EQUIVARIANT COHOMOLOGY WITH LOCAL COEFFICIENTS

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TO THE MEMORY OF MY MOTHER

AND

TO MY FATHER

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CONTENTS

INTRODUCTION

In this thesis we shall present an equivariant analogue of the Steenrod cohomology with local coefficients, and use this to develop an obstruction theory for equivariant fibrations. The work is inspired by a remark of tom Dieck [2] which says that a sensible translation of the classical obstruction theory to equivariant fibrations uses (meaning 'should use') equivariant cohomology with local coefficients.

The equivariant singular cohomology of Illman [9] is unsuitable for obstruction theory for equivariant sections of an equivariant fibration. The difficulty lies in connecting the obstruction cochains which arise from different fixed point sets, and the situation becomes no better even if we work with the Bredon cohomology [1]. Explicitly, if (B, A) is a G-complex pair, and $p : E \longrightarrow B$ is a G-fibration with fibre F, and if ϕ is an equivariant section over $B^n \bigcup A$, then, for two $(n+1)$-cells σ and σ' of B, the values $c_\phi(\sigma)$, $c_\phi(\sigma')$ of the obstruction cochain c_ϕ lie in different groups. In fact we do not have a coefficient system M such that both $c_\phi(\sigma)$ and $c_\phi(\sigma')$ belong to the same group $M(G/G_\sigma)$, even in the case when $G_\sigma = G_{\sigma'}$, as there is no canonical isomorphism between $\pi_n(F_\sigma)$ and $\pi_n(F_{\sigma'})$.

The main problem of the thesis is to extend the local cohomology of Steenrod [18], [20] to the category of G-spaces where G is a compact group, so that the resulting cohomology fits well into equivariant obstruction theory. We construct for a G-space X cohomology $H^*_G(X, M)$, where M is a suitable equivariant local coefficients system on X. The cohomology satisfies all the equivariant Eilenberg-Steenrod axioms, and can also be described in terms of the equivariant cellular structure when X is a G-CW-complex. Moreover, it reduces to the Steenrod cohomology with classical local coefficients system

[20] when G is trivial, and to the equivariant singular cohomology with contravariant coefficient system of Illman [9] when M is simple in certain sense and X is G-path connected. The key idea behind the construction of our cohomology lies in generalizing the classical fundamental groupoid to the equivariant fundamental groupoid of a G-space. The equivariant fundamental groupoid reduces to the classical fundamental groupoid when the group is trivial, and is the basis of the definition of a local coefficients system on a G-space. The local nature of this construction reflects the distinction of our cohomology theory with that of Illman. Whereas our coefficients system depends heavily on the space with which it is associated, the contravariant coefficient system of Illman is independent of the space.

We then build up an obstruction theory for equivariant section of G-fibration using our cohomology, when G is finite. This is accomplished by allowing the classical obstruction theory to dominate on each fixed point set X^H, and then piecing them together over the whole of X in a natural way. Note that if G is finite, then each X^H inherits an ordinary CW structure from the equivariant CW structure of X [13]. This is the basis of our obstruction theory. It seems that some kind of global description of the obstruction cochain, like that cosidered by Whitehead [20], is necessary to tackle the general case when G is compact.

Recently Møller [16] has built up an obstruction theory for a G-fibration by constructing certain groups for G-complexes which he calls the Bredon cohomology groups with local coefficients, where G is finite. ~~But unfortunately~~ However, these groups do not form contravariant functors, ~~Consequently~~ and therefore they do not provide an equivariant cohomology theory with local coefficients. However, if the coefficients system is simple, then the Møller groups do reduce to the Bredon cohomology groups. The present work was done independently without any knowledge of the work of Møller. In our approach, we have been able to present an equivariant singular cohomology theory with local coefficients and an equivariant cellular description of the theory. There are also other equivariant cohomology theories available in the literature, namely Matumoto [14], tom Dieck [2], Kozlowski [10], Willson [22], May [15], etc., but ~~none of these formulations can be used for an obstruction~~ theory ~~for an equivariant fibration~~. They seem to be unsuitable for our purpose, because they do not deal with equivariant local coeffici

As application of our obstruction theory we present the enumeration of regular homotopy classes of immersions of the projective space P^n into R^{2n}, and of the lens space L_p^{2n-1} into R^{4n-2}. We also consider the enumeration problem for immersions of the Grassman manifold $G_k(R^{n+k})$ into R^{2nk}. It may be noted that unlike the non-equivariant theory our computations avoids twisted coefficients. Moreover, our enumeration theorem for P^n, improves a result of Larmore and Rigdon [11].

The thesis is divided into five chapters, each of these is subdivided into sections. The principal statements are numbered consecutively in each section, and are variously labeled as definition, remark, lemma, proposition, theorem, or corollary.

Chapter 1 begins with the problem of attaching local coefficients on a G-space X. We introduce the concept of the equivariant fundamental groupoid $\Pi(X, G)$, and define an equivariant local coefficients system M on X as a functor from $\Pi(X, G)$ to the catrgory of abelian groups. Next, we introduce equivariant cohomology groups for G-spaces with local coefficients system. In Chapter 2, we show that the resulting cohomology satisfies the equivariant analogues of the Eilenberg-Steenrod axioms. Chapter 3 concerns the notion of cup product in the equivariant cohomology with local coefficients system, which makes the cohomology a graded ring. Then we give a cellular description of the cohomology groups for a G-CW-complex. In Chapter 4 we develop an obstruction theory for equivariant sections of G-fibrations, when the group is finite. We measure obstructions in extending equivariant sections as elements of cohomology groups with local coefficients system, and study their properties. In Chapter 5 we turn to the enumeration problems of regular homotopy classes of immersions as mentioned above.

CHAPTER 1

CONSTRUCTION OF EQUIVARIANT SINGULAR COHOMOLOGY WITH LOCAL COEFFICIENTS SYSTEM

1.0. Introduction.

In this chapter we shall introduce local coefficients system on a space with compact group action, and define singular Bredon-Illman cohomology with local coefficients system.

Throughout this chapter all spaces will be compactly generated, and G will be a compact group. We shall consider only closed subgroups of G. More generally, we may suppose, as in Illman [9], that any subgroup of G appearing in our discussion arises from a fixed orbit type family for G. Recall that an orbit type family (or an isotropy family, in the language of tom Dieck [2]) is a collection $\mathcal{F}$ of closed subgroups of G such that if $H \in \mathcal{F}$ and K is conjugate to H, then $K \in \mathcal{F}$ also. However, note that there will be no loss of generality if we base our arguments on the orbit type family consisting of all closed subgroups of G.

1.1. Equivariant fundamental groupoid.

1.1.1. DEFINITION. Let X be a G-space. Then define a category $\sum X = \sum(X, G)$ in the following way. The objects of the category are G-maps $x_H : G/H \longrightarrow X$, and a morphism from $x_H : G/H \longrightarrow X$ to $y_K : G/K \longrightarrow X$ is a pair (α, ω), where $\alpha : G/H \longrightarrow G/K$ is a G-map and $\omega : G/H \times I \longrightarrow X$ is a G-homotopy from x_H to $y_K \circ \alpha$. The composition of morphisms $(\alpha_1, \omega_1) : x_H \longrightarrow y_K$ and

$(\alpha_2, \omega_2) : y_K \longrightarrow z_L$ is given by $(\alpha_2 \circ \alpha_1, \omega) : x_H \longrightarrow z_L$ where $\omega : G/H \times I \longrightarrow X$ is

$$\omega(gH, t) = \begin{cases} \omega_1(gH, 2t) & 0 \leq t \leq 1/2 \\ \omega_2(\alpha_1(gH), 2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

The composition is well-defined. ~~and~~ It will be associative, and the morphism $(id_{G/H}, c) : x_H \longrightarrow x_H$ with the constant homotopy c ~~is~~ will be the identity after passing to the quotient by equivalence given in Definition 1.1.3 below.

Two morphisms $(\alpha_1, \omega_1), (\alpha_2, \omega_2) : x_H \longrightarrow y_K$ are *equivalent* if there exists a G-homotopy $j : G/H \times I \longrightarrow G/K$ from α_1 to α_2, and a G-homotopy $k : G/H \times I \times I \longrightarrow X$ from ω_1 to ω_2 such that

$$k(gH, 0, t) = x_H(gH), \text{ and } k(gH, 1, t) = y_K \circ j(gH, t).$$

(If G is finite, then $\alpha_1 = \alpha_2$ and j = constant homotopy.)

The equivalence relation is compatible with the composition defined above.

We shall denote the equivalence class of a morphism (α, ω) by $[\alpha, \omega]$.

1.1.2. REMARKS. If X is a point, then $\sum X$ reduces to the orbit category O_G, whose objects are homogeneous spaces G/H and morphisms are G maps between them.

Let $\eta_X : O_G \longrightarrow \mathcal{TOP}$ (category of spaces) be the fixed point sets system of Elmendorf [7]. Explicitly, η_X is a functor such that $\eta_X(G/H) = X^H$, and, for a G-map $\alpha : G/H \longrightarrow G/K$ given by a subconjugacy relation $g^{-1}Hg \subseteq K$, $\eta_X(\alpha) : X^K \longrightarrow X^H$ is left translation by g.

Recall that, since G is compact and H is closed, we have a canonical homeomorphism $a : Map_G(G/H, X) \longrightarrow X^H$ given by $a(f) = f(eH)$, with inverse given by $a^{-1}(x)(gH) = gx$. Therefore, we may identify an object $x_H : G/H \longrightarrow X$ in $\sum X$ with a point $x'_H = x_H(eH)$ in X^H. Then, if $\alpha : G/H \longrightarrow G/K$ is a morphism in O_G and $y_K : G/K \longrightarrow X$ is an object in $\sum X$, the point in X^H which corresponds to $y_K \circ \alpha : G/H \longrightarrow X$ is given by $y_K \circ \alpha(eH) = \eta_X(\alpha) \cdot y_K(eK)$. Thus a morphism $(\alpha , \omega) : x_H \longrightarrow y_K$ in $\sum X$ corresponds to a path $\langle \alpha , \omega \rangle$ in X^H from x'_H to $\eta_X(\alpha) \cdot y'_K$. Moreover, if two morphisms (α_1 , ω_1), $(\alpha_2 , \omega_2) : x_H \longrightarrow y_K$ are equivalent, then $\langle \alpha_1 , \omega_1 \rangle$ is freely homotopic to $\langle \alpha_2 , \omega_2 \rangle$ along the path $t \longrightarrow y_K \circ j(eH , t)$.

1.1.3. DEFINITION. For a G-space X, we define the ***equivariant fundamental groupoid*** $\Pi X = \Pi(X , G)$ to be the quotient category of $\sum X$ under the equivalence defined above.

Thus objects of ΠX are the same as those of $\sum X$, and a morphism $x_H \longrightarrow y_K$ is an equivalence class $[\alpha , \omega]$ where $(\alpha , \omega) : x_H \longrightarrow y_K$ is a morphism in $\sum X$.

1.1.4. REMARK. Although we call ΠX 'fundamental groupoid', it is not a groupoid in the categorical sense. Here a morphism may not be invertible. However, if G is compact Hausdorff, then any endomorphism $[\alpha , \omega] : x_H \longrightarrow x_H$ is invertible, with inverse $[\alpha^{-1} , \omega' \circ (\alpha^{-1} \times id_I)]$, where ω' is the inverse homotopy $\omega'(gH , t) = \omega(gH , 1-t)$. Note that in this case any G-map $\alpha : G/H \longrightarrow G/H$ is a homeomorphism.

The category is also known as 'discrete fundamental category' in the literature [2], [12].

If G is trivial, then ΠX reduces to the classical fundamental groupoid of X. Again, if G is finite, then for a fixed H, the objects x_H together with morphisms $x_H \longrightarrow y_H$ consisting of pairs $(id_{G/H} , [\omega])$, where $[\omega]$ is a G-homotopy class rel $G/H \times \partial I$

of G-homotopies $\omega : G/H \times I \longrightarrow X$ such that $\omega_0 = x_H$ and $\omega_1 = y_K$, constitute a subcategory of ΠX which is precisely the fundamental groupoid of X^H.

If X is a point, then ΠX reduces to the homotopy category hO_G of orbits G/H and G-homotopy classes of G-maps between them.

1.1.5. DEFINITION. We define a functor $\Phi : \Pi X \longrightarrow hO_G$ by sending $x_H : G/H \longrightarrow X$ to G/H, and sending $[\alpha , \omega]$ to the homotopy class $[\alpha]$ of α.

Then a G-map $f : X \longrightarrow Y$ defines a functor $\Pi f : \Pi X \longrightarrow \Pi Y$ in the following way.

$$\Pi f(x_H : G/H \longrightarrow X) = f o x_H,$$

$$\Pi f([\alpha , \omega] : x_H \longrightarrow y_K) = [\alpha , f o \omega] : f o x_H \longrightarrow f o y_K.$$

This functor has the property that $\Phi o \Pi f = \Phi$.

1.1.6. LEMMA. *A G-homotopy* $h : f \simeq f'$ *induces a natural equivalence* $h_{\#} : \Pi f \longrightarrow \Pi f'$.

Proof. Define

$$h_{\#}(x_H : G/H \longrightarrow X) = [id_{G/H} , h o (x_H \times id_I)] : f o x_H \longrightarrow f' o x_H.$$

Then, it is necessary to verify that for any morphism $[\alpha , \omega] : x_H \longrightarrow y_K$ in ΠX, the following diagram is commutative.

$$\begin{array}{ccc} f o x_H & \xrightarrow{h_{\#}(x_H)} & f' o x_H \\ {\scriptstyle \Pi f[\alpha , \omega]} \downarrow & & \downarrow {\scriptstyle \Pi f'[\alpha , \omega]} \\ f o y_K & \xrightarrow[h_{\#}(y_K)]{} & f' o y_K \end{array}$$

Write $(\alpha, \psi_1) = (id_{G/K}, h\circ(y_K \times id_I)) \circ (\alpha, f\circ\omega)$

and $(\alpha, \psi_2) = (\alpha, f'\circ\omega) \circ (id_{G/H}, h\circ(x_H \times id_I))$

where

$$\psi_1(gH, s) = \begin{cases} f\circ\omega(gH, 2s) = h(\omega(gH, 2s), 0), & 0 \le s \le \frac{1}{2} \\ h\circ(y_K \times id_I)(\alpha(gH), 2s-1) = h(y_K \circ \alpha(gH), 2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$\psi_2(gH, s) = \begin{cases} h\circ(x_H \times id_I)(gH, 2s) = h(x_H(gH), 2s), & 0 \le s \le \frac{1}{2} \\ f'\circ\omega(gH, 2s-1) = h(\omega(gH, 2s-1), 1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Then (α, ψ_1) becomes equivalent to (α, ψ_2) in view of the G-homotopies $j : G/H \times I \longrightarrow G/K$, and $k : G/H \times I \times I \longrightarrow Y$ given by

$$j(gH, t) = \alpha(gH)$$

$$k(gH, s, t) = \begin{cases} h(\omega(gH, 2s(1-t)), 2st), & 0 \le s \le \frac{1}{2} \\ h(\omega(gH, 1-2t(1-s)), 2s(1-t) + 2t - 1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Therefore, $h_{\#}$ is a natural transformation.

The inverse of $h_{\#}$, $h'_{\#}: \Pi f' \longrightarrow \Pi f$, is given by $h'_{\#}(x_H) = [id_{G/H}, h'\circ(x_H \times id_I)]$ where h' is the inverse homotopy of h. □

1.2. Equivariant local coefficients system.

1.2.1. DEFINITION. An *equivariant local coefficients system* of abelian groups (or simply, an equivariant local system) on a G-space X is a contravariant functor M from ΠX to the category $\mathcal{A}b$ of abelian groups.

If G is trivial, then M becomes the classical local coefficients system on X.

1.2.2. EXAMPLE. Let X be a G-space and $n > 1$. Then define a contravariant functor $\underline{M} : \sum X \longrightarrow \mathcal{A}b$ by setting $\underline{M}(x_H) = \pi_n(X^H, x'_H)$ for an object x_H in $\sum X$, where $x'_H = x_H(eH) \in X^H$, and setting

$$\underline{M}(\alpha, \omega) = \langle \alpha, \omega \rangle_*^{-1} \circ \pi_n(\eta_X(\alpha))$$

for a morphism $(\alpha, \omega) : x_H \longrightarrow y_K$ in $\sum X$. Here $\langle \alpha, \omega \rangle$ is the path in X^H from x'_H to $y_K \circ \alpha(eH)$, which is induced by (α, ω), and

$$\langle \alpha, \omega \rangle_* : \pi_n(X^H, x'_H) \longrightarrow \pi_n(X^H, y_K \circ \alpha(eH))$$

is the corresponding induced isomorphism, and

$$\pi_n(\eta_X(\alpha)) : \pi_n(X^K, y'_K) \longrightarrow \pi_n(X^H, \eta_X(\alpha) \cdot y'_K)$$

is the homomorphism induced by $\eta_X(\alpha)$ (see 1.1.2).

We shall show that if $(\alpha_1, \omega_1), (\alpha_2, \omega_2) : x_H \longrightarrow y_K$ are equivalent morphisms in $\sum X$, then $\underline{M}(\alpha_1, \omega_1) = \underline{M}(\alpha_2, \omega_2)$. For this purpose, suppose that we have G-homotopies $j : \alpha_1 \simeq \alpha_2$ and $k : \omega_1 \simeq \omega_2$ such that $k(gH, 0, t) = x_H(gH)$, and $k(gH, 1, t) = y_K \circ j(gH, t)$. Then $\sigma(t) = k(eH, 1, t)$ is a path in X^H from $y_K \circ \alpha_1(eH)$ to $y_K \circ \alpha_2(eH)$, and

$$\eta_X(\alpha_1) : (X^K, y'_K) \longrightarrow (X^H, y_K \circ \alpha_1(eH)$$

is freely homotopic to $\eta_X(\alpha_2) : (X^K, y'_K) \longrightarrow (X^H, y_K \circ \alpha_2(eH)$ along σ. Therefore, $\pi_n(\eta_X(\alpha_2)) = \sigma_* \circ \pi_n(\eta_X(\alpha_1)) : \pi_n(X^K, y'_K) \longrightarrow \pi_n(X^H, y_K \circ \alpha_2(eH))$. Again, the path $\langle \alpha_1, \omega_1 \rangle$ is freely homotopic to the path $\langle \alpha_2, \omega_2 \rangle$ along σ. Therefore $\langle \alpha_1, \omega_1 \rangle_*^{-1} = \langle \alpha_2, \omega_2 \rangle_*^{-1} \circ \sigma_* : \pi_n(X^H, y_K \circ \alpha_1(eH)) \longrightarrow \pi_n(X^H, x'_H)$. Consequently, we have

$$\begin{aligned} \underline{M}(\alpha_1, \omega_1) &= \langle \alpha_1, \omega_1 \rangle_*^{-1} \circ \pi_n(\eta_X(\alpha_1)) \\ &= \langle \alpha_2, \omega_2 \rangle_*^{-1} \circ \sigma_* \circ \pi_n(\eta_X(\alpha_1)) \\ &= \langle \alpha_2, \omega_2 \rangle_*^{-1} \circ \pi_n(\eta_X(\alpha_2)) = \underline{M}(\alpha_2, \omega_2). \end{aligned}$$

Thus the functor $\underline{M}$ passes onto the quotient ΠX and defines an equivariant local system $M : \Pi X \longrightarrow \mathcal{A}b$.

1.2.3. EXAMPLE. Suppose that $p : E \longrightarrow X$ is a G-fibration, that is, a G-map having equivariant homotopy lifting property with respect to G-spaces. Then, for every $H \subseteq G$, $p^H : E^H \longrightarrow X^H$ is an ordinary fibration. If each p^H has n-simple fibre, then there is an equivariant local system $\pi_n(\mathcal{F})$ on X defined as follows.

For each object $x_H : G/H \longrightarrow X$ in ΠX, set $\pi_n(\mathcal{F})(x_H) = \pi_n((p^H)^{-1}(x'_H))$, and for a morphism $[\alpha , \omega] : x_H \longrightarrow y_K$ in ΠX, set

$$\pi_n(\mathcal{F})([\alpha , \omega]) = \pi_n(\tau\langle \alpha , \omega\rangle^{-1} \circ \eta_E(\alpha)),$$

where $\eta_E(\alpha) : (p^K)^{-1}(y'_K) \longrightarrow (p^H)^{-1}(\eta_X(\alpha)\cdot y'_K)$ is the left translation induced by α, and $\tau\langle \alpha , \omega\rangle : (p^H)^{-1}(x'_H) \longrightarrow (p^H)^{-1}(\eta_X(\alpha)\cdot y'_K)$ is the homotopy equivalence obtained by translating fibres of p^H along the path $\langle \alpha , \omega\rangle$ (see [20], p. 185).

The functor $\pi_n(\mathcal{F})$ is well-defined, because if (α_1 , ω_1) is equivalent to (α_2 , ω_2) with G-homotopies $j : \alpha_1 \simeq \alpha_2$ and $k : \omega_1 \simeq \omega_2$, then $\tau\langle j_t , k_t\rangle^{-1} \circ \eta_E(j_t)$, $t \in I$, constitute a homotopy from $\tau\langle \alpha_1 , \omega_1\rangle^{-1} \circ \eta_E(\alpha_1)$ to $\tau\langle \alpha_2 , \omega_2\rangle^{-1} \circ \eta_E(\alpha_2)$, and therefore $\pi_n(\mathcal{F})([\alpha_1 , \omega_1]) = \pi_n(\mathcal{F})([\alpha_2 , \omega_2])$.

1.3. Simple equivariant local system.

1.3.1. DEFINITION. An equivariant local system $M : \Pi X \longrightarrow \mathcal{A}b$ is *simple* if M is independent of paths in the sense that $M([\alpha , \omega_1]) = M([\alpha , \omega_2])$ whenever $[\alpha , \omega_1]$ and $[\alpha , \omega_2]$ are morphisms in ΠX with the same source and target which are mapped under Φ onto the same morphism of orbits $[\alpha]$ in hO_G.

For example, if M_0 is a contravariant coefficient system of Illman [9], which is a contravariant functor $M_0 : hO_G \longrightarrow \mathcal{A}b$, then $M = M_0 \circ \Phi : \Pi X \longrightarrow \mathcal{A}b$ is a simple equivariant local system on X. Conversely we have

1.3.2. PROPOSITION. *If* M *is a simple equivariant local system on a* G*-path connected space* X*, then there exists a contravariant coefficient system* M_0 *such that, for every morphism* $[\alpha , \omega] : x_H \longrightarrow y_K$ *in* ΠX, $M([\alpha , \omega]) = M_0([\alpha])$.

Note that X is G-path connected if each fixed point set X^H is path connected.

Proof. Construct M_0 in the following way. Choose for each $H \subseteq G$ a point $x'_H \in X^H$ by means of a choice function x, and let $x_H : G/H \longrightarrow X$ be the corresponding G-map $x_H(gH) = g \cdot x'_H$. Then define $M_0(G/H) = M(x_H)$. In case $X^H = \emptyset$, we take $M_0(G/H) = \{0\}$, the trivial group.

Next choose for a G-map $\alpha : G/H \longrightarrow G/K$ a path σ_α in X^H from $x_H(eH)$ to $x_K \circ \alpha(eH)$. Such a path gives a morphism $(id_{G/H} , \omega_\alpha) : x_H \longrightarrow x_K \circ \alpha$ in $\sum X$ such that $\langle id_{G/H} , \omega_\alpha \rangle = \sigma_\alpha$, where $\omega_\alpha : G/H \times I \longrightarrow X$ is a G-homotopy from x_H to $x_K \circ \alpha$. Again, α determines a morphism $(\alpha , c) : x_K \circ \alpha \longrightarrow x_K$ in $\sum X$, where $c : G/H \times I \longrightarrow X$ is given by $c(gH , t) = x_K \circ \alpha(gH)$. Thus we have a morphism $[\alpha , c] \circ [id_{G/H} , \omega_\alpha] : x_H \longrightarrow x_K$ in ΠX. Define

$$M_0([\alpha]) = M([\alpha , c] \circ [id_{G/H} , \omega_\alpha]).$$

To see that $M_0([\alpha])$ is well-defined, first note that it is independent of the choice of ω_α (that is, σ_α), since M is simple. Next, suppose that $j : \alpha \simeq \alpha'$ is a G-homotopy $G/H \times I \longrightarrow G/K$. Let σ be the path in X^H from $x_K \circ \alpha(eH)$ to $x_K \circ \alpha'(eH)$ given by $\sigma(t) = x_K \circ j(eH , t)$. Then $\sigma_{\alpha'} = \sigma_\alpha * \sigma$ is a path in X^H from $x_H(eH)$ to $x_K \circ \alpha'(eH)$. As before, $\sigma_{\alpha'}$ gives a morphism $(id_{G/H} , \omega_{\alpha'}) : x_H \longrightarrow x_K \circ \alpha'$ in $\sum X$ such that $\langle id_{G/H} , \omega_{\alpha'} \rangle = \sigma_{\alpha'}$. Let $c' : G/H \times I \longrightarrow X$ be given by $c'(gH , t) = x_K \circ \alpha'(gH)$. Then we have a morphism

$$[\alpha', c'] \circ [id_{G/H}, \omega_{\alpha'}] : x_H \longrightarrow x_K$$

in ΠX.

Let

$$(\alpha, c) \circ (id_{G/H}, \omega_\alpha) = (\alpha, \omega) \text{ and } (\alpha', c') \circ (id_{G/H}, \omega_{\alpha'}) = (\alpha', \omega'),$$

where $\omega, \omega' : G/H \times I \longrightarrow X$ are given by

$$\omega(gH, s) = \begin{cases} \omega_\alpha(gH, 2s), & 0 \le s \le \frac{1}{2} \\ x_K \circ \alpha(gH), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$\omega'(gH, s) = \begin{cases} \omega_{\alpha'}(gH, 2s), & 0 \le s \le \frac{1}{2} \\ x_K \circ \alpha'(gH), & \frac{1}{2} \le s \le 1. \end{cases}$$

We shall show that (α, ω) is equivalent to (α', ω'). This will prove that $M_0([\alpha])$ is well-defined. For this purpose, consider a path σ_{j_t} in X^H from $x_H(eH)$ to $x_K \circ j_t(eH)$, which is the product of the path σ_α and a path in X^H from $x_K \circ \alpha(eH)$ to $x_K \circ j_t(eH)$ along σ. Then we have a morphism $(id_{G/H}, \omega_{j_t}) : x_H \longrightarrow x_K \circ j_t$ in $\sum X$, where $\langle id_{G/H}, \omega_{j_t} \rangle = \sigma_{j_t}$. Let, for each $t \in I$, $c_t : G/H \times I \longrightarrow X$ be the constant G-homotopy $c_t(gH, s) = x_K \circ j_t(gH)$. Then

$$(j_t, c_t) : x_K \circ j_t \longrightarrow x_K \text{ and } (j_t, c_t) \circ (id_{G/H}, \omega_{j_t}) : x_H \longrightarrow x_K$$

are morphisms in $\sum X$. Let $(j_t, c_t) \circ (id_{G/H}, \omega_{j_t}) = (j_t, k_t)$ where $k_t : G/H \times I \longrightarrow X$ is given by

$$k_t(gH, s) = \begin{cases} \omega_{j_t}(gH, 2s), & 0 \le s \le \frac{1}{2} \\ x_K \circ j_t(gH), & \frac{1}{2} \le s \le 1. \end{cases}$$

Let $k : G/H \times I \times I \longrightarrow X$ be defined by $k(gH, s, t) = k_t(gH, s)$. Then (α, k_0) is equivalent to (α', k_1) by the G-homotopies j and k. Moreover it is easy to see that (α, ω) is equivalent to (α, k_0) and (α', ω') is equivalent to (α', k_1). Thus (α, ω) is equivalent to (α', ω'). This completes the construction of $M_0 : hO_G \longrightarrow \mathcal{A}b$. Clearly we have $M([\alpha, \omega]) = M_0([\alpha])$ for every morphism $[\alpha, \omega] : x_H \longrightarrow y_K$ in ΠX. □

1.3.3. REMARK. The functor M_0 is independent of the choice function x up to equivalence. If $M_0' : hO_G \longrightarrow \mathcal{A}b$ is another functor obtained by a different choice function y, then there is a natural equivalence $T : M_0 \longrightarrow M_0'$.

Proof. For each object G/H in hO_G, choose a path u_H in X^H from y_H' to x_H'. This determines a morphism $[id_{G/H}, \omega_H] : y_H \longrightarrow x_H$ in ΠX so that $\langle id_{G/H}, \omega_H \rangle = u_H$. Then define

$$T(G/H) = M([id_{G/H}, \omega_H]) : M_0(G/H) \longrightarrow M_0'(G/H).$$

This is well-defined, since M is simple. Clearly this is an isomorphism with inverse given by $M([id_{G/H}, \omega_H'])$ where ω_H' is the inverse homotopy of ω_H.

Now for a morphism $[\alpha] : G/H \to G/K$ in hO_G construct, as in the proof of 1.3.2 a morphism $[\alpha, \omega] : x_H \longrightarrow x_K$ and a morphism $[\alpha, \omega'] : y_H \longrightarrow y_K$ so that $M_0([\alpha]) = M([\alpha, \omega])$ and $M_0'([\alpha]) = M([\alpha, \omega'])$. Then $[\alpha, \omega] \circ [id_{G/H}, \omega_H]$ and $[id_{G/K}, \omega_K] \circ [\alpha, \omega']$ are both morphisms from y_H to x_K. Since $\Phi([\alpha, \omega] \circ [id_{G/H}, \omega_H]) = \Phi([id_{G/K}, \omega_K] \circ [\alpha, \omega'])$ and M is simple, we have

$$\begin{aligned} T(G/H) \circ M_0([\alpha]) &= M([id_{G/H}, \omega_H]) \circ M([\alpha, \omega]) \\ &= M([\alpha, \omega] \circ [id_{G/H}, \omega_H]) \\ &= M([id_{G/K}, \omega_K] \circ [\alpha, \omega']) \end{aligned}$$

$$= M([\alpha , \omega']) \circ M([id_{G/K} , \omega_K])$$
$$= M'_0([\alpha]) \circ T(G/K)$$

showing that T is a natural transformation. □

1.4. Homomorphism and pull back.

We now introduce the notion of homomorphim between equivariant local systems, and define pull back of an equivariant local system by an equivariant map.

1.4.1. DEFINITION. A *homomorphism* $F : M \longrightarrow N$ between equivariant local systems M and N on X is a natural transformation between the functors.

If F is a natural equivalence, then it is called an *isomorphism*.

Recall from 1.1.5 that a G-map $f : X \longrightarrow Y$ induces a covariant functor $\Pi f : \Pi X \longrightarrow \Pi Y$.

1.4.2. DEFINITION. If $f : X \longrightarrow Y$ is G-map and M is an equivariant local system on Y, then we define the pull back of M by f, denoted by f^*M, to be the equivariant local system $M \circ \Pi f$ on X.

It follows from 1.1.6 that a G-homotopy $f \simeq f'$ induces an isomorphism $f^*M \longrightarrow f'^*M$.

Again, if M and N are equivariant local systems on X and $F : M \longrightarrow N$ is a homomorphism, then a G-map $f : Y \longrightarrow X$ induces a homomorphism $f^*F : f^*M \longrightarrow f^*N$ defined by

$$(f^*F)(x_H : G/H \longrightarrow Y) = F(f \circ x_H) : M(f \circ x_H) \longrightarrow N(f \circ x_H).$$

To see this note that if $[\alpha , \omega] : x_H \longrightarrow y_K$ is a morphism in ΠY, then

$\Pi f([\alpha , \omega]) = [\alpha , f\circ\omega] : f\circ x_H \longrightarrow f\circ y_K$ is a morphism in ΠX. Therefore, since F is natural, we have $F(f\circ x_H) \circ M([\alpha , f\circ\omega]) = N([\alpha , f\circ\omega]) \circ F(f\circ y_K)$; in other words, $f^*F(x_H) \circ f^*M([\alpha , \omega]) = f^*N([\alpha , \omega]) \circ f^*F(y_K)$. Thus $f^*F : f^*M \longrightarrow f^*N$ is a homomorphism. We shall call f^*F the pull back of F by f.

1.5. Equivariant cohomology.

Let Δ_n be the the standard n-simplex,

$$\Delta_n = \left\{ (x_0 , x_1 , \cdots , x_n) \in \mathbf{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 , x_i \geq 0 \right\}.$$

We shall denote the vertices of Δ_n by d_0, d_1, $\cdots$ d_n, and the j-th face operator by e_n^j, where $e_n^j : \Delta_{n-1} \longrightarrow \Delta_n$, $j = 0 , 1 , \cdots , n$, is defined by

$$e_n^j(x_0 , x_1 , \cdots , x_{n-1}) = (x_0 , x_2 , \cdots , x_{j-1} , 0 , x_j , \cdots , x_{n-1}).$$

Let H be a subgroup of G. A standard equivariant n-simplex of type H is the G-space $\Delta_n \times G/H$, where G acts trivially on the first factor.

Let X be a G-space. Then an equivariant singular n-simplex of type H in X is a G-map $\sigma : \Delta_n \times G/H \longrightarrow X$. The equivariant singular $(n-1)$-simplex $\sigma^{(j)} = \sigma \circ (e_n^j \times id_{G/H}) : \Delta_{n-1} \times G/H \longrightarrow X$ is called the j-th face of σ. If $\sigma : \Delta_n \times G/H \longrightarrow X$ is an equivariant singular n-simplex in X, then σ_H will denote the G-map $G/H \longrightarrow X$ defined by $\sigma_H(gH) = \sigma(d_0 , gH)$. Let M be an equivariant local system on X.

1.5.1. DEFINITION. We define $C_G^n(X ; M)$ to be the group of all functions c defined on equivariant singular n-simplexes $\sigma : \Delta_n \times G/H \longrightarrow X$ such that $c(\sigma) \in M(\sigma_H)$.

1.5.2. LEMMA. *If* $u : \Delta_q \longrightarrow \Delta_n$ *is a singular q-simplex in* Δ_n, *and* $\sigma : \Delta_n \times G/H \longrightarrow X$ *is an equivariant singular n-simplex in X, then there is an equivariant singular q-simplex* $\sigma(u) : \Delta_q \times G/H \longrightarrow X$ *in X, and a morphism* $\sigma(u)_* : \sigma_H \longrightarrow \sigma(u)_H$ *in* ΠX.

Proof. Define $\sigma(u)$ to be $\sigma \circ (u \times id_{G/H})$, and $\sigma(u)_*$ to be $[id_{G/H}, \alpha]$ where $\alpha : G/H \times I \longrightarrow X$ is the G-map given by

$$\alpha(gH, t) = \sigma((1-t)\cdot d_0 + t\cdot u(d_0), gH). \quad \square$$

Then $\sigma(e_n^j) : \Delta_{n-1} \times G/H \longrightarrow X$ is the j-th face $\sigma^{(j)}$ of σ. Note that $\sigma_H^{(j)} = \sigma_H$ for $j > 0$, and $\sigma(e_n^0)_* = \sigma_*^{(0)}$ is a morphism $\sigma_H \longrightarrow \sigma_H^{(0)}$.

1.5.3. DEFINITION. We define homomorphism

$$\delta : C_G^n(X ; M) \longrightarrow C_G^{n+1}(X ; M)$$

by

$$(\delta c)(\sigma) = M(\sigma_*^{(0)})(c(\sigma^{(0)})) + \sum_{j=1}^{n+1} (-1)^j c(\sigma^{(j)}),$$

where σ is an equivariant singular $(n+1)$-simplex in X.

These homomorphisms δ have the property $\delta \circ \delta = 0$. Thus we have a cochain complex $C_G(X ; M) = \left\{C_G^n(X ; M), \delta\right\}$.

1.5.4. Let $\sigma : \Delta_n \times G/H \longrightarrow X$ and $\sigma' : \Delta_n \times G/H' \longrightarrow X$ be two equivariant singular n-simplexes in X. Consider $\Delta_n \times G/H$ and $\Delta_n \times G/H'$ as trivial bundles over Δ_n, and suppose that $h : \Delta_n \times G/H \longrightarrow \Delta_n \times G/H'$ is a fibre preserving G-map such that $\sigma = \sigma' \circ h$. In this case we say that σ and σ' are *compatible* under h.

The map h induces a G-map $\overline{h} : G/H \longrightarrow G/H'$ given by

$$\overline{h}(gH) = p_2 \circ h(d_0, gH),$$

where p_2 is the projection onto the second factor. Then $\sigma = \sigma' \circ h$ implies that $\sigma_H = \sigma'_{H'} \circ \bar{h}$. Therefore, if $\mu : G/H \times I \longrightarrow X$ is the constant homotopy from σ_H to $\sigma'_{H'} \circ \bar{h}$, then we have a morphism $[\bar{h}, \mu] : \sigma_H \longrightarrow \sigma'_{H'}$ in ΠX. We shall denote this induced morphism by h_*.

1.5.5. DEFINITION. We define $S^n_G(X ; M)$ to be the subgroup of $C^n_G(X ; M)$ consisting of all functions c such that if σ and σ' are equivariant singular n-simplexes in X which are compatible under h, then $c(\sigma) = M(h_*)(c(\sigma'))$.

1.5.6. PROPOSITION. *If* $c \in S^n_G(X ; M)$, *then* $\delta c \in S^{n+1}_G(X ; M)$.

Proof. Let $\sigma : \Delta_{n+1} \times G/H \longrightarrow X$ and $\sigma' : \Delta_{n+1} \times G/H' \longrightarrow X$ be equivariant singular simplexes which are compatible under $h : \Delta_{n+1} \times G/H \longrightarrow \Delta_{n+1} \times G/H'$, and $h_* : \sigma_H \longrightarrow \sigma'_{H'}$ be the morphism induced by h. Then, for each $j = 0, 1, \cdots, n+1$, the G-map $h : \Delta_{n+1} \times G/H \longrightarrow \Delta_{n+1} \times G/H'$ determines fibre preserving G-map $h^j : \Delta_n \times G/H \longrightarrow \Delta_n \times G/H'$ by restricting h to the j-th face. Then the j-th faces $\sigma^{(j)}$ and $\sigma'^{(j)}$ are compatible under h^j. Clearly, the induced morphism $h^j_* : \sigma^{(j)}_H \longrightarrow \sigma'^{(j)}_{H'}$ is identical with h_* if $j > 0$.

1.5.7. LEMMA. For $j = 0$, *the morphisms* $h^0_* \circ \sigma^{(0)}_*$ *and* $\sigma'^{(0)}_* \circ h_*$ *from* σ_H *to* $\sigma'^{(0)}_{H'}$ *are equal, and therefore* $M(h^0_* \circ \sigma^{(0)}_*) = M(\sigma'^{(0)}_* \circ h_*)$.

Proof. First note that, by definition,

$$h^0_* \circ \sigma^{(0)}_* = [\bar{h}^0, \mu^0] \circ [id_{G/H}, \alpha] = [\bar{h}^0, \psi], \text{ where}$$

$$\psi(gH, s) = \begin{cases} \alpha(gH, 2s), & 0 \le s \le 1/2 \\ \mu^0(gH, 2s-1), & 1/2 \le s \le 1, \end{cases}$$

and $\sigma'^{(0)}_* \circ h_* = [\mathrm{id}_{G/H'}, \alpha'] \circ [\overline{h}, \mu] = [\overline{h}, \psi']$, where

$$\psi'(gH, s) = \begin{cases} \mu(gH, 2s), & 0 \le s \le 1/2 \\ \alpha'(\overline{h}(gH), 2s-1), & 1/2 \le s \le 1. \end{cases}$$

Here α, α' are defined as in the proof of 1.5.2, and μ, μ^0 are the constant G-homotopies as defined in 1.5.4.

Next, define $\sigma^t_H : G/H \longrightarrow X$ by $\sigma^t_H(gH) = \sigma((1-t)\cdot d_1 + t\cdot d_0, gH)$. Similarly define $\sigma'^t_{H'}: G/H' \longrightarrow X$ by replacing σ by σ' and H by H'. Then, we have the following morphisms

$$[\mathrm{id}_{G/H}, \alpha^t] : \sigma_H \longrightarrow \sigma^t_H,\ [\overline{h^t}, \mu^t] : \sigma^t_H \longrightarrow \sigma'^t_{H'},\ [\mathrm{id}_{G/H'}, \alpha'^t] : \sigma'^t_{H'} \longrightarrow \sigma'^{(0)}_{H'},$$

where $\alpha^t : G/H \times I \longrightarrow X$ is given by

$$\alpha^t(gH, s) = \sigma((s-st)\cdot d_1 + (1-s+st)\cdot d_0, gH),$$

$\overline{h^t} : G/H \longrightarrow G/H'$ is given by

$$\overline{h^t}(gH) = p_2 \circ h((1-t)\cdot d_1 + t\cdot d_0, gH),$$

$\mu^t : G/H \times I \longrightarrow X$ is the constant homotopy

$$\mu^t(gH, s) = \sigma^t_H(gH) = \sigma'^t_{H'} \circ \overline{h^t}(gH),$$

and $\alpha'^t : G/H' \times I \longrightarrow X$ is given by

$$\alpha'^t(gH', s) = \sigma'((1-t+st)\cdot d_1 + (t-st)\cdot d_0, gH').$$

Let $[\mathrm{id}_{G/H'}, \alpha'^t] \circ [\overline{h^t}, \mu^t] \circ [\mathrm{id}_{G/H}, \alpha^t] = [\overline{h^t}, k^t]$, where $k^t : G/H \times I \longrightarrow X$ is

$$k^t(gH, s) = \begin{cases} \alpha^t(gH, 4s), & 0 \le s \le 1/4 \\ \mu^t(gH, 4s-1), & 1/4 \le s \le 1/2 \\ \alpha'^t(\overline{h}^t(gH), 2s-1), & 1/2 \le s \le 1. \end{cases}$$

Now define $j : G/H \times I \longrightarrow G/H'$ by

$$j(gH, t) = p_2 \circ h((1-t)\cdot d_1 + t\cdot d_0, gH),$$

and $k : G/H \times I \times I \longrightarrow X$ by $k(gH, s, t) = k^t(gH, s)$. Then

$$k(gH, 0, t) = \alpha^t(gH, 0) = \sigma(d_0, gH) = \sigma_H(gH),$$

$$k(gH, 1, t) = \alpha'^t(\overline{h}^t(gH), 1) = \sigma'(d_1, \overline{h}^t(gH)) = \sigma'^{(0)}_{H'} \circ \overline{h}^t(gH) = \sigma'^{(0)}_{H'} \circ j(gH, t).$$

Note that for $t = 0$, $\sigma'^{0}_{H'} = \sigma'^{(0)}_{H'}$ and $[id_{G/H'}, \alpha'^0]$ is identity, and so the composition $[id_{G/H'}, \alpha'^t] \circ [\overline{h}^t, \mu^t] \circ [id_{G/H}, \alpha^t]$ is precisely equal to $[\overline{h}^0, \psi]$. Similarly for $t = 1$, the above composition reduces to $[\overline{h}, \psi']$. This shows that $(\overline{h}^0, \psi)$ is equivalent to $(\overline{h}, \psi')$. This completes the proof of the lemma. □

1.5.8. *Completion of the proof of Proposition* 1.5.6.

$$\begin{aligned} M(h_*)((\delta c)(\sigma')) &= M(h_*) \circ M(\sigma'^{(0)}_*)(c(\sigma'^{(0)})) + \sum_{j>0} (-1)^j M(h_*)(c(\sigma'^{(j)})) \\ &= M(\sigma^{(0)}_*) \circ M(h^0_*)(c(\sigma'^{(0)})) + \sum_{j>0} (-1)^j M(h^j_*)(c(\sigma'^{(j)})) \\ &= M(\sigma^{(0)}_*)(c(\sigma^{(0)})) + \sum_{j>0} (-1)^j c(\sigma^{(j)}) = (\delta c)(\sigma), \end{aligned}$$

since $c \in S^n_G(X; M)$. □

We have proved that $S_G(X; M) = \left\{S^n_G(X; M), \delta\right\}$ is a cochain complex.

1.5.9. DEFINITION. If A is a G-subspace of X, then $S_G(X, A; M)$ is the subcomplex of $S_G(X; M)$ consisting of all functions which kill equivariant singular simplexes in A.

1.5.10. DEFINITION. We define *the Bredon-Illman singular cohomology with local coefficients* by

$$H^n_G(X; M) = H^n(S_G(X; M)), \quad H^n_G(X, A; M) = H^n(S_G(X, A; M)).$$

1.5.11. Let $\mathcal{L}^2$ be the category whose objects are $(X, A; M)$, where (X, A) is a pair of G-spaces and M is an equivariant local system on X. A morphism $\phi = (\phi_1, \phi_2) : (X, A; M) \longrightarrow (Y, B; N)$ consists of a G-map $\phi_1 : (X, A) \longrightarrow (Y, B)$ and a homomorphism $\phi_2 : \phi_1{}^*N \longrightarrow M$ of local systems on X. The composition of two morphisms

$$\phi : (X, A; M) \longrightarrow (Y, B; N) \text{ and } \psi : (Y, B; N) \longrightarrow (Z, C; L)$$

is given by $w = (w_1, w_2)$, where $w_1 = \psi_1 \circ \phi_1$ and for any object x_H in ΠX

$$w_2(x_H : G/H \longrightarrow X) = \phi_2(x_H) \circ \psi_2(\phi_1 \circ x_H) : \phi_1{}^*\psi_1{}^*L(x_H) \longrightarrow M(x_H).$$

A morphism $\phi = (\phi_1, \phi_2) : (X; M) \longrightarrow (Y; N)$ induces a cochain map $\phi^\# : C_G(Y; N) \longrightarrow C_G(X; M)$ defined as follows. If $c \in C^n_G(Y; N)$ and $\sigma : \Delta_n \times G/H \longrightarrow X$, then $\phi^\#(c)(\sigma) = \phi_2(\sigma_H)(c(\phi_1 \circ \sigma))$.

1.5.12. PROPOSITION. *$\phi^\# : C_G(Y; N) \longrightarrow C_G(X; M)$ is a cochain map.*

Proof. Let $c \in C^n_G(Y; N)$ and $\sigma : \Delta_{n+1} \times G/H \longrightarrow X$. Then, since ϕ_2 is natural, $(\phi_1 \circ \sigma)^{(j)} = \phi_1 \circ \sigma^{(j)}$, $\sigma^{(j)}_H = \sigma_H$ for $j > 0$ and $(\Pi\phi_1)(\sigma^{(0)}_*) = (\phi_1 \circ \sigma)^{(0)}_*$

$$\begin{aligned}
\delta\phi^n(c)(\sigma) &= M(\sigma^{(0)}_*)\, \phi^n(c)(\sigma^{(0)}) + \sum_{j=1}^{n+1}(-1)^j \phi^n(c)(\sigma^{(j)}) \\
&= M(\sigma^{(0)}_*)\, \phi_2(\sigma^{(0)}_H)(c(\phi_1 \circ \sigma^{(0)})) + \sum_{j=1}^{n+1}(-1)^j\, \phi_2(\sigma^{(j)}_H)(c(\phi_1 \circ \sigma^{(j)})) \\
&= \phi_2(\sigma_H)\, \phi_1{}^*N(\sigma^{(0)}_*)(c((\phi_1 \circ \sigma)^{(0)})) + \sum_{j=1}^{n+1}(-1)^j\, \phi_2(\sigma_H)(c((\phi_1 \circ \sigma)^{(j)})) \\
&= \phi_2(\sigma_H)\, [N((\phi_1 \circ \sigma)^{(0)}_*)(c((\phi_1 \circ \sigma)^{(0)})) + \sum_{j=1}^{n+1}(-1)^j\, c((\phi_1 \circ \sigma)^{(j)})] \\
&= \phi_2(\sigma_H)\, \delta(c)(\phi_1 \circ \sigma) = \phi^{n+1}(\delta(c))(\sigma). \quad \square
\end{aligned}$$

1.5.13. PROPOSITION. *If* $c \in S^n_G(Y\,;N)$, *then* $\phi^{\#}(c) \in S^n_G(X\,;M)$.

Proof. Let σ and σ' be equivariant singular n-simplexes in X compatible under h with induced morphism $h_* = [\bar{h}\,,\mu] : \sigma_H \longrightarrow \sigma'_{H'}$. Then

$$M(h_*)(\phi^{\#}(c)(\sigma')) = M(h_*) \circ \phi_2(\sigma'_{H'})(c(\phi_1 \circ \sigma'))$$

$$= \phi_2(\sigma_H) \circ N([\bar{h}\,, \phi_1 \circ \mu])(c(\phi_1 \circ \sigma')) = \phi_2(\sigma_H)(c(\phi_1 \circ \sigma)) = \phi^{\#}(c)(\sigma)$$

since ϕ_2 is natural and $c \in S^n_G(Y\,;N)$. □

Thus we have a cochain map $\phi^{\#} : S_G(Y\,;N) \longrightarrow S_G(X\,;M)$. This cochain map induces a homomorphism $\phi^* : H^*_G(Y\,;N) \longrightarrow H^*_G(X\,;M)$. Similarly a morphism $\phi : (X\,, A\,;M) \longrightarrow (Y\,, B\,;N)$ induces homomorphism

$$\phi^* : H^*_G(Y\,, B\,;N) \longrightarrow H^*_G(X\,, A\,;M).$$

We have now all the necessary prerequisites to conclude that there is a sequence of contravariant functors H^*_G from $\mathcal{L}^2$ to the category of abelian groups. It may be noted that these functors reduce to the equivariant singular cohomology functors with contravariant coefficients system of Illman [9] when M is simple (in fact, this follows from a cochain isomorphism which may be constructed using 1.3.2), and to the singular cohomology functors with classical local coefficients system of Steenrod [20] when G is trivial.

1.5.14. REMARK. The equivariant homology with local coefficients may be treated in the same way using a covariant functor from ΠX to the category of abelian groups.

CHAPTER 2

PROPERTIES OF COHOMOLOGY WITH LOCAL COEFFICIENTS SYSTEM

2.0. Introduction.

In this chapter we prove that the sequence of functors $H_G^* : \mathcal{L}^2 \longrightarrow \mathcal{A}b$ introduced in Chapter 1 satisfy equivariant analogues of Eilenberg-Steenrod axioms.

As before, we assume G to be a compact group and all spaces to be compactly generated.

2.1. Exactness axiom.

Let $(X , A ; M)$ be an object in $\mathcal{L}^2$ and $i : A \longrightarrow X$ be the inclusion map. Then $(A ; i^*M)$ is an object in $\mathcal{L}^2$, which we shall write as $(A ; M)$. We have inclusion morphisms $i : (A ; M) \longrightarrow (X ; M)$ and $j : (X ; M) \longrightarrow (X , A ; M)$ in $\mathcal{L}^2$, which give rise to a short exact sequence of cochain complexes

$$0 \longrightarrow S_G(X , A ; M) \xrightarrow{j^{\#}} S_G(X ; M) \xrightarrow{i^{\#}} S_G(A ; M) \longrightarrow 0.$$

This will induce the following sequence of cohomology groups.

$$\cdots\cdots \longrightarrow H_G^{n-1}(A ; M) \xrightarrow{\delta^{n-1}} H_G^n(X , A ; M) \xrightarrow{j^*} H_G^n(X ; M) \xrightarrow{i^*} H_G^n(A ; M) \longrightarrow \cdots\cdots$$

2.1.1. THEOREM. *If* $(X, A; M)$ *is an object in* $\mathcal{L}^2$ *and* i, j *are the inclusions as described above, then the sequence*

$$\cdots\cdots \longrightarrow H_G^{n-1}(A; M) \xrightarrow{\delta^{n-1}} H_G^n(X, A; M) \xrightarrow{j^*} H_G^n(X; M) \xrightarrow{i^*} H_G^n(A; M) \longrightarrow \cdots\cdots$$

is exact and natural with respect to the morphisms in $\mathcal{L}^2$. □

2.2. Homotopy axiom.

We now prove the homotopy axiom for equivariant cohomology with local system.

Notation. In this section we shall write a homotopy $X \times I \longrightarrow Y$ as $I \times X \longrightarrow Y$ for a technical advantage.

2.2.1. DEFINITION. Two morphisms $\phi, \psi : (X, A; M) \longrightarrow (Y, B; N)$ are homotopic in $\mathcal{L}^2$ if there is a morphism

$$\lambda : (I \times X, I \times A; p^*M) \longrightarrow (Y, B; N)$$

such that $\lambda \circ i_0 = \phi$ and $\lambda \circ i_1 = \psi$, where $p : I \times X \longrightarrow X$ is the projection, and $i_0, i_1 : (X, A; M) \longrightarrow (I \times X, I \times A; p^*M)$ are given by the obvious inclusions $(X, A) \longrightarrow (I \times X, I \times A)$ and the identity natural transformation $i_k^* p^* M = M$, $k = 0, 1$.

If $\lambda = (\lambda_1, \lambda_2)$, then $\lambda_1 : (I \times X, I \times A) \longrightarrow (Y, B)$ is a G-homotopy from ϕ_1 to ψ_1 and $\lambda_2 : \lambda_1{}^*N \longrightarrow p^*M$ is a homomorphism such that $i_0{}^*\lambda_2 = \phi_2 : \phi_1{}^*N \longrightarrow M$ and $i_1{}^*\lambda_2 = \psi_2 : \psi_1{}^*N \longrightarrow M$.

2.2.2. Let $\tau^i_{n+1} : \Delta_{n+1} \longrightarrow I \times \Delta_n$, $0 \le i \le n$, be the linear singular $(n+1)$-simplexes in $I \times \Delta_n$ defined by setting

$$\tau^i_{n+1}(d_j) = (0, d_j) \text{ if } 0 \le j \le i, \text{ and } \tau^i_{n+1}(d_j) = (1, d_{j-1}) \text{ if } i+1 \le j \le n+1.$$

Then, if $\sigma : \Delta_n \times G/H \longrightarrow X$ is an equivariant singular n-simplex in X, denote the equivariant singular (n+1)-simplex

$$(id_I \times \sigma)\circ (\tau^i_{n+1} \times id_{G/H}) : \Delta_{n+1} \times G/H \longrightarrow I \times X$$

by $\tau^i_{n+1}(\sigma)$, $0 \le i \le n$. Then $\tau^i_{n+1}(\sigma)_H : G/H \longrightarrow I \times X$ satisfies the relation $\tau^i_{n+1}(\sigma)_H(gH) = (0\ ,\ \sigma_H(gH))$ for each i.

2.2.3. DEFINITION. We define a homomorphism

$$D : C^{n+1}_G(I \times X\ ;\ p^*M) \longrightarrow C^n_G(X\ ;\ M)$$

as follows. For $c \in C^{n+1}_G(I \times X\ ;\ p^*M)$ and $\sigma : \Delta_n \times G/H \longrightarrow X$,

$$D(c)(\sigma) = \sum_{i=0}^{n} (-1)^i\, c(\tau^i_{n+1}(\sigma)).$$

The definition makes sense, because

$$c(\tau^i_{n+1}(\sigma)) \in p^*M(\tau^i_{n+1}(\sigma)_H) = M(p\circ\tau^i_{n+1}(\sigma)_H) = M(\sigma_H).$$

We shall show that

2.2.4 $$D\delta + \delta D = i_1^{\#} - i_0^{\#}.$$

Let $\omega^i_n : \Delta_n \longrightarrow I \times \Delta_n$, $0 \le i \le n+1$, be the linear singular n-simplexes in $I \times \Delta_n$ defined by setting

$$\omega^i_n(d_j) = (0\ ,\ d_j) \text{ if } 0 \le j \le i-1 \ \text{ and } \ \omega^i_n(d_j) = (1\ ,\ d_j) \text{ if } i \le j \le n.$$

We shall need the following identities.

2.2.5 $$\tau^i_{n+1} \circ e^j_{n+1} = \begin{cases} (id_I \times e^j_n) \circ \tau^{i-1}_n & 0 \le j \le i-1 \le n-1 \\ \omega^i_n & 0 \le j = i \le n \\ \omega^{i+1}_n & 1 \le i+1 = j \le n+1 \\ (id_I \times e^{j-1}_n) \circ \tau^i_n & 2 \le i+2 \le j \le n+1. \end{cases}$$

Let $c \in C^{n+1}_G(I \times X\ ;\ p^*M)$ and $\sigma : \Delta_{n+1} \times G/H \longrightarrow X$ be an equivariant singular (n+1)-simplex in X. Then,

$$D\delta^{n+1}(c)(\sigma) = \sum_{i=0}^{n+1}(-1)^i\, \delta^{n+1}(c)(\tau^i_{n+2}(\sigma))$$

$$= \sum_{i=0}^{n+1}(-1)^i \left\{p^*M(\tau^i_{n+2}(\sigma)^{(0)}_*)\, c(\tau^i_{n+2}(\sigma)^{(0)}) + \sum_{j=1}^{n+2}(-1)^j\, c(\tau^i_{n+2}(\sigma)^{(j)})\right\}$$

$$= \sum_{i=1}^{n+1}(-1)^i \left\{p^*M(\tau^i_{n+2}(\sigma)^{(0)}_*)\, c(\tau^i_{n+2}(\sigma)^{(0)})\right\}$$

$$+ \sum_{0<j\leq i\text{-}1\leq n}(-1)^{i+j}\, c(\tau^i_{n+2}(\sigma)^{(j)}) + p^*M(\tau^0_{n+2}(\sigma)^{(0)}_*)\, c(\tau^0_{n+2}(\sigma)^{(0)})$$

$$+ \sum_{0<i\leq n+1} c(\tau^i_{n+2}(\sigma)^{(i)}) - \sum_{0\leq i\leq n+1} c(\tau^i_{n+2}(\sigma)^{(i+1)})$$

$$+ \sum_{2\leq i+2\leq j\leq n+2}(-1)^{i+j}\, c(\tau^i_{n+2}(\sigma)^{(j)})$$

$$= \sum_{i=1}^{n+1}(-1)^i\, p^*M(\tau^i_{n+2}(\sigma)^{(0)}_*)\, c\Big((id_I\times\sigma)\circ(\tau^i_{n+2}\times id_{G/H})\circ(e^0_{n+2}\times id_{G/H})\Big)$$

$$+ \sum_{0<j\leq i\text{-}1\leq n}(-1)^{i+j}\, c\Big((id_I\times\sigma)\circ(\tau^i_{n+2}\times id_{G/H})\circ(e^j_{n+2}\times id_{G/H})\Big)$$

$$+ p^*M(\tau^0_{n+2}(\sigma)^{(0)}_*)\, c\Big((id_I\times\sigma)\circ(\tau^0_{n+2}\times id_{G/H})\circ(e^0_{n+2}\times id_{G/H})\Big)$$

$$+ \sum_{0<i\leq n+1} c\Big((id_I\times\sigma)\circ(\tau^i_{n+2}\times id_{G/H})\circ(e^i_{n+2}\times id_{G/H})\Big)$$

$$- \sum_{0\leq i\leq n+1} c\Big((id_I\times\sigma)\circ(\tau^i_{n+2}\times id_{G/H})\circ(e^{i+1}_{n+2}\times id_{G/H})\Big)$$

$$+ \sum_{2\leq i+2\leq j\leq n+2}(-1)^{i+j}\, c\Big((id_I\times\sigma)\circ(\tau^i_{n+2}\times id_{G/H})\circ(e^j_{n+2}\times id_{G/H})\Big)$$

$$= \sum_{i=1}^{n+1}(-1)^i\, M(p\circ\tau^i_{n+2}(\sigma)^{(0)}_*)\, c\Big((id_I\times\sigma)\circ((id_I\times e^0_{n+1})\circ\tau^{i\text{-}1}_{n+1}\times id_{G/H})\Big)$$

$$+ \sum_{0<j\leq i\text{-}1\leq n}(-1)^{i+j}\, c\Big((id_I\times\sigma)\circ((id_I\times e^j_{n+1})\circ\tau^{i\text{-}1}_{n+1}\times id_{G/H})\Big)$$

$$+ M(p\circ\tau^0_{n+2}(\sigma)^{(0)}_*)\, c\Big((id_I\times\sigma)\circ(\omega^0_{n+1}\times id_{G/H})\Big)$$

$$+ \sum_{0<i\leq n+1} c\Big((id_I \times \sigma)\circ(\omega^i_{n+1} \times id_{G/H})\Big)$$

$$- \sum_{0\leq i\leq n+1} c\Big((id_I \times \sigma)\circ(\omega^{i+1}_{n+1} \times id_{G/H})\Big)$$

$$+ \sum_{2\leq i+2\leq j\leq n+2} (-1)^{i+j} c\Big((id_I \times \sigma)\circ((id_I \times e^{j-1}_{n+1}) \circ \tau^i_{n+1} \times id_{G/H})\Big),$$

by 2.2.5. From the definition of $p\circ\tau^i_{n+2}(\sigma)^{(0)}_*$ as given in Lemma 1.3.2 it is easy to see that

$$p\circ\tau^i_{n+2}(\sigma)^{(0)}_* = \begin{cases} \sigma^{(0)}_* & \text{if } i > 0 \\ id & \text{if } i = 0. \end{cases}$$

Therefore,

$$D\delta^{n+1}(c)(\sigma) = \sum_{i=1}^{n+1}(-1)^i M(\sigma^{(0)}_*)\, c\Big((id_I \times \sigma^{(0)}) \circ (\tau^{i-1}_{n+1} \times id_{G/H})\Big)$$

$$+ \sum_{0<j\leq i-1\leq n} (-1)^{i+j} c\Big((id_I \times \sigma^{(j)})\circ(\tau^{i-1}_{n+1} \times id_{G/H})\Big)$$

$$+ \sum_{0\leq i\leq n+1} c\Big((id_I \times \sigma)\circ(\omega^i_{n+1} \times id_{G/H})\Big) - \sum_{0\leq i\leq n+1} c\Big((id_I \times \sigma)\circ(\omega^{i+1}_{n+1} \times id_{G/H})\Big)$$

$$+ \sum_{2\leq i+2\leq j\leq n+2} (-1)^{i+j} c\Big((id_I \times \sigma^{(j-1)}) \circ (\tau^i_{n+1} \times id_{G/H})\Big).$$

The third line of the above sum equals

$c\Big((id_I \times \sigma)\circ(\omega^0_{n+1} \times id_{G/H})\Big) - c\Big((id_I \times \sigma)\circ(\omega^{n+2}_{n+1} \times id_{G/H})\Big) = i_1^{\#}(c)(\sigma) - i_0^{\#}(c)(\sigma)$.

Now changing the index i to i+1 on the first and the second line of the above sum, and the index j to j+1 in the fourth line of the sum, we see that the sum of the first, second and the fourth lines in the above sum equals

$$- M(\sigma^{(0)}_*) \sum_{i=0}^{n} (-1)^i c(\tau^i_{n+1}(\sigma^{(0)})) - \sum_{0<j\leq i\leq n} (-1)^{i+j} c(\tau^i_{n+1}(\sigma^{(j)}))$$

$$- \sum_{2 \le i+2 \le j+1 \le n+2} (-1)^{i+j} c(\tau^i_{n+1}(\sigma^{(j)}))$$

$$= - M(\sigma_*^{(0)}) \sum_{i=0}^{n} (-1)^i c(\tau^i_{n+1}(\sigma^{(0)})) - \sum_{i=0}^{n} \sum_{j=1}^{n+1} (-1)^{i+j} c(\tau^i_{n+1}(\sigma^{(j)}))$$

$$= - \delta^n D(c)(\sigma).$$

Therefore $D\,\delta^{n+1} + \delta^n D = i_1^{\#} - i_0^{\#}$.

2.2.6. PROPOSITION. *If* $c \in S_G^{n+1}(I \times X\,;\, p^*M)$ *then* $D(c) \in S_G^n(X\,;\,M)$.

Proof. Let σ and σ' be two equivariant singular n-simplexes in X compatible under $h : \Delta_n \times G/H \longrightarrow \Delta_n \times G/H'$. Now the G-map h determines unique fibre preserving G-map $h^i : \Delta_{n+1} \times G/H \longrightarrow \Delta_{n+1} \times G/H'$ such that $(id_I \times h)o(\tau^i_{n+1} \times id_{G/H}) = (\tau^i_{n+1} \times id_{G/H'})oh^i$, for each $0 \le i \le n$. Then, since $\sigma = \sigma' oh$, $\tau^i_{n+1}(\sigma) = \tau^i_{n+1}(\sigma')oh^i$, which means that, the equivariant singular (n+1)-simplexes $\tau^i_{n+1}(\sigma)$ and $\tau^i_{n+1}(\sigma')$ are compatible under h^i. Now if $\mu^i : G/H \times I \longrightarrow I \times X$ is the constant G-homotopy on $\tau^i_{n+1}(\sigma)_H$, then $\mu = p o \mu^i : G/H \times I \longrightarrow X$ is the constant G-homotopy on σ_H, since $p o \tau^i_{n+1}(\sigma)_H = \sigma_H$. Moreover, as $h^i_* = [\overline{h^i}\,,\,\mu^i]$, $h_* = [\overline{h}\,,\,\mu]$, and $p^*M = M o \Pi p$, we have

$$p^*M(h^i_*) = M o \Pi p([\overline{h^i}\,,\,\mu^i]) = M([\overline{h^i}\,,\,p o \mu^i]) = M([\overline{h}\,,\,\mu]) = M(h_*).$$

Then,

$$\begin{aligned} M(h_*)(D(c)(\sigma')) &= \sum_{i=0}^{n} (-1)^i M(h_*)c(\tau^i_{n+1}(\sigma')) \\ &= \sum_{i=0}^{n} (-1)^i p^*M(h^i_*)c(\tau^i_{n+1}(\sigma')) \\ &= \sum_{i=0}^{n} (-1)^i c(\tau^i_{n+1}(\sigma)) = D(c)(\sigma), \end{aligned}$$

as $c \in S_G^{n+1}(I \times X\,;\,p^*M)$. Thus $D(c) \in S_G^n(X\,;\,M)$. □

We get by some standard arguments the following theorem.

2.2.7. THEOREM. *If* $\phi , \psi : (X , A ; M) \longrightarrow (Y , B ; N)$ *are homotopic morphisms in* $\mathcal{L}^2$, *then*

$$\phi^* = \psi^* : H^n_G(Y , B ; N) \longrightarrow H^n_G(X , A ; M)$$

for all n.

2.3. Excision axiom.

In this section we shall prove the excision theorem for equivariant cohomology with local coefficients system. For a fixed subgroup H of G let $\Pi(\Delta_n , H)$ be the category whose objects are G-maps $x_H : G/H \longrightarrow \Delta_n \times G/H$ such that $p_2 o x_H = id : G/H \longrightarrow G/H$. Then $x_H(gH) = (x , gH)$ where $x \in \Delta_n$ and x is determined by $x = p_1 o x_H(eH)$. Thus objects in $\Pi(\Delta_n , H)$ can be identified with the points in Δ_n. A morphism from x_H to y_H in $\Pi(\Delta_n , H)$ is an equivalence class $[\omega]$ of G-homotopies $\omega : G/H \times I \longrightarrow \Delta_n \times G/H$ from x_H to y_H such that $p_2 o \omega$ is the constant G-homotopy of $id : G/H \longrightarrow G/H$. Two such G-homotopies ω and ω' are equivalent if there exists a G-homotopy $k : G/H \times I \times I \longrightarrow \Delta_n \times G/H$ from ω to ω' such that

$$k(gH , 0 , t) = x_H(gH) \text{ and } k(gH , 1 , t) = y_H(gH)$$

for $gH \in G/H$ and $p_2 o k_t$ is the constant G-homotopy of $id : G/H \longrightarrow G/H$. Then it is easy to see that the morphisms in $\Pi(\Delta_n , H)$ from x_H to y_H may be identified with homotopy classes of paths in Δ_n between points corresponding to x_H and y_H. Since Δ_n is contractible, between any two objects in $\Pi(\Delta_n , H)$ there exists a unique morphism.

We now have a functor $J : \Pi(\Delta_n , H) \longrightarrow \Pi(\Delta_n \times G/H)$ defined as follows. $J(x_H) = x_H$, and, for $[\omega] : x_H \longrightarrow y_H$, $J([\omega]) = [id_{G/H} , \omega]$. This is well-defined because if $[\omega] = [\omega']$ then $[id_{G/H} , \omega] = [id_{G/H} , \omega']$.

Let N be an equivariant local system on $\Delta_n \times G/H$, and $\overline{N} = N \circ J$. The functor $\overline{N}$ assigns to each $x \in \Delta_n$ a group $\overline{N}(x)$ so that the groups of different points of Δ_n are connected by uniquely defined isomorphisms. Put in a different way, $\overline{N}$ determines a transitive system of groups $\{\overline{N}(x)\}$ in the sense of Eilenberg-Steenrod [6 ,p.17]. Therefore, there exists a group $\overline{N}_H$ (the direct limit group) which is uniquely isomorphic to each $\overline{N}(x)$ under the projection $q_x : \overline{N}_H \longrightarrow \overline{N}(x)$ so that

2.3.1 $$q_{x,y} \circ q_x = q_y,$$

where $q_{x,y} : \overline{N}(x) \longrightarrow \overline{N}(y)$ is induced by the homotopy class of paths from y to x.

By an equivariant linear q-simplex in $\Delta_n \times G/H$, we shall mean an equivariant simplex of the form $w = u \times id_{G/H} : \Delta_q \times G/H \longrightarrow \Delta_n \times G/H$ where $u : \Delta_q \longrightarrow \Delta_n$ is a linear q-simplex in Δ_n. Then the objects w_H and $w_H^{(0)}$ in $\Pi(\Delta_n , H)$ are identified with the points $u(d_0)$ and $u(d_1)$ respectively. Also if $[u(d_0) , u(d_1)]$ denotes the unique morphism in $\Pi(\Delta_n , H)$ from $u(d_0)$ to $u(d_1)$, then it is easy to see from the definition of $W_*^{(0)}$ that $J([u(d_0) , u(d_1)]) = W_*^{(0)}$. Then it follows from 2.3.1

2.3.2 $$N(W_*^{(0)}) = \overline{N}([u(d_0) , u(d_1)]) = q_{u(d_0)} \circ q_{u(d_1)}^{-1}.$$

Let $L_G^q(\Delta_n \times G/H ; N)$ be the group of all functions c which are defined on the set of equivariant linear q-simplexes $w = u \times id_{G/H}$ such that $c(w) \in N(w_H) = \overline{N}(u(d_0))$. Then $L_G^*(\Delta_n \times G/H ; N)$ is a cochain complex with the coboundary as defined below.

Let $c \in L_G^q(\Delta_n \times G/H ; N)$ and $w = u \times id_{G/H}$ be an equivariant linear (q + 1)-simplex in $\Delta_n \times G/H$. Then define

$$\delta(c)(w) = N(w_*^{(0)})(c(u^{(0)} \times id_{G/H})) + \sum_{j=1}^{q+1} (-1)^j c(u^{(j)} \times id_{G/H})$$

Let $L^q(\Delta_n ; \overline{N}_H)$ be the group of ordinary cochains defined on linear q-simplexes

$u : \Delta_q \longrightarrow \Delta_n$ with values in the constant coefficient group $\overline{N}_H$. Define a homomorphism

$$\lambda^q : L^q_G(\Delta_n \times G/H\ ;\ N) \longrightarrow L^q(\Delta_n\ ;\ \overline{N}_H)$$

by $\lambda^q(c)(u) = q^{-1}_{u(d_0)}\, c(u \times id_{G/H})$, for $c \in L^q_G(\Delta_n \times G/H\ ;\ N)$ and $u : \Delta_q \longrightarrow \Delta_n$. Then λ^* is a cochain map. This can be seen as follows. For $c \in L^q_G(\Delta_n \times G/H\ ;\ N)$ and $u : \Delta_{q+1} \longrightarrow \Delta_n$

$$\delta^q \lambda^q(c)(u) = \lambda^q(c)\Big(\sum_{j=0}^{q+1}(-1)^j\, u^{(j)}\Big) = \sum_{j=0}^{q+1}(-1)^j\, \lambda^q(c)(u^{(j)})$$

$$= \sum_{j=0}^{q+1}(-1)^j\, q^{-1}_{u^{(j)}(d_0)}\, c(u^{(j)} \times id_{G/H})$$

$$= q^{-1}_{u(d_1)}\, c(u^{(0)} \times id_{G/H}) + \sum_{j=1}^{q+1}(-1)^j\, q^{-1}_{u(d_0)} c(u^{(j)} \times id_{G/H})$$

$$= q^{-1}_{u(d_0)} \circ N((u \times id_{G/H})^{(0)}_*)\, c(u^{(0)} \times id_{G/H}) + \sum_{j=1}^{q+1}(-1)^j\, q^{-1}_{u(d_0)} c(u^{(j)} \times id_{G/H})$$

$$= q^{-1}_{u(d_0)}\Big(\delta^q(c)(u \times id_{G/H})\Big) = \lambda^{q+1}\delta^q(c)(u),$$

since $u^{(j)}(d_0) = u(d_0)$, if $j > 0$ and $u^{(j)}(d_0) = u(d_1)$, if $j = 0$ and 2.3.2 holds. Therefore λ^* is a cochain map. The cochain map λ^* is actually an isomorphism. To see this, define

$$(\lambda^q)^{-1} : L^q(\Delta_n\ ;\ \overline{N}_H) \longrightarrow L^q_G(\Delta_n \times G/H\ ;\ N)$$

by

$$(\lambda^q)^{-1}(c')(u \times id_{G/H}) = q_{u(d_0)}(c'(u))$$

for $c' \in L^q(\Delta_n\ ;\ \overline{N}_H)$ and $u \times id_{G/H} : \Delta_q \times G/H \longrightarrow \Delta_n \times G/H$ a linear q-simplex. Then $(\lambda^*)^{-1}$ is the inverse of λ^*.

Recall that we have a subdivision chain map $\hat{sd}$, and a cochain homotopy $\hat{R}$, where

$$\hat{sd}^q : L^q(\Delta_n\ ;\ \overline{N}_H) \longrightarrow L^q(\Delta_n\ ;\ \overline{N}_H) \text{ and } \hat{R}^q : L^q(\Delta_n\ ;\ \overline{N}_H) \longrightarrow L^{q-1}(\Delta_n\ ;\ \overline{N}_H)$$

satisfy

2.3.3 $$\delta\, \hat{R} + \hat{R}\, \delta = id - \hat{sd}.$$

These induce homomorphisms

$$\overline{sd}^q : L^q_G(\Delta_n \times G/H ; N) \longrightarrow L^q_G(\Delta_n \times G/H ; N)$$

and
$$\overline{R}^q : L^q_G(\Delta_n \times G/H ; N) \longrightarrow L^{q-1}_G(\Delta_n \times G/H ; N)$$

defined respectively by $\overline{sd}^q = (\lambda^q)^{-1} \circ \hat{sd}^q \circ \lambda^q$ and $\overline{R}^q = (\lambda^{q-1})^{-1} \circ \hat{R}^q \circ \lambda^q$. Then $\overline{sd}$ is a cochain map. Now applying λ and λ^{-1} to both sides of 2.3.3 we get

2.3.4
$$\delta\, \overline{R} + \overline{R}\, \delta = \mathrm{id} - \overline{sd}.$$

We now define homomorphisms $sd^n : C^n_G(X ; M) \longrightarrow C^n_G(X ; M)$ in the following way. Let $\sigma : \Delta_n \times G/H \longrightarrow X$ be an equivariant n-simplex, and $\sigma^{\#} : C^n_G(X ; M) \longrightarrow C^n_G(\Delta_n \times G/H ; N) \subset L^n_G(\Delta_n \times G/H ; N)$ be the cochain map induced by σ, where $N = \sigma^* M$. Then define sd^n by

$$sd^n(c)(\sigma) = \overline{sd}^n\, (\sigma^{\#}(c))(i_n \times id_{G/H}),$$

where $i_n : \Delta_n \longrightarrow \Delta_n$ is the identity map. This is well-defined because, $\overline{sd}^n\, (\sigma^{\#}(c))(i_n \times id_{G/H})$ belongs to

$$\sigma^* M((i_n \times id_{G/H})_H) = M(\sigma \circ (i_n \times id_{G/H})_H) = M(\sigma_H).$$

Similarly define homomorphisms $R^n : C^n_G(X ; M) \longrightarrow C^{n-1}_G(X ; M)$ by

$$R^n(c)(\tau) = \overline{R}^n(\tau^{\#}(c))(i_{n-1} \times id_{G/H}),$$

where $\tau : \Delta_{n-1} \times G/H \longrightarrow X$ is an equivariant $(n-1)$-simplex. Then sd is again a cochain map. To verify this, let $\sigma : \Delta_{n+1} \times G/H \longrightarrow X$ be an equivariant singular $(n+1)$-simplex and $c \in C^n_G(X ; M)$. Then, as $\sigma^{\#}$ and $\overline{sd}$ are cochain maps

$$sd^{n+1}(\delta^n c)(\sigma) = \overline{sd}^{n+1}\, (\sigma^{\#}(\delta^n c))(i_{n+1} \times id_{G/H}) = \overline{sd}^{n+1}\, (\delta^n(\sigma^{\#}(c)))(i_{n+1} \times id_{G/H})$$

$$= \delta^n \overline{sd}^n\, (\sigma^{\#}(c))(i_{n+1} \times id_{G/H})$$

$$= \sigma^* M((i_{n+1} \times id_{G/H})^{(0)}_*)(\overline{sd}^n\, (\sigma^{\#}(c))(i^{(0)}_{n+1} \times id_{G/H}))$$

$$+ \sum_{j>0} (-1)^j\, \overline{sd}^n\, (\sigma^{\#}(c))(i^{(j)}_{n+1} \times id_{G/H}).$$

Now let us calculate the term

$$\overline{sd}^n\, (\sigma^{\#}(c))(i^{(j)}_{n+1} \times id_{G/H}).$$

We have by definition of $\bar{sd}^n$,

$$\bar{sd}^n\,(\sigma^{\#}(c))(i^{(j)}_{n+1}\times id_{G/H}) = \bar{sd}^n\,(\sigma^{\#}(c))(e^j_{n+1}\times id_{G/H})$$

$$= q_{e^j_{n+1}(d_0)}\,\hat{sd}^n\circ\lambda\,(\sigma^{\#}(c))(e^j_{n+1}) = q_{e^j_{n+1}(d_0)}\,\lambda\,(\sigma^{\#}(c))(\hat{sd}_n(e^j_{n+1})),$$

where $\hat{sd}_n : L_n(\Delta_{n+1}\,;\,\bar{N}_H) \longrightarrow L_n(\Delta_{n+1}\,;\,\bar{N}_H)$ is the subdivision chain map and

$N = \sigma^*M$. But $\hat{sd}_n(e^j_{n+1}) = \hat{sd}_n\circ\,(e^j_{n+1})_{\#}(i_n)$. Since $\hat{sd}$ is natural, we get

$$\hat{sd}_n\circ\,(e^j_{n+1})_{\#}(i_n) = (e^j_{n+1})_{\#}\circ\,\hat{sd}_n(i_n),$$

where $\hat{sd}_n$ in the right hand side of the above equality is the subdivision map $\hat{sd}_n : L_n(\Delta_n\,;\,\bar{N}_H) \longrightarrow L_n(\Delta_n\,;\,\bar{N}_H)$, $N = \sigma^*M$. But $sd_n(i_n) = \sum_s \pm\, u_s$, where $u_s : \Delta_n \longrightarrow \Delta_n$ are the linear injections onto the the n-simplexes of the barycentric subdivision of Δ_n, and the signs are chosen according to the matching of the orientations, and $u_s(d_0) = b_n$, b_n being the barycentre of Δ_n. Thus,

$$\bar{sd}^n\,(\sigma^{\#}(c))(i^{(j)}_{n+1}\times id_{G/H}) = q_{e^j_{n+1}(d_0)}\,\lambda\,(\sigma^{\#}(c))(\,\sum_s \pm\, e^j_{n+1}\circ\,u_s\,)$$

$$= q_{e^j_{n+1}(d_0)}(\,\sum_s \pm\,\lambda\,(\sigma^{\#}(c))(e^j_{n+1}\circ\,u_s\,))$$

$$= q_{e^j_{n+1}(d_0)}\;(\,\sum_s \pm\, q^{-1}_{e^j_{n+1}(b_n)}\;\sigma^{\#}(c)(e^j_{n+1}\circ\,u_s\times id_{G/H}))$$

$$= q_{e^j_{n+1}(d_0)}\;(\,\sum_s \pm\; q^{-1}_{e^j_{n+1}(u_s(d_0))}\;\sigma^{\#}(c)(e^j_{n+1}\circ\,u_s\times id_{G/H}))$$

$$= q_{e^j_{n+1}(d_0)}\;(\,\sum_s \pm\; q^{-1}_{e^j_{n+1}(u_s(d_0))}\;(\sigma^{(j)})^{\#}(c)(u_s\times id_{G/H}))$$

$$= \lambda^{-1}\circ\hat{sd}^n\circ\lambda((\sigma^{(j)})^{\#}(c))(i_n\times id_{G/H})$$

$$= \bar{sd}^n((\sigma^{(j)})^{\#}(c))(i_n\times id_{G/H}).$$

The last but one step follows by considering the transitive subsystem corresponding to the points of the j-th face of Δ_{n+1}. Therefore,

$$sd^{n+1}(\delta^n c)(\sigma) = \delta^n \overline{sd}^n\,(\sigma^{\#}(c))(i_{n+1} \times id_{G/H})$$

$$= \sigma^* M((i_{n+1} \times id_{G/H})_*^{(0)})(\overline{sd}^n\,(\sigma^{\#}(c))(i_{n+1}^{(0)} \times id_{G/H}))$$

$$+ \sum_{j>0} (-1)^j\,\overline{sd}^n\,(\sigma^{\#}(c))(i_{n+1}^{(j)} \times id_{G/H})$$

$$= M(\sigma_*^{(0)})\,\overline{sd}^n((\sigma^{(0)})^{\#}(c))(i_n \times id_{G/H}) + \sum_{j>0} (-1)^j\,\overline{sd}^n((\sigma^{(j)})^{\#}(c))(i_n \times id_{G/H})$$

$$= M(\sigma_*^{(0)})\,sd^n(c)((\sigma^{(0)}) + \sum_{j>0} (-1)^j\,sd^n(c)((\sigma^{(j)}) = \delta^n\,sd^n(c)(\sigma).$$

Thus we have proved that sd is a cochain map.

Next we shall show that

2.3.5 $$\delta^{n-1}R^n + R^{n+1}\delta^n = id - sd^n.$$

Let $\sigma : \Delta_n \times G/H \longrightarrow X$ be an equivariant n-simplex in X and $c \in C_G^n(X\,;\,M)$. Then $\sigma^{\#}(c) \in L_G^n(\Delta_n \times G/H\,;\,\sigma^*M)$ and so

$$\delta^{n-1}\,\overline{R}^n(\sigma^{\#}(c)) \in L_G^n(\Delta_n \times G/H\,;\,\sigma^*M).$$

We claim that

2.3.6 $$\delta^{n-1}\,\overline{R}^n(\sigma^{\#}(c))(i_n \times id_{G/H}) = \delta^{n-1}\,R^n(c)(\sigma).$$

This holds because

$$\delta^{n-1}\,R^n(c)(\sigma) = M(\sigma_*^{(0)})\,R^n(c)((\sigma^{(0)}) + \sum_{j=1}^{n} (-1)^j\,R^n(c)((\sigma^{(j)})$$

$$= M(\sigma_*^{(0)})\,\overline{R}^n\,((\sigma^{(0)})^{\#}(c))(i_{n-1} \times id_{G/H}) + \sum_{j=1}^{n} (-1)^j\,\overline{R}^n\,((\sigma^{(j)})^{\#}(c))(i_{n-1} \times id_{G/H})$$

$$= M(\sigma_*^{(0)})\,\overline{R}^n\,(\sigma^{\#}(c))(e_n^0 \circ i_{n-1} \times id_{G/H}) + \sum_{j=1}^{n} (-1)^j\,\overline{R}^n\,(\sigma^{\#}(c))(e_n^j \circ i_{n-1} \times id_{G/H})$$

$$= \sigma^* M((i_n \times id_{G/H})_*^{(0)})\, \overline{R}^n(\sigma^{\#}(c))(i_n \circ e_n^0 \times id_{G/H}) + \sum_{j=1}^{n} (-1)^j\, \overline{R}^n\, (\sigma^{\#}(c))(i_n \circ e_n^j \times id_{G/H})$$

$$= \sigma^* M((i_n \times id_{G/H})_*^{(0)})\, \overline{R}^n\, (\sigma^{\#}(c))(i_n^{(0)} \times id_{G/H}) + \sum_{j=1}^{n} (-1)^j\, \overline{R}^n\, (\sigma^{\#}(c))(i_n^{(j)} \times id_{G/H})$$

$$= \delta^{n-1}\, \overline{R}^n(\sigma^{\#}(c))(i_n \times id_{G/H}).$$

Now we have by 2.3.4

$$\delta^{n-1}\, \overline{R}^n(\sigma^{\#}(c)) + \overline{R}^{n+1}\delta^n(\sigma^{\#}(c)) = \sigma^{\#}(c) - \overline{sd}^n(\sigma^{\#}(c)).$$

Therefore by evaluating on $(i_n \times id_{G/H})$, and using 2.3.6 and the fact that $\sigma^{\#}$ is a cochain map, we get

$$\delta^{n-1}\, R^n(c)(\sigma) + R^{n+1}(\delta^n(c))(\sigma) = c(\sigma) - \overline{sd}^n(c)(\sigma).$$

Therefore 2.3.5 holds.

We shall now show that the cochain map sd and the homomorphism R pass onto the subcomplex $S_G(X\,;\,M)$.

2.3.7. PROPOSITION. (i) $sd^n(S_G^n(X\,;\,M)) \subseteq S_G^n(X\,;\,M)$,

(ii) $R^n(S_G^n(X\,;\,M)) \subseteq S_G^{n-1}(X\,;\,M)$.

Proof. Recall that $\hat{sd}_n$ is the subdivision chain map on the chain group $L_n(\Delta_n\,;\,\overline{N}_H)$ of linear n-simplexes with coefficients in the group $\overline{N}_H$, and, as mentioned before, $\hat{sd}_n(i_n) = \sum_j \pm u_j$, where $u_j : \Delta_n \longrightarrow \Delta_n$ are the linear injections onto the n-simplexes of the barycentric subdivision of Δ_n, so that $u_j(d_0) = b_n$ which is the barycentre of Δ_n.

Now let $c \in S_G^n(X\,;\,M)$, and σ, σ' be two equivariant n-simplexes of type H and H′ respectively in X compatible under $h : \Delta_n \times G/H \longrightarrow \Delta_n \times G/H'$, and $h_* : \sigma_H \longrightarrow \sigma'_H$ be the morphism induced by h. Then the G-map h determines fibre preserving G-maps $h_j : \Delta_n \times G/H \longrightarrow \Delta_n \times G/H'$ such that

$$h\circ(u_j\times id_{G/H}) = (u_j\times id_{G/H'})\circ h_j.$$

Since $\sigma = \sigma'\circ h$ we have then $\sigma(u_j) = \sigma'(u_j)\circ h_j$, where $\sigma(u_j) = \sigma\circ(u_j\times id_{G/H})$. In other words, the equivariant n-simplexes $\sigma(u_j)$ and $\sigma'(u_j)$ are compatible under h_j, and therefore

$$M(h_{j*})(c(\sigma'(u_j)) = c(\sigma(u_j)).$$

We shall need the following lemma.

2.3.8. LEMMA. $M(h_*)\circ q'_{b_n,d_0} = q_{b_n,d_0}\circ M(h_{j*})$.

Note that here $q_{b_n,d_0} = q_{d_0}\circ q_{b_n}^{-1}$, where

$$q_{d_0}:\overline{N}_H\longrightarrow\overline{N}(d_0) = M(\sigma_H),\quad q_{b_n}:\overline{N}_H\longrightarrow\overline{N}(b_n) = M(\sigma(u_j)_H),$$

and similarly for q'_{b_n,d_0} using σ'.

Proof. As described before, the equivariant simplexes σ and σ' determine transitive systems of groups $\overline{N}$ and $\overline{N}'$ respectively so that, for $x\in\Delta_n$,

$$\overline{N}(x) = \sigma^*M(x_H) = M(\sigma\circ x_H)\ \text{ and }\ \overline{N}'(x) = \sigma'^*M(x_{H'}) = M(\sigma'\circ x_{H'}),$$

where $x_H(gH) = (x\,,\,gH)$ and $x_{H'}(gH') = (x\,,\,gH')$. Again if $\alpha: I\longrightarrow\Delta_n$ is a path from x to y, and $\omega: G/H\times I\longrightarrow X$ and $\omega': G/H'\times I\longrightarrow X$ are given respectively by

$$\omega(gH\,,\,t) = \sigma(\alpha(t)\,,\,gH)\ \text{ and }\ \omega'(gH'\,,\,t) = \sigma'(\alpha(t)\,,\,gH'),$$

then we have morphisms

$$M([id_{G/H}\,,\,\omega]):\overline{N}(y)\longrightarrow\overline{N}(x)\ \text{ and }\ M([id_{G/H'}\,,\,\omega']):\overline{N}'(y)\longrightarrow\overline{N}'(x).$$

Since σ ans σ' are compatible under $h:\Delta_n\times G/H\longrightarrow\Delta_n\times G/H'$, there is a homomorphism from $\overline{N}'$ to $\overline{N}$. To see this, first define, for each $x\in\Delta_n$, a G-map $\overline{h}_x: G/H\longrightarrow G/H'$ by $\overline{h}_x(gH) = p_2\circ h(x\,,\,gH)$, and note that we have $\sigma\circ x_H = \sigma'\circ x_{H'}\circ\overline{h}_x$. Next define $\mu_x: G/H\times I\longrightarrow X$ to be the constant G-homotopy on $\sigma\circ x_H = \sigma'\circ x_{H'}\circ\overline{h}_x$. Then we have a morphism $h_{x*} = [\overline{h}_x\,,\,\mu_x]:\sigma\circ x_H\longrightarrow\sigma'\circ x_{H'}$, which induces a homomorphism $M(h_{x*}) : \overline{N}'(x) \longrightarrow \overline{N}(x)$.

These homorphisms constitute a natural transformation $\overline{N}' \longrightarrow \overline{N}$, because the morphisms

$(id_{G/H'}, \omega') \circ (\overline{h}_x, \mu_x)$ and $(\overline{h}_y, \mu_y) \circ (id_{G/H}, \omega) : \sigma \circ x_H \longrightarrow \sigma' \circ y_{H'}$ in $\sum X$ are equivalent under the G-homotopies

$$j : G/H \times I \longrightarrow G/H' \text{ and } k : G/H \times I \times I \longrightarrow X$$

given by

$$j(gH, t) = p_2 \circ h(\alpha(t), gH)$$

$$k(gH, s, t) = \begin{cases} \omega(gH, 2st), & 0 \le s \le 1/2 \\ \omega'(j(gH, t), 2s + 2t - 2st - 1), & 1/2 \le s \le 1. \end{cases}$$

Note that k is continuous, since $\sigma = \sigma' \circ h$.

Thus we have proved that

$$M(h_{x*}) \circ M([id_{G/H'}, \omega']) = M([id_{G/H}, \omega]) \circ M(h_{y*})$$

and the lemma follows from this by taking $x = d_0$ and $y = b_n$. Then $M([id_{G/H}, \omega])$ becomes q_{b_n, d_0}, and $M([id_{G/H'}, \omega'])$ becomes q'_{b_n, d_0}. □

2.3.9. *Completion of the proof of Proposition* 2.3.7.

Using the above information, one computes

$$\begin{aligned} M(h_*)(sd^n(c)(\sigma')) &= M(h_*)((\lambda^{-1} \circ \hat{sd}^n \circ \lambda)(\sigma'^{\#}(c))(i_n \times id_{G/H'})) \\ &= M(h_*) \circ q'_{d_0}(\lambda(\sigma'^{\#}(c))(\hat{sd}_n(i_n))) \\ &= M(h_*) \circ q'_{d_0}(\lambda(\sigma'^{\#}(c))(\sum_j \pm u_j)) \\ &= \sum_j \pm M(h_*) \circ q'_{d_0} \circ q'^{-1}_{u_j(d_0)}(\sigma'^{\#}(c)(u_j \times id_{G/H'})) \\ &= \sum_j \pm M(h_*) \circ q'_{b_n, d_0}(\sigma'^{\#}(c)(u_j \times id_{G/H'})) \end{aligned}$$

$$= \sum_j \pm q_{b_n, d_0} \circ M(h_{j*})(c(\sigma'(u_j)))$$

$$= q_{b_n, d_0} \sum_j \pm c(\sigma(u_j))$$

$$= q_{b_n, d_0} \left(\sum_j \pm \sigma^{\#}(c)(u_j \times id_{G/H}) \right)$$

$$= q_{d_0} \circ q_{b_n}^{-1} \left(\sum_j \pm \sigma^{\#}(c)(u_j \times id_{G/H}) \right)$$

$$= q_{d_0} \left(\sum_j \pm \lambda\, (\sigma^{\#}(c))(u_j) \right)$$

$$= q_{d_0}(\lambda\, (\sigma^{\#}(c))\, (\hat{sd}_n(i_n)))$$

$$= q_{d_0}(\hat{sd}^n \circ \lambda\, (\sigma^{\#}(c))\,(i_n))$$

$$= (\lambda^{-1} \circ \hat{sd}^n \circ \lambda)(\sigma^{\#}(c))(i_n \times id_{G/H})$$

$$= \bar{sd}^n(\sigma^{\#}(c))(i_n \times id_{G/H}) = sd^n(c)(\sigma).$$

Thus $sd^n(c)(\sigma) \in S^n_G(X\,;\,M)$ and the proof of Part (i) is complete.

Part (ii) can be proved similarly, by writing $\hat{R}_{n-1}(i_{n-1}) = \sum_j \pm v_j$, where $v_j : \Delta_n \longrightarrow \Delta_{n-1}$ are linear simplexes. □

Let $\mathfrak{U}$ be a family of G-subsets of X such that

$$X = \bigcup \left\{ \mathfrak{Int}U,\ u \in \mathfrak{U} \right\}.$$

An equivariant singular n-simplex $\sigma : \Delta_n \times G/H \longrightarrow X$ is said to be $\mathfrak{U}$-small if image of σ is contained in some member of $\mathfrak{U}$. Let $S^n_G(X\,;\,M\,;\,\mathfrak{U})$ denote the group of all functions c defined on equivariant n-simplexes σ which are $\mathfrak{U}$-small such that $c(\sigma) \in M(\sigma_H)$ and $M(h_*)c(\sigma') = c(\sigma)$ whenever σ and σ' are $\mathfrak{U}$-small simplexes of type H and H$'$ respectively, compatible under $h : \Delta_n \times G/H \longrightarrow \Delta_n \times G/H'$. Since faces of a $\mathfrak{U}$-small simplex are again $\mathfrak{U}$-small, these groups with the coboundary δ defined as in 1.5.3 lead us to a cochain complex

$$S_G(X, A; M; \mathfrak{U}) = \left\{S_G^n(X, A; M; \mathfrak{U}), \delta\right\}$$

for a pair of G-spaces.

2.3.10. PROPOSITION. *The inclusion map*

$$\alpha : S_G(X, A; M) \longrightarrow S_G(X, A; M; \mathfrak{U})$$

is a cochain homotopy equivalence.

Proof. An equivariant n-simplex $\sigma : \Delta_n \times G/H \longrightarrow X$ determines an ordinary n-simplex $\widetilde{\sigma} : \Delta_n \longrightarrow X$ where $\widetilde{\sigma}(x) = \sigma(x, eH)$. Then one can find integer m such that $(sd_n)^m(\widetilde{\sigma}) = \sum_k a_k \widetilde{\sigma}_k$, where $a_k \in \sigma^* M_H = \overline{N}_H$ and $\widetilde{\sigma}_k$ are $\mathfrak{U}$-small n-simplexes in X. Here $(sd_n)^m$ denotes the iteration of the subdivision chain map $sd_n : C_n(X; \overline{N}_H) \longrightarrow C_n(X; \overline{N}_H)$, m times. Let $\sigma_k : \Delta_n \times G/H \longrightarrow X$ be given by $\sigma_k(x, gH) = g \cdot \widetilde{\sigma}_k(x)$. Then σ_k are $\mathfrak{U}$-small. Let $m(\sigma)$ be the smallest integer such that each σ_k is $\mathfrak{U}$-small.

Let σ and τ be respectively equivariant n and $(n-1)$-simplexes in X. Now define homomorphisms

$$\beta : S_G^n(X, A; M; \mathfrak{U}) \longrightarrow S_G^n(X, A; M) \text{ and } D : S_G^n(X, A; M) \longrightarrow S_G^{n-1}(X, A; M)$$

by setting

$$\beta(c)(\sigma) = sd^{m(\sigma)}(c)(\sigma) + M(\sigma_*^{(0)}) \sum_{i=m(\sigma^{(0)})}^{m(\sigma)-1} (sd)^i \circ R(c)(\sigma^{(0)})$$
$$+ \sum_{j=1}^{n} (-1)^j \sum_{i=m(\sigma^{(j)})}^{m(\sigma)-1} (sd)^i \circ R(c)(\sigma^{(j)})$$

for $c \in S_G^n(X, A; M; \mathfrak{U})$ and

$$D(c)(\tau) = \sum_{i=0}^{m(\tau)-1} (sd)^i \circ R(c)(\tau),$$

for $c \in S^n_G(X, A; M)$, where $(sd)^0 = id$. Then $\delta D + D \delta = id - \beta$. This can be seen as follows. We have,

$$\delta(D(c))(\sigma) = M(\sigma_*^{(0)})(Dc(\sigma^{(0)})) + \sum_{j=1}^{n} (-1)^j (Dc)(\sigma^{(j)})$$

$$= M(\sigma_*^{(0)})\Big(\sum_{i=0}^{m(\sigma^{(0)})-1} (sd)^i \circ R(c)(\sigma^{(0)})\Big) + \sum_{j=1}^{n} (-1)^j \sum_{i=0}^{m(\sigma^{(j)})-1} (sd)^i \circ R(c)(\sigma^{(j)}).$$

On the other hand, by 2.3.5 we have

$$D(\delta c)(\sigma) = \sum_{i=0}^{m(\sigma)-1} (sd)^i \circ R(\delta c)(\sigma) = \sum_{i=0}^{m(\sigma)-1} (sd)^i \circ (c - sd(c) - \delta R(c))(\sigma)$$

$$= \sum_{i=0}^{m(\sigma)-1} \Big\{(sd)^i(c)(\sigma) - (sd)^{i+1}(c)(\sigma)\Big\} - \sum_{i=0}^{m(\sigma)-1} (sd)^i (\delta R(c))(\sigma)$$

$$= c(\sigma) - (sd)^{m(\sigma)}(c)(\sigma) - \sum_{i=0}^{m(\sigma)-1} \delta (sd)^i \circ R(c)(\sigma)$$

$$= c(\sigma) - (sd)^{m(\sigma)}(c)(\sigma) \quad - \sum_{i=0}^{m(\sigma)-1} \Big\{M(\sigma_*^{(0)})((sd)^i \circ R(c)(\sigma^{(0)}))$$

$$+ \sum_{j=1}^{n} (-1)^j ((sd)^i \circ R(c))(\sigma^{(j)})\Big\}.$$

Therefore,

$$(\delta D + D \delta)(c)(\sigma) = c(\sigma) - (sd)^{m(\sigma)}(c)(\sigma) - M(\sigma_*^{(0)}) \sum_{i=m(\sigma^{(0)})}^{m(\sigma)-1} (sd)^i \circ R(c)(\sigma^{(0)})$$

$$- \sum_{j=1}^{n} (-1)^j \sum_{i=m(\sigma^{(j)})}^{m(\sigma)-1} (sd)^i \circ R(c)(\sigma^{(j)})$$

$$= c(\sigma) - \beta(c)(\sigma).$$

The homomorphism β is a cochain map. For we have

$$\delta \, Dc + D \, \delta c = c - \beta c$$

or

$$\delta \, D \, \delta c + D \, \delta \, \delta c = \delta c - \beta \, \delta c$$

that is,

$$\delta \, D \, \delta c = \delta c - \beta \, \delta c.$$

Now $\delta \, \beta c = \delta c - \delta \, \delta \, Dc - \delta \, D \, \delta c = \delta c - \delta c + \beta \, \delta c = \beta \, \delta c$. Therefore, $\delta \, \beta = \beta \, \delta$. Since α is a cochain map, $\alpha \, \beta$ and $\beta \, \alpha$ are both cochain maps. Therefore, as $\alpha \, \beta = \mathrm{id}$ and $\beta \, \alpha = \beta$, β is a homotopy inverse of α. This completes the proof. □

By a standard argument we get

2.3.11. THEOREM. *Let* $(X , A ; M)$ *be an object in* $\mathfrak{L}^2$ *and* U *be an open* G*-subset of* X *such that* $\overline{U} \subset \mathfrak{I}\mathrm{nt}A$. *Then the inclusion* $i : (X - U , A - U ; M) \longrightarrow (X , A ; M)$ *induces isomorphism*

$$i^* : H^n_G(X , A ; M) \longrightarrow H^n_G(X - U , A - U ; M)$$

for all n.

2.4. Dimension axiom.

In this section we shall show that equivariant cohomology with local coefficients satisfies dimension axiom.

2.4.1. THEOREM. *If* M *is an equivariant local system on* G/H, *then*

$$H^n_G(G/H ; M) = \begin{cases} 0, & \text{if } n \neq 0 \\ M(\mathrm{id}_{G/H}), & \text{if } n = 0. \end{cases}$$

Proof. Let Σ be the set of equivariant simplexes $\omega_n : \Delta_n \times G/H \longrightarrow G/H$ of type H given by $\omega_n(x , gH) = gH$, for $n \geq 0$. Consider the cochain complex $D(G/H ; M) = \{D^n(G/H ; M) , \delta\}$, where $D^n(G/H ; M)$ is the group of all functions c defined on $\{\omega_n\}$ such that $c(\omega_n) \in M((\omega_n)_H)$ and δ is given as follows. First note that for

$\omega_n \in \Sigma$ $(\omega_n)_H : G/H \longrightarrow G/H$ is $\mathrm{id} : G/H \longrightarrow G/H$, and the j-th face $\omega_n^{(j)} = \omega_{n-1}$, $0 \leq j \leq n$. Define $\delta^n : D^n(G/H\,;M) \longrightarrow D^{n+1}(G/H\,;M)$ by

$$\delta^n(c)(\omega_{n+1}) = M((\omega_{n+1})_*^{(0)})\, c(\omega_{n+1}^{(0)}) + \sum_{j=1}^{n+1} (-1)^j\, c\,(\omega_{n+1}^{(j)}).$$

Since $(\omega_{n+1})_*^{(0)} : (\omega_{n+1})_H \longrightarrow (\omega_{n+1}^{(0)})_H = (\omega_n)_H$ is identity, δ takes the form

$$\delta^n(c)(\omega_{n+1}) = \sum_{j=0}^{n+1} (-1)^j\, c(\omega_n).$$

This means $\delta^n = 0$ if $n = 0$ or even, and δ^n is an isomorphism if n is odd. Consequently,

$$H^n(D(G/H\,;M)) = \begin{cases} 0, & \text{if } n \neq 0 \\ M(\mathrm{id}_{G/H}), & \text{if } n = 0. \end{cases}$$

Therefore the proof of the theorem will be complete if we show that the cochain complex $S_G(G/H\,;M)$ is isomorphic to $D(G/H\,;M)$.

Define $\alpha^n : S_G^n(G/H\,;M) \longrightarrow D^n(G/H\,;M)$ as follows.

$$\alpha^n(c)(\omega_n) = c(\omega_n) \in M((\omega_n)_H) = M(\mathrm{id}_{G/H})$$

for $c \in S_G^n(G/H\,;M)$. Note that $\alpha^{n+1}(\delta c)(\omega_{n+1}) = \delta c(\omega_{n+1}) = \sum_{j=0}^{n+1} (-1)^j\, c(\omega_n)$ and $\delta(\alpha^n c)(\omega_{n+1}) = \sum_{j=0}^{n+1} (-1)^j\, \alpha^n c(\omega_n) = \sum_{j=0}^{n+1} (-1)^j\, c(\omega_n)$. Thus $\alpha\,\delta = \delta\,\alpha$, and hence α is a cochain map.

Next we define homomorphisms $\beta^n : D^n(G/H\,;M) \longrightarrow S_G^n(G/H\,;M)$ in the following way. Let $\sigma : \Delta_n \times G/K \longrightarrow G/H$ be an equivariant n-simplex. Define a G-map $h(\sigma) : \Delta_n \times G/K \longrightarrow \Delta_n \times G/H$ by

$$h(\sigma)(x\,,gK) = (x\,,\sigma(x\,,gK)).$$

Then $\sigma = \omega_n \circ h(\sigma)$, and therefore σ and ω_n are compatible under $h(\sigma)$, and we have a morphism $h(\sigma)_* : \sigma_K \longrightarrow (\omega_n)_H$. Now define

$$\beta^n(c)(\sigma) = M(h(\sigma)_*)(c(\omega_n))$$

for $c \in D^n(G/H\ ;\ M)$. Then β is well-defined. For, let

$$\sigma : \Delta_n \times G/K \longrightarrow G/H \text{ and } \sigma' : \Delta_n \times G/K' \longrightarrow G/H$$

be equivariant n-simplexes compatible under the fibre preserving G-map $k : \Delta_n \times G/K \longrightarrow \Delta_n \times G/K'$, where as in Chapter 1, $\Delta_n \times G/K$ and $\Delta_n \times G/K'$ are considered as trivial bundles over Δ_n. Let $k_* = [\overline{k}\ ,\ \mu] : \sigma_K \longrightarrow \sigma'_{K'}$ be the morphism in $\Pi(G/H)$ induced by k . Here $\mu : G/K \times I \longrightarrow G/H$ is the constant G-homotopy from σ_K to $\sigma'_{K'} \circ \overline{k}$. As before, we have a morphism $h(\sigma')_* = [h(\overline{\sigma'})\ ,\ \mu'] : \sigma'_{K'} \longrightarrow (\omega_n)_H$, where $h(\overline{\sigma'}) : G/K' \longrightarrow G/H$ is the G-map

$h(\overline{\sigma'})(aK') = p_2 \circ h(\sigma')(d_0\ ,\ aK') = p_2(d_0\ ,\ \sigma'(d_0\ ,\ aK')) = \sigma'(d_0\ ,\ aK') = \sigma'_{K'}\quad(aK')$,

and $\mu' : G/K' \times I \longrightarrow G/H$ is the constant G-homotopy from $\sigma'_{K'}$ to $(\omega_n)_H \circ h(\overline{\sigma'})$. Then, $h(\sigma)_* = h(\sigma')_* \circ k_*$. To see this, recall that by our law of composition

$$h(\sigma')_* \circ k_* = [h(\overline{\sigma'}) \circ \overline{k}\ ,\ \psi] = [\sigma'_{K'} \circ \overline{k}\ ,\ \psi] = [\sigma_K\ ,\ \psi] = [h(\overline{\sigma})\ ,\ \psi],$$

where $\psi : G/K \times I \longrightarrow G/H$ is given by

$$\psi(gK\ ,\ t) = \begin{cases} \mu(gK\ ,\ 2t), & 0 \leq t \leq 1/2 \\ \mu'(\overline{k}(gK)\ ,\ 2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

$$= \begin{cases} \sigma_K(gK), & 0 \leq t \leq 1/2 \\ \sigma'_{K'}(\overline{k}(gK)), & 1/2 \leq t \leq 1 \end{cases}$$

But $\sigma'_{K'} \circ \overline{k} = \sigma_K$. Therefore, ψ is the constant G-homotopy from σ_K to $(\omega_n)_H \circ h(\overline{\sigma})$. Thus $[h(\overline{\sigma})\ ,\ \psi] = h(\sigma)_*$. Therefore,

$M(k_*)(\beta(c)(\sigma')) = M(k_*) \circ M(h(\sigma')_*)(c(\omega_n)) = M(h(\sigma')_* \circ k_*)c(\omega_n)$

$= M(h(\sigma)_*)c(\omega_n) = \beta(c)(\sigma)$. Consequently, $\beta(c) \in S^n_G(G/H\ ;\ M)$.

Next, note that β is a cochain map. This can be seen easily in the following way. First note that if $\sigma : \Delta_{n+1} \times G/K \longrightarrow G/H$ is an equivariant simplex, then the morphism $h(\sigma^{(0)})_* \circ \sigma_*^{(0)} : \sigma_K \longrightarrow (\omega_n)_H$ is given by

$$[h(\overline{\sigma}^{(0)}), \mu^0] \circ [id_{G/K}, \alpha] = [\sigma_K^{(0)}, \psi],$$

where $\mu^0 : G/K \times I \longrightarrow G/H$ is $\mu^0(gK, t) = \sigma_K^{(0)}(gK)$, for all $t \in I$, $\alpha : G/K \times I \longrightarrow G/H$ is the G-map $\alpha(gK, t) = \sigma((1-t)\cdot d_0 + t\cdot d_1, gK)$ and $\psi : G/K \times I \longrightarrow G/H$ is given by

$$\psi(gK, t) = \begin{cases} \alpha(gK, 2t), & 0 \leq t \leq 1/2 \\ \mu^0(gK, 2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

We claim that $h(\sigma^{(0)})_* \circ \sigma_*^{(0)} = (\omega_{n+1})_*^{(0)} \circ h(\sigma)_* = h(\sigma)_*$. Now

$$(\omega_{n+1})_*^{(0)} \circ h(\sigma)_* = [id_{G/H}, \alpha'] \circ [\sigma_K, \mu] = [\sigma_K, \psi'],$$

where $\alpha' : G/H \times I \longrightarrow G/H$ is $\alpha'(gH, t) = gH$, for all $t \in I$, and $\psi' : G/K \times I \longrightarrow G/H$ is given by

$$\psi'(gK, t) = \begin{cases} \mu(gK, 2t), & 0 \leq t \leq 1/2 \\ \alpha'(\sigma_K(gK), 2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

The proof of the fact $h(\sigma^{(0)})_* \circ \sigma_*^{(0)} = h(\sigma)_*$ is exactly similar to the proof of Lemma 1.5.7 (one has just to replace σ' by ω_{n+1} and h by $h(\sigma)$). Also observe that $h(\sigma)_* = h(\sigma^{(j)})_*$ if $j > 0$. Then for $c \in D^n(G/H ; M)$ and $\sigma : \Delta_{n+1} \times G/K \longrightarrow G/H$, we have

$$\begin{aligned}
\delta^n\beta^n(c)(\sigma) &= M(\sigma_*^{(0)})\, \beta^n(c)(\sigma^{(0)}) + \sum_{j=1}^{n+1} (-1)^j\, \beta^n(c)(\sigma^{(j)}) \\
&= M(\sigma_*^{(0)})\, M(h(\sigma^{(0)})_*)(c(\omega_n)) + \sum_{j=1}^{n+1} (-1)^j\, M(h(\sigma^{(j)})_*)(c(\omega_n)) \\
&= M(h(\sigma)_*)(c(\omega_n)) + \sum_{j=1}^{n+1} (-1)^j\, M(h(\sigma)_*)(c(\omega_n)) \\
&= M(h(\sigma)_*)\Big(\sum_{j=1}^{n+1} (-1)^j\, c(\omega_n)\Big) \\
&= M(h(\sigma)_*)\, \delta^n c(\omega_{n+1}) = \beta^{n+1}(\delta^n c)(\sigma).
\end{aligned}$$

This proves the fact that β is a cochain map.

The cochain map β is the inverse of α. For

$$\beta\,\alpha(c)(\sigma) = M(h(\sigma)_*)(\alpha(c)(\omega_n)) = M(h(\sigma)_*)(c(\omega_n)) = c(\sigma),$$

as σ and ω_n are compatible under $h(\sigma)$. Also

$$\alpha\,\beta(c)(\omega_n) = \beta(c)(\omega_n) = M(h(\omega_n)_*)\,c(\omega_n) = c(\omega_n),$$

since $h(\omega_n)_* = \mathrm{id}$. This completes the proof of dimension axiom. □

2.5. Other properties.

We mention here that the equivariant cohomology with local coefficients system is additive. We may also deduce the Mayer-Vietoris exact sequence for this cohomology theory.

2.5.1. THEOREM. *Let* X *be a* G*-space and* A *be a* G*-subspace of* X. *Let* M *be an equivariant local system on* X. *Suppose that* X *is the union of a family of mutually disjoint open* G*-subspaces* $\{X_\alpha\}$ *with* $A_\alpha = A \cap X_\alpha$, *and that* $M_\alpha = M \mid X_\alpha$. *Then*

$$H^n_G(X\,,\,A\,;\,M) \simeq \prod_\alpha H^n_G(X_\alpha\,,\,A_\alpha\,;\,M_\alpha).$$

The proof is obtained by using the standard properties of direct products.

2.5.2. THEOREM. *Let* X *be a* G*-space with an equivariant local system* M *on it, and* $X_1\,,\,X_2$ *be* G*-subspaces with* $X = \mathfrak{I}\mathrm{nt}X_1 \cup \mathfrak{I}\mathrm{nt}X_2$. *Then, the following Mayer-Vietoris sequence is exact.*

$$\cdots \longrightarrow H^{n-1}_G(X_1 \cap X_2\,;\,M) \longrightarrow H^n_G(X\,;\,M) \longrightarrow H^n_G(X_1\,;\,M) \oplus H^n_G(X_2\,;\,M) \longrightarrow \cdots$$

The proof follows immediately from Theorem 2.3.11 and the Barratt lemma [5, p. 2].

CHAPTER 3

CELLULAR DESCRIPTION OF THE COHOMOLOGY

3.0. Introduction.

In this chapter we introduce the notion of cup product in the equivariant cohomology with local coefficients system, which makes it a graded ring. We also give a cellular description of the cohomology groups.

3.1. CUP PRODUCT.

As in Chapter 1, G will be a compact group, and by a subgroup of G we shall mean a closed subgroup.

Let X be a G-space and M an equivariant local system of commutative rings with identity on X. For $\tau \in C_G^p(X ; M)$ and $\mu \in C_G^q(X ; M)$, define, $\tau \cup \mu \in C_G^{p+q}(X ; M)$ as follows.

$$(\tau \cup \mu)(\sigma) = \tau(\sigma(\alpha_p))\, M(\sigma(\beta_q)_*)\,(\mu(\sigma(\beta_q))),$$

where $\sigma : \Delta_{p+q} \times G/H \longrightarrow X$ is an equivariant $(p+q)$-simplex in X, $\alpha_p \times id_{G/H} : \Delta_p \times G/H \longrightarrow \Delta_{p+q} \times G/H$ is the equivariant front p-face, and $\beta_q \times id_{G/H} : \Delta_q \times G/H \longrightarrow \Delta_{p+q} \times G/H$ is the equivariant back q-face. Then, $(\tau \cup \mu)(\sigma) \in M(\sigma_H)$.

Clearly, this product is bilinear. The product is also associative. This can be seen as follows. Let $\eta \in C_G^r(X ; M)$, and $\sigma : \Delta_{p+q+r} \times G/H \longrightarrow X$ be any equivariant $(p+q+r)$-simplex. Then, observe that

$$(\sigma \circ (\alpha_{p+q} \times id_{G/H}))(\alpha_p) = \sigma(\alpha_p),\ (\sigma \circ (\beta_{q+r} \times id_{G/H}))(\beta_r) = \sigma(\beta_r),$$

$$(\sigma \circ (\alpha_{p+q} \times \mathrm{id}_{G/H}))(\beta_q) = (\sigma \circ (\beta_{q+r} \times \mathrm{id}_{G/H}))(\alpha_q),$$

and $$(\sigma \circ (\alpha_{p+q} \times \mathrm{id}_{G/H}))(\beta_q)_H = \sigma(\beta_{q+r})_H.$$

Also, note that the morphism $\sigma(\beta_r)_* : \sigma_H \longrightarrow \sigma(\beta_r)_H$ is equal to the composition of the morphisms $\sigma(\beta_{q+r})_* : \sigma_H \longrightarrow \sigma(\beta_{q+r})_H$ and

$$(\sigma \circ (\beta_{q+r} \times \mathrm{id}_{G/H}))(\beta_r)_* : \sigma(\beta_{q+r})_H \longrightarrow (\sigma \circ (\beta_{q+r} \times \mathrm{id}_{G/H}))(\beta_r)_H,$$

and that the morphism

$$(\sigma \circ (\alpha_{p+q} \times \mathrm{id}_{G/H}))(\beta_q)_* : \sigma(\alpha_{p+q})_H \longrightarrow (\sigma \circ (\alpha_{p+q} \times \mathrm{id}_{G/H}))(\beta_q)_H$$

is the same as $\sigma(\beta_{q+r})_*$. Using the above information, we get

$$((\tau \cup \mu) \cup \eta)(\sigma) = (\tau \cup \mu)\,(\sigma(\alpha_{p+q}))\; M(\sigma(\beta_r)_*)\; \eta(\sigma(\beta_r))$$

$$= \tau((\sigma \circ (\alpha_{p+q} \times \mathrm{id}_{G/H}))(\alpha_p))\, M((\sigma \circ (\alpha_{p+q} \times \mathrm{id}_{G/H}))(\beta_q)_*)\; \mu((\sigma \circ (\alpha_{p+q} \times \mathrm{id}_{G/H}))(\beta_q)$$
$$M(\sigma(\beta_r)_*)\; \eta(\sigma(\beta_r))$$

$$= \tau(\sigma(\alpha_p))\; M(\sigma(\beta_{q+r})_*) \mu((\sigma \circ (\beta_{q+r} \times \mathrm{id}_{G/H}))(\alpha_q))$$
$$M(\sigma(\beta_{q+r})_*) \circ\; M((\sigma \circ (\beta_{q+r} \times \mathrm{id}_{G/H}))(\beta_r)_*)\; \eta((\sigma \circ (\beta_{q+r} \times \mathrm{id}_{G/H}))(\beta_r))$$

$$= \tau(\sigma(\alpha_p))\; M(\sigma(\beta_{q+r})_*)\; ((\mu \circ \eta)(\sigma \circ (\beta_{q+r} \times \mathrm{id}_{G/H})))$$

$$= \tau(\sigma(\alpha_p))\; M(\sigma(\beta_{q+r})_*)\; (\mu \circ \eta)(\sigma(\beta_{q+r}))$$

$$= (\tau \cup (\mu \cup \eta))(\sigma).$$

This shows that the product is associative.

Let $I_X \in C^0_G(X\,;M)$ denote the 0-cochain, which assigns to every object x_H in ΠX, $1 \in M(x_H)$. Then it is clear that I_X is the identity for the above product. Standard argument shows that the product $\cup$ and the coboundary δ are related as follows.

3.1.1. PROPOSITION. *If* $\tau \in C^p_G(X\,;M)$ *and* $\mu \in C^q_G(X\,;M)$, *then*

$$\delta\,(\tau \cup \mu) = \delta\,\tau \cup \mu + (-1)^p\, \tau \cup \delta\,\mu.$$

3.1.2. PROPOSITION. *If* $\tau \in S_G^p(X ; M)$ *and* $\mu \in S_G^q(X ; M)$, *then*

$$\tau \cup \mu \in S_G^{p+q}(X ; M).$$

Proof. Let $\sigma : \Delta_{p+q} \times G/H \longrightarrow X$, $\sigma' : \Delta_{p+q} \times G/H' \longrightarrow X$ be equivariant $(p + q)$-simplexes in X compatible under the G-map

$$h : \Delta_{p+q} \times G/H \longrightarrow \Delta_{p+q} \times G/H'.$$

Then $\sigma = \sigma' \circ h$. The G-map h determines fibre preserving G-maps

$$h(\alpha) : \Delta_p \times G/H \longrightarrow \Delta_p \times G/H' \text{ and } h(\beta) : \Delta_q \times G/H \longrightarrow \Delta_q \times G/H'$$

by restricting h, respectively on the front p-face and back q-face of $\Delta_{p+q} \times G/H$, so that

$$(\alpha_p \times id_{G/H'}) \circ h(\alpha) = h \circ (\alpha_p \times id_{G/H})$$

and
$$(\beta_q \times id_{G/H'}) \circ h(\beta) = h \circ (\beta_q \times id_{G/H}).$$

Here, as mentioned before, for any n $\Delta_n \times G/H$ and $\Delta_n \times G/H'$ are considered as trivial bundles over Δ_n. Then,

$$\sigma'(\alpha_p) \circ h(\alpha) = \sigma' \circ (\alpha_p \times id_{G/H'}) \circ h(\alpha) = \sigma' \circ h \circ (\alpha_p \times id_{G/H}) = \sigma(\alpha_p)$$

$$\sigma'(\beta_q) \circ h(\beta) = \sigma' \circ (\beta_q \times id_{G/H'}) \circ h(\beta) = \sigma' \circ h \circ (\beta_q \times id_{G/H}) = \sigma(\beta_q).$$

Thus the equivariant p-simplexes $\sigma(\alpha_p)$ and $\sigma'(\alpha_p)$ are compatible under the G-map $h(\alpha)$, and the equivariant q-simplexes $\sigma(\beta_q)$ and $\sigma'(\beta_q)$ are compatible under the G-map $h(\beta)$. Therefore,

3.1.3 $\quad M(h(\alpha)_*)\, \tau(\sigma'(\alpha_p)) = \tau(\sigma(\alpha_p))$ and $M(h(\beta)_*)\, \mu(\sigma'(\beta_q)) = \mu(\sigma(\beta_q))$.

Now observe that $\sigma(\alpha_p)_H = \sigma_H$, and $\sigma'(\alpha_p)_{H'} = \sigma'_{H'}$, and that the morphism $h(\alpha)_* = [\,\overline{h(\alpha)}\,, \nu_\alpha] : \sigma(\alpha_p)_H \longrightarrow \sigma'(\alpha_p)_{H'}$ is equal to the morphism $h_* = [\,\overline{h}\,, \nu] : \sigma_H \longrightarrow \sigma'_{H'}$. Moreover, the morphism $\sigma'(\beta_q)_* \circ h_*$ is equal to the morphism $h(\beta)_* \circ \sigma(\beta_q)_*$. The proof of this fact is similar to the proof of Lemma 1.5.7.

Therefore,

$$M(h_*)(\tau \cup \mu)(\sigma') = M(h_*)\Big(\tau(\sigma'(\alpha_p))\, M(\sigma'(\beta_q)_*)\, \mu(\sigma'(\beta_q))\Big)$$

$$= M(h(\alpha)_*)\, \tau(\sigma'(\alpha_p))\, M(\sigma'(\beta_q)_* \circ h_*)\, \mu(\sigma'(\beta_q))$$
$$= \tau(\sigma(\alpha_p))\, M(h(\beta)_* \circ \sigma(\beta_q)_*)\, \mu(\sigma'(\beta_q))$$
$$= \tau(\sigma(\alpha_p))\, M(\sigma(\beta_q)_*) \circ M(h(\beta)_*)\, \mu(\sigma'(\beta_q))$$
$$= \tau(\sigma(\alpha_p))\, M(\sigma(\beta_q)_*)\, \mu(\sigma(\beta_q)) = (\tau \cup \mu)(\sigma).$$

This shows that $\tau \cup \mu \in S_G^{p+q}(X ; M)$. □

Now Proposition 3.1.1 implies that, for $\tau \in S_G^p(X ; M)$ and $\mu \in S_G^q(X ; M)$, the coboundary satisfies

$$\delta\,(\tau \cup \mu) = \delta\, \tau \cup \mu + (-1)^p\, \tau \cup \delta\, \mu.$$

Thus, $\oplus_p S_G^p(X ; M)$ is a graded ring with identity, the cocycle $\oplus_p Z_G^p(X ; M)$ is a subring and the coboundary $\oplus_p B_G^p(X ; M)$ is a two sided ideal in $\oplus_p Z_G^p(X ; M)$. Consequently, by passing onto the quotient, $\oplus_p H_G^p(X ; M)$ becomes a graded ring with unity, where the product

$$H_G^p(X ; M) \cup H_G^q(X ; M) \longrightarrow H_G^{p+q}(X ; M)$$

is given by $[\tau] \cup [\mu] = [\tau \cup \mu]$. This defines the cup product in cohomology.

For any G-subspace A of X, and $\tau \in S_G^p(X ; M)$, $\mu \in S_G^q(X , A ; M)$, the product $\tau \cup \mu \in S_G^{p+q}(X , A ; M)$. Thus we may define the cup product,

$$H_G^p(X ; M) \cup H_G^q(X , A ; M) \longrightarrow H_G^{p+q}(X , A ; M)$$

in the usual way.

3.2. Cellular description.

In this section we generalize a theorem of Milnor to equivariant cohomology with local coefficients system and prove some properties of the cohomology following Whitehead [20]. Finally, we use a spectral sequence argument to give a cellular description of the equivariant cohomology groups.

3.2.1. LEMMA. *If* $(X\,;X_1\,,X_2)$ *is a G-CW-complex triad, then we can find a G-open set* $X_1' \supset X_1$ *and a G-homotopy* $H: X\times I \longrightarrow X$ *satisfying*

(i) $H_0 = Id_X$, (ii) H *is stationary on* X_1, (iii) $H_1(X_1') \subset X_1$, (iv) $H(X_2\times I)\subset X_2$.

Proof (cf. [19] Lemma 7.4, p.100). Let $X_1^{-1} = X_1$ and $H^{-1}: X\times I \longrightarrow X$ be the stationary G-homotopy. Suppose that we have already constructed for each k, $-1\le k\le n$, a G-open neighbourhood X_1^k of X_1 in the $(k+1)$-skeleton $(X\,,X_1)_{k+1}$ with $X_1^k \cap (X\,,X_1)_k = X_1^{k-1}$, and a G-homotopy $H^k: X\times I\longrightarrow X$ satisfying

(a) $H_0^k = H_1^{k-1}$, (b) H^k is stationary on $(X\,,X_1)_k$, (c) $H_1^k(X_1^k)\subset X_1$,

(d) $H^k(X_2\times I)\subset X_2$.

Let the equivariant $(n+1)$-cells of $(X\,,X_1)$ be $\left\{e_\gamma^{n+1} : \gamma\in\Gamma\right\}$ and let f_γ^{n+1} be the characteristic map of e_γ^{n+1}. In D^{n+1}, let $D_0^{n+1} = \left\{x\in D^{n+1} : \|x\|\le 1/2\right\}$. Then $D^{n+1} - D_0^{n+1}$ is an open neighbourhood of S^n which can be contracted onto S^n : Define $K: D^{n+1}\times I\longrightarrow D^{n+1}$ by

$$K(x\,,t) = \begin{cases} (1+t)x & \|x\|\le \frac{1}{1+t} \\ \frac{x}{\|x\|} & \|x\|\ge \frac{1}{1+t}\end{cases} \qquad x\in D^{n+1},\ t\in I.$$

Now let

$$U_\gamma^{n+1} = \Big\{ f_\gamma^{n+1}(y\,,gH_\gamma) : f_\gamma^{n+1}(y/\|y\|\,,gH_\gamma)\in X_1^{n-1},\ (y\,,gH_\gamma)\in (D^{n+1}-D_0^{n+1})\times G/H_\gamma\Big\}.$$

Clearly U_γ^{n+1} is a G-open subset of e_γ^{n+1}, so if we take $X_1^n = X_1^{n-1}\cup\bigcup_{\gamma\in\Gamma} U_\gamma^{n+1}$, then X_1^n is a G-open neighbourhood of X_1 in $(X\,,X_1)_{n+1}$ with $X_1^n\cap(X\,,X_1)_n = X_1^{n-1}$. We define $H^n: (X\,,X_1)_{n+1}\times I\longrightarrow X$ by

$$H^n(x,t) = \begin{cases} H_1^{n-1}(x) & x \in (X, X_1)_n \\ H_1^{n-1}(f_\gamma^{n+1}(K(y,t), gH_\gamma)), & x = f_\gamma^{n+1}(y, gH_\gamma), \\ & (y, gH_\gamma) \in D^{n+1} \times G/H_\gamma,\ t \in I. \end{cases}$$

Then H^n is continuous and satisfies $H_0^n = H_1^{n-1}$ on $(X, X_1)_{n+1}$, so we can extend to X with $H_0^n = H_1^{n-1}$. Clearly (a), (b), (c) are satisfied. H^n also satisfies (d), since H^{n-1} does. Thus we can construct X_1^k, H^k, for all $k \geq -1$ by induction.

Now we take $X_1' = \bigcup_{k \geq -1} X_1^k$; then X_1' is open because $X_1' \cap e_\gamma^m = X_1^{m-1} \cap e_\gamma^m$ for all m, γ. We define $H : X \times I \longrightarrow X$ by

$$H(x,t) = \begin{cases} H^{r-1}(x, (r+1)(r(t-1)+1)) & \frac{r-1}{r} \leq t \leq \frac{r}{r+1},\ x \in X, \\ H^r(x, 1) & t = 1,\ x \in (X, X_1)_r. \end{cases}$$

Then H is continuous, $H_0 = H_0^0 = id_X$; also H is stationary on X_1, $H_1(X_1') \subset X_1$ and $H(X_2 \times I) \subset X_2$. □

3.2.2. PROPOSITION. *For any* G-CW-*complex triad* $(X ; X_1, X_2)$, *the inclusion* $i : (X_1, X_1 \cap X_2 ; M) \longrightarrow (X, X_2 ; M)$ *induces isomorphism*

$$i^* : H_G^n(X, X_2 ; M) \simeq H_G^n(X_1, X_1 \cap X_2 ; M),$$

where M *is an equivariant local system on* X.

Proof. Given any G-CW-complex triad $(X ; X_1, X_2)$ we choose a G-open neighbourhood X_1' of X_1 and a G-homotopy H as in Lemma 3.2.1. Let $r = H_1$. Then $r \mid X_1' : (X_1', X_1' \cap X_2) \longrightarrow (X_1, X_1 \cap X_2)$ is a G-homotopy inverse of $j : (X_1, X_1 \cap X_2) \longrightarrow (X_1', X_1' \cap X_2)$; so we have that $(r \mid X_1')^*$ is an isomorphism. Since $r \simeq id_X$, we see that

$$r^* : H_G^n(X, X_2 ; M) \longrightarrow H_G^n(X, X_2 ; M)$$

is identity. Moreover, $i'^* : H^n_G(X, X_2; M) \longrightarrow H^n_G(X'_1, X'_1 \cap X_2; M)$ is an isomorphism by excision, since $X - X_1$ is a G-open subset of X_2 and thus $X = X'_1 \cup (X - X_1) \subset \mathfrak{Int}X'_1 \cup \mathfrak{Int}X_2$. Since $i'^* \circ r^* = (r \mid X'_1)^* \circ i^*$, it follows that i^* is also an isomorphism. □

Following a standard argument, we get a relative Mayer-Vietoris exact sequence for a triad (cf. 2.5.2).

3.2.3. PROPOSITION. *If* $(X; X_1, X_2)$ *is a G-CW-complex triad and* X_3 *is a G-subset of* $X_1 \cap X_2$, M *an equivariant local system on* X, *then there is an exact sequence*

$$\cdots \longrightarrow H^{n-1}_G(X_1 \cap X_2, X_3; M) \xrightarrow{\Delta} H^n_G(X, X_3; M) \xrightarrow{\beta} H^n_G(X_1, X_3; M) \oplus H^n_G(X_2, X_3; M)$$
$$\xrightarrow{\alpha} H^n_G(X_1 \cap X_2, X_3; M) \rightarrow \cdots\cdots$$

where $\alpha(x, y) = i_1^*(x) - i_2^*(y)$, $\beta(x) = (i_3^*(x), i_4^*(x))$ *and* Δ *is the composite*

$$H^{n-1}_G(X_1 \cap X_2, X_3; M) \xrightarrow{\Delta_1} H^n_G(X_1, X_1 \cap X_2; M) \overset{j_1^*}{\cong} H^n_G(X, X_2; M) \xrightarrow{J_1^*} H^n_G(X, X_3; M).$$

Here $i_1, i_2, i_3, i_4, j_1,$ *and* J_1 *are inclusions*

$$i_1 : (X_1 \cap X_2, X_3) \longrightarrow (X_1, X_3), \ i_2 : (X_1 \cap X_2, X_3) \longrightarrow (X_2, X_3),$$
$$i_3 : (X_1, X_3) \longrightarrow (X, X_3), \ i_4 : (X_2, X_3) \longrightarrow (X, X_3),$$
$$j_1 : (X_1, X_1 \cap X_2) \longrightarrow (X, X_2), \ J_1 : (X, X_3) \longrightarrow (X, X_2),$$

and Δ_1 *is the coboundary for the triple* $(X_1, X_1 \cap X_2, X_3)$.

We shall need a slight variant of the Mayer-Vietoris exact sequence of Proposition 3.2.3.

3.2.4. THEOREM. *Let* $(X ; X_1, X_2)$ *be a G-CW-complex triad and* X_3 *a G-subset of* X, M *an equivariant local system on* X. *Then we have the following exact sequence*

$$\cdots\cdots \to H^n_G(X, X_3 ; M) \to H^n_G(X_1, X_1 \cap X_3 ; M) \oplus H^n_G(X_2, X_2 \cap X_3 ; M)$$
$$\to H^n_G(X_1 \cap X_2, X_1 \cap X_2 \cap X_3 ; M) \to H^{n+1}_G(X, X_3 ; M) \to \cdots\cdots .$$

Proof. Apply Proposition 3.2.3 for the triad $(X ; X_1', X_2')$ with $X_1' = X_1 \cup X_3$, $X_2' = X_2 \cup X_3$ and observe that $X_1' \cap X_2' = (X_1 \cap X_2) \cup X_3$. Also notice that

$$H^*_G(X_1', X_3 ; M) \simeq H^*_G(X_1, X_1 \cap X_3 ; M), \quad H^*_G(X_2', X_3 ; M) \simeq H^*_G(X_2, X_2 \cap X_3 ; M)$$

and
$$H^*_G(X_1' \cap X_2', X_3 ; M) \simeq H^*_G(X_1 \cap X_2, X_1 \cap X_2 \cap X_3 ; M)$$

by Proposition 3.2.2. □

3.2.5. DEFINITION. Let $M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \longleftarrow \cdots\cdots$ be an inverse system of abelian groups and homomorphisms. Let $M = \prod_{n=1}^{\infty} M_n$, and $d : M \longrightarrow M$ be the endomorphism given by

$$d(x_1, x_2, x_3, \cdots\cdots) = (x_1 - f_1(x_2), x_2 - f_2(x_3), \cdots\cdots).$$

Then Ker d is called the inverse limit of the inverse system $\{M_n\}$, and it is denoted by $\varprojlim M_n$. Moreover, Coker d $= M \mid d(M)$ is called the derived functor of the inverse limit functor and it is denoted by $\varprojlim^1 M_n$.

The generalization of Milnor's theorem to equivariant cohomology with local coefficients asserts

3.2.6. THEOREM. *Let* (X, A) *be a relative* G-CW-*complex, and let* $\{(X_n, A) : n = 0, 1, 2, \cdots\}$ *be an increasing sequence of* G-*subcomplexes such that* $\bigcup_{n=0}^{\infty} X_n = X$. *Let* M *be an equivariant local system on* X. *Then there is an exact sequence*

$$0 \longrightarrow \varprojlim{}^1 H_G^{q-1}(X_n, A; M) \xrightarrow{\alpha} H_G^q(X, A; M) \xrightarrow{\beta} \varprojlim H_G^q(X_n, A; M) \longrightarrow 0$$

in which β *is induced by the inclusion* $(X_n, A; M) \subset (X, A; M)$.

Proof. (cf. Whitehead [20], p.273). Let $\mathbf{R}^+$ be the set of non-negative real numbers which is given the structure of a G-CW-complex, where the vertices are the non-negative integers and 1-cells are the closed intervals $[n, n+1]$ with trival G-action. Let $L = \bigcup_{n=0}^{\infty} X_n \times [n, n+1]$, so that $(L, A \times \mathbf{R}^+)$ is a G-subcomplex of $(X \times \mathbf{R}^+, A \times \mathbf{R}^+)$ and the restriction to $(L, A \times \mathbf{R}^+)$ of the projection of $(X \times \mathbf{R}^+, A \times \mathbf{R}^+)$ on the first factor is a G-homotopy equivalence $p : (L, A \times \mathbf{R}^+) \longrightarrow (X, A)$. Let

$$L_1 = \bigcup_{i \geq 0} X_{2i} \times [2i, 2i+1], \text{ and } L_2 = \bigcup_{i \geq 0} X_{2i+1} \times [2i+1, 2i+2].$$

$$A_i = L_i \cap (A \times \mathbf{R}^+), \; i = 1, 2.$$

Then, as in 3.2.4, we have the following exact Mayer-Vietoris sequence

$$\cdots\cdots \xrightarrow{\lambda_{q-1}} H_G^{q-1}(L_1 \cap L_2, A_1 \cap A_2; \overline{M}) \to H_G^q(L, A \times \mathbf{R}^+; \overline{M})$$

$$\to H_G^q(L_1, A_1; \overline{M}) \oplus H_G^q(L_2, A_2; \overline{M}) \xrightarrow{\lambda_q} H_G^q(L_1 \cap L_2, A_1 \cap A_2; \overline{M}) \to \cdots$$

where $\overline{M} = p^* M$. But the additivty property implies that

$$H_G^q(L_1, A_1; \overline{M}) \simeq \prod_{i \geq 0} H_G^q(X_{2i}, A; M) \quad \text{and} \quad H_G^q(L_2, A_2; \overline{M}) \simeq \prod_{i \geq 0} H_G^q(X_{2i+1}, A; M),$$

so that

$$H_G^q(L_1, A_1; \overline{M}) \oplus H_G^q(L_2, A_2; \overline{M}) \simeq \prod_{n=0}^{\infty} H_G^q(X_n, A; M),$$

while

$$H^q_G(L_1 \cap L_2, A_1 \cap A_2; \overline{M}) \simeq \prod_{n=0}^{\infty} H^q_G(X_n, A; M).$$

Under these isomorphisms, the homomorphism λ_q, induced by the appropriate inclusions, corresponds to the endomorphism d of 3.2.5. Now the above exact sequence induces a short exact sequence

$$0 \longrightarrow \text{Coker } \lambda_{q-1} \longrightarrow H^q_G(L, A \times R^+; \overline{M}) \longrightarrow \text{Ker } \lambda_q \longrightarrow 0.$$

The middle group can be identified with $H^q_G(X, A; M)$ and the end groups with the appropriate $\varprojlim^1$ and $\varprojlim$. This completes the proof. □

Let (X, A) be a relative G-CW-complex, M an equivariant local system on X. Let X_n be the n-skeleton of (X, A), and $\{E^n_\alpha\}$ be the equivariant n-cells of (X, A). Let

$$h_\alpha : (\Delta_n \times G/H_\alpha, \dot{\Delta}_n \times G/H_\alpha) \longrightarrow (X_n, X_{n-1})$$

denote the characteristic map of E^n_α. Then $M_\alpha = h^*_\alpha M$ is an equivariant local system on $\Delta_n \times G/H_\alpha$.

3.2.7. THEOREM. *The homomorphisms*

$$H^q_G(X_n, X_{n-1}; M) \longrightarrow H^q_G(\Delta_n \times G/H_\alpha, \dot{\Delta}_n \times G/H_\alpha; M_\alpha)$$

induced by the characteristic maps h_α *represent the former group as direct product.*

Proof. Let $F_\alpha = \{h_\alpha(b_n, gH_\alpha) : gH_\alpha \in G/H_\alpha\}$, where b_n is the barycentre of Δ_n, and $F = \bigcup_\alpha F_\alpha$. Let $U = X_n - F$. Then U is a G-subset of X_n and X_{n-1} is a G-deformation retract of U. Therefore we have isomorphism

$$i^*_1 : H^q_G(X_n, U; M) \longrightarrow H^q_G(X_n, X_{n-1}; M).$$

Let $V = X_n - X_{n-1}$, $W = V \cap U$; then X_n is the union of the relatively G-open sets U, V and therefore we have isomorphism $i^*_2 : H^q_G(X_n, U; M) \longrightarrow H^q_G(V, W; M)$. Finally, let $V_\alpha = \text{Int } E^n_\alpha$, $W_\alpha = V_\alpha \cap U$. By the additivity Theorem 2.5.1, the homomorphisms

$$i_\alpha^* : H_G^q(V , W ; M) \longrightarrow H_G^q(V_\alpha , W_\alpha ; M)$$

represent the former group as direct product. The homomorphisms

$$H_G^q(X_n , X_{n-1} ; M) \longrightarrow H_G^q(\Delta_n \times G/H_\alpha , \dot{\Delta}_n \times G/H_\alpha ; M_\alpha)$$

$$H_G^q(X_n , U ; M) \longrightarrow H_G^q(\Delta_n \times G/H_\alpha , \Delta_n - \{b_n\} \times G/H_\alpha ; M_\alpha)$$

$$H_G^q(V , W ; M) \longrightarrow H_G^q(\mathfrak{Int}\ \Delta_n \times G/H_\alpha , (\mathfrak{Int}\ \Delta_n - \{b_n\}) \times G/H_\alpha ; M_\alpha)$$

induced by h_α, define homomorphisms

$$h_1 : H_G^q(X_n , X_{n-1} ; M) \longrightarrow \prod_\alpha H_G^q(\Delta_n \times G/H_\alpha , \dot{\Delta}_n \times G/H_\alpha ; M_\alpha)$$

$$h_2 : H_G^q(X_n , U ; M) \longrightarrow \prod_\alpha H_G^q(\Delta_n \times G/H_\alpha , \Delta_n - \{b_n\} \times G/H_\alpha ; M_\alpha)$$

$$h_3 : H_G^q(V , W ; M) \longrightarrow \prod_\alpha H_G^q(\mathfrak{Int}\ \Delta_n \times G/H_\alpha , (\mathfrak{Int}\ \Delta_n - \{b_n\}) \times G/H_\alpha ; M_\alpha).$$

The map

$$h_\alpha \mid \mathfrak{Int}\ \Delta_n \times G/H_\alpha : (\mathfrak{Int}\ \Delta_n \times G/H_\alpha , (\mathfrak{Int}\ \Delta_n - \{b_n\}) \times G/H_\alpha) \longrightarrow (V_\alpha , W_\alpha)$$

is a homeomorphism which induces isomorphism in the cohomology. Therefore we get an isomorphism

$$h_4 : \prod_\alpha H_G^q(V_\alpha , W_\alpha ; M) \longrightarrow \prod_\alpha H_G^q(\mathfrak{Int}\ \Delta_n \times G/H_\alpha , (\mathfrak{Int}\ \Delta_n - \{b_n\}) \times G/H_\alpha ; M_\alpha).$$

Let

$$j_1 : \prod_\alpha H_G^q(\Delta_n \times G/H_\alpha , \Delta_n\text{-}\{b_n\} \times G/H_\alpha ; M_\alpha) \longrightarrow \prod_\alpha H_G^q(\Delta_n \times G/H_\alpha , \dot{\Delta}_n \times G/H_\alpha ; M_\alpha)$$

be induced by appropriate inclusions. Similarly we have the homomorphism j_2 from

$$\prod_\alpha H_G^q(\Delta_n \times G/H_\alpha , \Delta_n - \{b_n\} \times G/H_\alpha ; M_\alpha)$$

to

$$\prod_\alpha H_G^q(\mathfrak{Int}\ \Delta_n \times G/H_\alpha , (\mathfrak{Int}\ \Delta_n - \{b_n\}) \times G/H_\alpha ; M_\alpha),$$

induced by appropriate inclusions. Let

$$i : H_G^q(V , W ; M) \longrightarrow \prod_\alpha H_G^q(V_\alpha , W_\alpha ; M)$$

be the homomorphism induced by i_α. Then we have the following commutativity relations : (a) $h_3 = h_4 \circ i$, (b) $h_3 \circ i_2 = j_2 \circ h_2$, (c) $j_1 \circ h_2 = h_1 \circ i_1$. Since j_1, j_2, i_1,

i_2, i, and h_4 are isomorphisms, so is h_1. This completes the proof of the theorem. □

3.2.8. COROLLARY. *If* X_n *is the n-skeleton of the relative G-CW- complex* (X , A) *and* M *an equivariant local system on* X, *then*

$$H^q_G(X_n , X_{n-1} ; M) = 0 \text{ if } q \neq n,$$

and $H^n_G(X_n , X_{n-1} ; M)$ *can be identified with the group of all functions* c *which assigns to each equivariant n-cell* $\sigma : \Delta_n \times G/H \longrightarrow X$ *of* (X , A) *an element* $c(\sigma) \in M(\sigma_H)$ *such that if* σ *and* σ' *are two equivariant n-cells of type* H *and* H' *respectively compatible under a fibre preserving G-map* $h : \Delta_n \times G/H \longrightarrow \Delta_n \times G/H'$ *then* $M(h_*)\, c(\sigma') = c(\sigma)$.

Proof. We have by Theorem 3.2.7

$$H^q_G(X_n , X_{n-1} ; M) \simeq \prod_\alpha H^q_G(\Delta_n \times G/H_\alpha , \dot{\Delta}_n \times G/H_\alpha ; M_\alpha)$$

where the product is taken over all equivariant n-cells E^n_α of (X , A), and $M_\alpha = \sigma^*_\alpha M$, σ_α being the characteristic map of E^n_α. Let $\Lambda_n = \left\{x \in \Delta_n : x_j = 0 \text{ for some } j > 0\right\}$ be the union of all faces of Δ_n except one. Set

$$\Delta^\alpha_n = \Delta_n \times G/H_\alpha, \quad \dot{\Delta}^\alpha_n = \dot{\Delta}_n \times G/H_\alpha \quad \text{and} \quad \Lambda^\alpha_n = \Lambda_n \times G/H_\alpha.$$

Then, we have the following long exact sequence of cohomology groups for the triple $(\Delta^\alpha_n , \dot{\Delta}^\alpha_n , \Lambda^\alpha_n)$

$$\cdots \to H^{q-1}_G(\Delta^\alpha_n , \dot{\Delta}^\alpha_n ; M_\alpha) \xrightarrow{j^*} H^{q-1}_G(\Delta^\alpha_n , \Lambda^\alpha_n ; M_\alpha) \xrightarrow{i^*} H^{q-1}_G(\dot{\Delta}^\alpha_n , \Lambda^\alpha_n ; M_\alpha)$$

$$\xrightarrow{\partial} H^q_G(\Delta^\alpha_n , \dot{\Delta}^\alpha_n ; M_\alpha) \to \cdots\cdots$$

where i^* and j^* are induced by the inclusions

$$i : (\dot{\Delta}^\alpha_n , \Lambda^\alpha_n ; M_\alpha) \subset (\Delta^\alpha_n , \Lambda^\alpha_n ; M_\alpha)$$

and $$j : (\Delta^\alpha_n , \Lambda^\alpha_n ; M_\alpha) \subset (\Delta^\alpha_n , \dot{\Delta}^\alpha_n ; M_\alpha).$$

Now, the map $(I \times \Delta^\alpha_n , I \times \Lambda^\alpha_n ; p^* M_\alpha) \longrightarrow (\Delta^\alpha_n , \Lambda^\alpha_n ; M_\alpha)$, given by

$(t, (x, g H_\alpha)) \longrightarrow ((1-t)x + t d_0, g H_\alpha)$, $x \in \Delta_n$, and the identity homomorphism between equivariant local systems, provide a homotopy equivalence between $(\Delta_n^\alpha, \Lambda_n^\alpha; M_\alpha)$ and $(d_0 \times G/H_\alpha, d_0 \times G/H_\alpha; M_\alpha)$ in $\mathcal{L}^2$. This means that $H_G^q(\Delta_n^\alpha, \Lambda_n^\alpha; M_\alpha) = 0$ for all q, and, by the above exact sequence,

$$H_G^{q-1}(\dot{\Delta}_n^\alpha, \Lambda_n^\alpha; M_\alpha) \xrightarrow{\partial} H_G^q(\Delta_n^\alpha, \dot{\Delta}_n^\alpha; M_\alpha)$$

is an isomorphism. Note further that the homomorphism

$$H_G^{q-1}(\dot{\Delta}_n^\alpha, \Lambda_n^\alpha; M_\alpha) \longrightarrow H_G^{q-1}((\dot{\Delta}_n - d_0) \times G/H_\alpha, (\Lambda_n - d_0) \times G/H_\alpha; M_\alpha)$$

induced by the inclusion

$$((\dot{\Delta}_n - d_0) \times G/H_\alpha, (\Lambda_n - d_0) \times G/H_\alpha; M_\alpha) \subset (\dot{\Delta}_n^\alpha, \Lambda_n^\alpha; M_\alpha)$$

is an isomorphism by excision. Moreover, there is a deformation retraction of $(\dot{\Delta}_n - d_0) \times G/H_\alpha$ onto $e_n^0(\Delta_{n-1}) \times G/H_\alpha$, which takes $(\Lambda_n - d_0) \times G/H_\alpha$ into $e_n^0(\dot{\Delta}_{n-1}) \times G/H_\alpha$, where $e_n^0 : \Delta_{n-1} \longrightarrow \Delta_n$ is the 0-th-face map. This provides a G-homotopy equivalence, and hence

$$H_G^{q-1}((\dot{\Delta}_n - d_0) \times G/H_\alpha, (\Lambda_n - d_0) \times G/H_\alpha; M_\alpha) \simeq H^{q-1}(\Delta_{n-1}^\alpha, \dot{\Delta}_{n-1}^\alpha; M_\alpha).$$

Consequently, $H_G^q(\Delta_n^\alpha, \dot{\Delta}_n^\alpha; M_\alpha) \simeq H_G^{q-1}(\Delta_{n-1}^\alpha, \dot{\Delta}_{n-1}^\alpha; M_\alpha)$ for $n > 0$. Now,

$$\begin{aligned}
H^q(\Delta_n \times G/H_\alpha, \dot{\Delta}_n \times G/H_\alpha; M_\alpha) &= H_G^q(\Delta_n^\alpha, \dot{\Delta}_n^\alpha; M_\alpha) \\
&\simeq H_G^{q-1}(\Delta_{n-1}^\alpha, \dot{\Delta}_{n-1}^\alpha; M_\alpha) \\
&\cdots\cdots\cdots \\
&\simeq H_G^{q-n}(\Delta_0^\alpha, \dot{\Delta}_0^\alpha; M_\alpha) \\
&= H_G^{q-n}(\Delta_0^\alpha; M_\alpha) \\
&= H_G^{q-n}(G/H_\alpha; M_\alpha) = 0, \text{ if } q \neq n.
\end{aligned}$$

This proves that $H_G^q(X_n, X_{n-1}; M) = 0$, if $q \neq n$, $n > 0$, and, for $n = 0$, $H_G^q(X_n, X_{n-1}; \mathcal{M})$ reduces to $\prod_\alpha H_G^q(G/H_\alpha; \mathcal{M})$, the product being taken over all equivariant 0-cells. Therefore, by Theorem 2.4.1, $H^q(X_n, X_{n-1}; \mathcal{M}) = 0$, if $q \neq 0$. The second part follows from the following fact

$$H^n_G(\Delta^\alpha_n , \dot{\Delta}^\alpha_n ; M_\alpha) = H^0_G(G/H_\alpha ; M_\alpha) = \sigma^*_\alpha M(\mathrm{id}_{G/H_\alpha}) = M((\sigma_\alpha)_{H_\alpha}).$$

This completes the proof. □

As a consequence, we get

3.2.9. COROLLARY. *If* $q \leq n < m$, *then* $H^q_G(X_m , X_n ; M) = 0$. *If* $q > m > n$, *then* $H^q_G(X_m , X_n ; M) = 0$.

3.2.10. PROPOSITION. *If* $q \leq n$, *then* $H^q_G(X , X_n ; M) = 0$.

This follows exactly as in [20], from Corollary 3.2.9 and Theorem 3.2.6.

Now, we are in a position to give a cellular description of the cohomology groups.

Let (X , A) be a relative G-CW-complex and M an equivariant local system on X. Consider the filtration

$$A = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset X$$

of the space X by its skeleta. Then, for the pair (X_p , X_{p-1}), we have the following exact sequence.

$$\cdots \to H^{p+q}_G(X_p , X_{p-1} ; M) \xrightarrow{j^*} H^{p+q}_G(X_p ; M) \xrightarrow{i^*} H^{p+q}_G(X_{p-1} ; M) \xrightarrow{\partial} H^{p+q+1}_G(X_p , X_{p-1} ; M) \to \cdots$$

Set $E^{p,q}_1 = H^{p+q}_G(X_p , X_{p-1} ; M)$, and $D^{p,q}_1 = H^{p+q}_G(X_p ; M)$. These provide a bigraded exact couple, where the homomorphisms of the exact couple are the homomorphisms induced by the appropriate inclusions and connecting homomorphisms. Thus, we have a spectral sequence $\{(E_r , d^r)\}$, where bidegree of d^r is $(r , 1-r)$ and $d^1 : E^{p,q}_1 \longrightarrow E^{p+1,q}_1$ is the coboundary operator of the triple $(X_{p+1} , X_p , X_{p-1})$

$$H^{p+q}(X_p , X_{p-1} ; M) \longrightarrow H^{p+q+1}(X_{p+1} , X_p ; M).$$

Therefore, we have a decreasing filtration

$$H^n_G(X, A; M) = J^{0,n} \supseteq J^{1,n-1} \supseteq \cdots\cdots \supseteq J^{p,n-p} \supseteq \cdots\cdots \supseteq J^{n+1,-1} = 0.$$

Note that, by Proposition 3.2.10 $H^n_G(X, X_n; M) = 0$, and hence, by cohomology exact sequence for the triple (X, X_n, A), it follows that

$$J^{n+1,-1} = \operatorname{Ker}[\, H^n_G(X, A; M) \longrightarrow H^n_G(X_n, A; M)\,] = 0.$$

The above spectral sequence converges to the bigraded module $\mathcal{G}\, H^*_G(X, A; M)$ associated to the graded module $H^*_G(X, A; M)$ with the filtration as described above. Thus, $E^{p,q}_\infty \simeq J^{p,q}/J^{p+1,q-1}$. Consequently, we get

3.2.11. THEOREM. *Let* (X, A) *be a relative* G-CW-complex *and* M *an equivariant local system on* X. *Let* $C^p(X, A; M) = H^p_G(X_p, X_{p-1}; M)$ *and* $\delta : C^p(X, A; M) \longrightarrow C^{p+1}(X, A; M)$ *be the coboundary operator of* (X_{p+1}, X_p, X_{p-1}). *Then* $C(X, A; M) = \{C^p(X, A; M), \delta\}$ *is a cochain complex and*

$$H^*_G(X, A; M) \simeq H^*(C(X, A; M)).$$

Proof. $C(X, A; M)$ is clearly a cochain complex. We have seen that $H^n_G(X_p, X_{p-1}; M) = 0$, if $n \neq p$. This implies $E^{p,q}_1 = 0$ if $q \neq 0$, $E^{p,0}_1 = C^p(X, A; M)$, $d^1 = \delta$. Therefore, $E^{p,q}_2 = 0$ if $q \neq 0$, and $E^{p,0}_2 = H^p(C(X, A; M))$. By induction, we have $E^{p,q}_r = 0$ if $q \neq 0$, and $E^{p,0}_r = E^{p,0}_2 = H^p(C(X, A; M))$ if $r \geq 2$. This implies that, $E^{p,q}_\infty = 0$ if $q \neq 0$, and $E^{p,0}_\infty = H^p(C(X, A; M))$. Since, $E^{p,q}_\infty \simeq J^{p,q}/J^{p+1,q-1}$, it follows that, $J^{p,q} = J^{p+1,q-1}$ if $q \neq 0$. Now,

$$H^n(C(X, A; M)) = E^{n,0}_\infty = J^{n,0}/J^{n+1,-1} = J^{n,0}$$

and

$$J^{n,0} = \operatorname{Ker}[\, H^n_G(X, A; M) \longrightarrow H^n_G(X_{n-1}, A; M)] = H^n_G(X, A; M),$$

since $H^n_G(X_{n-1}, A; M) = 0$, by Corollary 3.2.9. Thus,

$$H^n(C(X, A; M)) \simeq H^n_G(X, A; M).$$

This completes the proof. □

Finally, we prove a uniqueness theorem concerning equivariant cohomology on the category of relative G-CW-complex with equivariant local system.

Let $\mathcal{L}^2_C$ be the subcategory of $\mathcal{L}^2$ consisting of relative G-CW-complex with equivariant local systems. We have the following uniqueness theorem on the subcategory $\mathcal{L}^2_C$, (cf. [3]).

3.2.12. THEOREM. *If* $h = \{h^q\}$ *and* $k = \{k^q\}$ *are additive cohomology theories on* $\mathcal{L}^2_C$ *satisfying all the axioms, and if* $\Phi : h \longrightarrow k$ *is a natural transformation such that for every object* $(G/H ; M)$ *in* $\mathcal{L}^2_C$, $\Phi(G/H ; M)$ *is an isomorphism, then*

$$\Phi(X, A; \mathcal{M}) : h^q(X, A; \mathcal{M}) \longrightarrow k^q(X, A; \mathcal{M})$$

is an isomorphism for every object $(X, A; \mathcal{M})$ *in* $\mathcal{L}^2_C$.

Proof. Let us first prove the theorem for finite dimensional relative G-CW-complex. So, assume that h and k are additive cohomology theories on the category of finite dimensional relative G-CW-complex with equivariant local systems, and $\Phi : h \longrightarrow k$ is a natural transformation such that, for every object $(G/H ; M)$,

$$\Phi(G/H ; M) : h^*(G/H ; M) \longrightarrow k^*(G/H ; M)$$

is an isomorphism. We want to prove that, for every object $(X, A; M)$,

$$\Phi(X, A; M) : h^*(X, A; M) \longrightarrow k^*(X, A; M)$$

is an isomorphism, where (X, A) is a finite dimensional relative G-CW-complex. By the exact sequence for $(X, A; M)$ and five lemma, it is sufficient to consider the case when $\overset{\circ}{A}$ is empty. We have

$$h^q(X_n, X_{n-1}; M) \simeq \prod_\alpha h^{q-n}(G/H_\alpha; M),$$

and

$$k^q(X_n, X_{n-1}; M) \simeq \prod_\alpha k^{q-n}(G/H_\alpha; M),$$

the product being taken over all equivariant n-cells. By hypothesis, there is a natural isomorphism

$$h^{q-n}(G/H_\alpha; M) \simeq k^{q-n}(G/H_\alpha; M).$$

Therefore, $h^q(X_n, X_{n-1}; M) \simeq k^q(X_n, X_{n-1}; M)$. Now $\mathfrak{G}$ induces maps between cohomology exact sequences of $(X_n, X_{n-1}; M)$,

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & h^{q-1}(X_{n-1}; M) & \longrightarrow & h^q(X_n, X_{n-1}; M) & \longrightarrow & h^q(X_n; M) & \longrightarrow & h^q(X_{n-1}; M) & \longrightarrow \cdots \\
 & \downarrow \mathfrak{G} & & \downarrow \mathfrak{G} & & \downarrow \mathfrak{G} & & \downarrow \mathfrak{G} & \\
\cdots \longrightarrow & k^{q-1}(X_{n-1}; M) & \longrightarrow & k^q(X_n, X_{n-1}; M) & \longrightarrow & k^q(X_n; M) & \longrightarrow & k^q(X_{n-1}; M) & \longrightarrow \cdots
\end{array}$$

Also, note that $h^*(X_0; M) \simeq k^*(X_0; M)$. It follows, inductively, using five lemma, that, $\mathfrak{G}(X_n; M) : h^q(X_n; M) \simeq k^q(X_n; M)$. Since X is finite dimensional, we get $\mathfrak{G}(X; M) : h^q(X; M) \simeq k^q(X; M)$.

Now, we use Milnor's construction to prove the theorem for arbitrary relative G-CW- complexes with equivariant local system. Let $(X, A; M)$ be any object in $\mathcal{L}^2_C$. As in the proof of Theorem 3.2.6, let $L = \bigcup_{n=0}^{\infty} X_n \times [n, n+1]$ so that $(L, A \times \mathbf{R}^+)$ is a G-subcomplex of $(X \times \mathbf{R}^+, A \times \mathbf{R}^+)$. Let $p : L \longrightarrow X$ be the restriction of the projection map $X \times \mathbf{R}^+ \longrightarrow X$, and $\overline{M} = p^*M$. Then,

$$p : (L, A \times \mathbf{R}^+; \overline{M}) \longrightarrow (X, A; M)$$

is a homotopy equivalence in $\mathcal{L}^2_C$. As before, set

$$L_1 = \bigcup_{i \geq 0} X_{2i} \times [2i, 2i+1], \text{ and } L_2 = \bigcup_{i \geq 0} X_{2i+1} \times [2i+1, 2i+2].$$

Then, $L_1 \bigcap L_2 = \bigcup_{i \geq 0} X_i \times \{i\}$ and $L_1 \bigcup L_2 = L$. Note that L_1, L_2 and $L_1 \bigcap L_2$ are disjoint union of finite dimensional relative G-CW complexes and $(L ; L_1 , L_2)$ is a proper triad, so, we have a Mayer-Vietoris exact sequence for it. Now, suppose that $\mathfrak{S}$ is a natural transformation between cohomology theories h and k satisfying the conditions of the theorem. Then, $\mathfrak{S}$ induces maps between Mayer-Vietoris exact sequences and we have the following commutative diagram.

$$\begin{array}{ccccccc}
\cdots \to h^{q-1}(L_1, A_1 ; \overline{M}) \oplus h^{q-1}(L_2, A_2 ; \overline{M}) & \to & h^{q-1}(L_1 \bigcap L_2, A_1 \bigcap A_2 ; \overline{M}) & \to & h^q(L, A \times \mathbf{R}^+ ; \overline{M}) \to \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\cdots \to k^{q-1}(L_1, A_1 ; \overline{M}) \oplus k^{q-1}(L_2, A_2 ; \overline{M}) & \to & k^{q-1}(L_1 \bigcap L_2, A_1 \bigcap A_2 ; \overline{M}) & \to & k^q(L, A \times \mathbf{R}^+ ; \overline{M}) \to \cdots
\end{array}$$

Then from the first part of the proof and five lemma we get

$$\mathfrak{S} : h^q(L , A \times \mathbf{R}^+ ; \overline{M}) \simeq k^q(L , A \times \mathbf{R}^+ ; \overline{M}),$$

and hence

$$\mathfrak{S}(X , A ; M) : h^*(X , A ; M) \simeq k^*(X , A ; M),$$

as p is a homotopy equivalence. This completes the proof of the theorem. □

CHAPTER 4

OBSTRUCTION THEORY FOR G-FIBRATION

4.0. Introduction.

Having obtained a suitable equivariant cohomology with local coefficients system, we are now in a position to build up an obstruction theory for equivariant sections of G-fibrations, where G is a finite group. In this chapter, we measure obstructions for extending equivariant sections of a G-fibration as certain elements of the equivariant cohomology groups with local coefficients system and study their properties.

4.1. Lifting extension problem.

Throughout this chapter we shall assume that G is finite. In this case any G-CW-complex X becomes in a natural way a CW-complex with cellular G-action, and each X^H inherits a CW structure with n-skeleton $X^H_n = X^H \bigcap X_n$ such that equivariant cells $\sigma : \Delta_n \times G/H \longrightarrow X$ correspond bijectively to non-equivariant cells $\widetilde{\sigma} : \Delta_n \longrightarrow X^H$ under the assignment $\sigma(x , gH) = g \cdot \widetilde{\sigma}(x)$.

Let $p : E \longrightarrow B$ be a G-fibration. Then, for every subgroup H of G, $p^H : E^H \longrightarrow B^H$ is an ordinary fibration, where $p^H = p \mid E^H$. Assume that each $p^H : E^H \longrightarrow B^H$ has path connected base space and fibre. We consider the following problem.

Let (X , A) be a G-connected (that is, (X^H , A^H) is connected for every subgroup H of G) relative G-CW-complex, and $\phi : X \longrightarrow B$, $f : A \longrightarrow E$ be equivariant maps such that $p \circ f = \phi \mid A$. In this case we say that f is an equivariant partial lifting of

ϕ. Then the lifting extension problem consists of finding an equivariant map $\psi : X \longrightarrow E$ such that $p \circ \psi = \phi$ and $\psi \mid A = f$. We shall follow a stepwise extension process, which is similar to the Bredon's method [1] of extending equivariant maps. The main point of difference is that, while in Bredon's obstruction theory the coefficients for the obstruction lie in a fixed generic coefficient system (obtained by the homotopy groups of the fixed point sets of the target space), in the present case the coefficients form an equivariant local system.

Let $\sigma : \Delta_0 \times G/H \longrightarrow X$ be an equivariant 0-cell of (X , A). Choose a point $y \in (p^H)^{-1}(\phi\sigma(d_0 , eH))$ and define $f(\sigma(d_0 , eH)) = y$. Then extend f over the orbit by $f(\sigma(d_0 , gH)) = g{\cdot}y$. In this way f can be extended equivariantly over the 0-skeleton X_0 of (X , A), and we get a partial lifting $\psi_0 : X_0 \longrightarrow E$. Next suppose that $\sigma : \Delta_1 \times G/H \longrightarrow X$ is an equivariant 1-cell of (X , A). Consider the non-equivariant cell $\widetilde{\sigma} : \Delta_1 \longrightarrow X^H$ corresponding to σ, defined by $\widetilde{\sigma}(x) = \sigma(x , eH)$. Since fibre of p^H is 0-connected, we can extend ψ_0^H over the image of $\widetilde{\sigma}$, as in the non-equivariant case [20], and then extend it over the orbit as above. The partial liftings for the 1-cells of (X , A) fit together to define an equivariant partial lifting $\psi_1 : X_1 \longrightarrow E$ extending ψ_0.

Thus we have proved

4.1.1. THEOREM. *Let* $p : E \longrightarrow B$ *be a G-fibration such that for every* H, *the fibration* $p^H : E^H \longrightarrow B^H$ *has path connected base space and fibre. Let* (X , A) *be a relative G-CW-complex pair, which is G-connected, and let* $\phi : X \longrightarrow B$ *be an equivariant map. Then, any equivariant partial lifting* $f : A \longrightarrow E$ *of* ϕ *can be extended to an equivariant partial lifting* $\psi_1 : X_1 \longrightarrow E$ *of* ϕ.

Suppose, in general, that a lifting f is already given over X_n. We then consider the problem of extending f over X_{n+1}, $n \geq 1$. Assume that the fibre of $p^H : E^H \longrightarrow B^H$ is n-simple for each subgroup H of G. Then, as described in Example 1.2.3, we have an equivariant local system $\pi_n(\mathcal{F})$ on the G-space B. This induces an equivariant local system $\phi^*\pi_n(\mathcal{F})$ on X by ϕ. Let $\sigma : \Delta_{n+1} \times G/H \longrightarrow X$ be an equivariant (n + 1)-cell of (X , A), and $\tilde{\sigma} : \Delta_{n+1} \longrightarrow X^H$, $\tilde{\sigma}(x) = \sigma(x , eH)$, represent the corresponding (n + 1)-cell of (X^H , A^H). Then, as in [20], $f^H \circ \tilde{\sigma} \mid \dot{\Delta}_{n+1}$ defines a partial cross-section $k_H : \dot{\Delta}_{n+1} \longrightarrow W_H$ of the fibration $q_H : W_H \longrightarrow \Delta_{n+1}$ induced by $\phi^H \circ \tilde{\sigma}$. Since Δ_{n+1} is contractible, q_H is fibre homotopically trivial, and therefore W_H has the same homotopy type as the fibre $(q_H)^{-1}(d_0)$. Hence k_H represents a uniquely defined element $c^{n+1}(f)(\sigma)$ of

$$\pi_n((q_H)^{-1}(d_0)) = \pi_n((p^H)^{-1}(\phi \circ \sigma(d_0 , eH))) = \phi^*\pi_n(\mathcal{F})(\sigma_H).$$

Thus we have a function $c^{n+1}(f)$ defined on equivariant (n + 1)-cells

$$\sigma : (\Delta_{n+1} \times G/H , \dot{\Delta}_{n+1} \times G/H) \longrightarrow (X_{n+1} , X_n)$$

of (X , A) such that $c^{n+1}(f)(\sigma) \in \phi^*\pi_n(\mathcal{F})(\sigma_H)$.

4.1.2. PROPOSITION. *The cochain* $c^{n+1}(f)$ *belongs to*

$$C^{n+1}(X , A ; \phi^*\pi_n(\mathcal{F})) = H_G^{n+1}(X_{n+1} , X_n ; \phi^*\pi_n(\mathcal{F})).$$

Proof. Suppose $\sigma : \Delta_{n+1} \times G/H \longrightarrow X$ and $\rho : \Delta_{n+1} \times G/K \longrightarrow X$ are two equivariant (n + 1)-cells of (X , A) compatible under the fibre preserving G-map $h : \Delta_{n+1} \times G/H \longrightarrow \Delta_{n+1} \times G/K$. Then the induced morphism

$$h_* = [\overline{h} , \mu] : \sigma_H \longrightarrow \rho_K$$

gives a homomorphism

$$\phi^*\pi_n(\mathcal{F})(h_*) : \phi^*\pi_n(\mathcal{F})(\rho_K) \longrightarrow \phi^*\pi_n(\mathcal{F})(\sigma_H).$$

Recall that

$$\phi^*\pi_n(\mathfrak{F})(\rho_K) = \pi_n(\mathfrak{F})(\phi \circ \rho_K) = \pi_n((p^K)^{-1}(\phi \circ \rho_K)') = \pi_n((p^K)^{-1}(\phi \circ \rho(d_0 , eK))),$$

where $(\phi \circ \rho_K)'$ denotes the point in B^K corresponding to the G-map $\phi \circ \rho_K$. Now, since $\langle \overline{h} , \phi \circ \mu \rangle$ represents the constant loop at $(\phi \circ \sigma_H)'$, $\tau\langle \overline{h} , \phi \circ \mu \rangle = id$, and therefore

$$\phi^*\pi_n(\mathfrak{F})(h_*) = \pi_n(\tau\langle \overline{h} , \phi \circ \mu \rangle^{-1} \circ \eta_E(\overline{h})) = \pi_n(\eta_E(\overline{h})),$$

where

$$\eta_E(\overline{h}) : (p^K)^{-1}(\phi \circ \rho(d_0 , eK)) \longrightarrow (p^H)^{-1}(\phi \circ \sigma(d_0 , eH)),$$

as $\rho_K \circ \overline{h} = \sigma_H$. Now, as $\sigma = \rho \circ h$ and G is finite, we have $\eta_X(\overline{h}) \circ \widetilde{\rho} = \widetilde{\sigma}$. For,

$$\eta_X(\overline{h}) \circ \widetilde{\rho}(x) = \eta_X(\overline{h}) \circ \rho(x , eK) = \rho(x , \overline{h}(eH))$$

$$= \rho(x , \overline{h}_x(eH)) = \rho \circ h(x , eH) = \sigma(x , eH) = \widetilde{\sigma}(x),$$

where $\overline{h}_x : G/H \longrightarrow G/K$ is the G-map $\overline{h}_x(gH) = p_2 \circ h(x , gH)$ and $\overline{h} = \overline{h}_x$, as G is finite. Consequently, since f is a G-map,

$$\phi^*\pi_n(\mathfrak{F})(h_*)(c^{n+1}(f)(\rho)) = \pi_n(\eta_E(\overline{h}))([f^K \circ \widetilde{\rho} \mid \dot{\Delta}_{n+1}]) = [\eta_E(\overline{h}) \circ f^K \circ \widetilde{\rho} \mid \dot{\Delta}_{n+1}]$$

$$= [f^H \circ \eta_X(\overline{h}) \circ \widetilde{\rho} \mid \dot{\Delta}_{n+1}] = [f^H \circ \widetilde{\sigma} \mid \dot{\Delta}_{n+1}] = c^{n+1}(f)(\sigma).$$

This completes the proof of 4.1.2. □

We call the cochain $c^{n+1}(f)$ the equivariant obstruction to extending the equivariant partial lifting f.

4.1.3. PROPOSITION. *The G-map* f *can be extended to an equivariant partial lifting over* X_{n+1} *if and only if* $c^{n+1}(f) = 0$.

Proof. Suppose f has an extension over X_{n+1}. Let

$$\sigma : (\Delta_{n+1} \times G/H , \dot{\Delta}_{n+1} \times G/H) \longrightarrow (X_{n+1} , X_n)$$

be an equivariant (n + 1)-cell E^{n+1} of (X , A). Let $f' : E^{n+1} \longrightarrow E$ be an extension of $f \mid \dot{E}^{n+1}$. Then $f \circ \sigma \mid \dot{\Delta}_{n+1} \times G/H$ has the extension $f' \circ \sigma : \Delta_{n+1} \times G/H \longrightarrow E$, and therefore $f^H \circ \widetilde{\sigma} \mid \dot{\Delta}_{n+1}$ has an extension $(f')^H \circ \widetilde{\sigma} : \Delta_{n+1} \longrightarrow E^H$. Consequently, $k_H : \dot{\Delta}_{n+1} \longrightarrow W_H$ has an extension $k'_H : \Delta_{n+1} \longrightarrow W_H$. Therefore, by definition of $c^{n+1}(f)$, it follows that $c^{n+1}(f)(\sigma) = 0$.

Conversely, suppose that $c^{n+1}(f)(\sigma) = 0$ for every equivariant $(n + 1)$-cell E^{n+1} of (X , A) with characteristic map

$$\sigma : (\Delta_{n+1} \times G/H \, , \, \dot{\Delta}_{n+1} \times G/H) \longrightarrow (X_{n+1} \, , \, X_n).$$

Then k_H has an extension $k'_H : \Delta_{n+1} \longrightarrow W_H$. As described in [20], we can find an extension of $f^H \mid \widetilde{\sigma}(\dot{\Delta}_{n+1})$ to a partial lifting over $\widetilde{\sigma}(\Delta_{n+1})$. Then we can extend f equivariantly over the orbit. Patching together the extensions over each equivariant cell we get an equivariant partial lifting over X_{n+1}. This completes the proof of proposition 4.1.3. □

Recall that, since G is finite, for any G-space X the classical fundamental groupoid of each fixed point set X^H is a subcategory of ΠX. Thus any equivariant local system M on X induces an ordinary local coefficients system M_H on X^H by restricting M to the fundamental groupoid of X^H.

If (X , A) is a relative G-CW-complex and $A \subset X_0 \subset X_1 \subset \cdots \subset X$ is the filtration of X by skeleta, then, for any subgroup H of G, (X^H , A^H) is a relative CW-complex with the filtration $A^H \subset X_0^H \subset X_1^H \subset \cdots \subset X^H$ by its skeleta. For any equivariant local system M on X, we have

$$H^*(X^H , A^H ; M_H) \simeq H^*(C(X^H , A^H ; M_H)),$$

where $C(X^H , A^H ; M_H)$ is the cochain complex

$$C(X^H , A^H ; M_H) = \left\{C^p(X^H , A^H ; M_H) = H^p(X^H_p , X^H_{p-1} ; M_H) \, ; \, \delta_H\right\},$$

$\delta_H : H^p(X^H_p , X^H_{p-1} ; M_H) \longrightarrow H^{p+1}(X^H_{p+1} , X^H_p ; M_H)$ being the coboundary operator of the triple $(X^H_{p+1}, X^H_p, X^H_{p-1})$. Recall further that, $H^p(X^H_p, X^H_{p-1}; M_H)$ can be identified with the group of all functions c_H defined on p-cells of (X^H, A^H) such that $c_H(\widetilde{\sigma}) \in M_H(\widetilde{\sigma}(d_0))$ for any p-cell $\widetilde{\sigma} : \Delta_p \longrightarrow X^H$. Now given a cochain

$$c \in C^n(X, A; M)) = H^n_G(X_n, X_{n-1}; M),$$

define a cochain $c_H \in C^n(X^H, A^H; M_H) = H^p(X^H_n, X^H_{n-1}; M_H)$, for every subgroup H, as follows. Let $\widetilde{\sigma} : (\Delta_n, \dot{\Delta}_n) \longrightarrow (X^H_n, X^H_{n-1})$ be any n-cell of (X^H, A^H) and

$$\sigma : (\Delta_n \times G/H, \dot{\Delta}_n \times G/H) \longrightarrow (X_n, X_{n-1})$$

the corresponding equivariant n-cell of (X, A). Then set $c_H(\widetilde{\sigma}) = c(\sigma)$. This makes sense, because $M_H(\widetilde{\sigma}(d_0)) = M(\sigma_H)$. These non-equivariant cochains $\{c_H\}$, as H runs over subgroups of G are related as follows. Let

$$\sigma : \Delta_n \times G/H \longrightarrow X \text{ and } \tau : \Delta_n \times G/K \longrightarrow X$$

be equivariant n-cells of (X, A) compatible under a fibre preserving G-map $h : \Delta_n \times G/H \longrightarrow \Delta_n \times G/K$. Then we have a morphism

$$M(h_*) : M(\tau_K) \longrightarrow M(\sigma_H).$$

The corresponding non-equivariant cells $\widetilde{\sigma}$ and $\widetilde{\tau}$ are respectively cells of the fixed point subcomplexes X^H and X^K, and we have $\eta_X(\overline{h}) \circ \widetilde{\tau} = \widetilde{\sigma}$, as $\sigma = \tau \circ h$ and G is finite. Since σ and τ are compatible under h, we get $M(h_*)(c_K(\widetilde{\tau})) = c_H(\widetilde{\sigma})$. Moreover, if b denotes the coboundary $\delta c \in C^{n+1}(X, A; M)$ then $b_H = \delta_H c_H$.

The above observation can be used to show that the obstruction cochain is a cocycle.

4.1.4. PROPOSITION. $\delta\, c^{n+1}(f) = 0$.

Proof. Let $\tau : (\Delta_{n+2} \times G/H, \dot{\Delta}_{n+2} \times G/H) \longrightarrow (X_{n+2}, X_{n+1})$ be an equivariant (n + 2)-cell of (X, A), and $\widetilde{\tau}$ be the corresponding non-equivariant (n + 2)-

cell of (X^H, A^H). Then from the definition of the equivariant obstruction cochain it is easy to see that $c^{n+1}(f)_H$ is precisely equal to $c^{n+1}(f^H)$, which is the non-equivariant obstruction cochain to extending the partial lifting f^H of ϕ^H over X^H_{n+1}. Therefore $\delta\, c^{n+1}(f)(\tau) = (\delta\, c^{n+1}(f))_H(\widetilde{\tau}) = \delta_H\, c^{n+1}(f^H)(\widetilde{\tau}) = 0.$ as $\delta_H\, c^{n+1}(f^H) = 0.$ This shows that $c^{n+1}(f)$ is a cocycle. □

4.1.5. PROPOSITION. *If* f, f' *are equivariant partial liftings over* X_n *which are vertically G-homotopic (rel* A) *then* $c^{n+1}(f) = c^{n+1}(f')$.

Proof. Since f and f′ are vertically G-homotopic (rel A), then, for each equivariant cell $\sigma : (\Delta_{n+1} \times G/H, \dot{\Delta}_{n+1} \times G/H) \longrightarrow (X_{n+1}, X_n)$ of (X, A), the cross-sections of the fibration $q_H : W_H \longrightarrow \Delta_{n+1}$ defined respectively by $f^H \circ \widetilde{\sigma} \mid \dot{\Delta}_{n+1}$ and $(f')^H \circ \widetilde{\sigma} \mid \dot{\Delta}_{n+1}$ are vertically homotopic. In particular they are homotopic, and, therefore the elements assigned to the cell σ by the cochains $c^{n+1}(f)$ and $c^{n+1}(f')$ are equal. Since this is true for every equivariant cell σ, we get $c^{n+1}(f) = c^{n+1}(f')$. □

4.1.6. PROPOSITION. *If* $\psi : (X', A') \longrightarrow (X, A)$ *is an equivariant cellular map and* $f' = f \circ \psi \mid X'_n : X'_n \longrightarrow E$, *then* $c^{n+1}(f') = \psi^{\#}\, c^{n+1}(f)$.

Proof. Let $\sigma : \Delta_{n+1} \times G/H \longrightarrow X'_{n+1}$ be an equivariant $(n+1)$-cell of (X', A'), and $\widetilde{\sigma} : \Delta_{n+1} \longrightarrow X'^H_{n+1}$ be the corresponding non-equivariant cell. Now $c^{n+1}(f')(\sigma)$ is by definition equal to $c^{n+1}(f'^H)(\widetilde{\sigma})$, where $c^{n+1}(f'^H)$ is the non-equivariant obstruction to extending f'^H over X'^H_{n+1}. Therefore, by the corresponding result for the non-equivariant case, we get

$$\begin{aligned} c^{n+1}(f')(\sigma) &= c^{n+1}(f'^H)(\widetilde{\sigma}) = (\psi^H)^{\#}\, c^{n+1}(f^H)(\widetilde{\sigma}) \\ &= c^{n+1}(f^H)(\psi^H \circ \widetilde{\sigma}) = c^{n+1}(f)(\psi \circ \sigma) = \psi^{\#}\, c^{n+1}(f)(\sigma). \ \square \end{aligned}$$

Let $f_0, f_1 : X_n \longrightarrow E$ be equivariant partial liftings of ϕ, and let

$\lambda : X_{n-1} \times I \longrightarrow E$ be a vertical G-homotopy (rel A) between $f_0 \mid X_{n-1}$ and $f_1 \mid X_{n-1}$. These maps fit together to define an equivariant partial lifting

$$F : X_n \times \dot{I} \cup X_{n-1} \times I \longrightarrow E$$

of $\phi \circ p_1$, where $p_1 : X \times I \longrightarrow X$ is the projection onto the first factor. Then we define the equivariant difference cochain of f_0, f_1 with respect to λ to be the cochain

$$d^n(f_0, f_1; \lambda) \in C^n(X, A; \phi^* \pi_n(\mathcal{F})) = H^n_G(X_n, X_{n-1}; \phi^* \pi_n(\mathcal{F}))$$

such that for every equivariant n-cell σ of (X, A)

$$d^n(f_0, f_1; \lambda)(\sigma) = (-1)^n c^{n+1}(F)(\sigma \times i),$$

where $\sigma \times i$ denotes the $(n+1)$-cell of $X \times I$ corresponding to the n-cell σ. If λ is stationary so that $f_0 \mid X_{n-1} = f_1 \mid X_{n-1}$, then we write $d^n(f_0, f_1)$ instead of $d^n(f_0, f_1; \lambda)$. We shall also write $d^n(f_0, f_1; \lambda)$ simply as $d^n(F)$, when there is no chance of confusion.

It follows from the Proposition 4.1.3

4.1.7. PROPOSITION. *The G-map* $\lambda : X_{n-1} \times I \longrightarrow E$ *can be extended to a vertical G-homotopy* $\bar{\lambda} : X_n \times I \longrightarrow E$ *between* f_0 *and* f_1 *if and only if* $d^n(F) = 0$.

4.1.8. PROPOSITION. *The coboundary of* $d^n(F)$ *is given by*

$$\delta\, d^n(F) = c^{n+1}(f_1) - c^{n+1}(f_0).$$

Proof. Let $\sigma : (\Delta_{n+1} \times G/H, \dot{\Delta}_{n+1} \times G/H) \longrightarrow (X_{n+1}, X_n)$ be an equivariant $(n+1)$-cell of (X, A), and $\widetilde{\sigma}$ be the corresponding non-equivariant $(n+1)$-cell of (X^H, A^H). Then, as in 4.1.4

$\delta\, d^n(F)(\sigma) = (\delta\, d^n(F))_H(\widetilde{\sigma}) = \delta_H\, d^n(F)_H(\widetilde{\sigma}) = c^{n+1}(f_1^H)(\widetilde{\sigma}) - c^{n+1}(f_0^H)(\widetilde{\sigma})$

$= c^{n+1}(f_1)(\sigma) - c^{n+1}(f_0)(\sigma)$, by the property of the non-equivariant difference cochain [20]. This proves the result. □

4.1.9. PROPOSITION. *If* $f_0 : X_n \longrightarrow E$ *is an equivariant partial lifting of* ϕ, $\lambda : X_{n-1} \times I \longrightarrow E$ *a vertical G-homotopy of* $f_0 \mid X_{n-1}$ *to a G-map* $\alpha : X_{n-1} \longrightarrow E$, *and* $d \in C^n(X , A ; \phi^* \pi_n(\mathcal{F}))$, *then* α *can be extended to an equivariant partial lifting* $f_1 : X_n \longrightarrow E$ *of* ϕ *such that* $d^n(f^0 , f^1 ; \lambda) = d$.

Proof. Let $\sigma : (\Delta_n \times G/H , \dot{\Delta}_n \times G/H) \longrightarrow (X_n , X_{n-1})$ be an equivariant n-cell of (X , A), and $\tilde{\sigma}$ be the corresponding non-equivariant n-cell of (X^H , A^H). Now $f_0^H : X_n^H \longrightarrow E^H$ is a partial lifting of ϕ^H and $\lambda^H : X_{n-1}^H \times I \longrightarrow E^H$ is a vertical homotopy of $f_0^H \mid X_{n-1}^H$ to a map $\alpha^H : X_{n-1}^H \longrightarrow E^H$. Let

$$d_H \in C^n(X^H , A^H ; \phi^* \pi_n(\mathcal{F})_H) = H^n(X_n^H , X_{n-1}^H ; \phi^* \pi_n(\mathcal{F})_H)$$

be the cochain, defined as before, by $d_H(\tilde{\tau}) = d(\tau)$ for any n-cell $\tilde{\tau}$ of (X^H , A^H), τ being the corresponding equivariant n-cell of (X , A). Then by the corresponding result for non-equivariant case [20], we can extend α^H to a partial lifting $f_1(H) : X_n^H \longrightarrow E^H$ of ϕ^H such that $d_H^n(f_0^H , f_1(H) ; \lambda^H) = d_H$, where $d_H^n(f_0^H , f_1(H) ; \lambda^H)$ is the non-equivariant difference cochain. Now define f_1 on the image of $\tilde{\sigma}$ by $f_1(H)$, and then define over the orbit equivariantly. Define f_1 for every equivariant n-cell in this way. Then, for any equivariant n-cell $\tau : \Delta_n \times G/H \longrightarrow X_n$,

$$d^n(f_0 , f_1 ; \lambda)(\tau) = d^n(f_0 , f_1 ; \lambda)_H(\tilde{\tau}) = d_H^n(f_0^H , f_1(H) ; \lambda^H)(\tilde{\tau}) = d_H(\tilde{\tau}) = d(\tau).$$

Thus $d^n(f_0 , f_1 ; \lambda) = d$. □

We can prove a result similar to 4.1.6 in this situation also.

4.1.10. PROPOSITION. *Let* $\psi : (X' , A') \longrightarrow (X , A)$ *be a cellular G-map and let* $F : X_n \times \dot{I} \cup X_{n-1} \times I \longrightarrow E$ *be an equivariant partial lifting of* $\phi \circ p_1$. *Let*

$$F' : X'_n \times \dot{I} \cup X'_{n-1} \times I \longrightarrow E$$

be the composition $F' = F \circ (\psi \times id) \mid (X'_n \times \dot{I} \cup X'_{n-1} \times I)$. *Then,* $d^n(F') = \psi^{\#} d^n(F)$.

4.1.11. PROPOSITION. *Let* $f_i : X_n \longrightarrow E$ *be an equivariant partial lifting of* ϕ, $i = 0, 1, 2$, *and let* $\lambda_{01} : X_{n-1} \times I \longrightarrow E$, *and* $\lambda_{12} : X_{n-1} \times I \longrightarrow E$ *be vertical G-homotopies between the restrictions to* X_{n-1} *of* f_0 *and* f_1, *and of* f_1 *and* f_2, *respectively. Let* $\lambda_{02} : X_{n-1} \times I \longrightarrow E$ *be the G-homotopy of* f_0 *and* f_2 *defined by* λ_{01} *and* λ_{12}. *Then,*

$$d^n(f_0, f_2 ; \lambda_{02}) = d^n(f_0, f_1 ; \lambda_{01}) + d^n(f_1, f_2 ; \lambda_{12}).$$

Proof. Let $\tau : (\Delta_n \times G/H , \dot{\Delta}_n \times G/H) \longrightarrow (X_n , X_{n-1})$ be an equivariant n-cell of (X , A), and $\widetilde{\tau}$ be the corresponding non-equivariant n-cell of (X^H , A^H). Then, by the properties of non-equivariant difference cochain, we have

$$\begin{aligned} & d^n(f_0, f_1 ; \lambda_{01})(\tau) + d^n(f_1, f_2 ; \lambda_{12})(\tau) \\ &= d^n_H(f^H_0, f^H_1 ; \lambda^H_{01})(\widetilde{\tau}) + d^n_H(f^H_1, f^H_2 ; \lambda^H_{12})(\widetilde{\tau}) \\ &= d^n_H(f^H_0, f^H_2 ; \lambda^H_{02})(\widetilde{\tau}) = d^n(f_0, f_2 ; \lambda_{02})(\tau). \end{aligned}$$

This proves the result. □

4.1.12. PROPOSITION. *The cochain* $c^{n+1}(f)$ *is a coboundary if and only if the G-map* $f \mid X_{n-1}$ *can be extended to an equivariant partial lifting* $f' : X_{n+1} \longrightarrow E$ *of* ϕ.

Proof. Let $f' : X_{n+1} \longrightarrow E$ be an extension of $f \mid X_{n-1}$, and $\theta = f' \mid X_n$. Then, $\delta\, d^n(f, \theta) = c^{n+1}(\theta) - c^{n+1}(f)$; but $c^{n+1}(\theta) = 0$, since θ has an extension f', and so $c^{n+1}(f) = \delta(-\, d^n(f, \theta))$. Conversely, let $c^{n+1}(f) = \delta\, d$ with $d \in C^n(X , A ; \phi^* \pi_n(\mathcal{F}))$. Then, by Proposition 4.1.9, there exists a G-map $\theta : X_n \longrightarrow E$ which is an equivariant partial lifting of ϕ, and an extension of $f \mid X_{n-1}$ such that $d^n(f, \theta) = -\, d$. Then, $c^{n+1}(\theta) = c^{n+1}(f) + \delta\, d^n(f, \theta) = 0$. Therefore, θ has an extension, and hence $f \mid X_{n-1}$ has an extension to an equivariant partial lifting $f' : X_{n+1} \longrightarrow E$.

4.2. Primary obstruction.

Let $p : E \longrightarrow B$ be a G-fibration such that, for every subgroup H of G, $p^H : E^H \longrightarrow B^H$ has 0-connected base space and $(n-1)$-connected fibre (here $n \geq 1$; but if $n = 1$, then we assume that the fibre is 1-simple). Let (X, A) be a G-connected relative G-CW complex. Let $\phi : X \longrightarrow B$ be a G-map, and $f : A \longrightarrow E$ be an equivariant partial lifting of ϕ, i.e. $p \circ f = \phi \mid A$. Then we have

4.2.1. THEOREM. *The G-map f can be extended to an equivariant partial lifting $\psi_n : X_n \longrightarrow E$ of ϕ. If ψ_n^0 and ψ_n^1 are two such liftings, then $\psi_n^0 \mid X_{n-1}$ and $\psi_n^1 \mid X_{n-1}$ are vertically G-homotopic (rel A), and $c^{n+1}(\psi_n^0) \sim c^{n+1}(\psi_n^1)$.*

Proof. We have seen in Theorem 4.1.1 that f can be extended over X_1. Suppose $\psi_r : X_r \longrightarrow E$ is an extension of f to an equivariant lifting of ϕ $(1 \leq r \leq n-1)$. Then $c^{r+1}(\psi_r) \in C^{r+1}(X, A; \phi^* \pi_r(\mathcal{F}))$, and, since fibre is $(n-1)$-connected, we have $c^{r+1}(\psi_r) = 0$. Hence ψ_r can be extended over X_{r+1}. We proceed inductively to prove the first part of the theorem.

Let $\psi_n^0, \psi_n^1 : X_n \longrightarrow E$ be extensions of f. Then we can apply the result already proved, to the relative G-CW-complex $(X_n \times I, A \times I \bigcup X_n \times \dot{I})$, and the equivariant lifting F' defined over $A \times I \bigcup X_n \times \dot{I}$ by ψ_n^0, ψ_n^1 and the stationary homotopy $\psi_n^0 \mid A = f$, to deduce the existance of an equivariant partial lifting $F : X_{n-1} \times I \bigcup X_n \times \dot{I} \longrightarrow E$ extending F'. But this extension just defines a vertical G-homotopy (rel A) between $\psi_n^0 \mid X_{n-1}$ and $\psi_n^1 \mid X_{n-1}$.

The second part follows from the fact that

$$\delta\, d^n(\psi_n^0, \psi_n^1 ; F) = c^{n+1}(\psi_n^0) - c^{n+1}(\psi_n^1).$$

This completes the proof. □

By Theorem 4.2.1 the obstruction cocycle $c^{n+1}(\psi_n)$, for all possible equivariant partial liftings ψ_n extending f, lie in a single cohomology class $\gamma^{n+1} = \gamma^{n+1}(f) \in H_G^{n+1}(X, A; \phi^*\pi_n(\mathcal{F}))$. The class $\gamma^{n+1}(f)$ is called the equivariant primary obstruction to extending f. If A is empty then $\gamma^{n+1} \in H_G^{n+1}(X; \phi^*\pi_n(\mathcal{F}))$ is called the equivariant primary obstruction to lifting ϕ.

As a consequence of the results proved above we have

4.2.2. THEOREM. (i) *The G-map f can be extended to an equivariant partial lifting* $\psi_{n+1} : X_{n+1} \longrightarrow E$ *of* ϕ *if and only if* $\gamma^{n+1}(f) = 0$.

(ii) *If* $g : (X', A') \longrightarrow (X, A)$ *is an equivariant cellular map and* $f : A \longrightarrow E$ *is an equivariant partial lifting of* $\phi : X \longrightarrow B$, *then* $f \circ g \mid A' : A' \longrightarrow E$ *is an equivariant partial lifting of* $\phi \circ g$, *and*

$$\gamma^{n+1}(f \circ g \mid A') = g^* \gamma^{n+1}(f) \in H_G^{n+1}(X', A'; g^*\phi^*\pi_n(\mathcal{F})).$$

4.2.3. THEOREM (Extension). *Suppose that, for every subgroup* H, $p^H : E^H \longrightarrow B^H$ *has* q*-simple fibre, and*

$$H_G^{q+1}(X, A; \phi^*\pi_q(\mathcal{F})) = 0, \text{ whenever } n+1 \le q < \dim(X, A).$$

Then f *can be extended to an equivariant lifting* $\psi : X \longrightarrow E$ *of* ϕ *if and only if* $\gamma^{n+1}(f) = 0$.

Proof. If f is extendible, then $\gamma^{n+1}(f) = 0$. Conversely, suppose that $\gamma^{n+1}(f) = 0$. Let $\psi_n : X_n \longrightarrow E$ be an extension of f. Then $c^{n+1}(\psi_n)$ is a coboundary. Hence, by Proposition 4.1.11, $\psi_n \mid X_{n-1}$ has an extension $\psi_{n+1} : X_{n+1} \longrightarrow E$. Then we can define inductively a sequence of G-maps $\psi_q : X_q \longrightarrow E$ such that $\psi_{q+1} \mid X_{q-1} = \psi_q \mid X_{q-1}$ for all $q \ge n+1$. For suppose $\psi_q : X_q \longrightarrow E$ is an extension of f. Then, since $H_G^{q+1}(X, A; \phi^*\pi_q(\mathcal{F})) = 0$, $c^{q+1}(\psi_q)$ is a coboundary, and, therefore $\psi_q \mid X_{q-1}$ has an extension $\psi_{q+1} : X_{q+1} \longrightarrow E$. A G-map $\psi : X \longrightarrow E$ is then defined by $\psi \mid X_q = \psi_{q+1} \mid X_q$ for all $q \geq n+1$. This completes the proof. □

Let $\psi_0 , \psi_1 : X \longrightarrow E$ be equivariant liftings of ϕ which agree on A. Then, there is a vertical G-homotopy $\lambda : X_{n-1} \times I \longrightarrow E$ (rel A) between $\psi_0 \mid X_{n-1}$ and $\psi_1 \mid X_{n-1}$. Let $\psi_0' = \psi_0 \mid X_n$ and $\psi_1' = \psi_1 \mid X_n$. Since ψ_0' and ψ_1' have extensions over X, their obstructions vanish. Therefore, by Proposition 4.1.9, their difference cochain $d^n = d^n(\psi_0', \psi_1'; \lambda)$ is a cocycle. In fact, d^n corresponds, under the isomorphism

$$C^n(X , A ; \phi^*\pi_n(\mathcal{F})) \longrightarrow C^{n+1}(X \times I , X \times \dot{I} \cup A \times I ; p_1^*\phi^*\pi_n(\mathcal{F})),$$

to the obstruction to extending the equivariant partial lifting $F : X_n \times \dot{I} \cup X_{n-1} \times I \longrightarrow E$ of $\phi \circ p_1$, defined by the maps ψ_0', ψ_1' and λ. It follows that, the cohomology class $\delta^n(\psi_0 , \psi_1)$ of d^n depends only on ψ_0 and ψ_1 and not on the G-homotopy λ. The class $\delta^n(\psi_0 , \psi_1) \in H^n_G(X , A ; \phi^*\pi_n(\mathcal{F}))$ is called the equivariant primary difference of the liftings ψ_0 and ψ_1.

The following properties of the equivariant primary difference $\delta^n(\psi_0 , \psi_1)$ can be deduced from our previous results.

4.2.4. PROPOSITION. (i) *The* G-*maps* $\psi_0 \mid X_n , \psi_1 \mid X_n$ *are vertically* G-*homotopic (rel* A*) if and only if* $\delta^n(\psi_0 , \psi_1) = 0$.

(ii) *If* ψ_0 , ψ_1 *and* ψ_2 *are equivariant liftings of* ϕ *agreeing on* A, *then*

$$\delta^n(\psi_0 , \psi_2) = \delta^n(\psi_0 , \psi_1) + \delta^n(\psi_1 , \psi_2).$$

(iii) *If* $g : (X' , A') \longrightarrow (X , A)$ *is a cellular* G-*map, then*

$$\delta^n(\psi_0 \circ g , \psi_1 \circ g) = g^* \delta^n(\psi_0 , \psi_1) \in H^n_G(X' , A' ; g^* \phi^*\pi_n(\mathcal{F})).$$

4.2.5. THEOREM (Homotopy). *Suppose that, for every subgroup* H, $p^H : E^H \longrightarrow B^H$ *has* q-*simple fibre and*

$$H^q_G(X , A ; \phi^* \pi_q(\mathfrak{F})) = 0, \text{ whenever } n+1 \le q < 1+ \dim (X , A).$$

Then two equivariant liftings $\psi_0 , \psi_1 : X \longrightarrow E$ *of* $\phi : X \longrightarrow B$ *which agree on* A, *are vertically* G-*homotopic* (*rel* A) *if and only if* $\delta^n(\psi_0 , \psi_1) = 0$.

Proof. Define $F : X \times \dot{I} \bigcup A \times I \longrightarrow E$ by

$$F(x , t) = \psi_0(x) = \psi_1(x), \quad x \in A$$

$$\left.\begin{aligned} F(x , 0) &= \psi_0(x) \\ F(x , 1) &= \psi_1(x) \end{aligned}\right\} \; x \in X.$$

Then we can extend F to an equivariant partial lifting F_1 of $\phi \circ p_1$ over $(X \times I)_n$ and

$$\gamma^{n+1}(F) \in H^{n+1}_G(X \times I , A \times I ; {p_1}^* \phi^* \pi_n(\mathfrak{F}))$$

corresponds to $\delta^n(\psi_0 , \psi_1) \in H^n_G(X , A ; \phi^* \pi_n(\mathfrak{F}))$ under the isomorphism

$$H^n_G(X , A ; \phi^* \pi_n(\mathfrak{F})) \simeq H^{n+1}_G(X \times I , A \times I ; {p_1}^* \phi^* \pi_n(\mathfrak{F}))$$

induced by the obvious isomorphism on the cochain level. Then the theorem follows from Theorem 4.2.3 by applying it to the G-map F. □

4.2.6. THEOREM (Classification). *Suppose that*

(i) *for every subgroup* H *the fibre of* $p^H : E^H \longrightarrow B^H$ *is* q-*simple for* $n+1 \le q < 1+ \dim (X , A)$,

(ii) $H^q_G(X , A ; \phi^* \pi_q(\mathfrak{F})) = 0$, *whenever* $n+1 \le q < 1+ \dim (X , A)$,

(iii) $H^{q+1}_G(X , A ; \phi^* \pi_q(\mathfrak{F})) = 0$, *whenever* $n+1 \le q < \dim (X , A)$.

Let $\psi_0 : X \longrightarrow E$ *be an equivariant lifting of* $\phi : X \longrightarrow B$. *Then the correspondence* $\psi \longrightarrow \delta^n(\psi_0 , \psi)$ *sets up a bijection between the set of vertical* G-*homotopy classes* (*rel* A) *of equivariant liftings of* ϕ, *which agree with* ψ_0 *on* A, *and the group* $H^n_G(X , A ; \phi^* \pi_n(\mathfrak{F}))$.

Proof. For a lifting ψ of ϕ such that $\psi \mid A = \psi_0 \mid A$, let $\delta(\psi)$ denote $\delta^n(\psi_0, \psi)$. By Theorem 4.2.5, if ψ_1 and ψ_2 are liftings of ϕ extending $\psi_0 \mid A$, then ψ_1 is vertically G-homotopic to ψ_2 (rel A) if and only if $0 = \delta^n(\psi_1, \psi_2) = \delta^n(\psi_0, \psi_2) - \delta^n(\psi_0, \psi_1) = \delta(\psi_2) - \delta(\psi_1)$. Therefore δ is a one-one map.

Let $\xi \in H^n_G(X, A; \phi^*\pi_n(\mathcal{F}))$ and $d \in C^n(X, A; \phi^*\pi_n(\mathcal{F}))$ be a representative cocycle. There is an equivariant partial lifting $\psi : X_n \longrightarrow E$ such that

$$\psi \mid X_{n-1} = \psi_0 \mid X_{n-1} \text{ and } d^n(\psi_0 \mid X_n, \psi) = d.$$

Now, $0 = \delta\, d = \delta\, d^n(\psi_0 \mid X_n, \psi) = c^{n+1}(\psi) - c^{n+1}(\psi_0 \mid X_n) = c^{n+1}(\psi)$, as $\psi_0 \mid X_n$ has the extension ψ_0. Therefore ψ has an extension over X_{n+1}. Now if $q \geq n + 1$, then

$$H^{q+1}_G(X, X_n; \phi^*\pi_q(\mathcal{F})) = H^{q+1}_G(X, A; \phi^*\pi_q(\mathcal{F})) = 0.$$

By Theorem 4.2.3 ψ has an extension $\psi_1 : X \longrightarrow E$ and clearly $\delta^n(\psi_0, \psi_1) =$ the cohomology class of d. Thus $\xi = \delta(\psi_1)$. Hence every class can be realized. This completes the proof of the classification theorem. □

CHAPTER 5

ENUMERATION OF REGULAR HOMOTOPY CLASSES OF IMMERSIONS

5.0. Introduction.

In this final chapter we consider a problem of entirely different nature. Here we consider three enumeration problems of regular homotopy classes of immersions. The results are obtained by revitalizing an old technique, the primary classification theorem for sections of fibrations, in the equivariant setting. These involve application of the classification Theorem 4.2.6, and computations of equivariant cohomology of certain spaces with free group action and with simple equivariant local system.

5.1. Real projective space P^n.

We consider immersions of the real projective space P^n in $\mathbf{R}^{2n}$ $(n > 1)$. Let $V_n(\mathbf{R}^{2n})$ be the Stiefel manifold of n-frames in $\mathbf{R}^{2n}$, and, for a manifold X of dimension n, let $E(X) \longrightarrow X$ be the bundle associated to the tangent bundle of X with fibre $V_n(\mathbf{R}^{2n})$. Let $[P^n \subseteq \mathbf{R}^{2n}]$ denote the set of regular homotopy classes of immersions of P^n into $\mathbf{R}^{2n}$. Then, according to Hirsch [8], $[P^n \subseteq \mathbf{R}^{2n}]$ corresponds bijectively with the set of vertical homotopy classes of sections of $E(P^n) \longrightarrow P^n$.

Note that P^n is the orbit space of a free action of $\mathbf{Z}_2$ on S^n. Then the bundle $E(S^n) \longrightarrow S^n$ becomes a $\mathbf{Z}_2$-fibration, where the action of $\mathbf{Z}_2$ on $E(S^n)$ is given by

$$(-1)(x ; v_1 , \cdots , v_n) = (-x ; -v_1 , \cdots , -v_n),$$

$\left\{v_1 , v_2 , \cdots , v_n\right\} \in V_n(\mathbf{R}^{2n})$, and that on S^n is the antipodal action. Note that with this action, S^n is a $\mathbf{Z}_2$-complex with one equivariant cell in each dimension k, $0 \leq k \leq n$.

The bundle $E(P^n) \longrightarrow P^n$ is obtained from $E(S^n) \longrightarrow S^n$ by passing onto the quotient. Therefore, the vertical homotopy classes of sections of $E(P^n) \longrightarrow P^n$ are in a 1-1 correspondence with the vertical $\mathbb{Z}_2$-homotopy classes of $\mathbb{Z}_2$-equivariant sections of $E(S^n) \longrightarrow S^n$.

Since $V_n(\mathbb{R}^{2n})$ is $(n-1)$-connected, fibre of $E(S^n) \longrightarrow S^n$ is q-simple for all q and therfore we have an equivariant local system $\pi_n(\mathcal{F})$ on S^n induced by $E(S^n) \longrightarrow S^n$.

Let us now calculate the equivariant cohomology $H^n_{\mathbb{Z}_2}(S^n ; \pi_n(\mathcal{F}))$. First note that, as action of $\mathbb{Z}_2$ on S^n is free, the objects of the fundamental groupoid $\Pi(S^n , \mathbb{Z}_2)$ can be identified with points of S^n and, for any two objects x and y, morphisms from x to y are given by $(id , [\omega])$ and $(-id , [\omega'])$, where id in the first component is actually $id_{\mathbb{Z}_2/\{1\}}$. Note that $(id , [\omega])$ corresponds to the homotopy class of paths in S^n from x to y, whereas $(-id , [\omega'])$ corresponds to the homotopy class of paths from x to $-y$. Therefore the local coefficients system $\pi_n(\mathcal{F})$ is simple and assigns to each point the group $\pi_n(V_n(\mathbb{R}^{2n}))$. Let M denote a generic coefficient system determined by $\pi_n(\mathcal{F})$.

Case 1. When n is even.

In this case $\pi_n(V_n(\mathbb{R}^{2n})) = \mathbb{Z}$ and $M(\mathbb{Z}_2/\{1\}) = \mathbb{Z}$ has a natural $\mathbb{Z}_2$-module structure given by:

$$id = M(id_{\mathbb{Z}_2/\{1\}}) : \mathbb{Z} \longrightarrow \mathbb{Z} \text{ and } -id = M(-id_{\mathbb{Z}_2/\{1\}}) : \mathbb{Z} \longrightarrow \mathbb{Z}.$$

Thus in this case $H^n_{\mathbb{Z}_2}(S^n ; \pi_n(\mathcal{F}))$ is isomorphic to the classical equivariant cohomology of S^n with coefficients in the $\mathbb{Z}_2$-module $\mathbb{Z}$. Let us denote the generators of the cellular chain group $\mathcal{C}_k(S^n)$ corresponding to the two k-cells by e_k^+ and e_k^-, and let e_k^{+*} and

e_k^{-*} be the generators of the cochain group $\mathcal{C}^k(S^n ; \mathbf{Z})$ which are dual to e_k^+ and e_k^- respectively. Then the coboundary $\delta^k : \mathcal{C}^k(S^n ; \mathbf{Z}) \longrightarrow \mathcal{C}^{k+1}(S^n ; \mathbf{Z})$ is given by $\delta^k(e_k^{\pm *}) = e_{k+1}^{\pm *} + (-1)^{k+1} e_{k+1}^{\mp *}$. Let $E^*(S^n ; \mathbf{Z})$ denote the subcomplex of $\mathcal{C}^*(S^n ; \mathbf{Z})$ consisting of cochains which are equivariant with respect to $\mathbf{Z}_2$-action. Then, a cochain c of $\mathcal{C}^k(S^n ; \mathbf{Z})$ belongs to $E^k(S^n ; \mathbf{Z})$ if and only if $c((-1)\, e_k^{\pm}) = (-1)\, c(e_k^{\pm})$, or $c(e_k^+) = (-1)\, c(e_k^-)$. Thus, $E^k(S^n ; \mathbf{Z})$ is $\mathbf{Z}$ with generator $e_k^{+*} - e_k^{-*}$, and $\delta^k \mid E^k(S^n ; \mathbf{Z})$ is multiplication by 2 if k is even, and it is 0 if k is odd. It follows then, $H^n_{\mathbf{Z}_2}(S^n ; \pi_n(\mathcal{F})) = \mathbf{Z}$.

Case 2. When n is odd.

In this case $\pi_n(V_n(\mathbf{R}^{2n})) = \mathbf{Z}_2$. Thus M is the trivial $\mathbf{Z}_2$-module $\mathbf{Z}_2$. Now arguments similar to those in case 1, show that in this case

$$\mathcal{C}^k(S^n ; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2, \quad E^k(S^n ; \mathbf{Z}_2) = \mathbf{Z}_2,$$

and $\delta^k = 0$ for all k. Thus, $H^n_{\mathbf{Z}_2}(S^n ; \pi_n(\mathcal{F})) = \mathbf{Z}_2$.

Thus, all the conditions of Theorem 4.2.6 are satisfied, and we obtain

5.1.1. THEOREM. *If* n *is even, then* $[P^n \subseteq \mathbf{R}^{2n}] = \mathbf{Z}$, *and if* n *is odd,* $n > 1$, *then* $[P^n \subseteq \mathbf{R}^{2n}] = \mathbf{Z}_2$.

It may be noted that the case $n = 1$ is the theorem of Whitney [21] which says that $[S^1 \subseteq \mathbf{R}^2] = \mathbf{Z}$, and that Theorem 5.1.1 gives a mild improvement of a result of Larmore and Rigdon [11].

5.2. Lens space L_p^{2n-1}.

Here we consider the enumeration of regular homotopy classes of immersions of the lens space L_p^{2n-1} into $\mathbf{R}^{4n-2}$.

The lens space L_p^{2n-1} is the space of orbits $S^{2n-1}/\mathbf{Z}_p$ of the cyclic group $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$ (p an odd prime) acting freely on $S^{2n-1} \subset C^n$ by the action $k \cdot (z_0, \cdots, z_{n-1}) = (w^k z_0, \cdots, w^k z_{n-1})$ with $w = \exp(2\pi i/p)$. The CW-decomposition of S^{2n-1} compatible with this action is given by the cells,

$$e_r^{2k} = \left\{ z \in S^{2n-1} : z_j = 0 \text{ for } j > k, \ \arg(z_k) = 2\pi r/p \right\},$$

$$e_r^{2k+1} = \left\{ z \in S^{2n-1} : z_j = 0 \text{ for } j > k, \ 2\pi r/p < \arg(z_k) < 2\pi(r+1)/p \right\},$$

where $z = (z_0, \cdots, z_{n-1})$, $0 \le r < p$, $0 \le k < n$, and with suitable orientations of cells the boundaries are given by,

$$\partial(e_r^{2k}) = \sum_{j=0}^{p-1} e_j^{2k-1}, \quad \partial(e_r^{2k+1}) = e_r^{2k} - e_{r+1}^{2k} \quad (r \bmod p), \quad \text{(see Dold ([4]))}.$$

This provides a $\mathbf{Z}_p$-CW-complex structure on S^{2n-1} with one equivariant cell in each dimension $q = 0, 1, \cdots, 2n-1$, the action being $w \cdot e_r^q = e_{r+1}^q$.

The situation here is similiar to that considered in the first problem. Here also, the local coefficients system $\pi_{2n-1}(\mathcal{F})$ induced on S^{2n-1} by the $\mathbf{Z}_p$-fibration $E(S^{2n-1}) \longrightarrow S^{2n-1}$ is simple, where the $\mathbf{Z}_p$-action on $E(S^{2n-1})$ is induced by the $\mathbf{Z}_p$-action on S^{2n-1} and determines the trivial $\mathbf{Z}_p$-module $\mathbf{Z}_2$. Therefore we have

$$E^{2n-2}(S^{2n-1}; \mathbf{Z}_2) = E^{2n-1}(S^{2n-1}; \mathbf{Z}_2) = \mathbf{Z}_2,$$

and the coboundary $\delta^{2n-2} = 0$. Consequently, $H_{\mathbf{Z}_p}^{2n-1}(S^{2n-1}; \pi_{2n-1}(\mathcal{F})) = \mathbf{Z}_2$.

Now proceeding as before we obtain

5.2.1. THEOREM. $[L_p^{2n-1} \subseteq \mathbf{R}^{4n-2}] = \mathbf{Z}_2$.

5.3. Grassmann manifold $G_k(\mathbf{R}^{n+k})$.

In this last section we extend Theorem 5.1.1 by considering the enumeration of regular homotopy classes of immersions of the Grassmann manifold $G_k(\mathbf{R}^{n+k})$ in $\mathbf{R}^{2nk}$. Recall that the Grassmann manifold $G_k(\mathbf{R}^{n+k})$ of unoriented k-planes consists of all k-planes in $\mathbf{R}^{n+k}$ passing through the origin. Let $\widetilde{G}_k(\mathbf{R}^{n+k})$ be the Grassmann manifold of oriented k-planes in $\mathbf{R}^{n+k}$. Then $\widetilde{G}_k(\mathbf{R}^{n+k})$ is simply connected and a two fold covering of $G_k(\mathbf{R}^{n+k})$. If $X \in \widetilde{G}_k(\mathbf{R}^{n+k})$ then $-X$ denotes the same plane with opposite orientation. The mapping $A : \widetilde{G}_k(\mathbf{R}^{n+k}) \longrightarrow \widetilde{G}_k(\mathbf{R}^{n+k})$ given by $A(X) = -X$ is a homeomorphism, in fact an analytic diffeomorphism. Thus $\widetilde{G}_k(\mathbf{R}^{n+k})$ is a $\mathbf{Z}_2$-space and $G_k(\mathbf{R}^{n+k}) = \widetilde{G}_k(\mathbf{R}^{n+k})/\mathbf{Z}_2$. Pontryagin [17] has determined a cellular structure of $\widetilde{G}_k(\mathbf{R}^{n+k})$ which may be described as follows. A k-symbol σ is a monotone integer valued function on the set $\{1, 2, \cdots, k\}$ such that $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq n$, where $\sigma_i = \sigma(i)$. We shall write $\sigma_0 = 0$ and $\sigma_{k+1} = n$. To each such k-symbol $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k)$ one can associate two subsets $e(\sigma)_+$ and $e(\sigma)_-$ of $\widetilde{G}_k(\mathbf{R}^{n+k})$ such that $A(e(\sigma)_+) = e(\sigma)_-$, $A(e(\sigma)_-) = e(\sigma)_+$, and $e(\sigma)_+ \cap e(\sigma)_- = \emptyset$. The union $e(\sigma)_+ \cup e(\sigma)_-$ consists of k-planes X such that $\dim (X \cap \mathbf{R}^{\sigma_i + i}) = i$, $i = 1, 2, \cdots, k$. The subset $e(\sigma)_+$ is homeomorphic to $\mathbf{R}^{m(\sigma)}$ where $m(\sigma) = \sum_{i=1}^{k} \sigma_i$. The boundary $\overline{e(\sigma)}_+ - e(\sigma)_+ = \overline{e(\sigma)}_- - e(\sigma)_-$ is the union of cells $e(\tau)_+ \cup e(\tau)_-$ where τ runs over the symbols obtained from σ by replacing one σ_i by $\sigma_i - 1$ $1 \leq i \leq k$, provided the function thus defined is non-decreasing and non-negative. The cellular boundary is given as follows. For a k-symbol $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k)$ define

$$s(\sigma , i) = \sigma_i + i + k \text{ and } t(\sigma , i) = \sum_{j=1}^{k} w_j$$

where
$$w_j = \begin{cases} \sigma_j & \text{for } j \leq i \\ \sigma_i & \text{for } j \geq i. \end{cases} \qquad 1 \leq j \leq k.$$

Then

$$\partial e(\sigma_1 , \sigma_2 , \cdots , \sigma_k)_+ = \sum_{i=1}^{k} (-1)^{t(\sigma , i)} \Big\{ e(\sigma_1 , \sigma_2 , \cdots , \sigma_{i-1}, \sigma_i - 1, \sigma_{i+1}, \cdots, \sigma_k)_+$$
$$+ (-1)^{s(\sigma , i)} \; e(\sigma_1 , \sigma_2 , \cdots , \sigma_{i-1}, \sigma_i - 1, \sigma_{i+1}, \cdots, \sigma_k)_- \Big\} \text{ and}$$

$$\partial e(\sigma_1 , \sigma_2 , \cdots , \sigma_k)_- = \sum_{i=1}^{k} (-1)^{t(\sigma , i)} \Big\{ e(\sigma_1 , \sigma_2 , \cdots , \sigma_{i-1}, \sigma_i - 1, \sigma_{i+1}, \cdots, \sigma_k)_-$$
$$+ (-1)^{s(\sigma , i)} e(\sigma_1 , \sigma_2 , \cdots , \sigma_{i-1}, \sigma_i - 1, \sigma_{i+1}, \cdots, \sigma_k)_+ \Big\},$$ where only the meaningful symbols appear on the right hand side. Let $e(\sigma_1 , \sigma_2 , \cdots , \sigma_k)^*_+$ (resp. $e(\sigma_1 , \sigma_2 , \cdots , \sigma_k)^*_-$) denote the dual of $e(\sigma_1 , \sigma_2 , \cdots , \sigma_k)_+$ (resp. $e(\sigma_1 , \sigma_2 , \cdots , \sigma_k)_-$). Then it is easy to see that the the cellular coboundary is given by

$$\delta e(\sigma_1 , \sigma_2 , \cdots , \sigma_k)^*_+ = \sum_{i=1}^{k} (-1)^{t(\sigma , i)^*} \Big\{ e(\sigma_1 , \sigma_2 , \cdots , \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \cdots, \sigma_k)^*_+$$
$$+ (-1)^{s(\sigma , i)^*} \; e(\sigma_1 , \sigma_2 , \cdots , \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \cdots, \sigma_k)_- \Big\} \text{ and}$$

$$\delta e(\sigma_1 , \sigma_2 , \cdots , \sigma_k)^*_- = \sum_{i=1}^{k} (-1)^{t(\sigma , i)^*} \Big\{ e(\sigma_1 , \sigma_2 , \cdots , \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \cdots, \sigma_k)^*_-$$
$$+ (-1)^{s(\sigma , i)^*} e(\sigma_1 , \sigma_2 , \cdots , \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \cdots, \sigma_k)^*_+ \Big\},$$ where, as before, only the meaningful symbols appear on the right hand side, and, for a k-symbol $\sigma = (\sigma_1 , \sigma_2 , \cdots , \sigma_k)$,

$$s(\sigma, i)^* = \sigma_i + i + k + 1 \text{ and } t(\sigma, i)^* = \sum_{j=1}^{k} w_j$$

where
$$w_j = \begin{cases} \sigma_j & \text{for } j < i \\ \sigma_i + 1 & \text{for } j \geq i. \end{cases} \qquad 1 \leq j \leq k.$$

We now consider the enumeration of regular homotopy classes of immersions of $G_k(\mathbf{R}^{n+k})$ in $\mathbf{R}^{2nk}$. As before, let $E(G_k(\mathbf{R}^{n+k})) \longrightarrow G_k(\mathbf{R}^{n+k})$ be the bundle associated to the tangent bundle of $G_k(\mathbf{R}^{n+k})$ with fibre $V_{nk}(\mathbf{R}^{2nk})$. Then $E(\widetilde{G}_k(\mathbf{R}^{n+k})) \longrightarrow \widetilde{G}_k(\mathbf{R}^{n+k})$ is a $\mathbf{Z}_2$-fibration, where $\widetilde{G}_k(\mathbf{R}^{n+k})$ is equipped with a $\mathbf{Z}_2$-CW-complex structure provided by the $\mathbf{Z}_2$-equivariant CW-complex structure on $\widetilde{G}_k(\mathbf{R}^{n+k})$ as described above. Now arguments similar to the first two cases show that there is a 1-1 correspondence between $[G_k(\mathbf{R}^{n+k}) \subseteq \mathbf{R}^{2nk}]$ and $H^{nk}_{\mathbf{Z}_2}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\, \pi_{nk}(\mathcal{F}))$. It may be noted here that since $\pi_{nk}(\mathcal{F})$ is simple, it induces a $\mathbf{Z}_2$-module M, where M is the $\mathbf{Z}_2$-module $\mathbf{Z}$, in case nk is even, and M is the trivial $\mathbf{Z}_2$-module $\mathbf{Z}_2$ in case nk is odd. Let us denote the generators $e(n-1, n, \cdots, n)^*_{\pm}$ of $\mathcal{C}^{nk-1}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\, M)$ by $e^{nk-1}_{\pm}$, and the generators $e(n, n, \cdots, n)^*_{\pm}$ of $\mathcal{C}^{nk}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\, M)$ by $e^{nk}_{\pm}$. Then proceeding as in the first case we see that if nk is even then

$$E^{nk-1}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\, M) = E^{nk}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\, M) = \mathbf{Z},$$

$E^{nk-1}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\, M)$ is generated by $e^{nk-1}_{+} - e^{nk-1}_{-}$, and $E^{nk}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\, M)$ is generated by $e^{nk}_{+} - e^{nk}_{-}$. It follows from the formula of coboundary map that

$$\delta((e^{nk-1}_{+} - e^{nk-1}_{-})) = (-1)^{nk}\left\{1 + (-1)^{n+k}\right\}(e^{nk}_{+} - e^{nk}_{-}).$$

Therefore,

$$\delta : E^{nk-1}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\,M) \longrightarrow E^{nk}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\,M)$$

is the zero homomorphism.~~if n + k is odd and multiplication by 2 if n + k is even.~~

Similarly, if nk is odd then

$$E^{nk-1}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\,M) = E^{nk}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\,M) = \mathbf{Z}_2,$$

and the coboundary map

$$\delta : E^{nk-1}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\,M) \longrightarrow E^{nk}(\widetilde{G}_k(\mathbf{R}^{n+k})\,;\,M)$$

is the zero homomorphism.

As a consequence we obtain

5.3.1. THEOREM. *If* ~~n + k~~ nk *is odd then* $[G_k(\mathbf{R}^{n+k}) \subseteq \mathbf{R}^{2nk}] = \mathbf{Z}_2$, *and if* ~~n + k~~ nk *is even then* $[G_k(\mathbf{R}^{n+k}) \subseteq \mathbf{R}^{2nk}] = \mathbf{Z}_4$.

It may be noted that Theorem 5.1.1 follows from Theorem 5.3.1 when k = 1, and that Theorem 5.3.1 agrees with a special case of a more general result of Hirsch [8].

REFERENCES

[1] G.E. Bredon, *Equivariant Cohomology Theories,* Lecture Notes in Math., vol. 34, Springer-Verlag, Berlin, Heidelberg, New York, 1967.

[2] T. tom Dieck, *Transformation Groups,* Walter de Gruyter, Berlin, New York, 1987.

[3] A. Dold, *On General Cohomology,* Lectures given at the Nordic Summer School in Math., Aarhus Univ., 1968.

[4] A. Dold, *Lectures on Algebraic Topology,* Springer-Verlag, 1972.

[5] E. Dyer, *Cohomology Theories,* Benjamin, Inc., New York, Amsterdam, 1969.

[6] S. Eilenberg, and N. Steenrod, *Foundations of Algebraic Topology,* Princeton Univ. Press, 1952.

[7] A. D. Elmendorf, *Systems of fixed point sets,* Trans. Amer. Math. Soc., 277 (1983), 275-284.

[8] M. W. Hirsch, *Immersion of manifolds,* Trans. Amer. Math. Soc., 93 (1959), 242-278.

[9] S. Illman, *Equivariant singular homology and cohomology,* Mem. Amer. Math. Soc., 156 (1975).

[10] A. Kozlowski, *Equivariant bundles and cohomology,* Trans. Amer. Math. Soc., 296 (1986), 181-190.

[11] L.L. Larmore and R.D. Rigdon, *Enumerating immersions and embeddings of projective space,* Pacific J. Math., 64 (1971), 471-492.

[12] W. Lück, *Transformation groups and algebraic K-theory,* Lecture Notes in Maths., Vol. 1408, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[13] T. Matumoto, *On G-CW complexes and a theorem of J.H.C. Whitehead*, J. Fac. Sci. Univ. Tokyo, Sec. IA, Math., 18 (1971), 363-374.

[14] T. Matumoto, *Equivariant cohomology theories on G-CW complexes*, Osaka J. Math., 10 (1973), 51-68.

[15] J. P. May, *Equivariant homotopy and cohomology theory*, Contem. Math., 12 (1982).

[16] J. M. Møller, *On equivariant function spaces*, Pacific J. Math., 142 (1990), 103-119.

[17] L. Pontryagin, *Characteristic classes of differential manifolds*, Math. Sborn., 21 (1947), 233-284 [also in Amer. Math. Soc. Transl., (2), 32 (1950)].

[18] N. E. Steenrod, *Homology with local coefficients*, Ann. of Math., 44 (1943), 610-627.

[19] R. Switzer, *Algebraic Topology : Homotopy and Homology*, Springer, 1975.

[20] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Math. 61, Springer-Verlag, New York, Heidelberg, Berlin, 1978.

[21] H. Whitney, *On regular closed curves in the plane*, Compositio Math. 4 (1937), 276-284.

[22] S. Willson, *Equivariant homology theories on G-complexes*, Trans. Amer. Math. Soc., 212 (1975).