SOME CONTRIBUTIONS TO GENERALIZED INVERSES AND THE LINEAR COMPLEMENTARITY PROBLEM

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CONTENTS

CHAPTER I

INTRODUCTION

A generalized inverse (g-inverse) of a matrix A is a solution X to the matrix equation

$$A\,X\,A = A \qquad (1.1.1)$$

A g-inverse of A can be defined alternatively as a matrix X such that $x = Xb$ is a solution to the linear equation $Ax = b$ for any b that makes $Ax = b$ consistent. There is a vast literature on g-inverse. For a number of results on g-inverses and their applications one may refer to the well known books in the literature by Rao and Mitra (1971); and by Ben Israel and Greville (1974).

Another inverse that lies hidden in the definition of g-inverse is outer inverse. An outer inverse of a matrix A is a solution X to the matrix equation

$$X\,A\,X = X \qquad (1.1.2)$$

Ben Israel and Greville (1974) give some applications of outer inverse. A recent book by Getson and Hsuan (1988) lays emphasis on outer inverses and highlights their role in statistical applications.

Unless A is nonsingular, a g-inverse of A is not unique. Similarly an outer inverse of A is not unique unless $A = 0$. This has led to the introduction of a variety of inverses in the literature for various applications. However we have several results on characterization of these inverses available to us. These results enable us to understand the key variables that give rise to different types of inverses. Usually in the literature g-inverses and outer inverses are treated separately. In this thesis we introduce an integrated approach for studying both g-inverses and outer-inverses. This we accomplish by means of bordered matrices. Also we derive some new results using the new approach.

Given an n×n real matrix M and an n-dimensional real vector q, the linear complementarity problem (LCP) is a problem of computing a solution (w,z), if it exists, to the following system of equations:

$$w - Mz = q, \quad w, z \geq 0 \tag{1.1.3a}$$

$$w^T z = 0 \tag{1.1.3b}$$

where w^T is the transpose of w.

LCP is a unified formulation of linear and quadratic programming problems, and the problem of computing equilibrium strategies to a bimatrix game. It has application in several other areas too. (See for details Murty (1988)).

One of the important algorithms for LCP is due to Lemke (1965). However this algorithm cannot solve all LCPs. In this thesis we identify some classes of LCPs which are beyond the scope of Lemke's algorithm. Also we give new algorithms for solving these problems. This we achieve by exploiting a particular property of Lemke's algorithm in combination with the bordered matrix approach for g-inverses.

Now let us discuss briefly the results included in this thesis.

Characterization of g-inverses via bordered matrices

We shall confine ourselves to complex matrices for the sake of convenience especially for examining the eigen value properties of g-inverses/outer inverses. However the general results as obtained in Section 3 of Chapter 2 hold for matrices over a general field.

For an $m \times n$ matrix A and E,F,B of appropriate orders if $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ is square and nonsingular then let us denote the $n \times m$ leading submatrix of its inverse as A(E,F,B). (The symbol * stands for the conjugate transpose of the concerned matrix). However, whenever we write A(E,F,B) we shall presume that A(E,F,B) is well defined unless the context warrants its proof.

It may be interesting to observe that if A and G are conformable for multiplication then

$$\begin{bmatrix} A & I-AG \\ I & -G \end{bmatrix}^{-1} = \begin{bmatrix} G & I-GA \\ I & -A \end{bmatrix} \qquad (1.2.1)$$

which implies that if A is of order $m \times n$ then any matrix of order $n \times m$ can be represented as A(E,F,B). In particular, we see that any g-inverse or outer inverse of A can be represented as A(E,F,B) for some suitable choice of E,F and B.

A matrix X that satisfies both (1.1.1) and (1.1.2) is called a reflexive g-inverse of A. We see from Blattner (1962) that Condition 1 given below is sufficient for A(F,F,0) to be a reflexive g-inverse of A.

Condition 1: Given an $m \times n$ matrix A of rank r, let E and F be of order $m \times (m-r)$ and $n \times (n-r)$ respectively such that the column spaces of E and A are complementary to each other; and the column spaces of F and A* are complementary to each other.

We extend Blattner's result (vide Theorems 2.3.1 and 2.3.2) by showing that if A{1} is the set of all g-inverses of A then

$$A\{1\} = \{A(E,F,B) : E,F \text{ satisfy Condition 1}\} \qquad (1.2.2)$$

As regards outer inverses we show (vide Theorem 2.3.3) that if

$A\{2\}$ is the set of all outer inverses of A then

$$A\{2\} = \{A(E,F,0): E,F \text{ any matrices that make } A(E,F,0) \text{ well defined}\} \quad (1.2.3)$$

It may be appropriate here to recollect some of the important characterizations of g-inverse and outer inverse of A. They are:

$$A\{1\} = \{G + W - GAWAG : W \text{ arbitrary and } G \text{ is any particular element of } A\{1\}\} \quad (1.2.4)$$

$$A\{1\} = \{G + (I-GA)K+L(I-AG): K,L \text{ arbitrary and } G \text{ is any particular element of } A\{1\}\} \quad (1.2.5)$$

$$A\{1\} = \{Q^{-1}\begin{bmatrix} I & 0 \\ 0 & L \end{bmatrix} P^{-1} : P \text{ and } Q \text{ are nonsingular such that } A = P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q \text{ and } L \text{ arbitrary}\} \quad (1.2.6)$$

$$A\{1\} = \{Q^{-1}\begin{bmatrix} I & U \\ V & W \end{bmatrix}P^{-1} : P \text{ and } Q \text{ are two fixed non-singular matrices such that } A = P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q; \text{ and } U,V,W \text{ arbitrary}\} \quad (1.2.7)$$

If $A\{2\}_s$ denotes the set of all elements $A\{2\}$ with rank s then

$$A\{2\}_s = \{ YZ : ZAY = I_s\} \quad (1.2.8)$$

where I_s is the identity matrix of order s.

$$A\{2\} = \{U(V^*AU)_r^- V^*: U,V \text{ arbitrary and } (V^*AU)_r^- \text{ is a reflexive g-inverse of } V^*AU\} \quad (1.2.9)$$

The characterizations (1.2.4), (1.2.5), (1.2.6) and (1.2.8) are available in Ben Israel and Greville (1974); (1.2.7) in Boullion and Odell (1971); and (1.2.9) in Getson and Hsuan (1988).

As compared to the above characterizations the characterization via bordered matrices is advantageous in the following respects.

The bordered matrix approach enables us to represent both g-inverses and outer inverses in a single frame. This provides a better way of studying the two closely related inverses in a unified manner. Many of the questions raised for study in g-inverses are based on the knowledge of the relationship between a nonsingular matrix and its inverse. The bordered matrix approach seems to be more appropriate for such studies because we have only to compare a nonsingular matrix $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ with its inverse. If $A(E,F,B) \in A\{1\}$ then by pre and post multiplying $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ by suitable nonsingular matrices such that A remains unaffected in the final form, we get all possible g-inverses of A. This enables us to see the inter-relationship among various types of g-inverses. Further the bordered matrix $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ itself provies immediately some of the information like the row and column spaces, and rank of the g-inverse/outer inverse.

Inverses with spectral properties

By spectral properties we mean the properties involving eigen values and eigen vectors.

A vector x is called a principal eigen vector of grade p of a square matrix A corresponding to an eigen value λ of A if $(A-\lambda I)^{p-1}x \neq 0$ and $(A-\lambda I)^p x = 0$ where p is a positive integer. Following Ben Israel and Greville (1974) we call such a vector x a λ-vector of A of grade p.

If A is nonsingular it can be easily verified that A and $X = A^{-1}$ together have the following property.

Property 1: x is a λ-vector of A of grade p if and only if x is a λ^{-1}-vector of X of grade p.

In the case of a singular square matrix A we may try to examine the extent to which Property 1 can be satisfied by a g-inverse/outer inverse X of A. If A is singular then $\lambda = 0$ is an eigen value of A and so the closest approximation of Property 1 would be:

Property 2: For $\lambda \neq 0$, x is a λ-vector of A of grade p if and only if x is a λ^{-1}-vector of X of grade p; and x is a 0-vector of A of grade p if and only if x is 0-vector of X of grade p.

Any matrix X possessing Property 2 is called an S-inverse of A. It is well known that a g-inverse/outer inverse X of A is an S-inverse of A if and only if $X = A^{\#}$, the group inverse of A. (See Section 2 of Chapter 2 for definition of $A^{\#}$).

We note that $A^{\#}$ exists if and only if rank A = rank A^2. All singular matrices do not satisfy this rank condition. So, we relax Property 2 further and look for a g-inverse/outer inverse that satisfies

Property 3: For $\lambda \neq 0$, x is a λ-vector of A of grade p if and only if x is a λ^{-1}-vector of X of grade p; and x is a 0-vector of A if and only if x is a 0-vector of X without regard to grade.

Any matrix X satisfying the above property is defined as an S'-inverse of A. It has been proved (See Greville (1968), Boullion and Odell (1971) and also Ben Israel and Greville (1974) pp. 177) that $X \in A\{1\} \cup A\{2\}$ is an S'-inverse of A if and only if

$$X^D = (A^D)^{\#}$$

where the superfix D denotes the Drazin inverse of the concerned matrix. (See Section 2 of Chapter 2 for the definition of Drazin inverse). An S'-inverse belonging to $A\{1\} \cup A\{2\}$ exists for any square matrix A. In Section 4 of Chapter 2 we give its representation in terms of A(E,F,B).

There are other weaker properties which are of interest. For example

Property 4: For $\lambda \neq 0$, x is a λ-vector of A of grade p if and only if x is a λ^{-1}-vector of X of grade p.

Theorem 2.4.2 gives a sufficient condition for X = A(E,F,0) to have

Property 4. If X is a reflexive g-inverse of A the condition of Theorem 2.4.2 reduces to the condition that $R(F) = N(A^*)$, the null space of A^* in which case rank A = rank A^2. Such a reflexive g-inverse has been identified by Mitra (1968). In the above reflexive g-inverse if $R(E)=N(A^*)$ we get the g-inverse identified by Cline (1968).

Theorem 2.4.1 gives a sufficient condition for $X = A(E,F,B)$ to have

Property 5: For $\lambda \neq 0$, if x is a λ-vector of A of grade 1 then x is a λ^{-1}-vector of X of grade 1.

Inverses with nonnegative principal minors

Square matrices with positive (nonnegative) principal minors are known as P-(P_o-) matrices. These matrices, as we shall see in Chapter 3, have special significance in LCP. Also in a finite state Markov chain we see that the matrix I-T is a P_o-matrix where T is the transition probability matrix of the chain. The above matrix plays a pivotal role in the analysis of the Markov chain. Meyer (1975) used the group inverse of I-T to derive several results on the Markov chain. Later Hunter (1982) demonstrated the use of any g-inverse of I-T in the analysis of the chain. If A is a nonsingular P_o-matrix then A^{-1} is also a P_o-matrix, but it is not so in respect of a singular P_o-matrix and its g-inverse. Mohan, Neumann and Ramamurty (1984) investigated the conditions under which a reflexive g-inverse of I-T preserves the P_o- property. In this process we are led to the problem of obtaining conditions for a g-inverse (of any square matrix) to be a P_o-matrix.

By making use of the relationship between the minors of a nonsingular matrix and its inverse, we are able to express the minors of a g-inverse A(E,F,B) in terms of the minors of the nonsingular bordered matrix $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$. Thus, we derive necessary and sufficient conditions for a g-inverse to be a P_o-matrix. We derive some more results relating to minors.

It was proved by Mohan, Neumann and Ramamurty (1984) that in a finite state Markov chain the Drazin inverse of I-T is a P_o-matrix, but therein we do not find an interpretation for this result in the context of the Markov chain. We give in Section 7 of Chapter 2, Markov chain interpretation for the principal minors of any g-inverse of I-T when T is irreducible (See Section 5 of Chapter 2 for the definition of irreducibility).

Linear Complementarity Problem

Let us denote the LCP defined at (1.1.3) as (q,M). We see that all LCPs do not have solutions. For example the LCP (q,M) with $q = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has no solution. So, for a given M we are led to examine the set D(M) of all q for which (q,M) has at least one solution. If M is such that (q,M) has a solution for any q then M is called a Q-matrix and in such a case D(M) equals the entire Euclidean space. Also we note that the existence of a solution to (1.1.3a) ensures the existence of a solution to (q,M) if and only if D(M) is a convex set. A matrix M with D(M) convex is called a Q_o-matrix. We see that a Q-matrix is a Q_o-matrix too. A problem that has attracted the attention of many researchers is to characterize Q and Q_o-matrices. No constructive/efficient characterization of Q or Q_o-matrices have been obtained so far. (See the remarks of Aganagic and Cottle (1987)). Fredricksen, Watson and Murty (1986) respond to this problem but in a small measure by characterizing Q_o-matrices of order not greater than 3. Kelly and Watson (1979) propose a spherical geometric approach for characterizing Q-matrices. We may also see Watson (1976) in this connection. However several subclasses of Q and Q_o-matrices have been identified in the literature.

An LCP (q,M) may have more than one solution. So, in another direction of research, problems relating to the number of solutions to (q,M) is being studied. It is known that the LCP (q,M) has a unique solution if and only if M is a P-matrix. (See Samelson, Thrall and Wesler (1958)). Cottle and Stone (1983), introduce the class U of matrices M such that for q in the interior of D(M) (q,M) has a unique solution. Jeter and Pye (1987) study a subclass of U-matrices, called W-matrices. Stone (1986) studies matrices M for which (q,M) has the same number of solutions for all q in the interior of D(M). Kojima and Saigal (1979) show that if the principal minors of M are negative then for any q the number of solutions to (q,M) is 0, 1, 2 or 3. Apart from the above there are many other aspects that are being examined in LCP.

Obtaining algorithms for solving LCP is another important area of research. An algorithm for solving LCPs is said to process a particular LCP (q,M) if the algorithm is guaranteed to either determine that (q,M) has no solution, or find a solution for it after a finite amount of computational effort. We say an algorithm for LCPs processes a matrix M if the algorithm processes (q,M) for all q.

Some important algorithms for LCPs are due to Lemke (1965), Dantzig and Cottle (1967) and Chandrasekaran (1970). Lemke's algorithm is similar to the simplex algorithm for Linear Programming Problems (LPPs) and it can process most of the LCPs that arise in practice. The algorithm due to Dantzig and Cottle is also known as the principal pivot algorithm. It can process LCPs associated with P-matrices and positive semi-definite matrices. Chandrasekaran's algorithm can process LCPs associated with matrices whose off-diagonal elements are nonpositive, and this algorithm processes such problems in polynomial time. (That is, the number of steps taken by the algorithm to terminate is a polynomial in s where s is the size of the problem). The other two algorithms may take exponentially large member of steps to terminate.

Though Lemke's algorithm cannot process all LCPs we see that it can process many matrices that arise naturally in practical applications : for example,the co-positive plus matrices that include the matrices occuring in the LCP formulation of the Quadratic Programming Problem, and the L-matrices which include matrices that occur in the computation of equilibrium point of a bimatrix game through an LCP formulation. Further we note that it can process all the matrices that are processable by Chandrasekaran's algorithm and the algorithm due to Dantzig and Cottle (See Saigal (1971) and Eaves (1971a)). Many of the research results in LCP hinge on the properties of this algorithm.

In this thesis we observe some properties of Lemke's algorithm; have a look into the classes of matrices processable by the algorithm; and obtain some new results in LCP based on the observations. Also we identify two subclasses of Q_o-matrices by making use of a property of Lemke's algorithm in combination with g-inverse.

Organization of the thesis

The thesis is arranged into three chapters, Chapter 1 being the present one. Chapter 2 is on g-inverse and Chapter 3 is on the Linear Complementarity Problem. In Chapter 2 there are seven sections, Section 1 is an introduction to the Chapter, Section 2 introduces notation and preliminary, Section 3 deals with the bordered matrix approach for g-inverse

Section 4 gives some results on spectral inverses, Section 5 deals with the minors of g-inverses, Section 6 gives some results on principal minors of g-inverses of matrices with nullity one; and Section 7 deals with g-inverses of I-T where T is a Markov matrix.

Chapter 3 is divided into six sections, Section 1 is introduction to the chapter, Section 2 examines some properties of Lemke's algorithm, Section 3 gives an application of g-inverse for identifying some classes of Q_0-matrices, Section 4 presents two algorithms for processing the classes of matrices identified in Section 3, Section 5 is on solution rays to LCP, and Section 6 gives some results on matrices with nonpositive principal minors.

This thesis contains, among others, all the results that appear in Eagambaram (1988a & b), Eagambaram and Mohan (1987 a&b, and 1988).

For the sake of easy reference and completeness some of the definitions introduced in this chapter are repeated in the subsequent chapters as well.

CHAPTER 2

GENERALIZED INVERSES OF MATRICES

1. Introduction

There are a number of results available in the literature on characterization/computation of g-inverse of a matrix A, which is a solution X to the matrix equation $AXA = A$. Each result has its own advantage in understanding the properties of g-inverses. The bordered matrix approach for characterizing g-inverses, to which this chapter is devoted, was found useful for examining g-inverses with nonnegative principal minors. The same bordered matrix approach enables us to characterize outer inverses as well. (See Eagambaram (1988a & b)). Thus we come upon a unified way of looking at all g-inverses/outer inverses. In this chapter we give some applications of the bordered matrix approach in characterizing g-inverses/outer inverses.

In Section 2 we give the requisite preliminary. In Section 3 we prove results on characterization of g-inverses/outer inverses. In Section 4 we demonstrate the advantage of bordered matrices in identifying g-inverses/outer inverses with some spectral properties, and in Section 5 we characterize g-inverses/outer inverses with pre-specified principal minors. As a particular case, we concentrate on matrices of order n and rank n-1 in Section 6. Finally we give some interpretation of g-inverses in the context of Markov chain, in Section 7.

2. Preliminary

Let $C^{m\times n}$ denote the set of $m\times n$ complex matrices and $R^{m\times n}$ the set of $m\times n$ real matrices. When either m or n is zero $C^{m\times n}$ and $R^{m\times n}$ are taken to be empty. C^n and R^n denote $C^{n\times 1}$ and $R^{n\times 1}$ respectively. Let R^n_+ and R^n_-

stand for the sets of nonnegative and nonpositive vectors respectively, of R^n. The column space and null space of a matrix A are denoted as R(A) and N(A) respectively. For a square matrix A, det A and adj A denote the determinant and the adjoint of A respectively. When a matrix A is vacuous det A is taken as 1. The symbol $\subseteq$ means "subset of" and $\subset$ means "proper subset of". For an m×n matrix A and for $J \subseteq \{1,2,\ldots,m\}$, $K \subseteq \{1,2,\ldots,n\}$, A_{JK} stands for the submatrix of A obtained from the rows and columns indexed by J and K; $A_{J'K'}$ stands for $A_{(M-J)(N-K)}$; $A_{J.}$ denotes $A_{J\{1,2,\ldots,n\}}$ and $A_{.K}$ denotes $A_{\{1,2,\ldots,m\}K}$. The symbol $\oplus$ denotes the direct sum of two complementary subspaces of a vector space. For any set J, |J| denotes its cardinality.

If A is a nonsingular complex matrix then A and $X = A^{-1}$ satisfy

$$AXA = A \quad (1)$$

$$XAX = X \quad (2)$$

$$AX = (AX)^* \quad (3)$$

$$XA = (XA)^* \quad (4)$$

$$AX = XA \quad (5)$$

$$A^k = A^{k+1}X, \text{ for a nonnegative integer } k \quad (6)$$

where * denotes the conjugate transpose of the concerned matrix. Let $\{i,j,\ldots,k\} \subseteq \{1,2,\ldots,6\}$. For an arbitrary complex matrix A, a solution X that satisfies equations (i), (j),...,(k) above is called an {i,j,...,k} -inverse of A. This definition is due to Ben Isreal and Greville (1974).

We note that a {1}-inverse and {2}-inverse of A are the alternative names for a g-inverse and an outer inverse of A respectively. The well known Moore-Penrose inverse of A, denoted as A^+, is the unique {1,2,3,4}-inverse of A. The index of a square matrix A is the smallest nonnegative integer k such that rank A^{k+1} = rank A^k. If A is a square matrix of index k, then a {2,5,6}-inverse of A exists and it is denoted as A^D. A^D is called the Drazin inverse of A and it is unique. If the index of the square matrix A is 1, then A^D is a {1,2,5,6}-inverse of A and it is denoted as $A^{\#}$. $A^{\#}$ is known as the group inverse of A. A {1,3}-inverse of A is called a least square inverse of A and a {1,4}-inverse of A, a minimum norm inverse of A.

It is well known that $A \in C^{n\times n}$ with index k can be decomposed as

$$A = T\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}T^{-1} \qquad (2.2.1)$$

where C is nonsingular and N is such that $N^k = 0$. We may write T and T^{-1} as

$$T = [T_1 \quad T_2]$$

and

$$T^{-1} = \begin{bmatrix} \tilde{T}_1^* \\ \\ \tilde{T}_2^* \end{bmatrix} \qquad (2.2.2)$$

where the partition is conformable for multiplication in (2.2.1). We observe that

$$A^D = T\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1} \qquad (2.2.3)$$

The following notation is convenient for representing {i}-inverses, $i \in \{1,2\}$, in terms of bordered matrices:

Definition 2.2.1: Let $A \in C^{m\times n}$ and the matrices E,F and B be such that the bordered matrix $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ is square and nonsingular. If

$$\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}^{-1} = \begin{bmatrix} G & U \\ V^* & H \end{bmatrix}$$

where $G \in C^{n\times m}$, then G is denoted as A(E,F,B). Whenever we write A(E,F,B) it is presumed that A(E,F,B) is well defined unless the context necessitates its proof.

3. Generalized inverses via bordered matrices

Blattner (1962) showed that for $A \in C^{n\times n}$, of rank r, if $E,F \in C^{n\times(n-r)}$ are such that $R(A) \oplus R(E) = R(A^*) \oplus R(F) = C^n$, then A(E,F,0) exists and A(E,F,0) is a {1,2}-inverse of A. The following theorem extends the above result.

Theorem 2.3.1: Let $A \in C^{m\times n}$ be of rank r. Let $E \in C^{m\times(m-r)}$ and $F \in C^{n\times(n-r)}$ be such that $R(A) \oplus R(E) = C^m$ and $R(A^*) \oplus R(F) = C^n$. Then for an arbitrary $B \in C^{(n-r)\times(m-r)}$

(i) the matrix $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ is nonsingular,

(ii) the inverse of $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ is of the form $\begin{bmatrix} A(E,F,B) & U \\ V^* & 0 \end{bmatrix}$,

(iii) A(E,F,B) is a {1}-inverse of A,

(iv) $R(U) = N(A)$ and $R(V) = N(A^*)$,

(v) $N(A(E,F,B)) = R(E\hat{B})$ where $\hat{B}$ is any matrix such that $R(\hat{B}) = N(B)$

and

(vi) rank A(E,F,B) = rank A + rank B.

Proof: (i) It is obvious that $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ is a square matrix. Suppose $\begin{bmatrix} x \\ y \end{bmatrix}$ is a null vector of $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ where $x \in C^n$ and $y \in C^{m-r}$. Then

$$Ax + Ey = 0 \tag{2.3.1}$$

$$F^*x + By = 0 \tag{2.3.2}$$

$R(A) \oplus R(E) = C^m$ implies that $Ax = 0$ and $Ey = 0$. The columns of E being linearly independent we see that $y = 0$. Now (2.3.1) and (2.3.2) reduce to $\begin{bmatrix} A \\ F^* \end{bmatrix} x = 0$ which is true only when $x = 0$ because $R(A^*) \oplus R(F) = C^n$.

So, $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ is nonsingular.

(ii) Suppose that

$$\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}^{-1} = \begin{bmatrix} A(E,F,B) & U \\ V^* & H \end{bmatrix} \qquad (2.3.3)$$

We see that $AU + EH = 0$. Now arguing as in the proof of part (i) we infer that $H = 0$.

(iii) and (iv) are obvious.

(v) Let $x \in N(A(E,F,B))$.

$$\begin{bmatrix} A(E,F,B) & U \\ V^* & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ V^*x \end{bmatrix}$$

which leads to

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A & E \\ F^* & B \end{bmatrix} \begin{bmatrix} 0 \\ V^*x \end{bmatrix} = \begin{bmatrix} E\,V^*x \\ B\,V^*x \end{bmatrix}$$

We see that $x \in R(E)$. Let $x = Ey$. Then $BV^*x = BV^*Ey = By = 0$. This shows that $y \in R(\hat{B})$. Therefore $x \in R(E\hat{B})$. On the other hand if $x \in R(E\hat{B})$ then $x = E\hat{B}y$ for some y. Then from (2.3.3) we have

$$A(E,F,B)x = A(E,F,B)\ E\hat{B}y = -UB\hat{B}y = 0.$$

Thus part (v) is proved.

(vi) Without loss of generality we can assume that $\hat{B}$ is of full column rank.

Now,

$$\begin{aligned}\text{rank } A(E,F,B) &= m - \text{dimension } N(A(E,F,B)) \\ &= m - \text{rank } E\hat{B} \\ &= m - \text{number of columns of } E\hat{B} \\ &= m - \text{dimension of } N(B) \\ &= m - (m-r-\text{rank } B) \\ &= \text{rank } A + \text{rank } B\end{aligned}$$

Thus we complete the proof of the theorem. $\square$

Remark 2.3.1: In Theorem 2.3.1 it is easy to verify that $A(E,F,B) = A(E,F,0) - UBV^*$.

The following theorem proved by Eagambaram (1988a) asserts that any {1}-inverse of A can be obtained by the bordered matrix approach.

Theorem 2.3.2: Let $A \in C^{m\times n}$ be of rank r. Let G be a {1}-inverse of A. Then there exist $E \in C^{m\times(m-r)}$, $F \in C^{n\times(n-r)}$ and $B \in C^{(n-r)\times(m-r)}$ such that $A(E,F,B)$ is well defined and $A(E,F,B) = G$.

Proof: Let $\tilde{E} \in C^{m\times(m-r)}$ and $\tilde{F} \in C^{n\times(n-r)}$ be such that $R(\tilde{E}) = N(A^*)$ and $R(\tilde{F}) = N(A)$. Then $R(A) \oplus R(\tilde{E}) = C^m$ and $R(A^*) \oplus R(\tilde{F}) = C^n$. According to Theorem 2.3.1 $A(\tilde{E},\tilde{F},0)$ exists and it is a {1}-inverse of A. Let

$$\begin{bmatrix} A & \tilde{E} \\ \tilde{F}^* & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A(\tilde{E},\tilde{F},0) & \tilde{U} \\ \tilde{V}^* & 0 \end{bmatrix} \qquad (2.3.4)$$

It is known (for example see Rao and Mitra (1971), that, given a {1}-inverse $\tilde{G}$ of A, any other {1}-inverse $\hat{G}$ of A can be written as

$$\hat{G} = \tilde{G} + (I-\tilde{G}A)K + L(I-A\tilde{G})$$

where K and L are suitably chosen matrices. So, we can find matrices $\tilde{K}$ and $\tilde{L}$ such that

$$G = A(\tilde{E},\tilde{F},0) + (I-A(\tilde{E},\tilde{F},0)A)\tilde{K} + \tilde{L}(I-AA(\tilde{E},\tilde{F},0)) \qquad (2.3.5)$$

If

$$E = (I-AL)\tilde{E}$$

$$F = (I-A^* \tilde{K}^*)\tilde{F}$$

and $$B = \tilde{F}^* (\tilde{K}A\tilde{L} - \tilde{K}-\tilde{L})\tilde{E}$$

We see with the help of (2.3.4) and (2.3.5) that

$$\begin{bmatrix} A & E \\ F^* & B \end{bmatrix} \begin{bmatrix} G & \tilde{U} \\ \tilde{V}^* & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad (2.3.5^*)$$

Hence the theorem. □

The above two thorems are generalization of the corresponding results for square matrices which are published in Eagambaram (1988a).

Corollary 2.3.1: Let A,G,E,F and B be the matrices as in Theorem 2.3.2. Then

(i) $R(E) = R(I-AG)$

(ii) $R(F) = R(I-A^*G^*)$

and (iii) $B = -F^*GE$

Proof: (i) and (ii) follow from (2.3.5*). (iii) Follows from Remark 2.3.1 in conjunction with part (v) of Theorem 2.3.1. □

Remark 2.3.2: We see that for nonsingular matrices P and Q, $A(E,F,B) = A(EP, FQ, Q^*BP)$. Therefore, Corollary 2.3.1 shows that any matrices E,F and B satisfying (i), (ii) and (iii) of Corollary 2.3.1 yield $G = A(E,F,B)$. Further the choice of E and F determines B.

The following theorems characterize {2}-inverses.

Theorem 2.3.3: $G \in C^{n\times m}$ is a {2}-inverse of $A \in C^{m\times n}$ if and only if there exist matrices E and F such that $G = A(E,F,0)$.

Proof: It is obvious from the definition of A(E,F,0) that if G = A(E,F,0) then G is a {2}-inverse of A. On the other hand if G is a {2}-inverse of A, then A is a {1}-inverse of G and hence it follows from Theorem 2.3.2

that $G = A(E,F,0)$ for some E and F. □

Theorem 2.3.4: $A(E,F,0) = A(\tilde{E},\tilde{F},0)$ if and only if $R(E) = R(\tilde{E})$ and $R(F) = R(\tilde{F})$.

Proof: If $A(E,F,0) = A(\tilde{E},\tilde{F},0) = G$ then we see that $N(G) = R(E) = R(\tilde{E})$ and $N(G^*) = R(F) = R(\tilde{F})$. On the other hand if $R(E) = R(\tilde{E})$ and $R(F) = R(\tilde{F})$ then there exist nonsingular matrices P and Q such that $\tilde{E} = EP$ and $\tilde{F}^* = Q^*F^*$. Now

$$\begin{bmatrix} A & \tilde{E} \\ \tilde{F}^* & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Q^* \end{bmatrix} \begin{bmatrix} A & E \\ F^* & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$$

The above relationship shows that $A(E,F,0) = A(\tilde{E},\tilde{F},0)$. □

Remark 2.3.3: Theorem 2.3.4 shows that a {2}-inverse with a particular pair of column and row spaces is unique.

Now let us see how some well known g-inverses/outer inverses of A are represented in the form $A(E,F,B)$.

Theorem 2.3.5: $A(E,F,B)$ is a least square inverse of A (that is, $A(E,F,B)$ is a {1,3}-inverse of A) if and only if $R(E) = N(A^*)$.

Proof: In accordance with part (ii) of Theorem 2.3.1 let

$$\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}^{-1} = \begin{bmatrix} A(E,F,B) & U \\ V^* & 0 \end{bmatrix}$$

From the above we have: $A(E,F,B)$ is a {1,3}- inverse of A $\Longleftrightarrow$ EV^* is hermitian $\Longleftrightarrow$ $R(E) = R(V) = N(A^*)$ noting that $V^*E = I$. □

Corollary 2.3.1: $A(E,F,B)$ is a minimum norm inverse of A (that is, $A(E,F,B)$ is a {1,4}-inverse of A) if and only if $R(F) = N(A)$.

Proof: We note that if G is a {1,4}-inverse of A, then G* is a {1,3}-inverse of A*. Now the proof follows from Theorem 2.3.5.

□

Theorem 2.3.6: The {2}-inverse A(E,F,0) of A is the Moore-Penrose inverse A^+ if and only if R(E) = N(A*) and R(F) = N(A).

Proof: We note that $N(A^+) = N(A^*)$ and $N((A^+)^*) = N(A)$. Now the proof follows from Remark 2.3.3. □

Theorem 2.3.7: The {2}-inverse A(E,F,0) of A is the Drazin inverse A^D if and only if $R(E) = N(A^k)$ and $R(F) = N((A^k)^*)$ where k is the index of A.

Proof: We note from equations (2.2.1) - (2.2.3) that $N(A^D) = N(A^k)$ and $N((A^D)^*) = N((A^k)^*)$. The proof follows from Remark 2.3.3. □

For A(E,F,B) to be a {1}-inverse of A, Blattner's condition on E and F as given in Theorem 2.3.1 is a sufficient condition. So, in general A(E,F,B) need not be a {1}-inverse or a {2}- inverse. The following theorems give conditions for A(E,F,B) to be an {i}-inverse of A for $i \in \{1,2\}$.

Lemma 2.3.1: Let $A \in C^{m\times n}$ be of rank r. If E and F are of full column rank then A(E,F,B) is a {1}- inverse of A only if E and F satisfy the condition of Theorem 2.3.1.

Proof: Let

$$\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}^{-1} = \begin{bmatrix} G & U \\ V^* & H \end{bmatrix} \qquad (2.3.6)$$

where G is a {1}-inverse of A. We have,

$$AG + EV^* = I \qquad (2.3.7)$$

$$V^*A + HF^* = 0 \qquad (2.3.8)$$

Since E is of full column rank, from (2.3.7) we have $V^*A = 0$. Substituting this in (2.3.8) we get $HF^* = 0$. Again, F^* being of full row rank we find $H = 0$. From (2.3.6) we see that $AU = 0$ which shows that $R(U) \subseteq N(A)$. The R.H.S of (2.3.6) being a nonsingular matrix $\begin{bmatrix} U \\ 0 \end{bmatrix}$ and hence U is of full column rank. $F^*U = I$ implies that the number of columns of F is equal to the number of columns of U which is not greater than n-r. But then rank $\begin{bmatrix} A \\ F^* \end{bmatrix} = n$ implies that the order of F is $n\times(n-r)$. Similarly it follows that the order of E is $m\times(m-r)$. Now it is easily seen that E and F satisfy the conditions of Theorem 2.3.1. Hence the Lemma.

□

Theorem 2.3.8: Let I_m and I_n be the identity matrices of order m and n respectively. Then for $A \in C^{m\times n}$, A(E,F,B) is a {1}-inverse of A if and only if there exist nonsingular matrices $\begin{bmatrix} I_m & P_1 \\ 0 & P_2 \end{bmatrix}$ and $\begin{bmatrix} I_n & 0 \\ Q_1 & Q_2 \end{bmatrix}$ such that

$$\begin{bmatrix} I_m & P_1 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A & E \\ F^* & B \end{bmatrix} \begin{bmatrix} I_n & 0 \\ Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} A & E_1 & 0 \\ F_1^* & B_1 & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (2.3.9)$$

where E_1 and F_1 satisfy the conditions of Theorem 2.3.1.

Proof: Sufficiency: It is obvious from (2.3.9) that $A(E,F,B) = A(E_1,F_1,B_1)$ which is a {1}-inverse of A according to Theorem 2.3.1.

Necessity: Let A(E,F,B) be a {1}-inverse of A. From Lemma 2.3.1 we have that if E and F are of full rank then E and F themselves satisfy conditions of Theorem 2.3.1 and hence (2.3.9) holds with $P_1 = 0$, $Q_1 = 0$, P_2 and Q_2 identity matrices. Now if either E or F is not of full rank then we can obtain the relationship (2.3.9) by applying the fact that $A(E,F,B) = A(EP, FQ, Q^*BP)$ for nonsingular P, Q and Lemma 2.3.1. Hence the theorem.

□

Theorem 2.3.9: Let I_m and I_n be the identity matrices of order m and n respectively. Then for $A \in C^{n\times m}$, $A(\hat{E},\hat{F},\hat{B})$ is a {2}-inverse of A if and only if there exist nonsingular matrices $\begin{bmatrix} I_n & 0 \\ \hat{Q}_1 & \hat{Q}_2 \end{bmatrix}$ and $\begin{bmatrix} I_m & \hat{P}_1 \\ 0 & \hat{P}_2 \end{bmatrix}$ such that

$$\begin{bmatrix} I_m & 0 \\ \hat{Q}_1 & \hat{Q}_2 \end{bmatrix} \begin{bmatrix} A & \hat{E} \\ F^* & \hat{B} \end{bmatrix} \begin{bmatrix} I_m & \hat{P}_1 \\ 0 & \hat{P}_2 \end{bmatrix} = \begin{bmatrix} A & \hat{E}_1 & 0 \\ \hat{F}_1^* & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Proof: The proof follows by inverting the matrices on the L.H.S. and R.H.S. of (2.3.9). □

Theorems 2.3.3 - 2.3.7 have been included in Eagambaram (1988b).

It may be noted that it poses no difficulty in extending the results of this section to matrices over a general field.

4. Spectral inverses

In this section we identify some A(E,F,B) which have spectral properties comparable to those of A.

For $A \in C^{n\times n}$, a vector $x \in C^n$ is called a λ-vector of A of grade p if $(A-\lambda I)^p x = 0$ and $(A-\lambda I)^{p-1} x \neq 0$.

Theorem 2.4.1: Let $A \in C^{n\times n}$ and $G = A(E,F,B)$ with $R(F) \subseteq N((A^k)^*)$, where k is the index of A. If $\lambda \neq 0$ is an eigenvalue of A and x is an eigenvector of A corresponding to λ then λ^{-1} is an eigenvalue of G and x is an eigenvector of G corresponding to λ^{-1}.

Proof: Consider the form A as at (2.2.1) and of T at (2.2.2). Let $\lambda \neq 0$ be an eigenvalue of A and x be an eigenvector corresponding to λ.

We see that $x \in R(T_1)$. Since $R(F) \subseteq N((A^k)^*) = R(\tilde{T}_2)$ we have that $F^*x=0$. Therefore $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is an eigenvector of the nonsingular matrix $\begin{bmatrix} A & E \\ F^* & B \end{bmatrix}$ corresponding to the eigenvalue λ. Hence we see that λ^{-1} is an eigenvalue of G and x is an eigenvector of G corresponding to the eigenvalue λ^{-1}.

□

It may be noted that A(E,F,B), in general, need not be a generalized/outer inverse of A. Imposing the requisite conditions on E,F and B Theorem 2.4.1 can be stated for a generalized/outer inverse of A.

Theorem 2.4.2: Let $A \in C^{n\times n}$. Let the outer inverse $G = A(E,F,0)$ of A be such that $R(F) = N((A^k)^*)$, where k is the index of A. Then

for $\lambda \neq 0$, x is a λ^{-1}-vector of G of grade p if and only if x is a λ-vector of A of grade p.

Proof: Consider the decomposition of A as at (2.2.1) and of T as at (2.2.2). We see that $R(F) = R(\tilde{T}_2)$. So by virtue of Theorem 2.3.4 we have that $A(E,F,0) = A(E,\tilde{T}_2,0)$. Let

$$\begin{bmatrix} A & E \\ \tilde{T}_2^* & 0 \end{bmatrix}^{-1} = \begin{bmatrix} G & U \\ V^* & H \end{bmatrix} \qquad (2.4.1)$$

We find that

$$N(G^*) = R(\tilde{T}_2)$$

and so

$$R(G) = R(T_1) \qquad (2.4.2)$$

Further from (2.4.1) we get $V^*A + H\tilde{T}_2^* = 0$ which, in conjunction with the observation that $\tilde{T}_2^* A^k = 0$, shows that

$$R(V) \subseteq R(\tilde{T}_2). \qquad (2.4.3)$$

Let, for $\lambda \neq 0$, x be a λ-vector of A of grade p. It follows from the form of A at (2.2.1) that $x \in R(T_1)$. Since $\tilde{T}_2^* T_1 = 0$ we find that

$$\begin{bmatrix} A-\lambda I & E \\ \tilde{T}_2^* & -\lambda I \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} (A-\lambda I)x \\ 0 \end{bmatrix} \qquad (2.4.4)$$

It can be verified that $\tilde{T}_2^* Ax = 0$. So, by premultiplying (2.4.4) with $\begin{bmatrix} A-\lambda I & E \\ \tilde{T}_2^* & -\lambda I \end{bmatrix}$ p-1 times we see that $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is a λ-vector of $\begin{bmatrix} A & E \\ \tilde{T}_2^* & 0 \end{bmatrix}$ of grade . It follows that $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is a λ^{-1}-vector of $\begin{bmatrix} G & U \\ V^* & H \end{bmatrix}$ of grade p. By invoking (2.4.3) and applying the same technique as above with respect to $\begin{bmatrix} G & U \\ V^* & H \end{bmatrix}$ and λ^{-1}, we see that x is a λ^{-1}-vector of G of grade p.

Conversely for $\lambda \neq 0$, if x is a λ^{-1}-vector of G of grade p, then (2.4.2) implies that $x \in R(T_1)$. Now, by proceeding on the same lines as above we can show that x is λ-vector of A of grade p. Hence the Theorem. □

Corollary 2.4.1: (Mitra (1968)). Let $A \in C^{n\times n}$ be of index 1. Then a reflexive generalized inverse A(E,F,0) of A with R(F) = N(A*) satisfies Theorem 2.4.2. Further R(A(E,F,0)) = R(A).

Mitra (1968) denoted the above A(E,F,0) as A_c^-. He characterized A_c^- as a generalized inverse whose columns belong to the column space of A.

According to Ben Isreal and Greville (1974), $X \in C^{n\times n}$ is called an S'-inverse of $A \in C^{n\times n}$ if for $\lambda \neq 0$, x is a λ^{-1}-vector of X of grade p if and only if x is a λ-vector of A of grade p, and x is a 0-vector of X if and only if x is a 0-vector of A (without regard to grade). They have shown that a generalized/outer inverse X of $A \in C^{n\times n}$ is an S'-inverse of A if and only if

$$X^D = (A^D)^D$$

With the help of equations (2.2.1) - (2.2.3) we find that X is of the form

$$X = [T_1 T_2] \begin{bmatrix} C^{-1} & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \tilde{T}_1^* \\ \tilde{T}_2^* \end{bmatrix}$$

where M is a nilpotent matrix. Further, X is a generalized (outer) inverse of A if and only if M is a generalized (outer) inverse of N, where N is the nilpotent matrix in (2.2.1). In the common notation for generalized/outer inverse of N, if P_1, Q_1 and B are chosen such that $M = N(P_1, Q_1, B)$ is a nilpotent matrix, then we see that

$$X = A(T_2 P_1,\ \tilde{T}_2 Q_1,\ B).$$

The results of this section are available also in Eagambaram (1988b).

5. Generalized inverses with restrictions on minors.

A P_o(p)-matrix is a real square matrix whose principal minors are nonnegative (positive). An M-matrix is a P_o-matrix whose off-diagonal elements are nonpositive. A N_o(N)-matrix is a real square matrix whose principal minors are nonpositive (negative). A square matrix A is said to be reducible if there exists a permutation matrix P such that $\hat{A} = PAP^T$ is of the form $\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$.

Mohan, Neumann and Ramamurthy (1984) showed that if A is an irreducible M-matrix then A^+ and $A^\#$ are P_o-matrices. Also they showed that A^D is a P_o-matrix if A is an M-matrix. In the same paper they posed as an open question, the problem of obtaining conditions under which a g-inverse is a P_o-matrix. Eagambaram (1988a) provides a complete answer to this open question. In this section we see that the bordered matrix approach for obtaining g-inverses/outer inverses enables us to derive results which are more general than required for answering the open question of Mohan, Neumann and Ramamurthy (1984). That is, we derive necessary and sufficient conditions for obtaining a g-inverse/outer inverse with pre-specified minors. Also we derive some additional results en route.

Theorems 2.5.1 and 2.5.2 are well known. However for the sake of completeness we prove them here. Theorem 2.5.1 plays a pivotal role in this section.

Theorem 2.5.1: Let $C \in R^{n\times n}$ be nonsingular. Then for $J,K \subseteq \{1,2,\ldots,n\}$ with the same cardinality, we have

$$\det(C^{-1})_{J'K'} = \frac{(-1)^{\left(\sum_{j\in J} j + \sum_{k\in K} k\right)} \det C_{KJ}}{\det C} \tag{2.5.1}$$

Proof: Let us first prove the theorems for $J = K$. Without loss of generality we can assume that C is of the form

$$C = \begin{bmatrix} C_J & C_{JJ'} \\ C_{J'J} & C_{J'} \end{bmatrix}$$

If C_J is nonsingular it is easy to verify that

$$\det C = \det C_J \det(C_{J'} - C_{J'J}\, C_J^{-1}\, C_{JJ'}) \tag{2.5.2}$$

and

$$(C^{-1})_{J'} = (C_{J'} - C_{J'J}\, C_J^{-1}\, C_{JJ'})^{-1} \tag{2.5.3}$$

So, (2.5.1) with $J = K$ follows from (2.5.2) and (2.5.3). If C_J is singular $(C^{-1})_{J'}$ must also be singular. Otherwise by reversing the roles of C and C^{-1} in (2.5.2) and (2.5.3) we would arrive at the contradiction that

$$\det C_J = \frac{\det(C^{-1})_{J'}}{\det(C^{-1})} \neq 0$$

Therefore (2.5.1) holds when $J = K$. Now, let P and Q be permutation matrices such that C_{KJ} becomes a leading principal submatrix of P C Q. Let $L \subseteq \{1,2,\ldots,n\}$ be such that $(P\,C\,Q)_L = C_{KJ}$. Since (2.5.1) is found to hold for J = K, we see that

$$\det (P\,C\,Q)^{-1}_{L'} = \frac{\det(P\,C\,Q)_L}{\det P\,C\,Q}$$
$$= \frac{\det C_{KJ}}{\det C}\,(\det P.\ \det Q)^{-1}$$
$$= \frac{\det C_{KJ}}{\det C}\,(-1)^{(\sum_{j\in J} j + \sum_{k\in K} k)}$$

But

$$\det (P\,C\,Q)^{-1}_{L'} = \det(Q^T\,C^{-1}\,P^T)_{L'}$$
$$= \det\,(C^{-1})_{J'K'}.$$

Hence the theorem. □

Let us denote the set $\{1,2,\ldots,n\}$ by $\bar{N}$ and the set $\{1,2,\ldots,m\}$ by $\bar{M}$.

Theorem 2.5.2: $A \in R^{n\times n}$ is a P_o-matrix if and only if A+D is nonsingular for all positive diagonal matrix D.

Proof: We note that for $D = \mathrm{diag}(d_1,\ldots d_n)$

$$\det(A+D) = d_1\,d_2\ldots d_n + \sum_{J\subset \bar{N}} d(J)\,\det A_{J'} \tag{2.5.4}$$

where $d(J) = \prod_{j\in J} d_j$, $J \neq \phi$ and $d(\phi) = 1$.

So, if A is a P_o-matrix then A+D is nonsingular for all positive diagonal matrix D.

On the other hand if A is not a P_o-matrix then $\det A_J < 0$ for some $J \subseteq \bar{N}$. Let us assume without loss of generality that

$$A = \begin{bmatrix} A_J & A_{JJ'} \\ A_{J'J} & A_{J'} \end{bmatrix}$$

Since $\det A_J < 0$ there exists $d_1 > 0$ such that

$$\det (A_J + d_1 I_J) < 0.$$

Let $d_2 > 0$ be large enough so that

$$\det (A_{J'} + d_2 I_{J'} - A_{J'J}(A_J + d_1 I_J)^{-1} A_{JJ'}) > 0.$$

Then for $\tilde{D} = \begin{bmatrix} d_1 I_J & 0 \\ 0 & d_2 I_{J'} \end{bmatrix}$ we see that

$$\det (A+\tilde{D}) < 0.$$

But for a very large positive d

$$\det (A+\tilde{D} + dI) > 0.$$

Therefore there exists a $d_o > 0$ such that for $D = \tilde{D}+d_o I$, $\det(A+D) = 0$. This completes the proof. □

Theorem 2.5.3: Let $C \in R^{n\times n}$ be nonsingular. Then for $J \subseteq \bar{N}$ $(C^{-1})_J$ is a P_o-matrix if and only if $\det(C+D) \neq 0$ for every diagonal matrix D such that D_J is a positive diagonal matrix and $D_{J'} = 0$.

Proof: Let D be a diagonal matrix such that D_J is nonsingular and $D_{J'} = 0$ From (2.5.4) we have for $\tilde{d}_j = 1/d_j$ when $j \in J$ and $\tilde{d}_j = 0$ when $j \notin J$

$$\det((C^{-1})_J + \tilde{D}_J) = \prod_{j\in J} \tilde{d}_j + \sum_{K \subset J} \tilde{d}(K) \det(C^{-1})_{J-K} \qquad (2.5.5)$$

By virtue of Theorem 2.5.1, (2.5.5) becomes

$$\det((C^{-1})_J + \tilde{D}_J) = \prod_{j\in J} \tilde{d}_j + \sum_{K\subset J} \tilde{d}(K) \det C_{(J-K)'} (\det C)^{-1}$$

$$= (\prod_{j\in J} \tilde{d}_j)(\det C)^{-1}(\det C + \sum_{K \subset J} d(J-K) \det C_{(J-K)'})$$

$$= (\prod_{j\in J} \tilde{d}_j)(\det C)^{-1} \det (C+D).$$

So, from Theorem 2.5.2 we see that $(C^{-1})_J$ is a P_o-matrix if and only if $\det (C+D) \neq 0$ for all diagonal matrix D such that D_J is a positive diagonal matrix and $D_{J'} = 0$. □

If $(C^{-1})_J$ is a P_o-matrix for some $J \subseteq \bar{N}$ then $(C^{-1})_K$ is a P_o-matrix for all $K \subseteq J$. So, we have

Corollary 2.5.1: Let $C \in R^{n\times n}$ be nonsingular. Then for $J \subseteq \bar{N}$, $(C^{-1})_J$ is a P_o-matrix if and only if $\det(C+D) \neq 0$ for all nonnegative diagonal matrix D such that $D_{J'} = 0$.

Theorem 2.5.4: $A \in R^{n\times n}$ is an N_o-matrix if and only if $\det(A+D) \leq 0$ for all nonnegative diagonal matrix D with at least one zero diagonal entry.

Proof: Necessity: Since at least one diagonal entry of D is zero, the expansion of det (A+D) as in (2.5.4) gives

$$\det(A+D) = \sum_{J \subset \bar{N}} d(J) \det A_{J'}$$

where $d(J) = \prod_{j \in J} d_j$.

A is an N_o-matrix implies $\det A_{J'} \leq 0$ for all $J \subseteq \bar{N}$. Since D is nonnegative we see that $\det(A+D) \leq 0$.

Sufficiency: Suppose A is not an N_o-matrix. Then there exists a $J \subseteq \bar{N}$ such that $\det A_{J'} > 0$. Let the diagonal matrix D be such that for $d > 0$ $D_J = dI_J$ and $D_{J'} = 0$. Then the expansion $\det(A+D)$ as in (2.5.4) gives

$$\begin{aligned} \det(A+D) &= \sum_{K \subseteq J} d(K) \det A_{K'} \\ &= d(J) \left(\det A_{J'} + \sum_{K \subset J} \frac{d(K)}{d(J)} \det A_{K'}\right) \\ &= d^{|J|} \left(\det A_{J'} + \sum_{K \subset J} \frac{1}{d^{|J-K|}} \det A_{K'}\right) \end{aligned}$$

Taking d very large we see that $\det(A+D) > 0$.

Hence the theorem. □

Theorem 2.5.5: $A \in R^{n\times n}$ is a P_o-matrix if and only if $\det(A+D) \geq 0$ for all nonnegative diagonal matrix D.

Proof of the above theorems follows by arguing on the lines of the proof for Theorems 2.5.4. in conjection with Theorem 2.5.2.

Theorem 2.5.6: Let $C \in R^{n\times n}$ be nonsingular. Then for a given $J \subseteq \bar{N}$, $(C^{-1})_J$ is a P_o-matrix (N_o-matrix) if and only if for any nonnegative diagonal matrix D such that $D_{J'} = 0$

$$\frac{\det(C+D)}{\det C} \geq (\leq)\ 1.$$

Proof: We know, from (2.5.4), that

$$\frac{\det(C+D)}{\det C} = 1 + \sum_{\substack{K \subseteq \bar{N} \\ K \neq \phi}}' d(K)\ \left(\frac{\det C_{K'}}{\det C}\right) \tag{2.5.6}$$

We note that in (2.5.6) $d(K) = 0$ for all $K \not\subseteq J$, due to the definition of D.

Necessity: If $(C^{-1})_J$ is a P_o-matrix (N_o-matrix) then for $K \subseteq J$

$$\det(C^{-1})_K = \frac{\det C_{K'}}{\det C} \geq (\leq)\ 0 \tag{2.5.7}$$

In view of (2.5.7), (2.5.6) yields

$$\frac{\det\ (C+D)}{\det C} \geq (\leq)1$$

when $(C^{-1})_J$ is a P_o-matrix (N_o-matrix).

Sufficiency: Let us show that the condition $\frac{\det\ (C+D)}{\det C} \geq 1$ is sufficient for $(C^{-1})_J$ to be a P_o-matrix. The proof for the case of N_o-matrix is similar.

Suppose that $\det(C^{-1})_{\tilde{K}} < 0$ for some $\tilde{K} \subseteq J$. Define D such that $D_{\tilde{K}} = dI_{\tilde{K}}$ and $D_{\tilde{K}'} = 0$ where $d > 0$. Now from (2.5.4) we have

$$\frac{\det\ (C+D)}{\det C} = 1 + d(\tilde{K})\left(\frac{\det C_{\tilde{K}'}}{\det C} + \sum_{K \subset \tilde{K}} \frac{1}{d(\tilde{K}-K)} \frac{\det C_{K'}}{\det C}\right)$$

$$= 1 + d^{|\tilde{K}|}\left(\det\ (C^{-1})_{\tilde{K}} + \sum_{K \subset \tilde{K}} \frac{1}{d^{|\tilde{K}-K|}} \frac{\det C_{K'}}{\det C}\right)$$

Taking d very large positive we see that $\frac{\det (C+D)}{\det C} < 1$. Hence the Theorem.

□

Now we give the main results of this section.

Theorem 2.5.7: Let $A \in R^{m \times n}$. Let $J \subseteq \bar{M}$ and $K \subseteq \bar{N}$ be such that $|J| = |K|$. Then

$$\det A(E,F,B)_{K'J'} = \frac{\det \begin{bmatrix} A_{JK} & E_{J.} \\ (F_{K.})^T & B \end{bmatrix}}{\det \begin{bmatrix} A & E \\ F^T & B \end{bmatrix}}$$

The proof of the above theorem follows from the definition of A(E,F,B) and Theorem 2.5.1.

As a consequence of Theorem 2.5.7 we have

Theorem 2.5.8: Let $A \in R^{n \times n}$. Then A(E,F,B) is a P_o-matrix (N_o-matrix) if and only if

$$\frac{\det \begin{bmatrix} A_J & E_{J.} \\ (F_{J.})^T & B \end{bmatrix}}{\det \begin{bmatrix} A & E \\ F^T & B \end{bmatrix}} \geq (\leq)\ 0 \quad , \text{ for all } J \subseteq \bar{N} \qquad (2.5.8)$$

Theorem 2.5.9: Let $A \in R^{n \times n}$. Then A(E,F,B) is a P_o-matrix if and only if $\begin{bmatrix} A+D & E \\ F^T & B \end{bmatrix}$ is nonsingular for every positive diagonal matrix D.

The proof of the above theorem follows from Theorem 2.5.3.

Theorem 2.5.10: Let $A \in R^{n \times n}$ be a P_o-matrix. Then a {2}-inverse A(E,F,0) of A is a P_o-matrix if and only if

$$(A+D)\ R(A(E,F,0)) \cap N(A(E,F,0)) = \{0\}.$$

for every positive diagonal matrix D, where (A+D) R(A(E,F,0)) is the subspace

$$\{(A+D)x : x \in R(A(E,F,0)\}$$

Proof: Note that $R(A(E,F,0)) = N(F^T)$ and $N(A(E,F,0) = R(E)$. According to Theorem 2.5.2, A+D is nonsingular when D is a positive diagonal matrix. Since $\begin{bmatrix} A & E \\ F^T & 0 \end{bmatrix}$ is nonsingular the column vectors of E are linearly independent. Now it follows that $\begin{bmatrix} A+D & E \\ F^T & 0 \end{bmatrix}$ is singular if and only if there exists an $x \in N(F^T)$ such that $(A+D)x \in R(E)$. Therefore in view of Theorem 2.5.9 we see that the theorem stands proved. □

Ramamurthy and Mohan (1985) established the above theorem for the particular case where A(E,F,0) is a {1,2}-inverse of A.

The proof of Theorem 2.5.11 below is similar to that of Theorem 2.5.9 except that we invoke Corollary 2.5.1 instead of Theorem 2.5.3.

Theorem 2.5.11: Let $A \in R^{n \times n}$. Then A(E,F,B) is a P_o-matrix if and only if $\begin{bmatrix} A+D & E \\ F^T & B \end{bmatrix}$ is nonsingular for every nonnegative diagonal matrix D.

Corollary 2.5.2: Let $A \in R^{n \times n}$ and A(E,F,B) be a P_o-matrix. Then (A+D)(E,F,B) is a P_o-matrix for every nonnegative diagonal matrix D.

Proof: Since A(E,F,B) is a P_o-matrix from Theorem 2.5.11 we have that $\begin{bmatrix} A+D+\tilde{D} & E \\ F^T & B \end{bmatrix}$ is nonsingular for every nonnegative diagonal matrix $\tilde{D}$. But then by the same theorem we see that (A+D)(E,F,B) is a P_o-matrix.

The following theorem is an immediate consequence of Theorem 2.5.6.

Theorem 2.5.12: Let $A \in R^{n \times n}$. Then A(E,F,B) is a P_o-matrix (N_o-matrix) if and only if

$$\frac{\det \begin{bmatrix} A+D & E \\ F^T & B \end{bmatrix}}{\det \begin{bmatrix} A & E \\ F^T & B \end{bmatrix}} \geq (\leq) 1.$$

for every nonnegative diagonal matrix D.

6. <u>Some results on g-inverse of matrices of order n and rank n-1.</u>

Throughout this section A stands for a real $n \times n$ matrix of rank n-1; λ and $\pi \in R^n$ denote vectors spanning $N(A)$ and $N(A^T)$ respectively; α denotes a vector not in $R(A)$ and β a vector not in $R(A^T)$ so that $A(\alpha,\beta,c)$ is defined for all real c.

Theorem 2.6.1: For $A(\alpha,\beta,0)$ to be a P_o-matrix or N_o-matrix it is necessary that $\alpha_i\beta_i$ are of the same sign for all $i \in \bar{N}$.

The proof follows from equation (2.5.8) by taking all $J \subseteq \bar{N}$ with $|J| = 1$.

□

Corollary 2.6.1: A is a P_o-matrix or N_o-matrix only if $\lambda_i\pi_i$ are of the same sign for all $i \in \bar{N}$.

Proof: $A = A(\alpha,\beta,0)(\lambda,\pi,0)$. So, the proof follows from Theorem 2.6.1.

□

Corollary 2.6.2: The Moore-Penrose inverse of A or the group inverse of A (when it exists) is a P_o-matrix or N_o-matrix only if $\lambda_i\ \pi_i$ have the same sign for all $i \in \bar{N}$.

The proof is obtained by noting that $A(\pi,\lambda,0) = A^+$ and $A(\lambda,\pi,0) = A^\#$ (when it exists).

□

Theorem 2.6.2: Let adj $A_J \geq 0$ for all $J \subseteq \bar{N}$. Then

(i) $A(\alpha,\beta,0)$ exists for some α, $\beta \in R^n_+ \cup R^n_-$

(ii) all $A(\alpha,\beta,0)$ with α, $\beta \in R^n_+ \cup R^n_-$ are P_o-matrices.

Proof: (i)

$$\det \begin{bmatrix} A & \alpha \\ \beta^T & 0 \end{bmatrix} = -\beta^T(\text{adj } A)\alpha \qquad (2.6.1)$$

Since rank A = n-1, adj $A \geq 0$ implies that adj $A \neq 0$. So, we can choose $\alpha,\beta \in R^n_+ \cup R^n_-$ such that $-\beta^T(\text{adj } A)\alpha \neq 0$. For such α,β we see that $A(\alpha,\beta,0)$ exists.

(ii) When $\alpha,\beta \in R^n_+ \cup R^n_-$, for $J \subseteq \bar{N}$

$$\det \begin{bmatrix} A_J & \alpha_{J.} \\ (\beta_{J.})^T & 0 \end{bmatrix} = -(\beta_{J.})^T(\text{adj } A_J)\alpha_{J.} \qquad (2.6.2)$$

Now the proof follows from Theorem 2.5.8 in conjunction with (2.6.1) and (2.6.2) □

It is known that a nonsingular M-matrix is inverse-nonnegative. (See for example Berman and Plemmous (1979)).Since every principal submatrix A_J of an M-matrix A is an M-matrix and $A_J + \lambda I$ is nonsingular by Theorem 2.5.2, we see that adj $A_J \geq 0$. This property proves the following corollary to Theorem 2.6.2.

Corollary 2.6.3: If A is an M-matrix then for nonzero $\alpha,\beta \in R^n_+ \cup R^n_-$ $A(\alpha,\beta,0)$ is a P_o-matrix.

Corollary 2.6.4: If A is an M-matrix, then the Moore-Penrose inverse of A and the group inverse of A (when it exists) are P_o-matrices.

Proof: When A is an M-matrix, adj $A_J \geq 0$ for all $J \subseteq \bar{N}$. We know that $\lambda,\pi \in R^n_+ \cup R^n_-$ and so the proof follows from Theorem 2.6.2 by noting that the Moore-Penrose inverse and the group-inverse of A are $A(\pi,\lambda,0)$ and $A(\lambda,\pi,0)$ respectively. □

Corollary 2.6.4 was proved by Mohan, Neumann and Ramamurthy (1984); and Ramamurthy and Mohan (1985).

Theorem 2.6.3: Let A be a symmetric positive semi definite (PSD) matrix. Then

(i) $A(\alpha,\beta,0)$ with $\alpha=\beta$ exists, and

(ii) $A(\alpha,\alpha,0)$ is a P_o-matrix.

Proof: (i) Since A is a symmetric PSD matrix each of its principal submatrices is also a PSD matrix. Again, adj A_J is a PSD matrix for all $J \subseteq N$. Now

$$\det \begin{bmatrix} A & \alpha \\ \alpha^T & 0 \end{bmatrix} = -\alpha^T(\text{adj } A)\alpha \neq 0$$

because $R(A) = R(A^T)$ and $\alpha \notin R(A)$ implies that $\alpha \notin R(A^T)$. Therefore $A(\alpha,\alpha,0)$ exists for $\alpha \notin R(A)$.

(ii)

$$\frac{\det \begin{bmatrix} A_J & \alpha_{J.} \\ (\alpha_{J.})^T & 0 \end{bmatrix}}{\det \begin{bmatrix} A & \alpha \\ \alpha^T & 0 \end{bmatrix}} = \frac{\alpha_{J.}^T (\text{adj } A_J)\, \alpha_{J.}}{\alpha^T(\text{adj } A)\, \alpha} \geq 0, \quad J \subseteq \bar{N}$$

because $\alpha_{J.}^T (\text{adj } A)\alpha_{J.} \geq 0$ and for all $J \subseteq \bar{N}$ and $\alpha^T(\text{adj } A)\alpha > 0$. Hence the theorem. □

We observe that

$$\begin{bmatrix} A & \alpha \\ \beta^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A(\alpha,\beta,0) & (\beta^T\lambda)^{-1}\lambda \\ (\pi^T\alpha)^{-1}\pi^T & 0 \end{bmatrix} \tag{2.6.3}$$

for non zero $t \in R^1$

$$\begin{bmatrix} A & \alpha \\ \beta^T & t \end{bmatrix}^{-1} = \begin{bmatrix} A(\alpha,\beta,0)-t(\beta^T\lambda)^{-1}\lambda(\pi^T\alpha)^{-1}\pi^T & (\beta^T\lambda)^{-1}\lambda \\ (\pi^T\alpha)^{-1}\pi^T & 0 \end{bmatrix} \tag{2.6.4}$$

$$= \begin{bmatrix} (A-t^{-1}\alpha\ \beta^T)^{-1} & (\beta^T\lambda)^{-1}\lambda \\ (\pi^T\alpha)^{-1}\pi^T & 0 \end{bmatrix} \quad \text{for } t \neq 0 \qquad (2.6.5)$$

If $H \in R^{n\times n}$ is nonsingular, $H(\gamma,\delta,\theta)$ exists and

$$\begin{bmatrix} H & \gamma \\ \delta & \theta \end{bmatrix}^{-1} = \begin{bmatrix} H(\gamma,\delta,\theta) & \tilde{\lambda} \\ \tilde{\pi}^T & 0 \end{bmatrix}$$

then

$$H(\gamma,\delta,\theta) = H^{-1} - H^{-1}\gamma(\theta-\delta^T H^{-1}\gamma)^{-1}\ \delta^T \qquad (2.6.6)$$

The following theorem is a simple consequence of equations (2.6.3) and (2.6.4).

Theorem 2.6.4: If $\alpha, \pi \in R^n_+ \cup R^n_-$ then $A\ A(\alpha,\beta,t)$ is an M-matrix for all $t \in R^1$. Similarly, if $\beta,\lambda \in R^n_+ \cup R^n_-$ then $A(\alpha,\beta,t)A$ is an M-matrix for all $t \in R^1$.

Theorem 2.6.5: Let A be a P_o-matrix and its {1,2}-inverse $A(\alpha,\beta,0)$ also be a P_o-matrix. Let $\det \begin{bmatrix} A & \alpha \\ \beta & 0 \end{bmatrix} > 0$. Then $A(\alpha,\beta,t)$ exists for all $t \in R^1$ and $A(\alpha,\beta,t)$ is a P_o-matrix for all $t \geq 0$. Further if every proper principal minor of A is positive then $A(\alpha,\beta,t)$ is a P-matrix for all $t > 0$.

Proof: For all $J \subseteq \bar{N}$,

$$\det \begin{bmatrix} A_J & \alpha_{J.} \\ (\beta_{J.})^T & t \end{bmatrix} = \det \begin{bmatrix} A_J & \alpha_{J.} \\ (\beta_{J.})^T & 0 \end{bmatrix} + t \det A_J \qquad (2.6.7)$$

Now the existence of $A(\alpha,\beta,t)$ for $t \in R^1$ follows from (2.6.7); and that $A(\alpha,\beta,t)$ is a P_o-matrix for all $t \geq 0$ follows from (2.6.7) and Theorem 2.5.8. Where $\det A_J > 0$ for all proper subset J of $\bar{N}$ then from (2.6.7) we see that $A(\alpha,\beta,t)$ is a P-matrix for all $t > 0$. □

Remark 2.6.1: If A and $A(\alpha,\beta,0)$ are as at Theorem 2.6.5. Then from equations (2.6.4) and (2.6.5) we see that

(i) $A(\alpha,\beta,t) = A(\alpha,\beta,0) - t(\beta^T\lambda)^{-1}(\pi^T\alpha)^{-1}\lambda\pi^T$ is a P_o-matrix for all $t \geq 0$

(ii) $A - t\alpha\,\beta^T$ is a P_o-matrix for all $t \geq 0$.

Remark 2.6.2: As a Corollary to Remark 2.6.1, if A is an M-matrix and $\alpha,\beta \in R^n_+ \cup R^n_-$ we see in view of Corollary 2.6.3 and Corollary 2.6.4 that

(i) $A(\alpha,\beta,0) + t(\pi^T\lambda)\lambda\,\pi^T$ is a P_o-matrix for all $t \geq 0$

(ii) $A^+ + t(\pi^T\lambda)\lambda\pi^T$ is a P_o-matrix for all $t \geq 0$

(iii) When $A^\#$ exists $A^\# + t(\pi^T\lambda)\lambda\pi^T$ is a P_o-matrix for all $t \geq 0$.

(iv) If A is irreducible then every proper principal minor of A is positive and hence from Theorem 2.6.4 we see that the matrix pencils given at (i), (ii) and (iii) above are P-matrices for all $t > 0$. Further for sufficiently large positive t these matrix pencils are positive matrices because $(\pi^T\lambda)\lambda\pi^T$ is positive A being irreducible.

7. Generalized inverse and finite Markov chain.

A finite time-homogeneous Markov chain is a pair (T,π^o) where $T = [t_{ij}]$ is an n×n real nonnegative matrix with each row sum unity and π^o is an n-dimensional probability vector which is a nonnegative vector whose elements add up to unity. T is called the transition probability matrix (tpm) of the Markov chain. The set $\bar{N} = \{1,2,\ldots,n\}$ is known as the state space of the Markov chain. t_{ij} is the probability that the chain moves to state j from state i in one step. (For reference one may see Kemeny and Snell (1967)).

If $T^n = [t_{ij}^{(n)}]$, $t_{ij}^{(n)}$ is the probability that the chain visits state j at the nth step, given that it starts from state i initially. $\sum_{k=1}^{n} t_{ij}^{(k)}$ gives the expected number of visits to state j in n steps, given that the chain starts from state i.

The Markov chain (T, π^o) is said to be reducible if T is reducible. When T is irreducible then $A = I-T$ is an irreducible M-matrix and in this case A_J is nonsingular for all $J \subset \bar{N}$.

Mayer (1975) has highlighted the role of the group inverse of $A = I-T$ in obtaining several results relating to the Markov chain. These results are also available in Berman and Plemmons (1979). Later, Hunter (1982) demonstrated the use of any g-inverse in analysing irreducible Markov chains. In this section we give an interpretation for the principal minors of a g-inverse of A for an irreducible Markov chain. In Theorem 2.7.3 we show that the Moore Penrose inverse of I-T is a P_o-matrix under certain conditions.

Let T be irreducible and for $J \subset \bar{N}$, $J' = \bar{N} - J$, and $i,j \in J$, the (i,j)th element of $A_J^{-1} = \sum_0^\infty T_J^m$ gives the expected number of visits to state j before visiting any state in J', given that the chain starts initially from state i. If $\beta_i \alpha_j$ is the reward obtained whenever the state j is visited given that the chain starts from state i, then for $\sigma(J) = \sum_{j \in J} \pi_j^o \neq 0$ and

$$\theta^T = [\pi_1^o \beta_1, \ldots, \pi_n^o \beta_n]$$

$$\frac{\theta_{J.}^T A_J^{-1} \alpha_{J.}}{\sigma(J)} = \text{the expected total reward accummulated by the chain before it leaves } J \subset \bar{N}, \text{ given that it starts initially from } J. \qquad (2.7.1)$$

Theorem 2.7.1: Let $T \in R^{n \times n}$ and $e \in R^n$ with all components unity. Let $(T, \frac{1}{n} e)$ be an irreducible Markov chain. Let $A(\alpha,\beta,0) = G$ be a {1,2}-inverse of A. Let $\beta_i \alpha_i$ be the reward whenever state j is visited, given that the chain starts from state i. Then for $J \subset \bar{N}$, $J' = \bar{N} - J$ and $A = I-T$

$$\frac{\det G_{J'}}{\det A_J} \; \frac{\beta^T (\text{adj } A)\alpha}{|J|} = \text{the expected accummulated reward before the chain leaves } J, \text{ given that it starts from } J.$$

Proof: From Theorem 2.5.1 we have

$$\det G_{J'} = \frac{\det \begin{bmatrix} A_J & \alpha_{J.} \\ (\beta_{J.})^T & 0 \end{bmatrix}}{\det \begin{bmatrix} A & \alpha \\ \beta^T & 0 \end{bmatrix}} = \frac{\det A_J((\beta_{J.})^T A_J^{-1}\alpha_{J.})}{\beta^T(\text{adj } A)\alpha} \qquad (2.7.1^*)$$

So,

$$\frac{\det G_{J'}}{\det A_J} \frac{\beta^T(\text{adj } A)\alpha}{|J|} = \frac{(\beta_{J.})^T A_J^{-1}\alpha_{J.}}{|J|}$$

Now the proof follows from (2.7.1). □

Theorem 2.7.2: Let (T,π^o) be an irreducible Markov chain. Let $A = I-T$ and $(\pi^o)^T A = 0$. Then $A(e,\pi^o,0) = A^{\#}$ and for $J \subset \bar{N}$ with $\sigma(J) = \sum_{j\varepsilon J} \pi_j^o$

$$\frac{\det A_J^{\#}}{\det A_J} \frac{((\pi^o)^T(\text{adj } A)e)}{\sigma(J)}$$ = the expected number of steps taken by the process to leave J, given that the process starts from J.

Proof: The proof easily follows by noting that the expression on the L.H.S. above is identical with $((\pi_{J.}^o)^T A_J^{-1} e_{J.})(\sigma(J))^{-1}$

Mohan, Neumann and Ramamurthy (1984) showed that the Drazin inverse of an M-matrix is a P_o-matrix. They could not settle the question, whether the Moore-Penrose inverse of an M-matrix is a P_o-matrix. The following example shows that the Moore-Penrose inverse of an M-matrix need not be a P_o-matrix.

Example 2.7.1:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0.1 & 0 \\ -6 & 0 & -2 & 0 & 0.1 \end{bmatrix}$$

$$A^{+} = -(10.1804)^{-1} \begin{bmatrix} 0 & 3.92 & -3.92 & 8.16 & -3.76 \\ 0 & -16.5601 & 16.5601 & -23.98 & 16.04 \\ 0 & -11.4699 & 11.4699 & -23.98 & 16.04 \\ 0 & -8.801 & 8.801 & -15.204 & 10 \\ & 5.802 & -5.802 & 10 & -6.60\text{[illegible]} \end{bmatrix}$$

A is an M-matrix, but A^{+} is not a P_o-matrix.

It can be verified that any square matrix A can be reduced by using a suitable permutation matrix P to the form

$$PAP^{T} = \begin{bmatrix} B_{11} & 0 & 0 \cdots & 0 \\ B_{21} & B_{22} & 0 \cdots & 0 \\ \vdots & & & \\ B_{k1} & B_{k2} \cdots & B_{k3} \cdots & B_{kk} \end{bmatrix} \tag{2.7.2}$$

Where B_{ii}, $i = 1,2,\ldots,k$ is irreducible or 1×1 null matrix.

We see that if A = I−T where T is a Markov matrix, then in the reduced form (2.7.2) B_{11} is a singular M-matrix and B_{jj}, $j > 1$ is nonsingular if and only if $B_{ji} \neq 0$ for some $i < j$. So, in this case we can assume that A is the form

$$PAP^{T} = \begin{bmatrix} B_{11} & 0 \cdots 0 & & 0 \\ 0 & B_{22} \cdots 0 & & 0 \\ \vdots & & & \\ 0 & 0 & \cdots B_{k-1,k-1} & 0 \\ B_{k1} & B_{k2} \cdots & B_{kk-1} & B_{kk} \end{bmatrix} \tag{2.7.3}$$

where B_{ii}, $i = 1,2,\ldots,k-1$ is a singular irreducible M-matrix or a 1×1 null matrix, and B_{kk} is an M-matrix (see Berman and Plemmons (1979)).

Theorem 2.7.3: Let $A \in R^{n\times n}$ be an M-matrix of the form $A = I-T$ where T is a Markov matrix. Let P be a permutation matrix such that PAP^T is of the form (2.7.3). Then the Moore-Penrose inverse of A is a P_o-matrix, if

$$\text{rank } [B_{k1},\ldots,B_{kk-1}] \leq 1.$$

Proof: For simplicity, let us assume that k = 3 because for the general k the proof will be similar.

So, let

$$\hat{A} = PAP^T = \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

Let n_i be the order of B_{ii}, $i = 1,2,3$; and $n = n_1+n_2+n_3$.

<u>Case 1:</u> rank $[B_{31} \;\; B_{32}] = 0$

In this case $[B_{31} \;\; B_{32}] = [0 \;\; 0]$ and B_{33} is an irreducible singular M-matrix.

$$\hat{A}^+ = \begin{bmatrix} B_{11}^+ & 0 & 0 \\ 0 & B_{22}^+ & 0 \\ 0 & 0 & B_{33}^+ \end{bmatrix}$$

Since (from Corollary 2.6.4) B_{ii}^+, $i=1,2,3$ are P_o-matrices we see that $\hat{A}^+$ is a P_o-matrix and hence A^+ is a P_o-matrix.

<u>Case 2:</u> rank $[B_{31} \;\; B_{32}] = 1$.

We note that B_{33} is a nonsingular M-matrix. Since B_{11} and B_{22} are irreducible M-matrices, there exist positive vectors $\pi(1)$ and $\pi(2)$ such that $\pi^T(i)\, B_{ii} = 0$ $i = 1,2$. Let

$$\theta(i) = - B_{33}^{-1} B_{3i} \ell(i), \; i = 1,2$$

where $\ell(i)$ is the vector of dimension n_i with all its elements unity. Since $B_{3i} \leq 0$, $\theta(i)$, $i = 1,2$ are nonnegative vectors. Let

$$E = \begin{bmatrix} -\pi(1) & 0 \\ 0 & -\pi(2) \\ 0 & 0 \end{bmatrix}$$

and

$$F = \begin{bmatrix} \ell(1) & 0 \\ 0 & \ell(2) \\ \theta(1) & \theta(2) \end{bmatrix}$$

Then $R(E) = N(\hat{A}^T)$ and $R(F) = N(\hat{A})$. From Theorem 2.3.5 we see that $\hat{A}(E,F,0) = \hat{A}^+$

For $J \subset R^1$ and $k \in R^1$ let $J+k = \{j+k \mid j \in J\}$. Any $J \subseteq \bar{N}$ can be written as

$$J = J_1 \cup (J_2+n_1) \cup (J_3+n_1+n_2)$$

where $J_i \subseteq \{1,2,\ldots,n_i\}$, $i = 1,2,3$. When J_1 or J_2 is empty

$$\det \begin{bmatrix} \hat{A}_J & E_{J.} \\ (F_{J.})^T & 0 \end{bmatrix} = 0. \tag{2.7.4}$$

When neither J_1 nor J_2 is empty

$$\det \begin{bmatrix} \hat{A}_J & E_{J.} \\ (F_{J.})^T & 0 \end{bmatrix} = \det \begin{bmatrix} C_{11} & 0 & L_1 \\ 0 & C_{22} & L_2 \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

where C_{ii}, $i = 1,2$, is of the form

$$C_{ii} = \begin{bmatrix} \tilde{B}_{ii} & -\tilde{\pi}(i) \\ \tilde{\ell}^T(i) & 0 \end{bmatrix} \tag{2.7.5}$$

and

$$C_{33} = (B_{33})_{J_3};$$

$$\tilde{B}_{ii} = (B_{ii})_{J_i},$$

$$C_{3i} = [(B_{3i})_{J_3 J_i} \quad 0],$$

$$\tilde{\pi}(i) = (\pi(i))_{J_i}.$$

$$\tilde{\ell}(i) = (\ell(i))_{J_i}.$$

$$L_i = [0 \quad (\theta(i))_{J_3}]^T.$$

for i = 1,2.

Now

$$\det \begin{bmatrix} \hat{A}_J & E_{J.} \\ (F_{J.})^T & 0 \end{bmatrix} = \det \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & 0 \\ C_{31} & C_{32} & \tilde{C}_{33} \end{bmatrix}$$

where

$$\tilde{C}_{33} = C_{33} - C_{31} C_{11}^{-1} L_1 - C_{32} C_{22}^{-1} L_2 \qquad (2.7.6)$$

(2.7.6) is well-defined because C_{ii} as in expression (2.7.5) is nonsingular. It can be verified that

$$-C_{3i} C_{11}^{-1} L_i \geq 0, \quad i = 1,2.$$

Since rank C_{3i}, i = 1,2, is either 0 or 1 we see that $-C_{31}C_{11}^{-1}L_1 - C_{32}C_{22}^{-1}L_2$ can be written as $\gamma\delta^T$ for some $\gamma, \delta \in R_+^n$.

Therefore,

$$\det \begin{bmatrix} \hat{A}_J & E_{J.} \\ (F_{J.})^T & 0 \end{bmatrix} = \det C_{11} \det C_{22} \det \tilde{C}_{33}$$

$$= \det C_{11} \det C_{22} (\det C_{33} + \delta^T (\operatorname{adj} C_{33}) \gamma) > 0 \quad (2.7.7)$$

because $\det C_{ii} > 0$, i = 1,2,3. Thus (2.7.4) and (2.7.7) show that $\hat{A}^+$ is a P_o-matrix. Hence A^+ is a P_o-matrix. □

The above theorem is incorporated in Eagambaram (1988a).

CHAPTER 3

LINEAR COMPLEMENTARITY PROBLEM

1. Introduction

Given $M \in R^{n\times n}$ and $q \in R^n$ the linear complementarity problem (LCP), as already introduced in chapter 1, is the problem of obtaining a solution (w,z), if it exists, to the following system of equations:

$$w - Mz = q \quad w,z \geq 0 \tag{3.1.1a}$$

$$w^T z = 0 \tag{3.1.1b}$$

There are only two entities, namely, a matrix M and a vector q involved in defining an LCP; therefore it is denoted in short as (q,M).

If $M \in R^{n\times n}$ is such that (q,M) has a solution for all $q \in R^n$, then M is called a Q-matrix. If M is such that (q,M) has a solution whenever (3.1.1a) has a solution, then M is called a Q_o-matrix. One of the directions of research in LCP is to characterize Q and Q_o-matrices.

Lemke (1965) proposed an algorithm for solving LCP, which is akin to the simplex algorithm for solving the linear programming problem. The algorithm does not solve all (q,M) for arbitrary M and q. In another direction of research, an attempt is being made to identify the largest class of problems which can be solved by applying Lemke's algorithm. We make some observations in this regard in Section 2, after describing Lemke's algorithm.

We observe in Section 2 that Lemke's algorithm applied to (q,M) with the auxillary vector $d > 0$, gives rise to a set of solutions $(w(t),z(t),z_o(t))$ that satisfy

$$w - Mz - z_o d = q$$

Further, $(w(t), z(t), z_o(t))$ is continuous with respect to $t \in R^1_+$ and $||z(t)|| \to \infty$ as $t \to \infty$. The above result is an extension of Lemke's basic existence theorem of complementarity. (See Eaves (1971b)). Using the above property we identify in Section 3 two subclasses of Q_o-matrices via generalized inverses. Lemke's algorithm may not be applicable to some matrices belonging to these classes; but we give in Section 4 new algorithms that can solve (q,M) where M belongs to either of these two classes. In Section 5 we examine solution rays in the light of the remark of Cottle (1974). Some results on LCPs involving N_o-matrices are given in Section 6.

2. Lemke's algorithm

Lemke's algorithm seeks to obtain a solution to (q,M) with the help of the auxillary system of equations:

$$w - Mz - z_o d = q, \quad w, z, z_o \geq 0 \tag{3.2.1.a}$$

$$w^T z = 0 \tag{3.2.1b}$$

where d is a positive vector in R^n, $w, z \in R^n_+$ and $z_o \in R^1_+$. The algorithm generates a sequence of solutions (w^k, z^k, z^k_o), $k = 1,2,\ldots,$ to (3.2.1) and stops at a solution (w^r, z^r, z^r_o) when either

(i) $z^r_o = 0$ in which case a solution to (q,M) is (w^r, z^r)

or (ii) it is not possible by the rules of the algorithm to proceed to another solution to (3.2.1).

2. We require the following definitions for describing Lemke's algorithm.

Definition 3.2.1: For $M \in R^{n\times n}$, $C \in R^{n\times n}$ is called a complementary matrix of (I, -M) if $C_{.i} \in \{I_{.i}, -M_{.i}\}$, $i = 1,2,\ldots,n$. A nonsingular complementary matrix is called a complementary basis. Since a complementary matrix C is characterized by the set J of columns of -M participating in C, we denote C alternatively also as C(J).

Definition 3.2.2: A matrix $B \in R^{n\times n}$ is called an almost complementary basis (ACB) of (I, -M) with respect to $d > 0$ if B is a nonsingular matrix obtained by replacing one of the columns of a complementary matrix by -d. If B is obtained from $C(J)$ by replacing its i-th column by $-d$ then we denote B alternatively as $B(J,i)$.

Definition 3.2.3: For $A \in R^{m\times n}$ PosA is defined as

$$\text{PosA} = \{Ax : x \geq 0\}$$

We see that Pos A is a cone.

Definition 3.2.4: The cone PosC where C is a complementary matrix is called a complementary cone. A complementary cone PosC is said to be degenerate if $\det C = 0$. PosC is said to be blunt if there exists $0 \neq x \geq 0$ such that $Cx = 0$.

Definition 3.2.5: A vector y is said to be lexico positive if the first nonzero component of y is positive. We say that v is lexicographically greater than u if v-u is lexico positive. For a set S of linearly independent vectors, the unique vector $u^* \in S$ which is lexico-graphically less than any other vector in S is called the lexico minimum of S.

3. Now we describe Lemke's algorithm applied to (q,M) with the auxillary positive vector d.

<u>Step 1:</u> If $q \geq 0$ then $(w,z) = (q,0)$ is a solution to (q,M). The algorithm terminates.

Otherwise go to Step 2.

<u>Step 2:</u> Identify the index i that corresponds to the lexico minimum of row vectors $\frac{1}{d_k}[q_k, I_k]$, $k = 1,2,\ldots,n$.

Let $y = -M_{.i}$

$X = B(\phi, i)$

Put $\alpha = i$.

Go to Step 3.

Step 3: Let j be the index that corresponds to the lexico minimum of the row vectors $\{(X^{-1}y)_k\}^{-1}\ [(X^{-1}q)_k, (X^{-1})_{k.}]$ where k is such that $(X^{-1}y)_k > 0$.

If $(X^{-1}y)_k \leq 0$ for $k = 1,2,\ldots,n$

go to Step 5.

If $j = \alpha$ go to Step 4.

Otherwise let B be the ACB such that

$B_{.\alpha} = y$

$B_{.j} = -d$

$B_{.k} = X_{.k}$, $k \notin \{i,j\}$

Relabel X as old X

Redefine α, X and y as

$\alpha = j$

$X = B$

y = the single element of $\{I_{.j}\ ,\ -M_{.j}\} - \{(\text{old } X)_{.j}\}$

Go back to Step 3.

Step 4: Let C be the complementary basis obtained by replacing -d by y in X. $C^{-1}q$ gives rise to a solution to (q,M).

Step 5: The algorithm terminates without finding a solution.

4. Lemke's algorithm terminates in a finite number of steps. (See Murty (1988) for proof.)

5. Let B_k , $k = 1,2,\ldots r$ be the sequence of ACBs encountered by Lemke's algorithm. Let $\alpha(k) \in \{1,2,\ldots,n\}$ be such that $(B_k)_{.\alpha(k)}$ is the auxillary vector -d. Let $u_i \in \{w_i, z_i\}$ be the variable corresponding to the column $(B_k)_{.i}$ for $i \neq \alpha(k)$. Let $u_{\alpha(k)} \in \{w_{\alpha(k)}, z_{\alpha(k)}\}$ be the variable corresponding to the entering vector $y^k \in \{I_{.\alpha(k)}, -M_{.\alpha(k)}\}$. Then we observe that the triplet (w^k, z^k, z_o^k), $w^k, z^k \in R^n$, $z_o \in R^1$ is a solution to (3.2.1) where

$$w_i^k = (B_k^{-1}q)_i \quad \text{if } u_i = w_i,\ i \neq \alpha(k)$$
$$= 0 \quad \text{otherwise}$$

$$z_i^k = (B_k^{-1}q)_i \quad \text{if } u_i = z_i,\ i \neq \alpha(k)$$
$$= 0 \quad \text{otherwise}$$

$$z_o^k = (B_k^{-1}q)_{\alpha(k)}\,.$$

Let

$$\delta_k = \max\ \{\delta\ :\ B_k^{-1}(q-\delta y^k) \geq 0,\ \delta \geq 0\}$$

and $(\tilde{w}^k, \tilde{z}^k, \tilde{z}_o^k)$ be defined as

$$\tilde{w}_i^k = -(B_k^{-1}y^k)_i \quad \text{if } u_i = w_i,\ i \neq \alpha(k)$$
$$= 0 \quad \text{if } u_i \neq w_i$$

$$\tilde{z}_i^k = -(B_k^{-1}\ y^k)_i \quad \text{if } u_i = z_i,\ i \neq \alpha(k)$$
$$= 0 \quad \text{if } u_i \neq z_i$$

$$\tilde{w}_{\alpha(k)}^k = 1 \quad \text{if } u_{\alpha(k)} = w_{\alpha(k)}$$
$$= 0 \text{ otherwise}$$

$$\tilde{z}_{\alpha(k)}^k = 1 \quad \text{if } u_{\alpha(k)} = z_{\alpha(k)}$$
$$= 0 \quad \text{otherwise}$$

$$\tilde{z}_o^k = -\ (B_k^{-1}y^k)_{\alpha(k)}$$

Then we see that $(\tilde{w}^k, \tilde{z}^k, \tilde{z}_o^k) \neq (0,\ 0,\ 0)$ and

$$\tilde{w}^k - M\,\tilde{z}^k - \tilde{z}_o^k\, d = 0 \tag{3.2.2}$$

$$(w^k(\delta), z^k(\delta),\ z_o^k(\delta)) = (w^k,\ z^k,\ z_o^k) + \delta(\tilde{w}^k,\ \tilde{z}^k,\ \tilde{z}_o^k),\ 0 \leq \delta \leq \delta_k$$

is a solution to (3.2.1).

We see that

$$(w^k(\delta_k),\ z^k(\delta_k),\ z_o^k(\delta_k)) = (w^{k+1}(0),\ z^{k+1}(0),\ z_o^{k+1}(0)) \qquad (3.2.3)$$

6. If the algorithm terminates at the r-th iteration with a solution to (q,M) then δ_r is finite and $(w^r(\delta_r),\ z^r(\delta_r))$ is the solution obtained by the algorithm. If the algorithm terminates at the r-th iteration without obtaining a solution then we see that

$$(w^r,\ z^r,\ z_o^r) + \delta(\tilde{w}^r,\ \tilde{z}^r,\ \tilde{z}_o^r),\quad \delta \geq 0 \qquad (3.2.4)$$

is a solution ray to (3.2.1). The solution ray

$$(w^1,\ z^1,\ z_o^1) + \delta(d,0,1),\ \delta \geq 0 \qquad (3.2.5)$$

is called the primary ray of solution to (3.2.1) and (3.2.4) is called the secondary ray. It is well known that the primary and the secondary rays are different. (See Murty (1988)). We note that in the secondary ray (3.2.4) $(\tilde{w}^r,\ \tilde{z}^r,\ \tilde{z}_o^r) \geq 0$, and in the primary ray $z^1 = 0$ because $d > 0$.

7. The following argument shows that $\tilde{z}^r \neq 0$ in case of secondary ray termination.

Suppose $\tilde{z}^r = 0$. Since $(\tilde{w}^r,\ 0,\ \tilde{z}_o^r)$ satisfies (3.2.2) with $k = r$ we see that $\tilde{w}^r > 0$. Then $(w^r + \delta\tilde{w}^r)^T z^r = 0$ implies that $z^r = 0$. This implies that $(w^r,\ z^r,\ z_o^r) = (w^r,\ 0,\ z_o^r)$ lies on the primary ray which is a cotradiction. Therefore $\tilde{z}^r \neq 0$.

8. The secondary ray can be of two types: We call (3.2.4) a type I secondary ray if $\tilde{z}_o^r = 0$ and type II secondary ray if $\tilde{z}_o^r \neq 0$. Termination of the algorithm in type I secondary ray implies that $(\tilde{w}^r,\ \tilde{z}^r)$ is a nonzero solution to (0,M). Type II secondary ray termination implies that $(\tilde{w}^r,\ \tilde{z}^r)$ is a solution to $(\tilde{z}_o^r\, d,\ M)$.

9. Let B_k, $k = 1,2,\ldots,r$ be as introduced at para 5. Let C_k be the complementary matrix obtained from B_k by replacing $(B_k)_{.\alpha(k)}$ by y^k which is the entering vector at the k-th iteration. If the terminal complementary

matrix C_r be characterized by the set J so that $C_r = C(J)$ then it has been established (see Todd (1976)) that in the case of type II secondary ray termination det $M_J < 0$.

10. We say that (q,M) is processable by Lemke's algorithm if the secondary ray termination implies there is no solution to (q,M). We say $M \in R^{n \times n}$ is processable by Lemke's algorithm if (q,M) is processable for every $q \in R^n$.

11. The following well known lemma due to Farkas is useful in identifying classes of matrices processable by Lemke's algorithm.

Lemma 3.2.1: Given $M \in R^{n \times n}$ and $q \in R^n$ either (3.1.1a) has a solution or there exists a nonnegative vector π such that $-\pi^T M \geq 0$ and $\pi^T q < 0$; but not both.

12. Suppose that the ACB B_k obtained at the k-th iteration of the algorithm is such that

$$(B_k^{-1}[I, -M])_{\alpha(k)\cdot} \leq 0 \tag{3.2.6a}$$

and

$$z_o^k > 0 \tag{3.2.6b}$$

Then from the definition of z_o^k and Farkas' lemma it follows that (q,M) has no solution. We note that the above situation may occur before the algorithm terminates (i.e. $k < r$).

13. In view of the observation at para 9, we can preclude type II termination by imposing the condition that if (w,z) is a solution to (d,M) with $z \neq 0$ then

"for every L such that $\{j:z_j>0\} \subseteq L \subseteq \{j:w_j=0\}$ det $M_L > 0$" (3.2.6)

The above condition is due to Todd (1976).

14. In case of type I termination we see that for C_r as defined at para.9 PosC_r is a blunt cone and $q + z_o^r d$ lies on the cone PosC_r. If we

ensure that $Pos C_r$ lies on the boundary of $Pos(I,-M)$ then there is no feasible solution to (q,M). To do this we impose a stronger condition that if $Pos\ C$ is a blunt cone then there exists $0 \neq \pi \geq 0$ such that

$$\pi^T C = 0 \quad \text{and} \quad \pi^T [I, -M] \geq 0 \tag{3.2.7}$$

Condition (3.2.7) is due to Doverspike (1982).

15. For a given positive $d \in R^n$ let us define $\tau(d)$ as the class of matrices in $R^{n\times n}$ that satisfy the condition (3.2.6). Let $\hat{\Delta}$ be the class of matrices in $R^{n\times n}$ satisfying (3.2.7). Let $T(d) = \tau(d) \cap \hat{\Delta}$. We see that the matrices in $T(d)$ are processable by Lemke's algorithm with the auxillary vector d.

16. Garcia (1973) introduced the following classes of matrices:

$E_1(d)$: $M \in E_1(d)$ if (w, z), $z \neq 0$ is a solution to (d,M) then there exist vectors $0 \neq x \geq 0$ and $y \geq 0$ such that $y = -M^T x$, $x \leq z$ and $y \leq w$.

$E(d)$: $E_1(d) \cap E_1(0)$

It is easy to verify that if $M \in E(d)$, $d > 0$ then (d,M) has only one solution namely, $(d,0)$. If $M \in E_1(0)$ then it follows that all the blunt cones of $(I, -M)$ lie on the boundary of $Pos(I, -M)$. Therefore, for $d > 0$ $E_1(d) \subseteq \tau(d)$ and $E_1(0) \subseteq \hat{\Delta}$. Hence $E(d) \subseteq T(d)$.

17. The following classes of matrices are due to Eaves (1971a).

L_1 : $M \in L_1$ if for every $0 \neq x \geq 0$ there exists i such that $x_i > 0$ and $(Mx)_i \geq 0$.

L_2 : $M \in L_2$ if for some $0 \neq x \geq 0$, $Mx \geq 0$ and $x^T Mx = 0$ then there exist nonnegative diagonal matrices Λ and Ω such that $\Omega x \neq 0$ and $(\Lambda M + M^T \Omega)x = 0$.

L : $L_1 \cap L_2$.

It is easy to verify that $L_1 = \bigcup_{d>0} E_1(d)$ and $L_2 = E_1(0)$ and therefore $L \subseteq T(d)$ for all $d > 0$.

18. From para 15-17 we get the relationship that for $d > 0$

$L \subseteq E(d) \subseteq T(d)$

Doverspike (1982) shows that L_2 is a proper subset of $\hat{\Delta}$. Todd (1976) shows that $E_1(d)$, $d > 0$ is a proper subset of $\tau(d)$.

19. As is described at para.5-7 of this section the set of almost complementary solutions (w, z, z_o) generated by Lemke's algorithm applied to (q,M) with auxillary vector d is:

$$S = \{(w^k(\delta), z^k(\delta), z_o^k(\delta)), \; 0 \leq \delta \leq \delta_k, \; k = 1,2,\ldots,r\}.$$

Because of the relationship (3.2.3) we see that S is connected.

Let

$$t_k = \sum_{i=1}^{k} \delta_i, \quad k = 1,2,\ldots,r;$$

$$t_o = 0$$

and for $t_k \leq t \leq t_{k+1}$

$$(w(t), z(t), z_o(t)) = (w^k(t-t_k), z^k(t-t_k), z_o^k(t-t_k)) \qquad (3.2.8)$$

We see that $(w(t), z(t), z_o(t))$ is continuous with respect to $t \in [0, t_r)$ with $z(0) = 0$. We make use of the above form in proving the following theorem.

Theorem 3.2.1: For a given $c \geq 0$ and positive vectors $\pi, d \in R^n$ there exists a $\theta_o \in R$ such that there is a solution $(\bar{w}, \bar{z})$ to $(q+\theta_o d, M)$ with $\pi^T z = c$.

Proof: Case 1: There is a $\hat{\theta} \in R$ such that $(q+\hat{\theta}d, M)$ is not solvable by Lemke's algorithm using the artificial vector d.

Applying Lemke's algorithm to $(q+\hat{\theta}d, M)$ with the help of the artificial vector d we see that the algorithm terminates without finding a solution at the r-th iteration. From (3.2.4) we see that $||z^r(\delta)|| \to \infty$ as $\delta \to \infty$ because $\tilde{z}^r \neq 0$. (See para 7). So, the continuous function, $z(t)$ defined at (3.2.8) is such that $||z(0)|| = ||z^1|| = 0$ and $||z(t)|| \to \infty$ as $t \to \infty$. π being a positive vector, there is a $t^* \geq 0$ such that $\pi^T z(t^*) = c$. Now we have

$$w(t^*) - Mz(t^*) - z_o(t^*)d = q + \hat{\theta}d.$$

By taking $\theta_o = \hat{\theta} + z_o(t^*)$, $\bar{z} = z(t^*)$ and $\bar{w} = w(t^*)$ the theorem is established.

<u>Case 2</u>: Lemke's algorithm solves $(q+\theta d,\ M)$ for all $\theta \in R$ with the artificial vector d.

Let Lemke's algorithm produce solution to $(q+\theta d, M)$ at the r-th iteration. From para 6 we note that $(w^r(\delta_r), z^r(\delta_r))$ is the solution to $(q+\theta d, M)$ obtained by the algorithm. (Note that r depends on θ). We shall show that

$$||z^r(\delta_r)|| \to \infty \text{ as } \theta \to -\infty \qquad (3.2.9)$$

Suppose (3.2.9) is not true. $||q+\theta d|| \to \infty$ as $\theta \to -\infty$ implies that $||w^r(\delta_r) - M z^r(\delta_r)|| \to \infty$ and hence $||w^r(\delta_r)|| \to \infty$ as $\theta \to -\infty$. Let $||M z^r(\delta_r)|| < c_o$ for all $\theta \in R$. Let $\hat{\theta} \in R$ be such that $q+\hat{\theta}d < 0$ and the algorithm applied to solve $(q+\hat{\theta}d, M)$ terminate at the $\hat{r}$-th iteration with $||w^{\hat{r}}(\delta_{\hat{r}})|| > nc_o$. This implies that there exists an $i \in \{1,2,\ldots,n\}$ such that $(w^{\hat{r}}(\delta_{\hat{r}}))_i > c_o$. Since $|Mz^{\hat{r}}(\delta_{\hat{r}})|| < c_o$ we get a contradiction that $[w^{\hat{r}}(\delta_{\hat{r}}) - Mz^{\hat{r}}(\delta_{\hat{r}})]_i = (q+\hat{\theta}d)_i > 0$. Therefore (3.2.9) is established.

Since $\pi > 0$ we can find a $\theta^* \in R^1$ such that Lemke's algorithm applied to $(q+\theta^* d,\ M)$ terminates at the r^*-th iteration with $\pi^T z^{r^*}(\delta_{r^*}) > c$. Now from the definition of $z(t)$ at (3.2.8) it follows that $\pi^T z(0) = 0$ and $\pi^T z(t_{r^*}) > c$. $z(t)$ being continuous there exists a $\bar{t} \in [0,\ t_{r^*})$ such that $\pi^T z(\bar{t}) = c$. We have

$$w(\bar{t}) - Mz(\bar{t}) - z_o(\bar{t})d = q + \theta^* d.$$

By taking $\theta_o = \theta^* + z_o(\bar{t})$, $\bar{w} = w(\bar{t})$ and $\bar{z} = z(\bar{t})$ we establish the theorem.

□

We see that the continuous functions $z(t)$ and $\pi^T z(t)$ are more general than the functions identified under the basic existence theorem of complementarity which we explain in para 20 below.

20. Let $T \subseteq R^n$ be a convex set and let $f\colon T \to R^n$. Let for $x \in T$

$$S(x) = \{y: y \in T,\ y^T f(x) \leq z^T f(x),\ z \in T\}$$

We say that x is a stationary point of (f,T) if $x \in S(x)$. We note that if f be the affine function $f(x) = Mx+q$, $M \in R^{n\times n}$, $q \in R^n$ then x is a stationary point of (f, R^n_+) if and only if there is a $y \in R^n_+$ such that (y,x) solves (q,M). Further, if (y,x) solves (q,M) and $x^T d \leq k$ where d is any positive vector, then x is a stationary point of (f, D^n_k) where $D^n_k = \{y: y^T d \leq k\}$. The above definition and remarks, and the following theorem are available in Eaves (1971b).

Theorem 3.2.2: (Lemke): Let $f : R^n_+ \to R^n$ be the affine function $f(x) = Mx+q$. When Lemke's algorithm is applied to (q,M) with the auxillary vector $d > 0$, the algorithm constructs a piecewise affine function $X : R^1_+ \to R^n_+$ such that (i) each $X(t)$ is a stationary point of (f, D^n_k) where $k = d^T x(t)$ and (ii) given a real number k, there exists a t such that X(t) is a stationary point of (f, D^n_k).

We observe that Theorem 3.2.1, in a sense, is an extension of Theorem 3.2.2.

21. Let $C(J)$, $J \subseteq \{1,2,\ldots,n\}$ be a complementary basis of (I,-M). (See Definition 3.2.1). Then $\tilde{M} = -(C(J))^{-1}C(J')$ is called the principal pivot transform (PPT) of M with respect to C(J).

Theorem 3.2.3: Let $\tilde{M}$ be the PPT of M with respect to the complementary basis C(J). Then for $K \subseteq \{1,2,\ldots,n\}$

$$\det \tilde{M}_K = \frac{\det M_{K \Delta J}}{\det M_J}$$

where 'Δ' stands for the symmetric difference of two sets.

Proof: We note that $\begin{bmatrix} C(J') & C(J) \\ I & 0 \end{bmatrix}$ is nonsingular and

$$\begin{bmatrix} C(J') & C(J) \\ I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ -(C(J))^{-1} & \tilde{M} \end{bmatrix} \tag{3.2.10}$$

Applying Theorem 2.5.1 to (3.2.10) we get for $K \subseteq \{1,2,\ldots,n\}$

$$\det \tilde{M}_k = \frac{\det \begin{bmatrix} C(J') & (C(J))_{.K'} \\ I_{K'.} & 0 \end{bmatrix}}{\det \begin{bmatrix} C(J') & C(J) \\ I & 0 \end{bmatrix}}$$

$$= (-1)^{(n-1)|K|} \frac{\det \begin{bmatrix} (C(J'))_{.K'} & C(K \Delta J) \\ I_{K'} & 0 \end{bmatrix}}{(-1)^n (-1)^{|J|} \det M_J}$$

$$= \frac{(-1)^n (-1)^{|J|} \det M_{K \Delta J}}{(-1)^n (-1)^{|J|} \det M_J}$$

$$= \frac{\det M_{K \Delta J}}{\det M_J} .$$

□

3. Generalized Inverse and some classes of LCP

Let $M \in R^{n \times n}$ be of rank n-1 and M^- be a {1}-inverse of M. Then according to Theorem 2.3.2 there exist vectors g, h and a real number c such that $M^- = M(g,h,c)$ and

$$\begin{bmatrix} M & g \\ h^T & c \end{bmatrix}^{-1} = \begin{bmatrix} M^- & d \\ \pi^T & 0 \end{bmatrix} \tag{3.3.1}$$

where d and π are null vectors of M and M^T respectively.

Consider the equations relating to the LCP (q, M^-) :

$$w - M^- z = q, \quad w, z \geq 0 \tag{3.3.2a}$$

$$w^T z = 0 \tag{3.3.2b}$$

Premultiplying (3.3.2a) by M and re-arranging the terms in conjunction with (3.3.1) we get

$$z - Mw = -Mq + g\,\pi^T z, \quad z,w \geq 0 \tag{3.3.3a}$$

$$z^T w = 0 \tag{3.3.3b}$$

So, we see that if (w,z) is a solution to (q, M^-) then (z, w) is a solution to $(-Mq + g\,\pi^T z,\ M)$. By using this relationship and Theorem 3.2.1 we identify in this section two subclasses of Q_o-matrices of order n and rank n-1.

Theorem 3.3.1: Let $M \in R^{n\times n}$ of rank n-1 be such that $Md = 0$ and $\pi^T M = 0$ for vectors $d, \pi > 0$. Then M is a Q_o-matrix.

Proof: From Theorem 2.3.1 and Remark 2.3.2 we see that there exist vectors g and h such that

$$\begin{bmatrix} M & g \\ h^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} M^- & d \\ \pi^T & 0 \end{bmatrix} \tag{3.3.4}$$

If (w,z) is a solution to $(\bar{q}+\theta d,\ M^-)$ so that

$$w - M^- z = \bar{q}+\theta d \tag{3.3.5}$$

then we see that

$$z - Mw = -M\bar{q} + g(\pi^T z). \tag{3.3.6}$$

We shall show that

$$D(M) = \{q \in R^n | q = -M\bar{q} + cg,\ \bar{q} \in R^n,\ c \geq 0\}. \tag{3.3.7}$$

Since $g \notin R(M)$, we have

$$\{q \in R^n | q = -M\bar{q} + cg,\ \bar{q} \in R^n,\ c \in R^1\} = R^n$$

Suppose that for $q = -M\bar{q} + cg$, $\bar{q} \in R^n$; $c < 0$ and (q,M) has a solution (w,z) then

$$w - Mz = -M\bar{q} + cg \tag{3.2.8}$$

Multiplying (3.3.8) by π^T on the left we get

$$\pi^T w = c\, \pi^T g = c < 0$$

which is a contraction, because $\pi > 0$. Therefore

$$D(M) \subseteq \{q \in R^n | q = -M\bar{q} + cg,\ \bar{q} \in R^n,\ c \geq 0\}. \tag{3.3.9}$$

Now from Theorem 3.2.1 for $c \in R^1_+$ we get a $\theta_o \in R$ such that there is a solution (w,z) to $(\bar{q} + \theta_o d, M^-)$ with $\pi^T z = c$. In view of (3.3.5) and (3.3.6) we get

$$z - Mw = -M\bar{q} + cg. \tag{3.3.10}$$

(3.3.10) shows that every point q of the set on the right hand side of (3.3.9) is in D(M). This establishes (3.3.7). Since D(M) in (3.3.7) is a convex set M is a Q_o-matrix.

Remark 3.3.1: Every PPT of a Q_o-matrix is a Q_o-matrix. So we see that if there exists a PPT of M, say $\tilde{M}$, such that $\tilde{M}$ satisfies the conditions of Theorem 3.3.1, then M is a Q_o-matrix.

Remark 3.3.2: We see that if M satisfies the conditions of Theorem 3.3.1 then D(M) is a half space.

We now give examples to show that neither the rank condition nor the condition that $d, \pi > 0$ can be relaxed.

Example 3.3.1: The following example shows that the rank condition in Theorem 3.3.1 cannot be dropped

Let

$$M = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Rank (M) = 3-2 = 1. Note that $\pi^T = (1,2,1)$ and $d^T = (1,1,1)$ satisfy the condition $\pi^T M = 0$ and $Md = 0$.

Take $q = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$

It is easy to verify that q does not belong to any of the eight complementary cones. Hence $q \notin D(M)$. Note also that $\pi^T q > 0$. It is easy to check that $q \in Pos(I, -M)$. It follows that M is not a Q_o-matrix.

Example 3.3.2: The following example shows that the requirement that d be strictly positive in Theorem 3.3.1 cannot be dropped even when M has rank n-1. Let

$$M = \begin{bmatrix} 1 & -3 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

The rank of M = 2 = 3-1. Note that $\pi^T = [1, 2, 1]$ and $d^T = [1, 0, 1]$ are the unique vectors (unique upto scalar multiples) satisfying $\pi^T M = 0$ and $Md = 0$. Taking $q^T = [-1, 2, -2]$ we can verify that $q \notin D(M)$ but $q \in Pos(I, -M)$. Thus M is not a Q_o-matrix.

Example 3.3.3: The following example shows that the requirement that π be strictly positive in Theorem 3.3.1 cannot be relaxed even when M has rank n-1 and $d > 0$.

Let

$$M = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

The rank M = 1. We can take d^T as [1, 1]. The vector $\pi^T = [1, 0]$ satisfies $\pi^T M = 0$. It can be easily verified that M is not a Q_o-matrix.

Lemma 3.3.1: Let G be an n×n P-matrix. Given $d \geq 0$, $d \in R^n$, $q \in R^n$ let $(w(\theta), z(\theta))$ be a solution to $(q+\theta d, G)$ which exists and is unique because G is a P-matrix. Let $J = \{i: d_i > 0\}$. Then (i) there exists a $\theta^* \in R^1$ such that for $\theta \geq \theta^*$, $z_J(\theta) = 0$. (ii) $||z_J(\theta)|| \to \infty$ as $\theta \to -\infty$.

Proof: Let $(\bar{w}_{J'}, \bar{z}_{J'})$ be the unique solution to $(q_{J'}, A_{J'})$. Let $\theta^* = \min\{\theta : G_{JJ'} \bar{z}_{J'} + q_J + \theta d_J \geq 0\}$. Let $\bar{w}_J(\theta) = G_{JJ'} \bar{z}_{J'} + q_J + \theta d_J$. Then we note that $(w(\theta), z(\theta))$ with $w_J(\theta) = \bar{w}_J(\theta)$, $w_{J'}(\theta) = \bar{w}_{J'}$, $z_J(\theta) = 0$ and $z_{J'}(\theta) = \bar{z}_{J'}$ is a solution to $(q + \theta d, G)$ for all $\theta \geq \theta^*$. This proves (i).

Let C be a complementary basis corresponding to the solution to $(q+\theta d, G)$ for $\theta \in R^1$. Then

$$C^{-1}q + \theta C^{-1}d \geq 0 \tag{3.3.11}$$

Let

$$\theta_o = \inf\{\theta : C^{-1}q + \theta C^{-1}d \geq 0\}.$$

It follows then, that

$$C^{-1}q + \theta C^{-1}d \ngeq 0 \quad \text{for } \theta < \theta_o \tag{3.3.12}$$

Now let C_1 be the complementary basis corresponding to the solution to $(q + (\theta^*+\varepsilon)d, G)$ for a very small $\varepsilon > 0$ so that

$$\theta^* = \min\{\theta : C_1^{-1}q + \theta C_1^{-1}d \geq 0\}.$$

Put $\theta_1 = \theta^*$

Let C_2 be the complementary basis corresponding to the solution to $(q + (\theta_1 - \varepsilon)d, G)$ for a very small $\varepsilon > 0$. Let

$$\theta_2 = \inf\{\theta : C_2^{-1}q + \theta C_2^{-1}d \geq 0\}. \tag{3.3.13}$$

If $\theta_2 \neq -\infty$ we can proceed to obtain C_3 and θ_3 as at (3.3.13). The process of obtaining C_n and θ_n as described above cannot go on indefinitely because there are only 2^n distinct complementary bases and by (3.3.12) no two complementary bases obtained in the process can be identical. So, let the process stop at the m-th step with the complementary basis C_m and $\theta_m = -\infty$. This implies that

$$C_m^{-1}q + \theta C_m^{-1}d \geq 0 \quad \text{for all } \theta \leq \theta_{m-1} \tag{3.3.14}$$

(3.3.14) shows that

$$-C_m^{-1}d \geq 0 \tag{3.3.15}$$

Without loss of generality we can assume that

$d^T = [d_J^T, 0]$ and

$$G = \begin{bmatrix} G_J & G_{JJ'} \\ G_{J'J} & G_{J'} \end{bmatrix}$$

(3.3.14) leads to a solution $(\tilde{w}, \tilde{z})$ to the problem $(-d, G)$ with the complementary basis C_m. That is

$$\begin{bmatrix} \tilde{w}_J \\ \tilde{w}_{J'} \end{bmatrix} - \begin{bmatrix} G_J & G_{JJ'} \\ G_{J'J} & G_{J'} \end{bmatrix} \begin{bmatrix} \tilde{z}_J \\ \tilde{z}_{J'} \end{bmatrix} = \begin{bmatrix} -d_J \\ 0 \end{bmatrix} \qquad (3.3.16)$$

Since G is a P-matrix it is easily seen that in (3.3.16) $\tilde{z}_J \neq 0$. This implies that $(-C_m^{-1}d)_k > 0$ for some $k \in J$. This shows, in view of (3.3.14) that $||z(\theta)_J|| \to \infty$ as $\theta \to -\infty$.

Hence the lemma. □

Theorem 3.3.2: Let M be a square matrix of order n and rank n-1. Let d, π be nonnegative nonzero vectors in R^n such that $\pi^T M = 0$, $Md = 0$ and $J = \{i : d_i > 0\} = \{i : \pi_i > 0\}$. If M has a g-inverse G which is a P-matrix then M is a Q_0-matrix.

Proof: By Theorem 2.3.2 and Remark 2.3.2 we see that there exist $\alpha, \beta \in R^n$ and $x \in R^1$ such that

$$\begin{bmatrix} M & \alpha \\ \beta^T & x \end{bmatrix}^{-1} = \begin{bmatrix} G & d \\ \pi^T & 0 \end{bmatrix} \qquad (3.3.17)$$

We shall prove the theorem by showing that

$$D(M) = \{q : q = -M\bar{q} + \delta\alpha \text{ for } \bar{q} \in R^n,\ \delta \geq 0\}.$$

Proceeding as we did in the proof of Theorem 3.3.1 we see that

$$D(M) \subseteq \{q : q = -M\bar{q} + \delta\alpha,\ \bar{q} \in R^n,\ \delta \geq 0\}.$$

Now, let $q \in R^n$ be such that

$$q = -M\bar{q} + \delta\alpha$$

where $\bar{q} \in R^n$ and $\delta \geq 0$.

Look at the problem $(\bar{q} + \theta d, G)$. Since G is a P-matrix $(q+\theta d, G)$ has a unique solution for each θ and for each fixed $\bar{q}$. Let $(w(\theta), z(\theta))$ denote the solution to $(\bar{q} + \theta d, G)$. We have

$$w(\theta) - G\, z(\theta) = \bar{q} + \theta d.$$

From here it follows that

$$z(\theta) - Mw(\theta) = -M\bar{q} + \alpha\, \pi^T z(\theta) \qquad (3.3.18)$$

By the hypothesis of the theorem if $J = \{i : d_i > 0\}$ then $\pi_i > 0$ if and only if $i \in J$. By Lemma 3.2.1 we see that as $\theta \to -\infty$ there exists a $k \in J$ such that $z_k(\theta) \to \infty$. Thus $\pi^T z(\theta) \to \infty$ as $\theta \to -\infty$. Also there is a θ^* such that for $\theta > \theta^*$ $z_J(\theta) = 0$ which implies that for $\theta > \theta^*, \pi^T z(\theta) = 0$. Further by a result in parametric linear complementarity theory $z(\theta)$ and hence $\pi^T z(\theta)$ is a continuous function of θ, $-\infty < \theta < \infty$. (See for example Exercise 6.4 in Chapter 10 of Berman and Plemmons (1979)). These facts imply for a given $\delta \geq 0$ there is a $\hat{\theta}$ such that $\pi^T z(\hat{\theta}) = \delta$. (3.3.18) shows that $(z(\hat{\theta}), w(\hat{\theta}))$ is a solution to (q,M). Thus

$$D(M) = \{q : q = -M\bar{q} + \delta\alpha,\ \bar{q} \in R^n,\ \delta \geq 0\}.$$

It follows that D(M) is convex and hence M is a Q_o-matrix. □

Remark 3.3.3: Theorem 3.3.2 shows that a {2}-inverse $G(d, \pi, 0)$ of a P-matrix G is a Q_o-matrix if d and π are nonnegative such that $d_i > 0$ if and only if $\pi_i > 0$ for $i = 1,2,\ldots,n$.

We note that Theorem 3.3.1 is a generalization of Lemma 3 and Theorem 1 of Ramamurthy and Mohan (1987) who prove under assumptions stronger than those of Theorem 3.3.1, that M is a Q_o-matrix.

A P_o-matrix M with exactly one zero principal minor is called a P_1-matrix. We have

Corollary 3.3.1: (Cottle and Stone (1983). Let M be a P_1-matrix and not a Q-matrix. Then D(M) is a half space.

Proof: Let det $M_{J'} = 0$, and $J = \{1,2,\ldots n\} - J'$ where n is the order of M. If we take PPT with respect to the complementary basis C(J), then it follows from Theorem 3.2.3 that the PPT $\tilde{M}$ is of rank n-1 and every proper principal minor of $\tilde{M}$ is positive. If d is a null vector of $\tilde{M}$, then d does not have any zero component. If d has mixed signs, then $(I, -\tilde{M})$ does not have any blunt cone. In this case, $\tilde{M}$ being a P_o-matrix, we see that $\tilde{M}$ is a Q-matrix.(Also see Aganagic and Cottle (1979)). But then M should be a Q-matrix, which is a contradiction. Therefore we may assume that d > 0 and now it follows from Corollary 2.6.1 that there exists a vector $\pi > 0$ such that $\pi^T\tilde{M} = 0$. Therefore Remark 3.3.2 shows that $\tilde{M}$ and hence M is a Q_o-matrix with D(M) half space.

4. Algorithms

We present two algorithms in this section, Algorithm I and Algorithm II. Algorithm I solves (q,M) where M is a n×n matrix of rank (n-1) with positive vectors π and d satisfying $\pi^T M = 0$, $Md = 0$. Algorithm II solves (q,M) where M is a n×n matrix of rank (n-1) satisfying the conditions stated in Theorem 3.3.2.

Algorithm I:

Step 1: If $q \geq 0$, a solution to (q,M) is $w = q$, $z = 0$. The algorithm terminates. Otherwise go to step 2.

Step 2: For any two positive vectors α and β compute $\begin{bmatrix} M & \alpha \\ \beta & 0 \end{bmatrix}^{-1}$

Let it be $\begin{bmatrix} M^- & \bar{d} \\ \bar{\pi}^T & 0 \end{bmatrix}$. Go to step 3.

Step 3: Compute $c = \bar{\pi}^T q$. If $c < 0$ there is no solution to (q,M). The algorithm terminates. Otherwise go to step 4.

Step 4: Compute $\bar{q} = -M^{-}q$. If $c = 0$ let $\theta > 0$ be any number such that $\bar{q} + \theta\bar{d} \geq 0$. $w = 0$, $z = \bar{q} + \theta\bar{d}$ is a solution to (q,M). The algorithm terminates. If $c > 0$ go to step 5.

Step 5: Apply Lemke's algorithm to $(\bar{q}, M^{-})$ with the artificial vector $\bar{d}$. Let (w^k, z^k, z_o^k) be the almost complementary solution obtained at the k-th iteration. Compute $c_k = \bar{\pi}^T z^k$.

(a) If $c_k < c$ and the algorithm does not terminate in a secondary ray at the k-th iteration then continue with the (k+1)-th iteration.

(b) If $c_k < c$ and the algorithm temiantes in a secondary ray at the k-th iteration let $r = k$ and go to step 6.

(c) If $c_k \geq c$ then go to step 7.

Step 6: Let $(w^r, z^r, z_o^r) + \delta(\tilde{w}^r, \tilde{z}^r, \tilde{z}_o^r)$, $\delta \geq 0$ be the secondary as defined at (3.2.4). Let $\delta^* > 0$ be such that $\bar{\pi}^T(z^r + \delta^* \tilde{z}^r) = c$. $(z^r, w^r) + \delta^*(\tilde{z}^r, \tilde{w}^r)$ is a solution to (q,M). The algorithm terminates.

Step 7: If $c_k = c$ then (z^k, w^k) solves (q,M). If $c_k > c$ then let $(w^{k-1}(\delta), z^{k-1}(\delta), z_o^{k-1}(\delta))$ be as defined at para 5. Let δ^* be such that $\bar{\pi}^T z^{k-1}(\delta^*) = c$. Since $\bar{\pi}^T z^{k-1}(0) = \bar{\pi}^T z^{k-1} = c_{k-1} < c$ such a δ^* exists. $(z^{k-1}(\delta^*), w^{k-1}(\delta^*))$ solves (q,M). The algorithm terminates.

Theorem 3.4.1: Algorithm I either computes a solution to (q,M) or decides that there is no solution to (q,M) in a finite number of steps.

Proof: Step 3 of the algorithm decides if $q \in D(M)$ or not correctly in view of Theorem 3.3.1. Step 5 terminates after a finite number of iterations of Lemke's algorithm in one of the four possibilities a,b,c,or d: the algorithm terminates with a solution (w^k, z^k) to $(\bar{q}, M^{-})$ and $c_k = \bar{\pi}^T z^k < \bar{\pi}^T q = c$. However, in this case since $\beta^T M^{-} = 0$, $\beta^T \bar{q} = 0$. This implies that $\beta^T w^k = 0$ and hence $w^k = 0$. Thus $\bar{q} = -M^{-} z^k$. Therefore $(z^k, 0)$ solves $(-M\bar{q} + (\bar{\pi}^T z^k)\,\alpha,\ M)$. Thus we obtain

$$z^k = -M\bar{q} + (\bar{\pi}^T z^k)\,\alpha \geq 0$$

since $\bar{\pi}^T q > \bar{\pi}^T z^k$ it follows that $-M\bar{q} + (\bar{\pi}^T q)\ \alpha = q \geq 0$. However, by step (1) of the algorithm we reach step 5 only if $q \not\geq 0$. Thus case (d) does not arise. In each of the other cases we obtain a solution to (q,M) via steps 6 or 7. This completes the proof.

Example 3.4.1: Let $M = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $q = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Take $\alpha = \beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -0.25 & 0.25 & 0.50 \\ 0.25 & -0.25 & 0.50 \\ 0.50 & 0.50 & 0 \end{bmatrix}$$

We have $\bar{d} = \bar{\pi} = (0.50)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

According to step 3 of the algorithm, since $c = \bar{\pi}^T q = 0.50 > 0$ we go to step 5 and apply Lemke's algorithm to $(\bar{q}, M^-)$ where

$$M^- = \begin{bmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{bmatrix} \text{ and } \bar{q} = -M^- q = (0.75)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We obtain the following tableau:

TABLEAU

	w_1	w_2	z_1	z_2	z_0	$\bar{q}$
	1	0	0.25	-0.25	-0.50	0.75
	0	1	-0.25	0.25	-0.50	-0.75
w_1	1	-1	0.50	-0.50	0	1.50
z_o	0	-2	0.50	-0.50	1	1.50

Lemke's algorithm terminates at this stage with $c_k = 0$. From step 6 we see that $(1.50, 0, 0, 0, 1.50)^T + \delta(0.50, 0, 0, 1, 0.50)^T$, $\delta \geq 0$ is a solution to $w - M^- z - z_o\bar{d} = \bar{q}$. Now $\bar{\pi}^T z^k(\delta^*) = 0 + 0.50\ \delta^* = c = 0.50$ implies that $\delta^* = 1$. The solution to (q,M) is $([0, 1]^T, [2; 0]^T)$. Note that Lemke's algorithm does not solve (q,M).

Algorithm II:

We are given a vector $q \in R^n$, a matrix M, vectors α and β, and a real number x such that

$$\begin{bmatrix} M & \alpha \\ \beta^T & x \end{bmatrix}^{-1} = \begin{bmatrix} G & d \\ \pi^T & 0 \end{bmatrix}$$

where G is a P-matrix; d and π are vectors satisfying the hypothesis of Theorem 3.3.2. We are required to solve (q,M).

Step 1: If $q \geq 0$, $w = q$, $z = 0$ solves (q,M). The algorithm terminates. Otherwise go to step 2.

Step 2: Let $c = \pi^T q$. If $c < 0$ then (q,M) has no solution; the algorithm terminates. Otherwise go to step 3.

Step 3: Let $\bar{q} = -Gq$. Let d be such that $d_J > 0$ and $d_{J'} = 0$. Let $(\tilde{w}, \tilde{z})$ be the unique solution to $(q_{J'}, G_{J'})$. Choose θ large so that $\hat{w} = \bar{q}_J + G_{JJ'} \tilde{z} + \theta d_J > 0$. Let (w,z) be defined by $w_J = \hat{w}$, $w_{J'} = \tilde{w}$, $z_J = 0$, $z_{J'} = \tilde{z}$. (w,z) is a solution to $(\bar{q} + \theta d, G)$. If $c = 0$ go to step 4. Otherwise go to step 5.

Step 4: (z,w) solves (q,M) where w and z are as defined in step 3. The algorithm terminates.

Step 5: Let C_1 be the complementary basis corresponding to (w,z) in step 3, so that $C_1^{-1}\bar{q} + \theta C_1^{-1} d \geq 0$. Let $\theta_1 = \min\{\theta: C_1^{-1}\bar{q} + \theta C_1^{-1} d \geq 0\}$. Let $K = \{k : (C_1^{-1}q)_k + \theta_1 (C_1^{-1}d)_k = 0,\ (C_1^{-1}d)_k > 0\}$ and $i = \max\{k : k \in K\}$. Now we replace the i-th column of C_1 by the vector complementary to $(C_1)_{.i}$. Let the new complementary basis be C_2. We repeat the above procedure to get next basis C_3. For the basis C_k obtained in this process let $\theta_k = \min\{\theta: C_k^{-1}(\bar{q}+\theta d) \geq 0\}$. Let $c_k = \pi^T z^k$ where (w^k, z^k) solves $(\bar{q} + \theta_k d, G)$. If $c_k < c$ go on to the next basis. Otherwise go to step 6.

Step 6: If $(w(\theta), z(\theta))$ is the solution to $(\bar{q} + \theta d, G)$ then $z(\theta)$ is continuous with respect to θ. Therefore let $\theta^* \in (\theta_{k-1}, \theta_k]$ be such that

$\pi^T z(\theta^*) = c$. There $(z(\theta^*, w(\theta^*))$ solve (q,M). The algorith terminates.

Lemma 3.4.1: The complementary bases $c_1, c_2, \ldots$ of the above agorithm are distinct.

Proof: See Murty (1988, pp. 280-285).

□

Theorem 3.4.2: Algorithm II terminates in a finite number of steps.

Proof: This follows from Lemma 3.3.1, Theorem 3.3.2 and Lemma 3.4.1. (See also Murty (1988)).

□

Example 3.4.2: Consider

$$M = \begin{bmatrix} -6 & 3 & -1 \\ 6 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Note that rank $(M) = 2$. Here we can take $\pi^T = (1, 1, 0)$, $d^T = (1, 2, 0)$. Also we have

$$\begin{bmatrix} -6 & 3 & -1 & 11 \\ 6 & -3 & 1 & -10 \\ -2 & 1 & 0 & 3 \\ -5 & 3 & -1 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

where we have taken $\alpha^T = (11, -10, 3)$, $\beta^T = (-5, 3, -1)$ and $x = 9$.

Note that $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ is a P-matrix.

Algorithm II applies to this example, since all the hypothesis of Theorem 3.3.2 hold. Note that M is an N_o-matrix. For such a matrix Lemke's algorithm does not apply. See Saigal (1972). Also there is no known method other than enumeration methods, to solve the LCP with such a matrix.

Let

$$q = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$q \in D(M)$ because $\pi^T q = c = 1 > 0$. Hence algorithm II takes us to step 3 as $q \not\geq 0$.

$$\text{Now } \bar{q} = -Gq - \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

Algorithm II produces the following tableau.

	w_1	w_2	w_3	z_1	z_2	z_3	$\bar{q}$	d
w_1	1	0	0	-1	-2	0	0	1
w_2	0	1	0	1	-1	-1	2	2
w_3	0	0	1	-1	-2	-3*	-3	0
w_1	1	0	0	-1*	-2	0	0	1
w_2	0	1	-1/3	4/3	-1/3	0	3	2
z_3	0	0	-1/3	1/3	2/3	1	1	0

Now the above tableaux gives us a solution to $(\bar{q} + \theta d, G)$ whenever $\theta \geq 0$.

For $\theta \geq 0$ note that $z(\theta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We take $\theta_1 = 0$ and $\pi^T z(\theta_1) = c_1 = 0 < \pi^T q = 1$.

Hence we continue with the algorithm by introducing the complement of w_1, i.e. z_1 into the basis, since $w_1(\theta_1) = 0$. The * entry in tableaux 2 is used as pivot element.

	w_1	w_2	w_3	z_1	z_2	z_3	$\bar{q}$	d
z_1	-1	0	0	1	2	0	0	-1
w_2	4/3	1	-1/3	0	-3*	0	3	10/3
z_3	1/3	0	-1/3	0	0	1	1	1/3

The above tableau gives a solution to $(\bar{q} + \theta d, G)$ for $-9/10 \leq \theta \leq 0$. Hence $\theta_2 = -9/10$ and w_2 drops out. $\pi^T z(\theta_2) = 9/10 < c$. Hence we continue by pivoting in z_2

	w_1	w_2	w_3	z_1	z_2	z_3	$\bar{q}$	d
z_1	-1/9*	2/3	-2/9	1	0	0	2	11/9
z_2	-4/9	-1/3	1/9	0	1	0	-1	-10/9
z_3	1/3	0	-1/3	0	0	1	1	1/3

The above gives a solution to $(\bar{q} + \theta d, G)$ for $-18/11 \leq \theta \leq -9/10$. $\theta_3 = -18/11$, $\pi^T z(-18/11) = 9/11 < 1$. At $\theta = -18/11$, z_1 reduces to 0 and w_1 enters the basis.

	w_1	w_2	w_3	z_1	z_2	z_3	$\bar{q}$	d
w_1	1	-6	2	-9	0	0	-18	-11
z_2	0	-3	1	-4	1	0	-9	-6
z_3	0	2	1	3	0	1	7	4

The above tableau gives us a solution to $(q + \theta d, G)$ when $-7/4 < \theta < -18/11$. $\theta_4 = -7/4$, $\pi^T z(\theta_4) = 6/4 > c = 1$. Thus, the desired θ^* is in the interval $[-7/4, -18/11]$. We find that $-9 - 6\,\theta^* = 1$ gives us the solution $\theta^* = -10/6$. Now $w(\theta^*) = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix}$, $z(\theta^*) = \begin{bmatrix} 0 \\ 1 \\ 1/3 \end{bmatrix}$ solves $(\bar{q}+\theta^* d, G)$.

Thus $(z(\theta^*), w(\theta^*))$ solves (q, M). Hence the algorithm produces the solution $(z(\theta^*), w(\theta^*))$ for (q,M).(Note that algorithm I cannot be applied to this problem).

All the results of Section 3 and Section 4 are available in Eagambaram and Mohan (1987b).

5. Solution rays

If $(w + \lambda u, z + \lambda v)$ with $(u,v) \geq 0$, $v \neq 0$ is a solution to (q,M) for all $\lambda \geq 0$, then $(w,z) + \lambda(u,v)$, $\lambda \geq 0$ is called a solution ray to (q,M) at the solution (w,z) to (q,M).

The following lemmas are easy to verify.

Lemma 3.5.1: Let $q \in D(M)$. There is a solution ray to (q,M) at the solution (w,z) if and only if there is a complementary matrix C of $(I, -M)$ such that

C contains the vectors of the set $\{I_{.j} : w_j > 0\} \cup \{-M_{.j} : z_j > 0\}$ as its columns, $q \in$ PosC and PosC is a blunt cone.

Lemma 3.5.2: Let PosC be a blunt cone of $(I, -M)$. Then for $q \in$ PosC (q, M) has a solution ray.

Let $S(M) = \{q : (q,M)$ has a solution ray$\}$

and $S^*(M) = \{q : (q,M)$ has a solution ray at every solution (w,z) to $(q,M)\}$

We see that

$S^*(M) \subseteq S(M)$ = the union of all blunt cones of $(I, -M)$ (3.5.1)

Let $\tilde{D}(M)$ denote the boundary of $D(M)$.

Cottle (1974) proved Theorem 3.5.1 for a copositive plus matrix which is defined as the matrix M that satisfies the following conditions:

(i) $x^T Mx \geq 0$ for all $x \geq 0$ and

(ii) $x^T Mx = 0$, $x \geq 0$ implies that $(M + M^T)x = 0$.

Theorem 3.5.1: Let M be a copositive plus matrix. Then $S^*(M) = \tilde{D}(M)$.

After proving the above theorem Cottle remarks : "Ideally one would like a theorem which completely characterizes the class of matrices for which the result established here is valid". (See Cottle (1974) pp. 69).

In this section we give a nearly complete answer to Cottle's remark by obtaining a necessary and sufficient condition for $\tilde{D}(M) \subseteq S^*(M)$ when M is a Q_o-matrix. We prove that when $M \in T(d)$ or M satisfies the conditions of Theorem 3.3.1, $\tilde{D}(M) = S(M)$. We also show that $S^*(M) = \tilde{D}(M)$ when M satisfies the conditions of Theorem 3.3.1.

Theorem 3.5.2: Let $M \in R^{n\times n}$ be a Q_o-matrix. Then $\tilde{D}(M) \subseteq S^*(M)$ if and only if the following condition (α) holds.

Condition (α): Suppose C_1 is a $n\times k$ submatrix of a complementary matrix of $(I, -M)$ and suppose there is a $0 \neq x \geq 0$, $x \in R^n$ such that $x^T C_1 = 0$,

$-x^T M \geq 0$. Then there is a complementary matrix C of (I, -M) containing the columns of C_1 and a $0 \neq u \geq 0$, $u \in R^n$ such that $Cu = 0$.

Proof: Let M be a Q_o-matrix and condition (α) hold. Let $q \in \tilde{D}(M)$ and (w,z) be a solution to (q,M). We shall prove that condition (α) is sufficient by showing that there is a solution ray at (w,z).

Since $q \in \tilde{D}(M)$ and D(M) is a convex set there is a supporting hyperplane to D(M) containing q. In other words there exists a $0 \neq x \geq 0$ with

$$-x^T M \geq 0, \quad x^T q = 0 \qquad (3.5.2)$$

Let C_1 consist of the columns $\{I_{.j}: w_j > 0\} \cup \{-M_{.j}: z_j > 0\}$. Clearly C_1 is a submatrix of a complementary matrix of (I, -M) and from (3.5.2) it follows that

$$x^T C_1 = 0, \; -x^T M \geq 0, \quad 0 \neq x \geq 0 \qquad (3.5.3)$$

From Condition (α) we see that there is a complementry matrix C and a $0 \neq u \geq 0$, $u \in R^n$ such that $Cu = 0$ and C contains all the columns of C_1. This u gives rise to a solution ray at (w,z) by lemma 3.5.1.

Suppose now for each $q \in \tilde{D}(M)$ there is a solution ray at each solution to (q,M). We shall prove that Condition (α) holds.

Let C_1 be a n×k submatrix of a complementary matrix of (I, -M). Suppose there is a $0 \neq x \geq 0$, $x \in R^n$ such that $x^T C_1 = 0$, $-x^T M \geq 0$. Let $q = \sum_{j=1}^{k} (C_1)_{.j}$. Clearly $x^T q = 0$. From the properties of x and the fact that D(M) is convex it follows that $q \in \tilde{D}(M)$. Without loss of generality we can assume that

$$\{(C_1)_{.r}\} \in \{I_{.r}, -M_{.r}\}, \quad r = 1,2,\ldots,k.$$

Let for $r \in \{1,2,\ldots,k\}$

$$w_r = 1 \quad \text{if } (C_1)_{.r} = I_{.r}$$

$$= o \quad \text{otherwise}$$

$z_r = 1$ if $(C_1)_{.r} = -M_{.r}$

$= 0$ otherwise

Let $w_j = z_j = 0$ for $j \notin \{1,2,\ldots,k\}$. Then (w,z) is a solution to (q, M). By our hypothesis there is a solution ray $(w,z) + \lambda(\bar{u},\bar{v})$ to (q,M). Let C_2 be the submatrix of $(I, -M)$ defined as follows:

$I_{.j}$ is a column of C_2 if and only if $\bar{u}_j > 0$

$-M_{.j}$ is a column of C_2 if and only if $\bar{v}_j > 0$.

Let C be any complementary matrix of $(I, -M)$ containing the columns of C_1 and C_2. From the fact that $(w+\lambda\bar{u}, z+\lambda\bar{v})$ solves (q,M) for all $\lambda \geq 0$, it is clear that such a C exists. It is easy to see that for $u = \bar{u}+\bar{v}$ $Cu = 0$. This completes the proof of the theorem. □

Theorem 3.5.3: Let $M \in T(d)$, $d > 0$. Then $S(M) = \tilde{D}(M)$.

Proof: $T(d) = \tau(d) \cap \hat{\Lambda}$ (See para 15 Section 2). From the definition of $\hat{\Lambda}$ it follows that $S(M) \subseteq \tilde{D}(M)$. Let $q \in \tilde{D}(M)$. The point $q-d \notin D(M)$ otherwise we would get a contradiction that q is in the interior of $D(M)$ because $D(M)$ is convex. If we apply Lemke's algorithm to $(q-d, M)$ with the auxilary vector d the algorithm terminates in type I secondary ray termination. This implies that there exists a solution ray $(w,z) + \lambda(u,v)$, $\lambda \geq 0$ to the LCP $(q-d+\theta d,M)$ for some $\theta > 0$. (See para 6-8 of Section 2). Invoking lemma 3.5.1 we find that $q+(\theta-1)d$ lies in a blunt cone PosC. Since $M \in \hat{\Lambda}$, $\text{PosC} \subseteq \tilde{D}(M)$ and hence $q+(\theta-1)d \in \tilde{D}(M)$. This can happen only when $\theta = 1$. Therefore $q \in S(M)$. Hence $\tilde{D}(M) \subseteq S(M)$. This completes the proof.

□

Theorem 3.5.4: Let $M \in T(d)$, $d > 0$. Then $S^*(M) = \tilde{D}(M)$ if and only if Condition (α) of Theorem 3.5.2 holds.

Proof: From Theorem 3.5.3 we see that $S^*(M) \subseteq \tilde{D}(M)$. From Theorem 3.5.2 we find that condition (α) is necessary and sufficient for $\ddot{D}(M) \subseteq S^*(M)$. Hence the theorem. □

Remark 3.5.1: Doverspike (1982) gives an example of a matrix $M \in E(d)$, $d > 0$, a $q \in \tilde{D}(M)$ and a solution (w,z) to (q,M) such that at (w,z) there is no solution ray. So, condition (α) in Theorem 3.5.4 cannot be dispensed with.

Remark 3.5.2: It may be tempting to extend the result of Theorem 3.5.3 to the class of matrices which are processable by Lemke's algorithm. However, the following example of an Z-matrix which is processable by Lemke's algorithm (see Saigal (1970)), shows that a blunt cone Pos(-M) has nonempty intersection with the interior of R^4_+.

Example 3.5.1:

$$M = \begin{bmatrix} 2 & -2 & -2 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -4 & -2 \end{bmatrix}$$

Theorem 3.5.5: Let $M \in R^{n \times n}$ be of rank n-1 with $\pi^T M = 0$ and $Md = 0$ where π and d are positive vectors. Then (i) $M \in \hat{\Delta}$ and (ii) $S^*(M) = \tilde{D}(M)$.

Proof: (i) Suppose PosC(J), $J \subseteq \{1,2,\ldots,n\}$ is a blunt cone. Then there exists a vector $0 \neq x \geq 0$ such that $C(J)x = 0$. This implies that $My \geq 0$ for y such that $y_J = x_J$ and $y_{\bar{J}} = 0$. But we see that $\pi^T My = 0$ and so $My = 0$. The rank of M being n-1 we see that $y = d$. Therefore the only blunt cone of (I,-M) is Pos(-M). By Farkas' lemma we have, by virtue of $\pi^T M = 0$, that $Pos(-M) \subseteq \tilde{D}(M)$ and hence $M \in \hat{\Delta}$.

(ii) Suppose C_1 is a submatrix of a complementary matrix of (I,-M) such that there exists a vector $0 \neq x \geq 0$ satisfying $x^T C_1 = 0$ and $-x^T M \geq 0$. Then we find that

$-x^TMd=0$ implies that $x^TM=0$. The rank of M being n-1 we have that $x = \pi > 0$. So, C_1 consists of columns of -M only and hence M=C is a complementary matrix with Cd = 0. Therefore we see that Condition (α) of Theorem 3.5.2 is satisfied by M. M being a Q_o-matrix (vide Theorem 3.3.1) Theorem 3.5.2 shows that $S^*(M) = \tilde{D}(M)$. □

All the results of this section, excepting Theorem 3.5.5, are essentially taken from Eagambaram and Mohan (1987a).

6. LCP with N_o-matrices

In the process of examining the properties of Lemke's algorithm we could identify two subclasses of Q_o-matrices in Section 3. Further we found that Lemke's algorithm is not applicable to these classes of matrices. However we obtained algorithm via generalized inverse.

In this section we examine N_o-matrices in the light of Lemke's algorithm via principal pivot transforms. Saigal (1972) showed that an N-matrix with at least one positive entry is a Q-matrix and for obtaining a solution to (q,M) when M is a N-matrix, Lemke's algorithm can be applied to $(-M^{-1}q, M^{-1})$. A solution to $(-M^{-1}q, M^{-1})$ leads to a corresponding solution to (q,M). Here we show a similar result in respect of a nonsingular N_o-matrix.

Let us define

Condition (β): There exists a $J \subseteq \{1,2,\ldots,n\}$, $J \neq \phi$ such that $\det M_J \neq 0$ and $M_{.J}$ contains at least one positive entry.

It is easy to check that if $M \in R^{n\times n}$ is a Q-matrix then R^n is the union of all nondegenerate cones of (I,-M). If for all $\det M_J \neq 0$ $M_{.J} \leq 0$ then we see that the union of nondegenerate cones is contained in R^n_+. Therefore Condition (β) is a necessary condition for M to be a Q-matrix.

Lemma 3.6.1: Let $M \in R^{n\times n}$ be an N_o-matrix satisfying Condition (β). Then there exists a PPT $\tilde{M}$ of M and a positive vector $\tilde{d}$ such that $\tilde{M} \in \tau(\tilde{d})$.

Proof: Let $J \subseteq \{1,2,\ldots,n\}$, $J \neq \phi$ be such that det $M_J < 0$ and $M_{.J}$ contains at least one positive element. Let S be the union of the boundaries of all the complementary cones of $(I, -M)$. S is a closed set. We see that $(\text{Pos}C(J) - R^n_+) - S$ is an open set. Let $\tilde{M}$ be the PPT of M with respect to C(J). Then it can be verified that $\tilde{C}(K) = (C(J))^{-1}C(J \Delta K)$, $K \subseteq \{1,2,\ldots,n\}$ are the complementary matrices of $(I, -\tilde{M})$. So, Pos (C(J)) is transformed into R^n_+ and R^n_+ is transformed into Pos $(\tilde{C}(J))$. Let d be a vector in $(\text{Pos}(C(J)) - R^n_+) - S$. Then the PPT transforms d into $\tilde{d} > 0$. Again $d \notin R_+$ implies that $\tilde{d} \notin \text{Pos}(\tilde{C}(J))$. Also we see that $\tilde{d}$ does not lie on the boundary of any cone of $(I, -\tilde{M})$. Now, from Theorem 3.2.3 we see that

$$\det \tilde{C}(K) \geq 0 \text{ for all } K \neq J.$$

Therefore $\tilde{M} \in \tau(\tilde{d})$. □

Remark 3.6.1: If $M \in \hat{\Delta}$ and satisfies Condition (β) then there exist a PPT $\tilde{M}$ of M and a $\tilde{d} > 0$ such that $\tilde{M} \in T(\tilde{d})$, and therefore M is a Q_o-matrix.

A matrix M is said to be an R_o-matrix if there is no blunt cone of $(I, -M)$. Now we prove

Theorem 3.6.1: A nonsingular N_o-matrix $M \in R^{n\times n}$ with at least one positive entry is a Q-matrix if and only if M is an R_o-matrix.

Proof: Sufficiency: M satisfies Condition (β) and so by Lemma 3.6.1 $M^{-1} \in \tau(d)$ for some $d > 0$. If M is an R_o-matrix so is M^{-1}. Therefore $M^{-1} \in T(d)$. For any $q \in R^n$, Lemke's algorithm applied to (q, M^{-1}) does not end up with a secondary ray and hence obtains a solution to (q, M^{-1}). Therefore M^{-1} is a Q-matrix which implies M is also a Q-matrix.

Necessity: Suppose M is not an R_o-matrix. Then $\tilde{M} = M^{-1}$ is also not an R_o-matrix. Let Pos $\tilde{C}(K)$ be a blunt cone contained in $D(\tilde{M})$, which gives rise to a nonzero solution (u,v) to $(0,\tilde{M})$. This means

$$\begin{bmatrix} 0 \\ u_{\bar{K}} \end{bmatrix} - \begin{bmatrix} \tilde{M}_K & \tilde{M}_{K\bar{K}} \\ \tilde{M}_{\bar{K}K} & \tilde{M}_{\bar{K}} \end{bmatrix} \begin{bmatrix} v_K \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3.6.1)$$

where $K \subseteq \{1,2,\ldots,n\}$ and $\bar{K} = \{1,2,\ldots,n\} - K$. Let $q \in R^n$ be such that $q_K = -e(k)$ and $q_{\bar{K}} = e(\bar{K})$ where $e(L)$, $L \subseteq \{1,2,\ldots,n\}$ is an $|L|$-dimensional vector with all its elements unity. We claim that $(q, \tilde{M})$ has no solution. Suppose there is a solution (w,z), then

$$\begin{bmatrix} w_K \\ w_{\bar{K}} \end{bmatrix} - \begin{bmatrix} \tilde{M}_K & \tilde{M}_{K\bar{K}} \\ \tilde{M}_{\bar{K}K} & \tilde{M}_{\bar{K}} \end{bmatrix} \begin{bmatrix} z_K \\ z_{\bar{K}} \end{bmatrix} = \begin{bmatrix} -e(K) \\ e(\bar{K}) \end{bmatrix} \qquad (3.6.2)$$

From Theorem 3.2.3 we see that when L is a proper subset of $\{1,2,\ldots,n\}$, $\tilde{M}_L$ is a P_o-matrix. It is known that a P_o-matrix has sign non-reversal property. That is, if A is a P_o-matrix then for any nonzero vector x there exists an index i such that $x_i \neq 0$ and $x_i (Ax)_i \geq 0$. (See Berman and Plemmons (1979)). This shows that $z_K \neq 0$ otherwise we would get a contradiction that the P_o-matrix $\tilde{M}_{\bar{K}}$ violates the sign non-reversal property in respect of $z_{\bar{K}}$. Let α be the positive number such that $v_K - \alpha z_K \geq 0$ and at least one component of $v_K - \alpha z_K$ is zero. We see that $(u_{\bar{K}} - \alpha w_{\bar{K}} + \alpha e(\bar{K}))_i > 0$ whenever $(z_{\bar{K}})_i > 0$, $i \in \bar{K}$. Now from (3.6.2) we see that a proper principal submatrix of $\tilde{M}$ voilates the sign non-reversal property in respect of the vector $\begin{bmatrix} v_K - \alpha z_K \\ -\alpha z_{\bar{K}} \end{bmatrix}$. This contradicts the fact that every proper principal submatrix of $\tilde{M}$ is a P_o-matrix. Therefore the assumption that there is a solution to $(q, \tilde{M})$ is untenable.

Hence the theorem. □

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