for t sufficiently close to 0, $d^*_1(\theta^t(S)) = \alpha_1^{j(1)}$ and for t sufficiently close to 1, $d^*_1(\theta^t(S)) = \alpha_1^{j(2)}$. As t increases from 0 to 1, let $t_1, \ldots, t_k$ be the values[4] of t where $d^*_1(\theta^t(S))$ changes in value from $\alpha_1^{j(1)}$ to $\alpha_1^{k_1}$ to $\alpha_1^{k_2}$ to $\ldots$ to $\alpha_1^{j(2)}$. Let T be any coalition other than $\{2\}$ but which includes 2. By considering a suitable perturbation of θ and θ', (as noted previously), we can ensure that $d^*_1(\theta^t(T))$ does not change value at $t_1, \ldots, t_k$. By applying Property CA for values of t on either side of t_1

[4] The values $t_1, \ldots, t_k$ can be defined as the infimum of appropriate subintervals of $[0,1]$.

and sufficiently close to it, we can deduce that $\alpha_1^{j(1)} = \alpha_1^{k_1}$. Applying this argument repeatedly, we have $\alpha_1^{j(1)} = \alpha_1^{j(2)}$. By virtually identical arguments we can deduce that $\alpha_L^{j(1)} = \alpha_L^{j(2)} = \ldots = \alpha_L^{j(n-1)}$. ∎

An immediate consequence of Lemma 2.4.2 is that there exists an allocation vector, say α^1 with the property that $\sum_{i=1}^{r} \alpha_i^1 \geq \sum_{i=1}^{r} \alpha_i^k$ for all $r = 1, \ldots, n$ and for all $k = 1, \ldots, M$. Suppose $\mathbf{D}^A = ([\alpha^1], [\alpha^2], \ldots, [\alpha^M])$. We claim that Ω^A violates NTR property. To see this, pick α^k where $k \neq 1$. Let $\theta \in \mathbf{R}^n$ and let $\theta_{(1)} \geq \ldots \geq \theta_{(n)}$. Then

$$\sum_{i=1}^{n} \alpha_i^1 \theta_{(i)} - \sum_{i=1}^{n} \alpha_i^k \theta_{(i)}$$
$$= \sum_{i=1}^{n-1} (\alpha_i^1 - \alpha_i^k)(\theta_{(n-1)} - \theta_{(n)}) + \sum_{i=1}^{n-2} (\alpha_i^1 - \alpha_i^k)(\theta_{(n-2)} - \theta_{(n-1)})$$
$$+ \ldots + (\alpha_1^1 - \alpha_1^k)(\theta_{(1)} - \theta_{(2)}) \geq 0.$$

Clearly there does not exist a neighbourhood where α^k is efficient. Therefore NTR is violated and Ω^A must be simple.

Let $\mathbf{D}^A = ([\alpha])$. We show that Ω^A must satisfy Property $\mathbf{T}$. Let $a > b > c \geq 0$. and define $\theta, \theta' \in \mathbf{R}^n$ as follows: $\theta_1 = \theta_1' = b$; $\theta_2 = \ldots = \theta_n = c$; $\theta_2' = \ldots = \theta_n' = a$. Let $S \subseteq \mathbf{N}/\{1\}$ and define $\theta(S) \in \mathbf{R}^n$ in the usual way. Pick S such that $|S| = r - 1$. Clearly, there are $\binom{n-1}{r-1}$ such sets. In the profile $\theta(S)$, there are exactly $r - 1$ valuations greater than b. Therefore, $d_1^*(\theta(S)) = \alpha_r$. Applying Property CA and Theorem 2.3.1, we have

$$\sum_{S \subseteq \mathbf{N}/\{1\}} (-1)^{|S|} d_1^*(\theta(S)) = \sum_{r=1}^{n} (-1)^r \binom{n-1}{r-1} \alpha_r = 0$$

Thus, Property $\mathbf{T}$ is satisfied. This concludes the proof of the necessity part of the Theorem. ∎

Suppose that Ω^A with $\mathbf{D}^A = ([\alpha])$ is a simple allotment problem satisfying Property $\mathbf{T}$. We show that Ω^A is FB implementable. In order to do so, we will show that Ω^A satisfies Property $\mathbf{S}$ and then apply Theorem 2.3.1.

Let $\theta \in \mathbf{R}^n$ and let $j, l \in \mathbf{N}$ with $j \neq l$. Define the set $A_j(\theta_{-l}) = \{m \in \mathbf{N}/\{j, l\} \mid \theta_m > \theta_j \text{ or } \theta_j = \theta_m \text{ and } m < j\}$. The functions $f_{jl} : \Theta^{n-1} \rightarrow \mathbf{R}$ are constructed as follows:

for all $\theta \in \mathbf{R}^n$, $f_{jl}(\theta_{-l}) = \sum_{p=1}^{r}(-1)^{r-p}[\frac{(r-1)!(n-r-1)!}{(n-p)!(p-1)!}]\alpha_p = z_r$ where $|A_j(\theta_{-l})| = r$, $r = 1, \ldots, n-1$.

We claim that there exists an efficient rule d^* induced by Ω^A such that, for all $\theta \in \mathbf{R}^n$ and $j \in \mathbf{N}$, $d_j^*(\theta) = \sum_{l \neq j} f_{jl}(\theta_{-l})$. Let d^* be the following rule: for all $j \in \mathbf{N}$ and $\theta \in \mathbf{R}^n$, $d_j^*(\theta) = \alpha_r$ where $|\{m \in \mathbf{N}/\{j\} \mid \theta_m > \theta_j$ or $\theta_m = \theta_j$ and $m < j\}| = r - 1$ and $r = 1, 2, \ldots, n$. We consider two cases.

Case 1: $\theta \in \mathbf{R}^n$ is such that $d_j^*(\theta) = \alpha_r$, $r = 1, 2, \ldots, n-1$. Let $l \in [\{m \in \mathbf{N} \mid \theta_m > \theta_j$ or $\theta_m = \theta_j$ and $m < j\}]$. Observe that $|A_j(\theta_{-l})| = r - 1$. For l in the complement set, $|A_j(\theta_{-l})| = r$. Therefore,

$$\begin{aligned}
&\sum_{l \neq j} f_{jl}(\theta_{-l}) \\
&= (r-1)\sum_{p=1}^{r-1}[(-1)^{r-p-1}\frac{(r-2)!(n-r)!}{(p-1)!(n-p)!}\alpha_p] \\
&+(n-r)\sum_{p=1}^{r}(-1)^{r-p}[\frac{(r-1)!(n-r-1)!}{(p-1)!(n-p)!}\alpha_p] \\
&= \sum_{p=1}^{r-1}[\{(-1)^{r-p-1} + (-1)^{r-p}\}[\frac{(r-1)!(n-r)!}{(p-1)!(n-p)!}\alpha_p] + \alpha_r \\
&= \alpha_r.
\end{aligned}$$

Case 2: $\theta \in \mathbf{R}^n$ is such that $d_j^*(\theta) = \alpha_n$. Therefore,

$$\begin{aligned}
&\sum_{l \neq j} f_{jl}(\theta_{-l}) \\
&= (n-1)\sum_{p=1}^{n-1}[(-1)^{n-p-1}\frac{(n-2)!}{(p-1)!(n-p)!}\alpha_p] \\
&= \sum_{p=1}^{n-1}(-1)^{n-p-1}\binom{n-1}{n-p}\alpha_p \\
&= \sum_{p=1}^{n-1}(-1)^{n-p-1}\binom{n-1}{p-1}\alpha_p \\
&= \alpha_n \text{ (from Property } \mathbf{T}\text{)}.
\end{aligned}$$

Thus, Property **S** is satisfied and Ω^A is FB implementable. ■

REMARK 2.4.5 We can use the function f_{jl} and equation 2.3.5 to compute a set of balanced transfers. Without loss of generality, let $d_j^*(\theta) = \alpha_r$. This means, there are exactly $r - 1$ individuals ahead of individual j. Hence $n - r$

individuals are after j under the efficient decision rule. Thus, in state θ, $\theta_{(1)} \geq \ldots \geq \theta_{(r-1)} \geq \theta_{(r)} = \theta_j \geq \theta_{(r+1)} \geq \ldots \geq \theta_{(n)}$. Now it is easy to check that

$t_{(r)}(\theta) = \sum_{q \neq r} \alpha_q \theta_{(q)} - (n-1)\{\sum_{q=1}^{r-1} z_q \theta_{(q)} + \sum_{q=r+1}^{n} z_{q-1}\theta_{(q)}\}$, $r \in \{1, \ldots, n\}$. We can verify that the transfers add up to zero.

EXAMPLE 2.4.7 Consider, the simple allotment problem for three individuals $\Omega^A = \langle \mathbf{N} = \{1,2,3\}, \bar{\mathbf{D}}^A, \mathbf{R}_+ \rangle$, where $\bar{\mathbf{D}}^A = ([\bar{\alpha}_1 = \frac{2}{3}, \bar{\alpha}_2 = \frac{1}{3}, \bar{\alpha}_3 = 0])$. Thus, the decision matrix is of the form

$$\bar{\mathbf{D}}^A = \begin{pmatrix} 2/3 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 2/3 & 1/3 \\ 0 & 1/3 & 0 & 2/3 & 1/3 & 2/3 \end{pmatrix}$$

where the columns are the decisions. Note that $\bar{\alpha}_1 - 2\bar{\alpha}_2 + \bar{\alpha}_3 = 0$ and $(z_1 = \frac{1}{3}, z_2 = 0)$. This simple allotment problem can be thought of as an auction with three individuals, where the bidder with the highest valuation gets $\frac{2}{3}$ of the object. The bidder with the second highest valuation receives the remaining $\frac{1}{3}$ of that object. The person with lowest bid gets nothing. Observe, from the above specification of the transfer that $t_{(1)}(\theta) = -\frac{1}{3}\theta_{(2)}$, $t_{(2)}(\theta) = 0$ and $t_{(3)}(\theta) = \frac{1}{3}\theta_{(2)}$. We start by showing, that this transfer is dominant strategy incentive compatible. For proof, we consider a state $\theta = (\theta_1, \theta_2, \theta_3)$, where $\theta_1 > \theta_2 > \theta_3 > 0$. Here $d^*(\theta) = (d_1^*(\theta) = \frac{2}{3}, d_2^*(\theta) = \frac{1}{3}, d_3^*(\theta) = 0)$ and $(t_1(\theta) = -\frac{1}{3}\theta_2, t_2(\theta) = 0, t_3(\theta) = \frac{1}{3}\theta_2)$. Individual 3 is of the lowest type in state θ and receives a transfer of $\frac{1}{3}\theta_2$. If individual 3 announces $\hat{\theta}_3 \in (\theta_2, \theta_1)$, then $d_3^*(\theta_1, \theta_2, \hat{\theta}_3) = \frac{1}{3}$ and his transfer is $t_3(\theta_1, \theta_2, \hat{\theta}_3) = 0$. His benefit from this deviation is $\frac{1}{3}(\theta_3 - \theta_2) < 0$. Alternatively, if individual 3 announces $\tilde{\theta}_3 > \theta_1$, then $d_3^*(\theta_1, \theta_2, \tilde{\theta}_3) = \frac{2}{3}$ and his transfer is $t_3(\theta_1, \theta_2, \tilde{\theta}_3) = -\frac{1}{3}\theta_1$. The benefit from this deviation is $\frac{1}{3}(2\theta_3 - \theta_2 - \theta_1) < 0$. Thus individual 3 has no incentive to deviate. We can apply similar arguments to check that individuals, 2 nor 1 has any incentive to deviate. Therefore, this transfer scheme is strategyproof, satisfies efficiency of decision and budget balancedness.

For reasons, similar to the queueing model, this transfer scheme is strategyproof. Under this transfer scheme, the individual with the highest valuation pays a fraction of the second highest valuation to the person with lowest valuation. The fraction to be paid is the incremental benefit, in terms of decision, that the individual with highest valuation enjoys over and above the individual with the second highest valuation. However, in the allotment case, if the payment were to be as in the second price auction, to an outsider, then truth-telling would not be a dominant strategy in all states. An incentive problem for the second and third position would remain. If this money is paid to the individual with the lowest valuation, then the transfer scheme becomes incentive compatible. The reason is that the incremental benefit of the second individual over the third individual is the same as that of the first individual, over the second individual. So, the person getting the best decision ($\frac{2}{3}$) compensates the person getting the worst decision (0) by paying $\frac{1}{3}$ of the second highest valuation. This fraction $\frac{1}{3}$ is the common difference between any two neighbouring decisions. Once again, it is this spread of incentives over different decisions that help in achieving first best in this simple allotment problem.

For $|\mathbf{N}| = 3$, an allotment problem satisfying $\alpha_1 - \alpha_2 = \alpha_2 - \alpha_3$ is both necessary and sufficient for FB implementability. The condition means that the difference in neighbouring decisions is constant. This condition is sufficient though not necessary for $|\mathbf{N}| > 3$. The reason for this follows from the fact that Property **T** for $|\mathbf{N}| > 3$ is weaker than the constancy of neighbouring decisions. The next example for $|\mathbf{N}| = 4$ highlights this point.

EXAMPLE 2.4.8 Consider, the simple allotment problem for four individuals $\hat{\Omega}^A = \langle \mathbf{N} = \{1, 2, 3, 4\}, \hat{\mathbf{D}}^A, \mathbf{R}_+ \rangle$, where $\hat{\mathbf{D}}^A = ([\hat{\alpha}_1 = \frac{3}{4}, \hat{\alpha}_2 = \frac{1}{4}, \hat{\alpha}_3 = 0, \hat{\alpha}_4 = 0])$. Note that $\hat{\alpha}_1 - 3\hat{\alpha}_2 + 3\hat{\alpha}_3 - \hat{\alpha}_4 = 0$ and $(z_1 = \frac{1}{4}, z_2 = 0, z_3 = 0)$. Begin with the observation that $\frac{1}{2} = \hat{\alpha}_1 - \hat{\alpha}_2 \neq \hat{\alpha}_2 - \hat{\alpha}_3 = \frac{1}{4}$. Thus, constancy of neighbouring decisions is not necessary for Property **T**. The transfers are $t_{(1)}(\theta) = -\frac{1}{2}\theta_{(2)}$, $t_{(2)}(\theta) = 0$, $t_{(3)}(\theta) = \frac{1}{4}\theta_{(2)}$ and $t_{(4)}(\theta) = \frac{1}{4}\theta_{(2)}$. We can verify, by considering deviations, that this budget balanced transfer is dominant strategy incentive compatible. In this allotment problem, the highest type individual pays a fraction of the second highest valuation. This amount split equally as reward between the two individuals who get nothing. This fraction, that is to be paid equally, is the difference in the best and the second best decisions i.e. $\hat{\alpha}_1 - \hat{\alpha}_2$. Observe that $\hat{\alpha}_2 - \hat{\alpha}_3 = \hat{\alpha}_2 - \hat{\alpha}_4 = \frac{1}{4} = \frac{1}{2}\{\hat{\alpha}_1 - \hat{\alpha}_2\}$. So, the two individuals not getting the good receive a fraction $\frac{1}{2}\{\hat{\alpha}_1 - \hat{\alpha}_2\}$ of the second highest valuation, as compensation for not getting $\hat{\alpha}_2$ amount of the good.

Thus, budget balancedness of allotment problems depends crucially on the way the good is split. The first best implementable split of the good is captured by Property **T**. We make several other observations which are elementary consequences of Theorem 2.3.1.
(1) For $|\mathbf{N}| = 2$, the FB implementable simple allotment problem means, the decision matrix $\mathbf{D}^A$ is of the form $\mathbf{D}^A = ([\alpha] = [\alpha_1, \alpha_2])$, where $\alpha_1 + \alpha_2 = 1$. Moreover, Property **T** implies that $\alpha_1 - \alpha_2 = 0$. Solving for α_1 and α_2, we get $\alpha_1 = \alpha_2 = \frac{1}{2}$. Thus, the only FB implementable allotment problem is the one where $\mathbf{D}^A = ([\alpha] = [\frac{1}{2}, \frac{1}{2}])$. Observe that this problem is trivially FB implementable.
(2) In the case where $|\mathbf{N}| = 3$, the allotment vector must take the form $(\frac{1}{3} + \epsilon, \frac{1}{3}, \frac{1}{3} - \epsilon)$, where $0 \leq \epsilon \leq \frac{1}{3}$. So, for each ϵ, we have FB implementable simple allotment problem satisfying Property **T** with decision matrix of the form $\mathbf{D}^A_\epsilon = ([\alpha(\epsilon)] = [\frac{1}{3} + \epsilon, \frac{1}{3}, \frac{1}{3} - \epsilon])$. Observe that for $\epsilon = 0$, we have a decision matrix that is trivially FB implementable. For all $0 < \epsilon \leq \frac{1}{3}$, we have non-trivial FB implementable allotment problems.
(3) In the case where $|\mathbf{N}| = 4$, all vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ and $\alpha_1 - 3\alpha_2 + 3\alpha_3 - \alpha_4 = 0$ are FB implementable. The decision matrix is of the form $\mathbf{D}^A(\epsilon, \delta) = ([\alpha(\epsilon, \delta)] = [\epsilon, \frac{1}{2} - \frac{1}{3}\epsilon - \frac{2}{3}\delta, \frac{1}{2} - \frac{2}{3}\epsilon - \frac{1}{3}\delta, \delta])$, where $0 \leq \delta \leq \frac{1}{4}$ and for each δ we can define a feasible range of ϵ i.e. $\epsilon(\delta)$. For example, $\epsilon(0) \in [\frac{3}{8}, \frac{3}{4}]$, $\epsilon(\frac{1}{8}) \in [\frac{5}{16}, \frac{1}{2}]$ and $\epsilon(\frac{1}{4}) = \{\frac{1}{4}\}$. Observe that with $\delta = \epsilon(\frac{1}{4}) = \frac{1}{4}$, $\mathbf{D}^A(\frac{1}{4}, \frac{1}{4}) = ([\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}])$

is trivially FB implementable. For all $\delta \in [0, \frac{1}{4})$, we have non-trivial FB implementable allotment problems.
(4) $\mathbf{D}^A = ([2(n-1)/n(n-1), 2(n-2)/n(n-1), \ldots, 2/n(n-1), 0])$ for all $|\mathbf{N}| > 2$ is FB implementable. It is obvious that, $\mathbf{D}^A = ([1/n, 1/n, \ldots, 1/n])$ is trivially FB implementable for all $|\mathbf{N}| \geq 2$.
(5) The auction problem where $\mathbf{D}^A = ([1, 0, \ldots, 0])$ is not FB implementable.
(6) Finally, we note that the assumption of the NTR property is critical to our characterization result. Let $\mathbf{D}^A = ([\alpha])$ be a FB implementable allotment problem. Let β be another allotment vector with the property that α majorizes β (i.e. $\sum_{i=1}^{r} \alpha_i \geq \sum_{i=1}^{r} \beta_i$ for all $r = 1, \ldots n-1$). Now, consider the augmented problem $\bar{\mathbf{D}}^A = ([\alpha], [\beta])$. It is not difficult to show that $\bar{\mathbf{D}}^A$ will be FB implementable as well.

2.4.3 Public Good Problems

Recall that a public good decision is one where $\Theta = \mathbf{R}$ and the decision matrix comprises of the elements which are either 0 or 1. Since we shall impose the assumption of symmetry, the decision matrices under consideration can be expressed as a collection of vectors with different column sums. In other words, we can represent the typical decision matrix $\mathbf{D}^G$ as $\mathbf{D}^G = ([k_1], [k_2], \ldots, [k_s])$ where $k_1, k_2, \ldots, k_s$ are distinct integers lying between 0 and n and $[k_p]$ denotes the set of column vectors whose elements are either 0 or 1 and whose sum is k_p. Here, k_p denotes the number of individuals included; that is the vectors in the set $[k_p]$ represent situations where exactly $n - k_p$ individuals are excluded from the public good. Let $\mathbf{D}^g = \{[0], [1], \ldots, [n-1], [n]\}$.

DEFINITION 2.4.19 The public good decision problem Ω^g allows **full excludability** if $\Omega^g = \langle \mathbf{N}, \mathbf{D}^g, \mathbf{R} \rangle$

In the class of public good decision problems, that we consider, not all subsets of individuals may be excludable. In the extreme case of the classical pure public good problem the only subset which can be excluded is the entire set $\mathbf{N}$. The polar opposite is the case of full excludability where any subset can be excluded. The next result states that this is the only symmetric decision problem which is FB implementable.

THEOREM 2.4.4 Let Ω^G be a symmetric public good decision problem. If Ω^G is FB implementable, it must allow full excludability. Conversely, if it allows full excludability, then it is FB implementable.

PROOF: We first establish that if Ω^G is symmetric and FB implementable, then $\Omega^G = \Omega^g$. In order to do so, we record a mathematical Lemma from Walker [41].

The Cubical Array Lemma: Let $F : \Theta^n \to \mathbf{R}$. Let $S \subset \mathbf{N}$ and let $\theta, \theta' \in \Theta^n$. Define $\theta(S) \in \Theta^n$ where $\theta_j(S) = \theta_j$ if $j \in S$ and $\theta_j(S) = \theta_j'$ if $j \notin S$. Then F can be written as $F(\theta) = \sum_{j \in \mathbf{N}} h_j(\theta_{-j})$ for all $\theta \in \Theta^n$ for some functions $h_j : \Theta^{n-1} \to \mathbf{R}$, if and only if $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} F(\theta(S)) = 0$ for all $\theta, \theta' \in \Theta^n$.

We omit the proof of this Lemma which is fairly straightforward. It is clear that if the function F has a separable form, then it must satisfy an appropriate restriction on the nth order cross-partial derivative. The condition in the Lemma is the analogous of this derivative for finite changes. That this condition is also sufficient for the separability of F, is also quite intuitive.

Suppose $\Omega^G \neq \Omega^g$. Assume w.l.o.g. that $[k] \notin \mathbf{D}^G$ where k is some integer lying between 0 and n.

Step 1: $k \neq 0, n$.

Suppose $k = n$ (The case where $k = 0$ can be treated analogously). Let t be the greatest integer less than n such that $[t] \in \mathbf{D}^G$. Let θ be such that $\theta_j > 0$, $j = 1, 2, \ldots, n$. It is clear that $d_j^*(\theta) = 1$ if and only if θ_j is one of the t highest values in the set $\{\theta_1, \ldots, \theta_n\}$. We shall show by means of the Cubical Array Lemma that this function cannot satisfy Property S.

We first consider the case where n is odd. Let $m = (n+1)/2$. Let $\theta, \theta' \in \Theta^n$ be such that

$$\theta_1 > \theta'_n > \ldots > \theta'_{m+1} > \theta_m > \theta_{m+1} > \theta'_{m-1} > \ldots > \theta_n > \theta'_1 > 0$$

For all $S \subset \mathbf{N}$ and $j \neq m$, let $\theta_j(S) \in \Theta$ be such that $\theta_j(S) = \theta'_j$ if $j \in S$ and θ_j if $j \notin S$ and $\theta_m(S) = \theta_m = \theta_m{}'$. Our objective is to compute $\sum_{S \subseteq \mathbf{N}/\{m\}} (-1)^{|S|} d^*_m(\theta(S))$. Since d^*_j is either 0 or 1 for all j, it follows from the earlier observation that it is sufficient to consider only those sets S where θ_m is greater than at least $n-t+1$ values in $\theta_{-m}(S)$. Let $l \in [n-t+1, n-1]$. Suppose that θ_m is greater than exactly l values in $\theta_{-m}(S)$. Observe that there are $\binom{n-1}{l}$ such sets in S. Let $|\{j \in \{1, \ldots, m-1\} \mid \theta_m > \theta_j(S)\}| = q$ and $|\{j \in \{m+1, \ldots, n\} \mid \theta_m > \theta_j(S)\}| = r$, so that $q + r = l$. The profile θ and θ' have been picked in a manner such that $|S| = q + m + r - 1$. Observe that l is even if and only if $q + r$ is even. Therefore $(-1)^{|S|} = (-1)^{m+l-1}$. Hence

$$\sum_{S \subseteq n-\{m\}} (-1)^{|S|} d^*_m(\theta(S))$$
$$= \sum_{l=t-1}^{n-1} (-1)^{m+l-1} \binom{n-1}{l}$$
$$= (-1)^{m-1} \sum_{l=n-t+1}^{n-1} (-1)^l \binom{n-1}{l}$$
$$= (-1)^{m-1}(-1)^{n-t}\binom{n-2}{n-t} \text{ (by a well-known identity)}$$
$$\neq 0 \text{ (since } n > t).$$

In the case where n is even, we can take $m = n/2 + 1$ and by constructing similar profiles, obtain an almost identical expression as when n is odd. We omit the details.

Step 2: $k \notin \{1, \ldots, n-1\}$.

Suppose not, that is let $[k] \notin \mathbf{D}^G$. In view of Step 1 and the assumption that $\mathbf{D}^G$ has at least two columns, it follows that there exist integers t, r such that $0 \leq t \leq k \leq r \leq n$ and $[t], [r] \in \mathbf{D}^G$. In fact we can assume w.l.o.g that t and r are the greatest and smallest integers less than and greater than k respectively which have this property. We shall also assume that either $t \neq 0$

or $r \neq n$. In case, $t = 0$ and $r = n$, we have the standard pure public goods model where the impossibility result is well-known. We consider the case where $r \neq n$. The case where $t \neq 0$ is the symmetric analogue. Let $L = r - t$ (observe that $L \geq 2$) and pick $\alpha > 0$. Let θ and $\hat{\theta}$ satisfy the following:
$\theta_1 = \theta_2 = \ldots = \theta_{r-1} = \alpha$
$\theta_r = \ldots = \theta_{n-1} = (5/4 - L)\alpha$
$\theta_n = (5/2 - 2L)\alpha$
$\hat{\theta}_1 = \hat{\theta}_2 = \ldots = \hat{\theta}_n = 2\alpha$
$\hat{\theta}_r = \ldots = \hat{\theta}_{n-1} = (5 - 4L)\alpha$.
Observe that $0 > \theta_r > \theta_n > \hat{\theta}_r$. Let $S \subseteq \mathbf{N}$ and for all $j \neq n$, let $\theta_j(S) = \hat{\theta}_j$ if $j \in S$ and θ_j if $j \notin S$. Note that $\theta_n(S) = \theta_n = \theta_n'$. We now compute $d_n^*(\theta(S))$ for various $S \subseteq \mathbf{N}$.
(i) Since $[r] \in \mathbf{D}^G$ and only $r - 1$ valuations are positive it will always be efficient to exclude n unless θ_n is one of the r highest values in $(\theta(S))$, that is $d_n^*(\theta(S)) = 0$ unless $\{r+1, \ldots, n-1\} \subset S$.
(ii) Suppose $j \notin S$ where $j \in \{1, \ldots, r-1\}$. Then, the aggregate surplus obtained by selecting the appropriate element of $[t]$ is $2t\alpha$. The maximum surplus from selecting the appropriate element of $[r]$ is $(r-2)2\alpha + \alpha + (5/2 - 2L)\alpha$. But

$$2t\alpha - (r-2)2\alpha - \alpha - (5/2 - 2L)\alpha = 2\alpha(2 - L) - \alpha - (5/2 - 2L)\alpha = \alpha/2 > 0.$$

(iii) Suppose $|S| = \mathbf{N}/\{n\}$. Then it is efficient to include n. The aggregate surplus from including n is $(r-1)2\alpha + (5/2 - 2L)\alpha$ and from excluding him is $2t\alpha$. But

$$(r-1)2\alpha + (5/2 - 2L)\alpha - 2t\alpha = 2\alpha(L-1) + (5/2 - 2L)\alpha = \alpha/2 > 0.$$

Arguments (i)-(iii) establish that $d_n^*(\theta(S)) = 1$ if and only if $|S| = |\mathbf{N}/\{n\}|$. Thus,

$$\sum_{S \subseteq \mathbf{N}/\{n\}} (-1)^{|S|} d_n^*(\theta(S)) = (-1)^{n-1} \neq 0$$

Applying the Cubical Array Lemma, it follows that $d_n^*(\theta(S))$ does not satisfy Property **S**, that is Ω^G is not FB implementable. This establishes that for FB implementability we need $\Omega^G = \Omega^g$.

The sufficiency part of the Theorem is straightforward. It is easy to verify that an efficient decision rule associated with Ω^g is given by :
$\forall j \in \mathbf{N}$,

$$d_j^*(\theta) = \begin{cases} 1 & \text{if } \theta_j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

But d_j^* trivially satisfies Property **S**. In fact d^* induces truth-telling in dominant strategies without transfers. ■

In the case where full excludability is allowed, the public good acquires the characteristic of a private good. Moreover, since there are only two levels at which this good can be provided, it follows almost immediately that an efficient rule is strategyproof. The converse is non-trivial because we allow for all possible "intermediate levels" of excludability. There may be technological reasons which may make it infeasible to exclude or include some sets of individuals. We have briefly alluded to some of the situations that we have in mind, in Section 2. Once decision problems of this nature are considered, efficient rules become more complicated and the lack of FB implementability correspondingly harder to establish.

2.5 Individual Rationality

In this section consider the issue of individual rationality. In particular we try to identify the sub-class of decision problems that are individually rational within the class of FB implementable decision problems discussed earlier.

DEFINITION 2.5.20 A mechanism $\mathbf{M} = \langle d, \mathbf{t} \rangle$ is **individually rational** for a decision problem Ω, if for all $j \in \mathbf{N}$ and for all $\theta \in \Theta^n$.

$$U_j(d_j(\theta), t_j(\theta), \theta_j) \geq 0$$

We are assuming that all individuals have an outside option which is independent of their valuation and gives them a utility of zero. An individual who does not get an ex-post utility of at least zero will refuse to participate in the mechanism. We say that a decision is FB* **implementable** if it is FB implementable by a mechanism which satisfies individual rationality. We now examine whether the queueing, allotment and public goods problems are FB* implementable.

THEOREM 2.5.5 For $|N| \geq 3$, Ω^Q is not FB* implementable.

PROOF: We know that Ω^Q is FB implementable only for $|N| \geq 3$. Recall that the class of transfer that FB implements any decision problem Ω is of the form

$$t_j(\theta) = \sum_{l \neq j} \{d_l^*(\theta) - (n-1) f_{lj}(\theta_{-j})\}\theta_l + \gamma_j(\theta_{-j})$$

where $\sum_{j=1}^{n} \gamma_j(\theta_{-j}) = 0$. Our first step is to show that if the transfer is of the form

$$t_j(\theta) = \sum_{l \neq j} \{d_l^*(\theta) - (n-1) f_{lj}(\theta_{-j})\}\theta_l \qquad (2.5.6)$$

then individual rationality is violated.

Using 2.5.6 the transfer in state θ is of the form

$$t_{\mu(r)}(\theta) = \sum_{q=1}^{r-1} (\frac{q-1}{n-2})\theta_{\mu(q)} - \sum_{q=r+1}^{n} (\frac{n-q}{n-2})\theta_{\mu(q)} \qquad (2.5.7)$$

where $\theta_{\mu(1)} \geq \ldots \theta_{\mu(n)}$. We first show that the utility of the highest type individual in any state θ is the highest. For this we define

$\Delta^Q(r, r+1; \theta) =$

$U_{\mu(r)}(d^*_{\mu(r)}(\theta), t_{\mu(r)}(\theta), \theta_{\mu(r)}) - U_{\mu(r+1)}(d^*_{\mu(r+1)}(\theta), t_{\mu(r+1)}(\theta), \theta_{\mu(r+1)})$.

After substituting the values of $d^*_{\mu(.)}(\theta)$ and $t_{\mu(.)}(\theta)$ from 2.5.7 and simplifying we get

$$\Delta^Q(r, r+1; \theta) = \frac{(n-1)(n-r-1)}{(n-2)}(\theta_{\mu(r)} - \theta_{\mu(r+1)}) \qquad (2.5.8)$$

Since $\theta_{\mu(r)} \geq \theta_{\mu(r+1)}$, from 2.5.8 it is easy to check that $\Delta^Q(r, r+1; \theta) \geq 0$ for all $|\mathbf{N}| \geq 3$. Thus,

$\Delta^Q(1, r; \theta) = U_{\mu(1)}(d^*_{\mu(1)}(\theta), t_{\mu(1)}(\theta), \theta_{\mu(1)}) - U_{\mu(r)}(d^*_{\mu(r)}(\theta), t_{\mu(r)}(\theta), \theta_{\mu(r)}) \geq 0$ for all $r \neq 1$.

Let state θ be such that $\theta_{\mu(1)} = \theta_{\mu(2)} = \ldots = \theta_{\mu(n)} < 0$. The utility of highest type individual is

$U_{\mu(1)}(d^*_{\mu(1)}(\theta), t_{\mu(1)}(\theta), \theta_{\mu(1)}) = \frac{n-1}{2}\theta_{\mu(1)} < 0$.

In view of $\Delta^Q(1, r; \theta) \geq 0$, this proves that individual rationality is violated when transfers are of the form 2.5.6.

Now we prove that Ω^Q is not FB* implementable. The additional term in the general transfer function is $(\gamma_1(\theta_{-1}), \ldots, \gamma_n(\theta_{-n}))$ where $\sum_{j=1}^{n} \gamma_j(\theta_{-j}) = 0$. Now if there is some $m \in \mathbf{N}$ such that $\gamma_m(\theta_{-m}) > 0$, then there must exist at least one $l \in \mathbf{n}$ such that $\gamma_l(\theta_{-l}) < 0$ since $\sum_{j=1}^{n} \gamma_j(\theta_{-j}) = 0$. Since the utility of all individuals with the above specification of the state θ is negative without $(\gamma_1(\theta_{-1}), \ldots, \gamma_n(\theta_{-n}))$, the lth individual will get a negative payoff. This proves that Ω^Q is not FB* implementable. ■

We now turn to the allotment problems. We will show that there is a sub-class of FB implementable allotment problems which are also FB* implementable. Recall that for each FB implementable allotment problem Ω^A with decision matrix $\mathbf{D}^A = [(\alpha)]$, there exist an $(n-1) \times 1$ vector $z = (z_1, \ldots, z_{n-1})$, such that for all $r = 1, \ldots, n-1$, $\alpha_r = (n-r)z_r + (r-1)z_{r-1}$. The property that determines the sufficient sub-class of FB* implementable allotment problems is a condition on this vector z.

DEFINITION 2.5.21 For $|\mathbf{N}| \geq 3$ a FB implementable allotment problem Ω^A satisfies Property **Z** if $\hat{z}(r) = \sum_{q=1}^{r} z_q - r z_{r+1} \geq 0$[5] for all $r \in \{1, 2, \ldots, n-2\}$.

THEOREM 2.5.6 For $|\mathbf{N}| \geq 3$, if Ω^A satisfies Property **Z** then it is FB* implementable.

[5]Recall that $z_q = \sum_{p=1}^{q} (-1)^{q-p}[\frac{(q-1)!(n-q-1)!}{(n-p)!(p-1)!}]\alpha_p$ for all $q \in \{1, \ldots n-1\}$.

PROOF: We assume transfers to be

$$t_{(r)}(\theta) = \sum_{q \neq r} \alpha_q \theta_{(q)} - (n-1)\{\sum_{q=1}^{r-1} z_q \theta_{(q)} + \sum_{q=r+1}^{n} z_{q-1}\theta_{(q)}\} \tag{2.5.9}$$

where $\theta_{(1)} \geq \ldots \geq \theta_{(n)}$. We first show that $U_{(n)}(d^*{}_{(n)}(\theta), t_{(n)}(\theta), \theta_{(n)}) \geq 0$ is sufficient for FB* implementability of Ω^A.

To prove this step we define

$$\Delta^A(r, r+1; \theta) = U_{(r)}(d^*_{(r)}(\theta), t_{(r)}(\theta), \theta_{(r)}) - U_{(r+1)}(d^*_{(r+1)}(\theta), t_{(r+1)}(\theta), \theta_{(r+1)}).$$

After substituting for $d^*_{(.)}(\theta)$ and $t_{(.)}(\theta)$ and simplifying, we have $\Delta^A(r, r+1; \theta) = (n-1)z_r(\theta_{(r)} - \theta_{(r+1)})$. Since $z_r \geq 0$ and $\theta_{(r)} \geq \theta_{(r+1)}$ for all $\{1, \ldots, n\}$ we have $\Delta^A(r, r+1; \theta) \geq 0$ for all θ. This means that $U_{(n)}(d^*_{(n)}(\theta), t_{(n)}(\theta), \theta_{(n)}) \geq 0$ is sufficient for FB* implementability. We now show that this holds.

$$U_{(n)}(d^*_{(n)}(\theta), t_{(n)}(\theta), \theta_{(n)}) = \sum_{q=1}^{n-1}(\alpha_q - (n-1)z_q)\theta_{(q)} + \alpha_n \theta_{(n)} \tag{2.5.10}$$

After simplification, the RHS of 2.5.10 reduces to

$$\sum_{k=2}^{n-2} \hat{z}(r)(\theta_{(r)} - \theta_{(r+1)}) + (n-1)\bar{z}(\theta_{(n-1)} - \theta_{(n)}) + (n-1)\bar{z}\theta_{(n)}$$

where $\bar{z} = \sum_{k=1}^{n-1} z_k/(n-1) \geq 0$.

Given Property **Z** holds, $\{\theta_{(r)} - \theta_{(r+1)} \geq 0\}_{r=1}^{n-1}$ and $\theta_{(n)} \geq 0$ it follows that $U_{(n)}(d^*_{(n)}(\theta), t_{(n)}(\theta), \theta_{(n)}) \geq 0$ for all θ. This proves the Theorem. ∎

We now provide some applications of this result.

PROPOSITION 2.5.3 In the case where $|\mathbf{N}| = \{3, 4, 5\}$, all FB implementable allotment problems satisfy Property **Z** and are therefore FB* implementable.

PROOF: Given a FB implementable allotment problem Ω^A with $\mathbf{D}^A = ([\alpha])$ and $z_r = \sum_{p=1}^{r}(-1)^{r-p}[\frac{(r-1)!(n-r-1)!}{(n-p)!(p-1)!}]\alpha_p$, Property **Z** requires that $\hat{z}(r) = \sum_{q=1}^{r} z_q - rz_{r+1} \geq 0$ for all $r \in \{1, 2, \ldots n-2\}$. To prove the Proposition we consider each case (i.e. $|\mathbf{N}| = 3$, $|\mathbf{N}| = 4$ and $|\mathbf{N}| = 5$) separately.

Case 1:$|\mathbf{N}| = 3$

For a FB implementable allotment problem Ω^A with $\mathbf{D}^A = ([\alpha_1, \alpha_2, \alpha_3])$, $z_1 = \frac{\alpha_1}{2}$ and $z_2 = \frac{\alpha_3}{2}$. Given $\alpha_1 \geq \alpha_2 \geq \alpha_3$ it follows immediately that $z_1 - z_2 = \frac{\alpha_1 - \alpha_2}{2} \geq 0$. Hence for $|\mathbf{N}| = 3$, all FB implementable allotment problems satisfy Property **Z**.

Case 2:$|\mathbf{N}| = 4$

For a FB implementable allotment problem Ω^A with $\mathbf{D}^A = ([\alpha_1, \alpha_2, \alpha_3, \alpha_4])$, $z_1 = \frac{\alpha_1}{3}$, $z_2 = \frac{\alpha_2}{2} - \frac{\alpha_1}{6}$ and $z_3 = \alpha_3 - \alpha_2 + \frac{\alpha_1}{3} = \frac{\alpha_4}{3}$. Given $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$ it follows that $z_1 - z_2 = \frac{\alpha_1 - \alpha_2}{2} \geq 0$ and $z_1 + z_2 - 2z_3 = \frac{\alpha_1 + 3\alpha_2 - 4\alpha_4}{6} \geq 0$. Thus for $|\mathbf{N}| = 4$ all FB implementable allotment problems satisfy Property **Z**.

Case 3:$|\mathbf{N}| = 5$

For a FB implementable allotment problem Ω^A with $\mathbf{D}^A = ([\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5])$, $z_1 = \frac{\alpha_1}{4}$, $z_2 = \frac{\alpha_2}{3} - \frac{\alpha_1}{12}$, $z_3 = \frac{\alpha_3}{2} - \frac{\alpha_2}{3} + \frac{\alpha_1}{12}$ and $z_4 = \frac{\alpha_5}{4}$. Given $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5$ it follows that $z_1 - z_2 = \frac{\alpha_1 - \alpha_2}{3} \geq 0$, $z_1 + z_2 - 2z_3 = \alpha_2 - \alpha_3 \geq 0$ and $z_1 + z_2 + z_3 - 3z_4 = \frac{\alpha_1 + 2\alpha_3 - 3\alpha_4}{4} \geq 0$. Thus for $|\mathbf{N}| = 5$ all FB implementable allotment problems satisfy Property **Z**. ■

EXAMPLE 2.5.9 Consider the simple allotment problem of Example 4.7 where $\Omega^A = \langle \mathbf{N} = \{1, 2, 3\}, \bar{\mathbf{D}}^A, \mathbf{R}_+ \rangle$ where $\bar{\mathbf{D}}^A = ([\bar{\alpha}_1 = \frac{2}{3}, \bar{\alpha}_2 = \frac{1}{3}, \bar{\alpha}_3 = 0])$.

$$\bar{\mathbf{D}}^A = \begin{pmatrix} 2/3 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 2/3 & 1/3 \\ 0 & 1/3 & 0 & 2/3 & 1/3 & 2/3 \end{pmatrix}$$

Recall that the transfers are $t_{(1)}(\theta) = -\frac{1}{3}\theta_{(2)}$, $t_{(2)}(\theta) = 0$ and $t_{(3)}(\theta) = \frac{1}{3}\theta_{(2)}$. In this model,

$U_{(1)}(d^*_{(1)}(\theta), t_{(1)}(\theta), \theta_{(1)}) = \frac{2}{3}\theta_{(1)} - \frac{1}{3}\theta_{(2)} \geq 0,$

$U_{(2)}(d^*_{(2)}(\theta), t_{(2)}(\theta), \theta_{(2)}) = \frac{1}{3}\theta_{(2)} \geq 0$ and

$U_{(3)}(d^*_{(3)}(\theta), t_{(3)}(\theta), \theta_{(3)}) = \frac{1}{3}\theta_{(2)} \geq 0$.

Clearly, the utility of all individuals in all states $\theta \in \Theta^3$ is non-negative. Hence, the allotment problem is FB* implementable.

Finally we give one example to show that for $|\mathbf{N}| > 5$ there are FB implementable allotment problems that fails to satisfy Property **Z**.

EXAMPLE 2.5.10 Consider the following FB implementable allotment problem Ω^A for $|\mathbf{N}| = 6$ where $([\alpha_1 = \frac{250}{1122}, \alpha_2 = \frac{246}{1122}, \alpha_3 = \frac{218}{1122}, \alpha_4 = \frac{216}{1122}, \alpha_5 = \frac{192}{1122}, \alpha_6 = 0])$. Note that $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > \alpha_5 > \alpha_6 = 0$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6 = 1$ and $\alpha_1 - 5\alpha_2 + 10\alpha_3 - 10\alpha_4 + 5\alpha_5 - \alpha_6 = 0$. One can easily calculate that $(z_1 = \frac{50}{1122}, z_2 = \frac{49}{1122}, z_3 = \frac{40}{1122}, z_4 = \frac{48}{1122}, z_5 = 0)$. Clearly, $z_1 + z_2 + z_3 - 3z_4 = \frac{139-144}{1122} = -\frac{5}{1122} < 0$. This FB implementable allotment problem fails to satisfy Property **Z**.

One can construct many more examples of FB implementable allotment problems that fail to satisfy Property **Z** for $|\mathbf{N}| > 5$.

Finally, we show that the FB implementable public goods problem is also FB* implementable.

THEOREM 2.5.7 Ω^g is FB* implementable.

PROOF: It follows almost immediately from our remark that Ω^g can be FB implemented with zero transfers. Since all individual j is allowed access if and only if $\theta_j \geq 0$, we have $U_j(d^*_j(\theta), t_j(\theta), \theta_j) = d^*_j(\theta)\theta_j \geq 0$. Thus Ω^g is FB* implementable. ∎

2.6 Conclusion

In this essay we have extended the classical public goods problem to the larger class of public decision problems. We have shown that there exists members of this class where efficient outcomes can be obtained even when transfers are restricted to be balanced. The class of such problems is narrow but this is not surprising in view of the severity of the requirements imposed. One issue which remains to be addressed is the role of the linearity assumption in the possibility results. Do these results remain if for example we assume in the queueing model that costs are convex in waiting time? The next chapter deals with this question.

Chapter 3

Queueing Models

3.1 Introduction

This chapter develops and refines a line of research initiated in chapter two. One of the most striking examples of a first best implementable public decision is the queueing problem which is our concern in this chapter. In this model there is a server who has to serve a finite set of individuals. The server can serve one individual at a time. Thus, individuals have to wait in a queue. Waiting in a queue is costly for each individual. The server's objective is to order the individuals in a queue efficiently so as to minimise the aggregate waiting cost. If the cost of waiting in the queue is private information then an individual, if asked, will announce his costs strategically so as to get his job done as early as possible. Therefore, the principal in the queueing model has an incentive problem under incomplete information.

In the previous chapter we saw that if waiting costs are linear, it is possible to devise a scheme of balanced transfers that induce individuals to reveal their private information and attain efficiency. Suijs [36] proves a similar result in the context of his sequencing model. It is important to identify the reason why a possibility result holds in this model in contrast to the well-known impossibility result in the case of the public good model. In the latter model, an individual, by changing his announcement, affects the payoff of **all** individuals. It is this **severe** nature of this externality that leads to budget imbalance. In the first best implementable queueing model, the externality that can be imposed by an individual is more **subtle**. An individual with kth position in the queue, by changing his announcement can affect the decision of either individuals who precede him in the queue or those who succeed him. He cannot simultaneously affect the decisions of both the predecessor and the successor sets. Thus only the individuals getting the first position and last position in the queue can affect all other individuals by changing their announcements. This sort of externality is necessary for finding a Groves transfer where the individuals served earlier compensates for those served later in such a way that aggregate transfer is zero in all states.

Another important feature of any queueing model is that the incentive problem is "spread over" the queue positions and this helps in finding a balanced Groves transfer where the individuals being served "earlier" pay money to the individuals receiving "late" service. For example, with three individuals, a balanced Groves transfer in the queueing model is of the following type. The individual receiving the service first pays the waiting cost of the individual who is served second in the queue and this money goes to the individual who is served last. The first part of the transfer resembles that of the transfer in second price auction where the highest bidder pays the second highest bid. However, in the second price auction this money goes to an outsider like the principal. In the queueing model there is an incentive problem for all queue positions and so the payment of the second highest cost to the individual in the third queue position more then compensates him for the loss of getting the third queue position instead of the second queue position.

We attempt to answer the following question: *are there cost structures more general than the linear case where the "first best" can be attained?* We prove that for first best implementability it is both necessary and sufficient that preferences satisfy a certain combinatorial property and an independence property. The first property is a restriction on individual preferences and the second property is a restriction on the externality that an individual can impose on the remaining set of individuals. Preferences satisfy the independence property if, an individual, by changing his announcement, cannot change the relative queue positions of the remaining set of individuals. For instance, if there are n individuals then the relative queue positions of any set of $(n-1)$ individuals are independent of the queue position of the individual who is left out. This property captures the externality that an individual can impose on the remaining set of individuals. In spite of these requirements, apparently quite strong, there exists a fairly large class of queueing problems that are first best implementable. Given a broad class of first best implementable queueing models one can then explore the possibility of individual rationality, i.e. whether individuals would be willing to participate in the mechanism offered by the server. It can be shown that if the gross benefit from the service for all individuals is sufficiently high, then first best implementable queueing models satisfy individual rationality.

This chapter is organised as follows. In section two, the general queueing model is formalised and results on its first best implementability are derived. Section three deals with separable cost queueing models and its applications. Section four formalises a general class of first best implementable queueing problem. Section five is a discussion of discounted cost queueing model. The concluding section seven is preceded by an exploration of the possibility of individual rationality of first best queueing models.

3.2 The General Model

Let $\mathbf{N} = \{1, 2, \ldots, n\}$ be the set of individuals and $\theta_j(k)$ measure the cost of waiting k periods in the queue for individual j where $k \in \{1, \ldots, n\}$. The type of individual $j \in \mathbf{N}$ is the vector $\theta_j = (\theta_j(1), \ldots, \theta_j(n))$. Clearly, $\theta_j(k) \in \mathbf{R}_+$ for all $j \in \mathbf{N}$ and for all $k \in \{1, \ldots, n\}$[1]. It is assumed that all individuals dislike waiting i.e. $0 \leq \theta_j(1) \leq \theta_j(2) \leq \ldots \leq \theta_j(n)$. Let $\bar{\Theta}$ be the

[1] $\mathbf{R}_+$ represents the non-negative orthant of $\mathbf{R}$.

largest domain satisfying this condition. For all $j \in \mathbf{N}$, $\theta_j \in \bar{\Theta}$, the utility of each individual j is assumed to be quasi-linear and is of the form:

$$U_j(k, t_j; \theta_j) = v_j - \theta_j(k) + t_j$$

where $v_j(> 0)$ is the benefit derived by individual j from the service and t_j is the transfer that individual j receives.

The server's aim is to achieve efficiency or minimise the aggregate cost. A permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ of the set $\mathbf{N}$ represents a particular queue. Thus, $\sigma_j = k$ indicates that individual j has the kth position in the queue. Let Σ be the set of all possible permutations of $\mathbf{N}$. Given a permutation or a queue $\sigma = (\sigma_1, \ldots, \sigma_n)(\in \Sigma)$, the cost of any individual $j \in \mathbf{N}$ is $\theta_j(\sigma_j)$. A state of the world is $\theta = (\theta_1, \ldots, \theta_n) \in \bar{\Theta}^n$ where θ_j is a $1 \times n$ vector.

DEFINITION 3.2.22 Given a state θ, a queue $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is **efficient** if $\sigma^* \in argmin_{\sigma \in \Sigma} \sum_{j \in \mathbf{N}} \theta_j(\sigma_j)^2$.

Efficiency in this context is an assignment problem that gives each individual exactly one queue position and each queue position to exactly one individual in such a way that the aggregate cost is minimised[3].

If the server knows $\theta = (\theta_1, \ldots, \theta_n)$ then he can calculate the efficient queue. However, if θ_j is private information for individual j, the server's problem then is to design a **mechanism** that will elicit this information truthfully. Formally, a mechanism $\mathbf{M}$ is a pair $\langle \sigma, \mathbf{t} \rangle$ where $\sigma : \bar{\Theta}^n \to \Sigma$ and $\mathbf{t} \equiv (t_1, \ldots, t_n) : \bar{\Theta}^n \to \mathbf{R}^n$. This problem is called a **general queueing problem** under incomplete information and is written as $\Omega = \langle \mathbf{N}, \bar{\Theta} \rangle$. Under $\mathbf{M} = \langle \sigma, \mathbf{t} \rangle$, given all others' announcement θ_{-j}, the utility of individual j of type θ_j when his announcement is $\theta_j{}'$ is given by
$U_j(\sigma_j(\theta_j{}', \theta_{-j}), t_j(\theta_j{}', \theta_{-j}), \theta_j) = v_j - \theta_j(\sigma_j(\theta_j{}', \theta_{-j})) + t_j(\theta_j{}', \theta_{-j})$.

[2]Observe that there can be states with more than one efficient queue. So we have an efficiency correspondence.

[3]This is a subtle optimization problem. An algorithm which computes efficiency is the Hungarian method which can be found in Bapat and Raghavan [2].

DEFINITION 3.2.23 $\Omega = \langle \mathbf{N}, \bar{\Theta} \rangle$ is **implementable** if there exists an *efficient rule*[4] $\sigma^* : \bar{\Theta}^n \to \Sigma$ and a mechanism $\mathbf{M} = \langle \sigma^*, \mathbf{t} \rangle$ such that for all $j \in \mathbf{N}$, for all $(\theta_j, \theta_j') \in \bar{\Theta}^2$, and for all $\theta_{-j} \in \bar{\Theta}^{n-1}$,

$$U_j(\sigma_j^*(\theta), t_j(\theta); \theta_j) \geq U_j(\sigma_j^*(\theta_j', \theta_{-j}), t_j(\theta_j', \theta_{-j}); \theta_j)$$

This definition says that for any given θ_{-j}, individual j cannot benefit by reporting anything other than his true type. In other words, truth-telling is a dominant strategy for all individuals. Moreover, this truth-telling leads to efficient queue.

DEFINITION 3.2.24 $\Omega = \langle \mathbf{N}, \bar{\Theta} \rangle$ is **first best implementable** or **FB implementable**, if there exists a mechanism $\mathbf{M} = \langle \sigma^*, \mathbf{t} \rangle$ which implements it and such that, for all $\theta \in \bar{\Theta}^n$, $\sum_{j \in \mathbf{N}} t_j(\theta) = 0$.

Thus, a queueing problem is first-best implementable if, it can be implemented in a manner such that aggregate transfers are zero in every state of the world. In such problems, incomplete information does not impose any welfare cost. In the next section the question of FB implementability of the general queueing model is analysed.

3.2.1 Characterization Results

In this sub-section the necessary and sufficient conditions relating to the FB implementability of the general queueing model are derived. As a preliminary step to the main results, some more definitions and notations are introduced that will be extensively used in this section.

DEFINITION 3.2.25 A mechanism $\mathbf{M} = \langle \sigma, \mathbf{t} \rangle$ is a Groves mechanism if for all $j \in \mathbf{N}$, the transfer is of the form

$$t_j(\theta) = -\sum_{l \neq j} \theta_l(\sigma_l^*(\theta)) + \gamma_j(\theta_{-j}) \tag{3.2.1}$$

[4] An efficient rule is a single valued selection from the efficiency correspondence.

In a Groves mechanism the transfer of any individual $j \in \mathbf{N}$ in any state θ is the negative of minimum cost i.e. $-\sum_{l \in \mathbf{N}} \theta_l(\sigma_l^*(\theta))$ plus the cost of individual j and a constant $\gamma_j(\theta_{-j})$. The utility of individual j with a Groves transfer is his gross benefit v_j less the minimum cost in state θ plus the constant. It is well known that such a transfer results in dominant strategy incentive compatibility because the servers' objective of minimising the aggregate cost is now an objective of individual j as well and this is true for all $j \in \mathbf{N}$.

According to a well known result of Holmström (see Holmström [21]), decision problems with smoothly connected domains are implementable if and only if the mechanism is a Groves mechanism. In more precise terms, **convex domains** are smoothly connected (see Theorem (2) in Holmström [21]). It can be easily checked that the domain under consideration in the general queueing model satisfy Holmström's definition of "convex" domains. Hence it is implementable if and only if the mechanism is a Groves mechanism.

Let $C(\sigma^*(\theta');\theta) = \sum_{j \in \mathbf{N}} \theta_j(\sigma_j^*(\theta'))$ where, as stated earlier, $\sigma^*(\theta')$ is an efficient queue for the announced state θ'. Thus, $C(\sigma^*(\theta');\theta)$ is the minimum aggregate cost with respect to the announced state θ' when the actual state is θ. For notational simplicity define $C(\theta) \equiv C(\sigma^*(\theta);\theta)$ to be the minimum aggregate cost with respect to the actual state θ when the announced state is also θ.

REMARK 3.2.6 From the definition of efficiency it follows that for all θ and θ', $C(\theta) \leq C(\sigma^*(\theta');\theta)$.

DEFINITION 3.2.26 $\Omega = \langle \mathbf{N}, \bar{\Theta} \rangle$ satisfies the **Combinatorial Property (or CP)** if for all $j \in \mathbf{N}$, for all $\theta_j \in \bar{\Theta}$,

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \theta_j(k) = 0 \qquad (3.2.2)$$

This property is a combinatorial condition on the domain of preferences. The meaning of this property will become explicit from the following discussion. For individual j with type $\theta_j = (\theta_j(1), \ldots, \theta_j(n))$ define the first order difference at queue position $k \in \{1, \dots, n-1\}$ as $\Delta(1)\theta_j(k) = \theta_j(k+1) - \theta_j(k)$. Thus, the first order difference at k represents the increase in queueing cost if individual j is moved from kth position to $(k+1)$th position. In particular, the first order difference at queue position 1 is $\Delta(1)\theta_j(1) = \theta_j(2) - \theta_j(1)$. Similarly, the second order difference at queue position 1 is $\Delta^2(1)\theta_j(1) = \Delta(1)[\Delta(1)\theta_j(1)] = \Delta(1)[\theta_j(2) - \theta_j(1)] = \theta_j(3) - 2\theta_j(2) + \theta_j(1)$. One can similarly derive $\Delta^3(1)\theta_j(1)$, $\Delta^4(1)\theta_j(1)$ and so on. It can be quite easily checked from (3.2.2) that

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\theta_j(k) = \Delta^{n-1}(1)\theta_j(1) = 0.$$

Thus a type θ_j of individual j satisfies CP if the $(n-1)$th order difference at queue position 1 is zero. CP is analogous to $(n-1)$th order derivative at queue position 1. CP implies and is implied by some kind of separability to be discussed later in Proposition 3.2.4.

To define the next property one needs to introduce some more notations and definitions. Consider a queueing model Ω. Define, by eliminating $l \in \mathbf{N}$, the **l-reduced queueing model** of Ω to be $\Omega_{N-l} = \langle \mathbf{N} - l, \bar{\Theta} \rangle$. In any state $\theta_{-l} \in \Theta^{n-1}$, let $\sigma^*(\theta_{-l})$ be the efficient queue in Ω_{N-l}. In other words,

$$\sigma^*(\theta_{-l}) \in argmin_{\tilde{\sigma} \in \Sigma(N-l)} \sum_{j \neq l} \theta_j(\tilde{\sigma}_j)$$

where $\Sigma(N-l)$ is the set of all possible permutations of $\{1, \dots, n-1\}$ and $\tilde{\sigma}_j$ is the position of individual $j (\neq l)$ in the particular queue $\tilde{\sigma} \in \Sigma(N-l)$. In short, Ω_{N-l} is a l-reduced queueing model of $(n-1)$ individuals obtained from Ω by excluding $l \in \mathbf{N}$.

For Ω, define $P(\sigma^*(\theta), j) = \{p \in \mathbf{N}/\{j\} \mid \sigma_j^*(\theta) > \sigma_p^*(\theta)\}$ to be the predecessor set of individual j in state θ. In other words, under the efficient queue $\sigma^*(\theta)$ in state θ, $P(\sigma^*(\theta), j)$ is the set of individuals receiving the service before individual j. Similarly, for Ω_{N-l}, define $P(\sigma^*(\theta_{-l}), j) = \{p \in \mathbf{N}/\{j, l\} \mid \sigma_j^*(\theta_{-l}) > \sigma_p^*(\theta_{-l})\}$ to be the predecessor set of individual j in state θ_{-l}. Under the efficient queue $\sigma^*(\theta_{-l})$ in state θ_{-l}, $P(\sigma^*(\theta_{-l}), j)$ is the set of individuals receiving the service before individual j.

DEFINITION 3.2.27 Ω satisfies **Independence Property (or IP)** if for all states $\theta \in \bar{\Theta}^n$, for all $j \in \mathbf{N}$, for all $l \in \mathbf{N}/\{j\}$ and for all $\theta_{-l} \in \bar{\Theta}^{n-1}$,

$$P(\sigma^*(\theta_{-l}), j) = \begin{cases} P(\sigma^*(\theta), j) & \text{if } l \notin P(\sigma^*(\theta), j) \\ P(\sigma^*(\theta), j)/\{l\} & \text{if } l \in P(\sigma^*(\theta), j) \end{cases}$$

This property means that if in state θ, $\sigma_j^*(\theta) = k$, then $\sigma_j^*(\theta_{-l}) = k$ for all $l \notin P(\sigma^*(\theta), j)$ and $\sigma_j^*(\theta_{-l}) = k - 1$ for all $l \in P(\sigma^*(\theta), j)$. Another way of stating IP is the following: Consider Ω and a pair $\{j, l\} \in \mathbf{N}$. If $\sigma_j^*(\theta) < \sigma_l^*(\theta)$ in state θ, then $\sigma_j^*(\theta_{-i}) < \sigma_l^*(\theta_{-i})$ for all $i \in \mathbf{N}/\{j, l\}$. If this condition holds for all pair of individuals and for all states in Ω then it is easy to check that Ω satisfies IP. This condition says that if individual j's position in the queue is less than that of individual l in some state θ, then j's queue position must continue to remain less than that of l's position in all Ω_{N-i} that includes both j and l. This condition must hold for all pair $\{j, l\} \in \mathbf{N}$ and for all states $\theta \in \bar{\Theta}^n$. IP eliminates the possibility that an individual $l \in \mathbf{N}/\{j\}$, who is a predecessor (successor) of individual j in state θ is a successor (predecessor) of individual j in state θ_{-i} for some i-reduced queueing model Ω_{N-i} where $i \in \mathbf{N}/\{j, l\}$. Thus, IP guarantees that the externality imposed by an individual (i in the above argument) is not severe enough to change the relative queue position of the remaining set of individuals.

The separability implied by the combinatorial property and the link between the combinatorial property (or CP) and the independence property (or IP) is captured by the following Proposition and the explanation following it.

PROPOSITION 3.2.4 $\Omega = \langle \mathbf{N}, \bar{\Theta} \rangle$ satisfies CP, if and only if for each $\theta_j \in \bar{\Theta}$ there exists a unique vector $H_j = \{h_j(1), \ldots, h_j(n-1)\}$ such that for all $k \in \{1, \ldots, n\}$,

$$\theta_j(k) = (n-k)h_j(k) + (k-1)h_j(k-1). \qquad (3.2.3)$$

PROOF: Given a $n \times 1$ vector of type for individual $j \in \mathbf{N}$, $\theta_j \in \bar{\Theta}$ in Ω satisfying CP, we define a $(n-1) \times 1$ vector $H_j = \{h_j(1), \ldots, h_j(n-1)\}$ such that for all $k \in \{1, \ldots, n-1\}$,

$$h_j(k) = \sum_{r=1}^{k} (-1)^{k-r} \frac{(k-1)!(n-k-1)!}{(r-1)!(n-r)!} \theta_j(r) \tag{3.2.4}$$

First we check using (3.2.4) that for all $k \in \{1, \ldots, n-1\}$, (3.2.4) holds. Then we check for $k = n$ this condition holds **only if** Ω satisfies CP.

$(n-k)h_j(k) + (k-1)h_j(k-1)$

$= (n-k) \sum_{r=1}^{k} (-1)^{k-r} \frac{(k-1)!(n-k-1)!}{(r-1)!(n-r)!} \theta_j(r) + (k-1) \sum_{r=1}^{k-1} (-1)^{k-r-1} \frac{(k-2)!(n-k)!}{(r-1)!(n-r)!} \theta_j(r)$

$= \sum_{r=1}^{k-1} \{(-1)^{k-r} + (-1)^{k-r-1}\} \frac{(k-1)!(n-k)!}{(r-1)!(n-r)!} \theta_j(r) + \theta_j(k)$

$= \theta_j(k)$ (because $(-1)^{k-r} + (-1)^{k-r-1} = 0$).

For $k = n$,

$(n-k)h_j(k) + (k-1)h_j(k-1)$

$= (n-1)h_j(n-1)$

$= (n-1) \sum_{r=1}^{n-1} (-1)^{n-1-r} \frac{(n-2)!}{(r-1)!(n-r)!} \theta_j(r)$

$= \sum_{r=1}^{n-1} (-1)^{n-1-r} \frac{(n-1)!}{(r-1)!(n-r)!} \theta_j(r)$

$= \sum_{r=1}^{n-1} (-1)^{n-1-r} \binom{n-1}{r-1} \theta_j(r)$

$= \theta_j(n)$ (from CP).

The last step not only proves the necessity of CP but also guarantees that for θ_j, H_j is unique.

To prove the other part of the Lemma it is easy to see that if $\theta_j(k) = (n-k)h_j(k) + (k-1)h_j(k-1)$ for all $k = 1, \ldots, n$ then

$\sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \theta_j(k)$

$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \{(n-k)h_j(k) + (k-1)h_j(k-1)\}$

$= (n-1)\{\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n-2}{k-1}h_j(k) + \sum_{k=2}^{n}(-1)^{k-1}\binom{n-2}{k-2}h_j(k-1)\}$

$= 0.$ ∎

Consider Ω and some state θ where individual $j \in \mathbf{N}$ gets the kth queue position. Note that in state θ, out of the remaining $(n-1)$ individuals in $\mathbf{N}/\{j\}$, there are $(n-k)$ individuals receiving the service after individual j and there are $(k-1)$ individuals receiving the service before individual j. Consider Ω_{N-l} for all $l \in \mathbf{N}/\{j\}$. Now if an individual l receiving the service after individual j, i.e. $l \notin P(\sigma^*(\theta), j)$, were to be eliminated, then from IP it follows that j retains the kth queue position in Ω_{N-l}. If, on the other hand, an individual l who was receiving the service before individual j, i.e. $l \in P(\sigma^*(\theta), j)$, is eliminated from the queue then, from IP it follows that the queue position of individual j changes from k in Ω to $(k-1)$ in Ω_{N-l}. If the vector H_j in Proposition 3.2.4 replaces θ_j for reduced queueing models $\{\Omega_{N-l}\}_{l \neq j}$, the cost of kth queue position for individual j i.e. $\theta_j(k)$, in Ω for state θ, can now be represented as the sum of costs in $(n-1)$ reduced queueing models. Here individual j has cost $h_j(k)$ in $(n-k)$ of the reduced queueing models. These are reduced models Ω_{N-l} such that $l \notin P(\sigma^*(\theta), j)$ and total number of such reduced models is $|\mathbf{N} - [P(\sigma^*(\theta), j) \cup \{j\}]| = n - k$. Similarly, individual j has a cost of $h_j(k-1)$ in $(k-1)$ of these models. These are reduced models Ω_{N-l} such that $l \in P(\sigma^*(\theta), j)$ and total number of such reduced models is $|P(\sigma^*(\theta), j)| = k-1$. Observe that this will give $\theta_j(k) = (n-k)h_j(k) + (k-1)h_j(k-1)$ which follows from CP as established in Proposition 3.2.4.

REMARK 3.2.7 Consider an individual j and a profile $\theta_j \in \bar{\Theta}$ satisfying CP. From Proposition 3.2.4 it follows that there exists a unique vector H_j such that for all $k \in \{1, \ldots, n\}$, $\theta_j(k) = (n-k)h_j(k) + (k-1)h_j(k-1)$. Using $\theta_j(1) \leq \theta_j(2) \leq \ldots \leq \theta_j(n)$ one obtains the following restriction on the elements of H_j. $h_j(1) \leq h_j(r) \leq h_j(n-1)$ for all $r \in \{2, \ldots, n-2\}$. One cannot comment on the ordering of the set of elements belonging to the set $\{h_j(2), \ldots, h_j(n-2)\}$.

REMARK 3.2.8 Consider Ω and $\{\theta_j, \theta_l\} \in \bar{\Theta}^2$ such that $\theta_j(k+1) - \theta_j(k) \geq \theta_l(k+1) - \theta_l(k)$ for all $k \in \{1, \ldots, n\}$. Note that $\theta_j(k+1) - \theta_j(k) \geq \theta_l(k+1) - \theta_l(k)$ implies that if in some state individuals j and l are assigned queue positions k and $k+1$, then it is more efficient to serve individual j ahead of individual l because the marginal cost of shifting individual j from queue position k to queue position $k+1$ is no less than the same marginal cost for individual l. If this condition is true for all k then $\sigma_j^*(\theta_j, \theta_l, \theta_{-j-l}) < \sigma_l^*(\theta_j, \theta_l, \theta_{-j-l})$ for all $\theta_{-j-l} \in \bar{\Theta}^{n-2}$. One obvious implication of this observation is that $\sigma_j^*(\theta_j, \theta_l, \theta_{-j-l-i}) < \sigma_l^*(\theta_j, \theta_l, \theta_{-j-l-i})$ for all $i \in \mathbf{N}/\{j, l\}$ and for all $\theta_{-j-l-i} \in \bar{\Theta}^{n-3}$. Another useful implication is the following. Consider a state θ where for all pairs $\{j, l\}$, if $\theta_j(2) - \theta_j(1) \geq \theta_l(2) - \theta_l(1)$, then $\theta_j(k+1) - \theta_j(k) \geq \theta_l(k+1) - \theta_l(k)$ for all $k \in \{2, \ldots, n-1\}$. In state θ, if for some pair $\{j, l\}$, $\sigma_j^*(\theta) < \sigma_l^*(\theta)$ then from the construction of state θ it follows that $\sigma_j^*(\theta_{-i}) < \sigma_l^*(\theta_{-i})$ for all $i \in \mathbf{N}/\{j, l\}$. Therefore, in state θ, Ω satisfies IP. This remark will be used in some of the results to follow.

In the case of $|\mathbf{N}| = 3$, it is possible to show that CP implies IP. Unfortunately, for $|\mathbf{N}| > 3$ this is no longer true.

PROPOSITION 3.2.5 $\Omega = \langle \mathbf{N} = \{1, 2, 3\}, \bar{\Theta} \rangle$ satisfies CP $\Rightarrow$ Ω satisfies IP.

PROOF: $\Omega = \langle \mathbf{N} = \{1, 2, 3\}, \bar{\Theta} \rangle$ satisfies CP implies that the second order difference is zero. Thus $\Delta(1)\theta_j(1) \equiv \theta_j(2) - \theta_j(1) = \theta_j(3) - \theta_j(2) \equiv \Delta(1)\theta_j(2)$ for all $j = 1, 2, 3$. Therefore, for a pair of preferences $\{\theta_j, \theta_l\} \in \bar{\Theta}^2$, if $\theta_j(2) - \theta_j(1) \geq \theta_l(2) - \theta_l(1)$, then $\theta_j(3) - \theta_j(2) \geq \theta_l(3) - \theta_l(2)$. Using Remark 3.2.8 it immediately follows that Ω satisfies IP. ■

The next example is to show that for $|\mathbf{N}| > 3$, there is no relationship between CP and IP. Specifically, it shows that if Ω satisfies CP, it may not satisfy IP.

EXAMPLE 3.2.11 Consider the general queueing model for four individuals, i.e. $\Omega = \langle \mathbf{N} = \{1, 2, 3, 4\}, \bar{\Theta} \rangle$. Let the state $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ be of the following form: $\theta_1 = (1, 23, 45, 67)$, $\theta_2 = (3, 12, 27, 48)$, $\theta_3 = (1, 4, 9, 16)$ and $\theta_4 = (1, 7, 13, 19)$. Here, $\sigma_j^*(\theta) = j$ for all $j = 1, 2, 3, 4$. We find that state θ satisfies CP i.e. for all $j = 1, 2, 3, 4$,

$$\sum_{k=1}^{4}(-1)^{k-1}\binom{3}{k-1}\theta_j(k) = \binom{3}{0}\theta_j(1) - \binom{3}{1}\theta_j(2) + \binom{3}{2}\theta_j(3) - \binom{3}{3}\theta_j(4) = 0.$$

Now consider the $\{1\}$-reduced queueing model $\Omega_{N-1} = \langle\{2, 3, 4\}, \bar{\Theta}\rangle$. In this reduced model we consider the first three elements of the vectors θ_2, θ_3 and θ_4. Here $\sigma_2^*(\theta_{-1}) = 1$, $\sigma_3^*(\theta_{-1}) = 3$ and $\sigma_4^*(\theta_{-1}) = 2$. Therefore, $P(\sigma^*(\theta_{-1}), 3) = \{2, 4\} \neq P(\sigma^*(\theta), 3)/\{1\} = \{2\}$ and $P(\sigma^*(\theta_{-1}), 4) = \{2\} \neq P(\sigma^*(\theta), 4)/\{1\} = \{2, 3\}$. Thus for state θ, IP is violated.

THEOREM 3.2.8 $\Omega = \langle \mathbf{N}, \bar{\Theta}\rangle$ is FB implementable **if only if** it satisfies CP and IP.

Before proving the Theorem a Lemma due to Walker [41] is stated below. Consider two profiles $\theta = (\theta_1, \ldots, \theta_n)$ and $\theta' = (\theta'_1, \ldots, \theta'_n)$. Define for $S \subseteq \mathbf{N}$, a type $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$. Thus for each $S \subseteq \mathbf{N}$, we have a state $\theta(S) = (\theta_1(S), \ldots, \theta_n(S))$.

LEMMA 3.2.3 Ω is FB implementable only if for all $(\theta, \theta') \in \bar{\Theta}^n \times \bar{\Theta}^n$, $\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S)) = 0$.

It is quite easy to see why Lemma 3.2.3 is necessary for FB implementability. Given the Groves transfer, balancedness requires that $(n-1)C(\theta) = \sum_{j\in\mathbf{N}}\gamma_j(\theta_{-j})$[5]. For any two profiles θ and θ' one can now easily check that $\sum_{S\subseteq\mathbf{N}}(-1)^{|S|}C(\theta(S)) = \frac{1}{(n-1)}\sum_{j\in\mathbf{N}}\sum_{S\subseteq\mathbf{N}}(-1)^{|S|}\gamma_j(\theta_{-j}(S)) = 0$. It is quite clear that if the function C has a separable form , then it must satisfy an appropriate restriction on the nth order cross partial derivative. The condition in the Lemma is analogous of this derivative for finite changes.

PROOF OF THEOREM(3.2.8):

[5] Adding the Groves transfer of all individuals and setting it to zero gives this condition. This condition in a more general framework was derived by Holmström [22].

Necessity: We prove the necessity part of the Theorem in two steps. In the first step we prove that Ω is FB implementable only if it satisfies CP. In the second step we prove that Ω satisfying CP is FB implementable only if it satisfies IP.

Step 1: We start with a given type for individual 1 (i.e. θ_1) and construct θ_{-1} and θ'. Then we apply Lemma 3.2.3 to derive the result. Consider individual 1 and any announcement $\theta_1 = (\theta_1(1), \ldots, \theta_1(k), \ldots, \theta_n(n))$. Define real numbers $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \underline{\theta}\}$ such that $0 = \epsilon_1 < \epsilon_2 < \ldots < \epsilon_n$ and $0 \leq \underline{\theta} \leq \theta_1(1)$. Consider two states $\theta = (\theta_1, \ldots, \theta_n)$ and $\theta' = (\theta'_1, \ldots, \theta'_n)$ of the following type: $\theta_j(k) = \theta_1(k) + k\epsilon_j$ and $\theta'_j(k) = \underline{\theta}$, for all $j \in \mathrm{N}$ and for all $k = 1, \ldots, n$. Therefore $\theta_j = (\theta_1(1) + \epsilon_j, \theta_1(2) + \epsilon_j, \ldots, \theta_1(n) + n\epsilon_j)$ and $\theta'_j = (\underline{\theta}, \underline{\theta}, \ldots, \underline{\theta})$ for all $j \in \mathrm{N}$. Consider any two queue positions k and $k+1$ and any two individuals j and $j+1$ with types θ_j and θ_{j+1}, respectively. Note that from the construction of θ, on the one hand, it follows that if individual j gets the kth position and $(j+1)$th individual gets the $(k+1)$th position, then the costs for these two positions add up to $\{2\theta_1(k)+k\epsilon_j+(k+1)\epsilon_{j+1}\}$. If, on the other hand, the positions of j and $(j+1)$ are interchanged then the costs add up to $\{2\theta_1(k)+(k+1)\epsilon_j+k\epsilon_{j+1}\}$. Clearly the former cost exceeds the latter [6] and holds for all $k = 1, \ldots, n-1$. Thus the queue that minimises the aggregate cost requires that, $\sigma^*_j(\theta) > \sigma^*_{j+1}(\theta)$ for all $j = 1, \ldots, n-1$. This implies that the efficient queue in state θ is $\sigma^*(\theta) = (\sigma^*_1(\theta) = n, \ldots, \sigma^*_j(\theta) = n - j + 1, \ldots, \sigma^*_n(\theta) = 1)$. In state θ' any queue is efficient because the costs of all individuals are identical.

Now consider profiles $\theta(S) = (\theta_1(S), \ldots, \theta_n(S))$ where $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$. For all $s \in S$, the efficient queue position is behind all $j \notin S$, i.e. $\sigma^*_s(\theta(S)) \in \{n - |S| + 1, \ldots, n\}$. This is because the queueing costs of all individuals $j \notin S$, in all queue positions strictly exceed the queueing costs of all individual $s \in S$. Moreover, given θ_1, from the construction of θ_{-1} and from the argument given for the efficient queue in state θ it follows that if $\{j, l\} \notin S$ and $j < l$, then $\sigma^*_j(\theta(S)) > \sigma^*_l(\theta(S))$.

[6] This is because from the construction it follows that $\epsilon_{j+1} > \epsilon_j$ for all $j = 1, \ldots n-1$.

Consider the sum $\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S))$. Observe first that, for all $l \in \mathbf{N}/\{1\}$ with type θ_l, if there exists m sets $T^1,\ldots,T^m$ of size $|T^1|,\ldots,|T^m|$ respectively with $T^{\hat{m}} \subseteq \mathbf{N}/\{l\}$ for all $\hat{m} = 1,\ldots,m$, for which individual l's position is $k(\in \{1,2,\ldots,n\})$, then $\sum_{\hat{m}=1}^{m}(-1)^{|T^{\hat{m}}|}\theta_l(k) = 0$. Therefore, the sum $\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S))$ is independent of all elements in the set of vectors $\{\theta_2,\ldots,\theta_n\}$. Also observe that the terms containing $\underline{\theta}$ in $\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S))$ is given by $-n\{\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\}\underline{\theta} = 0$. Therefore, $\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S))$ is also independent of $\underline{\theta}$. All these observations imply that:

$$\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S)) = \sum_{S\subseteq \mathbf{N}/\{1\}}(-1)^{|S|}\theta_1(\sigma_1^*(\theta_1,\theta_{-1}(S))).$$

For individual 1 with type θ_1, $\sigma_1^*(\theta_1,\theta_{-1}(S)) = n - |S|$ for all $S \subseteq \mathbf{N}/\{1\}$. Thus

$$\begin{aligned}
&\sum_{S\subseteq \mathbf{N}/\{1\}}(-1)^{|S|}\theta_1(\sigma_1^*(\theta_1,\theta_{-1}(S)))\\
&= \sum_{|S|=0}^{n-1}(-1)^{|S|}\binom{n-1}{|S|}\theta_1(n-|S|)\\
&= \sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\theta_1(n-k+1)\\
&= \sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{n-k}\theta_1(k)\\
&= (-1)^{n-2}\{\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\theta_1(k)\}.
\end{aligned}$$

An application of Lemma 3.2.3 yields the result for individual 1. We can obtain the same result for all $j \in \mathbf{N}/\{1\}$.

Step 2: For $|\mathbf{N}| = 3$, Ω satisfying CP is FB implementable only if it satisfies IP follows from Proposition 3.2.5. For Ω with $|\mathbf{N}| > 3$, consider any two cost vectors $\theta_1 = (\theta_1(1),\ldots,\theta_1(n))$ and $\theta_2 = (\theta_2(1),\ldots,\theta_2(n))$ for individuals 1 and 2 respectively. Let $z = max[max\{\Delta(1)\theta_1(k)\}_{k\neq n}, max\{\Delta(1)\theta_2(k)\}_{k\neq n}]$. Consider the real numbers $\{\epsilon_3,\ldots,\epsilon_n,\underline{\theta}\}$ such that $z < \epsilon_3 < \ldots < \epsilon_n$ and $\underline{\theta} \in [0, min\{\theta_1(1),\theta_2(1)\}]$. Define $\theta_j(k) = k\epsilon_j$ for all $k \in \{1,\ldots,n\}$ and for all $j \in \{3,\ldots,n\}$. Also define $\theta'_j(k) = \underline{\theta}$ for all $j \in \mathbf{N}$ and for all $k \in \{1,\ldots,n\}$.

From the construction it follows that $\Delta(1)\theta_j(k) > max\{\Delta(1)\theta_1, \Delta(1)\theta_2(k)\}$ for all $j \in \mathbf{N}/\{1,2\}$. Now consider profiles $\theta(S) = (\theta_1(S), \ldots, \theta_n(S))$ where $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$. For all $s \in S$, the efficient queue position is behind all $j \notin S$, i.e. $\sigma^*_s(\theta(S)) \in \{n - |S| + 1, \ldots, n\}$. This is because the queueing costs of all individuals $j \notin S$, in all queue positions strictly exceed the queueing costs of all individual $s \in S$. From the construction of $\{\theta_3, \ldots, \theta_n\}$ it follows that for all $\{j, l\} \in \mathbf{N}/\{1,2\}$ and for all $S \in \mathbf{N}/\{j, l\}$, with $j < l$, $\sigma^*_j(\theta_j, \theta_l, \theta_{-j-l}(S)) > \sigma^*_l(\theta_j, \theta_l, \theta_{-j-l}(S))$. Observe that for all $S \in \mathbf{N}/\{1,2\}$, individuals 1 and 2 are immediate neighbours with any one of 1 and 2 having queue position $n - |S|$ and the other having queue position $n - |S| - 1$. If, on the one hand, $\theta_1(n - |S| - 1) + \theta_2(n - |S|) \leq \theta_1(n - |S|) + \theta_2(n - |S| - 1)$ then $\sigma^*_1(\theta_1, \theta_2, \theta_{-1-2}(S)) < \sigma^*_2(\theta_1, \theta_2, \theta_{-1-2}(S))$. If, on the other hand, $\theta_1(n - |S| - 1) + \theta_2(n - |S|) \geq \theta_1(n - |S|) + \theta_2(n - |S| - 1)$ then $\sigma^*_1(\theta_1, \theta_2, \theta_{-1-2}(S)) > \sigma^*_2(\theta_1, \theta_2, \theta_{-1-2}(S))$. Define, $Z(k, k+1) = min\{\theta_1(k) + \theta_2(k+1), \theta_1(k+1) + \theta_2(k)\}$ for all $k \in \{1, \ldots n-1\}$. Making use of the above observations and the definition of $Z(k, k+1)$ we get

$$\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} C(\theta(S)) = \sum_{k=1}^{n-1} (-1)^{n-k-1} \binom{n-2}{k-1} [Z(k, k+1) - \theta_1(k) - \theta_2(k)] \tag{3.2.5}$$

For Ω to be FB implementable it is necessary from Lemma 3.2.3 that the RHS of equation (3.2.5) is zero for all $\{\theta_1, \theta_2\} \in \bar{\Theta}^2$ and satisfying CP. This crucially depends on the terms with $Z(k, k+1)$ in the RHS of equation (3.2.5). We claim that the RHS of (3.2.5) is zero **if and only if** either one of the two following conditions is satisfied then Ω is FB implementable.

1. $Z(k, k+1) = \theta_1(k) + \theta_2(k+1)$ for all $k \in \{1, \ldots, n-1\}$

2. $Z(k, k+1) = \theta_1(k+1) + \theta_2(k)$ for all $k \in \{1, \ldots, n-1\}$.

We first prove the **if** part of this claim. If condition (1) holds i.e. $Z(k, k+1) = \theta_1(k) + \theta_2(k+1)$ for all $k \neq n$, then by substituting (1) in (3.2.5) and simplifying it we get

$$\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S)) = \sum_{k=1}^{n-1}(-1)^{n-k-1}\binom{n-2}{k-1}\{\Delta(1)\theta_2(k)\} = 0.$$

The last step follows from CP[7]. Similarly one can show that if condition (2) holds then

$$\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S)) = \sum_{k=1}^{n-1}(-1)^{n-k-1}\binom{n-2}{k-1}\{\Delta(1)\theta_1(k)\} = 0.$$

To prove the **only if** part of this claim we first assume that the claim is not true. This implies that there exists θ_1 and θ_2 for individuals 1 and 2 such that

(*i*) $\Delta(1)\theta_1(1) \geq \Delta(1)\theta_2(1)$,

(*ii*) $\exists$ a set $T \subset \{2,3,\ldots,n-1\}$ such that $\Delta(1)\theta_1(r) < \Delta(1)\theta_2(r)$ for all $r \in T$ and

(*iii*) $\exists\, p \in \{1,\ldots,n-1\}/T$ such that $\Delta(1)\theta_1(p) > \Delta(1)\theta_2(p)$.

Using conditions (*i*) – (*ii*) in (3.2.5) and simplifying it we get

$$\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S)) = \sum_{r\in T}(-1)^{n-r-1}\binom{n-2}{r-1}\{\Delta(1)\theta_1(r)-\Delta(1)\theta_2(r)\} \quad (3.2.6)$$

From Lemma 3.2.3 we know that for FB implementability we need that the RHS of (3.2.6) must be zero. Now if $\Delta(1)\theta_2(k) = a > 0$ for all $k \in \{1,\ldots,n-1\}$ then construct a profile $\hat{\theta}_1$ such that $\Delta(1)\hat{\theta}_1(k) = (2k+1)b$ with $b \in (\frac{a}{2n-3}, \frac{a}{2n-1})$. Applying the same type of construction i.e. $(2n+1)b < \epsilon_3 < \ldots < \epsilon_n$ and $\underline{\theta} \in [0, min\{\hat{\theta}_1, \theta_2\}]$ and defining θ_j for all $j \in \{3.\ldots.n\}$ and θ'_j for all $j \in \mathbf{N}$ in the same way as before we get

$$\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S)) = \Delta(1)\theta_2(n-1) - \Delta(1)\hat{\theta}_1(n-1) = a - (2n-1)b \neq 0.$$

---

[7] If $\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\theta_j(k) = 0$, then $\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n-2}{k-1}\{\theta_j(k+1) - \theta_j(k)\} = 0$.

If $\Delta(1)\theta_2(k)$ is not a constant for all $k \neq n$, then consider $\bar{\theta}_2$ such that $\bar{\theta}_2(k) = \theta_2(k)$ for all $k \in \{1, \ldots, n-2\}$ and select $\{\bar{\theta}_2(n-1), \bar{\theta}_2(n)\}$ in such a way that $\bar{a} = \Delta(1)\bar{\theta}_2(n-1) > max\{\Delta(1)\bar{\theta}_2(k)\}_{k \neq n-1}$. Again, define $\bar{\theta}_1$ such that $\bar{\theta}_1(k) = k\epsilon_1$ where $max\{\Delta(1)\bar{\theta}_2(k)\}_{k\neq n-1} < \epsilon_1 < \bar{a}$. Again by applying the same type of construction we get

$$\sum_{S\subseteq \mathbf{N}} (-1)^{|S|}C(\theta(S)) = \Delta(1)\bar{\theta}_2(n-1) - \Delta(1)\bar{\theta}_1(n-1) = \bar{a} - \epsilon_1 \neq 0.$$

Therefore, for a preference satisfying condition $(i) - (iii)$ and Lemma 3.2.3, we can find a preference in its neighbourhood that fails to satisfy Lemma 3.2.3. Thus, Ω satisfying CP is FB implementable **only if** either (1) holds or condition (2) holds. Since the selection of individuals 1 and 2 for the above construction was arbitrary, it follows that Ω satisfying CP is FB implementable **only if** for all $j \neq l$,
either (a) $\theta_j(k+1) - \theta_j(k) \geq \theta_l(k+1) - \theta_l(k)$ for all $k \neq n$
or (b) $\theta_j(k+1) - \theta_j(k) \leq \theta_l(k+1) - \theta_l(k)$ for all $k \neq n$.
This condition means that the descending order of $\{\theta_j(2) - \theta_j(1)\}_{j=1}^{n}$ determines the efficient queue i.e. if $\theta_j(2) - \theta_j(1) \geq \theta_l(2) - \theta_l(1)$ in some state θ, then $\sigma_j^*(\theta) < \sigma_l^*(\theta)$. Using Remark 3.2.8 we get Ω satisfies IP. The logic is quite simple, if for example, θ_j, θ_l are such that $\theta_j(k+1) - \theta_j(k) \geq \theta_l(k+1) - \theta_l(k)$ for all $k \in \{1, \ldots, n-1\}$, then individual j is served ahead of individual l for all eliminations of $i \in \mathbf{N}/\{j,l\}$. This proves Step 2.
Sufficiency: Consider the sum $\sum_{l\neq j} h_j(\sigma_j^*(\theta_{-l}))$ in state θ for individual $j \in \mathbf{N}$. From IP we get

$\sum_{l\neq j} h_j(\sigma_j^*(\theta_{-l}))$
$= (n - \sigma_j^*(\theta))h_j(\sigma_j^*(\theta)) + (\sigma_j^*(\theta) - 1)h_j(\sigma_j^*(\theta) - 1)$
$= \theta_j(\sigma_j^*(\theta))$ (from condition (3.2.3) in Proposition 3.2.4).

Now consider a particular Groves mechanism $\hat{\mathbf{M}} = \langle \sigma^*, \hat{t} \rangle$ where $\hat{\gamma}_j(\theta_{-j}) = (n-1)\sum_{l\neq j} h_l(\sigma_l^*(\theta_{-j}))$. Then it follows that

$\sum_{j\in\mathbf{N}} \hat{\gamma}_j(\theta_{-j})$

$= (n-1) \sum_{j \in \mathbf{N}} \sum_{l \neq j} h_l(\sigma_l^*(\theta_{-j}))$

$= (n-1) \sum_{j \in \mathbf{N}} \{\sum_{l \neq j} h_j(\sigma_j^*(\theta_{-l}))\}$

$= (n-1) \sum_{j \in \mathbf{N}} \theta_j(\sigma_j^*(\theta))$

$= (n-1)C(\theta)$.

This implies that for all $\theta \in \hat{\Theta}^n$, $\sum_{j \in \mathbf{N}} \hat{t}_j(\theta) = -(n-1)C(\theta) + \sum_{j \in \mathbf{N}} \hat{\gamma}_j(\theta_{-j}) = 0$.
■

This section dealt with the restrictions required for FB implementability of the general queueing model. The next few sections restrict the cost of each individual to have a common functional form.

3.3 Separable Cost Models

In this section a class of queueing models, called separable cost queueing models, are considered. For separable cost queueing models, $\theta_j(k)$ satisfies the following conditions:

1. $\theta_j(k) = f(k)g(\theta_j)$ for all $j \in \mathbf{N}$, for all $k \in \{1, 2, \ldots, n\}$ and for all $\theta_j \in \Theta$. Here Θ is assumed to be an interval in $\mathbf{R}_+$.

2. $g(\theta_j) > 0$ for all $\theta_j \in \Theta$ and $g(\theta_j)$ is continuous and strictly increasing in θ_j.

3. Finally, $f(k) \geq f(k-1)$ for all $k \in \{2, \ldots, n\}$.

The first condition multiplicatively separates the cost of each individual for each position into two functions. The first function f depends on the queue position. The second function g depends on the type (or cost parameter i.e. θ_j) of an individual. The second condition is a restriction on the type function. The third condition restricts the queueing cost function f to be non-decreasing in queue positions. The second and third conditions together imply that $\theta_j(k+1) \geq \theta_j(k)$ for all $k \in \{1, \ldots, n-1\}$. The cost parameter (i.e. θ_j for $j \in \mathbf{N}$) is private information.

Each pair of functions (f, g) together with $\mathbf{N}$ and type space Θ, defines a **separable cost queueing problem** $\hat{\Omega} = \langle \mathbf{N}, \Theta, (f, g) \rangle$. A major benefit of such a specification is that the efficiency condition is completely transparent in this context. For a $\hat{\Omega}$, the queue $\sigma^*(\theta) \in \Sigma$ **is efficient** in state θ if for all $j \neq l$, $\theta_j > \theta_l \Rightarrow \sigma_j^*(\theta) < \sigma_l^*(\theta)$. Ties can be broken in many ways. A particular way of breaking ties, that will be followed in this paper, is to consider the natural ordering i.e. if $\theta_j = \theta_j$ and $j < l$ then $\sigma_j^*(\theta) < \sigma_l^*(\theta)$[8]. The next Proposition is related to IP of $\hat{\Omega}$.

PROPOSITION 3.3.6 $\hat{\Omega}$ satisfies IP.

PROOF: Consider $\hat{\Omega} = \langle \mathbf{N}, \Theta, (f, g) \rangle$ and an individual $j \in \mathbf{N}$ with queue position $\sigma_j^*(\theta)$ in state θ. If $l \in P(\sigma^*(\theta), j)$ then $\sigma_j^*(\theta_{-l}) = \sigma_j^*(\theta) - 1$ and $P(\sigma^*(\theta_{-l}), j) = P(\sigma^*(\theta), j)/\{l\}$ in $\hat{\Omega}_{N-l} = \langle \mathbf{N} - l, \Theta, (f, g) \rangle$. This is because individual l is a predecessor of j in $\hat{\Omega} = \langle \mathbf{N}, \Theta, (f, g) \rangle$. Also and because according to the definition of efficiency and the same tie breaking rule assumption, individual j's queue position with respect to all other individuals $\mathbf{N}/\{j, l\}$ remains unchanged in $\hat{\Omega}_{N-l} = \langle \mathbf{N} - l, \Theta, (f, g) \rangle$. Recall that the definition of efficiency for any separable cost queueing model $\hat{\Omega}$ depends only on the order of types of individuals. They remain invariant for the set of $n - 1$ individuals included in any $\hat{\Omega}_{N-l} = \langle \mathbf{N}/\{l\}, \Theta, (f, g) \rangle$. If, on the other hand, $l \in \mathbf{N}/\{P(\sigma^*(\theta), j) \cup j\}$ then $\sigma_j^*(\theta_{-l}) = \sigma_j^*(\theta)$ and $P(\sigma^*(\theta_{-l}), j) = P(\sigma^*(\theta), j)$ in $\hat{\Omega}_{N-l} = \langle \mathbf{N} - l, \Theta, (f, g) \rangle$. This is because individual l is a successor of individual j in $\hat{\Omega} = \langle \mathbf{N}, \Theta, (f, g) \rangle$ and according to the definition of efficiency, individual j's queue position with respect to all other individuals $\mathbf{N}/\{j, l\}$ remains unchanged in $\hat{\Omega}_{N-l} = \langle \mathbf{N} - l, \Theta, (f, g) \rangle$. ■

[8]Note that the definition of efficient queue depends only on a pairwise comparison of individual types. In other words, if $\theta_j > \theta_l$, then for all $\theta_{-j-l} \in \Theta^{n-2}$, $\sigma_j^*(\theta) < \sigma_l^*(\theta)$. Also note that given the domain specification, there are states for which more than one ordering is efficient. So we have an efficiency correspondence for all such states. The tie breaking rule guarantees that in all states where more than one ordering is efficient, the decision picked is unique. Thus, a tie breaking rule guarantees a single valued selection of ordering decision from the efficiency correspondence.

The remainder of this section will deal with the question of FB implementability of the class of separable cost queueing model. The combinatorial property (or CP) is both necessary and sufficient for FB implementability of the class of separable cost queueing models. Note that $\hat{\Omega} = \langle \mathbf{N}, \Theta, (f, g) \rangle$ satisfies CP if

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} f(k) = 0.$$

REMARK 3.3.9 From condition (3.2.3) it follows that $\hat{\Omega}$ satisfies CP, if and only if there exists a unique vector $H = \{h(1), \ldots, h(n-1)\}$ such that for all $k \in \{1, \ldots, n\}$,

$$f(k) = (n-k)h(k) + (k-1)h(k-1) \tag{3.3.7}$$

where $h(k) = \sum_{r=1}^{k} (-1)^{k-r} \frac{(k-1)!(n-k-1)!}{(r-1)!(n-r)!} f(r)$. The other observation that follows from condition (3.3.7) is

$$\sum_{k=1}^{n} f(k) = n \sum_{k=1}^{n-1} h(k) \tag{3.3.8}$$

Condition (3.3.8) will be useful in deriving later results.

PROPOSITION 3.3.7 $\hat{\Omega} = \langle \mathbf{N}, \Theta, (f, g) \rangle$ is FB implementable **if and only if** the cost function satisfies CP.

PROOF: To prove the necessary part of the Proposition we first construct two profiles and then apply Lemma 3.2.3. Let the two states θ and θ' be of the following form: $\theta'_1 > \theta'_2 > \ldots > \theta'_n > \theta_1 > \theta_2 > \ldots > \theta_n$. Now for all $S \subseteq \mathbf{N}$, we consider profiles $\theta(S) = (\theta_1(S), \ldots, \theta_j(S), \ldots, \theta_n(S))$ where $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$.

For all $S \subseteq \mathbf{N}/\{1\}$ with profiles $(\theta_1, \theta_{-1}(S))$, $\sigma^*_1(\theta_1, \theta_{-1}(S)) = N - |S|$ and for all $S \subseteq \mathbf{N}/\{n\}$ with profiles $(\theta'_n, \theta_{-n}(S))$, $\sigma^*_n(\theta'_n, \theta_{-n}(S)) = |S| + 1$. Therefore,

$$\sum_{S\subseteq \mathbf{N}/\{1\}}(-1)^{|S|}C_1(\theta_1,\theta_{-1}(S)) = \sum_{|S|=0}^{n-1}(-1)^{|S|}\binom{n-1}{|S|}f(n-|S|)g(\theta_1)$$

and

$$\sum_{S\subseteq \mathbf{N}/\{n\}}(-1)^{|S|}C_n(\theta'_n,\theta_{-n}(S)) = \sum_{|S|=0}^{n-1}(-1)^{|S|}\binom{n-1}{|S|}f(|S|+1)g(\theta'_n).$$

For all $x_j \in \{\theta_2,\dots,\theta_n,\theta'_1,\dots,\theta'_{n-1}\}$, if the sets $\{m_1,\dots,m_p\}$, all subsets of $S/\{j\}$, are such that $\sigma^*_j(x_j,\theta_{-j}(m_q)) = k$, for all $q \in \{1,\dots,p\}$, then $\sum_{q=1}^{p}(-1)^{m_q} = 0$. Therefore, $\sum_{S\subseteq \mathbf{N}/\{j\}}(-1)^{|S|}C(x_j,\theta_{-j}(S)) = 0$.

Combining all these observations we get

$$\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(\theta(S)) = \{g(\theta_1)-g(\theta'_n)\}\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n-1}{k-1}f(k).$$

Applying Lemma 3.2.3 and using $g(\theta_1)\neq g(\theta'_n)$ in the above equation we get

$$\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n-1}{k-1}f(k) = 0 \tag{3.3.9}$$

The sufficiency follows quite easily from Theorem 3.2.8. ■

Consider a queueing model $\hat{\Omega} = \langle \mathbf{N},\Theta,(f,g)\rangle$. For convenience consider the "inverse" of the queue σ^*. That is, given $\theta\in\Theta^n$, let μ be a permutation such that $\theta_{\mu(1)}\geq\ldots\geq\theta_{\mu(n)}$. Furthermore, if $g(\theta_j) = g(\theta_l)$ and $j<l$, then $j\in P(\sigma^*(\theta),l)$. The explicit form of the transfer can be obtained from the following condition:

$$t_{\mu(k)}(\theta) = \sum_{q=1}^{k-1}\{(n-1)h(q)-f(q)\}g(\theta_{\mu(q)}) + \sum_{q=k+1}^{n}\{(n-1)h(q-1)-f(q)\}g(\theta_{\mu(q)}) \tag{3.3.10}$$

where $h(q) = \sum_{r=1}^{q}(-1)^{q-r}\frac{(q-1)!(n-q-1)!}{(r-1)!(n-r)!}f(r)$ for all $q\in\{1,\dots,n-1\}$.

The existence of FB implementable $\hat{\Omega}$ is already established in the previous chapter for $f(k)=k$. The question of the existence of other FB implementable separable cost queueing models is analysed in the next section.

3.3.1 Applications

In this section the existence of a fairly large class of FB implementable separable cost queueing models is established. We start by defining a broad class of queueing cost function.

DEFINITION 3.3.28 $\tilde{f}_a^{n-2}$ is called a **polynomial cost function of degree** $n-2$ if

1. $\tilde{f}_a^{n-2}(k) = \sum_{p=1}^{n-2} a_p k^p$, for all $k \in \{1,\dots,n\}$.

2. $\tilde{f}_a^{n-2}(k) \geq \tilde{f}_a^{n-2}(k-1)$, for all $k \in \{2,\dots,n\}$.

It is important to observe that the class of polynomial cost depends crucially on the specification of the vector $a = (a_1,\dots,a_{n-2})$. Let $\tilde{\Omega}_a^{n-2} = \langle \mathrm{N}, \Theta, (\tilde{f}_a^{n-2}, g)\rangle$ be a particular separable cost queueing model with polynomial cost $\tilde{f}_a^{n-2}$. Also let $\tilde{\mathbf{\Omega}}^{n-2}$ be the class polynomial cost queueing models. Observe that from Proposition 3.3.6 it follows that all polynomial cost queueing models $\tilde{\Omega}_a^{n-2} \in \tilde{\mathbf{\Omega}}^{n-2}$ satisfy IP. One can now introduce some special cases of the class of polynomial cost queueing models. One such special case is the factorial cost queueing model.

DEFINITION 3.3.29 $f^{[m]}$ is called a **factorial cost function of degree m** if for all $m \in \{1,\dots n-2\}$,

$$f^{[m]}(k) = [k]_m = k(k-1)\dots(k-m+1).$$

Here m represents the queue position from which the $f^{[m]}$ is non-zero i.e.

$$f^{[m]}(k) = \begin{cases} \frac{k!}{(k-m)!} & \text{if } k \geq m \\ 0 & \text{otherwise.} \end{cases}$$

One can easily verify that for all $k \in \{2,\dots,n\}$,

$$f^{[m]}(k) - f^{[m]}(k-1) = \frac{m(k-1)!}{(k-m)!} \geq 0.$$

Let $\hat{\Omega}^{[m]} = \langle \mathbf{N}, \Theta, (f^{[m]}, g) \rangle$ be a separable cost queueing model with factorial cost of degree $m \leq n-2$.

Another type of polynomial cost queueing model is the standard cost queueing model.

DEFINITION 3.3.30 f^m is said to be a **standard cost function of degree m** if for all $m \in \{1, \ldots n-2\}$, $f^m(k) = k^m$.

One can easily that for all $k \in \{2, \ldots, n\}$,

$$f^m(k) - f^m(k-1) = \{k^{m-1} + k^{m-2}(k-1) + \ldots + (k-1)^{m-1}\} > 0.$$

Let $\hat{\Omega}^m = \langle \mathbf{N}, \Theta, (f^m, g) \rangle$ be a separable cost queueing model with standard queueing cost f^m of degree $m \leq n-2$. Notice that $f^{[1]} = f^1$ i.e. factorial cost function of degree one and standard cost function of degree one are identical.

REMARK 3.3.10 Following remarks can be made about the polynomial cost $\tilde{f}_a^{n-2}$,

1. By selecting appropriate values of a_p for all $p = 1, \ldots, n-2$, one can get factorial cost of any degree $m \leq n-2$. For example, with $|\mathbf{N}| = 4$, $a_2 = 1$ and $a_1 = -1$ we get $\tilde{f}_a^2(k) = k^2 - k = k(k-1) = f^{[2]}(k)$. In general, $\tilde{f}_a^{n-2}$ is a factorial cost of degree $m \leq n-2$ if $a_p = s(m,p)$ for all $p = 1, \ldots, m$ and $a_p = 0$ otherwise. $s(m,p)$ for all $p = 1, \ldots, m$, are Stirling number of the first kind[9].

2. A polynomial cost $\tilde{f}_a^{n-2}$ is a standard cost of degree m if $a_m = 1$ and $a_p = 0$ for all $p \neq m$.

THEOREM 3.3.9 $\tilde{\Omega}_a^{n-2} \in \tilde{\Omega}^{n-2}$ is FB implementable.

[9] A Stirling number of the first kind, $s(m,p)$, is defined as the coefficient of x^p in the expansion of $[x]_p = x(x-1)\ldots(x-p+1)$, i.e. $[x]_p = \sum_{p=1}^{m} s(m,p)x^p$. For further references see Tomescu and Melter [38].

We state and prove two Lemma that will be used in proving Theorem 3.3.9.

LEMMA 3.3.4 $\hat{\Omega}^{[m]}$ is FB implementable.

PROOF: To prove this Lemma we will have to show that $\hat{\Omega}^{[m]}$ satisfies CP. From the definition of $f^{[m]}$, it follows that

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}f^{[m]}(k)$$
$$=\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\frac{k!}{(k-m)!}$$
$$=[n-1]_{m-1}(-1)^{m-1}\sum_{k=m}^{n}(-1)^{k-m}\binom{n-m}{k-m}\{(k-m)+m\}$$
$$=[n-1]_{m}(-1)^{m}\sum_{k=m+1}^{n}(-1)^{k-m-1}\binom{n-m-1}{k-m-1}$$
$$+m[n-1]_{m-1}(-1)^{m-1}\sum_{k=m}^{n}(-1)^{k-m}\binom{n-m}{k-m}$$
$$=0.$$ ■

LEMMA 3.3.5 $\hat{\Omega}^{m}$ is FB implementable.

PROOF: To prove this Lemma we use the following mathematical identity

$$k^m=\sum_{q=1}^{m}S(m,q)[k]_q \tag{3.3.11}$$

where $[k]_q = k(k-1)\dots(k-q+1)$ and $S(m,q)$ are Stirling number of the second kind[10].

[10] A Stirling number of the second kind $S(m,q)$, is defined as the coefficient of $[x]_q$ in the expansion of x^q, i.e., $x^q=\sum_{q=0}^{m}S(m,q)[x]_q$. Stirling number of the second kind are such that $S(m,1)=S(m,m)=1$. Moreover, these numbers are unimodal i.e. they satisfy one of the following formulae:

1. $1=S(m,1)<S(m,2)<\dots<S(m,M(m))>S(m,M(m)-1)\dots>S(m,m)=1$

   or

2. $1=S(m,1)<S(m,2)<\dots<S(m,M(m)-1)=S(m,M(m))>\dots>S(m,m)=1$

and $M(m+1)=M(m)$ or $M(m+1)=M(m)+1$ where $M(m)=max\{q \mid S(m,q)$ is maximum; $1\leq q\leq m\}$. For a better understanding see Tomescu and Melter [38].

From (3.3.11), it follows that

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}f^{m}(k)$$
$$=\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\sum_{q=1}^{m}S(m,q)f^{[q]}(k)$$
$$=\sum_{q=1}^{m}S(m,q)\{\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}f^{[q]}(k)\}$$
$= 0$ (from Lemma 3.3.4). ■

PROOF OF THEOREM 3.3.9: To prove this Theorem we will have to show that $\tilde{\Omega}_a^{n-2}$ satisfies CP. Given the form of the cost function we get

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\tilde{f}_a^{n-2}(k)$$
$$=\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\{\sum_{p=1}^{n-2}a_p f^{p}(k)\}$$
$$=\sum_{p=1}^{n-2}a_p\{\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}f^{p}(k)\}$$
$= 0$ (from Lemma 3.3.5). ■

The remaining part of this section deals with examples of different polynomial cost queueing models with $|\mathbf{N}| = 4$. The first two examples are factorial cost queueing models of degree one and two. The third example is a standard cost queueing model of degree two. The final example is a polynomial cost queueing model of degree two.

EXAMPLE 3.3.12 Consider $\hat{\Omega}^{[1]} = \langle \mathbf{N} = \{1,2,3,4\}, \Theta, (f^{[1]}, g)\rangle$ where the queueing cost function is of the form $f^{[1]}(k) = k$, for all $k = 1,2,3,4$. Condition (3.3.7) gives $h^{[1]}(k) = \sum_{r=1}^{k}(-1)^{k-r}\frac{(k-1)!(n-k-1)!}{(r-1)!(n-r)!}f^{[1]}(r)$. Elementary computation gives $\mathbf{H}^{[1]} = \{h^{[1]}(1) = \frac{1}{3}, h^{[1]}(2) = \frac{5}{6}, h^{[1]}(3) = \frac{4}{3}\}$.

Now consider a state $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ such that $\theta_1 \geq \theta_2 \geq \theta_3 \geq \theta_4$. This means that $\sigma_j^*(\theta) = j$, for all $j = 1,2,3,4$. We can see that for all $j \neq l$,

$$\sigma_j^*(\theta_{-l}) = \begin{cases} j & \text{if } \sigma_j^*(\theta) < \sigma_l^*(\theta) \\ j-1 & \text{if } \sigma_j^*(\theta) > \sigma_l^*(\theta). \end{cases}$$

Consider the Groves transfer, as defined in condition (3.3.10). Computation of the transfers give

$$\begin{aligned}
t_1(\theta) &= -\{f^{[1]}(2)g(\theta_2) + f^{[1]}(3)g(\theta_3) + f^{[1]}(4)g(\theta_4)\} \\
&\quad +(n-1)\{h^{[1]}(1)g(\theta_2) + h^{[1]}(2)g(\theta_3) + h^{[1]}(3)g(\theta_4)\} \\
&= -g(\theta_2) - \tfrac{1}{2}g(\theta_3), \\
t_2(\theta) &= -\{f^{[1]}(1)g(\theta_1) + f^{[1]}(3)g(\theta_3) + f^{[1]}(4)g(\theta_4)\} \\
&\quad +(n-1)\{h^{[1]}(1)g(\theta_1) + h^{[1]}(2)g(\theta_3) + h^{[1]}(3)g(\theta_4)\} \\
&= -\tfrac{1}{2}g(\theta_3), \\
t_3(\theta) &= -\{f^{[1]}(1)g(\theta_1) + f^{[1]}(2)g(\theta_2) + f^{[1]}(4)g(\theta_4)\} \\
&\quad +(n-1)\{h^{[1]}(1)g(\theta_1) + h^{[1]}(2)g(\theta_2) + h^{[1]}(3)g(\theta_4)\} \\
&= \tfrac{1}{2}g(\theta_2), \\
t_4(\theta) &= -\{f^{[1]}(1)g(\theta_1) + f^{[1]}(2)g(\theta_2) + f^{[1]}(3)g(\theta_3)\} \\
&\quad +(n-1)\{h^{[1]}(1)g(\theta_1) + h^{[1]}(2)g(\theta_2) + h^{[1]}(3)g(\theta_3)\} \\
&= \tfrac{1}{2}g(\theta_2) + g(\theta_3).
\end{aligned}$$

Note that $\sum_{j=1}^{4} t_j(\theta) = 0$. To write an explicit form of the transfers for each state $\theta \in \Theta$, consider the "inverse" of the order σ^*, suppose μ is a permutation such that

$$\theta_{\mu(1)} \geq \theta_{\mu(2)} \geq \theta_{\mu(3)} \geq \theta_{\mu(4)}.$$

The transfers are

$t_{\mu(1)}(\theta) = -g(\theta_{\mu(2)}) - \frac{1}{2}g(\theta_{\mu(3)})$,

$t_{\mu(2)}(\theta) = -\frac{1}{2}g(\theta_{\mu(3)})$,

$t_{\mu(3)}(\theta) = \frac{1}{2}g(\theta_{\mu(2)})$ and

$t_{\mu(4)}(\theta) = \frac{1}{2}g(\theta_{\mu(2)}) + g(\theta_{\mu(3)})$. Observe that $\sum_{k=1}^{4} t_{\mu(k)}(\theta) = 0$.

The reason why these transfers are incentive compatible is intuitive. The individuals with first and second positions in the queue compensates the individuals with third and last positions in the queue in such a way that truth-telling is a dominant strategy for all the individuals. The amount that the first individual pays exceeds the amount paid by the second individual by $g(\theta_{\mu(2)})$. So, by moving in the second position the first individual with type $\theta_{\mu(1)}$ cannot benefit because his reduction in payment will be $g(\theta_{\mu(2)})$ and his increase in queueing cost will be $g(\theta_{\mu(1)})(> g(\theta_{\mu(2)}))$. Similarly, the individual having second position in the queue cannot benefit by moving ahead in the queue. In which case, he will have to pay $g(\theta_{\mu(1)})$ more and his reduction in cost will be $g(\theta_{\mu(2)})(< g(\theta_{\mu(1)}))$. One can, by applying similar arguments, check that these transfers are dominant strategy incentive compatible for all individuals.

EXAMPLE 3.3.13 Consider $\hat{\Omega}^{[2]} = \langle \mathbf{N} = \{1,2,3,4\}, \Theta, (f^{[2]}, g)\rangle$ where the queueing cost function is of the form $f^{[2]}(k) = k(k-1)$, for all $k = 1,2,3,4$. Using condition (3.3.7) we get $h^{[2]}(r) = \sum_{r=1}^{m}(-1)^{m-r}\frac{(m-1)!(n-m-1)!}{(r-1)!(n-r)!}f^{[2]}(r)$. Elementary computation gives $\mathbf{H}^{[2]} = \{h^{[2]}(1) = 0, h^{[2]}(2) = 1, h^{[2]}(3) = 4\}$. Consider a state $\theta \in \Theta^4$ and $\theta_{\mu(1)} \geq \theta_{\mu(2)} \geq \theta_{\mu(3)} \geq \theta_{\mu(4)}$ [11]. The transfers are:

$t_{\mu(1)}(\theta) = -2g(\theta_{\mu(2)}) - 3g(\theta_{\mu(3)})$,

$t_{\mu(2)}(\theta) = -3g(\theta_{\mu(3)})$,

$t_{\mu(3)}(\theta) = g(\theta_{\mu(2)})$ and

$t_{\mu(4)}(\theta) = g(\theta_{\mu(2)}) + 6g(\theta_{\mu(3)})$. Adding the transfers give $\sum_{k=1}^{4} t_{\mu(k)}(\theta) = 0$. By considering deviations one can find that truth-telling is a dominant strategy for all individuals.

Observe that the factorial cost of degree 1 is of the same form as the standard cost of degree 1, i.e. $f^{[1]}(k) = f^{1}(k) = k$. An example of a standard cost queueing model for $|\mathbf{N}| = 4$ with $m = 2$ is given below.

EXAMPLE 3.3.14 Consider $\hat{\Omega}^{2} = \langle \mathbf{N} = \{1,2,3,4\}, \Theta, (f^{2}, g)\rangle$ where the queueing cost function is of the form $f^{2}(k) = k^2$, for all $k = 1,2,3,4$. Observe that from Proposition 3.3.5 it follows that

[11] μ is the inverse of σ^* as defined in the previous example.

$f^2(k) = k^2 = S(2,1)k + S(2,2)k(k-1)$,

where $\{S(2,2), S(2,1)\}$ are Stirling numbers of the second kind satisfying $S(2,1) = S(2,2) = 1$[12].

From condition (3.3.7) we know that $h^2(r) = \sum_{r=1}^{m} (-1)^{m-r} \frac{(m-1)!(n-m-1)!}{(r-1)!(n-r)!} f^2(r)$.

By substituting the factorial cost representation as explained above we observe that

$h^2(r) = h^{[1]}(r) + h^{[2]}(r)$. Thus, $\mathbf{H}^2 = \{h^2(1) = \frac{1}{3}, h^2(2) = \frac{11}{6}, h^2(3) = \frac{16}{3}\}$.

For a state $\theta \in \Theta^4$ with $\theta_{\mu(1)} \geq \theta_{\mu(2)} \geq \theta_{\mu(3)} \geq \theta_{\mu(4)}$, the explicit form of the transfers are:

$t_{\mu(1)}(\theta) = -3g(\theta_{\mu(2)}) - \frac{7}{2}g(\theta_{\mu(3)})$,

$t_{\mu(2)}(\theta) = -\frac{7}{2}g(\theta_{\mu(3)})$,

$t_{\mu(3)}(\theta) = \frac{3}{2}g(\theta_{\mu(2)})$ and

$t_{\mu(4)}(\theta) = \frac{3}{2}g(\theta_{\mu(2)}) + 7g(\theta_{\mu(3)})$. Check that $\sum_{k=1}^{4} t_{\mu(k)}(\theta) = 0$. With these transfers it is clear that the individuals getting first and second positions in the queue compensate the individuals getting third and fourth positions in the queue in such a way that truth-telling is a dominant strategy for all individuals.

EXAMPLE 3.3.15 Consider $\tilde{\Omega}^2_a = \langle \mathrm{N} = \{1,2,3,4\}, \Theta, (\tilde{f}^2_a, g)\rangle$ where the queueing cost function is of the form $\tilde{f}^2_a(k) = a_1k + a_2k^2$, for all $k = 1,2,3,4$[13]. Observe that

[12]For $m = 3$, $f^3(k) = k^3 = S(3,1)k + S(3,2)k(k-1) + S(3,3)k(k-1)(k-2)$ where $S(3,1) = S(3,3) = 1$ and $S(3,2) = 3$. For other ms' one can similarly represent standard cost as a weighted sum of factorial costs of degrees $\{1,\ldots,m\}$ where the weights are Stirling numbers of the second kind.

[13]Note that $\tilde{\Omega}^2_a = \langle \mathrm{N} = \{1,2,3,4\}, \Theta, (\tilde{f}^2_a, g)\rangle$ is a polynomial cost queueing model if $\tilde{f}^2_a(k+1) - \tilde{f}^2_a(k) \geq 0$ for all $k = 1,2,3$. Therefore one of the following conditions must satisfied.

1. $a_1 < 0 \Rightarrow a_1 + 3a_2 \geq 0$
2. $a_1 = 0 \Rightarrow a_2 \geq 0$ and
3. $a_1 > 0 \Rightarrow a_1 + 7a_2 \geq 0$.

$$\begin{aligned}
&\tilde{f}_a^2(k)\\
&= a_1 f^1(k) + a_2 f^2(k)\\
&= a_1 f^{[1]}(k) + a_2\{S(2,1)f^{[1]}(k) + S(2,2)f^{[2]}(k)\}\\
&= \{a_1 + a_2\} f^{[1]}(k) + a_2 f^{[2]}(k).
\end{aligned}$$

From condition (3.3.7) we know that $\tilde{h}_a^2(k) = \sum_{r=1}^{k}(-1)^{k-r}\frac{(k-1)!(n-k-1)!}{(r-1)!(n-r)!}\tilde{f}_a^2(r)$. By substituting the factorial cost representation it is quite easy to observe that $\tilde{h}_a^2(k) = \{a_1 + a_2\}h^{[1]}(k) + a_2 h^{[2]}(k)$ for all $k = 1, 2, 3$.

Thus $\tilde{\mathbf{H}}_a^2 = \{\tilde{h}_a^2(1) = \frac{1}{3}(a_1 + a_2), \tilde{h}_a^2(2) = \frac{5}{6}a_1 + \frac{11}{6}a_2, \tilde{h}_a^2(3) = \frac{4}{3}a_1 + \frac{16}{3}a_2\}$.

Consider a state $\theta \in \Theta^4$ and $\theta_{\mu(1)} \geq \theta_{\mu(2)} \geq \theta_{\mu(3)} \geq \theta_{\mu(4)}$. Here the transfers are:

$$\begin{aligned}
t_{\mu(1)}(\theta) &= -(a_1 + 3a_2)g(\theta_{\mu(2)}) - \tfrac{1}{2}(a_1 + 7a_2)g(\theta_{\mu(3)}),\\
t_{\mu(2)}(\theta) &= -\tfrac{1}{2}(a_1 + 7a_2)g(\theta_{\mu(3)}),\\
t_{\mu(3)}(\theta) &= \tfrac{1}{2}(a_1 + 3a_2)g(\theta_{\mu(2)}) \text{ and}\\
t_{\mu(4)}(\theta) &= \tfrac{1}{2}(a_1 + 3a_2)g(\theta_{\mu(2)}) + (a_1 + 7a_2)g(\theta_{\mu(3)}).
\end{aligned}$$

Adding up the transfers for all $k = \{1, 2, 3, 4\}$ gives $\sum_{k=1}^{4} t_{\mu(k)}(\theta) = 0$.

The analysis of the class of separable cost queueing models in this section suggests the existence of a fairly large class of FB implementable separable cost queueing models. This FB implementable class increases with the number of individuals. For example, factorial cost and standard cost queueing models are of degree $m \leq n-2$. The class of polynomial cost queueing model are also of degree $n-2$. So degree of $n-2$ plays an important role in FB implementability of queueing models. This is because CP requires that the $(n-1)$th order difference at queue position 1 must be zero. Thus polynomial costs of degree more than $n-2$ are **not** FB implementable.

3.4 A General Class

A more general class of queueing models that are FB implementable is considered in this section. This class is defined by the following property.

DEFINITION 3.4.31 Ω satisfies Property **G** if for all $j \in \mathbf{N}$ and for all $k \in \{1, \ldots, n\}$, $\theta_j(k)$ satisfies the following conditions:

1. $\theta_j(k) = \sum_{p=1}^{M} f^p(k)g^p(\theta_j) + \beta_j(\theta_j)$ for all $j \in \mathbf{N}$, for all $k \in \{1, \ldots, n\}$, for all $\theta_j \in \Theta$ and for all $p \in \{1, \ldots, M\}$.

2. For all $p \in \{1, \ldots, M\}$, $g^p(\theta_j) > 0$ for all $\theta_j \in \Theta$ and $g^p(\theta_j)$ is continuous and strictly increasing in θ_j.

3. For all $p \in \{1, \ldots, M\}$, $f^p(k) \geq f^p(k-1)$ for all $k \in \{2, \ldots, n\}$ and $\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}f^p(k) = 0$.

Let $\mathbf{\Omega}^G$ be the class of queueing models satisfying Property **G**. Observe that a queueing model $\Omega^G \in \mathbf{\Omega}^G$ with $M = 1$ and $\beta_j(\theta_j) = 0$ for all $\theta_j \in \Theta$ and for all $j \in \mathbf{N}$, is a first best implementable separable cost queueing model. In the next result it is proved that this class of queueing models is FB implementable.

THEOREM 3.4.10 $\Omega^G \in \mathbf{\Omega}^G$ is FB implementable.

PROOF: To prove this Theorem we first argue that $\Omega^G \in \mathbf{\Omega}^G$ satisfies IP. This follows from the fact that the efficient queue in any $\Omega^G \in \mathbf{\Omega}^G$ depends on the ordering of the types as was the case for separable cost queueing models. Hence by following same arguments as in Proposition 3.3.6, one can prove that $\Omega^G \in \mathbf{\Omega}^G$ satisfies IP.

The next step is to specify a Groves transfer and show that $\Omega^G \in \mathbf{\Omega}^G$ is balanced for all states $\theta \in \Theta^n$. Observe that, for all $p \in \{1, \ldots, M\}$, $\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}f^p(k) = 0$ implies from condition (3.3.7) that for all $p \in$

$\{1,\ldots,M\}$, there exists a unique vector $\mathbf{H}^p = \{h^p(1),\ldots,h^p(n-1)\}$ such that for all $k \in \{1,\ldots,n\}$, $f^p(k) = (n-k)h^p(k) + (k-1)h^p(k-1)$. Given $\Omega^G \in \mathbf{\Omega}^G$ satisfy IP, by following the sufficiency argument in Proposition 3.3.7 we get that for all $\theta \in \Theta^n$ and for all $p \in \{1,\ldots,M\}$, $f^p(\sigma_j^*(\theta)) = \sum_{l\neq j} h^p(\sigma_j^*(\theta_{-l}))$.

Now consider a particular Groves mechanism $\hat{\mathbf{M}} \equiv \langle \sigma^*, t\rangle$ where

$$\hat{\gamma}_j(\theta_{-j}) = \sum_{l\neq j}\{\beta_l(\theta_l) + (n-1)\sum_{p=1}^{M} g^p(\theta_l)h^p(\sigma_l^*(\theta_{-j}))\} \tag{3.4.12}$$

Then it follows that

$$\begin{aligned}
&\sum_{j\in\mathbf{N}} \hat{\gamma}_j(\theta_{-j})\\
&= (n-1)\sum_{j\in\mathbf{N}}\sum_{p=1}^{M}\sum_{l\neq j} g^p(\theta_l)h^p(\sigma_l^*(\theta_{-j})) + (n-1)\sum_{j\in\mathbf{N}}\beta_j(\theta_j)\\
&= (n-1)\sum_{j\in\mathbf{N}}\sum_{p=1}^{M} g^p(\theta_j)\{\sum_{l\neq j} h^p(\sigma_j^*(\theta_{-l}))\} + (n-1)\sum_{j\in\mathbf{N}}\beta_j(\theta_j)\\
&= (n-1)\sum_{j\in\mathbf{N}}\{\sum_{p=1}^{M} f^p(\sigma_j^*(\theta))g^p(\theta_j) + \beta_j(\theta_j)\}.\\
&= (n-1)C(\theta).
\end{aligned}$$

This implies that for all $\theta \in \hat{\Theta}^n$, $\sum_{j\in\mathbf{N}} t_j(\theta) = -(n-1)C(\theta) + \sum_{j\in\mathbf{N}}\hat{\gamma}_j(\theta_{-j}) = 0$.

■

One can easily verify the following results:

- $\Omega^G \in \mathbf{\Omega}^G$ satisfies CP.
- The class of FB implementable separable cost queueing models is a special case of $\mathbf{\Omega}^G$.

3.5 A Discounted Cost Model

In all the previous sections, the queueing models that were considered had a cost specification that was increasing over time. Discounting is another standard way of evaluating costs or benefits which accrue over time. For example, in repeated games, one way to analyse benefits of an individual over time is to study the discounted payoff of the individual. Similarly, in some bargaining models, the cost of delay is measured in terms of a constant discount rate. One can think of many other situations where discounting is a standard way of measuring the cost of delay. Therefore, another way of specifying costs in a queueing model is to consider a decrease in benefit from the service over time. The general model specified in section two is general enough to include this model as a special case in the following way.

DEFINITION 3.5.32 A queueing model $\Omega^d = \langle \mathbf{N}, [0,1] \rangle$ is called a **discounted cost model** if for all $j \in \mathbf{N}$ and for all $k \in \{1, \ldots, n\}$, $\theta_j(k) = (1 - \theta_j^k)v_j$ where $\theta_j \in [0,1]$.

The utility of an individual j in Ω^d is of the form $U_j(k, t_j; \theta_j) = \theta_j^k v_j + t_j$. This form of the utility is obtained by substituting $\theta_j(k) = (1 - \theta_j^k)v_j$ in the general queueing model. Here $\theta_j \in [0,1]$ represents the type of individual j which is private information. One can check that $\theta_j(k+1) - \theta_j(k) = \theta_j^k(1 - \theta_j)v_j \geq 0$ for all $k \in \{1, \ldots, n-1\}$.

It is quite easy to observe that for Ω^d, the domain specified satisfies Holmström's definition of "convex domains" and hence can be implemented only by Groves mechanism. For discounted cost queueing models, CP means that for all $j \in \mathbf{N}$, $\theta_j(1 - \theta_j)^n = 0$ i.e. $\theta_j \in \{0, 1\}$. Thus for all $\theta_j \in (0,1)$ and for all $j \in \mathbf{N}$, CP is not satisfied. The next Proposition looks at the question of FB implementability of Ω^d.

PROPOSITION 3.5.8 $\Omega^d = \langle \mathbf{N}, [0,1] \rangle$ is not FB implementable.

PROOF: To prove this Proposition we will consider two states and apply Lemma 3.2.3. Consider a particular individual $m \in \mathbf{N}$ such that $v_m \leq v_j$ for all $j \in \mathbf{N}/\{m\}$. Let $\theta_m = \frac{1}{3}$, $\theta_j = \frac{1}{2}$ for all $j \in \mathbf{N}/\{m\}$ and $\bar{\theta}_j = 0$ for all $j \in \mathbf{N}$. Consider two states $\theta = (\theta_1, \ldots, \theta_n)$ and $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_n)$. Elementary calculation yields $\theta_m^k(1 - \theta_m)v_m < \theta_j^k(1 - \theta_j)v_j$ for all $k \in \{1, \ldots, n\}$ and for all $j \in \mathbf{N}/\{m\}$. Therefore, $n = \sigma_m^*(\theta) > \sigma_j^*(\theta)$ for all $j \in \mathbf{N}/\{m\}$. Now for all $S \subseteq \mathbf{N}$, we consider profiles $\theta(S) = (\theta_1(S), \ldots, \theta_j(S), \ldots, \theta_n(S))$ where $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \bar{\theta}_j = 0$ if $j \in S$. For all $S \subseteq \mathbf{N}/\{m\}$ with profiles $(\theta_m, \theta_{-m}(S))$, $\sigma_m^*(\theta_m, \theta_{-m}(S)) = n - |S|$. Therefore,

$$\sum_{S \subseteq \mathbf{N}/\{m\}} (-1)^{|S|} C_m(\theta_m, \theta_{-m}(S)) = \sum_{|S|=0}^{n-1} (-1)^{|S|} \binom{n-1}{|S|} \theta_m^{(n-|S|)} v_m$$

For all $x_j \in \{\theta_2, \ldots, \theta_n\}/\{\theta_m\}$, if the sets $\{m_1, \ldots, m_p\}$, all subsets of $S \subseteq \mathbf{N}/\{j\}$, are such that $\sigma_j^*(x_j, \theta_{-j}(m_q)) = k$, for all $q \in \{1, \ldots, p\}$, then $\sum_{q=1}^{p} (-1)^{m_q} = 0$. Therefore, $\sum_{S \subseteq \mathbf{N}/\{j\}} (-1)^{|S|} C(x_j, \theta_{-j}(S)) = 0$.

Combining all these observations and the fact that $\bar{\theta}_j = 0$ for all $j \in \mathbf{N}$ we get

$$\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} C(\theta(S))$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n-1}{k-1} \theta_m^{n-k+1}$$

$$= \theta_m(\theta_m - 1)^n = \tfrac{1}{3}(\tfrac{1}{3} - 1)^n \neq 0. \quad \blacksquare$$

From the constructions in the previous Proposition it is easy to see that if θ is such that $\theta_j \in \{0, 1\}$ for all $j \in \mathbf{N}$, then Ω^d is FB implementable. Therefore discounted cost queueing model cannot be FB implemented simply because it fails to satisfy CP. This result confirms the importance of CP as a necessary condition for FB implementability of any queueing model.

3.6 Individual Rationality

This section deals with the identification of the sub-class of individually rational queueing models within the class of FB implementable queueing models discussed earlier.

DEFINITION 3.6.33 A mechanism $\mathbf{M} \equiv \langle \sigma, \mathbf{t} \rangle$ is **individually rational** for a decision problem Ω, if for all $j \in \mathbf{N}$ and for all $\theta \in \Theta^N$,

$$U_j(d_j(\theta), t_j(\theta), \theta_j) \geq 0$$

This definition means that each individual has an outside option independent of his valuation which gives him a utility of zero. An individual will refuse to participate in the mechanism if he does not get an ex-post utility of at least zero.

A queueing problem is said to be FB* **implementable** if it is FB implementable by a mechanism which satisfies individual rationality.

PROPOSITION 3.6.9 $\Omega = \langle \mathbf{N}, \bar{\Theta} \rangle$ satisfying CP and IP is **not** FB* implementable.

PROOF: Consider a state θ satisfying CP and IP such that $\theta_j(1) > v_j$ for all $j \in \mathbf{N}$. Clearly,

$$U_j(\sigma_j^*(\theta), t_j(\theta), \theta_j) - t_j(\theta) = v_j - \theta_j(\sigma_j^*(\theta)) < 0$$

for all $j \in \mathbf{N}$. Balancedness imply $\sum_{j \in \mathbf{N}} t_j(\theta) = 0$. If $t_j(\theta) < 0$ for some $j \in S \subset \mathbf{N}$ then $U_j(\sigma_j^*(\theta), t_j(\theta), \theta_j) < 0$ for all $j \in S$ and hence individual rationality is not satisfied. So for all $j \in \mathbf{N}$, $t_j(\theta) \geq 0$. Therefore, for balancedness we need $t_j(\theta) = 0$ for all $j \in \mathbf{N}$. If $t_j(\theta) = 0$ for all $j \in \mathbf{N}$ then $U_j(\sigma_j^*(\theta), t_j(\theta), \theta_j) < 0$ for all $j \in \mathbf{N}$. Thus $\Omega = \langle \mathbf{N}, \bar{\Theta} \rangle$ is **not** FB* implementable. ■

The general queueing model is not FB* implementable simply because the cost of an individual can be so high as to exceed his benefit from the service. However, for $\Omega^G \in \mathbf{\Omega}^G$ and separable cost queueing models where an individual's cost parameter θ_j belongs to an interval $\Theta \equiv [\underline{\theta}, \bar{\theta}]$, one can find sufficient condition under which FB* implementability can be achieved. To establish this result for $\Omega^G \in \mathbf{\Omega}^G$, consider $\bar{\beta}_j \geq \beta_j(\theta)$ for all $\theta \in \Theta \equiv [\underline{\theta}, \bar{\theta}]$ and for all $j \in \mathbf{N}$. Let $\bar{\beta} \geq \bar{\beta}_j$ and let $\underline{v} \leq v_j$ for all $j \in \mathbf{N}$.

PROPOSITION 3.6.10 If $\Omega^G \in \mathbf{\Omega}^G$ satisfies

$$\underline{v} \geq \sum_{p=1}^{M} \{[g^p(\bar{\theta}) - \frac{n-1}{n} g^p(\underline{\theta})] \sum_{r=1}^{n} f^p(r)\} + \bar{\beta},$$

then it is FB* implementable.

PROOF: Consider $\Omega^G \in \mathbf{\Omega}^G$ and also the Groves mechanism M with the Groves transfer obtained from condition (3.4.12). Further we take an individual with queue position $k \in \{1, \dots, n\}$ in state θ. The utility of the individual is given by

$$\begin{aligned}
&U_{\mu(k)}(k, t_{\mu(k)}, \theta_{\mu(k)}) \\
&= v_{\mu(k)} - \sum_{p=1}^{M} f^p(k) g^p(\theta_{\mu(k)}) - \beta_{\mu(k)}(\theta_{\mu(k)}) + t_{\mu(k)}(\theta) \\
&= v_{\mu(k)} - \sum_{p=1}^{M} f^p(k) g^p(\theta_{\mu(k)}) - \sum_{p=1}^{M} \sum_{r \neq k} f^p(r) g^p(\theta_{\mu(r)}) - \beta_{\mu(k)}(\theta_{\mu(k)}) \\
&\quad +(n-1) \sum_{p=1}^{M} \{ \sum_{q<k} h^p(q) g^p(\theta_{\mu(q)}) + \sum_{q>k} h^p(q-1) g^p(\theta_{\mu(q)}) \} \\
&\geq v_{\mu(k)} - \sum_{p=1}^{M} g^p(\bar{\theta}) \sum_{r=1}^{n} f^p(r) + (n-1) \sum_{p=1}^{M} g^p(\underline{\theta}) \sum_{r=1}^{n-1} h^p(r) - \bar{\beta} \\
&= v_{\mu(k)} - \sum_{p=1}^{M} g^p(\bar{\theta}) \sum_{r=1}^{n} f^p(r) + \tfrac{n-1}{n} \sum_{p=1}^{M} g^p(\underline{\theta}) \sum_{r=1}^{n} f^p(r) - \bar{\beta} \text{ (from (3.3.8))} \\
&= v_{\mu(k)} - \sum_{p=1}^{M} \{ g^p(\bar{\theta}) - \tfrac{n-1}{n} g^p(\underline{\theta}) \} \sum_{r=1}^{n} f^p(r) - \bar{\beta} \\
&\geq \underline{v} - [\sum_{p=1}^{M} \{ g^p(\bar{\theta}) - \tfrac{n-1}{n} g^p(\underline{\theta}) \} \sum_{r=1}^{n} f^p(r) + \bar{\beta}] \\
&\geq 0.
\end{aligned}$$

■

A similar result follows for the first best implementable separable cost queueing model.

Corollary 3.6.1 If a FB implementable $\hat{\Omega} = \langle \mathbf{N}, \Theta, (f, g) \rangle$ satisfies

$$\underline{v} \geq \{[g(\bar{\theta}) - \frac{n-1}{n} g(\underline{\theta})] \sum_{r=1}^{n} f(r)\},$$

then it is FB* implementable.

The prove of this corollary is immediate from the fact that a first best implementable separable cost queueing model is a special case of Ω^G with $M = 1$ and $\beta_j(\theta) = 0$ for all $\theta_j \in \Theta$ and for all $j \in \mathbf{N}$.

From this section one can conclude that with a very general cost structure one cannot FB* implement the queueing model. However, for the separable cost queueing models and general FB implementable class of queueing models one can find lower bounds on the benefit, sufficient for FB* implementability.

3.7 Conclusion

This chapter provides a complete characterization of FB implementability of queueing models. For a queueing model to be first best implementable, it is necessary and sufficient that the type vector of each individual satisfies a certain combinatorial property called CP and that the externality that can be imposed by an individual on the remaining set of individuals satisfies the independence property. The class of queueing models analysed in this chapter are implementable only by Groves mechanism. Therefore, the first best implementability of a queueing problem reduces to the problem of finding appropriate balanced Groves mechanism. The independence property allows for a particular type of separability which matches the separability obtained from the combinatorial property. This chapter identifies a fairly large class of first best implementable queueing models. For completeness, the question of individual rationality of the FB implementable queueing models is analysed in section six. The broad conclusion from this analysis is that if the benefit derived by each individual from the service is sufficiently high, then a FB implementable queueing model satisfies individual rationality.

Chapter 4

Sequencing Models

4.1 Introduction

In a sequencing model there is a large multi-unit firm with each unit in need of the facility provided by a particular repair and maintenance unit. The unit can service only one unit at any given time. Therefore, units which are not attended to, incur a cost for the time they are down. In this framework, the firm's role is like that of a planner who wants to serve the units by forming a queue that minimises the total cost of waiting. Each unit's cost parameter is private information. The objective of the firm is to determine the order in which the units are to be serviced. The presence of private information implies that the firm has an incentive problem. The sequencing model as an incentive problem was studied by Dolan [10]. He provided a mechanism which was incentive compatible but not budget balancing. This model was also analysed by Suijs [36] where he assumed costs to be linear over time. He proved that under this assumption, it is possible to design a mechanism satisfying truth-telling in dominant strategies, efficiency or aggregate cost minimisation and budget-balancedness. He further conjectured that linearity of the costs is crucial for this result.

In this chapter a more general class of cost functions is analysed. In particular, this chapter identifies the class of cost functions that are necessary and sufficient for first best implementability among a particular sub-class of weakly convex functions. The main result is that first best implementability is achieved only if the cost function is linear in time. Thus, we prove the Suijs conjecture for a broad class of models. If the cost function is linear then the queue position of any two units is independent of the preferences revealed by the other units. It is this independence that is crucial for first best implementability. For non-linear cost, independence of this type is not satisfied for all profiles.

This chapter is arranged in the following way: in section two, the model is developed. Section three is the main section of this essay where, among other things, the necessary and sufficient condition for first best implementability is derived. Section four concludes the chapter.

4.2 The Model

Let $\mathbf{N} \equiv \{1, 2, \ldots, n\}$ be the set of units of a multi-unit firm. Each unit $j \in \mathbf{N}$ have a cost parameter $\theta_j \in \Theta$ which belongs to an interval in the non-negative orthant $\mathbf{R}_+$ of the real line $\mathbf{R}$ and a servicing cost s_j that belongs to the positive orthant $\mathbf{R}_{++}$ of the real line. Let $C(\tau; \theta_j) = \theta_j F(\tau) + \beta_j$ measure the cost of waiting $\tau(\in \mathbf{R}_+)$ periods in the queue for unit $j \in \mathbf{N}$ with cost parameter θ_j. The server's aim is to find an efficient queue i.e. a queue that minimises the aggregate cost. By means of a permutation σ of $\mathbf{N}$ one can describe the position of each unit in the queue. Specifically, $\sigma_j = k$ indicates that unit j has the kth position in the queue. Given a permutation or a queue $\sigma = (\sigma_1, \ldots, \sigma_n)$ the cost of unit $j \in \mathbf{N}$ is $C(\tau(\sigma_j); \theta_j)$. Let Σ be the set of all possible permutations of $\mathbf{N}$. For a particular queue $\sigma \in \Sigma$, define $P(\sigma, j) = \{p \in \mathbf{N}/\{j\} \mid \sigma_j > \sigma_p\}$ to be the predecessor set of j. Given a queue $\sigma \in \Sigma$ and a unit j,

$$C(S_j(\sigma); \theta_j) = \theta_j F(S_j(\sigma)) + \beta_j \tag{4.2.1}$$

measures the cost of waiting in the queue for unit j where $S_j(\sigma) = \sum_{l \in P(\sigma, j)} s_l$ −

s_j.

The following assumptions are made about the cost function F in (4.2.1).

ASSUMPTION 1 $F'(\tau) > 0$ and $F''(\tau) \geq 0$ for all $\tau \in \mathbf{R}_{++}$.

For all real numbers $\{a, b\}$ with $a > b$, define the function f as follows:

$$f(x) = \frac{F(a+b+x) - F(b+x)}{F(a+b+x) - F(a+x)} \tag{4.2.2}$$

ASSUMPTION 2 The function $f(x)$ is non-decreasing in $x \in \mathbf{R}_+$.

The first assumption restricts F to be twice differentiable with positive first derivative and non-negative second derivative. Observe that if $F''(\tau) = 0$ for all τ i.e if the function is linear, then from (4.2.2) it follows that $f(x) = \frac{a}{b}$. Therefore, $f'(x) = 0$ for all $x \geq 0$. It is not difficult to argue the converse i.e. if $f'(x) = 0$ for all $x \geq 0$, then the function F is linear. For $f'(x) > 0$, it is necessary that the function F is **strictly convex**. In the class of strictly convex functions, functions of the form $F(\tau) = a_r\tau^r$ for all real number $r \geq 2$, satisfies Assumptions 1 and 2 provided $a_r > 0$. If the cost function F is a polynomial of the form $F(\tau) = \sum_{i=0}^{k} a_i\tau^i$, then F satisfies Assumptions 1 and 2 with some restrictions on the co-efficients $\{a_1, a_2, \ldots, a_k\}$. For example, if $k = 2$ i.e. $F^*(\tau) = a_0 + a_1\tau + a_2\tau^2$ then F^* satisfies Assumptions 1 and 2 for all $a_1 \geq 0$ and $a_2 > 0$. For $k = 3$ i.e. $F^{**}(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3$, F^{**} satisfies Assumptions 1 and 2 if and only if $a_1 \geq 0$, $a_3 > 0$ and $2a_2^2 \geq 3a_1a_3$. Let $\mathbf{F}$ be the class of cost function satisfying Assumptions 1 and 2. Observe that $\mathbf{F}$ is a sub-class of weakly convex F functions.

The utility of unit $j \in \mathbf{N}$ with cost parameter θ_j is given by

$$U_j(\sigma_j, t_j; \theta_j) = v_j - C(S_j(\sigma); \theta_j) + t_j$$

where v_j is the benefit derived by unit j from the service and t_j is the transfer that it receives.

For a given $s = (s_1, \ldots, s_n)$, let $\theta = (\theta_1, \ldots, \theta_{j-1}, \theta_j, \theta_{j+1}, \ldots, \theta_n)$ be a state of the world or a profile and let (θ'_j, θ_{-j}) be another profile of the form $(\theta_1, \ldots, \theta_{j-1}, \theta'_j, \theta_{j+1}, \ldots, \theta_n)$ where both θ and (θ'_j, θ_{-j}) belong to Θ^n. Consider the problem of the server whose objective is to minimise the aggregate cost of waiting in the queue. A queue σ^*, given s, is **efficient** or minimises aggregate waiting cost if

$$\sigma^* \in argmin_{\sigma \in \Sigma} \sum_{j \in \mathbf{N}} C(S_j(\sigma); \theta_j).$$

Throughout this analysis the servicing time vector $s = (s_1, \ldots, s_n)$ is assumed to be common knowledge. If the server also knows $\theta = (\theta_1, \ldots, \theta_n)$ then he can calculate the efficient queue and service the units accordingly. However, as θ_j is private information to unit j, the server's problem then is to design a **mechanism** that will elicit this information truthfully. Formally, a mechanism $\mathbf{M}$ is a pair $\langle \sigma, \mathbf{t} \rangle$ where $\sigma : \Theta^n \to \Sigma$ and $\mathbf{t} \equiv (t_1, \ldots, t_n) : \Theta^n \to \mathbf{R}^n$. A **sequencing problem** under incomplete information is written as $\Omega = \langle \mathbf{N}, F, \Theta \rangle$ where $\mathbf{N}$ is the number of units of a firm in need of the facility, F represents the cost of each unit of the firm which takes identical functional form for all units $j \in \mathbf{N}$ and Θ is the type space of each unit representing the cost parameter. Under $\mathbf{M} = \langle \sigma, \mathbf{t} \rangle$, given all others' announcement θ_{-j}, the utility of unit j of type θ_j when its announcement is θ_j' is given by
$U_j(\sigma_j(\theta_j', \theta_{-j}), t_j(\theta_j', \theta_{-j}), \theta_j) = v_j - C(S_j(\sigma(\theta_j', \theta_{-j})); \theta_j) + t_j(\theta_j', \theta_{-j})$.

DEFINITION 4.2.34 A sequencing problem $\Omega = \langle \mathbf{N}, F, \Theta \rangle$ is said to be **implementable** if there exists an efficient rule $\sigma^* : \Theta^n \to \Sigma$ and a mechanism $\mathbf{M} = \langle \sigma^*, \mathbf{t} \rangle$ such that for all $j \in \mathbf{N}$, for all $(\theta_j, \theta_j') \in \Theta^2$, and for all $\theta_{-j} \in \Theta^{n-1}$,

$$U_j(\sigma_j^*(\theta), t_j(\theta); \theta_j) \geq U_j(\sigma_j^*(\theta_j', \theta_{-j}), t_j(\theta_j', \theta_{-j}); \theta_j)$$

This definition says that for any given θ_{-j}, unit j cannot benefit by reporting anything other than it's true type. In other words, truth-telling is a dominant strategy for all units. Moreover, implementability also means that in each state the queue selected is an efficient one.

DEFINITION 4.2.35 A sequencing problem $\Omega = \langle \mathbf{N}, F, \Theta \rangle$ is **first best implementable** or **FB implementable** if there exists a mechanism $\mathbf{M} = \langle \sigma^*, \mathbf{t} \rangle$ which implements it and such that, for all $\theta \in \Theta^n$, $\sum_{j \in \mathbf{N}} t_j(\theta) = 0$.

Thus, a sequencing problem is first-best implementable if, it can be implemented in a manner such that aggregate transfers are zero in every state of the world. In such problems, incomplete information does not impose any welfare cost.

Define the minimum cost function $C : \Theta^n \times \Theta^n \to \mathbf{R}$. For a state θ' with announcement θ,

$$C(\sigma^*(\theta); \theta') = \sum_{j \in \mathbf{N}} C_j(S_j(\sigma_j^*(\theta)); \theta_j')$$

where $\sigma^*(\theta) \in argmin_{\sigma \in \Sigma} \sum_{j \in \mathbf{N}} C_j(S_j(\sigma); \theta_j)$. For simplicity of notation let $C(\theta) \equiv C(\sigma^*(\theta); \theta)$. In other words, $C(\theta)$ represents the minimum cost when announced state θ is also the true state.

DEFINITION 4.2.36 A mechanism $\mathbf{M} = \langle \sigma, \mathbf{t} \rangle$ is a Groves mechanism if, for all $j \in \mathbf{N}$ and for all $\theta \in \Theta^n$,

$$t_j(\theta) = -C(\theta) + C_j(S_j(\sigma^*(\theta)); \theta_j) + \gamma_j(\theta_{-j}) \tag{4.2.3}$$

In a Groves mechanism the transfer of any unit $j \in \mathbf{N}$ in any state θ is the negative of minimum cost $C(\theta)$ plus the cost of unit j and a constant $\gamma_j(\theta_{-j})$. The utility of unit j with a Groves transfer is its benefit v_j less the minimum cost in state θ plus the constant. It is well known that such a transfer results in dominant strategy incentive compatibility because the servers' objective of minimising the aggregate cost is now an objective of unit j as well and this is true for all $j \in \mathbf{N}$.

REMARK 4.2.11 A sequencing model Ω is implementable if and only if the mechanism is a Groves mechanism. This result is not new in the literature because under relatively weak assumptions on the domain of preferences.

Groves mechanisms have been shown (by Green and Laffont [14], Holmström [22] and more recently by Suijs [36]) to be the only class that satisfies the implementability condition. The domain of any sequencing model with $F \in \mathbf{F}$ satisfies Holmström's definition of "convex" domains and Suijs' definition of "graph connected" domains. Thus it follows from Theorem 2 of Holmström and Theorem 3.2 of Suijs that sequencing models are implemented uniquely by Groves mechanism.

The main difficulty with Groves mechanisms are that they are not balanced for a broad class of public decision problems. The question of whether or not Groves mechanism can be FB implemented for sequencing models is addressed in the next section.

4.3 Main Result

In this section the primary objective is to look at the question of first best implementability of a particular class of sequencing models. Let $\mathbf{\Omega}^C$ be the class of sequencing models with the cost function $F \in \mathbf{F}$. The main objective in this section is to completely characterize the class of sequencing models $\mathbf{\Omega}^C$. In order to pursue this goal, the analysis that follows in the next few paragraphs is necessary.

In a sequencing model with non-linear cost function, it is quite difficult to find an algorithm for calculating the efficient queue. However, if costs are linear, there exists an easy algorithm to calculate the efficient queue. Consider the sequencing model $\Omega^L = \langle \mathbf{N}, F^L, \Theta \rangle \in \mathbf{\Omega}^C$, where $|\mathbf{N}| \geq 3$ and F^L is linear. Consider a particular queue $\sigma = (\sigma_1, \ldots, \sigma_n)$ with j and l as immediate neighbours[1] in the queue with j preceding l i.e. $\sigma_l = \sigma_j + 1$. If their positions are interchanged, then the total cost changes by an amount $\{\theta_l s_j - \theta_j s_l\}$ regardless of their positions compared to all other units. This interchange will lead to a reduction in cost if $\{\theta_l s_j - \theta_j s_l\} > 0$ and will lead to

[1] Units j and l are said to be **immediate neighbours** in a queue $\sigma = (\sigma_1, \ldots, \sigma_n)$ if $\sigma_j = k$ implies that either $\sigma_l = k + 1$ or $\sigma_l = k - 1$.

an increase in cost if $\{\theta_l s_j - \theta_j s_l\} < 0$. Using this observation it is now quite easy to see that for any given $s = (s_1, \ldots, s_n)$ and any given state $\theta \in \Theta$, the efficient queue can be obtained by considering the **urgency index** $u_j = \frac{\theta_j}{s_j}$ for all $j \in \mathbf{N}$. In particular, given s and θ, $\sigma^*(\theta) \in \Sigma$ is efficient if and only if for all $j \neq l$, $u_j > u_l \iff \sigma_j^*(\theta) < \sigma_l^*(\theta)$. Ties can be broken by considering the natural ordering i.e. if $u_j = u_l$ then $\sigma_j^*(\theta) < \sigma_l^*(\theta)$ if $j < l$[2]. Consider two units j and l and assume without loss of generality $u_j > u_l$. Then unit j will always stay ahead of unit l independent of the preferences announced by all other units. Linearity implies that the queue position of any two units j and l is **independent** of the preferences announced by all other units. Suijs [36] conjectured that it is this independence that drives first best implementability in Ω^L.

In non-linear cost models, the primary hurdle is to obtain an algorithm for finding efficient queue. The assumptions made about the cost function, in the previous section, is primarily to reduce the difficulties, that one may encounter, while calculating the efficient queue in a sequencing model with non-linear cost. The following observation can be made about the efficient queue of any sequencing model $\Omega^* \in \mathbf{\Omega}^C/\{\Omega^L\}$.

OBSERVATIONS:

1. If $s_j \leq s_l$ and $\theta_j \geq \theta_l$ with at least one strict inequality, then $\sigma_j^*(\theta) < \sigma_l^*(\theta)$ for all $\theta_{-j-l} \in \Theta^{n-2}$ and for all given s_{-j-l}. The proof of this observation follows quite easily from the fact that $F \in \mathbf{F}/\{F^L\}$ is weakly convex.

2. Let $s_j < s_l$, $\theta_j < \theta_l$ and $\theta_j F(s + s_j + s_l) + \theta_l F(s + s_l) > \theta_l F(s + s_j + s_l) + \theta_j F(s + s_j)$. The third inequality means the following: If a set of units $\tilde{P} \subset \mathbf{N}/\{j, l\}$ with $s = \sum_{p \in \tilde{P}} s_p$ are served before units j and l, then it is more efficient to serve j before unit l (i.e. $\sigma_l = \sigma_j + 1$). Now consider a set $\hat{P}$ such that $\tilde{P} \subset \hat{P} \subseteq \mathbf{N}/\{j, l\}$ and $\hat{s} = \sum_{p \in \hat{P}} s_p > s$. If

[2]See Curiel, Pederzoli and Tijs [6].

the set of units $\hat{P}$ are served before units j and l, then serving unit j before unit l will continue to remain more efficient simply because from Assumption (2) and the above inequalities we know that

$$\frac{\theta_l}{\theta_j} < \frac{F(s+s_j+s_l)-F(s+s_j)}{F(s+s_j+s_l)-F(s+s_l)} \leq \frac{F(\hat{s}+s_j+s_l)-F(\hat{s}+s_j)}{F(\hat{s}+s_j+s_l)-F(\hat{s}+s_l)}$$

for all $0 \leq s < \hat{s}$. Note that the first two inequalities i.e. $s_j < s_l$ and $\theta_j < \theta_l$ is crucial for this argument to go through.

THEOREM 4.3.11 For $|\mathbf{N}| \geq 3$, $\Omega^* \in \mathbf{\Omega}$ is first best implementable **if and only if** $\Omega^* = \Omega^L$.

Before going into the proof of this Theorem, a Lemma (due to Walker [41]) is stated. For this Lemma, some more notations are introduced. Consider two profiles $\theta = (\theta_1, \ldots, \theta_n)$ and $\theta' = (\theta'_1, \ldots, \theta'_n)$. Define for $S \subseteq \mathbf{N}$, a type $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$. Therefore, for each $S \subseteq \mathbf{N}$, a state $\theta(S)$ is of the form $(\theta_1(S), \ldots, \theta_n(S))$.

LEMMA 4.3.6 Ω is FB implementable only if for all $(\theta, \theta') \in \Theta^n \times \Theta^n$, $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} C(\theta(S)) = 0$.

Given the form of the transfer (4.2.3), balancedness requires that the minimum aggregate cost is $(n-1)$ type separable i.e. $(n-1)C(\theta) = \sum_{j \in \mathbf{N}} \gamma_j(\theta_{-j})$. Thus it follows that for any two profiles θ and θ', $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} C(\theta(S)) = \frac{1}{(n-1)} \sum_{j \in \mathbf{N}} \sum_{S \subseteq \mathbf{N}} (-1)^{|S|} \gamma_j(\theta_{-j}(S)) = 0$. It is quite clear that if the function C has a separable form , then it must satisfy an appropriate restriction on the nth order cross partial derivative. The condition in Lemma 4.3.6 is analogous with this derivative for finite changes.

PROOF OF THEOREM 4.3.11: For the sufficiency part of the Theorem see Suijs [36]. To prove the necessity part of the Theorem we start by assuming that the claim in the Theorem is false i.e. there exists a first best implementable $\Omega^* \in \mathbf{\Omega}^C/\{\Omega^L\}$ with the cost function $F(\neq F^L)$. From Assumption 2, we know that there exists $0 < \alpha_1 < \alpha_2 < \alpha_3$ such that

$$\tilde{w}(\alpha_2, \alpha_3) \equiv \frac{F(\alpha_2+\alpha_3)-F(\alpha_2)}{F(\alpha_2+\alpha_3)-F(\alpha_3)} < \frac{F(\alpha_1+\alpha_2+\alpha_3)-F(\alpha_1+\alpha_2)}{F(\alpha_1+\alpha_2+\alpha_3)-F(\alpha_1+\alpha_3)} \equiv \hat{w}(\alpha_1, \alpha_2, \alpha_3).$$

Consider any set of real numbers $\{\alpha_4, \alpha_5, \ldots, \alpha_n\}$ satisfying $\alpha_3 < \alpha_4 < \ldots < \alpha_n$. Let the servicing time vector $s = (s_1 = \alpha_1, s_2 = \alpha_2 = 2, s_3 = \alpha_3, s_4 = \alpha_4, \ldots, s_n = \alpha_n)$. Consider $\{\beta_2, \beta_3\} \in \Theta \times \Theta$ satisfying $\beta_2 < \beta_3$ and $\tilde{w}(\alpha_2, \alpha_3) < \{\beta_3\}/\{\beta_2\} < \hat{w}(\alpha_1, \alpha_2, \alpha_3)$. Consider real numbers $\hat{x}$ and $\epsilon = 1/\hat{x}$ such that $\hat{x}^{\frac{1}{n}} = [F^*(\sum_{j=1}^{n} s_j) - F^*(s_1)]/[F^*(s_2) - F^*(s_1)]$. Using $\alpha, \beta, \hat{x}, \epsilon$ we construct two profiles θ and θ' such that $\theta = (\theta_1 = \hat{x}, \theta_2 = \beta_2, \theta_3 = \beta_3, \theta_4 = \epsilon, \ldots, \theta_n = \epsilon)$ and $\theta' = (\theta'_1 = \epsilon, \theta'_2 = \beta_3, \theta'_3 = \beta_2, \theta'_4 = \hat{x}, \ldots, \theta'_n = \hat{x})$[3]. Now for each $S \subseteq \mathbf{N}$, consider the profile $\theta(S) = (\theta_1(S), \ldots, \theta_j(S), \ldots, \theta_n(S))$ where $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$.

The following observations can be made about any state $\theta(S)$. Units having a very high cost parameter of $\hat{x}$ are served before units 2 and 3 and the units having a very low cost parameter of ϵ are served after units 2 and 3 in $\sigma^*(\theta(S))$. Let $X(\theta(S)) \subseteq \mathbf{N}/\{2,3\}$ be a set such that $\theta_j(S) = \hat{x}$ for all $j \in X(\theta(S))$ and $E(\theta(S)) \subseteq \mathbf{N}/\{2,3\}$ be another set such that $\theta_j(S) = \epsilon$ for all $j \in E(\theta(S))$. For all pairs $\{j, l\} \in X(\theta(S))$, $\sigma^*_j(\theta(S)) < \sigma^*_l(\theta(S))$ if $j < l$. The reason follows quite easily from Observation 1. Note that $\theta_j(S) = \theta_l(S) = \hat{x}$ and $j < l$ implies that $s_j < s_l$. By the same reasoning it follows that for all pairs $\{j, l\} \in E(\theta(S))$, $\sigma^*_j(\theta(S)) < \sigma^*_l(\theta(S))$ if $j < l$. From the construction of β_2 and β_3 it follows that in a state $\theta(S)$ such that $X(\theta(S)) = \phi$, $\sigma^*_3(\theta_2, \theta_3, \theta_{-2-3}(S)) = 1 < \sigma^*_2(\theta_2, \theta_3, \theta_{-2-3}(S)) = 2$. Observe that there is only one state where $X(\theta(S)) = \phi$-the state with $S = \{1\}$. From the constructions of β_2, β_3 and states θ and θ' and from Observation 2 it follows that for all states $\theta(S)$ other than with $S = \{1\}$, $\sigma^*_2(\theta(S)) < \sigma^*_3(\theta(S))$. Therefore, for all states where $X(\theta(S)) \neq \phi$, $\sigma^*_2(\theta(S)) < \sigma^*_3(\theta(S))$. Thus if $|X(\theta(S))| = k$, then $\sigma^*_2(\theta(S)) = k + 1$ and $\sigma^*_3(\theta(S)) = k + 2$ for all

[3] For $|\mathbf{N}| = 3$, this construction is $\theta = (\theta_1 = \hat{x}, \theta_2 = \beta_2, \theta_3 = \beta_3)$ and $\theta' = (\theta'_1 = \epsilon, \theta'_2 = \beta_3, \theta'_3 = \beta_2)$.

$k \in \{1, \ldots, n-2\}$.

Using the above observations and applying Lemma 4.3.6 on θ and θ' for the servicing vector s and after simplification we get

$\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} C(\theta(S)) = \beta_2[F^*(\alpha_2) - F^*(\alpha_2+\alpha_3)] + \beta_3[F^*(\alpha_2+\alpha_3) - F^*(\alpha_3)] > 0.$
Therefore, Lemma 4.3.6 is violated. This is a contradiction. ∎

We now give an explicit form of the Groves transfer that first best implements $\Omega^L = \langle \mathbf{N}, F^L, \Theta \rangle$ with $F^L(\tau) = a_0 + a_1\tau$ for all $\tau \in \mathbf{R}_+$. One of the reasons we do so is because this is not provided in Suijs [36]. Consider the sequencing model Ω^L. Define, by eliminating $l \in \mathbf{N}$, the **l-reduced sequencing model** of Ω^L to be $\Omega^L_{N-l} = \langle \mathbf{N}-l, F^L, \Theta \rangle$. In any state $\theta_{-l} \in \Theta^{n-1}$, let $\sigma^*(\theta_{-l})$ be the efficient queue in Ω^L_{N-l}. In other words,

$$\sigma^*(\theta_{-l}) \in argmin_{\tilde{\sigma} \in \Sigma(N-l)} \sum_{j \neq l} \theta_j(\tilde{\sigma}_j)$$

where $\Sigma(N-l)$ is the set of all possible permutations of $\{1, \ldots, n-1\}$ and $\tilde{\sigma}_j$ is the position of unit $j (\neq l)$ in the particular queue $\tilde{\sigma} \in \Sigma(N-l)$. In short, Ω^L_{N-l} is a l-reduced sequencing model of $(n-1)$ units obtained from Ω^L by excluding $l \in \mathbf{N}$. For Ω^L_{N-l}, define $P(\sigma^*(\theta_{-l}), j) = \{p \in \mathbf{N}/\{j, l\} \mid \sigma^*_j(\theta_{-l}) > \sigma^*_p(\theta_{-l})\}$ to be the predecessor set of unit j in state θ_{-l}[4]. Under the efficient queue $\sigma^*(\theta_{-l})$ in state θ_{-l}, $P(\sigma^*(\theta_{-l}), j)$ is the set of units receiving the service before unit j.

Consider a mechanism $\mathbf{M}$ that implements Ω^L. The transfer must be of the form (4.2.3) for all $j \in \mathbf{N}$ and for all $\theta \in \Theta^n$. For FB implementability we need

$$(n-1)C(\theta) = \sum_{j \in \mathbf{N}} \gamma_j(\theta_{-j}) \tag{4.3.4}$$

for all $\theta \in \Theta^n$.

Now we want to find a function γ_j such that condition (4.3.4) is satisfied. Let $p(r) \subseteq P(\sigma^*(\theta_{-l}, j)$ such that $|p(r)| = r$. Define $z_{jr} : \Theta^{n-1} \to \mathbf{R}$ and

[4] Recall that for Ω^L, $P(\sigma^*(\theta), j) = \{p \in \mathbf{N}/\{j\} \mid \sigma^*_j(\theta) > \sigma^*_p(\theta)\}$ is the predecessor set of unit j in state θ. In other words, under the efficient queue $\sigma^*(\theta)$ in state θ, $P(\sigma^*(\theta), j)$ is the set of units receiving the service before unit j.

$h_j : \Theta^{n-1} \to \mathbf{R}$ as follows:
for all $j \in \mathbf{N}$, for all $l \neq j$, for all $r \in \{1, \ldots, n-1\}$ and for all $\theta_{-l} \in \Theta^{n-1}$,

$$z_{jr}(\theta_{-l}) = \sum_{p(r-1) \subseteq P(\sigma^*(\theta_{-l}), j)} \{ \sum_{q \in p(r-1)} s_q + s_j \}$$

and

$$h_j(\theta_{-l}) = \sum_{r=1}^{m} (-1)^{m-r} \frac{(n-m-1)!(m-r)!}{(n-r)!} z_{jr}(\theta_{-l})$$

where $\sigma_j^*(\theta_{-l}) = m \in \{1, \ldots, n-1\}$. We first show that $\sum_{l \neq j} h_j(\theta_{-l}) = S_j(\sigma^*(\theta))$. This can be done by considering two cases.

Case (1): $\sigma_j^*(\theta) \neq n$. Here we can have two sub-cases- the first possibility being that the excluded unit l is not a predecessor of j in state θ. The second possibility being that the unit excluded is a predecessor of j in state θ. In the first sub-case observe that

$$\sum_{l \notin P(\sigma^*(\theta), j)} h_j(\theta_{-l})$$
$$= \sum_{l \notin P(\sigma^*(\theta), j)} \sum_{r=1}^{m} (-1)^{m-r} \tfrac{(n-m-1)!(m-r)!}{(n-r)!} z_{jr}(\theta_{-l})$$
$$= (n-m) \sum_{r=1}^{m} (-1)^{m-r} \tfrac{(n-m-1)!(m-r)!}{(n-r)!} \sum_{p(r-1) \subseteq P(\sigma^*(\theta), j)} \{ \sum_{q \in p(r-1)} s_q + s_j \}.$$

From the urgency index criterion for finding the efficient queue in Ω^L it follows that for all $l \notin P(\sigma^*(\theta), j)$, the predecessor set of unit j in Ω^L_{N-l} is $P(\sigma^*(\theta_{-l}), j) = P(\sigma^*(\theta), j)$.
Therefore,

(1) $$\sum_{l \notin P(\sigma^*(\theta), j)} h_j(\theta_{-l})$$
$$= \sum_{r=1}^{m} (-1)^{m-r} \tfrac{(n-m)!(m-r)!}{(n-r)!} \sum_{p(r-1) \subseteq P(\sigma^*(\theta), j)} \{ \sum_{q \in p(r-1)} s_q + s_j \}.$$

In the second sub-case observe that

$$\sum_{l \in P(\sigma^*(\theta), j)} h_j(\theta_{-l})$$
$$= \sum_{l \in P(\sigma^*(\theta), j)} \sum_{r=1}^{m-1} (-1)^{m-r-1} \tfrac{(n-m)!(m-r-1)!}{(n-r)!} z_{jr}(\theta_{-l})$$

$= \sum_{r=1}^{m-1} (-1)^{m-r-1} \frac{(n-m)!(m-r-1)!}{(n-r)!} (m-r) \sum_{p(r-1) \subseteq P(\sigma^*(\theta), j)} \{ \sum_{q \in p(r-1)} s_q + s_j \}.$

The last step follows from the following relation

$$\sum_{l \in P(\sigma^*(\theta), j)} z_{jr}(\theta_{-l}) = (m-r) \sum_{p(r-1) \subseteq P(\sigma^*(\theta), j)} \{ \sum_{q \in p(r-1)} s_q + s_j \}.$$

Therefore,

(2) $\sum_{l \in P(\sigma^*(\theta), j)} h_j(\theta_{-l})$

$= \sum_{r=1}^{m-1} (-1)^{m-r-1} \frac{(n-m)!(m-r)!}{(n-r)!} \sum_{p(r-1) \subseteq P(\sigma^*(\theta), j)} \{ \sum_{q \in p(r-1)} s_q + s_j \}.$

Adding (1) and (2) we get

$\sum_{l \neq j} h_j(\theta_{-l})$

$= \sum_{r=1}^{m-1} (-1)^{m-r-1} \{1-1\} \frac{(n-m)!(m-r)!}{(n-r)!} \sum_{p(r-1) \subseteq P(\sigma^*(\theta), j)} \{ \sum_{q \in p(r-1)} s_q + s_j \}$

$+ \sum_{p(m-1) \subseteq P(\sigma^*(\theta), j)} \{ \sum_{q \in p(m-1)} s_q + s_j \}.$

$= \{ \sum_{q \in P(\sigma^*(\theta), j)} s_q + s_j \}$

The last step follows from $p(m-1) = P(\sigma^*(\theta), j)$. Thus $\sum_{l \neq j} h_j(\theta_{-l}) = S_j(\sigma^*(\theta))$.

Case (2): $\sigma_j^*(\theta) = n$.

Here $l \in P(\sigma^*(\theta), j)$, for all $l \neq j$. Therefore, it follows that

$\sum_{l \notin P(\sigma^*(\theta), j)} h_j(\theta_{-l})$

$= \sum_{r=1}^{n-1} (-1)^{n-r-1} \frac{(n-m-1)!(m-r)!}{(n-r)!} \sum_{p(r-1) \subseteq P(\sigma^*(\theta), j)} \{ \sum_{q \in p(r-1)} s_q + s_j \}$

$= \{ \sum_{l \neq j} s_l + s_j \} = S_j(\sigma^*(\theta)).$

Define, $\tilde{h}_j(\theta_{-l}) = \frac{a_0}{n-1} + a_1 h(\theta_{-l})$. Therefore, from elementary calculation it follows that $\sum_{l \neq j} \tilde{h}_j(\theta_{-l}) = a_0 + a_1 S_j(\sigma^*(\theta)) = F^L(S_j(\sigma^*(\theta)))$.

Now consider a mechanism $\mathbf{M}^*$ with $\gamma_j = \gamma_j^*$ such that for all $j \in \mathbf{N}$ and for all θ_{-j}, $\gamma_j^*(\theta_{-j}) = (n-1) \sum_{l \neq j} \tilde{h}_l(\theta_{-j}) \theta_l$. Here,

$\sum_{j \in \mathbf{N}} \gamma_j^*(\theta_{-j})$
$= (n-1) \sum_{j \in \mathbf{N}} \sum_{l \neq j} \tilde{h}_l(\theta_{-j})\theta_l$
$= (n-1) \sum_{j \in \mathbf{N}} \sum_{l \neq j} \tilde{h}_j(\theta_{-l})\theta_j$
$= (n-1) \sum_{j \in \mathbf{N}} \{a_0 + a_1 S_j(\sigma^*(\theta))\}\theta_j$ (from $\sum_{l \neq j} \tilde{h}_j(\theta_{-l}) = a_0 + a_1 S_j(\sigma^*(\theta))$)
$= (n-1)C(\theta)$. Thus, the mechanism $\mathbf{M}^*$ satisfies condition (4.3.4).
The construction of γ_j will be more explicit from the next example.

EXAMPLE 4.3.16 Consider $\Omega^L = \langle \mathbf{N} = \{1,2,3,4\}, F^L, \Theta \rangle$. Therefore, $F^L(\tau) = a_0 + a_1\tau$ for all $\tau > 0$. Let $\frac{\theta_1}{s_1} \geq \frac{\theta_2}{s_2} \geq \frac{\theta_3}{s_3} \geq \frac{\theta_4}{s_4}$. Given the servicing time vector $s = (s_1, s_2, s_3, s_4)$, it is easy to verify using Observation 1 that for the profile $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, $\sigma_j^*(\theta) = j$ for all $j \in \mathbf{N}$. This follows from the urgency index definition of efficiency. Using the form of h_j we get

(1)$h_1(\theta_1, \theta_2, \theta_3) = h_1(\theta_1, \theta_2, \theta_4) = h_1(\theta_1, \theta_3, \theta_4) = \frac{1}{3}s_1$.

(2)$h_2(\theta_1, \theta_2, \theta_3) = h_2(\theta_1, \theta_2, \theta_4) = \frac{1}{2}(s_1 + s_2) - \frac{1}{6}s_1$ and $h_2(\theta_2, \theta_3, \theta_4) = \frac{1}{3}s_2$.

(3)$h_3(\theta_1, \theta_2, \theta_3) = (s_1 + s_2 + s_3) - \frac{1}{2}(s_1 + s_3) - \frac{1}{2}(s_2 + s_3) + \frac{1}{3}s_3$,
$h_3(\theta_2, \theta_3, \theta_4) = \frac{1}{2}(s_2 + s_3) - \frac{1}{6}s_3$ and $h_3(\theta_1, \theta_3, \theta_4) = \frac{1}{2}(s_1 + s_3) - \frac{1}{6}s_3$.

(4)$h_4(\theta_1, \theta_2, \theta_4) = (s_1 + s_2 + s_4) - \frac{1}{2}(s_1 + s_4) - \frac{1}{2}(s_2 + s_4) + \frac{1}{3}s_4$,
$h_4(\theta_1, \theta_3, \theta_4) = (s_1 + s_3 + s_4) - \frac{1}{2}(s_1 + s_4) - \frac{1}{2}(s_3 + s_4) + \frac{1}{3}s_4$,
$h_4(\theta_2, \theta_3, \theta_4) = (s_2 + s_3 + s_4) - \frac{1}{2}(s_2 + s_4) - \frac{1}{2}(s_3 + s_4) + \frac{1}{3}s_4$.

Elementary calculation gives $\sum_{l \neq j} h_j(\theta_{-l}) = S_j(\sigma^*(\theta))$ for all $j \in \mathbf{N}$. Now we can define $\tilde{h}_j(\theta_{-l}) = \frac{a_0}{3} + a_1 h_j(\theta_{-l})$ and check that

$$\sum_{j \neq l} \tilde{h}_j(\theta_{-l}) = a_0 + a_1 S_j(\theta) = F^L(S_j(\theta)).$$

Now it is quite easy to check that $\sum_{j=1}^{n} t_j(\theta) = 0$.

The result derived in this chapter confirms Suijs conjecture that linearity is crucial for a sequencing model to be first best implementable when the cost function is restricted to a sub-class of weakly convex functions. In any sequencing situation, there are a finite set of decisions. The decisions are the set of all possible permutation of the finite set of the number of units. Each permutation represents a queue. One can make a comparative study of the sequencing model Ω^L with that of the classic incentive problem of non-excludable public goods where the set of decisions is also finite. In the public goods problem the decision is whether or not to produce the public good. The public goods problem is not first best implementable because the budget balancedness condition cannot be satisfied in all states of the world. The reason for budget imbalance is the externality that an individual can impose on the remaining set of individuals. Here, an individual, by changing his announcement can change the decision of all other individuals. While, for Ω^L, the externality that can be imposed by a unit on the remaining set of units is more subtle. If a unit is allotted a position k in the queue in some state, then by changing its cost parameter the unit can either change the cost of the units served before it (i.e. its predecessor set) or the cost of the units served after it (i.e. its successor set). The unit cannot simultaneously affect both the predecessor and the successor sets. This sort of externality, which is certainly less severe than the externality in public goods problem, is crucial for first best implementability of sequencing models.

4.4 Conclusion

In this chapter a necessary and sufficient condition for FB implementability for a class of sequencing models is derived. It has been shown that for first best implementability, it is both necessary and sufficient that the cost function is linear. A natural question which arises from our analysis is the following: are the two assumptions on the cost function necessary for our characterization result? We intend to answer this question in future research.

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