Studies in Strategic Probability

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Thesis submitted to the Indian Statistical Institute
in partial fulfilment of the requirements
for the award of the degree of Doctor of Philosophy
CALCUTTA
1998

Acknowledgements

I would like to take this opportunity to convey my highest regard, my deepest gratitude to my supervisor, Professor B.V.Rao. He, with his deep understanding, keen interest and fresh vigour, helped me constantly and guided me at every stage of this research work. Amid all his troubles, his selfless pursuit of knowledge has always been to me a constant source of inspiration. I owe him more thanks than I can ever express. I gratefully acknowledge his kind permission to include our joint works in this thesis.

At the same time, I want to thank Professor S.M.Srivastava. I am indebted to him in many ways. He was always ready with his help whenever I needed it. I remember with gratitude many interesting and helpful discussions I had with him. He taught me a good deal of Mathematics and instructed me in Latex which proved very useful at the time of typing this thesis.

I am also grateful to Professor K.P.S.B.Rao. A portion of this work was prepared while I was a research fellow under his supervision in 1983.

I want to record my thanks to all my professors whose teachings and illuminating discussions ushered me into the mysterious world of Mathematics. I am indebted to all my friends and colleagues whose constant help and support proved most valuable at various points of time.

I thank Mr. Samir Kumer Chakraborty for the pains he took in carefully typing a substantial portion of this thesis when it appeared in the form of technical reports. This made my task of typing considerably lighter.

I thank ISI for providing congenial academic atmosphere and all possible facilities necessary for carrying out research work.

Finally I would like to thank my parents and my husband who have always been enthusiastic about my work and anxiously awaited its completion. I also thank my children who allowed their mother, however grudgingly, to pursue her own studies without paying them as much attention as they deserved.

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Introduction

Finitely additive measures, according to S. Bochner, are more interesting, more difficult to handle and more important than countably additive ones. (See the foreward by Dorothy Maharam Stone in [2]). De Finetti, one of the noted advocates of finitely additive probability put forth a strong case in its favour in his work, published as early as 1930. Since then many eminent mathematicians contributed to this area, bringing into focus various aspects of the theory of finitely additive probability. It was argued by some that countable additivity is not an integral part of probability concept but is rather in the nature of a regularity hypothesis, assumed most often to make mathematics tractable. Kolmogorov, in his book 'Foundations of the Theory of Probability' put down countable additivity as an additional axiom to define probability, which, in his own words, is assumed arbitrarily, since the models which satisfy this axiom have been found expedient in researches of the most diverse sort. See [25]. As was observed by De Finetti ([10],p.119), "No one has given a real justification of countable additivity (other than just taking it as a 'natural extension' of finite additivity); indeed many authors do also take into account cases in which it does not hold, but they consider them separately, not as absurd, but nonetheless 'pathological', outside the 'normal' theory."

Although there is much to say in favour of the theory of finitely additive probability it cannot be denied that it is not developed as systematically as its brother—the theory of countably additive probability. One obvious reason , of course is that it has failed to generate the same amount of activity among the researchers, as the other one has done, the operation in this field being tedious and cumbersome with scope, as yet very limited. One major drawback of finitely additive probability is the difficulty to construct a suitable, well-behaved product measure on an appropriate σ-field, rich enough to support interesting stochastic processes. Of course, we are concerned only with discrete-parameter stochastic processes. It may be mentioned here that continuous-time stochastic processes in finitely additive setup is an area which still remains unexplored. Of course, we must mention here that Kallianpur and Karandikar [22] have successfully explored finitely additive approach to white noise filtering.

The first rigorous attempt to construct a suitable product probability in finitely additive setup was made and a nice technique was prescribed by Dubins and Savage [6] in their book 'How to Gamble If You Must'. Their object was to develop the theory of optimal gambling in the finitely additive setup. Here they introduced the finitely additive product probability on a product space good enough for their purpose. More precisely, they dealt with the field of clopen sets on infinite product spaces. This, they called strategic probability. This foundation of strategic setup was extended by Purves and Sudderth [28] in their seminal paper 'Some finitely additive Probability'. Here they obtained a natural extension of strategic probability to the Borel σ-field of product space and supplied all the details it needed in this beautiful structure for other researchers to come and embellish it with their ideas. The following few years—a period which extended almost to a decade—witnessed most enthusiastic research activity in this area. The works mentioned above paved the way to develop a substantial part of classical probability theory in this setup. Here are some of the highlights: The strong law of large numbers was treated in Purves and Sudderth [28], Chen [5]; the law of iterated logarithm in Chen [4]; the central limit theorem in Ramakrishnan [32], Karandikar [23]; markov chains and potential theory in Ramakrishnan [30], [31], [33]; random walks in Karandikar [24]; martingales in Dubins and Savage [6], Purves and Sudderth [28]; Komlos type theorems in Halevy and Bhaskara Rao [18]. More recently—as pointed out to us by J.K. Ghosh—Heath and Sudderth [19],[20], Lane and Sudderth [26] have advocated that it is beneficial to use finitely additive priors in some problems of statistical inference. In fact the prescription of [20] is simple: A Bayesian, who seeks to avoid incoherent inferences, might be advised to abandon improper countably additive priors and use only finitely additive priors (and face the consequences concerning posteriors).

In the course of our investigations we come across some interesting results which render strategic setup a character of its own, having some peculiar traits which do not exist in the countably additive framework. Some entirely new concepts emerge. As discovered by Ramakrishnan, the concept of communication among states leads in this setup to two inequivalent notions, namely—weak communication and strong communication (see [30] for details). As we shall see later the concept of recurrence also has two inequivalent analogues in this setup, namely—weak recurrence and strong recurrence. Moreover, random walks in this setup could be indeed purely nonatomic in sharp contrast to the countably additive setup. It immediately follows that the symmetric σ-field of i.i.d. random variables could be purely nonatomic. In this context it is very interesting to note that the fundamental paper of Hewitt and Savage [21] discusses finitely additive versions of their zero one law. As they say, their paper can be viewed as an abbreviation of two papers—one finitely additive version and the other countably additive version.

Briefly the organization of the thesis is as follows. Each chapter starts with an introduction. In the first chapter of this thesis we discuss recurrence and transience of random walks in the strategic framework. In the second chapter we consider Blackwell's problem of atomicity in the context of strategic random walks. The third chapter is devoted to an analogue of the Hewitt–Savage zero-one law in the strategic setup. In the fourth chapter we discuss the completeness of L_p-spaces, first over general finitely additive measure spaces and then over strategic probability spaces.

0.1 Preliminaries

The Basic Setup: To define strategic probability we start with an arbitrary non-empty set I. Let $H = I^\infty$, where I^∞ is the one-sided product of countably many copies of I. More precisely, H is the collection of all sequences $(h_1, h_2, \ldots)$ where each $h_i \in I$. We equip H with the product of discrete topologies. We will refer to the elements of H as histories. Let I^* be the set of all finite sequences of elements of I, including the empty one. Let Γ be the set of all finitely additive probabilities defined on all subsets of I. The term 'probability' for us will mean a nonnegative, finitely additive, normalised set function. We will explicitly say 'countably additive probability' in case the function is also countably additive.

Definition (0.1.1): A strategy σ is a function on I^* into Γ. Thus corresponding to a strategy σ we have a family of probabilities denoted by $\sigma(<>),\ldots,\sigma(<i_1,i_2,\ldots,i_k>),\ldots$etc. $\sigma(<>)$ is the probability corresponding to the empty sequence which is called initial distribution. For $p,\ q \in I^*$, pq will stand for the element of I^* whose terms consist of the terms of p followed by the terms of q. For $p \in I^*$ and $h \in H$, ph will stand for the element of H whose terms consist of the terms of p followed by the terms of h. If $A \subset H$, $Ap = \{h \in H : ph \in A\}$ and $pA = \{h \in H : h = ph', h' \in A\}$. A strategy σ is said to be an independent strategy if for all $p \in I^*$, $\sigma(p)$ depends only on the length of p, not on its terms. In other words, a strategy σ is independent if there exists a sequence of elements γ_0, $\gamma_1,\ldots \in \Gamma$ such that $\sigma(<>) = \gamma_0$, and in general for $p \in I^*$ of length k, $\sigma(p) = \gamma_k$. A strategy σ is said to be i.i.d if there is one $\gamma \in \Gamma$ such that for any $p \in I^*$, $\sigma(p) = \gamma$.

Definition (0.1.2): If σ is a strategy and $p \in I^*$, the conditional strategy σ given p, denoted by $\sigma[p]$ is the strategy defined by $\sigma[p](q) = \sigma(pq)$ for all $q \in I^*$.

In [6], Dubins and Savage obtain, corresponding to every strategy, a finitely additive probability measure on the field of clopen subsets of H. This probability is also denoted by σ. They define this probability using induction on the structure of clopen sets. To describe the basic property of this probability we need the following definitions:

Definition (0.1.3): A stop rule s is a function on H into $\mathbb{N}$ (the set of positive integers) such that if $s(h) = n$ and h' agrees with h through the first n coordinates, then $s(h') = n$.

If s is a stop rule and $h \in H$ then $p_s(h)$ is the finite sequence $(h_1, \ldots, h_m)$ where $m = s(h)$. In particular if $s \equiv n$ then $p_s(h)$ is denoted by $p_n(h)$.

Definition (0.1.4): A clopen set K of H is said to be determined by stop rule s if

$$Kp_s(h) = \begin{cases} H & \text{for } h \in K \\ \phi & \text{for } h \in K^c \end{cases}$$

For every clopen set K of H there exists a stop rule s such that K is determined by s. A proof of this fact can be found in [6]. Given a strategy σ, using induction on the structure of stop rules one can define $\sigma(K)$ for every clopen set K. It then follows that $\sigma(K)$ so defined satisfies

$$\sigma(K) = \int \sigma[p_s(h)](Kp_s(h))d\sigma(h).$$

In [28], Purves and Sudderth have shown that this probability defined for clopen sets induces in a canonical way a probability on a field $\mathcal{A}(\sigma)$ of subsets of H including $\mathcal{B}$, the σ-field of Borel subsets of H. On any open set U the induced probability σ is defined as

$$\sigma(U) = sup\{\sigma(K) : K \text{ clopen subset of } U\}.$$

It has been shown in [28] that the induced probability is unique subject to certain regularity conditions. σ is characterised by the following three properties:

C(i) For any Borel $A \subset H$, $\sigma(A) = \int \sigma[x](Ax)d\sigma$.

C(ii) If $O \subset H$ is open then

$$\sigma(O) = sup\{\sigma(K) : K \text{ clopen } K \subset O\}.$$

C(iii) If $A \subset H$ is Borel and $\epsilon > 0$ then there is a closed set C and open set O such that $C \subset A \subset O$ and $\sigma(O \setminus C) < \epsilon$.

Below we give some useful results regarding σ which we need in the sequel.

(P1) (Cor.4.1 p.265 in [28]) If A is Borel and s a stop rule then,

$$\sigma(A) = \int \sigma[p_s(h)](Ap_s(h))d\sigma(h)$$

(P2) (Lemma 5.2 p.266 in [28]) If $A^1, A^2, \ldots$ are Borel and $\sigma[p_n(h)](A^n p_n(h)) = 0$ for all n and h then $\sigma(\cup A^n) = 0$.

(P3) (Lemma 1 p.273 in [28]) Let $\{L_n\}_{n \geq 1}$ be a sequence of clopen subsets of H and $\{t_n\}$ a sequence of strictly increasing stop rules (i.e. $t_m(h) > t_n(h)$ for all $m > n$ and for all $h \in H$) and $\{\alpha_n\}$ a sequence of real numbers such that

(i) L_n is determined by t_n for all $n \in \mathbb{N}$,

(ii) $\sigma(L_1) \geq (\leq)\alpha_1$ and for all $h \in H$,

$$\sigma[p_{t_n}(h)](L_{n+1}p_{t_n}(h)) \geq (\leq)\alpha_{n+1} \quad n \in \mathbb{N}.$$

Then $\sigma(\cap_{n=1}^{\infty} L_n) \geq (\leq)\Pi_{n=1}^{\infty}\alpha_n$. Most often we have occasion to use only the first inequality.

(P4) (Theorem 5.2 p.269 [28]) If $A^1, A^2, \ldots$ are increasing Borel sets then

$$\sigma(\cup A^n) = \sup_s \sigma(A^s)$$

where the supremum is taken over all stop rules s and A^s denotes $\{h \in H : h \in A^{s(h)}\}$.

(P5) (Theorem 7.2 p.275 [28]) Let σ be induced by an i.i.d. strategy. Then we have the following result: If $A \subset N$ then

$$\sigma\{h : \frac{1}{n}\sum_{k=1}^{n} 1_A(h_k) \to \gamma(A)\} = 1.$$

This is a strong law of large numbers.

(P6) (Theorem 3.1 p.34 [29]) Let σ be induced by an independent strategy. Then we have the following result: If B is a tail Borel set then $\sigma(B) = 0$ or 1. Recall that a set $B \subset H$ is a tail set provided $Bp = Bq$ whenever $p, q \in Seq$ have the same length.

This is Kolmogorov $0 - 1$ Law.

(P7) (Theorem 4.1 p.35 [29]) If B is a Borel set then

$$\sigma\{h : \sigma[p_n(h)](Bp_n(h)) \to 1_B(h)\} = 1$$

This is Levy $0 - 1$ Law.

Chapter 1

Random Walks: Recurrence and Transience

1.1 Introduction

In this chapter we discuss recurrence and transience of random walks in the finitely additive strategic setup. The theory of general Markov chains in this setup has been developed by Ramakrishnan in [30] and [31]. We find that the implementation of these results to the present context of random walks needs some interesting calculations. Soon it transpired that the techniques of Karandikar [23] and [24] are more suitable. The results presented in this chapter may be regarded as a continuation of both Ramakrishnan ([30] and [31]) as well as Karandikar ([23] and [24]).

In §2 we start with various observations concerning random variables in a finitely additive setting, culminating in an elementary proof of Theorem 1.2 of Karandikar [24]. §3 cites some preliminaries which we will use in the sequel. We take up the study of random walks in

§4. In §5 we discuss the possibilities of extending some of our results to higher dimensions.

1.2 Preliminaries

A finitely additive probability space is a triplet $(H, \mathcal{C}, \mu)$ where $\mathcal{C}$ is a field of subsets of a set H and μ is a finitely additive probability on $\mathcal{C}$. For $A \subset H$ let $\mu^*(A) = \inf\{\mu(C) : C \in \mathcal{C}, C \supset A\}$, and $\mu_*(A) = 1 - \mu^*(A^c)$. We shall assume, as in [24], that $\mathcal{C}$ is complete in the sense : $A \subset H$ and $\mu^*(A) = \mu_*(A)$ implies that $A \in \mathcal{C}$. In what follows, $\mathbb{R}$ is the set of real numbers. Recall (see [8]) that if X and s_n for $n \geq 1$ are real valued functions on H, say that $s_n \xrightarrow{\mu} X$ (s_n converges to X in μ probability) in case $\mu^*(|X - s_n| > \epsilon) \longrightarrow 0$ for each $\epsilon > 0$. Let,

$$\mathcal{E} = \{X : X = \sum_{i=1}^{n} a_i I_{A_i}, A_i \in \mathcal{C},\ a_i \in \mathbb{R},\ n \geq 1\}$$

Elements of $\mathcal{E}$ are called simple functions. Let,

$$\mathcal{L} = \{X : \text{There exists a sequence } s_n \in \mathcal{E} \text{ such that } s_n \xrightarrow{\mu} X\}$$

Elements of $\mathcal{L}$ are called random variables.

Remark (1.2.0) : Note that if $X \in \mathcal{L}$ then the distribution of X is tight in the sense that for each $\epsilon > 0$ there is a number a with $\mu^*(|X| > a) < \epsilon$.

For $s \in \mathcal{E}$, recall that $\int s d\mu = \Sigma a_i \mu(A_i)$. We say that $X \in \mathcal{L}$ is μ integrable (or integrable) if there is a sequence $\{s_n\}$ in $\mathcal{E}$ such that $s_n \xrightarrow{\mu} X$ and $\int |s_n - s_m| d\mu \longrightarrow 0$ as $n, m \longrightarrow \infty$. In such a case $\int X d\mu$ is defined – unambiguously, as is easy to see – as $\lim_n \int s_n d\mu$. It is also denoted by $E_\mu(X)$.

Definition (1.2.1) : Let $X \in \mathcal{L}$. Say that a real number c is a continuity point of X in case $\mu^*(c-\frac{1}{n} < X \leq c+\frac{1}{n}) \longrightarrow 0$ as $n \longrightarrow \infty$.

This is equivalent to saying that if $a_n \uparrow c$ and each $a_n < c$, $b_n \downarrow c$ and each $b_n > c$ then $\mu^*(a_n < X \leq b_n) \longrightarrow 0$ as $n \longrightarrow \infty$. The set of continuity points of X is denoted by $C(X)$.

Remark (1.2.2) : If $c \in C(X)$ then the set $(X = c)$ is in $\mathcal{C}$ and moreover, $\mu(X = c) = 0$. However the converse need not be true. First let us recall that a finitely additive probability μ defined on the power set of a countable set H is said to be diffuse if $\mu\{h\} = 0$ for all $h \in H$. Let H be the set of nonnegative integers, $\mathcal{C}$ the powerset, μ any diffuse probability, X is the function $X(0) = 0$ and $X(n) = \frac{1}{n}$ for $n \geq 1$ then $0 \notin C(X)$ but the set $(X = 0) \in \mathcal{C}$ and $\mu(X = 0) = 0$.

Of course if $c \in C(X)$ then the sets $(X \leq c), (X \geq c)$ etc. are all in $\mathcal{C}$ as shown in the next lemma.

Lemma 1.2.3 *Let $X \in \mathcal{L}$ and $c \in C(X)$ then $(X \leq c) \in \mathcal{C}$.*

<u>Proof.</u> : Since $\mathcal{C}$ is complete, it suffices to show that $\mu^*(X \leq c) = \mu_*(X \leq c)$; equivalently we show that $\mu^*(X \leq c) + \mu^*(X > c) = 1$. Since the inequality $\mu^*(A) + \mu^*(A^c) \geq 1$ always holds, we need to show that $\mu^*(X \leq c) + \mu^*(X > c) \leq 1$. Fix $\epsilon > 0$. As $c \in C(X)$, fix an integer $N \geq 1$ such that $\mu^*(c - \frac{1}{N} < X \leq c + \frac{1}{N}) < \frac{\epsilon}{4}$. As $X \in \mathcal{L}$ fix $s_n \in \mathcal{E}$ such that $s_n \xrightarrow{\mu} X$. Fix an integer n_o so that $\mu^*(|s_{n_o} - X| > \frac{1}{2N}) < \frac{\epsilon}{4}$. Get $A \in \mathcal{C}$ such that $A \supset (|s_{n_o} - X| > \frac{1}{2N})$ and $\mu(A) < \epsilon/4$. Clearly,

$$\begin{aligned}
(X \leq c) &\subset (s_{n_o} \leq c + \tfrac{1}{2N}) \cup A \\
(X > c) &\subset (s_{n_o} > c - \tfrac{1}{2N}) \cup A, \\
(c - \tfrac{1}{2N} < s_{n_o} < c + \tfrac{1}{2N}) &\subset (c - \tfrac{1}{N} < X < c + \tfrac{1}{N}) \cup A.
\end{aligned}$$

So that

$$\begin{aligned} & \mu^*(X \le c) + \mu^*(X > c) \\ \le & \ 1 + \mu(c - \tfrac{1}{2N} < s_{n_o} < c + \tfrac{1}{2N}) + 2\mu(A) \\ \le & \ 1 + \mu^*(c - \tfrac{1}{N} < X < c + \tfrac{1}{N}) + 3\mu(A) \\ \le & \ 1 + \tfrac{\epsilon}{4} + \tfrac{3\epsilon}{4}. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the proof is complete. ∎

As in the countably additive case, we have the following

Lemma 1.2.4 *For $X \in \mathcal{L}$, the complement of $C(X)$ is at most countable.*

Proof. : For any real number c, letting $j(c) = \inf_n\{\mu^*(c - \frac{1}{n} < X \le c + \frac{1}{n})\}$ one shows that for any fixed integer k the set $A_k = \{c : j(c) > \frac{1}{k}\}$ has at most $k-1$ points. ∎

Definition (1.2.5) : For $X \in \mathcal{L}$ we define

$$\begin{aligned} \tilde{F}_X(c) &= \mu(X \le c) \quad \text{for } c \in C(X) \\ F_X(c) &= \mu^*(X \le c) \quad \text{for } c \in \mathbb{R}. \end{aligned}$$

We think of F_X as the distribution function of X. Clearly F_X extends $\tilde{F}_X$.

Remark (1.2.6) : At each point in $C(X)$, F is continuous, whereas, for points outside $C(X)$ nothing can be said in general. This statement is illustrated by the following example. But before that we need some definitions. Let $a \in \mathbb{R}$. A probability δ is said to be diffuse to the right of a at a if $\delta(a, a+\epsilon) = 1 \ \forall \epsilon > 0$. A probability δ is said to be diffuse to the left of a at a if $\delta(a-\epsilon, a) = 1 \ \forall \epsilon > 0$. Now let H be the real line $\mathbb{R}$, $\mathcal{C}$ be the power set, $\mu = \sum_{i=1}^4 \frac{1}{4}\delta_i$ where δ_1 is a diffuse probability to the right of $+1$ at $+1$; δ_2 is a diffuse probability to the left of -1 at -1; δ_3 is a diffuse probability to the right of 0 at 0; and δ_4 is a diffuse probability to the left of 0 at 0. Let $X(\omega) = \omega$. Then it is easy to see that F_X is not left continuous at -1, not right continuous at +1, neither left continuous nor right continuous at 0.

Of course, the function F_X as defined above does not, in any sense, determine the distribution of the random variable X. However, our objective is not to study these distribution functions as such, but to relate them to countably additive probabilities.

Definition (1.2.7) : For $X \in \mathcal{L}$ and $a \in \mathbb{R}$ define

$$F^*(a) = \inf_{b>a} F_X(b) = \inf_{\substack{b>a \\ b\in C(X)}} \tilde{F}_X(b)$$

It is not difficult to verify that the two infimums above are indeed equal. F^* is a right continuous distribution function in the usual sense. F^* is called the associate of F_X. If Y is a random variable with distribution function F^* we say that Y is an associate of X.

Remark (1.2.8) : If $c \in C(X)$ then $F^*(c) = F(c)$ and c is a point of continuity of F^*. However if $c \notin C(X)$ then F^* is not left continuous at c.

The association between F_X and F^* is clarified by the following Lemma.

Lemma 1.2.9 *For any bounded continuous fuction* $g : \mathbb{R} \longrightarrow \mathbb{R}$

$$E_\mu(g(X)) = \int g dF^*.$$

<u>Proof.</u> : For each positive integer $n \geq 1$ choose a number $k_n > 0$ such that $\mu^*(|X| > k_n) < \frac{1}{2^n}$. This is possible since X is tight (see remark (1.2.0)). Moreover we can assume that $k_n \in C(X)$. By the uniform continuity of g on $[-k_n, k_n]$ choose $\delta_n > 0$ such that for x, y

in $[-k_n, k_n]$ if $|x-y| < \delta_n$ then $|g(x)-g(y)| < \frac{1}{2^n}$. Choose a partition $-k_n = x_o^n < x_1^n < \ldots < x_{l_n}^n = k_n$ with $|x_i^n - x_{i+1}^n| < \delta_n$ for all i. There is no loss in assuming that each of these points is in $C(X)$.

Define

$$g_n(x) = \begin{cases} g(x_i^n) & \text{if } x \in [x_i^n, x_{i+1}^n), 0 \le i \le l_n - 1 \\ 0 & \text{if } |x| > k_n \end{cases}$$

Then it is easy to see that $g_n oX \xrightarrow{\mu} goX$. Also, $\mu(g_n(x)) = \int g_n dF^*$.

Taking limits and using the Dominated Convergence theorem (see [8]) we get the result. ■

Higher dimensions (1.2.10) : The extension to multidimensions is straightforward which will be explained briefly now. Let $X_1, X_2, \ldots, X_k$ be k random variables on $(H, \mathcal{C}, \mu)$. Say that $\underset{\sim}{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ is a continuity point of the random vector $\underset{\sim}{X} = (X_1, X_2, \ldots, X_k)$ if $\mu^*(a_i - \frac{1}{n} < X_i \le a_i + \frac{1}{n},\ i = 1, \ldots, k)$ converges to 0 as $n \longrightarrow \infty$. Obviously, if for one i, a_i is a continuity point of X_i, then $\underset{\sim}{a}$ is a continuity point of $\underset{\sim}{X}$. There are only countably many noncontinuity points for the random vector $\underset{\sim}{X}$. Define $F_{\underset{\sim}{X}}(a_1, a_2, \ldots, a_k) = \mu^*(X_1 \le a_1, \ldots, X_k \le a_k)$. If $\underset{\sim}{a}$ is a continuity point of $\underset{\sim}{X}$ then $F_{\underset{\sim}{X}}$ is indeed continuous at $\underset{\sim}{a}$.

For two points $\underset{\sim}{b} = (b_1, b_2, \ldots, b_k)$ and $\underset{\sim}{a} = (a_1, a_2, \ldots, a_k)$ of $\mathbb{R}^k$ we use the notation $\underset{\sim}{b} > \underset{\sim}{a}$ if $b_i \ge a_i\ \forall i$ and $b_i > a_i$ for at least one i.

Define F^* as follows :

$$F^*_{\underset{\sim}{X}}(\underset{\sim}{a}) = \inf_{\underset{\sim}{b} > \underset{\sim}{a}} F_{\underset{\sim}{X}}(\underset{\sim}{b}) = \inf_{\substack{\underset{\sim}{b} > \underset{\sim}{a} \\ \underset{\sim}{b} \in C(X)}} F_{\underset{\sim}{X}}(\underset{\sim}{b})$$

$F^*_{\underset{\sim}{X}}$ is a distribution function on $\mathbb{R}^k$ in the usual sense. This is called the associate of $F_{\underset{\sim}{X}}$.

Moreover, for every bounded continuous function $g : \mathbb{R}^k \longrightarrow \mathbb{R}$

$$E(g(X_1, X_2, \dots, X_k)) = \int g dF^*_{\underset{\sim}{X}} \dots (1)$$

Given $(X_1, X_2, \dots, X_k)$ there is only one right continuous distribution function $F^*_{\underset{\sim}{X}}$ satisfying (1) for every bounded continuous function g, namely the one constructed above.

If $Y_1, \dots, Y_k$ are random variables defined on some countably additive probability space having joint distribution $F^*_{\underset{\sim}{X}}$ then we refer to $(Y_1, \dots, Y_k)$ as an associate of $(X_1, \dots, X_k)$. We can now extend the notion of associate to a sequence of random variables. Let $\underset{\sim}{Y} = (Y_i \; ; i \geq 1)$ be a sequence of random variables defined on a countably additive probability space. Let $\underset{\sim}{X} = (X_i \; ; i \geq 1)$ be a sequence of random variables defined on a finitely additive probability space $(H, \mathcal{C}, \mu)$. We call $\underset{\sim}{Y}$ an associate of $\underset{\sim}{X}$ if for every k, $(Y_1, \dots, Y_k)$ is an associate of $(X_1, \dots, X_k)$. Suppose for example that $\bigcup_{n \geq 1} \sigma(X_1, \dots, X_n) = \mathcal{F}$ (say) is included in $\mathcal{C}$ and μ' is another finitely additive probability on $(H, \mathcal{C})$ agreeing with μ on $\mathcal{F}$. Then any associate of $(X_n; \; n \geq 1)$ on $(H, \mathcal{C}, \mu)$ is also an associate of $(X_n; \; n \geq 1)$ on $(H, \mathcal{C}, \mu')$. However the properties of the sequence $(X_n \; ; n \geq 1)$ under μ may be entirely different from those under μ'. It is not difficult to construct such examples, see for instance ([24] p.196). This forces one to restrict attention to regular setups. We shall work in the more popular strategic setting.

1.3 Strategic Setting

For the basic framework and the necessary properties of this setup see §0.1.

We shall now return to the concept of associates introduced at the end of §1. First note that every sequence (X_n) has at least one associate. Consider the strategic probability σ induced by an independent strategy, (i.e., $\sigma(<>) = \gamma_1, \ldots, \sigma(< i_1, i_2, \ldots, i_k >) = \gamma_{k+1}, \ldots$, etc.) and the random variables $(X_n;\ n \geq 1)$ on $(I^\infty, \mathcal{P}^\infty(I), \sigma)$. X_1, $X_2, \ldots$ are said to be independent if $X_n(h)$ depends only on the nth coordinate h_n of h. It follows from (P3) in §0.1 that if X_i are independent then

$$\sigma(X_i \in A_i,\ \forall\, i \geq 1) = \Pi_{i=1}^{\infty} \gamma_i(X_i \in A_i).$$

An independent sequence of random variables is said to be identically distributed (abbreviated as i.i.d.) if for each $n \geq 1$ and for each $x \in \cap_n C(X_n)$, we have $\gamma_n(X_n \leq x) = \gamma_1(X_1 \leq x)$. This is equivalent to saying that F_n^* does not depend on n, where F_n^* is the associate of F_{X_n}.

In what follows, we will consider coordinate random variables $(X_n;\ n \geq 1)$ on $(I^\infty, \mathcal{P}^\infty(I), \sigma)$. Obviously, $(X_n;\ n \geq 1)$ are independent (resp. i.i.d.) if σ is induced by an independent (resp. i.i.d.) strategy. If $(Y_n; n \geq 1)$ is any associate of (X_n) then it is clear that (Y_n) is a sequence of independent (resp. i.i.d.) random variables if (X_n) are independent (resp. i.i.d.). One would like to deduce results about the sequence (X_n) using the known classical results for its associate (Y_n). One such technique, developed by Karandikar is the following :

Proposition 1.3.1([24]) : *Let* $\underset{\sim}{X} = (X_n)$ *be a sequence of i.i.d. random variables defined on* $(I^\infty, P^\infty(I), \sigma)$. *Let* $\underset{\sim}{Y} = (Y_n)$ *be an* *associate of* (X_n). *Let* $A \in \mathcal{B}(\mathbb{R}^\infty)$ *be such that*

$$\begin{array}{rcl} (x_1, x_2, \ldots) & \in & A, \\ (x'_1, x'_2, \ldots) & \in & \mathbb{R}^\infty, \\ \textit{and } \sum |x_i - x'_i| & < & \infty \\ \Longrightarrow \ (x'_1, x'_2, \ldots) & \in & A \end{array} \qquad (*)$$

Then $\underset{\sim}{X} \in A$ *a.s.* σ *iff* $\underset{\sim}{Y} \in A$ *a.s.*

Here are some examples of sets satisfying the condition $(*)$.

Lemma 1.3.2 *Let*

$$A = \{(x_1, x_2, \ldots) \in \mathbb{R}^\infty : \textit{For each } k, \ (\sum_{i=k}^{n} x_i : n \geq k) \textit{ is dense in } \mathbb{R}\}$$

Then A *satisfies* $(*)$

(Note that in the definition of A we could have taken $k = 1$ i.e. we could have considered partial sums starting from x_1 and had we done that we would have got the same A. But we prefer this form of A because to verify that A satisfies $(*)$ this form proves more convenient.)
Proof. : Clearly A is a Borel set in $\mathbb{R}^\infty$. Let $\underset{\sim}{x} = (x_1, x_2, \ldots) \in A$ and $\Sigma|x_i - x'_i| < \infty$ and $k \geq 1$. Fix any $a \in \mathbb{R}$ and $\epsilon > 0$. Choose k_ϵ so that $\sum_{i=k_\epsilon}^{\infty} |x_i - x'_i| < \epsilon/4$. If $k \geq k_\epsilon$ choose n_k so that $\sum_{i=k}^{n_k} x_i \in (a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2})$. If $k < k_\epsilon$ choose n_k so that $\sum_{i=k}^{n_k} x_i \in (a - l - \frac{\epsilon}{2}, a - l + \frac{\epsilon}{2})$ where $l = \sum_{i=k}^{k_\epsilon}(x'_i - x_i)$. In either case, $\sum_{i=k}^{n_k} x'_i = \sum_{i=k}^{n_k} x_i + \sum_{i=k}^{n_k}(x'_i - x_i) \in (a - \epsilon, a + \epsilon)$. This shows that $(\sum_{i=k}^{n} x'_i : n \geq k)$ is dense in $\mathbb{R}$. That is $(x'_1, x'_2, \ldots) \in A$.
■

Lemma 1.3.3 *Let* $A = \{(x_1, x_2, \ldots) \in \mathbb{R}^\infty :$ *For each* k, $|\sum_{i=k}^{n} x_i| \longrightarrow \infty$ *as* $n \longrightarrow \infty.\}$
Then A *satisfies* $(*)$.

Proof. : Clearly A is a Borel set in $\mathbb{R}^\infty$. Let $\underset{\sim}{x} = (x_1, x_2, \ldots) \in A$. Assume that $\sum |x_i - x_i'| = c < \infty$. Fix any $k \geq 1$.

$$\sum_{i=k}^{n} x_i' = \sum_{i=k}^{n} x_i + \sum_{i=k}^{n} (x_i' - x_i).$$

Since the second term on the right side is bounded between $-c$ and $+c$, and the first term in modulus tends to ∞ as $n \longrightarrow \infty$, we conclude that $|\sum_k^n x_i'| \longrightarrow \infty$ as $n \longrightarrow \infty$. That is, $(x_1', x_2', \ldots) \in A$. ■

Lemma 1.3.4 *Fix* $l \neq 0$. *Let*

$$A = \{(x_1, x_2, \ldots) \in \mathbb{R}^\infty \; : \; (\forall \epsilon > 0)(\exists k_\epsilon)(\forall k \geq k_\epsilon) (\forall r \in \mathbb{Z})[\text{For infinitely many values of } n, \; \textstyle\sum_k^n x_i \in (rl - \epsilon, rl + \epsilon)]\}$$

Then A *satisfies* $(*)$.

Proof. : Clearly A is a Borel set in $\mathbb{R}^\infty$. Let $(x_1, x_2, \ldots) \in A$. Assume that $\sum |x_i - x_i'| < \infty$. Fix $\epsilon > 0$. Choose k_1 such that $\sum_{k_1}^{\infty} |x_i - x_i'| < \epsilon/2$. Choose k_2 such that for all $k \geq k_2$ and for all $r \in \mathbb{Z}$ we have for infinitely many n. $\sum_k^n x_i \in (rl - \frac{\epsilon}{2}, rl + \frac{\epsilon}{2})$. Put k_ϵ to be the maximum of k_1 and k_2. Now if we take any $k \geq k_\epsilon$ and any $r \in \mathbb{Z}$

$$\sum_k^n x_i' = \sum_k^n x_i + \sum_k^n (x_i' - x_i) \in (rl - \epsilon, rl + \epsilon)$$

for infinitely many values of n. ■

1.4 Random Walks

We consider random walks on the real line. Accordingly let γ be a finitely additive probability on $\mathbb{R}$. We consider $(\mathbb{R}^\infty, P^\infty(\mathbb{R}), \sigma)$ where σ is the strategic probability induced by γ so that the coordinate random variables $X_1, X_2, \cdots$ are i.i.d. Of course $S_o = 0$, and for $n \geq 1$, $S_n = \sum_1^n X_i$ is the random walk being considered. We refer to this random walk as $RW(\gamma)$.

Definition 1.4.1 : For a random walk $RW(\gamma)$, its state space S is defined as follows : In case γ is tight, S is the closed semigroup generated by the closed support of $\tilde{\gamma}$. In case γ is not tight, we take $S = \mathbb{R}$ (for reasons that will become clear later). (For definition of tightness see Remark 1.2.0). The elements of S are called the states of the random walk. A state a, is said to be weakly recurrent if for each $\epsilon > 0$, $\sigma(S_n \in (a - \epsilon, a + \epsilon)\text{i.o.}) = 1$. A state a is said to be strongly recurrent if $\sigma(S_n \in (a - \epsilon, a + \epsilon) \text{ i.o for each } \epsilon > 0) = 1$.

If a state is not weakly recurrent, it is called transient. A random walk is called weakly recurrent/strongly recurrent/transient if all states are so.

It will follow from our results that all states in a random walk are of the same type. We shall first consider random walks induced by tight probabilities. Accordingly let γ be a tight probability on $\mathbb{R}$. Let $\tilde{\gamma}$ be its associate. Recall that a countably additive probability μ on $(\mathbb{R}, \mathcal{B})$ is said to be lattice if there is a $d > 0$ such that $\mu\{nd : n \in \mathbb{Z}\} = 1$. Otherwise it is nonlattice.

Lemma 1.4.2 *Let $\tilde{\gamma}$ be nonlattice. Then $RW(\gamma)$ is strongly recurrent iff $RW(\tilde{\gamma})$ is recurrent.*

Proof. : Apply 1.3.2 and 1.3.3 ■

Lemma 1.4.3 *Let $\tilde{\gamma}$ be lattice. Then $RW(\gamma)$ is weakly recurrent iff $RW(\tilde{\gamma})$ is recurrent.*

Proof. : Let $\tilde{\gamma}$ be concentrated on L say, where L is the set of all multiples of l. Here $l > 0$. Assume that $RW(\tilde{\gamma})$ is recurrent. So we have $\sum_{i=1}^{n} Y_i = a$ i.o. for all $a \in L$ almost surely. Let A be as in 1.3.4. Thus $(Y_1, Y_2, \dots) \in A$ a.s. As a consequence $(X_1, X_2, \dots) \in A$ a.s. σ. However this does not immediately imply the weak recurrence of $RW(\gamma)$, the reader should note the appearance of k depending on ϵ in the definition of A. We shall complete the argument as follows : Fix $\epsilon > 0$. Let $C_i = \{X_i \in \bigcup_{r \in \mathbb{Z}} (rl - \frac{\epsilon}{2^{i-1}} \ , \ rl + \frac{\epsilon}{2^{i+1}})\}$. By tightness of γ we have $\sigma(C_i) = 1$ for $i \geq 1$. Since each C_i is a clopen set, noting that $(X_i, i \geq 1)$ is an independent sequence and that C_i depends on X_i only, using **(P3)** in §0.1 we conclude that $\sigma(\bigcap_1^\infty C_i) = 1$. Note that on $\bigcap C_i$, $\sum_{i=1}^{k} X_i \in (rl - \frac{\epsilon}{2}, rl + \frac{\epsilon}{2})$ for some $r \in Z$. Now fix any $r \in Z$

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{k_{\epsilon/2}-1} X_i + \sum_{i=k_{\epsilon/2}}^{n} X_i$$

On $\bigcap C_i$ the first sum on the right side is in $(r_o l - \frac{\epsilon}{2}, r_o l + \frac{\epsilon}{2})$ for some $r_o \in \mathbb{Z}$. By definition of A, a.s. σ, the second sum belongs to $((r - r_o)l - \frac{\epsilon}{2}, (r - r_o)l + \frac{\epsilon}{2})$ for infinitely many values of n. As a consequence a.s. σ, $\sum_1^n X_i \in (rl - \epsilon, r1 + \epsilon)$ for infinitely many values of n. This shows that $RW(\gamma)$ is weakly recurrent.

Conversely, if $RW(\tilde{\gamma})$ is not recurrent, i.e., $RW(\tilde{\gamma})$ is transient, then 1.3.3 shows that $RW(\gamma)$ is not weakly recurrent. This completes the proof. ■

Lemma 1.4.4 *Suppose γ is not countably additive and $\tilde{\gamma}$ is lattice. Then $RW(\gamma)$ is not strongly recurrent.*

Proof. : Suppose $\tilde{\gamma}$ is concentrated on the lattice $L = \{rl : r \in \mathbb{Z}\}$; $l > 0$.

Let $d_i = \inf\,(|X_i - rl| : r \in \mathbb{Z})$ and $K_1 = [X_1 \in \{\bigcup_{r\in\mathbb{Z}}(rl - \delta, rl + \delta) - L\}]$ where $0 < \delta < l/4$. Of course, d_i depends on the history h.

Since γ is not countably additive (and is tight) it follows that $\sigma(K_1) > 0$. For $i \geq 2$ let

$$K_i = \{X_i \in \bigcup_{r\in\mathbb{Z}}(rl - \frac{d_1}{2^{i+1}}, rl + \frac{d_1}{2^{i+1}})\}.$$

Clearly $K_1, K_2, \ldots$ are all clopen sets.

Note that for $h \in K_1$, $d_1(h) > 0$ which implies that $\sigma[h_1](K_2 h_1) = 1$ and in general $\sigma[h_1 \ldots h_n](K_{n+1} h_1 \ldots h_n) = 1$. An application of **(P3)** in §0.1 now yields that $\sigma(\bigcap_{i=1}^{\infty} K_i) = \sigma(K_1) > 0$. Now notice that on $\bigcap_{i=1}^{\infty} K_i$, $\sum_{i=1}^{n} X_i = rl + \sum_{i=1}^{n} \pm d_i$ for some $r \in \mathbb{Z}$. Thus on $\bigcap K_i$, $|S_n - rl| > \frac{d_1}{2}$ for all n.

This implies that the random walk is not strongly recurrent. ■

Thus, in case γ is tight, (1.4.2) and (1.4.3) show that $RW(\gamma)$ is transient iff $RW(\tilde{\gamma})$ is transient. Of course, in the lattice case, $RW(\gamma)$ is never strongly recurrent unless γ is countably additive. We shall now proceed to consider the case when γ is not tight. First we shall restrict ourselves to random walks on $\mathbb{Z}$, the set of integers. Accordingly let us consider an i.i.d. strategic probability σ on $(\mathbb{Z}^{\infty}, \mathbb{P}^{\infty}(\mathbb{Z}))$ induced by a γ on $\mathbb{Z}$. In what follows $\delta_{+\infty}$ stands for any finitely additive probability on $\mathbb{Z}$ such that $\delta_{+\infty}(n, \infty) = 1$ for each $n \geq 1$. Similarly $\delta_{-\infty}$ stands for any finitely additive probability on $\mathbb{Z}$ such that $\delta_{-\infty}(-\infty, -n) = 1$ for each $n \geq 1$. In other words, δ_{∞} is any diffuse probability concentrated near $+\infty$ and $\delta_{-\infty}$ is any diffuse probability concentrated near $-\infty$.

Lemma 1.4.5 *Let $\gamma = a\delta_{-\infty} + b\delta_{+\infty} + c\mu$ where $a > 0$, $b > 0$, $a + b + c = 1$ and μ is a countably additive probability on $\mathbb{Z}$. Then $RW(\gamma)$ is transient i.e. $|S_n| \longrightarrow \infty$ a.s. σ and $RW(\gamma)$ is oscillating i.e., a.s. σ, S_n changes sign infinitely often.*

Proof. : Fix an $\epsilon > 0$. Choose ϵ_n, $n \geq 1$ such that $\prod_{n=1}^{\infty}(1-\epsilon_n) > 1-\epsilon$. Since μ is tight, fix L_n, $n \geq 1$ such that $\mu(-L_n, +L_n) > 1 - \epsilon_n$ for $n \geq 1$. In what follows, for notational convenience h_0 is taken as 0. For each positive integer $n \geq 1$ we will fix a suitable positive integer m_n later. For now, define, for $n \geq 1$, the sets

$$A_n = \{h = (h_1, h_2, \ldots) : h_n \in (-L_n, L_n) \text{ or } |h_n| > \exp(\sum_{i=0}^{n-1} |h_i| + \sum_{i=1}^{m_n} L_i\}$$

By choice of L_n it is clear that $\sigma(A_n) > 1 - \epsilon_n$ and also for each h, $\sigma[p_{n-1}(h)](A_n p_{n-1}(h)) > 1 - \epsilon_n$. Let $A = \bigcap A_n$. By **(P3)** in §0.1, $\sigma(A) \geq \prod_1^{\infty}(1 - \epsilon_n) > 1 - \epsilon$. Let,

$$\begin{aligned} Y_i(h) &= 1 \text{ if } h_i \in (-L_i, L_i) \\ &= 0 \textit{ otherwise} \end{aligned}$$

Put, $c_i = c\mu(-L_i, L_i) = \gamma(-L_i, L_i)$ and $Z_i = Y_i - c_i$.

Then clearly $E(Z_i) = 0$, Z_i depends only on the ith coordinate, (Z_i) are independent. By the SLLN (see **(P5)**, §0.1)

$$\sigma[\frac{1}{n}\Sigma Z_i \longrightarrow 0] = 1.$$

$c_i \longrightarrow c$ (by the tightness of μ) so that $\frac{c_1 + \cdots + c_n}{n} \longrightarrow c$. As a consequence $\sigma[\frac{1}{n}\sum_1^n Y_i \longrightarrow c] = 1$. Choose $\eta > 0$ so that $c + \eta < 1$ and for simplicity assume that $c+\eta$ is a rational number.

$$\sigma[\frac{1}{n}\sum_1^n Y_i < c+\eta \text{ for large } n] = 1. \ (*)$$

Let B be the event in brackets above.

Let $1-c-\eta = \frac{p}{q}$ where p, q are positive integers (recall that $c+\eta$ is rational). Put $m_n = q(n+1)$. We shall now show that with this choice of m_n, we shall have $|X_n| \longrightarrow \infty$ on $A \bigcap B$. To see this, fix $h \in A \bigcap B$. Thus for each k either $h_k \in (-L_k, L_k)$ or $|h_k| > \exp(\sum_0^{k-1}|h_i| + \sum_{i=1}^{m_k} L_i)$. Let,

$$N_n(h) = \mathrm{Card}\{k \le n : |h_k| > \exp(\sum_0^{k-1}|h_i| + \sum_{i=1}^{m_k} L_i\}$$

$$M_n(h) = \max\{k \le n : |h_k| > \exp(\sum_0^{k-1}|h_i| + \sum_{i=1}^{m_k} L_i\}$$

If $h \in B$, then for sufficiently large $n, N_n(h) > (1-c-\eta)n$. This follows from the definitions of the random variables Y_i and the event B. Obviously $M_n(h) \ge N_n(h)$. Thus suppressing h—for notational ease—we have $q(M_n+1) \ge q(N_n+1) \ge n$. Also, for $i = M_n+1, \ldots, n$, $h_i \in (-L_i, L_i)$ so that

$$|\sum_{i=1}^n h_i| \ge |h_{M_n}| - \sum_{i=0}^{M_n-1}|h_i| - \sum_{i=M_n+1}^{n}|h_i|$$

$$\ge \exp(\sum_0^{M_n-1}|h_i| + \sum_1^{q(M_n+1)} L_i) - \sum_0^{M_n-1}|h_i| - \sum_{M_n+1}^{n}|h_i|$$

(by definition of M_n)

$$\ge \exp(\sum_0^{M_n-1}|h_i| + \sum_1^{q(M_n+1)} L_i) - \sum_0^{M_n-1}|h_i| - \sum_1^{n} L_i$$

(because for $h \in A$ if $M_n < i \leq n$, then $h_i \in (-L_i, L_i)$).

$$\longrightarrow \infty \text{ as } n \longrightarrow \infty \quad (\text{because} \quad q(M_n + 1) \geq n.)$$

Thus for $h \in A \bigcap B$, $|S_n(h)| \longrightarrow \infty$. Since $\sigma(A \bigcap B) > 1 - \epsilon$ and $\epsilon > 0$ is arbitrary we have shown that $RW(\gamma)$ is transient. Two more applications of the SLLN shows that S_n changes sign infinitely often, completing proof of the Lemma. ■

Lemma 1.4.6 *Let $\gamma = b\delta_{+\infty} + c\mu$ with $b > 0$, $c \geq 0$, $b + c = 1$ and μ a countably additive probability. Then $RW(\gamma)$ diverges to $+\infty$.*

Proof. : We proceed as in Lemma 1.4.5. Fix an $\epsilon > 0$ and for each $n \geq 1$ an $\epsilon_n > 0$ so that $\prod_{n=1}^{\infty}(1 - \epsilon_n) > (1 - \epsilon)$. For $n \geq 1$ fix L_n so that $\mu(-L_n, L_n) > (1 - \epsilon_n)$. Define

$$A_n = \{h = (h_1, h_2, \ldots) : h_n \in (-L_n, L_n) \text{ or } h_n > \exp(\sum_{i=1}^{m_n} L_i)\},$$

where m_n are suitably chosen positive integers. We can now proceed as in the proof of the earlier lemma with appropriate modifications.■

We shall now summarize the conclusions of 1.4.2-1.4.6. In what follows $\delta_{+\infty}$ stands for any finitely additive probability on $\mathbb{R}$ with $\delta_{+\infty}(n, \infty) = 1$ for all $n \geq 1$. Similarly $\delta_{-\infty}$ stands for any finitely additive probability on $\mathbb{R}$ with $\delta_{-\infty}(-\infty, -n) = 1$ for all $n \geq 1$.

Theorem 1.4.7 *Consider the random walk $RW(\gamma)$ on the real line $\mathbb{R}$ where $\gamma = a\delta_{-\infty} + b\delta_{-\infty} + c\mu$ with $a, b, c \geq 0$: $a + b + c = 1$: and μ a tight probability on $\mathbb{R}$.*

(i) If $c = 1$ and $\tilde{\mu}$ is nonlattice, then $RW(\gamma) = RW(\mu)$ is strongly recurrent iff $RW(\tilde{\mu})$ is recurrent.

(ii) *If $c = 1$ and $\tilde{\mu}$ is lattice, then $RW(\gamma) = RW(\mu)$ is weakly recurrent iff $RW(\tilde{\mu})$ is recurrent. It is never strongly recurrent unless μ is countably additive.*

(iii) *If $c < 1$ and $a = 0$, then $RW(\gamma)$ diverges to $+\infty$ a.s. σ.*

(iv) *If $c < 1$ and $b = 0$, then $RW(\gamma)$ diverges to $-\infty$ a.s. σ.*

(v) *If $c < 1$, $a > 0$ and $b > 0$, then $RW(\gamma)$ changes sign infinitely often and diverges in modulus to ∞ a.s. σ.*

(1.4.8) Some Remarks:

1. At the end of §2, we remarked about the necessity of restricting attention to some kind of regular setups. Here is a specific example : Consider on $\mathbb{R}$ the finitely additive probability $\gamma = \frac{1}{3}\delta_{-1} + \frac{1}{3}\delta_{+1} + \frac{1}{3}\lambda$ where λ is the Lebesgue measure on $[-\frac{1}{2}, \frac{1}{2}]$; δ_{+1} is a finitely additive probability giving mass one to each open interval $(1, 1 + \frac{1}{n})$ for $n \geq 1$; δ_{-1} is a finitely additive probability giving mass one to each interval $(-1, -1 + \frac{1}{n})$ for $n \geq 1$. On $\mathbb{R}^\infty$, let $\mathcal{C}$ be the field of finite disjoint unions of rectangles $F_1 \times \cdots \times F_n \times \mathbb{R} \times \mathbb{R} \times \cdots$ where each F_i is a finite disjoint union of left open right closed intervals in $\mathbb{R}$. On $\mathcal{C}$ define μ as the obvious product probability with each coordinate equipped with γ. Clearly $\tilde{\gamma}$ is nonlattice and $RW(\tilde{\gamma})$ is recurrent. If A is the set of all sequences in $\mathbb{R}^\infty$ having dense partial sums then $\mu^*(A) = 1$ and $\mu_*(A) = 0$ so that we can get an extension – say μ' – of μ with $\mu'(A) < 1$. With such a μ' the coordinate random variables are i.i.d. in any reasonable definition (see for example [24] Section 3). However the random walk $RW(\gamma)$ considered on this space is not recurrent.

2. Notice that the γ in the earlier remark is tight. Similar examples can be constructed with nontight γ also.

3. We have chosen the strategic setting as our regular setup. In [24] another notion – namely, regularity – was introduced. Some of our results – especially Lemmas 1.4.2—1.4.4 dealing with the case of tight γ hold good even in this setup.

1.5 Extension to Higher Dimensions

In this section, we show that some of our results obtained in the last few sections can be extended to higher dimensions with a slight modification in the argument. But unlike the one-dimensional case, we do not have here a neat theory covering all possible cases. The reason is, in one-dimension we have a neat dichotomy, namely, the state space of $RW(\tilde{\gamma})$ can be either lattice or the whole space. In higher dimension the nature of the state space can be more complicated depending on γ.

First we will consider the strategic probability σ induced by a tight γ. Below we give a version of Karandikar's theorem (see Proposition 1.3.1 or [24]) which we will apply to d-dimensional random vectors.

Proposition 1.5.1 *Suppose that* $(\tilde{X}_n)$ *is a sequence of i.i.d. random vectors defined on* $(I^\infty, \mathcal{P}^\infty(I), \sigma)$ *where* $I = \mathbb{R}^d$. *Let* $A \in \mathcal{B}(I^\infty)$ *be such that *for all* $\mathbf{x} = (\tilde{x}_1, \tilde{x}_2, \ldots)$ *and* $\mathbf{x}' = (\tilde{x}'_1, \tilde{x}'_2, \ldots) \in I^\infty$, $\mathbf{x} \in A$ *and* $\sum_{i=1}^\infty \|\tilde{x}_i - \tilde{x}'_i\| < \infty$ *imply* $\mathbf{x}' \in A$.* *Then* $\tilde{\mathbf{X}} \in A$ *a.s.* σ_γ *iff* $\tilde{\mathbf{Y}} \in A$ *a.s. where* $\tilde{\mathbf{Y}}$ *is the countably additive associate of* $\tilde{\mathbf{X}}$.

As in one dimensional case, we refer to the property of A that has been marked by $*$ above, as $(*)$. In what follows, $S(\tilde{\gamma})$ denotes the support of $\tilde{\gamma}$ in $\mathbb{R}^d$. For a fixed $\epsilon > 0$ let $B(S(\tilde{\gamma}), \epsilon)$ be the ϵ-neighbourhood of $S(\tilde{\gamma})$ i.e., $\underset{\sim}{x} \in B(S(\tilde{\gamma}), \epsilon) \longleftrightarrow$ for some $\underset{\sim}{y} \in S(\tilde{\gamma}), ||\underset{\sim}{x} - \underset{\sim}{y}|| < \epsilon$.

Note that if $(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots)$ be a sequence of i.i.d. random vectors, each having distribution γ then for any $\epsilon > 0$ and for all i, $\underset{\sim}{X}_i \in B(S(\tilde{\gamma}), \epsilon)$ a.s. γ.

Below we give the analogues of Lemma 1.3.2, 1.3.3 and 1.3.4 in d-dimensional setting where $d \geq 2$.

Fix a countable dense set $\{s_1, s_2, \ldots\}$ in the closed group generated by $S(\tilde{\gamma})$ which will be denoted by $Gr(\tilde{\gamma})$.

Lemma 1.5.2 *Let*

$$A = \{(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots) \in I^\infty : \textit{ For each } k. (\sum_{i=k}^{n} \underset{\sim}{x}_i : n \geq k) \textit{ is dense in } \mathbb{R}^d\}$$

Then A satisfies $(*)$.

Lemma 1.5.3 *Let*

$$A = \{(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots) \in I^\infty : \textit{ For each } k, ||\sum_{i=k}^{n} \underset{\sim}{x}_i|| \longrightarrow \infty \textit{ as } n \longrightarrow \infty\}$$

Then A satisfies $(*)$.

Lemma 1.5.4 *Let*

$$\begin{aligned} A = \{(x_1, x_2, \ldots) \in I^\infty \; : \; & (\forall \epsilon > 0)(\exists k_\epsilon)(\forall k \geq k_\epsilon) \\ & (\forall m \in \mathbb{N}) \textit{ [For infinitely many values of } n. \\ & \textstyle\sum_{i=k}^{n} \underset{\sim}{x}_i \in B(s_m, \epsilon)]\} \end{aligned}$$

Then A satisfies $(*)$.

The proofs are same as in one dimensional case.

The next proposition establishes a correspondence between $RW(\gamma)$ and $RW(\tilde{\gamma})$. We will first consider two-dimensional random walks.

Proposition 1.5.5: *$RW(\gamma)$ is weakly recurrent iff $RW(\tilde{\gamma})$ is recurrent.*

<u>Proof.</u>: Assume $RW(\tilde{\gamma})$ is recurrent. Let A be as in Lemma 1.5.4. Since $RW(\tilde{\gamma})$ is recurrent $(\underset{\sim}{Y_1}, \underset{\sim}{Y_2}, \ldots) \in A$ a.s where $(\underset{\sim}{Y_1}, \underset{\sim}{Y_2}, \ldots)$ is the countably additive associate of $(\underset{\sim}{X_1}, \underset{\sim}{X_2}, \ldots)$. Therefore by Proposition 1.5.1, $(\underset{\sim}{X_1}, \underset{\sim}{X_2}, \ldots) \in A$ a.s. σ_γ. To prove that $S_k = \sum_{i=1}^{k} \underset{\sim}{X_i}$, $k \geq 1$ is weakly recurrent we will argue as follows.

Fix $a_0 \in Gr(\tilde{\gamma})$ and $\epsilon > 0$. By Lemma 1.5.4. corresponding to $\frac{\epsilon}{4}$ there exists a positive integer $k_{\frac{\epsilon}{4}}$ such that $\forall k \geq k_{\frac{\epsilon}{4}}$ for infinitely many values of n, $\sum_{i=k}^{n} \underset{\sim}{x_i} \in B(s_m, \frac{\epsilon}{4})$ $\forall m \in \mathbb{N}$. Now, $\sum_{i=1}^{k} \underset{\sim}{X_i} = \sum_{i=1}^{k_{\frac{\epsilon}{4}}} \underset{\sim}{X_i} + \sum_{i=k_{\frac{\epsilon}{4}}+1}^{k} \underset{\sim}{X_i}$.

The second part of the sum visits $B(s_m, \frac{\epsilon}{4})$ infinitely often $\forall m \in \mathbb{N}$ by Lemma 1.5.4. To tackle the first part, set

$$C_i^\epsilon = \{\underset{\sim}{X} = (\underset{\sim}{X_1}, \underset{\sim}{X_2}, \ldots) : \underset{\sim}{X_i} \in B(S(\tilde{\gamma}), \frac{\epsilon}{2^{i+1}})\}$$

Note that $\sigma_\gamma(C_i^\epsilon) = 1$ and C_i^ϵ's are clopen and independent for $i = 1, 2, \ldots$. Therefore we have $\sigma_\gamma(\cap C_i^\epsilon) = 1$ by **(P3)** in §0.1 .

On $D = (\cap C_i^\epsilon) \cap \underset{\sim}{X}^{-1}(A)$, $\sum_{i=1}^{k_{\frac{\epsilon}{4}}} \underset{\sim}{X_i} \in B(a, \frac{\epsilon}{2})$ for some $a \in Gr(\tilde{\gamma})$......(1)

$(s_1, s_2, \ldots)$ being dense in $Gr(\tilde{\gamma})$ $\exists m_0$ s.t. $B(s_{m_0}, \frac{\epsilon}{4}) \subset B(a_0 - a, \frac{\epsilon}{2})$.

Now, we have already observed that $\sum_{i=k_{\frac{\epsilon}{4}}+1}^{k} \underset{\sim}{X_i} \in B(s_{m_0}, \frac{\epsilon}{4})$ infinitely often since $\underset{\sim}{X} \in A$. So consequently $\sum_{i=k_{\frac{\epsilon}{4}}+1}^{k} \underset{\sim}{X_i} \in B(a_0 - a, \frac{\epsilon}{2})$ infinitely often........(2)

Hence from (1) and (2) it follows that $\sum_{i=1}^{k} X_i \in B(a_0, \epsilon)$ infinitely often which shows that $RW(\gamma)$ is weakly recurrent completing the proof. To prove the converse apply Lemma 1.5.3. ■

Remark (1.5.6): It is easy to see that if one of the marginals of $\tilde{\gamma}$ is lattice then $RW(\gamma)$ cannot be strongly recurrent unless the corresponding marginal of γ is countably additive. But it is possible to construct a γ using transition function $p(x, \cdot)$ which is lattice for each x and finitely additive but $RW(\gamma)$ is strongly recurrent as the following example shows:

Example (1.5.7): Let γ_1 be a convex combination of the uniform measure on $(-\frac{1}{2}, \frac{1}{2})$ and a diffuse measure at the left of 1 and at the right of -1. Let $p(x, \cdot)$ be diffuse and $\frac{1}{2}$ at $\sqrt{1-x^2}$ and $\frac{1}{2}$ at $-\sqrt{1-x^2}$ for all x. Define γ on $\mathcal{P}(\mathbb{R}^2)$ as follows:

$$\gamma(A) = \int p(x, A_x) d\gamma_1(x)$$

$RW(\tilde{\gamma})$ is recurrent and the marginals of $\tilde{\gamma}$ are nonlattice. Moreover, the state space of $RW(\tilde{\gamma})$ is entire $\mathbb{R}^2$. Therefore we can apply Lemma 1.5.2. to conclude that $RW(\gamma)$ is strongly recurrent.

In $\mathbb{R}^d$, $d \geq 3$ the random walks in countably additive setup are always transient. Likewise d-dimensional random walks ($d \geq 3$) in strategic setting are always transient as can be shown by an application of Lemma 1.5.3.

Remark (1.5.8): If γ is a probability on $\mathbb{R}^2$ which has a marginal with a nontrivial part diffuse at $+\infty$ (or $-\infty$) then it can be shown that $RW(\gamma)$ is transient.

This fact can be stated more precisely as follows:

Theorem 1.5.9 *Let γ be a probability on $\mathbb{R}^2$ and γ_1 and γ_2 be its* *marginals, i.e.* $\gamma_1(A) = \gamma(A \times \mathbb{R})$ *and* $\gamma_2(A) = \gamma(\mathbb{R} \times A)$. *One of them, say* γ_1 *has the following form*

$$\gamma_1 = a\delta_{-\infty} + b\delta_{\infty} + c\mu$$

where $a, b, c \geq 0$ *and* $a + b + c = 1$ *and* μ *is tight. Then*

1. *If* $c < 1$ *and* $a > 0, b > 0$ *then for any positive integer* N, $S_n(h)$ *visits both* $(-\infty, -N) \times \mathbb{R}$ *and* $(N, \infty) \times \mathbb{R}$ *infinitely often a.s.* σ_γ.

2. *If* $c < 1$ *and* $a = 0$ *then for any positive integer* N, $S_n(h) \in (N, \infty) \times \mathbb{R}$ *eventually a.s.* σ_γ.

3. *If* $c < 1$ *and* $b = 0$ *then for any positive integer* N, $S_n(h) \in (-\infty, -N) \times \mathbb{R}$ *eventually a.s.* σ_γ.

The proof goes along the same line as the proof of Lemma 1.4.5. and 1.4.6. A slight modification is necessary to fit it into the two-dimensional framework. The sets A_n which play the main role in the argument in Lemma 1.4.5 would be defined here as follows:

$$A_n = \{h = ((h_1, h_1'), (h_2, h_2'), \ldots) \ : \quad h_n \in (-L_n, L_n) \quad \text{or} \quad |h_n| > \exp(\textstyle\sum_{i=0}^{n-1} |h_i| + \sum_{i=1}^{m_n} L_i)\}$$

The notations have the same meanings as in Lemma 1.4.5. Now A_ns are subsets of $(\mathbb{R}^2)^\infty$. Since A_n is finite dimensional we can show by direct computation that $\sigma_\gamma(A_n) > (1 - \epsilon_n)$. After this exactly the same proof follows except that here we define $Y_i(h)$ as:

$$Y_i(h) = \begin{cases} 1 & \text{if } (h_i, h_i') \in (-L_i, L_i) \times \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

If we assume that γ_2 is of the form discussed above then similar results hold with obvious changes in coordinates.

If both γ_1 and γ_2 are of this form then the random walk is transient and depending on γ the partial sums belong to any of the four quadrants eventually with probability one, moving farther and farther away from the two axes. Below we give a simple example of γ which induces a random walk such that the partial sums avoid the second quadrant and visit first, third and fourth quadrants infinitely often, gradually moving away from the two axes.

Example (1.5.10): Let $\gamma_1 = \frac{1}{2}\delta_{-\infty} + \frac{1}{2}\delta_{\infty}$. Let $p(\cdot,\cdot)$ be a transition function defined as follows:

$$p(x,\cdot) = \begin{cases} \delta_{-\infty} & \text{if } x < 0 \\ \frac{1}{2}\delta_{-\infty} + \frac{1}{2}\delta_{\infty} & \text{if } x \geq 0 \end{cases}$$

Let $\gamma(A) = \int p(x, Ax)d\gamma_1(x)$

Then obviously $\gamma((-\infty,0)\times(0,\infty)) = 0$ and $\gamma_2 = \frac{3}{4}\delta_{-\infty} + \frac{1}{4}\delta_{\infty}$. To verify this, observe that

$$\gamma_2(A) = \gamma(\mathbb{R}\times A) = \int p(x,A)d\gamma_1.$$

If $A = (n,\infty)$ then $p(x,A) = 0$ if $x < 0$ and $\frac{1}{2}$ if $x \geq 0$. Therefore, $\gamma_2(A) = \frac{1}{4}$. Similarly, if $A = (-\infty,-n)$ then $\gamma_2(A) = \frac{3}{4}$. Now, it can be easily verified that the partial sums visit all quadrants except the second one infinitely often and move away from the two axes almost surely.

Clearly, we can construct γ in a similar fashion so that the induced random walk avoids any number of the quadrants we like and visits the rest infinitely often moving away from the two axes steadily. Also it is easy to see that using a suitable $p(x,\cdot)$, γ can be so constructed that the random walk moves towards $(\pm\infty, \pm\infty)$ along appropriately chosen fixed unbounded curves. Since we do not have a complete and comprehensive picture of all possible scenarios we will not go into the details here.

So far we have discussed one particular aspect of random walks in strategic setup, namely, their recurrence and transience and observed some peculiar characteristics which distinguish them from their countably additive counterparts. In the next chapter we will look into another aspect of random walks , namely, their atomicity. We analyse the behaviour of strategic random walks in that context and contrast them to their countably additive counterparts.

Chapter 2

Purely Nonatomic Random Walks

2.1 Introduction

In this chapter we concentrate on another aspect of strategic random walks, namely their atomicity. Here our aim is to describe a class of random walks on integers which are purely nonatomic. This is in contrast to the countably additive setup where all random walks are simply atomic— a very well-known result proved by David Blackwell. Our result partially answers a question raised by Ramakrishnan in his thesis.

2.2 Preliminaries

Here are some definitions and notations which we will use in the sequel. We mainly follow ([3], [30], [33]).

We will work on the set of integers $\mathbb{Z}$. γ is a finitely additive probability on $\mathbb{Z}$ and σ is the markov strategy with initial distribution γ, corresponding to the random walk induced by γ. More precisely, $\sigma(<>) = \gamma$, $\sigma(< i_1, i_2, \ldots i_n >) = \gamma(A - i_n)$, $n \geq 1$.($<>$ denotes the empty sequence). We shall denote this random walk by $RW(\gamma)$. σ also stands for the strategic measure ([28], or see Preliminaries in §0.1) induced on $\mathbb{Z}^\infty$, the space of histories $h = (h_1, h_2, \cdots)$. On $\mathbb{Z}^\infty$, the shift operator is denoted by S, and is defined as follows: $S(h_1, h_2, \ldots) = (h_2, h_3, \ldots)$. In what follows, only Borel subsets of $\mathbb{Z}^\infty$ are considered. A set A is invariant if it is invariant under S, i.e., if $S^{-1}(A) = A$. The class of invariant Borel sets form a σ-field called invariant σ-field which we will denote by $\mathcal{I}$. We call an invariant set A a σ-atom if A is an atom with respect to σ restricted to $\mathcal{I}$, i.e., if $\sigma(A) > 0$ and for all invariant $B \subset A$, $\sigma(B) = 0$ or $\sigma(A \setminus B) = 0$. $RW(\gamma)$ is atomic in case there is a sequence of σ atoms $A_1, A_2, \cdots$ which are disjoint and $\sum \sigma(A_i) = 1$. $RW(\gamma)$ is simply atomic in case it is atomic and there is exactly one σ atom. $RW(\gamma)$ is purely nonatomic in case there are no σ atoms, equivalently, given any invariant set A with $\sigma(A) > 0$, there is an invariant set $B \subset A$ with $0 < \sigma(B) < \sigma(A)$. All these notions and their detailed analysis is due to D. Blackwell [3] in the countably additive case. Ramakrishnan ([30], [33]) discussed them in the strategic setup. In particular he showed that $\mathbb{Z}^\infty$ can be decomposed into countably many σ atoms and a nonatomic part. He showed

Theorem 2.2.1: (S. Ramakrishnan [30]). *$RW(\gamma)$ is simply atomic if either i) γ is a 0-1 valued or ii) γ has a nontrivial countably additive part not concentrated at 0 or iii) γ has a nontrivial translation invariant part.*

(The reader should note that the part (ii) of Theorem 2.2.1. is slightly different from what has been stated by Ramakrishnan in his thesis. We have added the condition that the countably additive part of γ must not concentrate at zero for $RW(\gamma)$ to be simply atomic).

Of course if i) holds then σ itself is 0-1 valued. In the other two cases one shows that bounded harmonic functions are constants. He raised the question whether every $RW(\gamma)$ is simply atomic as in [3]. We shall exhibit purely nonatomic $RW(\gamma)$. Recall that γ is diffuse in case γ has no nontrivial countably additive part – equivalently, $\gamma\{n\} = 0$ for each $n \in \mathbb{Z}$. Thus in case γ is not diffuse, and the countably additive part of γ is not concentrated at 0, Theorem 2.2.1 implies that $RW(\gamma)$ is simply atomic.

2.3 Main Results

Definitions 2.3.1: Let seq(Z) be the set of finite sequences of integers including the empty sequence. $T \subset \text{seq}(\mathbb{Z})$ is called a tree if i) $s \in T$ and t is an initial segment of s imply $t \in T$ and ii) if $s \in T$ then there is a $y \in \mathbb{Z}$ such that $sy \in T$. A tree T is called disjointed in case $sy \in T$ and $ty \in T$ imply $s = t$. For any tree T its body [T] is the set of histories all of whose initial segments are in T.

Theorem 2.3.2: *Suppose γ is a diffuse probability on $\mathbb{Z}$ which is not 0-1 valued. If there exists a disjointed tree T such that $\sigma([T]) = 1$. Then $RW(\gamma)$ is purely nonatomic.*

Proof: Since γ is nontrivial, fix $A \subset \mathbb{Z}$ with $\gamma(A) = \alpha$, $0 < \alpha < 1$.

Set,

$$\begin{aligned} A_0 &= \{h : h_1 \in A\} \cap [T] \\ A_1 &= \{h : h_1 \notin A\} \cap [T] \\ B_0 &= \cup_{-\infty}^{\infty} S^n A_0 \quad \text{and} \quad B_1 = \cup_{-\infty}^{\infty} S^n A_1 \end{aligned}$$

Using the fact that T is disjointed it is easy to verify that B_0 *and* B_1 are disjoint. To see this, first observe that the first coordinates of A_0 and A_1 are different and both A_0 and A_1 are subsets of $[T]$ where T is a disjointed tree. Therefore, it follows that

$$\forall\ h \in A_0,\ h' \in A_1 \text{ and } \forall\ m,\ n \in \mathbf{N},\ h_n \neq h'_m \ldots\ldots(*)$$

Now, B_0 and B_1 are invariantisations of A_0 and A_1 respectively. Therefore, for any $h \in B_0$ there exists some $\tilde{h} \in A_0$ such that $h_n = \tilde{h}_n$ for all sufficiently large n. The same is true if we replace B_0 and A_0 by B_1 and A_1 respectively. Since A_0 and A_1 have the property marked by $(*)$ above, clearly B_0 and B_1 are disjoint. Moreover $\sigma([T]) = 1$ implies that $\sigma(B_0) = \sigma(A_0) = \alpha$ and $\sigma(B_1) = \sigma(A_1) = 1 - \alpha$. For $i = 0, 1$ and $j = 0, 1$ put

$$A_{ij} = \{h : h_1 \in A^i \text{ and } h_2 - h_1 \in A^j\} \cap [T]$$

where $A^0 = A$ and $A^1 = A^c$

$$B_{ij} = \bigcup_{-\infty}^{\infty} S^n A_{ij}$$

Then B_{ij}, $i = 0, 1$, $j = 0, 1$; are disjoint and $\sigma(B_{ij}) = \alpha^i \alpha^j$ where $\alpha^0 = \alpha$ and $\alpha^1 = 1 - \alpha$. Proceeding in this way we can get, for any given $\epsilon > 0$, a decomposition of $\mathbb{Z}^\infty$ into finitely many invariant sets of σ measure smaller than ϵ. This shows that $RW(\gamma)$ is purely nonatomic. ∎

Specific situations where the above theorem applies are included in the following:

Theorem 2.3.3. *Suppose γ is diffuse and not 0-1 valued on $\mathbb{Z}$. Suppose that for some sequence of positive integers* $0 < \alpha_1 < \alpha_2 <$

$\cdots$ *we have* $\alpha_{k+1} - \alpha_k \to \infty$ *and* $\gamma\{\pm\alpha_i : i \geq 1\} = 1$. *Then* $RW(\gamma)$ *is purely nonatomic.*

Proof : Let $g_k = \alpha_{k+1} - \alpha_k, k = 1, 2, \ldots\ldots$ By our assumption $g_k \to \infty$ i.e. for any fixed $N > 1$ there exists $n_0 \geq 1$ such that $g_k \geq N$ for all $k \geq n_0$. Let $f(N)$ be the first n_0 for which this happens.

Now, set $A = \{\pm\alpha_1, \pm\alpha_2, \ldots\}$ and

$$\begin{aligned} B \;=\; \{h : |h_1| \geq \alpha_1 : \;\; \forall i \geq 1 \;\; & h_{i+1} - h_i \in A \\ \text{and} \;\; & |h_{i+1} - h_i| > \alpha_{f(N_i)+1} \\ \text{where} \;\; & N_i > 2\textstyle\sum_{j<i} |h_j - h_{j-1}|\} \end{aligned}$$

To show that $\sigma(B) = 1$ observe that $B = \cap_{n\geq 1} B_n$ where B_n is defined as follows.

$$h \in B_n \longleftrightarrow Bp_n(h) \neq o$$

B_n is clopen, being a finite dimensional cylinder set and clearly $\sigma[p_{n-1}(h)](B_n p_{n-1}(h)) = 1$ for all $h \in B_{n-1}$. So by **P3** in §0.1. $\sigma(B) = 1$. Let T be the set of all initial segments of histories in B. T is a tree. As B is a closed set (intersection of clopen sets) $[T] = B$. To complete the proof, in view of Theorem 2.3.2, we need to show that if $sy \in T$ and $ty \in T$ then $s = t$; in the notation of Theorem 2.3.2. In other words, we need to show that for any two histories h and h' in B, $h_n = h'_m$ implies that $n = m$ and $h_i = h'_i$ for $i = 1, 2, \ldots, n$. This in turn is equivalent to showing that $h_n = h'_m$ implies $h_i - h_{i-1} = h'_i - h'_{i-1}$ for $i = 1, 2, \ldots, n$ taking $h_0 = 0$. Before proving this, observe a simple fact which we will use in the argument:

For any history $h \in B$, and for all $n > 1$, h_n and $h_n - h_{n-1}$ have the same sign because of the following two relations (1) and (2) :

(1) $|h_n - h_{n-1}| > 2\sum_{i<n} |h_i - h_{i-1}|$

(2) $h_n = (h_n - h_{n-1}) + (h_{n-1} - h_{n-2}) + \cdots + h_1$

Now take any two histories h and $h' \in B$. Assume $h_n = h'_m$ for some n and m. This we can write in the following form which we will use afterwards:

(3) $(h_n - h_{n-1}) + (h_{n-1} - h_{n-2}) + \cdots + h_1 = (h'_m - h'_{m-1}) + (h'_{m-1} - h'_{m-2}) + \cdots + h'_1$

If possible, suppose $h_n - h_{n-1} > h'_m - h'_{m-1}$. We will assume both are of the same sign because otherwise h_n and h'_m are of opposite signs and cannot be equal. Moreover, without any loss they may be assumed to be positive. Now, both $h_n - h_{n-1}$ and $h'_m - h'_{m-1}$ are in A. Set $h_n - h_{n-1} = \alpha_{k_1}$ and $h'_m - h'_{m-1} = \alpha_{k_2}$. $\alpha_{k_1} > \alpha_{k_2}$ by our assumption. But

$$\alpha_{k_1} - \alpha_{k_2} \geq g_{k_1 - 1} > 2 \sum_{i \leq n-1} |h_i - h_{i-1}|$$

Also,

$$\alpha_{k_1} - \alpha_{k_2} \geq g_{k_2} > 2 \sum_{i \leq m-1} |h'_i - h'_{i-1}|$$

Therefore, $\alpha_{k_1} - \alpha_{k_2} = h_n - h_{n-1} - (h'_m - h'_{m-1}) > \sum_{i \leq n-1} |h_i - h_{i-1}| + \sum_{i \leq m-1} |h'_i - h'_{i-1}|$ ....(*)

This leads to the following sequence of inequalities:

$$\begin{aligned}
|h_n| &= |h_1 + (h_2 - h_1) + \cdots (h_n - h_{n-1})| \\
&\geq |h_n - h_{n-1}| - \textstyle\sum_{i \leq n-1} |h_i - h_{i-1}| \\
&> (h'_m - h'_{m-1}) + \textstyle\sum_{i \leq m-1} |h'_i - h'_{i-1}| \\
&\geq |(h'_m - h'_{m-1}) + (h'_{m-1} - h'_{m-2}) + \cdots + h'_1| \\
&= |h'_m|
\end{aligned}$$

The strict inequality we have in the third line follows from (*).

This shows that the assumption $h_n - h_{n-1}$ is greater than $h'_m - h'_{m-1}$ leads to a contradiction. Similarly we can show that $h'_m - h'_{m-1}$ cannot be greater than $h_n - h_{n-1}$. Therefore they have to be equal.

Now, cancelling them from both sides of the equation (3) we will argue as before for the pair $h_{n-1} - h_{n-2}$ and $h'_{m-1} - h'_{m-2}$ to conclude that they must be equal. Proceeding in this way we finally show that $h_n = h'_m$ implies $m = n$ and $(h_i - h_{i-1}) = (h'_i - h'_{i-1})$ for $i = 1, 2, \ldots\ldots\ldots, n$, thus proving our assertion that T is a disjointed tree. Hence $RW(\gamma)$ is purely nonatomic. ■

In the next section we will give some examples to illustrate the above result. But before that, we would like to draw the readers' attention to some interesting facts about simply atomic random walks. Note that though we have a fairely large class (and we conjecture that it is exhaustive too) of purely non-atomic random walks, the number of cases we know of simply atomic random walks are indeed very small in comparison. Moreover, the method we have used so far to prove that a random walk is simply atomic is by verification that the bounded harmonic functions are constants. This is the well-known method due to Blackwell who made use of it in countably additive setup. But as Ramakrishnan has pointed out (see Theorem 15.12 in [30]) for finitely additive γ the condition that the bounded harmonic functions are constants is only sufficient, not necessary for $RW(\gamma)$ to be simply atomic. Below we give a necessary and sufficient condition:

Theorem 2.3.4: *$RW(\gamma)$ is simply atomic if and only if*

$$(\sigma \times \sigma)_* \{(h, h') : \exists N \ \forall n \geq N, \ m \geq N \ h_n \neq h'_m\} = 0.$$

Here $\sigma \times \sigma$ is the product probability defined in the obvious way on the field generated by the Borel rectangles of $Z^{\infty} \times Z^{\infty}$. $(\sigma \times \sigma)_*$ is the usual inner measure. (For definition see Preliminaries of Chapter 1.)

Proof: Let us denote the set $\{(h, h') : \exists N \ \forall n \geq N, \ m \geq N \ h_n \neq h'_m\}$ by D.

If $(\sigma \times \sigma)_*(D) > 0$ then by definition of inner measure there exists a Borel rectangle $B_1 \times B_2 \subset D$ such that $(\sigma \times \sigma)(B_1 \times B_2) > 0$ which implies $\sigma(B_1) > 0$ and $\sigma(B_2) > 0$. Since $B_1 \times B_2 \subset D$, $h \in B_1$ and $h' \in B_2$ imply $\exists N \; \forall \; n \geq N, m \geq N. h_n \neq h'_m$. In other words, if $h \in B_1$ and $h' \in B_2$ then after some stage no coordinate of h can equal any coordinate of h'. In particular, B_1 and B_2 are disjoint. Moreover, their invariantisations are also disjoint. So we get two disjoint invariant sets both having positive probability. This shows that $RW(\gamma)$ is not simply atomic.

Conversely, if $RW(\gamma)$ is not simply atomic then we have two disjoint invariant sets B_1 and B_2 both having positive probability. Since B_1 and B_2 are invariant, corresponding to them there are two subsets ξB_1 and ξB_2 of $\mathbf{Z}$ such that

$$\begin{aligned} B_1 &\sim \{h : h_n \in \xi B_1 \text{ eventually}\} &= \tilde{B}_1(say) \\ B_2 &\sim \{h : h_n \in \xi B_2 \text{ eventually}\} &= \tilde{B}_2(say) \end{aligned}$$

(Here $A \sim B$ denotes $\sigma(A \Delta B) = 0$.) For details see Lemma 15.2 in [30]. B_1, B_2 being disjoint we can assume without loss that ξB_1 and ξB_2 are disjoint. Otherwise we can eliminate the common portion from both ξB_1 and ξB_2. This will not change the probabilities of $\tilde{B}_1$ and $\tilde{B}_2$. We will verify this as follows. Set $E = \xi B_1 \cap \xi B_2$. Let $\tilde{E} = \{h : h_n \in E \quad \text{eventually}\}$ Then clearly, $\tilde{E} \subset \tilde{B}_1 \cap \tilde{B}_2$. We will show that $\tilde{E} \subset (\tilde{B}_1 \Delta B_1 \cup \tilde{B}_2 \Delta B_2)$ and this will imply that $\sigma(\tilde{E}$ [illegible] To show this, let $h \in \tilde{E}$. Then $h \in \tilde{B}_1 \cap \tilde{B}_2$. If $h \in B_1$ then $h \notin B_2$ since they are disjoint and therefore $h \in \tilde{B}_2 \setminus B_2$. On the other hand, if $h \notin B_1$ then $h \in \tilde{B}_1 \setminus B_1$. This shows that $h \in (\tilde{B}_1 \Delta B_1 \cup \tilde{B}_2 \Delta B_2$ [illegible] we claimed and thus without any loss we may assume that ξB [illegible] and ξB_2 are disjoint. Therefore, if $h \in \tilde{B}_1$ and $h' \in \tilde{B}_2$, $\exists N \; \forall n > N$, [illegible] N, $h_n \in \xi B_1$, $h'_m \in \xi B_2$ — hence, $h_n \neq h'_m$. So, we have $\tilde{B}$ [illegible] consequently, $(\sigma \cdot \sigma)_r(D) > 0$. ∎

2.4 Examples and Remarks

Example (2.4.1) : If γ is diffuse, not 0-1 valued and for some $k \geq 2$, $\gamma\{\pm n^k : n > 1\} = 1$ then Theorem 2.3.3 applies and $RW(\gamma)$ is purely nonatomic.

Example (2.4.2) : Let a be an integer ≥ 2. Suppose γ is diffuse, not 0-1 valued, and $\gamma\{\pm a^n : n \geq 1\} = 1$ then again Theorem 2.3.3 applies and $RW(\gamma)$ is purely nonatomic.

Example (2.4.3) : Suppose γ is diffuse, not 0-1 valued on $\mathbb{Z}$ and $\gamma\{n^n : n \geq 1\} = 1$. Then $RW(\gamma)$ is purely nonatomic. Apply Theorem 2.3.3.

Remark (2.4.4). It is known ([30],[33]) that in case bounded γ-harmonic functions are constant, then $RW(\gamma)$ is simply atomic. It is interesting to note that the converse is not true.

For instance if $\gamma\{2^n : n \geq 1\} = 1$ and

$$h(m) = 2^{-n_1} + 2^{-n_2} + \cdots + 2^{-n_k} \quad \text{if} \quad m = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}, \; n_1 > n_2 > \cdots$$

on $\mathbb{Z}^+ = \{1, 2, \cdots\}$; then $RW(\gamma)$ on $\mathbb{Z}^+$ is simply atomic as soon as γ is 0-1 valued. However h is always a bounded γ harmonic function. As soon as γ is not concentrated at a point, h is nonconstant. However for any $c > 0$ $\gamma(h > c) = 0$. Thus h is in a sense almost surely zero.

Example (2.4.5) : Let a, b be integers and $A = \{a - \ldots$

Define for $n \geq 1$ and $B \subset \mathbb{Z}$

$$\begin{aligned} \mu_n(B) &= \tfrac{1}{n}\text{Card}\{k : a+kb \in B.\ 1 \le k \le n\} \\ \text{and } \gamma_0(B) &= \ell(\mu_1(B), \mu_2(B), \cdots) \end{aligned}$$

where ℓ is a Banach limit. Then $RW(\gamma_0)$ is simply atomic. In fact if $\gamma = c\gamma_0 + (1-c)\gamma_1$, $0 < c \le 1$, γ_1 any probability then $RW(\gamma)$ is simply atomic. The idea of the proof is similar to that used in Proposition 16.5 in [30]. For the sake of completeness we give below a brief outline of the argument.

First we show that γ_0 is invariant under a translation by b i.e. $\gamma_0(A) = \gamma_0(A+b)$, $A \subset \mathbf{Z}$. Let $A \subset \mathbf{Z}$. $Card\{k : a+kb \in A. 1 \le k \le n\} = Card\{k : a+kb \in A+b. 2 \le k \le n+1\}$. Therefore, $\mu_n(A) = \frac{1}{n}Card\{k : a+kb \in A, 1 \le k \le n\} = \frac{1}{n}Card\{k : a+kb \in A+b. 2 \le k \le n+1\}$. Now,

$$\begin{aligned} \mu_{n+1}(A+b) &= \tfrac{1}{n+1}Card\{k : a+kb \in A+b, 1 \le k \le n+1\} \\ &= \tfrac{1}{n+1}Card[\{a+b\} \cap A+b] + \tfrac{n}{n+1}\mu_n(A) \end{aligned}$$

The first term on the right side is at most $\frac{1}{n+1}$ and hence tends to zero as n tends to ∞. So taking Banach limit ℓ on both sides and using the fact that Banach limit is shift invariant we have $\ell\{\mu_n(A+b)\} = \ell\{\mu_{n+1}(A+b)\} = \ell\{\frac{n}{n+1}\mu_n(A)\}$. But $\frac{n}{n+1}\mu_n(A) = \mu_n(A) + \frac{1}{n+1}\mu_n(A)$. The second term on the right side tends to zero as n tends to ∞. Therefore, we have $\ell\{\frac{n}{n+1}\mu_n(A)\} = \ell\{\mu_n(A)\}$. Hence, $\gamma_0(A) = \gamma_0(A+b)$. Now, coming back to the proof that $RW(\gamma)$ is simply atomic. it suffices to show that any bounded harmonic function with respect to γ is constant, (see Theorem 15.12 in [30] or [33]). So let f be a bounded harmonic function with respect to γ. Since γ_0 is invariant under a translation by b we have

$$\int f(i+j)d\gamma_0(j) = \int f(i+j+b)d\gamma_0(j). \ldots (*)$$

changing the variable from j to $j' = j + b$. Set $g(i) = f(i+b) - f(i)$. Since g is harmonic with respect to γ

$$\begin{aligned} g(i) &= \int g(i+j)d\gamma(j) \\ &= c\int g(i+j)d\gamma_0(j) + (1-c)\int g(i+j)d\gamma_1(j) \\ &= (1-c)\int g(i+j)d\gamma_1(j) \end{aligned}$$

(The third equality follows from $(*)$). Since $1 - c < 1$ and g is bounded the above equality implies that $g(i) = 0$ for all i. Therefore, $f(i+b) = f(i)$ i.e. f is periodic with period b. If for all proper subgroups $J \subset I$, $\gamma(J) < 1$ then by corollary 16.3 [30] f is constant and we are done. If on the other hand, there exists a proper subgroup $J \subset I$ such that $\gamma(J) = 1$ then we will work with J and γ restricted to J and arrive at the same conclusion. For details see Theorem 16.6 [30]. ∎

Remark (2.4.6). The purely nonatomic $RW(\gamma)$ provide examples of failure of the Hewitt-Savage zero one law. Failure of the Hewitt-Savage $0-1$ law in the strategic set up was first noted in [29].

So far, we have considered random walks induced by γ which are supported on subsets of $\mathbb{Z}$. But the same idea as we used in the proof of Theorem 2.3.3 can be implemented to show that there are purely nonatomic random walks $RW(\gamma)$ with γ not necessarily having $\mathbb{Z}$ as its support. Below we give an example of such a random walk.

Example (2.4.7). Let γ be a probability diffuse at 0 from the right and concentrated on the sequence $\mathcal{S} = \{\frac{1}{n}\}_{n\geq 1}$. To show that $RW(\gamma)$ is purely nonatomic consider the following set

$$\begin{aligned} B = \{ h = (h_1, h_2, \cdots) \ : \ & h_{i+1} - h_i \in \mathcal{S}, \\ & h_{i+1} - h_i < (h_i - h_{i-1})^2 \ \forall i \} \end{aligned}$$

Then clearly $\sigma(B) = 1$. As in Theorem 2.3.3 let T be the set of initial segments of histories in B. Then T is a tree and $[T] = B$ as B is closed. To prove that T is disjointed just notice that for any finite sequence of positive integers arranged in increasing order $n_1 < n_2 < \cdots < n_k$ with the additional property that $n_{i-1} > n_i^2$ we have the following inequality

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} < \frac{1}{n_1} + \frac{1}{n_1^2} + \frac{1}{(n_1^2)^2} + \cdots < \frac{1}{n_1^2 - 1} < \frac{1}{n_1 - 1}$$

....(*)

Now suppose two partial sums $\sum_{i=1}^{k} \frac{1}{n_i}$ and $\sum_{i=1}^{l} \frac{1}{m_i}$ are equal. We want to show that in such a case $k = l$ and that the two equations are term by term equal. If not, to get a contradiction assume that $\frac{1}{n_1} < \frac{1}{m_1}$ i.e. $n_1 > m_1$. So $m_1 \leq n_1 - 1$. But then from (*) we get $\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} < \frac{1}{n_1 - 1} < \frac{1}{m_1}$. So the partial sums cannot be equal. For the same reason $\frac{1}{n_1} > \frac{1}{m_1}$ is also impossible. So we must have $\frac{1}{n_1} = \frac{1}{m_1}$. We complete the proof in a routine way.

Remark (2.4.8). We conjecture that for every γ, $RW(\gamma)$ must either be simply atomic or purely nonatomic. More specifically, we conjecture that $RW(\gamma)$ is purely nonatomic if γ is concentrated on a sequence with diverging gap as in Theorem 2.3.3. On the other hand, if γ gives mass strictly less than 1 to all such sequences then $RW(\gamma)$ is simply atomic. We have not been able to prove it.

We have already got some classes of γ which induce simply atomic random walks [see Example 2.4.3 and [30], §16]. Below we define another interesting class of γ:

Let γ_1 be a diffuse probability concentrated on a sequence of integers $\mathcal{S}_1 = \{\alpha_n\}_{n>0}$. Let γ_2 be a translate of γ_1 concentrated on $\mathcal{S}_2 = \{\alpha_n + r\}_{n>0}$ where r is any integer, so that $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint.

Let $\gamma = \alpha\gamma_1 + (1-\alpha)\gamma_2$, $0 < \alpha < 1$. We conjecture that random walks induced by γ belonging to this class are simply atomic. But we are unable to prove it.

The above discussion gives us some idea about the atomicity of strategic random walks. The discussion shows that in respect of atomicity the contrast between strategic random walks and their countably additive counterparts is remarkably pronounced. The fact that countably additive random walks are simply atomic is a direct consequence of Hewitt–Savage 0-1 law. As our investigation shows, strategic random walks could be purely nonatomic. Thus the Hewitt–Savage 0-1 law would drastically fail. This leads naturally to two questions regarding the Hewitt–Savage 0-1 law: Firstly, are there natural conditions so that a symmetric Borel set satisfying those conditions obeys

Hewitt Savage 0-1 law for any γ. Secondly, are there reasonable conditions on γ so that every symmetric Borel set obeys Hewitt Savage 0-1 law. We take up these problems in the next chapter.

Chapter 3

Hewitt – Savage Zero-One Law

3.1 Introduction

In the countably additive theory of probability the Hewitt – Savage $0-1$ law states the following : In a product space with independent identical components, every symmetric set has probability either zero or one. This theorem has diverse applications, especially in random walks, potential theory and U-statistics. In the context of finitely additive probabilities Purves and Sudderth observed in [29] that the Hewitt Savage $0-1$ law fails in this setup. As we have seen in Chapter 2, this failure can indeed be spectacular – the symmetric σ field could be purely nonatomic. In this chapter we restrict our attention to product spaces where the component space is a countable set. Let γ be a finitely additive probability on this set. We show that the Hewitt -Savage 0-1 law holds for the infinite product measure $\gamma \times \gamma \times \ldots$

(which will be denoted by σ_γ in the sequel) if and only if γ takes at most two values when restricted to subsets of the set $\{i : \gamma(i) = 0\}$. Moreover, when this does not hold, the symmetric σ-field is indeed nonatomic.

The organization of this chapter is as follows : In §2 we describe an alternative way (Theorem 3.2.1) to select a point in the infinite product space with distribution σ_γ. In §3 we use the construction of §2 to prove the main result (Theorem 3.3.1) described in the earlier paragraph. We conclude with some remarks in the last section.

For notations and basic concepts in this chapter see §0.1.

3.2 An Identification of σ_γ

Given a finitely additive probability γ on N, define the countably additive measure γ_1 on N by setting $\gamma_1(i) = \gamma(i)$ and set $\gamma_2 = \gamma - \gamma_1$. Then γ_2 is purely finitely additive and $\gamma = \gamma_1 + \gamma_2$. This is nothing but the Hewitt- Yosida - Kakutani decomposition (see [36] and [34]). Set $\mathsf{N}_\infty = \mathsf{N} \cup \{\infty\}$. λ denotes the countably additive probability on N_∞ defined by $\lambda(i) = \gamma_1(i) = \gamma(i)$ for $i \in \mathsf{N}$ and $\lambda(\infty) = \gamma_2(\mathsf{N})$. μ denotes the finitely additive probability on N defined by $\mu(A) = \frac{\gamma_2(A)}{\gamma_2(\mathsf{N})}$. This μ is just γ_2, normalized. Throughout this section we assume that $0 < \gamma_1(\mathsf{N}) < 1$, so that λ is not point mass at ∞ and μ is well defined.

We are going to use μ and λ to select a sequence of integers with distribution σ_γ. To get started, here is the method for selecting a single integer z_1: first use μ to select x_1 and then λ to select y_1. If $y_1 \neq \infty$, set $z_1 = y_1$, and if $y_1 = \infty$, set $z_1 = x_1$. Then the probability that $z_1 \in E$ is $\gamma(E)$.

To select an infinite sequence of integers with distribution σ_γ, first use μ (repeatedly and independently) to select $x_1, x_2, \ldots$ and then use λ (again repeatedly and independently) to select $y_1, y_2, \ldots$. Let x be the first sequence and y the second one. Let $T(x, y)$ be the sequence obtained by replacing the first occurrence of ∞ in y by x_1, the second occurrence of ∞ in y by x_2, and so on, until all the infinities have been replaced. Then $T(x, y)$ has distribution σ_γ. This is the content of Theorem 3.2.1 below.

To state the Theorem, we need some notation. Set $H_\infty = \mathbb{N}_\infty \times \mathbb{N}_\infty \times \ldots$ equipped with the product topology, where $\mathbb{N}_\infty$ has the discrete topology.

Set $H^1 = H \times H_\infty$. Points in H^1 are denoted by (x, y), where $x \in H$ and $y \in H_\infty$. Let σ_μ be the finitely additive probability on H induced by the i.i.d. strategy μ. Let σ_λ be the usual countably additive product measure $\lambda \times \lambda \times \ldots$. For any Borel set C in H^1 and any $x \in H$, let C_x be the section of C at x, namely, $C_x = \{y \in H_\infty : (x, y) \in C\}$. As σ_λ is countably additive, observe that $\sigma_\lambda(C_x)$ is a measurable function of x on H. Consequently, the expression $\int \sigma_\lambda(C_x) d\sigma_\mu(x)$ is well defined. Denote this by $\sigma'(C)$. Then σ' is a finitely additive probability on H^1.

Here is the precise formulation of the selection procedure mentioned above.

Theorem 3.2.1 *For Borel $B \subset H$, $\sigma'(T^{-1}B) = \sigma_\gamma(B)$.*

We start making a series of observations leading to the proof. For any infinite sequence $v = (v_1, v_2, \ldots)$ we let $v_{(1)} = (v_2, v_3, \ldots)$.
$\underline{1}^o$. For any Borel set $S \subset H^1$ and $i \in \mathbb{N}$ define

$$iS = \{(x, y) \in H^1 : y_1 = i \ \& \ (x, y_{(1)}) \in S\}$$

Similarly we can define iS for any Borel set $S \subset H_\infty$ as follows:

$iS = \{y \in H_\infty : y_1 = i \ \& \ y_{(1)} \in S\}$

Claim : $\sigma'(iS) = \lambda(i)\sigma'(S)$.

To see this observe that $(iS)_x = iS_x$ so that

$$\sigma'(iS) = \int \sigma_\lambda(iS)_x d\sigma_\mu(x) = \lambda(i)\int \sigma_\lambda(S_x)d\sigma_\mu(x) = \lambda(i)\sigma'(S)$$

$\underline{2}^o$. Claim : $\sigma'(\cup_{i\in N} iS) = \sum_{i\in N}\sigma'(iS)$.

To see this, note that,

$$\sigma_\lambda(\cup iS_x) = \sum_i(\lambda(i))\sigma_\lambda(S_x)$$

for any $x \in H$ by the countable additivity of λ. Now

$$\begin{aligned}\sigma'(\cup iS) &= \int \sigma_\lambda(\cup iS_x)d\sigma_\mu(x)\\ &= \int(\sum\lambda(i))\sigma_\lambda(S_x)d\sigma_\mu(x)\\ &= (\sum\lambda(i))\sigma'(S) = \sum\sigma'(iS)\end{aligned}$$

where 1^o is used in the last equality.

More generally we have.

$\underline{3}^o$ Claim : For each $i \in$ N, let S_i be a Borel subset of H^1. Then

$$\sigma'(\cup_{i\in N} iS_i) = \sum_{i\in N}\sigma'(iS_i)$$

To see this fix any integer $k > 1$. Proceeding as in 2^o we get

$$\begin{aligned}\sigma'(\cup iS_i) &= \int\sum_i\lambda(i)\sigma_\lambda((S_i)_x)d\sigma_\mu(x)\\ &\geq \int\sum_{i=0}^k\lambda(i)\sigma_\lambda((S_i)_x)d\sigma_\mu(x)\\ &= \sum_{i=0}^k\sigma'(iS_i)\end{aligned}$$

where we used the finite additivity of the integral and 1^o for the last equality. k being arbitrary we get

$$\sigma'(\cup iS_i) \geq \sum\sigma'(iS_i)$$

$iS = \{y \in H_\infty : y_1 = i \ \& \ y_{(1)} \in S\}$
Claim : $\sigma'(iS) = \lambda(i)\sigma'(S)$.
To see this observe that $(iS)_x = iS_x$ so that

$$\sigma'(iS) = \int \sigma_\lambda(iS)_x d\sigma_\mu(x) = \lambda(i) \int \sigma_\lambda(S_x) d\sigma_\mu(x) = \lambda(i)\sigma'(S)$$

$\underline{2}^o$. Claim : $\sigma'(\cup_{i\in\mathbb{N}} iS) = \sum_{i\in\mathbb{N}} \sigma'(iS)$.
To see this, note that,

$$\sigma_\lambda(\cup iS_x) = \sum_i (\lambda(i))\sigma_\lambda(S_x)$$

for any $x \in H_\infty$ by the countable additivity of λ. Now

$$\begin{aligned} \sigma'(\cup iS) &= \int \sigma_\lambda(\cup iS_x) d\sigma_\mu(x) \\ &= \int (\sum \lambda(i))\sigma_\lambda(S_x) d\sigma_\mu(x) \\ &= (\sum \lambda(i))\sigma'(S) = \sum \sigma'(iS) \end{aligned}$$

where 1^o is used in the last equality.
More generally we have,

$\underline{3}^o$ Claim : For each $i \in \mathbb{N}$, let S_i be a Borel subset of H^1. Then

$$\sigma'(\cup_{i\in\mathbb{N}} iS_i) = \sum_{i\in\mathbb{N}} \sigma'(iS_i)$$

To see this fix any integer $k > 1$. Proceeding as in 2^o we get

$$\begin{aligned} \sigma'(\cup iS_i) &= \int \textstyle\sum_i \lambda(i)\sigma_\lambda((S_i)_x) d\sigma_\mu(x) \\ &\geq \int \textstyle\sum_{i=0}^k \lambda(i)\sigma_\lambda((S_i)_x) d\sigma_\mu(x) \\ &= \textstyle\sum_{i=0}^k \sigma'(iS_i) \end{aligned}$$

where we used the finite additivity of the integral and 1^o for the last equality. k being arbitrary we get

$$\sigma'(\cup iS_i) \geq \sum \sigma'(iS_i)$$

To show the reverse inequality, fix $\epsilon > 0$ and an integer k such that $\sum_{k+1<i<\infty} \lambda(i) < \epsilon$. Observe that $\sum_{k+1<i<\infty} \lambda(i)\sigma_\lambda((S_i)_x) < \epsilon$ for each x so that proceeding as above

$$\begin{aligned} \sigma'(\cup iS_i) &\le \textstyle\sum_{i=0}^{k} \sigma'(iS_i) + \epsilon \\ &\le \textstyle\sum_{i=0}^{\infty} \sigma'(iS_i) + \epsilon \end{aligned}$$

ϵ being arbitrary the proof is complete.

$\underline{4}^o$. For $i \in \mathbb{N}$, $B_i \subset H$ be Borel sets and $B = \cup_{i\in\mathbb{N}}(iB_i)$. Then

$$\sigma'\{(x,y) \in T^{-1}B : y_1 = \infty\} = \lambda(\infty) \int \sigma'(T^{-1}B_i)d\mu(i)$$

To see this denote by C the set in braces on the left side. Observe that $y \in C_x$ iff $y_1 = \infty$ and $T(x_{(1)}, y_{(1)}) \in Bx_1$ (Recall that $Bx_1 = \{h \in H : x_1 h \in B\}$). Thus $y \in C_x$ iff $y_1 = \infty$ and $y_{(1)} \in (T^{-1}Bx_1)_{x(1)}$. Thus

$$\begin{aligned} \sigma'(C) &= \textstyle\int \sigma_\lambda(C_x)d\sigma_\mu(x) \\ &= \lambda(\infty) \textstyle\int \sigma_\lambda(T^{-1}Bx_1)_{x(1)}d\sigma_\mu(x) \\ &= \lambda(\infty) \textstyle\int \sigma'(T^{-1}Bx_1)d\mu(x_1) \\ &= \lambda(\infty) \textstyle\int \sigma'(T^{-1}B_i)d\mu(i) \end{aligned}$$

where in the third equality we used C(i) in §0.1 applied to the function (instead of sets), $f(x) = \sigma_\lambda(T^{-1}B)_x$. For the last equality note that $Bi = B_i$.

$\underline{5}^o$. For any clopen set $\Gamma \subset H$, $\sigma'(T^{-1}\Gamma) = \sigma_\gamma(\Gamma)$.

The proof is by induction on the rank of the clopen set Γ. Of course if $\Gamma = \phi$ or H this is trivial. Suppose then that $\Gamma = \cup(i\Gamma_i)$ and $\sigma_\gamma(\Gamma_i) = \sigma'(T^{-1}\Gamma_i)$. We show that $\sigma_\gamma(\Gamma) = \sigma'(T^{-1}\Gamma)$.

$$\begin{aligned}
\sigma_\gamma(\Gamma) &= \int \sigma_\gamma(\Gamma i)d\gamma(i) && \text{(by } C(i) \text{ in §0.1)} \\
&= \int \sigma_\gamma(\Gamma_i)d\gamma(i) && (since,\ \Gamma i = \Gamma_i) \\
&= \textstyle\sum_i \sigma_\gamma(\Gamma_i)\lambda(i) + \lambda(\infty)\int \sigma_\gamma(\Gamma_i)d\mu(i) \\
&= \textstyle\sum_i \sigma'(T^{-1}\Gamma_i)\lambda(i) + \lambda(\infty)\int \sigma'(T^{-1}\Gamma_i)d\mu(i) \\
&\quad \text{(by induction on the rank of } \Gamma) \\
&= \sigma'(\cup_i i T^{-1}\Gamma_i) + \sigma'\{(x,y) : y_1 = \infty \quad \& \quad T(x,y) \in \Gamma\} \\
&\quad \text{(by } 3^o \text{ and } 4^o) \\
&= \sigma'(T^{-1}\Gamma) && (\sigma' \text{ being finitely additive})
\end{aligned}$$

$\underline{6}^o$. For any open set $U \subset H$, $\sigma'(T^{-1}U) \geq \sigma_\gamma(U)$.

Indeed fix $\epsilon > 0$ and by the regularity property of strategic probability (see §0.1), get clopen $\Gamma \subset U$ with $\sigma_\gamma(U) \leq \sigma_\gamma(\Gamma) + \epsilon$. Note that $T^{-1}\Gamma \subset T^{-1}U$ so that

$$\sigma_\gamma(U) \leq \sigma_\gamma(\Gamma) + \epsilon = \sigma'(T^{-1}\Gamma) + \epsilon \quad \text{by } 5^o$$

$$\leq \sigma'(T^{-1}U) + \epsilon$$

ϵ being arbitrary this proves the stated inequality.

$\underline{7}^o$. If $V \subset H^1$ is open and $\epsilon > 0$ then there is a clopen set $C \subset V$ such that $\sigma'(V) \leq \sigma'(C) + \epsilon$.

To see this, fix clopen sets $L_i \subset H$ and $M_i \subset H_\infty$ such that $V = \cup_i(L_i \times M_i)$. Put $V_n = \cup_{i<n}(L_i \times M_i)$ so that V_n are clopen and $V_n \uparrow V$. Set

$$A_n = \{x : \sigma_\lambda((V_n)_x) > \sigma_\lambda(V_x) - \frac{\epsilon}{2}\}$$

As σ_λ is countably additive, $A_n \uparrow H$ and hence $\sigma_\mu(\cup_n A_n) = 1$. By (P4) in §0.1 get a stoptime τ with $\sigma_\mu(A_\tau) > 1 - \epsilon/2$. Define

$$\begin{aligned}
C &= \{(x,y) : (x,y) \in V_{\tau(x)}\} \\
&= \cup_n[V_n \cap \{(x,y) : \tau(x) = n\}]
\end{aligned}$$

Then $C \subset V$, C is clopen. Note that if $x \in A_\tau$ then $\sigma_\lambda(C_x) \geq \sigma_\lambda(V_x) - \epsilon/2$. Moreover for any x, $\sigma_\lambda(C_x) \geq \sigma_\lambda(V_x) - 1$. We will use the last inequality for $x \notin A_\tau$. As a result

$$\begin{aligned} \sigma'(C) &= \int \sigma_\lambda(C_x) d\sigma_\mu(x) \\ &= \int_{A_\tau} \sigma_\lambda(C_x) d\sigma_\mu(x) + \int_{A_\tau^c} \sigma_\lambda(C_x) d\sigma_\mu(x) \\ &\geq \int \sigma_\lambda(V_x) d\sigma_\mu(x) - \tfrac{\epsilon}{2}\sigma_\mu(A_\tau) - \sigma_\mu(A_\tau^c) \\ &\geq \sigma'(V) - \epsilon \end{aligned}$$

To proceed further we introduce a hypothesis, which will be relaxed later.

($\perp$) : $\underline{\mathrm{N} = \mathrm{N}_1 \cup \mathrm{N}_2,\ \mathrm{N}_1 \cap \mathrm{N}_2 = \emptyset,\ \mu(\mathrm{N}_1) = 1}$, $\lambda(\mathrm{N}_2 \cup \{\infty\}) = 1$.

8^o. Assume ($\perp$). For any open $U \subset H$; $\sigma'(T^{-1}U) \leq \sigma_\gamma(U)$.

To see this, temporarily denote by H_1, the set of sequences of points from N_1. Denote by H_2, the set of sequences of points from $\mathrm{N}_2 \cup \{\infty\}$ with infinitely many occurrences of ∞. Set $D = H_1 \times H_2 \subset H^1$. Let R be the set of sequences of points of N having infinitely many occurrences of elements from N_1. Then by the usual SLLN, $\sigma_\lambda(H_2) = 1$.

Consequently $\sigma'(D) = 1$. By strategic SLLN—see (P5) in §0.1—$\sigma_\gamma(R) = 1$. Moreover T is a homeomorphism on D onto R. Fix $\epsilon > 0$. $T^{-1}U$ being open in H^1, use 7^o to get a clopen set $\Gamma \subset T^{-1}U$ with

$$\sigma'(T^{-1}U) \leq \sigma'(\Gamma) + \epsilon$$

Note that

$$\begin{aligned} \sigma'(\Gamma) &= \sigma'(\Gamma \cap D), \qquad \text{as } \sigma'(D) = 1 \\ &\leq \sigma'(T^{-1}T(\Gamma \cap D)) \end{aligned}$$

As Γ is closed in H^1, $\Gamma \cap D$ is closed in D so that $T(\Gamma \cap D)$ is closed in R. Say $T(\Gamma \cap D) = C \cap R$ where C is closed in H.

$$\begin{aligned}
\sigma'(\Gamma) &\leq \sigma'(T^{-1}(C \cap R)) && \\
&\leq \sigma'(T^{-1}C) && \\
&\leq \sigma_\gamma(C) && \text{by} \quad 6^o \\
&= \sigma_\gamma(C \cap R) && \text{as} \quad \sigma_\gamma(R) = 1 \\
&= \sigma_\gamma(T(\Gamma \cap D)) && \\
&\leq \sigma_\gamma(U) && \text{as} \quad \Gamma \subset T^{-1}U
\end{aligned}$$

Thus $\sigma'(T^{-1}U) \leq \sigma_\gamma(U) + \epsilon$. Since ϵ is arbitrary we are done.

Combining 6^o and 8^o we immediately obtain:

$\underline{9}^o$. Assume ($\perp$). For any open $U \subset H$, $\sigma'(T^{-1}U) = \sigma_\gamma(U)$.

This leads us to a special case of the theorem, which we state as a Lemma.

Lemma 3.2.2 *Assume* ($\perp$). *For any Borel* $B \subset H$. $\sigma'(T^{-1}B) = \sigma_\gamma(B)$.

Proof : Suffices to show that $\sigma_\gamma(B) \geq \sigma'(T^{-1}B)$. To this end, fix $\epsilon > 0$. By C(ii) in §0.1, take open $U \supset B$ with $\sigma_\gamma(B) \geq \sigma_\gamma(U) - \epsilon$. Then we have,

$$\begin{aligned}
\sigma_\gamma(B) &\geq \sigma_\gamma(U) - \epsilon & \\
&= \sigma'(T^{-1}U) - \epsilon & \text{by } 9^0 \\
&\geq \sigma'(T^{-1}B) - \epsilon & \text{as } T^{-1}U \supset T^{-1}B
\end{aligned}$$

Since ϵ is arbitrary the proof is complete. ■

To remove the assumption ($\perp$) we need to work a little more. Suppose o is a map on $\mathbb{Z}$ onto $\mathbb{N}$. Let η be a finitely additive probability on $\mathbb{Z}$ and γ be defined on $\mathbb{N}$ by $\gamma(A) = \eta(o^{-1}A)$. Let σ_η and σ_γ be the strategic measures on $\mathbb{Z}^\infty$ and $\mathbb{N}^\infty = H$ respectively. Define o_∞ on $\mathbb{Z}^\infty$ by

$$o_\infty(x_1, x_2, \cdots) = (o(x_1), o(x_2), \cdots).$$

It is natural to expect that as in the countably additive case, $\sigma_\gamma(B) = \sigma_\eta(o_\infty^{-1}B)$ for all Borel sets $B \subset H$. We have not been able to establish this. We show that if o is a finite-to-one map — for every i, $o^{-1}\{i\}$ is finite — then this is indeed correct. The general proof eludes us.

Lemma 3.2.3 *Let o be a finite-to-one map. Then for every Borel set $B \subset H$*

$$\sigma_\eta(o_\infty^{-1}(B)) = \sigma_\gamma(B)$$

Proof : Since both σ_η and σ_γ are strategic measures, it is easy to see by using induction on the rank of clopen sets, the desired equality holds when B is a clopen set in H. Proceeding as in 6^o we can establish that $\sigma_\gamma(U) \leq \sigma_\eta(o_\infty^{-1}U)$ for all open sets $U \subset H$. Note that o_∞ takes open sets to open sets. As o is finite-to-one, a simple argument shows that o_∞ takes closed sets to closed sets as well. Thus for any clopen set $\Gamma \subset \mathbb{Z}^\infty$, $o_\infty(\Gamma)$ is a clopen set in H. Now take any open set $U \subset H$ and fix $\epsilon > 0$. Then there exists a clopen set $\Gamma \subset o_\infty^{-1}(U)$ with $\sigma_\eta(o_\infty^{-1}U) \leq \sigma_\eta(\Gamma) + \epsilon$. Note that $\sigma_\eta(\Gamma) \leq \sigma_\eta(o_\infty^{-1}o_\infty\Gamma) = \sigma_\gamma(o_\infty\Gamma) \leq \sigma_\gamma(U)$ where the equality is a consequence of the fact that $o_\infty\Gamma$ is clopen. ϵ being arbitrary we deduce that $\sigma_\eta(o_\infty^{-1}U) \leq \sigma_\gamma(U)$. This establishes the result for B an open set. Now proceed as in Lemma 3.2.2 to complete the proof. ∎

Proof of Theorem 3.2.1 : The main idea is to shift the purely finitely additive part of γ to the negative integers so that $(\perp)$ holds. Then we will apply Lemma 3.2.2. Finally we bring back the mass from negative integers. These three steps are achieved with the help of the maps ψ_∞, $\overline{T}$ and o_∞ as detailed below.

Given γ on $\mathbb{N}$ define $\overline{\gamma}$ on $\mathbb{Z}$ by $\overline{\gamma}(A) = \gamma_1(A) + \gamma_2(-A)$ where $-A = \{-x : x \in A\}$. Define $\overline{\mu}$ and $\overline{\lambda}$ for $\overline{\gamma}$ just as μ, λ were defined for γ. Define σ'' on $\mathbb{Z}^\infty \times \mathbb{Z}^\infty_\infty$ with $\sigma_{\overline{\mu}}$ and $\sigma_{\overline{\lambda}}$ just as σ' was defined on $\mathbb{N}^\infty \times \mathbb{N}^\infty_\infty$ with σ_μ and σ_λ.

Define the map $\psi_\infty : \mathbb{N}^\infty \times \mathbb{N}^\infty_\infty \to \mathbb{Z}^\infty \times \mathbb{Z}^\infty_\infty$ by

$$\psi_\infty(x, y) = (-x, y)$$

where $-x = (-x_1, -x_2, \ldots)$ if $x = (x_1, x_2, \ldots)$. Then it is immediate that $\sigma''(B) = \sigma'(\psi_\infty^{-1} B)$ for each Borel set $B \subset \mathbb{Z}^\infty \times \mathbb{Z}^\infty_\infty$.

Define the map $\overline{T} : \mathbb{Z}^\infty \times \mathbb{Z}^\infty_\infty \to \mathbb{Z}^\infty$ just as T was defined from $\mathbb{N}^\infty \times \mathbb{N}^\infty_\infty$ to $\mathbb{N}^\infty$. Lemma 3.2.2 applies now to yield that $\sigma_{\overline{\gamma}}(B) = \sigma''(\overline{T}^{-1} B)$ for each Borel set $B \subset \mathbb{Z}^\infty$. Lemma 3.2.2, though stated for $\mathbb{N}$ applies to $\mathbb{Z}$ as well. Finally, define the map $\phi : \mathbb{Z} \to \mathbb{N}$ by $\phi(x) = \mid x \mid$. Lemma 3.2.3 now yields that for each Borel $B \subset \mathbb{N}^\infty$, $\sigma_\gamma(B) = \sigma_{\overline{\gamma}}(\phi_\infty^{-1} B)$.

Thus for each Borel $B \subset H$, $\sigma_\gamma(B) = \sigma_{\overline{\gamma}}(\phi_\infty^{-1} B) = \sigma''(\overline{T}^{-1} \phi_\infty^{-1} B) = \sigma'(\psi_\infty^{-1} \overline{T}^{-1} \phi_\infty^{-1} B)$. Observe that $T = \phi_\infty \circ \overline{T} \circ \psi_\infty$ to complete the proof. ■

3.3 Hewitt-Savage $0 - 1$ Law

A permutation π of $\{1, 2, \ldots\}$ is called a finite permutation if $\pi(n) = n$ for all sufficiently large n. If $h = (h_1, h_2, \ldots) \in H$ and π is a finite permutation then $h_\pi = (h_{\pi(1)}, h_{\pi(2)}, \ldots)$. A Borel set $B \subset H$ is called symmetric if $h_\pi \in B$ whenever $h \in B$ and π is a finite permutation.

Let γ be a finitely additive probability on $\mathbb{N}$ and as usual σ_γ the strategic measure on $H = \mathbb{N}^\infty$ induced by the i.i.d. strategy γ. Let $A = \{i \in \mathbb{N} : \gamma(i) = 0\}$. Suppose γ restricted to A takes more than two values. Say $A_o \subset A$ and $0 < \gamma(A_o) < \gamma(A)$. Then, generalizing a construction of [29] we can exhibit a symmetric Borel set $S \subset H$ such that $0 < \sigma_\gamma(S) < 1$ as follows : Let H_1 be the subset of H consisting of those histories in which elements of A occur at infinitely many coordinate places and they occur in increasing order of magnitude. Note that $\sigma_\gamma(H_1) = 1$. Let S_1 be the subset of H_1 consisting of those histories in which the first occurrence from A is from A_o. Define for all $n \geq 1$, $B_n = \{h \in H_1 : h_i \notin A \text{ for } i < n \text{ and } h_n \in A_o\}$. Then $S_1 = \cup_{n \geq 1} B_n$. Direct computation shows that for $n \geq 1$, $\sigma_\gamma(B_n) = [1-\gamma(A)]^{n-1}\gamma(A_o)$ so that $\sigma_\gamma(S_1) \geq \frac{\gamma(A_o)}{\gamma(A)}$. Similar computation shows $\sigma_\gamma(H_1 \backslash S_1) \geq \frac{\gamma(A \backslash A_o)}{\gamma(A)}$. It follows that equality must hold at both the places. Thus if S is the symmetrization of S_1 then $0 < \sigma_\gamma(S) = \sigma_\gamma(S_1) < 1$. If $\beta = \max\{\frac{\gamma(A_o)}{\gamma(A)}, \frac{\gamma(A \backslash A_o)}{\gamma(A)}\}$ then we have a decomposition of H into two symmetric sets, each having σ_γ measure $\leq \beta$. Since $\beta < 1$ and the above construction can be extended by taking into account the first finite number of occurrences of points of A, we get the following : given $\epsilon > 0$ there is a decomposition of H into symmetric sets each having σ_γ measure $\leq \epsilon$.

Now suppose γ restricted to A is trivial – that is γ assumes at most two values on subsets of A. In this case we show that the Hewitt-Savage $0-1$ law holds. Main idea is the following : Suppose $S \subset H$ is a symmetric Borel set. Then for $x \in H$, $(T^{-1}S)_x$ is a symmetric Borel set in H_∞ so that $\sigma_\lambda(T^{-1}S)_x = 0$ or 1. Let $E = \{x : \sigma_\lambda((T^{-1}S)_x) = 1\}$. In case γ is countably additive on A^c then of course $\gamma \mid A$ is its purely finitely additive part and hence μ is $0-1$ valued, so is σ_μ. Thus $\sigma_\mu(E)$ is either 0 or 1. Accordingly $\sigma_\gamma(S)$ is 0 or 1. The problem becomes more difficult if γ is not countably additive on A^c or, equivalently, if γ_1 and γ_2 are not supported by disjoint sets. In that case μ may not be two valued even though $\mu|A$ is so and therefore the above argument is not applicable. But observe that we are interested only in the value of $\sigma_\mu(E)$ and if we can show that $\sigma_\mu(E)$ is either 0 or 1 we are done. That is what we are going to show next. If $A = \phi$, that is, if γ gives positive mass to every singleton then a simple calculation shows that E is a tail set in H so that $\sigma_\mu(E)$ is either 0 or 1 (see P6 in §0.1) as we want.

But in general situation when A is not necessarily empty E may not be a tail set. However we can apply Levy 0-1 law (see P7 §0.1) to conclude that $\sigma_\mu(E) = 0$ or 1. In the countably additive case the martingale theoretic proof of the Hewitt-Savage $0 - 1$ law is well known. See [27]. Here is our main Theorem :

Theorem 3.3.1 *Let γ be a finitely additive probability on* $\mathbb{N}$ *and* $A = \{i : \gamma(i) = 0\}$.

a) If γ restricted to A is trivial (assumes at most two values) then for any symmetric Borel set $S \subset H$. $\sigma_\gamma(S)$ is either 0 or 1.

b) If γ restricted to A is nontrivial (assumes more than two values) then for any $\epsilon > 0$ there is a finite partition of H into symmetric Borel sets each having σ_γ measure $< \epsilon$.

<u>Proof</u>. Part (b) was already established above. We shall prove (a). We can and shall assume that γ is not countably additive. We use the notation of §2. In particular μ, λ, σ', T are as discussed there. Let $S \subset H$ be a symmetric Borel set.

We first observe that for $x \in H$, $(T^{-1}S)_x$ is a symmetric set in H_∞. Let $y = (y_1, y_2, \ldots) \in (T^{-1}S)_x$. Let $i < j$. Let $\tilde{y}$ be obtained by permuting the coordinates y_i and y_j in y. We show that $\tilde{y} \in (T^{-1}S)_x$.

If both y_i and y_j are ∞ then of course $\tilde{y} = y \in (T^{-1}S)_x$. In the other case, a simple calculation shows that, $T(x, \tilde{y})$ is obtained by a finite permutation of $T(x, y)$ so that $T(x, \tilde{y}) \in S$ and hence $\tilde{y} \in (T^{-1}S)_x$. As a consequence, for each $x \in H$, $\sigma_\lambda((T^{-1}S)_x)$ is either 0 or 1. Let now

$$E = \{x \in H : \sigma_\lambda((T^{-1}S)_x) = 1\}$$

Here are some properties of E.

1°. If $x^0 = (x_1^0, x_2^0, \ldots) \in E$, $k \geq 1$, $x_k^0 \in A^c$ and $\tilde{x^0}$ is obtained from x^0 by deleting x_k^0, then $\tilde{x^0} \in E$.

To show $\tilde{x^0} \in E$ we only need to show that $\sigma_\lambda((T^{-1}S)_{\tilde{x^0}}) > 0$.

Let $M = \{y \in H_\infty \ : \ y_i = \infty \ \text{ for all } \ i \leq k\}$. Then $\sigma_\lambda(M) > 0$ and hence so is $\sigma_\lambda(M_o)$ where $M_o = (T^{-1}S)_x \cap M$. Let

$$M_1 = \{y \in H_\infty : y_k = x_k^0 \ \& \ \overline{y} \in M_o\}$$

where

$$\begin{aligned} \overline{y}_i &= y_i \text{ if } i \neq k \\ &= \infty \text{ if } i = k \end{aligned}$$

Then $\sigma_\lambda(M_1) > 0$ and it is easy to verify that $M_1 \subset (T^{-1}S)_{\tilde{x^0}}$.

2°. If $x = (x_1, x_2, \ldots) \in E$; $k \geq 1$; $a \in A^c$ and $\tilde{x}$ is obtained from x by inserting a just before x_k then $\tilde{x} \in E$.

Proceed as earlier and take

$$M = \{y \in H_\infty : y_i = \infty \ \text{ for } \ i \leq k-1, \ y_k = a\}$$

and

$$M_1 = \{y \in H_\infty : y_k = \infty \ \& \ \overline{y} \in M_o\}$$

where

$$\begin{aligned} \overline{y}_i &= y_i \quad \text{if } i \neq k \\ &= a \quad \text{if } i = k \end{aligned}$$

Now to show that $\sigma_\mu(E)$ is either 0 or 1 we argue as follows :

For any $p \in Seq$, let $\mid p \mid$ denote the length p. Now fix a $p \in Seq$ and let its length be n. We have the following relation:

$$\sigma_\mu(Ep) = \int_{i \in A^c} \sigma_\mu(Epi)d\mu(i) + \int_{i \in A} \sigma_\mu(Epi)d\mu(i)$$

For $i \in A^c$, $Epi = Ep$ by 1^o and 2^o. So we have

$$\sigma_\mu(Ep)\mu(A) = \int_{i \in A} \sigma_\mu(Epi)d\mu(i)$$

Since μ is two-valued on A, this implies that $\mu\{i \in A : \mid \sigma_\mu(Ep) - \sigma_\mu(Epi) \mid > \frac{\epsilon}{2^{n-1}}\} = 0$ where $\epsilon > 0$ is any arbitrary number fixed beforehand. (Recall that $n = \mid p \mid$). As noted above, $Ep = Epi$ for all $i \in A^c$. Thus,

$$\mu\{i : \mid \sigma_\mu(Ep) - \sigma_\mu(Epi) \mid > \frac{\epsilon}{2^{n-1}}\} = 0.$$

Let I_p denote the set in braces and K_p the set of all those histories whose first coordinate is in I_p so that $\sigma_\mu(K_p) = 0$. This is all done for a fixed $p \in Seq$. Now having done this for each fixed $p \in Seq$ define

$$F_k = \cup_{\mid p \mid = k}(pK_p)$$

and

$$F = \cup_{k=0}^{\infty} F_k$$

Note that $Ep = E$ when $\mid p \mid = 0$.

Also observe that $\sigma_\mu(F_k p_k(x)) = 0$ for all k and x. Now by property P2 in §0.1 we have

$$\sigma_\mu(F) = 0.$$

This can be restated as,

3^o. $\sigma_\mu\{x : \forall n \geq 0 \mid \sigma_\mu(Ep_n(x)) - \sigma_\mu(Ep_{n+1}(x)) \mid \leq \frac{\epsilon}{2^{n-1}}\} = 1$

To complete the proof of the theorem fix $x \in F^c$ such that $\sigma_\mu(Ep_n(x)) \to 1_F(x)$. This is possible by Levy $0-1$ law (see P7 §0.1). Combining this with 3^o we get $|\sigma_\mu(E) - 1_F(h)| \leq \epsilon$. If, to start with, $0 < \sigma_\mu(E) < 1$, then an appropriate choice of ϵ would give rise to a contradiction. ■

3.4 Remarks.

1. Lemma 3.2.3 of §2 is perhaps true for general ϕ not necessarily finite-to-one valued.

2. There is perhaps a more direct way to prove Theorem 3.3.1 without going through the detour as we did.

3. S. Ramakrishnan [30] proved that if S is a G_δ set in H which is a countable intersection of symmetric open sets then $\sigma_\gamma(S)$ is either zero or one for any γ. It is quite likely that for a large class of sets the $0-1$ law holds whatever be γ.

   To be more precise, we can define a hierarchy of symmetric Borel sets as follows: Let

$$S_1(H) = \{U \subset H : U \text{ open and symmetric}\}$$

$$P_1(H) = \{F \subset H : F \text{ closed and symmetric}\};$$

   and for any countable ordinal α, define by induction,

$$S_\alpha(H) = (\cup_{\beta<\alpha} P_\beta(H))_\sigma$$

   and

$$P_\alpha(H) = (\cup_{\beta<\alpha} S_\beta(H))_\delta$$

Recall that for a class of sets $\mathcal{A}$, $\mathcal{A}_\sigma$ denotes the class $\{A : A = \cup_{n=1}^{\infty} A_n, A_n \in \mathcal{A}\}$ and $\mathcal{A}_\delta$ denotes the class $\{A : A = \cap_{n=1}^{\infty} A_n, A_n \in \mathcal{A}\}$

Ramakrishnan proved in his thesis that whatever be γ, sets in $P_\alpha(H)$ (and hence in $S_\alpha(H)$) have σ_γ probability 0 or 1, if $\alpha \leq 2$. Now a natural question that arises is, whether the sets in $P_\alpha(H)$ enjoy the same property even for $\alpha > 2$. We don't know the answer. It is worth noting that this hierarchy does not include all symmetric Borel sets. To see this, let $\overline{\mathcal{S}}$ be the σ-field generated by $\{S_\alpha(H) \cup P_\alpha(H)\}$. Then $\overline{\mathcal{S}}$ is countably generated—as pointed out to us by S.M.Srivastava—since the class of symmetrized basic open sets generates this σ-field. But clearly $\mathcal{S}$, the σ-field of symmetric Borel sets is not countably generated. Therefore, $\mathcal{S}$ strictly contains $\overline{\mathcal{S}}$. In other words there are symmetric Borel sets B such that B cannot be written in the form of countable union or intersection of the sets from $P_\alpha(H)$ and $S_\alpha(H)$.

4. The sets of interest in the context of random walks are G_δ sets of the form $S = \{h : \sum_1^n h_i \in A \ i.o.\}$ for some $A \subset \mathbb{N}$ (see [30], [33], and also, chapter 2). It is interesting to note that such a set S is a countable intersection of symmetric open sets iff A is of the form $\{i : i \geq k\}$ for some k. Indeed, suppose $S = \cap U_n$, where U_n is symmetric open for all n. Let $i \in A$. Then $(i, 0, 0, \ldots) \in S$ and hence in U_n for all n. Since U_n is open there exists a basic open set which contains $(i, 0, 0, \ldots)$ and is a subset of U_n. In other words, there exists an m, depending on n, such that $(\underbrace{i, 0, 0, \ldots, 0}_{m})H \subset U_n$. Therefore, $(\underbrace{i, 0, 0, \ldots, 0}_{m}, 1, 0, 0, \ldots) \in U_n$.

Since U_n is symmetric as well, after performing a finite number of permutations we have $(1, i, 0, \ldots) \in U_n$. This being true for all n, the point $(1, i, 0, \ldots) \in S$. Therefore, $i + 1 \in A$. If k denotes the least element in A then A is of the form we described above. But in that case S is already open.

5. Let $A = \{i : \gamma(i) = 0\}$. If $\gamma \mid A$ takes more than two values then the Hewitt-Savage $0-1$ law does not hold. However the Kolmogorov $0-1$ law holds. As a consequence in this case there are symmetric sets which are not equivalent to any tail set under σ_γ. Of course if $\gamma \mid A$ is atmost two-valued then the Hewitt-Savage $0-1$ law holds. So trivially, any symmetric set is equivalent to a tail set under σ_γ. Thus tail σ-field is σ_γ equivalent (in the obvious sense) to the symmetric σ-field iff $\gamma \mid A$ is atmost two-valued.

6. If A_n consists of all histories with at least one coordinate smaller than n then A_n is a symmetric set and $A_n \uparrow H$. If γ is diffuse (that is every singleton gets γ mass zero) then clearly $\sigma_\gamma(A_n) = 0$ showing that σ_γ is not countably additive on the symmetric σ field. Contrast this with the fact that σ_γ is countably additive on the tail σ field (Theorem 2 of [29]). It can be shown that σ_γ is countably additive on the symmetric σ field iff $\gamma\{i : \gamma(i) = 0\} = 0$. Towards proving this, first let $\gamma\{i : \gamma(i) = 0\} > 0$. As before, we call the set in braces A. Enumerate A as $\{i_1, i_2, \ldots\}$. Let A_{i_n} consist of all histories which have at least one cordinate coming from A and minimum of them $\leq i_n$. Then clearly A_{i_n} is symmetric. Let $\overline{A}$ be the set of all histories which have no coordinate coming from A. Then $\overline{A}$ is also symmetric and by strategic SLLN (see P5 in §0.1), $\sigma_\gamma(\overline{A}) = 0$.

Obviously, $A_{i_n} \uparrow \overline{A}^c$. Since γ is diffuse on A, $\sigma_\gamma(A_{i_n}) = 0$ but $\sigma_\gamma(A^c) = 1$ showing that σ_γ is not countably additive on the symmetric σ-field. Conversely, if $\gamma(A) = 0$, without loss of generality we may assume $\gamma(i) > 0$ for all i. We will show that σ_γ is countably additive on the symmetric σ-field. Let $B_1, B_2, \ldots$ be a sequence of symmetric sets. It suffices to show that

$$\sigma_\gamma(B_i) = 0 \ \forall\, i \longrightarrow \sigma_\gamma(\cup B_i) = 0.$$

For that it suffices to show, by P2 in §0.1, that $\forall\ p \in Seq$, $\sigma_\gamma(B_ip) = 0$. If possible suppose, $\sigma_\gamma(B_ip) > 0$ for some i and some $p = (p_1, p_2, \ldots, p_n)$. But then, $\sigma_\gamma(B_i) \geq \gamma(p_1)\gamma(p_2)\ldots\gamma(p_n)\sigma_\gamma(B_ip)$ as $B_i \supset pB_ip$. Since $\gamma(p_i) > 0$ by our assumption, $\sigma_\gamma(B_i) > 0$ which is a contradiction.

7. As noted earlier, $\mathbb{N}$ is taken for convenience but the theorems hold good for any countable set.

Chapter 4

Completeness of L_p-Spaces

4.1 Introduction

The fact that L_p-spaces over countably additive measures are nice Banach spaces has various useful applications. But finitely additive measures are not so well-behaved. Eventhough the L_p norms are defined in the usual way when the underlying measure is finitely additive, in general L_p-spaces fail to be complete. In [16] Green gave necessary and sufficient conditions for $L_p(X, \mathcal{F}, \mu)$, $1 \leq p < \infty$, to be a complete normed linear space for a positive bounded finitely additive measure space $(X, \mathcal{F}, \mu)$. But in [1], V. Aversa and K.P.S.B.Rao have shown that the necessity part of Green's result is not correct. In §3 of this chapter we give a complete solution of this problem. In §4 we consider the completeness of L_∞-spaces. §5 gives an application of the completeness of L_p-spaces. In §6-§11, we study in some detail the completeness of L_p-spaces supported on a general probability space and analyse its interrelation with the structure of the underlying probability γ. This study yields a number of interesting results which we believe are useful. These sections are concerned with the completeness of the space $L_1(\gamma)$ of integrable functions over a finitely additive nonnegative bounded measure γ defined on a σ field of subsets of a set. In the mathematical literature ([2], [8], [21], [36]) the domain for such a γ is a field of sets. Our interest is not in finitely additive measures per se, but in the development of probability. The natural domain for γ then is a σ field and that is what we take in those sections.

4.2 Preliminaries

We need the following definitions and notations in the sequel. We will consider a measure space $(X, \mathcal{F}, \mu)$. For the definitions of measure space, measurable functions (random variables), integrable functions and convergence in measure see preliminaries in Chapter 1. Only we should bear in mind that in §3 and §4 of this chapter we will consider general finitely additive measures, not just probabilities.

In what follows, all measures μ considered are assumed to be nonnegative. But that restriction is superfluous in case of the results obtained in §3, §4 and §5 because the defition of L_p-spaces over a general measure space involves only the total variation of the measure. For $f, g : X \to \Xi$ we write $f \sim g$ if $f = g$ a.s. $[\mu]$ i.e. if $\mu^*\{x : |f(x) - g(x)| > \epsilon\} = 0$ for all $\epsilon > 0$.

Denote by $\mathcal{L}_0(\mu)$ the linear space of all measurable functions and let $L_0(\mu) = \mathcal{L}_0(\mu)/\sim$.

For $0 < p < \infty$ put

$$\mathcal{L}_p(\mu) = \{f \in L_0(\mu) : |f|^p \text{ is integrable}\}$$

and $L_p(\mu) = \mathcal{L}_p(\mu)/\sim$.

The space $L_p(\mu)$ is equipped with $||\cdot||_p$ where $||f-g||_p$ defines an invariant metric on $L_p, p > 0$ and is a norm on L_p for $p \ge 1$. $||\cdot||_p$ is defined as follows:

$$||f||_p = \begin{cases} \int |f|^p d\mu & \text{if } 0 < p < 1 \\ (\int |f|^p d\mu)^{1/p} & \text{if } 1 \le p < \infty \end{cases}$$

For $p = 0$, we say $f_n \to f$ in L_p if

$$\forall\ \epsilon > 0\ \ \mu^*(|f_n - f| > \epsilon) \to 0.$$

In other words, $f_n \to f$ in L_0 is, by definition, equivalent to the concept of $f_n \to f$ in μ measure.

A function $f : X \to \mathbb{R}$ is called essentially bounded if there exists a positive real number k such that

$$\mu^*\{x : |f(x)| > k\} = 0.$$

Denote by $\mathcal{L}_\infty(\mu)$ the linear space of all essentially bounded measurable functions on X and put

$$L_\infty(\mu) = \mathcal{L}_\infty(\mu)/\sim .$$

The space $L_\infty(\mu)$ is equipped with the norm

$$||f||_\infty = inf\{k > 0 : \mu^*\{x : |f(x)| > k\} = 0\}.$$

A sequence $\{A_n\}$, where $A_n \subset X$, is said to be μ-Cauchy if $\mu^*(A_n \Delta A_m) \to 0$ as $n, m \to \infty$.

4.3 Completeness of L_p, $0 \le p < \infty$

We begin this section by proving three lemmas which will be useful in proving our main theorem, Theorem 4.3.4.

Lemma 4.3.1. *If $A_n \subset X$ and the sequence $\{I_{A_n}\}$ converges to f in μ, then $f = I_A$ a.s. $[\mu]$ for some $A \subset X$ and $\mu^*(A_n \Delta A) \to 0$.*

Proof: We shall show that there is a set $A \subset X$ such that

$$\mu^*\{x : |f(x) - I_A(x)| > 1/k\} = 0 \ \forall \ k \geq 1.$$

Consider

$$B_k = \{x : f(x) \in (-\infty, -1/k) \cup (1/k, 1 - 1/k) \cup (1 + 1/k, \infty)\}, \ k > 3.$$

Since $B_k \subset \{x : |f(x) - I_{A_n}(x)| > 1/k\}$ for all n, we have $\mu^*(B_k) = 0$. Let $A = \{x : 1/2 < f(x) < 1 + 1/2\}$. Then $\{x : |f(x) - I_A(x)| > 1/k\} \subset B_k$ for all $k \geq 3$. Therefore $\mu^*\{x : |f(x) - I_A(x)| > 1/k\} = 0$ for all $k \geq 1$. Thus $f = I_A$ a.s. $[\mu]$. Hence $\{I_{A_n}\}$ converges to I_A in μ or, equivalently, $\mu^*(A_n \Delta A) \to 0$.

■

Lemma 4.3.2 *Suppose for every μ-Cauchy sequence $\{A_n\} \subset \mathcal{F}$ with $\mu(A_n) < \infty$ there exists $A \subset X$ with $\mu^*(A_n \Delta A) \to 0$. Then for every sequence $\{B_n\} \subset \mathcal{F}$ there exists $B \subset X$ such that*

(i) $\mu^*(B_n \setminus B) = 0$ for all n

(ii) $\mu^*(B) \leq \sum_{n=1}^{\infty} \mu(B_n)$

Proof: If $\sum_{n=1}^{\infty} \mu(B_n) = \infty$ the assertion of the lemma is trivially true with $B = \cup B_n$. So without loss of generality assume that $\sum_{n=1}^{\infty} \mu(B_n) < \infty$.

Let $A_k = \cup_{n=1}^{k} B_n$. Then $\mu(A_k) < \infty$ and $\{A_k\}$ is a μ-Cauchy sequence in $\mathcal{F}$ since $\mu(A_k \Delta A_{k+1}) \leq \mu(B_{k+1})$ and $\mu(A_k \Delta A_{k+l}) \leq \sum_{n=k}^{k+l-1} \mu(A_n \Delta A_{n+1})$. Take $B \subset X$ with $\mu^*(A_k \Delta B) \to 0$. Now, $\mu^*(A_k \setminus B) \leq \mu^*(A_k \Delta B)$. But $\mu^*(A_k \setminus B)$ is an increasing sequence of non-negative real numbers. It follows that $\mu^*(A_k \setminus B) = 0$ for all k. This yields (i).

For (ii), notice that

$$\begin{aligned}\mu^*(B) &\leq \mu^*(A_k \cup A_k\Delta B)\\ &\leq \mu(A_k) + \mu^*(A_k\Delta B)\\ &\leq \textstyle\sum_{n=1}^{k} \mu(B_n) + \mu^*(A_k\Delta B)\end{aligned}$$

Since $\mu^*(A_k\Delta B) \to 0$ we get (ii). ■

Lemma 4.3.3. *Let $0 < p < \infty$ and let $\{f_n\}$ be a Cauchy sequence in $L_p(\mu)$ which converges to f in μ. Then $f \in L_p(\mu)$ and $||f_n - f||_p \to 0$.*

Proof: We shall show that if $\{f_n\}$ is a Cauchy sequence in $L_p(\mu)$, $0 < p < \infty$, then it satisfies the following two conditions:

(i) The measures λ_n on $\mathcal{F}$ defined as

$$\lambda_n(F) = \int_F |f_n|^p d\mu, F \in \mathcal{F}$$

are uniformly absolutely continuous with respect to μ, i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that $\lambda_n(E) < \epsilon$ for all n whenever $\mu(E) < \delta$.

(ii) For each $\epsilon > 0$, there exists $E_\epsilon \in \mathcal{F}$ such that $\mu(E_\epsilon) < \infty$ and $\lambda_n(E_\epsilon^c) < \epsilon$ for all n.

The assertion follows from this by Theorem 4.6.10 in [2]. (In fact, that theorem is formulated in [2] for $1 \leq p < \infty$ but the proof can be easily adapted to the case where $0 < p < 1$).

We first prove (i) and (ii) for $0 < p < 1$. Fix $\epsilon > 0$. Since $\{f_n\}$ is a Cauchy sequence in $L_p(\mu)$, there exists $N > 1$ such that $\int |f_n - f_m|^p d\mu < \epsilon/2$ for all $n, m \geq N$.

Now,

$$\int_E |f_n|^p d\mu \leq \int_E |f_n - f_N|^p d\mu + \int_E |f_N|^p d\mu \text{ for all } n \geq N$$

Since $f_1, f_2, \ldots, f_N \in L_p(\mu)$, there exists $\delta > 0$ such that $\lambda_1(E), \ldots, \lambda_N(E) < \epsilon/2$ whenever $\mu(E) < \delta$,(see ([2]), Theorem 4.4.13 (xi)).

It follows that $\lambda_n(E) < \epsilon$ whenever $\mu(E) < \delta$ and n is arbitrary. This proves (i) for $0 < p < 1$. With the same notation, there exists $E_\epsilon \in \mathcal{F}$ such that $\mu(E_\epsilon) < \infty$ and $\lambda_n(E_\epsilon^c) < \epsilon/2$ for $n = 1, 2, \ldots, N$ (See [2], Lemma 4.4.15). This yields (ii) for $0 < p < 1$. For $1 \leq p < \infty$ the same argument goes through except that we use the inequality

$$\left(\int_E |f_n|^p d\mu\right)^{1/p} \leq \left(\int_E |f_n - f_N|^p d\mu\right)^{1/p} + \left(\int E|f_N|^p d\mu\right)^{1/p}$$

■

Now, we are ready to prove our main theorem.

Theorem 4.3.4. *Let $0 \leq p < \infty$. Then $L_p(\mu)$ is complete if and only if for every μ-Cauchy sequence $\{A_n\} \subset \mathcal{F}$ with $\mu(A_n) < \infty$ there exists $A \subset X$ with $\mu^*(A_n \Delta A) \to 0$.*

(However it is easy to see that the condition $\mu(A_n) < \infty$ is not essential. We assume it for convenience.)

Proof: Necessity. Let $\{A_n\} \subset \mathcal{F}$ be a μ-Cauchy sequence with $\mu(A_n) < \infty$. Then $\{I_{A_n}\}$ is a Cauchy sequence in $L_p(\mu)$. Hence, by our assumption of completeness, there exists $f \in L_p(\mu)$ such that $||I_{A_n} - f||_p \to 0$. By Theorem 4.6.10 of [2], $\{I_{A_n}\}$ converges to f in μ. This yields, in view of lemma 4.3.1, the desired conclusion.

Sufficiency. Case $p = 0$. Let $\{f_n\}$ be a Cauchy sequence in $L_0(\mu)$. By passing to a subsequence, we may assume that

$$\mu^*\{x \in X : |f_n(x) - f_{n+1}(x)| > 2^{-n}\} < 2^{-n}.$$

Since f_n's are measurable, we may assume, without loss of generality, that f_n's are simple functions to ensure that $\{x \in X :$

$|f_n(x) - f_{n+1}(x)| > 2^{-n}\} \in \mathcal{F}$. Indeed, we can do this without any loss of generality because of the following reasons. $\forall n$, $f_n \in L_0$ and the set of simple functions is dense in L_0. Therefore, given any sequence of positive reals $\{\epsilon_n\}$ with $\epsilon_n \to 0$, we can get a sequence of simple functions $\{s_n\}$ such that $\mu^*(|f_n - s_n| > \epsilon_n) < \epsilon_n$. Easy to see that if $s_n \to f$ in μ then $f_n \to f$ in μ.

Set

$$A_n = \{x \in X : |f_n(x) - f_{n+1}(x)| > 2^{-n}\}$$

Since we have assumed that $f_n's$ are simple functions $A_n \in \mathcal{F}$. Now,

$$\sum_{n=k}^{\infty} \mu(A_n) < \sum_{n=k}^{\infty} 2^{-n} = 1/2^{k-1}, \ \forall\ k.$$

Now, for each fixed $k \geq 1$ consider the sequence $\{A_n\}_{n \geq k}$. Then, by our assumption and lemma 4.3.2 we can get a set $B_k \subset X$ such that

(i) $\mu^*(A_n \setminus B_k) = 0$ for all $n \geq k$

(ii) $\mu^*(B_k) \leq \sum_{n=k}^{\infty} \mu(A_n) < 1/2^{k-1}$

Without loss of generality, we may assume that $B_k \subset B_{k-1}$ for all k (because, otherwise, take B_1, $B_1 \cap B_2$.... etc.) Let $B = \cap_k B_k$. Then clearly, $\mu^*(B) = 0$.

We shall now define our function f to which $\{f_n\}$ converges in μ.

If $x \in B$, define $f(x)$ arbitrarily. If x does not belong to B, let $k(x)$ be the smallest k such that $x \in B_k^c$. Note that if x does not belong to $\cup_{i=n}^{m-1} A_i$ and $m > n$, then

(iii) $|f_n(x) - f_m(x)| \leq \sum_{i=n}^{m-1} 2^{-i} < 1/2^{n-1}$

Now, consider the following two cases:

Case 1. $x \notin \cup_{n \geq k(x)} A_n$. Then, by (iii), $\{f_n(x)\}$ is a Cauchy sequence of real numbers. Define

$$f(x) = lim_{n \to \infty} f_n(x).$$

Case 2. $x \in \cup_{n > k(x)} A_n$. Let $n(x)$ be the smallest $n \geq k(x)$ such that $x \in A_n$. Define $f(x) = f_{n(x)}(x)$.

To prove that $\{f_n\}$ converges in μ to f define

$$C_k = B \cup (\cup_{n=1}^{k} A_n \setminus B_1) \cup (\cup_{n=2}^{k} A_n \setminus B_2) \cup .... \cup (A_k \setminus B_k) \cup B_k$$

In view of (i) and (ii) we have $\mu^*(C_k) \leq \mu^*(B_k) < 1/2^{k-1}$

We claim that x does not belong to C_k implies $|f(x) - f_k(x)| \leq 1/2^{k-1}$.

Indeed, if $x \notin C_k$ we have $k(x) \leq k$. If x is as in case 1, then x does not belong to $\cup_{n \geq k} A_n$, and so in view of (iii)

$$|f(x) - f_k(x)| = lim_{n \to \infty} |f_n(x) - f_k(x)| \leq 1/2^{k-1}$$

Now, let x be as in case 2. Since x does not belong to C_k, we have $k(x) \leq k < n(x)$. Hence, by (iii) $|f(x) - f_k(x)| \leq 1/2^{k-1}$ and so the claim is proved. Since $\mu^*(C_n) \to 0$ as $n \to \infty$, we have $f_n \xrightarrow{\mu} f$. This proves the sufficiency for $p = 0$.

Case $0 < p < \infty$. As easily seen, a Cauchy sequence in $L_p(\mu)$ is also a Cauchy sequence in $L_0(\mu)$. Hence the assertion follows from the case $p = 0$ and lemma 4.3.3. ■

Remark (4.3.5): Note that the argument given above goes through even if we work with a seemingly weaker version of the conditions given in lemma 4.3.2. To be more precise, we can prove, using the same argument that $L_p(X, \mathcal{F}, \mu)$ is complete if for every sequence $\{B_n\} \subset \mathcal{F}$ and for any preassigned $\epsilon > 0$, there exists $B \subset X$, depending on ϵ such that

(i) $\mu^*(B_n \setminus B) = 0$ for all n

(ii) $\mu^*(B) \leq \sum_{n=1}^{\infty} \mu(B_n) + \epsilon$.

We sum up the above discussion in the following theorem:

Theorem 4.3.6 *The following are equivalent :*

1. *$L_p(X, \mathcal{F}, \mu)$ is complete for $0 < p < \infty$.*

2. *For every μ-Cauchy sequence $\{A_n\} \subset \mathcal{F}$ with $\mu(A_n) < \infty$ there exists $A \subset X$ with $\mu^*(A_n \Delta A) \to 0$.*

3. *For every sequence of sets $\{B_n\} \subset \mathcal{F}$ there exists a set $B \subset X$ such that*

   (a) $\mu^*(B_n \setminus B) = 0 \ \ \forall n$ and

   (b) $\mu^*(B) \leq \sum_{n=1}^{\infty} \mu(B_n)$.

4. *For every sequence of sets $\{B_n\} \subset \mathcal{F}$ and any preassigned positive number ϵ there exists a set $B \subset X$ depending on ϵ and satisfies the following:*

   (a) $\mu^*(B_n \setminus B) = 0 \ \forall \ n$ and

   (b) $\mu^*(B) \leq \sum_{n=1}^{\infty} \mu(B_n) + \epsilon$.

Proof: We have already proved (1)$\Longleftrightarrow$(2) (Theorem 4.3.4.) and (2) $\Longrightarrow$(3) (Lemma 4.3.2.). (3)$\Longrightarrow$(4) is trivial. To show (4)$\Longrightarrow$(1) we modify the argument given in the sufficiency part of theorem 4.3.4 as follows:

Fix a sequence of positive numbers $\{\epsilon_k\}$ such that $\epsilon_k \downarrow 0$. For each $k \geq 1$ select a set B_k' so that

(a) $\mu^*(A_n \setminus B_k') = 0$ for all $n \geq k$

(b) $\mu^*(B_k') \leq \sum_{n=k}^{\infty} \mu(A_n) + \epsilon_k$.

This can be done by using the present hypothesis for the sequence of sets $\{A_n\}_{n\geq k}$. Now repeat the same argument as given in the sufficiency part of Theorem 4.3.4. with B_k replaced by B'_k throughout and this leads to (1).

This completes the proof. ∎

Further, if we assume $\mathcal{F}$ to be a σ-field then the above Theorem takes the following pleasing form which we will use afterwards:

Theorem 4.3.7 *Suppose γ is a finitely additive measure on a σ-field $\mathcal{F}$. Then following are equivalent:*

1. *$L_1(\gamma)$ is complete.*

2. *$(\mathcal{F}, d)$ is complete where d is the usual pseudometric on $\mathcal{F}$ given by $d(A, B) = \mu(A\Delta B) \wedge 1$. (Notice that if μ is bounded we define $d(A, B)$ simply as $\mu(A\Delta B)$.)*

3. *Given any sequence of sets $\{A_n\}$ in $\mathcal{F}$, there exists a set $A \in \mathcal{F}$ such that, $\mu(A_n \setminus A) = 0$ for each n; and $\mu(A) \leq \sum \mu(A_n)$.*

4. *Given any sequence of sets $\{A_n\}$ in $\mathcal{F}$ and $\epsilon > 0$, there exists a set $A \in \mathcal{F}$ such that*

$$\mu(A_n \setminus A) = 0 \text{ for each } n; \text{ and } \mu(A) \leq \sum \mu(A_n) + \epsilon.$$

5. *Given any sequence of sets $\{A_n\}$ in $\mathcal{F}$ there is a sequence of sets $\{B_n\}$ in $\mathcal{F}$ such that*

$$B_n \subset A_n, \ \mu(A_n \setminus B_n) = 0 \ \text{ for each } n;$$

$$\text{and } \ \mu(\cup B_n) \leq \sum \mu(B_n) = \sum \mu(A_n).$$

6. *Given a sequence* $\{A_n\}$ *of pairwise disjoint sets in* $\mathcal{F}$, *there exists a sequence* $\{B_n\}$ *in* $\mathcal{F}$ *such that,*

$$B_n \subset A_n, \quad \mu(A_n \setminus B_n) = 0 \quad \textit{for each } n;$$

$$\textit{and } \mu(\cup B_n) = \sum \mu(B_n) = \sum \mu(A_n).$$

7. *Given an increasing sequence* $\{A_n\}$ *of sets in* $\mathcal{F}$, *there exists a set* $A \in \mathcal{F}$ *such that*

$$\mu(A_n \setminus A) = 0 \ \textit{for each } n; \ \textit{and } \mu(A) = \lim_n \mu(A_n).$$

Remark (4.3.8). It is interesting—but not surprising—to note that the conditions for completeness of $L_p(X, \mathcal{F}, \mu)$ are independent of p as long as $p < \infty$. So from now on, we will only consider $L_1(X, \mathcal{F}, \mu)$ and when the underlying measure space is fixed, we will just call it L_1.

Proposition 4.3.9. *Suppose* $L_1(X, \mathcal{F}, \mu)$ *is complete and* $\mathcal{F}$ *is a* σ*-field and* μ *is bounded. Then for any* μ*-Cauchy sequence* $\{A_n\} \subset \mathcal{F}$ *we can find* $\tilde{A} \in \mathcal{F}$ *such that* $\mu(A_n \Delta \tilde{A}) \to 0$.

Proof: Since L_1 is complete there exists $A \subset X$ such that $\mu^*(A_n \Delta A) \to 0$. Using the fact that $\mathcal{F}$ is a σ-field we shall show that we can choose $\tilde{A}$ from $\mathcal{F}$ such that $\mu(A_n \Delta \tilde{A}) \to 0$. In $\mathcal{F}$ find $\tilde{A}$ containing A such that

$$\mu^*(A) = \mu(\tilde{A}).$$

This is clearly possible since $\mathcal{F}$ is a σ-field. Also, since $A_n \Delta A_m = A_n^c \Delta A_m^c$, $\{A_n^c\}$ is a μ-Cauchy sequence and there exists a $B \subset X$ such that $\mu^*(A_n^c \Delta B) \to 0$. We can take $B = A^c$ and as before we can get $\tilde{B}$ in $\mathcal{F}$, $\tilde{B}$ containing A^c such that $\mu^*(A^c) = \mu(\tilde{B})$.

We claim that $\mu(A_n) \to \mu(\tilde{A})$ and $\mu(A_n^c) \to \mu(\tilde{B})$. To prove the claim notice that $A_n \subset A \cup (A_n \Delta A)$. Therefore,

$$\mu(A_n) \leq \mu^*(A) + \mu^*(A_n \Delta A)$$

Since $\mu^*(A_n \Delta A) \to 0$ we have, taking limits on both sides of the above inequality.

$$lim_{n \to \infty} \mu(A_n) \leq \mu^*(A) \ldots (*)$$

Again, as $A \setminus (A \setminus A_n) \subset A_n$ and hence $A \subset A_n \cup (A \setminus A_n)$, we have

$$\mu^*(A) \leq \mu(A_n) + \mu^*(A \setminus A_n).$$

Taking limits on both sides we get

$$\mu^*(A) \leq lim \mu(A_n) \ldots (**).$$

$(*)$ and $(**)$ together give $lim \mu(A_n) = \mu^*(A) = \mu(\tilde{A})$ as required.The same argument yields the result for A_n^c. Now we want to show that $\mu(\tilde{A} \cap \tilde{B}) = 0$. For that observe, $\tilde{A}$ contains A and $\tilde{B}$ contains A^c. So.

$$\begin{aligned}
\mu(X) &= \mu(\tilde{A} \cup \tilde{B}) \\
&= \mu(\tilde{A}) + \mu(\tilde{B}) - \mu(\tilde{A} \cap \tilde{B}) \\
&= lim(\mu(A_n) + \mu(A_n^c)) - \mu(\tilde{A} \cap \tilde{B}) \\
&= \mu(X) - \mu(\tilde{A} \cap \tilde{B})
\end{aligned}$$

Therefore, $\mu(\tilde{A} \cap \tilde{B}) = 0$.

But $A_n \Delta \tilde{A} \subset (A_n \Delta A) \cup (\tilde{A} \cap \tilde{B})$. So it immediately follows that $\mu(A_n \Delta \tilde{A}) \leq \mu^*(A_n \Delta A)$ and hence converges to 0. ∎

Example (4.3.10). Let $X = [0, 1]$, $\mathcal{F}$ be the field generated by all open sets of X and μ be any finite countably additive measure on $\mathcal{F}$.

Then the σ-field generated by $\mathcal{F}$ is the Borel σ-field $\mathcal{B}_X$ and μ can be extended as a finite countably additive regular measure $\tilde{\mu}$ on $\mathcal{B}_X$. Since $\tilde{\mu} = \mu^*|\mathcal{B}_X$ and since any μ-Cauchy sequence of sets in $\mathcal{F}$ is $\tilde{\mu}$-Cauchy sequence of sets in $\mathcal{B}_X$ it follows that $L_1(X, \mathcal{F}, \mu)$ is complete.

Example (4.3.11). Let $X = [0, 1)$, let

$$\mathcal{F} = \{\cup_{i=1}^{n}[a_i, b_i) : n \in \mathbb{N}, a_i < b_i, a_i, b_i \in (0, 1)\}$$

and μ be the Lebesgue measure restricted to $\mathcal{F}$.

Then $(X, \mathcal{F}, \mu)$ does not satisfy the condition of Theorem 4.3.4 and hence $L(X, \mathcal{F}, \mu)$ is not complete. Indeed, let $\{r_1, r_2, ...\}$ be an enumeration of the rationals in [0, 1) and put

$$A_n = [r_n - 2^{-(n+2)}, r_n + 2^{-(n+2)}) \cap X; \ n = 1, 2, ...$$

Then $A_n \in \mathcal{F}$ and $\sum_{n=1}^{\infty} \mu(A_n) < 1/2$.

Suppose $(X, \mathcal{F}, \mu)$ satisfies the conditions of Theorem 4.3.4. Then there exits a subset $A \subset X$ such that

(i) $\mu^*(A_n \setminus A) = 0$ for all n

(ii) $\mu^*(A) \leq 1/2$.

Conditions (i) implies that $A_n \cap A$ is nonempty for all $n \geq 1$. Hence A is dense in X, which contradicts (ii)

Remark (4.3.12). In the above two examples the underlying measures are essentially the same. In Example 4.3.10. we can take μ to be the Lebesgue measure λ restricted to $\mathcal{F}$, the field generated by all open subsets of X. As the example shows, $L_1(X, \mathcal{F}, \lambda|\mathcal{F})$ is complete. On the other hand, in the Example 4.3.11., though we work with the Lebesgue measure λ restricted to the field $\mathcal{F}$, where

$\mathcal{F} = \{\cup_{i=1}^{n}[a_i, b_i) : n \in \mathbb{N}, a_i < b_i, a_i, b_i \in (0, 1)\}$, $L_1(X, \mathcal{F}, \lambda|\mathcal{F})$ is not complete. The reason is that in the second example the field is too small. This shows that the underlying field plays an important role in determining the completeness of L_p-spaces. This happens because the value of the outer measure μ^* depends on the field where μ is defined.

Example (4.3.13). Let $X = \mathbb{N}$, the set of positive integers and let $\mathcal{F} = \mathcal{P}(\mathbf{N})$, the power set of $\mathbb{N}$. Define μ on $\mathcal{F}$ as follows:

$$\mu(A) = \begin{cases} \sum_{n \in A} 2^{-n} & \text{if } A \text{ is finite} \\ 2 - \sum_{n \in A^c} 2^{-n} & \text{if } A^c \text{ is finite} \end{cases}$$

Extend μ to $\mathcal{F}$ as a positive real-valued measure. Then $(X, \mathcal{F}, \mu)$ does not satisfy the conditions of Theorem 4.3.4. Indeed, put $A_n = \{1, 2, \ldots, n\}$ and suppose

$$\mu(A_n \setminus A) \to 0 \text{ for some } A \subset X.$$

Then $A = X$, whence $\mu(A \setminus A_n) > 1$.

4.4 Completeness of L_∞

The following theorem shows that L_∞ is always complete in contrast to the L_p-spaces for $p < \infty$.

Theorem 4.4.1. *The space $L_\infty(X, \mathcal{F}, \mu)$ is complete for every finitely additive measure space $(X, \mathcal{F}, \mu)$.*

Proof: Let $\{f_n\}$ be a Cauchy sequence of functions in $L_\infty(\mu)$. We shall define a function $f \in L_\infty$ such that $f_n \to f$ in L_∞. By passing to a subsequence, if necessary, we may assume that

$$||f_n - f_{n+1}||_\infty < 1/2^n$$

Let

$$A_n = \{x : |f_n(x) - f_{n+1}(x)| > 1/2^n\}, \ n \geq 1.$$

Then $\mu^*(A_n) = 0$ for all n. Moreover, as in the proof of Theorem 4.3.4, if for some $m > n$, x does not belong to $\cup_{i=n}^{m-1} A_i$, then $|f_n(x) - f_m(x)| < 1/2^{n-1}$. So, if for a given x, $n(x)$ is the smallest n such that $x \in A_n$ then for any $k < n(x)$,

$$|f_k(x) - f_{n(x)}(x)| < 1/2^{k-1} \ldots (+).$$

This inequality we will use in our argument later.

Now we are ready to define our desired function f.

Case 1. $x \in (\cup_{n=1}^{\infty} A_n)^c$. Then $\{f_n(x)\}$ is a Cauchy sequence of real numbers. Define $f(x) = lim_{n \to \infty} f_n(x)$

Case 2. $x \in \cup_{n=1}^{\infty} A_n$. Let $n(x)$ be the smallest n such that $x \in A_n$. Define $f(x) = f_{n(x)}(x)$.

To show that $f_n \to f$ in L_∞, define $H_n = \cup_{k=1}^{n} A_k$. Since $\mu^*(A_k) = 0$ for any k, $\mu^*(H_n) = 0$. We claim that $x \in (H_n)^c$ implies

$$|f_n(x) - f(x)| \leq 1/2^{n-1}.$$

This is clear if x is as in case 1. Let x be as in case 2. Since $x \notin H_n$, we have $n(x) > n$ and so by (+),

$$|f_n(x) - f_{n(x)}(x)| = |f_n(x) - f_{n(x)}| \leq 1/2^{n-1}.$$

Thus the claim is proved. Now, since $\mu^*(H_n) = 0$, it follows immediately that $f_n \to f$ in μ and f is essentially bounded. Therefore, in view of Proposition 4.6.13 in ([2]), $f \in L_\infty$. Moreover, $||f_n - f||_\infty \to 0$. Thus f is as desired. ∎

4.5 An Application of Completeness of L_p-Spaces

In this section we slightly digress from our main theme, namely, the study of interrelation between the completeness of L_p-spaces and the underlying measure spaces. This we will take up again in the subsequent sections. Here we will consider a nonnegative bounded measure μ on a σ-field $\mathcal{F}$. We will show that if ν is a bounded signed measure on $\mathcal{F}$ such that $\nu << \mu$ i.e., ν is absolutely continuous with respect to μ then the exact Radon-Nikodym derivative for ν with respect μ exists if and only if $L_1(X, \mathcal{F}, \mu)$ is complete. Before proving this result we will give some definitions.

Definition (4.5.1.) ν is absolutely continuous with respect to μ if given $\epsilon > 0$, there exists a $\delta > 0$ such that $\mu(A) < \delta$ implies $|\nu(A)| < \epsilon$. ν is said to have an exact Radon-Nikodym dervative with respect to μ if

$$\exists f \in L_1(\mu) \ \ such\ \ that \ \ \forall A \subset X, \ \ \nu(A) = \int_A f d\mu$$

Proposition 4.5.2. *Let μ be a positive bounded measure on $(X, \mathcal{F})$ where $\mathcal{F}$ is a σ-field. Then $L_1(\mu)$ is complete if and only if for every bounded signed measure ν which is absolutely continuous with respect to μ, there exists an exact Radon-Nikodym derivative f for ν with respect to μ.*

Proof: Sufficiency. Suppose for all bounded absolutely continuous $\nu << \mu$, an exact Radon-Nikodym derivative with respect to μ exists. We shall show that $L_1(\mu)$ is complete.

Take any $L_1(\mu)$ Cauchy-sequence of functions $\{f_n\}$ so that $\int |f_n - f_m| d\mu \to 0$ as $n, m \to \infty$. Define $\mu_n(A) = \int_A f_n d\mu$. Observe that

$$|\mu_n(A) - \mu_m(A)| = \left| \int_A f_n d\mu - \int_A f_m d\mu \right| \leq \int |f_n - f_m| d\mu \to 0.$$

Therefore, $\mu_n(A) \to \nu(A)$ (some real number) for all $A \in \mathcal{F}$. By finitely-additive version of Vitali-Hahn-Saks theorem (see [2], Theorem 8.8.4) ν is a bounded finitely-additive measure on $\mathcal{F}$.

Claim: $\nu << \mu$. Fix $\epsilon > 0$. Since each μ_n is absolutely continuous with respect to μ, μ_n is uniformly absolutely continuous with respect to μ (see [2], Theorem 8.8.4.) i.e., there exists a $\delta > 0$ and n_0 such that $\mu(A) < \delta \longrightarrow |\mu_n(A)| < \epsilon/2$ for all $n \geq n_0$. Therefore, $|\nu(A)| \leq \epsilon/2$ which proves the claim. By our assumption there exists a function $f \in L_1(\mu)$ such that $\nu(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$. It is easy to see that $f_n \to f$ in $L_1(\mu)$. Observe that

$$\left| \int_A (f_n - f) d\mu \right| = |\mu_n(A) - \nu(A)| \to 0 \quad \text{for all } A \in \mathcal{F}.$$

We shall show that this convergence is uniform over A. Indeed, since $\{f_n\}$ is $L_1(\mu)$-Cauchy, given $\epsilon > 0$ there exists a positive integer N_ϵ, such that $\int |f_n - f_m| d\mu < \epsilon/2$ for all $n, m \geq N_\epsilon$. Since $|\int_A (f_n - f) d\mu| \to 0$ for all $A \in \mathcal{F}$, there exists a positive integer n_A such that $|\int_A (f_n - f) d\mu| < \epsilon/2$ for all $n \geq n_A$. Choose $n_A > N_\epsilon$.

Now,

$$\begin{aligned} |\textstyle\int_A (f_n - f) d\mu| &= |\textstyle\int_A (f_n - f_{n_A} + f_{n_A} - f) d\mu| \\ &\leq |\textstyle\int_A (f_n - f_{n_A}) d\mu| + |\textstyle\int_A (f_{n_A} - f) d\mu| \\ &< \epsilon \quad \text{for all } n \geq N_\epsilon \end{aligned}$$

Since N_ϵ is independent of A, this proves that $|\int_A (f_n - f) d\mu| \to 0$ uniformly over A. Therefore, $\int |f_n - f| d\mu \to 0$ by Theorem 4.6.14 in [2].

Necessity. Let $L_1(\mu)$ be complete. Let ν be bounded and $\nu << \mu$. Then by Radon-Nikodym theorem for finitely additive measures (see [2], Theorem 6.3.4.), we know that for all $\epsilon > 0$ there exists a simple function f_ϵ such that $|\nu(A) - \int_A f_\epsilon d\mu| < \epsilon$ for all $A \in \mathcal{F}$. So, taking $\epsilon = 1, 1/2, \ldots, 1/n, \ldots etc.$ we get a sequence of functions $\{f_n\}$, which are simple and hence in $L_1(\mu)$ such that $\int_A (f_n - f_m)d\mu \to 0$ uniformly over A. Therefore $\int |f_n - f_m|d\mu \to 0$ by Theorem 4.6.14 in [2] and $L_1(\mu)$ being complete there exists a function $f \in L_1(\mu)$ such that $\int |f_n - f|d\mu \to 0$. Easy to see that f is an exact Radon-Nikodym derivative for ν with respect to μ. ∎

4.6 Further Results on Completeness of L_p-Spaces

In what follows, we consider a nonnegative finitely additive bounded set function γ defined on a σ field $\mathcal{F}$ of subsets of a space Ω. Even though our motivation and interest is only in probabilities, it is convenient to deal with bounded measures. Also, we will consider only $L_1(\gamma)$.as we have already seen (see remark(4.3.8)) that the completeness of $L_1(\gamma)$ is equivalent to the completeness of $L_p(\gamma)$ for any p with $1 \leq p < \infty$.

Our starting point is Theorem 4.3.7. As a useful consequence of the criteria given in the Theorem we have the following Theorem:

Theorem 4.6.1 *a) Suppose $\gamma = \gamma_1 + \gamma_2$*

i) If $L_1(\gamma)$ is complete then so are $L_1(\gamma_1)$ and $L_1(\gamma_2)$.

ii) If $L_1(\gamma_1)$ and $L_1(\gamma_2)$ are complete then it is not necessary that $L_1(\gamma)$ be complete.

iii) If $L_1(\gamma_1)$ and $L_1(\gamma_2)$ are complete and γ_1, γ_2 are supported on disjoint sets then $L_1(\gamma)$ is complete.

(b) Suppose $\Omega_o \in \mathcal{F}$, $0 < \gamma(\Omega_o) < \gamma(\Omega)$. Let γ_1, γ_2 denote restrictions of γ to Ω_o and $\Omega - \Omega_o$ respectively. Then $L_1(\gamma)$ is complete iff $L_1(\gamma_1)$ and $L_1(\gamma_2)$ are complete.
(c) Suppose $L_1(\gamma)$ is complete. $\mathcal{F}_o$ be a sub σ field of $\mathcal{F}$ which includes all γ null sets that are in $\mathcal{F}$. Let γ_o be γ restricted to $\mathcal{F}_o$. Then $L_1(\gamma_o)$ is complete.

Proof: (c) and (aiii) follow from criterion 6 of Theorem 4.3.7. (b) follows from (ai) and (aiii). To observe (aii) take $\Omega = \{1, 2, \cdots\}$, $\mathcal{F} =$ power set of Ω, γ_1 is the countably additive measure, $\gamma_1(n) = \frac{1}{2^n}$, $n = 1, 2, \cdots$; γ_2 is diffuse 0-1 valued measure on $\mathcal{F}$ giving 0 to singletons; $\gamma = \gamma_1 + \gamma_2$. Then both $L_1(\gamma_1)$ and $L_1(\gamma_2)$ are complete. However criterion 6 of Theorem 4.3.7 fails for the sequence of sets $A_n = \{n\}$ to show that $L_1(\gamma)$ is not complete. Finally, we prove (ai) as follows: Towards verifying criterion 6 of Theorem 4.3.7, suppose $\{A_n\}$ is a sequence of disjoint sets in $\mathcal{F}$. Since $L_1(\gamma)$ is complete, get a sequence $\{B_n\}$ as stated there. In particular, for each n, $\gamma_1(A_n \setminus B_n) = 0 = \gamma_2(A_n \setminus B_n)$. Moreover

$$\gamma(\cup B_n) = \gamma_1(\cup B_n) + \gamma_2(\cup B_n)$$

$$\sum \gamma(B_n) = \sum \gamma_1(B_n) + \sum \gamma_2(B_n)$$

By choice of $\{B_n\}$, the left sides of the equations above are same. So must be the right sides. But $\gamma_1(\cup B_n) \geq \sum \gamma_1(B_n)$ and $\gamma_2(\cup B_n) \geq \sum \gamma_2(B_n)$ so that equality must hold at both places. In other words the same sequence $\{B_n\}$ verifies that criterion 6 of Theorem 4.3.7 holds for both γ_1 as well as γ_2.

■

Remark (4.6.2). Theorem 4.6.1(ai) can equivalently be stated as follows. If $L_1(\gamma)$ is complete and $\gamma_1 \leq \gamma$ (inequality being setwise) then $L_1(\gamma_1)$ is also complete.

4.7 Yosida-Hewitt Decomposition

Recall that a finitely additive positive measure μ on $(\Omega, \mathcal{F})$ is said to be purely finitely additive in case λ is a positive countably additive measure and $\lambda(A) \leq \mu(A)$ for all $A \in \mathcal{F}$ implies that $\lambda \equiv 0$.

The celebrated decomposition theorem due to Yosida and Hewitt [36] (see also [34]) says that any finitely additive positive measure γ on $(\Omega, \mathcal{F})$ can be decomposed as $\gamma = \gamma_1 + \gamma_2$ where γ_1 is countably additive and γ_2 is purely finitely additive. Moreover such a decomposition is unique.

Theorem 4.7.1 *Let $\gamma = \gamma_1 + \gamma_2$ be the Yosida-Hewitt decomposition of γ. $L_1(\gamma)$ is complete iff γ_1, γ_2 are supported on disjoint sets and $L_1(\gamma_2)$ is complete.*

Proof : If γ_1, γ_2 satisfy the conditions of the Theorem, then $L_1(\gamma)$ is complete in view of Theorem 4.6.1 (aiii). Conversely let us assume that $L_1(\gamma)$ is complete. The idea is the following. We shall express $\Omega = A \cup B \cup C$ where $A, B, C \in \mathcal{F}$ and are pairwise disjoint, $\gamma_1(A) = 0$, $\gamma_2(B) = 0$, and if $S \in \mathcal{F}$, $S \subset C$ then $\gamma_1(S) > 0$ iff $\gamma_2(S) > 0$.

Assume for a moment that such a decomposition exists. We claim that γ_2 (and hence γ_1) is null on C. If not, γ_2 being purely finitely additive we can get $C_n \subset C$, $C_n \uparrow C$, $\lim_n \gamma_2(C_n) < \gamma_2(C)$. Since $L_1(\gamma)$ is complete we can use criterion 7 of Theorem 4.3.7 and get $\tilde{C}$ such that $\gamma(C_n \setminus \tilde{C}) = 0$ for each n and $\gamma(\tilde{C}) = \lim_n \gamma(C_n)$. In particular, $\gamma_1(C_n \setminus \tilde{C}) = 0$ for each n so that $\gamma_1(\tilde{C}) \geq \lim_n \gamma_1(C_n)$. In case $\gamma_2(C \setminus \tilde{C}) = 0$ we have $\gamma_2(\tilde{C}) \geq \gamma_2(C) > \lim_n \gamma_2(C_n)$ implying that $\gamma(\tilde{C}) > \lim_n \gamma(C_n)$, a contradiction. Thus $\gamma_2(C \setminus \tilde{C}) > 0$. But then $\gamma_1(C \setminus \tilde{C}) > 0$. Since $C_n \setminus \tilde{C} \uparrow C \setminus \tilde{C}$ and γ_1 is countably additive we conclude that $\gamma_1(C \setminus \tilde{C}) = \lim_n \gamma_1(C_n \setminus \tilde{C}) = 0$, again a contradiction. Thus γ_2 must be null on C. So must be γ_1. Thus γ_1 and γ_2 are supported on B and A respectively. The proof is completed by using Theorem 4.6.1(b).

We shall now proceed to exhibit the stated decomposition. Consider,

$$\mathcal{C} = \{S \in \mathcal{F} : \gamma_1(S) > 0 \text{ and } \gamma_2(S) = 0\}$$
$$\text{and } \beta = \sup\{\gamma_1(S) : S \in \mathcal{C}\}$$

Completeness of $L_1(\gamma)$ now allows us to show that this supremum is indeed attained. To see this, pick $S_n \in \mathcal{C}$, $\gamma_1(S_n) \uparrow \beta$. Since $\mathcal{C}$ is closed under finite unions we can assume that S_n increases with n. By criterion 7 of Theorem 4.3.7, get B such that $\gamma(B) = \beta$ and for each n, $\gamma(S_n \setminus B) = 0$. There is no loss to assume that $B \subset \cup_n S_n$. First observe that $\gamma_1(B) + \gamma_2(B) = \gamma(B) = \beta$. Secondly, $\gamma(S_n \setminus B) = 0$ and hence $\gamma_1(S_n \setminus B) = 0$ for each n, so that $\gamma_1(B) \geq \gamma_1(S_n)$ for all n, implying that $\gamma_1(B) \geq \beta$. These two observations show that $\gamma_1(B) = \beta$ and $\gamma_2(B) = 0$.

In an analogous manner, consider

$$\mathcal{D} = \{S \in \mathcal{F} : \gamma_2(S) > 0 \;\&\; \gamma_1(S) = 0\}$$

Since γ_1 is countably additive, $\mathcal{D}$ is closed under countable unions and hence there is a set A such that $\gamma_1(A) = 0$ and $\gamma_2(A) = \sup\{\gamma_2(S) : S \in \mathcal{D}\}$. By construction it is clear that $\gamma_1(A \cap B) = 0 = \gamma_2(A \cap B)$. Thus we can assume $A \cap B = o$. Set $C = \Omega \setminus \{A \cup B\}$ to complete the proof. ∎

Remark (4.7.2). Since γ_1 in the Theorem above is countably additive clearly $L_1(\gamma_1)$ is complete. That is why in the theorem it was enough to state the condition that $L_1(\gamma_2)$ be complete. Note how the completeness of $L_1(\gamma)$ played a crucial role in obtaining the decomposition. Look at the example given in the proof of Theorem 4.6.1.a(ii).

4.8 Discrete Measures

Recall that a finitely additive nonnegative measure γ on $(\Omega, \mathcal{F})$ is said to be discrete if $\gamma = \sum a_i \delta_i$ where each δ_i is a 0-1 valued measure and $a_i > 0$. Since we are considering only bounded measures, clearly $\sum a_i < \infty$.

Theorem 4.8.1 *Let $\gamma = \sum a_i \delta_i$ be discrete. $L_1(\gamma)$ is complete iff δ_i are uniformly singular, that is, there are pairwise disjoint sets $A_i \in \mathcal{F}$ with $\delta_i(A_i) = 1$ for each i.*

Proof : If $\{\delta_i\}$ are uniformly singular then criterion 6, Theorem 4.3.7 applies to show that $L_1(\gamma)$ is complete. To prove the converse, assume that $\{\delta_i\}$ are not uniformly singular. Let us say that δ_i can be separated if there is a set A_i in $\mathcal{F}$ such that $\delta_i(A_i) = 1$ and $\delta_j(A_i) = 0$ for

$j \neq i$. If each δ_i can be separated, witnessed by say A_i, then setting $B_n = A_n \setminus \cup_{i<n} A_i$, we observe that $\delta_i(B_i) = 1$ for each i; showing that δ_i are uniformly singular. Thus there is a δ_i which cannot be separated. By renaming if necessary we shall assume that δ_1 cannot be separated. We shall construct a sequence of pairwise disjoint sets $(A_n)_{n>1}$ such that (i) if $j > 1$ then for some i, $\delta_j(A_i) = 1$ and (ii) for each i, $\delta_1(A_i) = 0$. If this is done, then we claim that criterion 6 of Theorem 4.3.7 fails for this sequence. Indeed, suppose we have sets $B_i \subset A_i$, $\gamma(A_i \setminus B_i) = 0$ for each i. By properties (i) and (ii) we have $\sum_{i>1} \gamma(B_i) = \sum_{i>1} a_i$. If $\delta_1(\cup B_i) = 0$ then property (i) implies that δ_1 can be separated which is not the case. Thus $\delta_1(\cup B_i) = 1$ implying that $\gamma(\cup B_i) = \sum_{i\geq 1} a_i > \sum \gamma(B_i)$.

We shall now proceed to exhibit sets A_i as stated. Pick B_o such that $\delta_1(B_o) = 1$. Set $A_1 = B_o^c$ and $S_1 = \{j > 1 : \delta_j(B_o) = 1\}$. S_1 is infinite because δ_1 can not be separated. Pick the first integer $j_1 \in S_1$ and write $B_o = B_1 \cup A_2$ disjoint union with $\delta_1(B_1) = 1$ and $\delta_{j_1}(A_2) = 1$. To do this, just note that δ_i being 0-1 valued they are pairwise singular. Then $S_2 = \{j > 1 : \delta_j(B_1) = 1\}$ is again infinite. Proceed inductively by picking the first j in S_n at the nth stage. This completes the proof of the Theorem. ■

The following theorem, a slight extension of Theorem 4.8.1, will be needed later. Theorem 4.8.1 corresponds to the case when γ_0 is absent.

Theorem 4.8.2 *Suppose* $\gamma = \sum_{i\geq 0} a_i\gamma_i$ *with* $a_i > 0$, $\sum a_i < \infty$. *Assume that* γ_i, $i \geq 1$ *are 0-1 valued. If* $L_1(\gamma)$ *is complete then* γ_i, $i \geq 1$ *are uniformly singular.*

Proof : Apply Theorem 4.6.1(ai) and Theorem 4.8.1. ■

4.9 Sobczyk-Hammer Decomposition

Recall that a finitely additive nonnegative measure γ on $(\Omega, \mathcal{F})$ is said to be strongly continuous if given $\epsilon > 0$, there is a finite decomposition $\Omega = \cup A_i$ where $A_i \in \mathcal{F}$ and $\gamma(A_i) < \epsilon$ for each i. The well known decomposition theorem due to Sobczyk and Hammer [35] says that any finitely additive positive measure $\gamma = \gamma_1 + \gamma_2$ where γ_1 is discrete and γ_2 is strongly continuous. Further such a decomposition is unique.

Theorem 4.9.1 *Let $\gamma = \gamma_1 + \gamma_2$ be the Sobczyk-Hammer decomposition of γ. $L_1(\gamma)$ is complete iff $L_1(\gamma_1), L_1(\gamma_2)$ are complete and γ_1, γ_2 are supported on disjoint sets.*

Proof : The if part is a consequence of Theorem 4.6.1. To prove the converse, assume that $L_1(\gamma)$ is complete. By Theorem 4.6.1 again, $L_1(\gamma_1)$ and $L_1(\gamma_2)$ are also complete. We only need to show now that γ_1, γ_2 are supported on disjoint sets.

First assume that γ_1 is 0-1 valued. By using strong continuity of γ_2, one can obtain for each n, a set A_n such that $\gamma_1(A_n^c) = 1$ and $\gamma_2(A_n^c) < \frac{1}{n}$. We can also assume that A_n increases with n. $\lim_n \gamma(A_n) = \lim_n \gamma_2(A_n) = \gamma_2(\Omega)$. By Theorem 4.3.7(7) get $B \subset \cup A_n$ with $\gamma(B) = \gamma_2(\Omega)$ and $\gamma(A_n \setminus B) = 0$ for each n. In particular $\gamma_2(A_n \setminus B) = 0$ for each n so that $\gamma_2(B) \geq \gamma_2(A_n \cap B) = \gamma_2(A_n)$ which increases to $\gamma_2(\Omega)$. Thus $\gamma_2(B) = \gamma_2(\Omega)$. But since $\gamma(B) = \gamma_2(\Omega)$ we conclude that $\gamma_1(B) = 0$. In other words γ_1, γ_2 are supported on B^c and B respectively.

To treat the general case, assume that the discrete part $\gamma_1 = \sum_{i\geq 1} a_i\delta_i$, $a_i > 0$, $\sum a_i < \infty$, δ_i 0-1 valued. By Theorem 4.8.2, δ_i are uniformly singular so that Ω can be written as disjoint union $\cup_{i\geq 1} A_i$ with $\delta_i(A_i) = 1$ for each i. By Theorem 4.7.1, $L_1(a_i\delta_i + \gamma_2)$ is complete for each i and hence by earlier para we can get $B_i \subset A_i$ such that $\delta_i(B_i) = 1$ and $\gamma_2(B_i) = 0$. Since $L_1(\gamma)$ is complete, by Theorem 4.3.7(6) we can get $C_i \subset B_i$ with $\gamma(B_i \setminus C_i) = 0$ for each i and $\gamma(\cup C_i) = \sum \gamma(C_i)$. In particular for each i, $\delta_i(B_i \setminus C_i) = 0$ so that $\delta_i(C_i) = \delta_i(B_i) = 1$. Since $\gamma_2(C_i) = 0$ and $\gamma_1(C_i) = a_i$ for each i, we have

$$\sum \gamma(C_i) = \sum \gamma_1(C_i) + \sum \gamma_2(C_i) = \sum a_i$$

$$\gamma(\cup C_i) = \gamma_1(\cup C_i) + \gamma_2(\cup C_i) \geq \sum a_i + \gamma_2(\cup C_i)$$

Since left sides are equal we conclude that $\gamma_2(\cup C_i) = 0$. In other words γ_1 is supported on $\cup C_i$ and γ_2 is supported on its complement, as claimed. ∎

Combining Theorems 4.7.1, 4.8.1, 4.8.2 and 4.9.1 we obtain the following two versions of the main characterization theorem.

Theorem 4.9.2 *Let γ be a finitely additive probability on $(\Omega, \mathcal{F})$, $\mathcal{F}$ being a σ-field of subsets of Ω. Then $L_1(\gamma)$ is complete iff Ω has a decomposition, $\Omega = \Omega_o \cup \Omega_1 \cup \Omega_2$, where $\Omega_i \in \mathcal{F}$ for $0 \leq i \leq 2$ such that (i) γ restricted to Ω_o is countably additive (ii) γ restricted to Ω_1 is discrete and is a combination of uniformly singular 0-1 probabilities and (iii) γ restricted to Ω_2 is strongly continuous and its L_1 space is complete.*

Theorem 4.9.3 *Let γ be a finitely additive probability on $(\Omega, \mathcal{F})$. $\mathcal{F}$ being a s-field. Then $L_1(\gamma)$ is complete iff Ω has a decomposition $\Omega = \cup_{i\geq 0}\Omega_i$, where each $\Omega_i \in \mathcal{F}$ such that (i) γ restricted to Ω_o is countably additive (ii) γ restricted to each Ω_i, $i \geq 2$ is at most two valued and (iii) γ restricted to Ω_1 is strongly continuous and its L_1 space is complete and (iv) for each $A \in \mathcal{F}$, $\gamma(A) = \sum_{i=0}^{\infty} \gamma(\Omega_i \cap A)$.*

In Theorem 4.9.2 we had a finite decomposition so that the last condition of Theorem 4.9.3 was not imposed. We conclude this section with a few remarks.

Remark (4.9.4). Here is an example of a sequence of 0-1 measures which are not uniformly singular. $\Omega = \{0, 1, 2, \cdots\}$. For $i \geq 2$, let γ_i be a diffuse 0-1 measure concentrated on the powers of ith prime. Let

$$\mathcal{C} = \{A : \gamma_i(A) = 1 \text{ for all but finitely many } i \geq 2\}.$$

Extend $\mathcal{C}$ to an ultrafilter and denote by γ_1 the corresponding 0-1 measure. Then $\{\gamma_i : i \geq 1\}$ are not uniformly singular. Note that all these γ_i are purely finitely additive. If we did not want this, we could have taken point masses and any diffuse 0-1 measure. Also note that all these γ_i are defined on power set of Ω. The same construction can be carried out on any $(\Omega, \mathcal{F})$ provided $\mathcal{F}$ is infinite.

Remark (4.9.5). Given any sequence of 0-1 measures $\{\gamma_i\}$ on $(\Omega, \mathcal{F})$ there exists an infinite subsequence which is uniformly singular. We inductively construct a sequence of disjoint sets $A_i \geq 1$ and indices n_i, $i \geq 1$ such that $\gamma_{n_i}(A_i) = 1$ for all i. Just make sure that at the

kth stage infinitely many γ_i are concentrated on the complement of $\cup_{i\leq k} A_i$.

Remark (4.9.6). If $\gamma = \sum \frac{1}{2^i}\gamma_i$ where γ_i are as in Remark 4.9.4, then $L_1(\gamma)$ is not complete though γ is defined on the power set of Ω.

Remark (4.9.7). Given $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is infinite it is always possible to obtain strongly continuous γ on $\mathcal{F}$ such that $L_1(\gamma)$ is not complete. $\mathcal{F}$ being infinite, the general case can be reduced to $\Omega = N$ and $\mathcal{F}$ is power set. This is what we treat. Let μ be any extension of density charge defined on arithmetic progressions. μ is clearly strongly continuous. Fix a decomposition of N into disjoint sets A^n, $n \geq 1$ with $\mu(A^n) > 0$ for each n. Let

$$\mathcal{F} = \{B : \mu(B \cap A^n) = \mu(A^n) \text{ for all but finitely many } n\}$$

To start with, let A_0^n and A_1^n be a decomposition of A_n into two disjoint sets of positive μ-measure. For each k, $1 \leq k \leq n$ and each sequence $(\epsilon_1, \cdots, \epsilon_k)$ of 0's and 1's fix a subset $A^n_{\epsilon_1\cdots\epsilon_k}$ of A^n with positive μ measure such that $A^n_{\epsilon_1\cdots\epsilon_k}$ is the disjoint union of $A^n_{\epsilon_1\cdots\epsilon_k 0}$ and $A^n_{\epsilon_1\cdots\epsilon_k 1}$. This can be done by the strong continuity of μ. For $k \geq 1$ define $B_{\epsilon_1\cdots\epsilon_k} = \cup_{n\geq k} A^n_{\epsilon_1\cdots\epsilon_k}$. Extend $\mathcal{F}$ restricted to $B_{\epsilon_1\cdots\epsilon_k}$ to ultrafilter on $\cup_{n\geq k} A^n$. Let $\eta_{\epsilon_1\cdots\epsilon_k}$ be the associated 0-1 measure supported on $\cup_{n\geq k} A^n$, defined on power set of N. For $k \geq 1$, let η_k be the average of the 2^k measures $\eta_{\epsilon_1\cdots\epsilon_k}$ obtained varying $(\epsilon_1 \cdots \epsilon_k)$ over sequences of 0's and 1's of length k. Fix any Banach limit ℓ and set $\eta(A) = \ell\{\eta_k(A) : k \geq 1\}$ for $A \subset N$. Let $\gamma = \frac{1}{2}\eta + \frac{1}{2}\mu$. It is not difficult to see that γ is strongly continuous. Since $\eta_k(A_n) = 0$ for $k > n$, it follows that for any n, $\eta(A_n) = 0$. Using this, one can argue that criterion 6 of Theorem 4.3.7 fails for the sequence $\{A_n\}$ so that $L_1(\gamma)$ is not complete.

Remark (4.9.8). We shall see later examples of strongly continuous γ for which $L_1(\gamma)$ is complete. However, we are unable to decide if there are extensions of density charges for which the L_1 space is complete.

4.10 Finite Strategic Products

When dealing with finitely additive measures, product measures are not in general well defined on product σ fields. However, as we have already seen, there is one situation where the product probability measures are well defined by successive integration, namely strategic probabilities. So far we have considered strategic probabilities only on infinite product spaces. But in the same way we can define strategic probabilities on finite product spaces as well. As in the case of infinite product spaces here also we need to consider probability measures defined on power sets. So then, for $1 \leq i \leq k$, let γ_i be a finitely additive probability defined on power set of Ω_i. Let $\Omega = \otimes_{i=1}^{k}\Omega_i$. On power set of Ω define

$$\gamma(A) = \int \cdots \int 1_A(x_1, \cdots, x_k) d\gamma_k(x_k) \cdots d\gamma_1(x_1)$$

This γ is called the strategic product of the γ_i, $1 \leq i \leq k$. Carefully note the order of integration. For more on this, see [6]. The reader should note that even though the probabilities are defined on power sets, their L_1 spaces may still be incomplete, see Remarks 4.9.6 and 4.9.7 of the previous section.

Theorem 4.10.1 *Let γ be the strategic product of γ_i, $1 \le i \le k$.*

(a) If for each i, $L_1(\gamma_i)$ is complete then so is $L_1(\gamma)$.

(b) If $L_1(\gamma)$ is complete then so is $L_1(\gamma_1)$.

(c) $L_1(\eta^k)$ is complete iff $L_1(\eta)$ is complete. Here η^k is the k fold strategic product of η.

Proof : (c) is immediate from (a) and (b). To prove (b) Let f_n be Cauchy in $L_1(\gamma_1)$. Set $\tilde{f}_n(x_1, \cdots, x_k) = f_n(x_1)$ on Ω. Clearly $\tilde{f}_n$ is Cauchy in $L_1(\gamma)$ so that, by hypothesis, we can get a limit $\tilde{f} \in L_1(\gamma)$. Set $f(x) = \int \tilde{f}(x, x_2, \cdots, x_k) d\gamma_k(x_k) \cdots d\gamma_2(x_2)$. It is easy to check that $f_n \to f$ in $L_1(\gamma_1)$. Indeed,

$$\begin{aligned}
& \int |f_n(x) - f(x)| d\gamma_1(x) \\
= & \int | \int (f_n(x, x_2, \dots x_k) - \tilde{f}(x, x_2, \dots x_k)) d\gamma_k \dots d\gamma_2 | d\gamma_1(x) \\
\le & \int (\int |\tilde{f}_n - \tilde{f}|(x, x_2, \dots x_k)) d\gamma_1(x) \\
= & \int |\tilde{f}_n - \tilde{f}| d\gamma \longrightarrow 0
\end{aligned}$$

This shows that $L_1(\gamma_1)$ is complete. We shall now prove (a) using ideas from Theorem 4.3.4. For simplicity we consider $k = 2$. Let $\{f_n\}$ be Cauchy in $L_1(\gamma_1 \times \gamma_2)$. We shall assume that $0 \le f_n \le 1$ for each n — see Theorem 4.3.7(2). By passing to a subsequence if necessary, we shall assume that

$$\gamma_1\{x_1 : \int | f_n(x_1, x_2) - f_{n+1}(x_1, x_2) | d\gamma_2(x_2) > \frac{1}{2^n}\} < \frac{1}{2^n}$$

Let the set in braces be denoted by A_n. Since $L_1(\gamma_1)$ is complete, Theorem 4.3.7(3) applied to the sequence of sets $\{A_n : n \ge k\}$ gives us a set B_k such that,

$$\gamma_1(B_k) \le \frac{1}{2^{k-1}} \text{ and } \gamma_1(A_n \setminus B_k) = 0 \text{ for all } n \ge k.$$

By taking successive intersections if necessary, we shall assume that B_k decreases with k. Let $B_\infty = \cap_k B_k$. Clearly $\gamma(B_\infty) = 0$. Set $f(x_1, x_2) = 0$ if $x_1 \in B_\infty$. If $x_1 \notin B_\infty$, let $k(x_1)$ be the first integer k such that $x_1 \notin B_k$. If $x_1 \notin \cup_{n \geq k(x_1)} A_n$ then clearly, the sequence of functions $g_n(x_2) = f_n(x_1, x_2)$, defined for $n \geq k(x_1)$ is a Cauchy sequence in $L_1(\gamma_2)$ and hence has an L_1 limit say g depending on x_1, of course. Set $f(x_1, x_2) = g(x_2)$. If $x_1 \in \cup_{n \geq k(x_1)} A_n$ then choose the first integer n — say $n(x_1)$ — such that $x_1 \in A_n$ and $n \geq k(x_1)$. Set $f(x_1, x_2) = f_{n(x_1)}(x_1, x_2)$. This defines f on all of $\Omega_1 \times \Omega_2$. Note that if

$$C_k = (\cup_{n=1}^{k} A_n \setminus B_1) \cup (\cup_{n=2}^{k} A_n \setminus B_2) \cup \cdots \cup (A_k \setminus B_k) \cup B_k$$

then by choice of the sets B_k, the first k sets that appear on the right side above are γ_1 null so that

$$\gamma_1(C_k) = \gamma_1(B_k) \leq \frac{1}{2^{k-1}};$$

Suppose $x_1 \notin C_k$, then

$$\begin{aligned} \int |f_{n(x_1)}(x_1, x_2) - f_k(x_1, x_2)|\, d\gamma_2(x_2) &\leq \tfrac{1}{2^{k-1}}, \quad \text{for } x_1 \in \cup_{n \geq k(x_1)} A_n \\ \int |f_k(x_1, x_2) - f(x_1, x_2)|\, d\gamma_2(x_2) &\leq \tfrac{1}{2^{k-1}}, \quad \text{for } x_1 \notin \cup_{n \geq k(x_1)} A_n \end{aligned}$$

Using the above three inequalities, and performing another integration w.r.t. γ_1 one obtains that $f_n \to f$ in $L_1(\gamma_1 \times \gamma_2)$. This completes the proof of the Theorem.

Remark (4.10.2). If $L_1(\gamma_1 \times \gamma_2)$ be complete it is not necessary that $L_1(\gamma_2)$ be complete. In fact if γ_1 is 0-1 valued and not countably additive then $L_1(\gamma_1 \times \gamma_2)$ is complete, whatever be γ_2. To see this, we shall verify Theorem 4.3.7(4). Let $\{A_n\}$ be a sequence of subsets of $\Omega_1 \times \Omega_2$ and $\epsilon > 0$. Fix a partition $N_1, N_2, \cdots$ of Ω_1 such that

$\gamma_1(N_i) = 0$ for each i. Fix $n \geq 1$. Observe that $\gamma_1\{x : \gamma_2(A_n)_x < \gamma(A_n) + \frac{\epsilon}{2^{n+1}}\} = 1$ where $\gamma = \gamma_1 \times \gamma_2$. Let the set in braces be denoted by C_n. Define

$$B_n = \{(x, y) : x \in C_n, \ x \notin \cup_{k<n} N_k\} \cap A_n$$

We will now show that these sets verify Theorem 4.3.7(4). Since $A_n \setminus B_n \subset C_n \cup (\cup_{k<n} N_k) \times \Omega_2$, $\gamma(A_n \setminus B_n) = 0$. To see $\gamma(\cup B_n) \leq \sum \gamma(B_n)$ we first observe that

$$\gamma(\cup B_n) = \int \gamma_2(\cup B_n)_x d\gamma_1(x)$$

and for any fixed x there is exactly one k such that $x \in N_k$. So if $n \geq k$, then $(B_n)_x$ is empty. Therefore, for all x there exists a k, depending on x, such that $\gamma_2(\cup_{n=1}^{\infty} B_n)_x = \gamma_2(\cup_{n=1}^{k} B_n)_x \leq \sum_{n=1}^{k} \gamma_2(B_n)_x$. Hence,

$$\begin{aligned} \gamma(\cup_{n=1}^{\infty} B_n) & \\ &= \int \gamma_2(\cup_{n=1}^{\infty} B_n)_x d\gamma_1(x) \\ &\leq \int \textstyle\sum_{n=1}^{\infty} \gamma_2(B_n)_x d\gamma_1(x) \\ &\leq \int \textstyle\sum_{n=1}^{\infty} \gamma_2(A_n)_x 1_{C_n}(x) d\gamma_1(x) \\ &< \int \textstyle\sum_{n=1}^{\infty} (\gamma(A_n) + \frac{\epsilon}{2^{n+1}}) d\gamma_1 \\ &= \textstyle\sum_{n=1}^{\infty} \gamma(A_n) + \frac{\epsilon}{2} \end{aligned}$$

This verifies Theorem 4.3.7(4) and we are done. The same proof works even if γ_1 is a purely finitely additive probability which is a combination of uniformly singular 0-1 valued measures. The same argument shows that if γ is the product $\otimes_{i=1}^{k} \gamma_i$ and $L_1(\gamma)$ is complete then $L_1(\gamma_i)$ need not be complete for any $i > 1$.

4.11 Infinite Strategic Products

For each $n \geq 1$, let γ_n be a finitely additive probability defined on the power set of Ω. Let $H = \Omega^\infty$. Let σ be the strategic probability defined on the Borel σ-field of H and induced by independent strategy γ_n (for details see §0.1). This σ is called the strategic product of γ_n and will also be denoted by $\odot_{n=1}^{\infty}\gamma_n$. In case each $\gamma_i = \gamma$, where γ is a fixed finitely additive probability then σ, induced by i.i.d. strategy γ will be denoted by γ^∞. For this setup our results are only fragmentary. However, using the arguments given in the last section we can prove the following theorem:

Theorem 4.11.1 *Let σ be the strategic product of γ_i, $1 \leq i < \infty$ i.e. $\sigma = \odot_{i=i}^{\infty}\gamma_i$*

(a) If $L_1(\sigma)$ is complete then so is $L_1(\gamma_1)$

(b) If γ_1 is a combination of 0-1 valued uniformly singular purely finitely additive probabilities then whatever be γ_i for $i > 1$. $L_1(\sigma)$ is complete.

(c) If each γ_i is a fixed finitely additive probability γ which is a combination of 0-1 valued, uniformly singular and purely finitely additive probabilities then $L_1(\sigma)$ is complete.

Proof: For (a) see Theorem 4.10.1(b) and for (b) see Remark 4.10.2. Finally (c) is a special case of (b).

Using (c) we can construct an example of strongly continuous γ for which L_1 space is complete.

Example: Let δ_∞ (resp. $\delta_{-\infty}$) be a 0-1 valued, purely finitely additive probability concentrated on the set of positive (resp. negative) integers and let $\gamma = \frac{1}{2}\delta_\infty + \frac{1}{2}\delta_{-\infty}$. Then $\sigma = \gamma^\infty$ is strongly continuous and $L_1(\sigma)$ is complete by Theorem 4.11.1(c).

We have seen in the last section that on an infinite σfield we can always construct a strongly continuous probability whose L_1-space is not complete. Therefore one may ask the following question:

Is it true that given an infinite σfield we can always construct a strongly continuous probability on it such that its L_1-space is complete? The question remains open.

If we take $\Omega = \mathbb{Z}$,we can prove part (c) of Theorem 4.11.1 for a more general class of γ.

Theorem 4.11.2 *Let $\Omega = \mathbb{Z}$ and $\gamma = \sum_{i \geq 0} a_i \gamma_i$ where $a_i \geq 0$. $\forall\, i$ and $\sum_{i \geq 0} a_i = 1$. γ_0 is countably additive and γ_i is 0-1 valued for all $i \geq 1$ so that there exists a sequence of disjoint sets $\{A_i\}_{i \geq 0}$ such that $\gamma_i(A_i) = 1$. Then $L_1(\sigma)$ is complete where $\sigma = \gamma^\infty$.*

For the proof of the Theorem we need the following lemma.

Lemma 4.11.3. *Let X and Y be two polish spaces and $\mathcal{B}_X$ and $\mathcal{B}_Y$ be their respective Borel σ-fields. Let μ and ν be two finitely additive probabilities on $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively. Suppose T is a continuous and measure preserving map on X into Y. If there exists a Borel $D \subset X$ with $\mu(D) = 1$ such that $T : D \to T(D)$ is a homeomorphism and $T(D)$ is Borel in Y then $L_1(\mu)$ is complete implies $L_1(\nu)$ is complete.*

(In this lemma one could replace homeomorphism by Borel isomorphism. However, we stated it in the form in which it will be used later.)

<u>Proof</u>: Let $\{A_n\}$ be a ν-Cauchy sequence of sets in $\mathcal{B}_Y$. Consider $B_n = T^{-1}A_n$. Since T is measure preserving (i.e. $\nu(A) = \mu(T^{-1}A)$), $\{B_n\}$ is μ-Cauchy. By our assumption $L_1(\mu)$ is complete. Therefore there exists a $B \in \mathcal{B}_X$ such that $\mu(B_n \Delta B) \to 0$ (by Theorem 4.3.7(2)). Clearly, $\mu((D \cap B_n)\Delta(D \cap B)) = \mu(D \cap (B_n \Delta B)) \to 0$. Let $T(D \cap B) =$

A. Then $A \subset T(D)$. Since $T|D$ is a homeomorphism, A is a Borel subset of $T(D)$ and hence $A \in \mathcal{B}_Y$ as by our assumption $T(D) \in \mathcal{B}_Y$. Now, $\nu(A_n \Delta A) = \mu(T^{-1}A_n \Delta T^{-1}A) = \mu(B_n \Delta T^{-1}A) = \mu(D \cap (B_n \Delta T^{-1}A)) = \mu((D \cap B_n)\Delta(D \cap T^{-1}A)) = \mu((D \cap B_n)\Delta(D \cap B)) \to 0$. The third equality follows since $\mu(D) = 1$ and the last equality follows from our assumption that T is a homeomorphism and hence is one-one when restricted to D. ■

Proof of Theorem 4.11.2. Let $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$, $H = \mathbb{Z}^\infty$ and $H_\infty = (\mathbb{Z}_\infty)^\infty$. Let $H^1 = H \times H_\infty$. Then recall from chapter 3 that we can define a probability σ' on Borel σ-field of H^1 such that $\sigma' = \sigma_\mu \times \sigma_\lambda$, where $\mu = \frac{\sum_{i \geq 1} a_i \gamma_i}{1 - a_0}$ and $\lambda|\mathbb{Z} = a_0 \gamma_0$ and $\lambda(\{\infty\}) = 1 - a_0$. Also we can have a continuous map $T : H^1 \to H$ such that T is measure preserving, i.e., $\sigma'(T^{-1}A) = \sigma(A)$. By our assumption, γ_0 and $\sum_{i \geq 1} a_i \gamma_i$ are supported on disjoint sets. Therefore, as in 8^0 of §3.2 (see p.57) we will denote by H_1 the set of sequences of the points from the support of μ. We denote by H_2 the set of sequences of the points from the support of λ with infinitely many occurrences of ∞. Finally we set $D = H_1 \times H_2$ and R be the set of sequences of integers having infinitely many occurrences of elements from the support of μ. We have already seen in 8^0 of §3.2 (p.57) that $T|D$ is a homeomorphism and $TD = R$. Therefore, this T satisfies the hypothesis of Lemma 4.11.3. Also note that $L_1(H^1, \mathcal{B}_{H^1}, \sigma')$ is complete. Indeed, $H^1 = H \times H_\infty$ and $\sigma' = \sigma_\mu \times \sigma_\lambda$. $L_1(H, \mathcal{B}_H, \sigma_\mu)$ is complete by Theorem 4.11.1.(c) and $L_1(H_\infty, \mathcal{B}_{H_\infty}, \sigma_\lambda)$ is complete because σ_λ is countably additive. Therefore by the argument given in Theorem 4.10.1.(a) it follows that $L_1(H^1, \mathcal{B}_{H^1}, \sigma')$ is complete. Then by Lemma 4.11.3. $L_1(H, \mathcal{B}_H, \sigma)$ is complete and we are done. ■

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for $i = 1, 2, ..., m$ and $\alpha = 1, 2, ..., n$.

The *corrector matrices*, $\boldsymbol{M}_\varepsilon$, are defined as follows,

$$\boldsymbol{M}_\varepsilon e_i^\alpha = \boldsymbol{D}\wp_\varepsilon(\underline{\zeta}_i^{\alpha,\varepsilon}). \tag{4.4.3}$$

Then, $\boldsymbol{B}^*$ is given by the formula

$$\boldsymbol{B}^* = \lim_{\varepsilon \to 0} \chi_\varepsilon \boldsymbol{M}_\varepsilon^t \boldsymbol{B}_\varepsilon \boldsymbol{M}_\varepsilon \text{ in } D'(\Omega). \tag{4.4.4}$$

Let $F^1(\underline{\theta})$ be defined as follows.

$$F^1(\theta) = \frac{1}{2}\int_\Omega \boldsymbol{B}^* \boldsymbol{D}\underline{u} \,.\, \boldsymbol{D}\underline{u}\, dx$$

where $\underline{u} = \underline{u}(\underline{\theta})$ is the solution of

$$\left.\begin{aligned} -\text{div}(\boldsymbol{A}^*\boldsymbol{D}\underline{u}) + K\underline{u} &= \chi\underline{f} + \underline{\theta} \text{ in } \Omega, \\ \underline{u} &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{4.4.5}$$

It can be verified, as in the scalar case, by first introducing and homogenizing the state-adjoint state systems of equations that $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ defined above are the coefficients for the homogenized system. This implies, as in the scalar case, that the energies converge to an appropriate energy in which the matrices $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ appear naturally. All these show that F^1 defined above will satisfy (2.1.19) for a suitably modified Lemma 2.1.1. So, this and the discussion in Section 4.1, where F^2 and U_{ad} have been defined, show that $\underline{\theta}^*$ is the minimizer of the functional $F^1 + F^2$ over the domain U_{ad}.

In the *periodic case*, it is possible to give an explicit formula for $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$. Let S be a closed subset Y with Lipschitz boundary. We then define a periodically perforated domain as in Chapter 3 and we assume that Ω_ε is a connected set. Let $\boldsymbol{A} \in M_n^m(a, b, \mathbb{R}^n)$ and $\boldsymbol{B} \in M_n^m(c, d, \mathbb{R}^n)$ be Y-periodically defined block matrices and $\boldsymbol{B}$ is assumed to be symmetric. Define the sequences $\boldsymbol{A}_\varepsilon$ and $\boldsymbol{B}_\varepsilon$ as follows:

$$\boldsymbol{A}_\varepsilon(x) = \boldsymbol{A}(\frac{x}{\varepsilon}),\ \boldsymbol{B}_\varepsilon(x) = \boldsymbol{B}(\frac{x}{\varepsilon}).$$

We consider the homogenization of the problem $(\mathbf{P}_\varepsilon)$ defined with these coefficients on periodically perforated domains. To obtain the homogenized coefficients we need

$\chi_j(y)$ will denote the characteristic function of Y_j (j=1, 2) extended Y-periodically to all of R^N. Then, $\chi_\varepsilon^j(\frac{x}{\varepsilon})$ is clearly the characteristic function of Ω_ε^j; this will simply be denoted by χ_ε^j. $\Gamma_{1,2}^\varepsilon = \partial\Omega_1^\varepsilon \bigcap \partial\Omega_2^\varepsilon \bigcap \Omega$ will denote the interface of Ω_1^ε with Ω_2^ε which is interior to Ω and, $\Gamma_{1,2} = \partial Y_1 \bigcap \partial Y_2 \bigcap Y$ will denote the corresponding interface in the reference cell. We also set $\Omega_3^\varepsilon \equiv \Omega_2^\varepsilon$, $Y_3 \equiv Y_2$, and $\chi_3 \equiv \chi_2$, to be used to simplify notation at times.

Let $\mu_j : R^N \times R^N \longrightarrow R^N$ ($j = 1, 2, 3$) be Carathéodory functions, Y-periodic in the first and continuous in the second variable, for which there exist positive constants k, C, c_0 and $1 < p < \infty$ such that for every ξ, $\eta \in R^N$ and a.e. $y \in Y$

$$|\mu_j(y\,,\,\xi)| \;\le\; C|\,\xi\,|^{p-1} + k \tag{6.2.1}$$

$$(\mu_j(y\,,\,\xi) - \mu_j(y\,,\,\eta))\,.\,(\xi - \eta)) \;\ge\; 0 \tag{6.2.2}$$

$$\mu_j(y\,,\,\xi)\,.\,\xi \;\ge\; c_0|\,\xi\,|^p - k. \tag{6.2.3}$$

Let $c_j \in C_\sharp(Y)$ ($j = 1, 2, 3$) be continuous Y-periodic functions on R^N such that

$$0 < c_0 \le c_j \le C. \tag{6.2.4}$$

The exact microscopic model for diffusion in a partially fissured medium is given by the system

$$c_1(\frac{x}{\varepsilon})\frac{\partial u_1^\varepsilon}{\partial t} - \operatorname{div}\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u_1^\varepsilon\right) \;=\; 0 \quad \text{in } \Omega_1^\varepsilon \tag{6.2.5}$$

$$c_2(\frac{x}{\varepsilon})\frac{\partial u_2^\varepsilon}{\partial t} - \operatorname{div}\mu_2\left(\frac{x}{\varepsilon}\,,\,\nabla u_2^\varepsilon\right) \;=\; 0 \quad \text{in } \Omega_2^\varepsilon \tag{6.2.6}$$

$$c_3(\frac{x}{\varepsilon})\frac{\partial u_3^\varepsilon}{\partial t} - \varepsilon\operatorname{div}\mu_3\left(\frac{x}{\varepsilon}\,,\,\varepsilon\nabla u_3^\varepsilon\right) \;=\; 0 \quad \text{in } \Omega_2^\varepsilon \tag{6.2.7}$$

$$\alpha u_2^\varepsilon + \beta u_3^\varepsilon \;=\; u_1^\varepsilon \quad \text{on } \Gamma_{1,2}^\varepsilon \tag{6.2.8}$$

$$\alpha\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u_1^\varepsilon\right)\,.\,\nu_1^\varepsilon \;=\; \mu_2\left(\frac{x}{\varepsilon}\,,\,\nabla u_2^\varepsilon\right).\nu_1^\varepsilon \tag{6.2.9}$$

$$\beta\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u_1^\varepsilon\right)\,.\,\nu_1^\varepsilon \;=\; \varepsilon\mu_3\left(\frac{x}{\varepsilon}\,,\,\nabla\varepsilon u_3^\varepsilon\right).\nu_1^\varepsilon \tag{6.2.10}$$

where the last two conditions hold on $\Gamma_{1,2}^\varepsilon$. We have the homogeneous Neumann condition on the external boundary

$$\mu_1\left(\frac{x}{\varepsilon},\, \nabla u_1^\varepsilon\right).\nu = 0 \text{ on } \partial\Omega_1^\varepsilon \cap \partial\Omega \tag{6.2.11}$$
$$\mu_2\left(\frac{x}{\varepsilon},\, \nabla u_2^\varepsilon\right).\nu = 0 \text{ on } \partial\Omega_2^\varepsilon \cap \partial\Omega \tag{6.2.12}$$
$$\mu_3\left(\frac{x}{\varepsilon},\, \varepsilon\nabla u_3^\varepsilon\right).\nu = 0 \text{ on } \partial\Omega_2^\varepsilon \cap \partial\Omega \tag{6.2.13}$$

where ν denotes the outward normal on $\partial\Omega$.

The system is completed by the initial conditions

$$u_j^\varepsilon(0\,,.) = u_j^0 \in L^2(\Omega),\ 1 \le j \le 3. \tag{6.2.14}$$

$u_1^\varepsilon(x,t)$ is the flow in the fissures Ω_1^ε with the flux given by $-\mu_1\left(\frac{x}{\varepsilon}, \nabla u_1^\varepsilon\right)$. The flow in the matrix has two components: $u_2^\varepsilon(x,t)$ with the flux $-\mu_2\left(\frac{x}{\varepsilon}, \nabla u_2^\varepsilon\right)$, is the usual flow through the matrix and; the slow scale flow $u_3^\varepsilon(x,t)$ with flux $-\varepsilon\mu_3\left(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon\right)$, leading to local storage in the matrix. The "total flow " in the matrix is $\alpha u_2^\varepsilon + \beta u_3^\varepsilon$, where $\alpha + \beta = 1$ with $\alpha \ge 0, \beta > 0$. (6.2.8) represents the continuity of flow across the interface and (6.2.9), (6.2.10) determine the partition of flux across the interface.

We now describe the variational formulation needed to study the well posedness of the Cauchy problem. The *state space* is the Hilbert space

$$H_\varepsilon \equiv L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon) \times L^2(\Omega_2^\varepsilon)\ (= L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon)^2)$$

equipped with the inner product

$$([u_1,\, u_2,\, u_3], [\phi_1,\, \phi_2,\, \phi_3])_{H_\varepsilon} = \sum_{j=1}^{3} \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})\, u_j(x)\, \phi_j(x)\, dx.$$

Define the *energy space*

$$B_\varepsilon \equiv H_\varepsilon \cap \{[\vec{u}] \in W^{1,p}(\Omega_1^\varepsilon) \times W^{1,p}(\Omega_2^\varepsilon)^2 : u_1 = \alpha u_2 + \beta u_3 \text{ on } \Gamma_{1,2}^\varepsilon\}$$

where $\vec{u} = (u_1, u_2, u_3)$. B_ε is a Banach space with the norm

$$\|\, [u_1,\, u_2,\, u_3]\|_{B_\varepsilon} \equiv \sum_{j=1}^{3} \|\, \chi_j^\varepsilon\, u_j\, \|_{L^2(\Omega)} + \sum_{j=1}^{3} \|\, \chi_j^\varepsilon\, \nabla\, u_j\, \|_{L^p(\Omega)}.$$

Define the operator $A_\varepsilon : B_\varepsilon \longrightarrow B_\varepsilon'$ (where B_ε' denotes the dual of B_ε) by,

$$A_\varepsilon([u_1, u_2, u_3])([\phi_1, \phi_2, \phi_3]) \equiv \sum_{j=1}^{2} \int_{\Omega_j^\varepsilon} \mu_j(\tfrac{x}{\varepsilon}, \nabla u_j) . \nabla\phi_j \, dx + \int_{\Omega_2^\varepsilon} \mu_3(\tfrac{x}{\varepsilon}, \varepsilon\nabla u_3) . \varepsilon\nabla\phi_3 \, dx$$

for $[u_1, u_2, u_3]$, $[\phi_1, \phi_2, \phi_3] \in B_\varepsilon$.

Let $V_\varepsilon \equiv \{\overrightarrow{u^\varepsilon} \in L^p([0,T]; B_\varepsilon) : (\overrightarrow{u^\varepsilon})' \in L^q([0,T]; B_\varepsilon')\}$, q being $p/(p-1)$.

For $\varepsilon > 0$, the Cauchy problem is equivalent to finding a solution $\overrightarrow{u^\varepsilon} \in V_\varepsilon$ to the problem

$$\frac{d\overrightarrow{u^\varepsilon}}{dt} + A_\varepsilon \overrightarrow{u^\varepsilon} = 0 \text{ in } L^q([0,T]; B_\varepsilon') \tag{6.2.15}$$

$$\overrightarrow{u^\varepsilon}(0) = \overrightarrow{u^0} \text{ in } H_\varepsilon \tag{6.2.16}$$

and this problem is well-posed, thanks to the conditions (6.2.1)-(6.2.3) (cf. Showalter [34]). We end with an identity(cf. [14]),

$$\frac{1}{2}\left\|\overrightarrow{u^\varepsilon}(T)\right\|_{H_\varepsilon}^2 - \frac{1}{2}\left\|\overrightarrow{u^\varepsilon}(0)\right\|_{H_\varepsilon}^2 + \int_0^T A_\varepsilon(\overrightarrow{u^\varepsilon})(\overrightarrow{u^\varepsilon})dt = 0. \tag{6.2.17}$$

6.3 Homogenization

The micro-model presented in the previous section was homogenized in [14], using two-scale convergence; we recall the main results.

In this case, the definition of two-scale convergence (cf. [1], [14]) is the following.

Definition 6.3.1 *A function, $\psi(t,x,y) \in L^q([0,T] \times \Omega, C_\sharp(Y))$, which is Y-periodic in y and satisfies*

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \psi\left(t, x, \frac{x}{\varepsilon}\right)^q dx\, dt = \int_0^T \int_\Omega \int_Y \psi(t,x,y)^q \, dy\, dx\, dt$$

is called an admissible *test function.* ■

Definition 6.3.2 *A sequence f^ε in $L^p([0,T] \times \Omega))$ two-scale converges to a function $f(t,x,y) \in L^p([0,T] \times \Omega \times Y)$ if for any admissible test function $\psi(t,x,y)$,*

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega f^\varepsilon(t,x)\, \psi\left(t, x, \frac{x}{\varepsilon}\right) dx\, dt = \int_0^T \int_\Omega \int_Y f(t,x,y)\, \psi(t,x,y)\, dy\, dx\, dt$$

We write $f^\varepsilon \xrightarrow{2-s} f$. ■

Remark 6.3.1 *The space of admissible functions used in the definition of two-scale convergence differs from the one used in Chapter 3. But this is justified by Remark 3.2.3. Two-scale convergence is also obtainable for sequences in L^p spaces, $1 < p < \infty$ (cf. Allaire [1]).* ■

Proposition 6.3.1 *[14] Let $\overrightarrow{u^\varepsilon}$ be the solution of the Cauchy problem (6.2.5)-(6.2.14). The following estimate holds*

$$\sum_{j=1}^{2} \|\chi_j^\varepsilon \nabla u_j^\varepsilon\|_{p,\Omega_T}^p + \|\chi_2^\varepsilon \varepsilon \nabla u_3^\varepsilon\|_{p,\Omega_T}^p \le \frac{C}{2c} \sum_{j=1}^{3} \|u_j^0\|_{2,\Omega}^2 . \blacksquare \tag{6.3.1}$$

Proposition 6.3.2 *[14] Let $\overrightarrow{u^\varepsilon}$ be the solution of the Cauchy problem (6.2.5)-(6.2.14). There exist functions u_j in $L^p([0,T]; W^{1,p}(\Omega))$, $j = 1, 2$ and functions U_j in $L^p([0,T] \times \Omega; W_\sharp^{1,p}(Y_j)/R)$, $j = 1, 2, 3$ such that, for a subsequence of $\overrightarrow{u^\varepsilon}$, (to be indexed by ε again) the following hold:*

$$\begin{aligned}
\chi_j^\varepsilon u_j^\varepsilon &\xrightarrow{2-s} \chi_j(y) u_j(t,x), \quad j = 1, 2, \\
\chi_2^\varepsilon u_3^\varepsilon &\xrightarrow{2-s} \chi_2(y) U_3(t,x,y), \\
\chi_j^\varepsilon \nabla u_j^\varepsilon &\xrightarrow{2-s} \chi_j(y)(\nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)), \quad j = 1, 2, \\
\varepsilon \chi_2^\varepsilon \nabla u_3^\varepsilon &\xrightarrow{2-s} \chi_2(y) \nabla_y U_3(t,x,y), \\
\chi_j^\varepsilon \mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon) &\xrightarrow{2-s} \chi_j(y) \mu_j(y, \nabla_x u_j + \nabla_y U_j), \quad j = 1, 2, \\
\chi_2^\varepsilon \mu_3(\frac{x}{\varepsilon}, \varepsilon \nabla u_3^\varepsilon) &\xrightarrow{2-s} \chi_2(y) \mu_3(y, \nabla_y U_3), \\
\chi_j^\varepsilon u_j^\varepsilon(T,x) &\xrightarrow{2-s} \chi_j(y) u_j(T,x), \quad j = 1, 2, \\
\chi_2^\varepsilon u_3^\varepsilon(T,x) &\xrightarrow{2-s} \chi_2(y) U_3(T,x,y) \text{ and}, \\
u_1(t,x) &= \alpha u_2(t,x) + \beta U_3(t,x,y) \text{ for all } y \in \Gamma_{1,2}. \quad \blacksquare
\end{aligned}$$

Proposition 6.3.3 *[14] The functions u_1, u_2, U_1, U_2, U_3 satisfy the homogenized sys-*

tem

$$
\begin{aligned}
&-\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j} c_j(y)u_j\frac{\partial\phi_j}{\partial t}\,dy\,dx\,dt - \int_0^T\int_\Omega\int_{Y_2} c_3(y)U_3\frac{\partial\Phi_3}{\partial t}\,dy\,dx\,dt\\
&-\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)u_j^0\,\phi_j(0,x)\,dy\,dx - \int_\Omega\int_{Y_2} c_3(y)u_3^0\,\Phi_3(0,x,y)\,dy\,dx\\
&+\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j} \mu_j(y,\nabla_x u_j+\nabla_y U_j).(\nabla_x\phi_j+\nabla_y\Phi_j)\,dy\,dx\,dt\\
&+\int_0^T\int_\Omega\int_{Y_2}\mu_3(y,\nabla_y U_3).(\nabla_y\Phi_3)\,dy\,dx\,dt = 0
\end{aligned}
\tag{6.3.2}
$$

for all

$$
\begin{aligned}
\phi_j(t,x) &\in L^p([0,T];W^{1,p}(\Omega)),\ j=1,2\\
\Phi_j(t,x,y) &\in L^p([0,T]\times\Omega;W^{1,p}_\sharp(Y_j)),\ j=1,2,3
\end{aligned}
$$

satisfying

$$
\begin{aligned}
\frac{\partial\phi_j}{\partial t} &\in L^q([0,T];W^{-1,q}(\Omega)),\ j=1,2\\
\frac{\partial\Phi_j}{\partial t} &\in L^q([0,T]\times\Omega;(W^{1,p}_\sharp(Y_j))'),\ j=1,2,3
\end{aligned}
$$

$$\beta\Phi_3(t,x,y)=\phi_1(t,x)-\alpha\phi_2(t,x)\ \textit{for all}\ y\in\Gamma_{1,2}\ \textit{and,}$$

$$\phi_1(T,x)=\phi_2(T,x)=\Phi_3(T,x,y)=0.\quad\blacksquare$$

The strong form of the homogenized problem has the following description. The *state space* is $H\equiv L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega\times Y_2)$ equipped with the scalar product

$$
\begin{aligned}
(\overrightarrow{\psi},\overrightarrow{\phi})_H &= \sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\psi_j(x)\phi_j(x)\,dy\,dx\\
&\quad+\int_\Omega\int_{Y_2} c_3(y)\Psi_3(x,y)\Phi_3(x,y)\,dy\,dx
\end{aligned}
$$

for every $\overrightarrow{\psi}=[\psi_1,\psi_2,\Psi_3]$, $\overrightarrow{\phi}=[\phi_1,\phi_2,\Phi_3]\in H$. Define the *energy space*,

$$
\begin{aligned}
B\equiv\{[\phi_1,\phi_2,\Phi_3]\in\ & H\cap W^{1,p}(\Omega)\times W^{1,p}(\Omega)\times L^2(\Omega;W^{1,p}_\sharp(Y_2)/R)\\
:\ & \beta\Phi_3(x,y)=\phi_1(x)-\alpha\phi_2(x,y)\text{ for all }y\in\Gamma_{1,2}\}
\end{aligned}
$$

and the corresponding evolution space $V\equiv L^p([0,T];B)$.

Proposition 6.3.4 *[14] $\vec{u} = [u_1, u_2, U_3] \in V$ and is the solution of the strong homogenized system,*

$$\left(\int_{Y_1} c_1(y)\,dy\right)\frac{\partial u_1}{\partial t}(t,x) + \frac{1}{\beta}\frac{\partial}{\partial t}\left(\int_{Y_2} c_3(y)U_3(t,x,y)\,dy\right) = \mathrm{div}_x\left(\int_{Y_1} \mu_1(y, \nabla_x u_1 + \nabla_y U_1)\,dy\right) \quad (6.3.3)$$

$$\left(\int_{Y_2} c_2(y)\,dy\right)\frac{\partial u_2}{\partial t}(t,x) - \frac{\alpha}{\beta}\frac{\partial}{\partial t}\left(\int_{Y_2} c_3(y)U_3(t,x,y)\,dy\right) = \mathrm{div}_x\left(\int_{Y_2} \mu_2(y, \nabla_x u_2 + \nabla_y U_2)\,dy\right) \quad (6.3.4)$$

$$c_3(y)\frac{\partial U_3(t,x,y)}{\partial t} - \mathrm{div}_y\, \mu_3(y, \nabla U_3(t,x,y)) = 0 \quad (6.3.5)$$

where $U_3(t,x,y)$ and $\mu_3(y, \nabla_y U_3(t,x,y)).\nu$ are Y-periodic and,

$$\beta U_3(t,x,y) = u_1(t,x) - \alpha u_2(t,x) \text{ for } y \in \Gamma_{1,2} \quad (6.3.6)$$

with boundary conditions

$$\int_{Y_1} \mu_1(y, \nabla_x u_1 + \nabla_y U_1)\,dy.\nu = 0 \text{ on } \partial\Omega \quad (6.3.7)$$

$$\int_{Y_2} \mu_2(y, \nabla_x u_2 + \nabla_y U_2)\,dy.\nu = 0 \text{ on } \partial\Omega \quad (6.3.8)$$

and initial conditions

$$u_j(0,x) = u_j^0(x) \; j = 1,2; \; U_3(0,x,y) = u_3^0(x). \quad (6.3.9)$$

The functions $U_j(t,x,y)$ solve the cell problems,

$$\mathrm{div}_y\, \mu_j(y, \nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)) = 0 \text{ for } y \in Y_j \quad (6.3.10)$$

$$\mu_j(y, \nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)).\nu = 0 \text{ on } \Gamma_{1,2} \text{ and} \quad (6.3.11)$$

Y-periodic on $\Gamma_{2,2}$, for $j = 1, 2$. In the above, t, x are treated as parameters and the cell equations are solved. ■

For $\xi \in R^N$, define the following functions;

$$\lambda_j(\xi) = \int_{Y_j} \mu_j(y, \xi + \nabla_y V_j^\xi(y)\, dy, \ j = 1, 2 \tag{6.3.12}$$

where V_j^ξ is the Y-periodic solution of

$$\begin{aligned} \operatorname{div}_y \mu_j(y, \xi + \nabla_y V_j^\xi(y)) &= 0 \text{ in } Y_j \qquad (6.3.13) \\ \mu_j(y, \xi + \nabla_y V_j^\xi(y)).\nu &= 0 \text{ on } \Gamma_{1,2} \qquad (6.3.14) \end{aligned}$$

Then, because of (6.3.10), (6.3.11), the right hand sides in (6.3.3), (6.3.4) can be replaced by the functions $\operatorname{div}_x \lambda_1(\nabla_x u_1(t,x))$ and $\operatorname{div}_x \lambda_2(\nabla_x u_2(t,x))$ respectively. Also the left hand sides of (6.3.7), (6.3.8) can be replaced by $\lambda_1(\nabla_x u_1).\nu$ and $\lambda_2(\nabla_x u_2).\nu$ respectively.

Remark 6.3.2 *We note that the functions λ_j can be interpreted as the integrands in the Γ – limit of the functionals*

$$F_{j,\varepsilon}(\nabla v) = \int_\Omega \chi_j^\varepsilon \mu_j(\frac{x}{\varepsilon}, \nabla v)\, dx\,.$$

In fact, $\Gamma - \lim F_{j,\varepsilon}(\nabla v) = \int_\Omega \lambda_j(\nabla v)\, dx$ (cf. Dal Maso [16]). Further, the functions $\lambda_j,\ j = 1, 2$ satisfy conditions (6.2.1)-(6.2.3) for the same p but maybe for different constants $\tilde{k}, \tilde{C}, \tilde{c}_0$ (cf. [17], [10]). ■

Proposition 6.3.5 *[14] The following* energy identity *holds*

$$\begin{aligned} \frac{1}{2}\sum_{j=1}^{2} \int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\, dy\, dx + \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\, dy\, dx \\ -\frac{1}{2}\sum_{j=1}^{2} \int_\Omega\int_{Y_j} c_j(y)|u_j^0(x)|^2\, dy\, dx - \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|u_3^0(x)|^2\, dy\, dx \\ +\sum_{j=1}^{2} \int_0^T \int_\Omega\int_{Y_j} \mu_j(y, \nabla_x u_j + \nabla_y U_j).(\nabla_x u_j + \nabla_y U_j)\, dy\, dx\, dt \\ + \int_0^T \int_\Omega\int_{Y_2} \mu_3(y, \nabla_y U_3).\nabla_y U_3\, dy\, dx\, dt = 0. \end{aligned}$$ ■

6.4 Correctors

We now prove corrector results for the gradient of flows under stronger hypotheses on μ_j's than (6.2.1)-(6.2.3). Let $k_1, k_2 > 0$ be constants and assume now that the μ_j's are Carathéodory functions, Y-periodic in the ~~second~~ first variable, satisfying for $\xi, \eta \in R^N$ with $|\xi| + |\eta| > 0$ and a.e. $y \in Y$:

$$\mu_j(y,0) = 0, \tag{6.4.1}$$

$$|\mu_j(y,\xi) - \mu_j(y,\eta)| \leq k_1(|\xi| + |\eta|)^{p-2}|\xi - \eta|, \tag{6.4.2}$$

$$(\mu_j(y,\xi) - \mu_j(y,\eta)).(\xi - \eta) \geq k_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2. \tag{6.4.3}$$

Remark 6.4.1 *Note that (6.4.1) and (6.4.2) imply*

$$|\mu_j(y,\xi)| \leq k_1|\xi|^{p-1} \tag{6.4.4}$$

and, (6.4.1) and (6.4.3) imply

$$\mu_j(y,\xi).\xi \geq k_2|\xi|^p. \tag{6.4.5}$$

Thus, the new hypotheses are indeed stronger than the original hypotheses on μ_j's. Moreover,

$$(\mu_j(y,\xi) - \mu_j(y,\eta)).(\xi - \eta) \geq k_2|\xi - \eta|^p \quad \text{if } p \geq 2 \tag{6.4.6}$$

$$|\mu_j(y,\xi) - \mu_j(y,\eta)| \leq k_1|\xi - \eta|^{p-1} \quad \text{if } 1 < p < 2. \tag{6.4.7}$$

These inequalities follow from (6.4.3) and (6.4.2) and triangle inequality in R^N. ■

Remark 6.4.2 *An example of μ_j satisfying (6.4.1)- (6.4.3) is $\mu_j = |\xi|^{p-2}\xi$, i.e. the corresponding diffusion operator is the p-Laplacian. Let Γ, γ be positive constants. The following class of functions, $f \in C^0(\overline{\Omega} \times R^N; R^N) \cap C^1(\Omega \times R^N \setminus \{0\}; R^N)$, which satisfy condition (6.4.1) and*

$$\sum_{i,j=1}^{N} |\frac{\partial f_j}{\partial \eta_i}|(x,\eta) \leq \Gamma|\eta|^{p-2}$$

$$\sum_{i,j=1}^{N} |\frac{\partial f_j}{\partial \eta_i}|(x,\eta)\xi_i\xi_j \geq \gamma|\eta|^{p-2}|\xi|^2$$

for all $x \in \Omega, \eta \in R^N \setminus \{0\}$ and $\xi \in R^N$, also satisfy (6.4.1)-(6.4.3) (cf. Damascelli [18]). ■

Let $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$ be the solution of the Cauchy problem (6.2.5)- (6.2.14) and let $[u_1, u_2, U_1, U_2, U_3]$ be as in Section 6.3. We will denote $[0,T] \times \Omega$ by Ω_T. Define the sequence of functions

$$\xi_j(t,x,y) \equiv \chi_j(y)(\nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)), \ j = 1,2, \tag{6.4.8}$$

$$\xi_3(t,x,y) \equiv \chi_2(y)\nabla_y U_3(t,x,y) \tag{6.4.9}$$

and let,

$$\xi_j^\varepsilon(t,x) \equiv \xi_j(t,x,\frac{x}{\varepsilon}), \ j = 1,2,3. \tag{6.4.10}$$

The main theorems of this Chapter are the following:

Theorem 6.4.1 *Let ξ_j^ε's be as above and assume that the functions $\nabla_y U_j, \ j = 1,2,3$ are admissible (cf. Definition 6.3.1), then*

$$\overline{\lim}_{\varepsilon \to 0} \left\| \chi_j(\frac{x}{\varepsilon}) \left(\nabla u_j^\varepsilon(t,x) - \xi_j^\varepsilon(t,x)\right)\right\|_{p,\Omega_T} \longrightarrow 0, \ j=1,2,$$

$$\overline{\lim}_{\varepsilon \to 0} \left\| \chi_2(\frac{x}{\varepsilon}) \left(\varepsilon\nabla u_3^\varepsilon(t,x) - \xi_3^\varepsilon(t,x)\right)\right\|_{p,\Omega_T} \longrightarrow 0. \blacksquare$$

Theorem 6.4.2 *Under the same assumptions as in Theorem 6.4.1*

$$\overline{\lim}_{\varepsilon \to 0} \left\| \chi_j(\frac{x}{\varepsilon}) \left(\mu_j\left(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon\right) - \mu_j\left(\frac{x}{\varepsilon}, \xi_j^\varepsilon(t,x)\right)\right)\right\|_{q,\Omega_T} \longrightarrow 0, \ j=1,2,$$

$$\overline{\lim}_{\varepsilon \to 0} \left\| \chi_2(\frac{x}{\varepsilon}) \left(\mu_3\left(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon\right) - \mu_3\left(\frac{x}{\varepsilon}, \xi_3^\varepsilon(t,x)\right)\right)\right\|_{q,\Omega_T} \longrightarrow 0. \blacksquare$$

Remark 6.4.3 *Theorem 6.4.1 shows that $\chi_j(\frac{x}{\varepsilon})\nabla_x(u_j(t,x) + \varepsilon U_j(t,x,\frac{x}{\varepsilon}))$ strongly approximates $\chi(\frac{x}{\varepsilon})\nabla u_j^\varepsilon(t,x)$ for $j = 1,2$ in $L^p([0,T] \times \Omega)$, whereas Proposition 6.3.2 only implies that these two sequences have the same two-scale limit and hence, the same weak limit in L^P. Similarly, for the third component of the flow. Theorem 6.4.2 is about a strong approximation for the flux terms. The utility of the corrector results* *lie in the fact that the approximations involve the homogenized Cauchy problem and cell problems which are computationally simpler compared to the original Cauchy problem. In this context, it is desirable to get order of ε estimates for the corrector results and, this is still open.* ■

We first prove a few lemmas yielding some limits and estimates required in proving Theorems 6.4.1 and 6.4.2.

Henceforth, M will denote a generic constant which does not depend on ε, but probably on p, k_1, k_2, c_0, C, and the L^2 norm of the initial vector $\overrightarrow{u^0}$. Let $0 < \kappa < 1$ be a constant and $\Phi_j(t,x,y)$ be admissible test functions such that

$$\sum_{j=1}^{3} \|\nabla_y U_j - \Phi_j\|^p_{p,[0,T]\times\Omega\times Y_j} \leq \kappa.$$

Note that,

$$\Phi_j(t,x,\frac{x}{\varepsilon}) \xrightarrow{2-s} \Phi_j(t,x,y)$$

for j=1,2,3. Define the functions:

$$\eta_j^\varepsilon(t,x) = \chi_j(\frac{x}{\varepsilon})(\nabla_x u_j(t,x) + \Phi_j(t,x,\frac{x}{\varepsilon})),\ j = 1,2 \tag{6.4.11}$$

$$\eta_3^\varepsilon(t,x) = \chi_2(\frac{x}{\varepsilon})\Phi_3(t,x,\frac{x}{\varepsilon}). \tag{6.4.12}$$

Then we note that the functions $\eta_j^\varepsilon(t,x)$ and $\mu_j^\varepsilon(\frac{x}{\varepsilon},\eta_j^\varepsilon(t,x))$ arise from admissible test functions and we have the following two-scale convergence (cf. [14]),

$$\begin{aligned}
\eta_j^\varepsilon &\xrightarrow{2-s} \chi_j(y)(\nabla_x u_j(t,x) + \Phi_j(t,x,y)) \doteq \eta_j(t,x,y),\ j = 1,2,\\
\eta_3^\varepsilon &\xrightarrow{2-s} \chi_2(y)\Phi_3(t,x,y) \doteq \eta_3(t,x,y),\\
\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon) &\xrightarrow{2-s} \chi_j(y)\,\mu_j(y\,,\,\eta_j(t,x,y)),\ j = 1,2,3.
\end{aligned}$$

Lemma 6.4.1 *(cf. Lemma 3.1 [17]) Let $1 < p < 2$ and $\phi_1, \phi_2 \in L^p(\Omega_T)^N$. Then,*

$$\begin{aligned}
\|\phi_1 - \phi_2\|^p_{p,\Omega_T} \leq &\left[\int_0^T\int_\Omega |\phi_1 - \phi_2|^2(|\phi_1| + |\phi_2|)^{p-2}\chi\,dx\,dt\right]^{\frac{p}{2}}\\
&\times\left[\int_0^T\int_\Omega (|\phi_1| + |\phi_2|)^p\,dx\,dt\right]^{\frac{2-p}{2}}
\end{aligned}$$

where χ denotes the characteristic function of the set

$$\{(t,x) \in [0,T] \times \Omega : |\phi_1|(t,x) + |\phi_2|(t,x) > 0\}.$$

Proof: Multiply and divide the integrand in left hand side by $(|\phi_1| + |\phi_2|)^{(2-p)p/2}$ and apply Hölder's inequality to get the result. ■

Lemma 6.4.2

$$\sum_{j=1}^{2} \|\chi_j(y)(\nabla_x u_j + \nabla_y U_j)\|_p^p + \|\chi_2(y)\nabla_y U_3\|_p^p \le \frac{C}{2k_2} \sum_{j=1}^{3} \|u_j^0\|_{2,\Omega}^2$$

Proof: Follows from the energy identity (Proposition 6.3.5) and (6.4.5). ■

Lemma 6.4.3 *Let $\xi_i, \eta_i, \xi_i^\varepsilon, \eta_i^\varepsilon, i = 1,2,3$ be functions as defined above. Then,*

$$\overline{\lim}_{\varepsilon\to 0} \int_0^T \int_{\Omega_i^\varepsilon} \left(\mu_i(\frac{x}{\varepsilon}, \nabla u_i^\varepsilon) - \mu_i(\frac{x}{\varepsilon}, \eta_i^\varepsilon)\right).(\nabla u_i^\varepsilon - \eta_i^\varepsilon)\, dx\, dt$$
$$\le \textstyle\sum_{j=1}^3 \int_0^T \int_\Omega \int_{Y_j} (\mu_j(y,\xi_j) - \mu_j(y,\eta_j)).(\xi_j - \eta_j)\, dy\, dx\, dt$$

for i=1,2 and

$$\overline{\lim}_{\varepsilon\to 0} \int_0^T \int_{\Omega_2^\varepsilon} \left(\mu_3(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon) - \mu_3(\frac{x}{\varepsilon}, \eta_3^\varepsilon)\right).(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\, dx\, dt$$
$$\le \textstyle\sum_{j=1}^3 \int_0^T \int_\Omega \int_{Y_j} (\mu_j(y,\xi_j) - \mu_j(y,\eta_j)).(\xi_j - \eta_j)\, dy\, dx\, dt\,.$$

Proof: Denote the integrals appearing in the left hand sides of the above relations by $l_1^\varepsilon, l_2^\varepsilon$ and l_3^ε respectively. Then for i=1,2,3, using (6.2.17), we obtain,

$$\begin{aligned}
l_i^\varepsilon &\le \sum_{j=1}^{3} l_j^\varepsilon \\
&= \frac{1}{2}\sum_{j=1}^{3} \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^0(x)|^2\, dx - \frac{1}{2}\sum_{j=1}^{3} \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(T,x)|^2\, dx \\
&\quad - \sum_{j=1}^{2} \int_0^T \int_{\Omega_j^\varepsilon} \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon).(\nabla u_j^\varepsilon - \eta_j^\varepsilon)\, dx\, dt \\
&\quad - \int_0^T \int_{\Omega_j^\varepsilon} \mu_3(\frac{x}{\varepsilon}, \eta_3^\varepsilon).(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\, dx\, dt \\
&\quad - \sum_{j=1}^{2} \int_0^T \int_{\Omega_j^\varepsilon} \mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon).\eta_j^\varepsilon\, dx\, dt - \int_0^T \int_{\Omega_2^\varepsilon} \mu_3(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon).\eta_3^\varepsilon\, dx\, dt
\end{aligned}$$

We now use the two-scale convergence properties of various functions discussed so far to pass to the limit. We get,

$$\begin{aligned}\overline{\lim}_{\varepsilon\to 0}\sum_{j=1}^{3} l_j^\varepsilon \;=\;& \frac{1}{2}\sum_{j=1}^{3}\int_\Omega\int_{Y_j} c_j(y)|u_j^0(x)|^2\,dy\,dx \\ &- \underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(T,x)|^2\,dx \\ &- \sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\eta_j).(\xi_j-\eta_j)\,dy\,dx\,dt \\ &- \sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\xi_j).\eta_j\,dy\,dx\,dt\end{aligned}$$

The right hand side can be written as

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{3}\int_\Omega\int_{Y_j} c_j(y)|u_j^0(x)|^2\,dy\,dx - \underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx \\ &\quad+\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\left(\mu_j(y,\xi_j)-\mu_j(y,\eta_j)\right).(\xi_j-\eta_j)\,dy\,dx\,dt \\ &\quad-\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\xi_j).\xi_j\,dy\,dx\,dt\end{aligned}$$

which, using Proposition 6.3.5 to replace the last expression, is nothing but,

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\,dy\,dx + \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\,dy\,dx \\ &\quad- \underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx \\ &\quad+\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\left(\mu_j(y,\xi_j)-\mu_j(y,\eta_j)\right).(\xi_j-\eta_j)\,dy\,dx\,dt\end{aligned}$$

However, by standard arguments,

$$\begin{aligned}&\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\,dy\,dx + \int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\,dy\,dx \\ &\quad\le \underline{\lim}_{\varepsilon\to 0}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx\end{aligned}$$

This completes the proof. ■

Lemma 6.4.4 *Let ξ_j, η_j, κ be as before. Then,*

$$\sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} (\mu_j(y,\xi_j) - \mu_j(y,\eta_j)) .(\xi_j - \eta_j)\, dy\, dx\, dt \le M\kappa^{\delta(p)}$$

where

$$\delta(p) = \begin{cases} 1 & \text{if } 1 < p < 2, \\ \frac{2}{p} & \text{if } p \ge 2. \end{cases}$$

Proof: Let the left hand side of the estimate be denoted by S.

Case 1: $1 < p < 2$. Using (6.4.7) we get,

$$\begin{aligned} S &\le \sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} |(\mu_j(y,\xi_j) - \mu_j(y,\eta_j))||(\xi_j - \eta_j)|\, dy\, dx\, dt \\ &\le k_1 \sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} |\xi_j - \eta_j|^p\, dy\, dx\, dt \\ &\le M\kappa \end{aligned}$$

Case 2: $2 \le p$. Using (6.4.2) and Hölder's inequality we get,

$$\begin{aligned} S &\le \sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} |\mu_j(y,\xi_j) - \mu_j(y,\eta_j)||\xi_j - \eta_j|\, dy\, dx\, dt \\ &\le k_1 \sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} |\xi_j - \eta_j|^2 (|\xi_j| + |\eta_j|)^{p-2}\, dy\, dx\, dt \\ &\le k_1 \sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^2 \left(\int_0^T \int_\Omega \int_{Y_j} (|\xi_j| + |\eta_j|)^p\, dy\, dx\, dt \right)^{\frac{p-2}{p}} \\ &\le k_1 \sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^2 \, (\|\xi_j\|_p + \|\eta_j\|_p)^{p-2} \\ &\le k_1 \left(\sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^p \right)^{\frac{2}{p}} \left(\sum_{j=1}^{3} (\|\xi_j\|_p + \|\eta_j\|_p)^p \right)^{\frac{p-2}{p}} \\ &\le k_1 \left(\sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^p \right)^{\frac{2}{p}} \left(\sum_{j=1}^{3} (2\,\|\xi_j\|_p + \|\xi_j - \eta_j\|_p)^p \right)^{\frac{p-2}{p}} \\ &\le k_1 2^{\frac{(p-2)(p-1)}{p}} \left(\sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^p \right)^{\frac{2}{p}} \left(\sum_{j=1}^{3} \left(2^p\, \|\xi_j\|_p^p + \|\xi_j - \eta_j\|_p^p \right) \right)^{\frac{p-2}{p}} \end{aligned}$$

Therefore, by the estimate for the second term proved in Lemma 6.4.2, we get the result.■

Theorem 6.4.3

$$\overline{\lim}_{\varepsilon\to 0} \left\|\chi_j(\frac{x}{\varepsilon})\ (\nabla u_j^\varepsilon(t,x) - \eta_j^\varepsilon(t,x))\right\|_{p,\Omega_T}^p \le M\kappa^{r(p)}\ ,$$
$$\overline{\lim}_{\varepsilon\to 0} \left\|\chi_2(\frac{x}{\varepsilon})\ (\varepsilon\nabla u_3^\varepsilon(t,x) - \eta_3^\varepsilon(t,x))\right\|_{p,\Omega_T}^p \le M\kappa^{r(p)}$$

where

$$r(p) = \begin{cases} \frac{p}{2} & \text{if } 1 < p < 2, \\ \frac{2}{p} & \text{if } p \ge 2. \end{cases}$$

Proof: Case 1: $1 < p < 2$. We use Lemma 6.4.1 with the functions $\chi_j^\varepsilon \nabla u_j^\varepsilon$ and η_j^ε, $j = 1,2$ to get,

$$\|\chi_j^\varepsilon\nabla u_j^\varepsilon - \eta_j^\varepsilon\|_{p,\Omega_T}^p$$
$$\le (\int_0^T\int_{\Omega_j^\varepsilon} |\nabla u_j^\varepsilon - \eta_j^\varepsilon|^2(|\nabla u_j^\varepsilon| + |\eta_j^\varepsilon|)^{p-2}\,dx\,dt)^{\frac{p}{2}}(\int_0^T\int_{\Omega_j^\varepsilon}(|\nabla u_j^\varepsilon| + |\eta_j^\varepsilon|)^p\,dxdt)^{\frac{2-p}{2}}$$

Therefore, using strong monotonicity (6.4.3), we get,

$$\|\chi_j^\varepsilon\nabla u_j^\varepsilon - \eta_j^\varepsilon\|_{p,\Omega_T}^p \le k\left(\int_0^T\int_{\Omega_j^\varepsilon}\left(\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon) - \mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)\right).(\nabla u_j^\varepsilon - \eta_j^\varepsilon)\ dx\ dt\right)^{\frac{p}{2}}$$
$$\times(\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p + \|\eta_j^\varepsilon\|_p^p)^{\frac{2-p}{2}}$$

where $k = 2^{\frac{(p-1)(2-p)}{2}}/k_2^{\frac{p}{2}}$. Similarly,

$$\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon\|_{p,\Omega_T}^p \le k\left(\int_0^T\int_{\Omega_2^\varepsilon}\left(\mu_3(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon) - \mu_3(\frac{x}{\varepsilon},\eta_3^\varepsilon)\right).(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\ dx\ dt\right)^{\frac{p}{2}}$$
$$\times(\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p + \|\eta_3^\varepsilon\|_p^p)^{\frac{2-p}{2}}$$

Let,

$$S_1^\varepsilon = \sum_{j=1}^2 \|\chi_j^\varepsilon\nabla u_j^\varepsilon - \eta_j^\varepsilon\|_{p,\Omega_T}^p + \|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon\|_{p,\Omega_T}^p\ ,$$
$$S_2^\varepsilon = \sum_{j=1}^2\int_0^T\int_{\Omega_j^\varepsilon}\left(\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon) - \mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)\right).(\nabla u_j^\varepsilon - \eta_j^\varepsilon)\,dx\,dt$$
$$+\int_0^T\int_{\Omega_2^\varepsilon}\left(\mu_3(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon) - \mu_3(\frac{x}{\varepsilon},\eta_3^\varepsilon)\right).(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\,dx\,dt \text{ and },$$
$$S_3^\varepsilon = \sum_{j=1}^2\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p + \|\eta_j^\varepsilon\|_p^p + \|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p + \|\eta_3^\varepsilon\|_p^p.$$

Then, from the estimates for the individual terms in S_1^ε and a simple application of Hölder's inequality in $\mathbb{R}^3$, $S_1^\varepsilon \leq k(S_2^\varepsilon)^{\frac{p}{2}} \times (S_3^\varepsilon)^{\frac{2-p}{2}}$.

Note that η_j^ε arise from admissible test functions. Therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{j=1}^{3}\|\eta_j^\varepsilon\|_p^p &= \sum_{j=1}^{3}\|\eta_j\|_{p,[0,T]\times\Omega\times Y}^p \\ &\leq \sum_{j=1}^{3}2^{p-1}(\|\xi_j\|_{p,[0,T]\times\Omega\times Y}^p + \sum_{j=1}^{3}\|\eta_j-\xi_j\|_{p,[0,T]\times\Omega\times Y}^p) \\ &\leq M\end{aligned}$$

where the last estimate follows from Lemma 6.4.2. Also by (6.2.17) and (6.4.5), we get,

$$\sum_{j=1}^{2}\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p + \|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p \leq \frac{1}{2k_2}\left\|\overrightarrow{u^\delta}\right\|_{H_\varepsilon}^2 \leq M$$

From this we conclude that, $\overline{\lim}_{\varepsilon\to 0}S_3^\varepsilon \leq M$. Therefore, taking $\limsup$ as $\varepsilon \to 0$ and using Lemmas 6.4.3 and 6.4.4, we get

$$\overline{\lim}_{\varepsilon\to 0}S_1^\varepsilon \leq M\kappa^{\frac{p}{2}}.$$

This concludes the proof in this case.

Case 2: $2 \leq p$. From (6.4.6), we get,

$$|\nabla u_j^\varepsilon - \eta_j^\varepsilon|^p \leq \tfrac{1}{k_2}\left(\mu_j(\tfrac{x}{\varepsilon},\nabla u_j^\varepsilon) - \mu_j(\tfrac{x}{\varepsilon},\eta_j^\varepsilon)\right).(\nabla u_j^\varepsilon - \eta_j^\varepsilon)$$

Therefore, by integrating with respect to t in $[0,T]$ and x in Ω_j^ε, we get,

$$\|\chi_j^\varepsilon\nabla u_j^\varepsilon - \eta_j^\varepsilon\|_{p,\Omega_T}^p \leq \tfrac{1}{k_2}\int_0^T\int_{\Omega_j^\varepsilon}(\mu_j(\tfrac{x}{\varepsilon},\nabla u_j^\varepsilon) - \mu_j(\tfrac{x}{\varepsilon},\eta_j^\varepsilon)).(\nabla u_j^\varepsilon - \eta_j^\varepsilon)\,dx\,dt$$

Similarly,

$$\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon\|_{p,\Omega_T}^p \leq \tfrac{1}{k_2}\int_0^T\int_{\Omega_2^\varepsilon}(\mu_3(\tfrac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon) - \mu_3(\tfrac{x}{\varepsilon},\eta_3^\varepsilon)).(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\,dx\,dt$$

We note that if S_1^ε and S_2^ε are defined as in the previous case, then $S_1^\varepsilon \leq \frac{1}{k_2}S_2^\varepsilon$. Passing to the limit, as before, we reach our conclusions. ■

Theorem 6.4.4

$$\overline{\lim}_{\varepsilon\to 0} \left\| \chi_j^\varepsilon \mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon) - \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon) \right\|_{q,\Omega_T}^q \leq M\kappa^{s(p)}$$
$$\overline{\lim}_{\varepsilon\to 0} \left\| \chi_2^\varepsilon \mu_3(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon) - \mu_3(\frac{x}{\varepsilon}, \eta_3^\varepsilon) \right\|_{q,\Omega_T}^q \leq M\kappa^{s(p)}$$

where

$$s(p) = \begin{cases} \frac{p}{2} & \text{if } 1 < p < 2, \\ \frac{2}{p-1} & \text{if } p \geq 2. \end{cases}$$

Proof: We will prove only the first of these estimates, the other is proved similarly.

If $1 < p < 2$, by (6.4.2) and triangle inequality in R^N, we get,

$$\int_0^T \int_{\Omega_j^\varepsilon} |\mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon) - \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon)|^q \, dx \, dt \leq k_1 \int_0^T \int_{\Omega_j^\varepsilon} |\nabla u_j^\varepsilon - \eta_j^\varepsilon|^{q(p-1)} \, dx \, dt$$

Since $q(p-1) = p$, using the Theorem 6.4.3, the estimate follows easily. Let $2 \leq p$. Then,

$$\int_0^T \int_{\Omega_j^\varepsilon} |\mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon) - \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon)|^q \, dx \, dt$$
$$\leq k_1 \int_0^T \int_{\Omega_j^\varepsilon} |\nabla u_j^\varepsilon - \eta_j^\varepsilon|^q \left(|\nabla u_j^\varepsilon| + |\eta_j^\varepsilon|\right)^{(p-2)q} \, dx \, dt$$

The right hand side, by Hölder's inequality,

$$\leq k_1 2^{p-1} \left(\int_0^T \int_{\Omega_j^\varepsilon} |\nabla u_j^\varepsilon - \eta_j^\varepsilon|^p \, dx \, dt\right)^{\frac{1}{p-1}} \left(\int_0^T \int_{\Omega_j^\varepsilon} (|\nabla u_j^\varepsilon|^p + |\eta_j^\varepsilon|^p) \, dx \, dt\right)^{\frac{p-2}{p-1}}$$
$$\leq M \left\| \chi_j^\varepsilon \nabla u_j^\varepsilon - \eta_j^\varepsilon \right\|_{p,\Omega_T}^{\frac{p}{p-1}}.$$

So, again using Theorem 6.4.3, we get the desired result. ■

Proof of Theorems 6.4.1 and 6.4.2: Since, $\nabla_y U_j$'s are assumed to be admissible test functions, we can take $\Phi_j \equiv \nabla_y U_j$. Thus, κ can be taken arbitrarily small and therefore, Theorem 6.4.1 follows from Theorem 6.4.3. Similarly, Theorem 6.4.2 follows from Theorem 6.4.4. ■

Remark 6.4.4 *The functions $\nabla_y U_j(t, x, y)$ will be admissible if we have C^1 regularity of U_j in the variable y. Even if the functions $\nabla_y U_j$ are not admissible, Theorems 6.4.3 and 6.4.4 are corrector results in their own right.* ■

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